Dissertation

# On the Construction of Wavelets and Multiwavelets for General Dilation Matrices

Anne Kopsch

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# On the Construction of Wavelets and Multiwavelets for General Dilation Matrices

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# Zusammenfassung


Wavelets haben große Bedeutung im Bereich der Angewandten Mathematik erlangt, wo sich Konzepte der Numerik, der Signalanalyse sowie der Funktionalanalysis verknüpfen. Der Begriff Wavelet wurde von A. Grossman und J. Morlet Anfang der 1980er Jahre eingeführt und ist eine Übersetzung des französischen Wortes "ondelette", welches "kleine Welle" bedeutet. Wavelets werden zur Konstruktion von Basen des Raumes der quadratintegrierbaren Funktionen $L_2(\mathbb{R}^n)$ verwendet. Diese Basen entstehen durch skalieren, dilatieren und translatieren einer endlichen Menge $\{\psi_i\}_{i\in I}$ von Funktionen, die auch Mother Wavelets genannt werden:

$$\{\psi_{i,j,k}(x) := 2^{jn/2}\psi_i(2^j x - k), i \in I, j \in \mathbb{Z}, k \in \mathbb{Z}^n\}.$$

Da sich Signale für gewöhnlich als Funktionen $f \in L_2(\mathbb{R}^n)$ modellieren lassen, können wir solche Basen nutzen, um eine Waveletdarstellung des Signals $f$ zu erhalten:

$$f(x) = \sum_{i\in I}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n} c_{i,j,k}\psi_{i,j,k}(x). \tag{0.1}$$

Die Mother Wavelets können so konstruiert werden, dass sie exponentiell abfallend sind oder einen kompakten Träger haben. Folglich können wir eine Darstellung erhalten, die lokal in der Zeit ist und bei der sich kleine Änderungen des Signals nur auf wenige Koeffizienten $c_{i,j,k}$ auswirken. Dies ist ein Vorteil gegenüber der Fouriertransformation, die nur im Frequenzbereich lokalisiert ist. Andere wünschenswerte Eigenschaften sind eine hohe Anzahl an verschwindenden Momenten, welche eine dünne Darstellung von $f$ zur Folge hat oder Orthogonalität, welche zu einer einfachen Berechnung der Koeffizienten $c_{i,j,k}$ führt.

Ein Beispiel für eine Waveletbasis ist die sogenannte Haar-Basis, welche von A. Haar im Jahr 1910 konstruiert wurde, siehe [43]. Das Mother Wavelet ist definiert durch

$$\psi(x) := \begin{cases} 1 \quad , & 0 \le x < \frac{1}{2}, \\ -1 \, , & \frac{1}{2} \le x < 1, \\ 0 \quad , & \text{sonst.} \end{cases}$$

Für weitere Beispiele verweisen wir auf [75, 60, 29].

Um Waveletbasen zu konstruieren, nutzt man einen systematischen Ansatz, der unter dem Namen Multiskalenanalyse bekannt ist. Dieser Ansatz wurde von S. Mallat und Y. Meyer im Jahr 1986 entwickelt, siehe [57, 62].

**Definition 0.1.**
Eine Folge abgeschlossener Unterräume $\{S_j\}_{j\in\mathbb{Z}}$, $S_j \subset L_2(\mathbb{R})$, heißt *Multiskalenanalyse*, wenn die folgenden Bedingungen erfüllt sind:

(M1) $\ldots \subset S_{j-1} \subset S_j \subset S_{j+1} \subset \ldots,$

(M2) $\overline{\bigcup_{j=-\infty}^{\infty}} S_j = L_2(\mathbb{R}),$

(M3) $\bigcap_{j=-\infty}^{\infty} S_j = \{0\},$

(M4) $f \in S_j \Leftrightarrow f(2\,\cdot) \in S_{j+1},$

(M5) es existiert ein Generator $\varphi \in S_0$, auch *Skalierungsfunktion* genannt, dessen Translate $\{\varphi(\cdot - k), k \in \mathbb{Z}\}$ eine Riesz-Basis für $S_0$ bilden.

Dass die ganzzahligen Translate von $\varphi$ eine *Riesz-Basis* für $S_0$ bilden sollen bedeutet, dass

$$S_0 := \overline{\operatorname{span}\{\varphi(\cdot - k), k \in \mathbb{Z}\}} \tag{0.2}$$

und dass die ganzzahligen Translate von $\varphi$ *stabil* sind. Stabil heißt, es existieren Konstanten $C_1, C_2 > 0$, sodass

$$C_1\|c\|_{\ell_2(\mathbb{Z})} \leq \left\| \sum_{k\in\mathbb{Z}} c_k\varphi(\cdot - k) \right\|_{L_2(\mathbb{R})} \leq C_2\|c\|_{\ell_2(\mathbb{Z})}, \quad c = \{c_k\}_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z}).$$

Die Bedingungen (M1), (M4) und (M5) stellen sicher, dass die Skalierungsfunktion $\varphi$ *verfeinerbar* ist. Das bedeutet $\varphi$ erfüllt die *Verfeinerungsgleichung*

$$\varphi(x) = \sum_{k\in\mathbb{Z}} a_k\varphi(2x - k) \tag{0.3}$$

mit der *Maske* $a = \{a_k\}_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$. Aufbauend auf dem Konzept der Multiskalenanalyse werden Wavelets als Basen der Komplementräume $W_j := S_{j+1} \ominus S_j, j \in \mathbb{Z}$, konstruiert. Aus den Bedingungen (M4) und (M5), bzw. (0.2), ergibt sich, dass sich das Wavelet $\psi$ durch die Gleichung

$$\psi(x) = \sum_{k\in\mathbb{Z}} b_k\varphi(2x - k) \tag{0.4}$$

bestimmen lässt.

Das Konzept der Multiskalenanalyse können wir leicht ins multivariate Setting übertragen. Dazu betrachten wir das Tensorprodukt von $n$ univariaten Multiskalenanalysen. In diesem Fall hat der Generator die Form

$$\varphi(x) := \varphi(x_1) \cdot \varphi(x_2) \cdot \ldots \cdot \varphi(x_n) \qquad \text{für } x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Man nennt ihn *separabel*. Für weitere Informationen zum Tensorproduktansatz verweisen wir auf [58]. Ein Nachteil dieses Ansatzes ist die Fokussierung auf die Koordinatenachsen. Diese Fokussierung kann Probleme in Anwendungen wie der Bildanalyse verursachen, bei der es darauf ankommt, mehr als nur horizontale und vertikale Strukturen zu erkennen. Einen Ausweg stellt die Wahl eines *nicht separablen* Generators dar. Wir verweisen an dieser Stelle auf die Lehrbücher [62, 29, 18, 81, 64], welche eine gute Übersicht zum Thema univariate und multivariate Wavelets ermöglichen.

Darüber hinaus ist es vorteilhaft, das Konzept der Multiskalenanalyse zu verallgemeinern. In (0.1) ist es wünschenswert, die notwendigen Berechnungen zu reduzieren. Eine Möglichkeit dies zu tun, ist die Anzahl an Mother Wavelets, bzw. die Kardinalität von $I$, zu reduzieren. Dies erreicht man durch das Einbinden von expandierenden Matrizen mit ganzzahligen Einträgen in die multivariate Variante von Bedingung (M4). Diese Matrizen nennt man *Dilations-* oder *Skalierungsmatrizen* und deren Determinante bestimmt die Anzahl an Mother Wavelets. Es gilt die Formel $|\det M| - 1$, siehe [61]. In der Literatur wurde der Fall $M = 2I$ über die Jahre eingehend studiert. Bei dieser Skalierungsmatrix werden $2^n - 1$ Mother Wavelets benötigt. Wir sehen, dass ein Anstieg der Raumdimension $n$ einen exponentiellen Anstieg der benötigten Anzahl an Mother Wavelets zur Folge hat. Aus diesem Grund ist man daran interessiert, eine Skalierungsmatrix mit kleiner Determinante in den Konstruktionsprozess einzubinden. Ein weiterer Vorteil von allgemeinen Skalierungsmatrizen ist, dass sich die oben beschriebene Fokussierung auf die Koordinatenachsen durch Einbau einer Rotationskomponente abmildern lässt.

Neben der Wahl eines geeigneten Dilationsparameters ist die Wahl eines geeigneten Generators wichtig. Das liegt daran, dass das Wavelet einen Großteil seiner Eigenschaften vom Generator erbt. Jedoch gibt es spezielle Kombinationen von Eigenschaften, wie orthogonal und interpolierend zu sein oder orthogonal und symmetrisch mit kleinem Träger und Approximationsordnung größer eins zu sein, die ein Generator alleine nicht besitzen kann, siehe [29, 53, 74]. Dieses Problem lässt sich lösen, indem man die Anzahl an Generatoren erhöht, siehe beispielsweise [53, 40, 66]. In diesem Zusammenhang spricht man von einer *Multiskalenanalyse mit Vielfachheit*. Diese wurde in [3, 41, 46] Anfang der 1990er Jahre eingeführt. Wavelets, die mithilfe einer solchen Multiskalenanalyse konstruiert werden, nennt man *Multiwavelets*. Wir bemerken, dass ein Anstieg der Generatoren auch einen Anstieg der Anzahl benötigter Mother Wavelets zur Folge hat. Insgesamt benötigen wir $(|\det M| - 1)N$ Mother Wavelets, wobei $N$ für die Anzahl an Generatoren steht, siehe [16, 82]. Für $N > 1$ ist diese Formel jedoch nicht für den Tensorproduktansatz gültig. In Beispiel 3.1 dieser Arbeit zeigen wir auf, dass im Fall $M = 2I$ sogar $(2^n - 1)N^2$ Mother Wavelets benötigt werden. Für weitere Informationen zum Thema Multiwavelets verweisen wir den Leser auf [52, 67].

Abhängig von den gewünschten Eigenschaften der (Multi)Waveletbasis, ist der Konstruktionsprozess mehr oder weniger restriktiv. Um eine kompakt getragene, orthornomale Waveletbasis zu erhalten, benötigt man einen kompakt getragenen Generator mit orthonormalen Translaten. Es ist notwendig, dass die Translate orthonormal sind, denn ansonsten würde das Durchführen einer Orthogonalisierungsprozedur den Verlust des kompakten Trägers verursachen. Die Konstruktion eines solchen Generators ist alles andere als trivial wie I. Daubechies in [28] für den Fall $n = 1$ aufzeigt. Wie in [21] nachgewiesen, ist bereits eine Anpassung ihrer Methode für die Dimension 2 mit großen Einschränkungen verbunden. Das führt uns zu den sogenannten Pre-Wavelets, welche von G. Battle eingeführt wurden, siehe [7] und Definition 1 in [8]. Eine Waveletbasis ist eine *Pre-Waveletbasis*, falls

$$\langle \psi_{i,j,k}, \psi_{i',j',k'} \rangle_{L_2(\mathbb{R}^n)} = 0 \quad \text{für } j \neq j', \quad j, j' \in \mathbb{Z}, i, i' \in I \,\text{endlich}, k, k' \in \mathbb{Z}^n,$$

erfüllt ist. Das bedeutet, es liegt nur Orthogonalität zwischen verschiedenen Skalen vor. Aus diesem Grund ist es eine schwächere Bedingung als die Orthonormalitätsbedingung

$$\langle \psi_{i,j,k}, \psi_{i',j',k'} \rangle_{L_2(\mathbb{R}^n)} = \delta_{i,i'}\, \delta_{j,j'}\, \delta_{k,k'}, \quad i, i' \in I \,\text{endlich}, \; j, j' \in \mathbb{Z}, k, k' \in \mathbb{Z}^n.$$

Das hat den Vorteil, dass wir an Flexibilität gewinnen und der kompakte Träger eines Generators ohne große Einschränkungen erhalten werden kann.

Aufbauend auf den vorherigen Ausführungen ist diese Arbeit durch die folgenden zwei Fragestellungen motiviert:

(Q1) Was sind Minimalanforderungen, sodass eine Konstruktion von Pre-Waveletbasen und Pre-Multiwaveletbasen noch möglich ist?

(Q2) Wie können wir die Anzahl an Mother-Wavelets minimieren?

Frage (Q1) ist von theoretischem Interesse. Sie zielt darauf ab, die Grenzen der Theorie aufzuzeigen. Frage (Q2) ist von theoretischem und praktischem Interesse. Reduziert man die Anzahl an Mother Wavelets, so werden weniger Berechnungen in Anwendungen benötigt. Darüber hinaus sind wir in dieser Arbeit hauptsächlich daran interessiert, kompakt getragene Pre-(Multi)Wavelets ausgehend von kompakt getragenen Generatoren zu konstruieren. Die Kombination von kompaktem Träger und einer minimalen Anzahl an Mother Wavelets ist unter anderem in der Signal- und Bildverarbeitung von besonderem Interesse. Hier ist es so, dass die Wavelet-Zerlegung eines Signals mittels Hochpass- und Tiefpassfiltern erfolgt, wobei diese aus den Koeffizienten der Verfeinerungsgleichung (0.3) bzw. der Funktionalgleichung des

Wavelets (0.4) gewonnen werden, siehe [57]. Diese Gleichungen haben für allgemeine Skalierungsmatrizen die Form

$$\varphi(x) = \sum_{k\in\mathbb{Z}^n} a_k \varphi(Mx-k)$$

sowie

$$\psi_i(x) = \sum_{k\in\mathbb{Z}^n} b_{i,k} \varphi(Mx-k), \quad 1 \leq i \leq \det M - 1.$$

Der kompakte Träger der Generatoren reduziert für gewöhnlich die Länge der Folge $\{a_k\}_{k\in\mathbb{Z}^n}$ und daher die Länge des Tiefpassfilters. Der kompakte Träger des Wavelets hingegen reduziert für gewöhnlich die Länge der Folge $\{b_{i,k}\}_{k\in\mathbb{Z}^n}, 1 \leq i \leq \det M - 1$, und daher die Länge des Hochpassfilters. Zusätzlich resultiert eine Minimierung von $\det M - 1$, bzw. eine Minimierung der Anzahl von Mother Wavelets, in einer Minimerung der Anzahl an Hochpassfiltern insgesamt.

Im Hinblick auf die Frage (Q1) reduzieren wir die Annahmen, die in der Definition der Multiskalenanalyse getroffen werden, auf die multivariaten Versionen von (M1), (M2) und (M3). In diesem Zusammenhang sprechen wir von einer verallgemeinerten Multiskalenanalyse. Anzumerken ist, dass diese Definition einer verallgemeinerten Multiskalenanalyse nicht mit der in der Literatur geläufigen Definition übereinstimmt, siehe [6, 59]. Hier ist eine verallgemeinerte Multiskalenanalyse eines separablen Hilbertraums $\mathcal{H}$ als Folge abgeschlossener Räume $\{S_j\}_{j\in\mathbb{Z}}$ in $\mathcal{H}$ definiert, sodass Folgendes gilt:

(1) $S_j \subset S_{j+1}$,

(2) $\overline{\bigcup_{j=-\infty}^{\infty} S_j} = \mathcal{H}$,

(3) $\bigcap_{j=-\infty}^{\infty} S_j = \{0\}$,

(4) $\delta(S_j) = S_{j+1}$,

(5) $S_0$ ist invariant unter der Anwendung von $\Upsilon$,

wobei $\Upsilon$ eine abzählbare, abelsche Gruppe unitärer Operatoren auf $\mathcal{H}$ ist und $\delta$ ein unitärer Operator auf $\mathcal{H}$ ist, sodass $\delta^{-1}\,\Upsilon\,\delta \subset \Upsilon$ gilt. Mit anderen Worten: Die Autoren in [6] haben Translation und Dilation durch allgemeinere Operatoren ersetzt. Darüber hinaus ist einer der Hauptunterschiede zu unserer Definition die Existenz der Annahme (4). Daher sprechen wir hier von einer *stationären Multiskalenanalyse*, während unsere Definition als *nicht stationäre Multiskalenanalyse* klassifiziert werden kann.

Im Hinblick auf die Frage (Q2) binden wir eine Skalierungsmatrix in unser Setting ein. Dies erfolgt durch das Hinzufügen der Annahme, dass die Räume $\{S_j\}_{j\in\mathbb{Z}}$ $M^{-j}$-shift-invariante Räume sein sollen. Zusammenfassend erhalten wir die nachfolgende Definition.

**Definition 0.2.**
Eine Folge abgeschlossener, $M^{-j}$-shift-invarianter Unterräume $\{S_j\}_{j\in\mathbb{Z}}$, $S_j \subset L_2(\mathbb{R}^n)$, heißt *verallgemeinerte Multiskalenanlayse*, wenn die folgenden Bedingungen erfüllt sind:

(M1) $\ldots \subset S_{j-1} \subset S_j \subset S_{j+1} \subset \ldots,$

(M2) $\overline{\bigcup_{j=-\infty}^{\infty} S_j} = L_2(\mathbb{R}^n),$

(M3) $\bigcap_{j=-\infty}^{\infty} S_j = \{0\}.$

Dieser Ansatz ist inspiriert durch Resultate von C. de Boor, R. A. DeVore und A. Ron. In [30] konstruierten sie kompakt getragene und stabile multivariate Pre-Wavelets. Ihre Konstruktionen passen in unser Setting für den Spezialfall $M = 2I$. Aus diesem Grund sind wir bestrebt, verallgemeinerte Konstruktionsprozeduren für allgemeine Skalierungsmatrizen zu finden, welche uns Pre-Wavelets und Pre-Multiwavelets liefern. Neben [30] gibt es weitere Publikationen, die rund um das Thema Pre-(Multi)Wavelets und Skalierungsmatrizen erschienen sind. Das Hauptinteresse galt hier der stationären Multiskalenanalyse mit einem Generator und $M = 2I$. In diesem Setting haben sowohl Micchelli [63] als auch Chui und Wang [20] die Existenz von kompakt getragenen univariaten Pre-Wavelets untersucht. Dabei nahmen sie an, dass der Generator $\varphi$ ganzzahlige stabile Translate besitzt und sowohl verfeinerbar als auch kompakt getragen ist. Im multivariaten Setting beschäftigten sich Jia and Micchelli [49] mit der Existenz von stabilen Pre-Wavelets. Die zugrundeliegenden Annahmen an den Generator waren seine Verfeinerbarkeit mit einer Maske $a$ in $\ell_1(\mathbb{Z}^n)$, Stabilität seiner ganzzahligen Translate und $\sum_{k\in\mathbb{Z}^n} |\varphi(\cdot - k)| \in L_2(T^n)$. Des Weiteren konstruierten sie kompakt getragene Pre-Wavelets unter der Annahme, dass $\varphi$ kompakt getragen ist. Wir merken an, dass diese Ergebnisse auf der Erweiterbarkeit einer endlichen Menge von Laurentpolynomen und dem Quillen-Suslin Theorem beruhen, siehe [68, 77]. In [54] erweiterte M.-J. Lai die Annahmen von Micchelli, bzw. Chui und Wang, und konstruierte kompakt getragene und stabile Pre-Wavelets. Darüber hinaus beschäftigte sich R. A. Zalik mit der Konstruktion von Pre-Multiwavelets im Fall einer stationären Multiskalenanalyse und einer allgemeinen Skalierungsmatrix. In [82] leitete er in dem soeben beschriebenen Setting eine explizite Darstellung der Fouriertransformation von stabilen Pre-Multiwavelets her.

Im Gegensatz zu den vorgenannten Publikationen, reduzieren wir die anfänglichen Annahmen durch Verwendung der verallgemeinerten Multiskalenanalyse, siehe Definition 0.2. Bei der Konstruktion von Pre-Wavelets in Kapitel 6 ist unsere Hauptannahme

$$\operatorname{supp}\hat{\varphi}_j = \mathbb{R}^n, \quad j \in \mathbb{Z},$$

wobei $\varphi_j$ der Generator des Raumes $S_j$ sei und $\hat{\varphi}_j$ die Fouriertransformierte dieses Generators. Dies ist keine besonders restriktive Annahme. Im Hinblick auf praktische Anwendungen ist man daran interessiert, mit gut lokalisierten Funktionen zu arbeiten, das heißt Funktionen mit kompaktem Träger. In diesem Fall besagt das Paley-Wiener Theorem, dass der Träger der Fouriertransformierten ganz $\mathbb{R}^n$ ist. Außerdem ist zu betonen, dass wir nicht annehmen, dass eine Verfeinerungsgleichung der Form

$$\varphi_j = \sum_{k\in\mathbb{Z}^n} a_k \, \varphi_{j+1}(\cdot + M^{-(j+1)}k) \tag{0.5}$$

gilt, wobei $a$ eine Folge in $\ell_2(\mathbb{Z}^n)$ ist. Wir hingegen arbeiten mit einer Gleichung der Form

$$\hat{\varphi}_j = A\,\hat{\varphi}_{j+1}, \tag{0.6}$$

wobei $A$ $2\pi(M^T)^{j+1}\mathbb{Z}^n$-periodisch ist. Diese folgt direkt aus der Schachtelung der $M^{-j}$-shift-invarianten Räume, siehe Kapitel 4. Die Gleichung (0.5) ist nicht äquivalent zu der Gleichung (0.6). Angenommen $\varphi_{j+1}$ besitzt $M^{-(j+1)}\mathbb{Z}^n$-stabile Translate, dann ist (0.5) äquivalent zu

$$\hat{\varphi}_j = A\hat{\varphi}_{j+1}, \quad A := \sum_{k\in\mathbb{Z}^n} a_k e^{i\left\langle \cdot, -M^{-(j+1)}k\right\rangle} \in L_2(\widetilde{C}_{j+1}),$$

wobei $\widetilde{C}_{j+1} := (M^T)^{j+1}[-\pi,\pi)^n$, siehe [35]. Im Gegensatz dazu könnten wir die Funktion $A$ in (0.6) so wählen, dass sie nicht lokal quadrat-integrierbar auf $\widetilde{C}_{j+1}$ ist. Wenn es um die Konstruktion von Pre-Multiwavelets geht, nimmt (0.6) die Form

$$\begin{pmatrix} \hat{\varphi}_1^j \\ \vdots \\ \hat{\varphi}_N^j \end{pmatrix} = \widetilde{A} \begin{pmatrix} \hat{\varphi}_1^{j+1} \\ \vdots \\ \hat{\varphi}_N^{j+1} \end{pmatrix}$$

an. Hierbei ist $\widetilde{A}$ eine Matrix mit $2\pi(M^T)^{j+1}\mathbb{Z}^n$-periodischen Funktionen als Einträgen und $\varphi_1^j, \ldots, \varphi_N^j$ sind die Generatoren des Raumes $S_j, j \in \mathbb{Z}$. Daneben haben wir zwei Hauptannahmen. Die erste Annahme besteht darin, dass spezielle Translate der Generatoren in $\Phi_j := \{\varphi_1^j, \dots, \varphi_N^j\}$ eine Basis für den Raum $S_{j+1}(\Phi_{j+1})$ liefern. Die zweite Annahme ist eine Gleichung der Form

$$\begin{pmatrix} \hat{\varphi}_1^{j+1} \\ \vdots \\ \hat{\varphi}_N^{j+1} \end{pmatrix} = \Gamma \begin{pmatrix} \hat{\varphi}_1^{j} \\ \vdots \\ \hat{\varphi}_N^{j} \end{pmatrix}, \quad j \in \mathbb{Z},$$

wobei $\Gamma$ eine quadratische Matrix ist, welche fast überall nicht-singulär ist und $2\pi(M^T)^{j+1}\mathbb{Z}^n$-periodische Einträge besitzt. Weitere Details sind in Kapitel 7 zu finden.

Diese Arbeit ist wie folgt aufgebaut: In Kapitel 1 führen wie Lebesgue- und Hilberträume sowie fundamentale Hilfsmittel ein, die wir stets benötigen werden. In Kapitel 2 stellen wir dem Leser zunächst das klassische multivariate Wavelet-Setting vor, bei dem man mit einem Generator und der Skalierungsmatrix $M = 2I$ arbeitet. Im Hinblick auf die Fragen (Q1) und (Q2) verallgemeinern wir dieses Setting in Kapitel 3. Insbesondere führen wir hier das Konzept der verallgemeinerten Multiskalenanalyse ein. Die Räume in dieser Art Multiskalenanalyse sind so gewählt, dass sie $M^{-j}$-shift-invariant sind, wobei $M$ einer allgemeinen Skalierungsmatrix entspricht. Dabei unterscheiden wir, ob die Räume von einem oder von mehreren Generatoren erzeugt werden. In Kapitel 4 werden wir beide Arten von shift-invarianten Räumen charakterisieren und wir werden analysieren, wann diese die Bedingungen (M2) und (M3) erfüllen. Anschließend diskutieren wir in Kapitel 5 orthogonale Projektionen von Funktionen des Raumes $L_2(\mathbb{R}^n)$ auf shift-invariante Räume und wie man eine explizite Darstellung ihrer Fouriertransformation herleiten kann. Dies ist wichtig für die nachfolgenden zwei Kapitel, die der eigentlichen Konstruktion von Pre-Wavelets und Pre-Multiwavelets gewidmet sind. In Kapitel 6 beschäftigen wir uns mit der Konstruktion von kompakt getragenen Pre-Wavelets. Wir verdeutlichen unseren Konstruktionsprozesss durch ein Beispiel, bei dem wir exponentielle Box-Splines als Generatoren für die Räume $S_j, j \in \mathbb{Z}$, wählen. Der Vorteil von exponentiellen Box-Splines im Zusammenhang mit nicht-stationären Multiskalenanalysen ist, dass sie für die Reproduktion von trigonometrischen Funktionen verwendet werden können, siehe [23]. Im Anschluss diskutieren wir die Konstruktion von stabilen Pre-Wavelets. In Kapitel 7 verallgemeinern wir unsere Konstruktionsprozedur für kompakt getragene Pre-Wavelets um kompakt getragene Pre-Multiwavelets zu erhalten. Genauer gesagt konstruieren wir Pre-Multiwavelets für den Fall, dass die Räume $S_j, j \in \mathbb{Z}$, endlich viele Generatoren besitzen und wir konstruieren kompakt getragene Pre-Multiwavelets für den Fall, dass wir zwei oder drei Generatoren pro Raum $S_j$ haben. Wir beobachten, dass dieser Zuwachs an Generatoren von einem Zuwachs an Annahmen an die Generatoren begleitet wird. Neben der Verallgemeinerung der Konstruktionsprozedur verallgemeinern wir des Weiteren auch das Beispiel mit den exponentiellen Box Splines aus dem vorherigen Kapitel. Abschließend ziehen wir ein Fazit und diskutieren offene Probleme und Ideen für die zukünftige Forschung.

Zusammenfassend lassen sich die Fragen (Q1) und (Q2) wie folgt beantworten:

(Q1) Während des Verallgemeinerungsprozesses der Resultate in [30] für eine allgemeine Skalierungsmatrix und endlich viele Generatoren, begegnen wir einigen Schwierigkeiten. Eine der größten Schwierigkeiten tritt auf, wenn es darum geht sicherzustellen, dass die Determinante einer Grammatrix, welche sich aus speziellen Translaten der Generatoren von $S_0$ ergibt, ungleich 0 ist. Für nähere Informationen verweisen wir auf die Abschnitte 6.1.1 und 7.1.1. Für einen Generator ist dies unter einer relativ schwachen Annahme möglich, siehe Korollar 6.1.9. Für zwei und drei Generatoren hingegen müssen bereits komplexere Abschätzungen erfüllt sein, siehe (7.4) und (7.5). Für endlich viele Generatoren verschärft sich die Situation weiter, siehe (7.1). Daher müssen wir eine eher starke Annahme an die Generatorenmengen der Räume $S_j$ hinzufügen. Darüber hinaus ist es alles andere als trivial, die Konstruktionsprozedur einer kompakt getragenen Basis in das Setting mit zwei bzw. drei Generatoren pro Raum $S_j$ zu verallgemeinern, siehe Theorem 7.1.10. Dies weist alles darauf hin, dass es nicht möglich erscheint, die Annahmen in der Definition der verallgemeinerten Multiskalenanalyse noch weiter zu reduzieren.

(Q2) Wir schaffen es, eine allgemeine Skalierungsmatrix in unseren Konstruktionsprozess einzubauen. Daher ist es möglich, die Anzahl an benötigten Mother Wavelets zu minimieren, indem man eine Skalierungsmatrix mit Determinante $\pm 2$ auswählt.

# Contents

# Introduction

Wavelets have become a very powerful tool in the field of applied mathematics where concepts of numerics, signal analysis and functional analysis are brought together. The term "wavelet" was first introduced by A. Grossman and J. Morlet in the early 1980s and it is a translation of the french word "ondelette" which means "little wave". Wavelets are used to construct bases for the space of square integrable functions $L_2(\mathbb{R}^n)$. These bases are obtained by scaling, dilating and shifting a finite set $\{\psi_i\}_{i\in I}$ of functions, also referred to as *mother wavelets*:

$$\{\psi_{i,j,k}(x) := 2^{jn/2}\psi_i(2^j x - k), i \in I, j \in \mathbb{Z}, k \in \mathbb{Z}^n\}.$$

Since signals are usually modeled by a function $f \in L_2(\mathbb{R}^n)$, we can use such bases to obtain a wavelet representation of a signal $f$:

$$f(x) = \sum_{i\in I}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n} c_{i,j,k}\psi_{i,j,k}(x). \tag{0.7}$$

The mother wavelets can be constructed such that they are exponentially decaying or compactly supported. Hence, we may obtain a representation which is local in the time domain and small changes in the signal affect only a few coefficients $c_{i,j,k}$. This is an advantage in comparison to the Fourier Transform which is only localized in the frequency domain. Other desirable properties are a high number of vanishing moments which results in a sparse representation of $f$ or orthogonality which leads to an easy computation of the coefficients $c_{i,j,k}$.

An example of a wavelet basis is the so-called Haar-basis which was constructed by A. Haar in 1910, see [43]. The mother wavelet is given by

$$\psi(x) := \begin{cases} 1 \ , & 0 \le x < \frac{1}{2}, \\ -1 \, , & \frac{1}{2} \le x < 1, \\ 0 \ , & \text{otherwise.} \end{cases}$$

For more examples we refer the reader to [75, 60, 29].

In order to construct wavelet bases one uses a systematic approach called multiresolution analysis which was developed by S. Mallat and Y. Meyer in 1986, see [57, 62].

**Definition 0.3.**
A sequence of closed subspaces $\{S_j\}_{j\in\mathbb{Z}}$, $S_j \subset L_2(\mathbb{R})$, is called *multiresolution analysis* if the following conditions are fulfilled:

(M1) $\ldots \subset S_{j-1} \subset S_j \subset S_{j+1} \subset \ldots,$

(M2) $\overline{\bigcup_{j=-\infty}^{\infty} S_j} = L_2(\mathbb{R}),$

(M3) $\bigcap_{j=-\infty}^{\infty} S_j = \{0\},$

(M4) $f \in S_j \Leftrightarrow f(2\,\cdot) \in S_{j+1},$

(M5) there exists a generator $\varphi \in S_0$, also called *scaling function*, whose translates $\{\varphi(\cdot - k), k \in \mathbb{Z}\}$ provide a Riesz basis for $S_0$.

By saying that the integer translates of $\varphi$ provide a *Riesz basis* for $S_0$, we mean that

$$S_0 := \overline{\operatorname{span}\{\varphi(\cdot - k), k \in \mathbb{Z}\}}$$

and that the integer translates of $\varphi$ are *stable*, i.e., there exist constants $C_1, C_2 > 0$ such that

$$C_1 \|c\|_{\ell_2(\mathbb{Z})} \leq \left\| \sum_{k \in \mathbb{Z}} c_k \varphi(\cdot - k) \right\|_{L_2(\mathbb{R})} \leq C_2 \|c\|_{\ell_2(\mathbb{Z})}, \quad c = \{c_k\}_{k \in \mathbb{Z}} \in \ell_2(\mathbb{Z}).$$

Conditions (M1), (M4) and (M5) ensure that the scaling function $\varphi$ is *refinable*. This means that $\varphi$ satisfies the *refinement equation*

$$\varphi(x) = \sum_{k \in \mathbb{Z}} a_k \varphi(2x - k), \tag{0.8}$$

with the *mask* $a = \{a_k\}_{k \in \mathbb{Z}} \in \ell_2(\mathbb{Z})$. Based on the concept of multiresolution analysis, wavelets are constructed such that they provide bases for the complement spaces $W_j := S_{j+1} \ominus S_j, j \in \mathbb{Z}$. Hence, conditions (M4) and (M5) yield that the wavelet $\psi$ can be determined by the equation

$$\psi(x) = \sum_{k \in \mathbb{Z}} b_k \varphi(2x - k). \tag{0.9}$$

The concept of multiresolution analysis can be easily transferred to the multivariate setting by considering the tensor product of $n$ univariate multiresolution analyses. Then the generator is of the form

$$\varphi(x) := \varphi(x_1) \cdot \varphi(x_2) \cdot \ldots \cdot \varphi(x_n) \quad \text{for } x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Such a generator is called *separable*. For more details concerning the tensor product approach see [58]. However, this approach leads to a bias towards the coordinate axes. This bias can cause problems in applications like image analysis where it is important to detect not only horizontal or vertical structures. To overcome this problem we can work with a *non-separable* generator. We refer the reader to the textbooks [62, 29, 18, 81, 64] for a good overview of univariate and multivariate wavelets.

It is beneficial to generalize the concept of multiresolution analysis. In (0.7) it is desirable to reduce computations to a minimum. One possibility is to reduce the number of mother wavelets which is equivalent to reducing the cardinality of $I$. This can be done by incorporating expanding integer matrices $M$ into the multivariate counterpart of condition (M4). These matrices are called *dilation* or *scaling matrices* and their determinant is related to the number of mother wavelets by the formula $|\det M| - 1$, see [61]. In literature the case $M = 2I$ was intensively studied over the years. This choice for the dilation matrix leads to $2^n - 1$ mother wavelets. We observe that as we increase the spatial dimension $n$, the number of mother wavelets increases exponentially. Hence, one is interested in incorporating a dilation matrix with a small determinant into the construction procedure. Another advantage of general scaling matrices is that the bias towards the coordinate axes can be reduced by incorporating a rotation component.

Besides the choice of an appropriate dilation parameter, the choice of an appropriate generator is important. This is due to the fact that wavelets inherit most of their properties from their generator. But there are specific combinations of properties like being orthogonal and interpolating that a single generator cannot possess. Another example is orthogonality and symmetry with small support and approximation order greater than one, see [29, 53, 74]. This problem can be overcome by an increase of the number of generators, see, e.g., [53, 40, 66]. In this context we talk about *multiresolution analyses with multiplicity* which were introduced in [3, 41, 46] in the early 1990s. Wavelets which are constructed with such multiresolution analyses are called *multiwavelets*. However, more generators lead to more mother wavelets. In total, we need $(|\det M| - 1)N$ mother wavelets where $N$ denotes the number of generators, see [16, 82]. This formula does not remain valid for the tensor product approach if $N > 1$. In Example 3.1 of this work we derive that $(2^n - 1)N^2$ mother wavelets are required in case $M = 2I$. For more information on multiwavelets we refer the reader to [52, 67].

In general, the construction procedure of a (multi)wavelet basis is more or less restrictive depending on the desired properties of the basis. In order to obtain a compactly supported and orthonormal wavelet basis, a compactly supported generator with orthonormal shifts is needed. Orthonormality of the shifts is necessary because otherwise an orthogonalization procedure would cause the loss of the desired compact support. The construction of such a generator is far from being trivial as I. Daubechies illustrates in [28] for $n = 1$. As shown in [21], an adaptation of her method for two dimensions is already accompanied by major restrictions. This brings us to the so-called pre-wavelets which were introduced by G. Battle, see [7] and Definition 1 in [8]. A wavelet basis is called *pre-wavelet basis* if

$$\langle \psi_{i,j,k}, \psi_{i',j',k'} \rangle_{L_2(\mathbb{R}^n)} = 0 \quad \text{for } j \neq j', \quad j, j' \in \mathbb{Z}, i, i' \in I \text{ finite}, k, k' \in \mathbb{Z}^n,$$

is satisfied, i.e., we only have orthogonality between different scales. Hence, it is a weaker condition in comparison to the orthonormality condition

$$\langle \psi_{i,j,k}, \psi_{i',j',k'} \rangle_{L_2(\mathbb{R}^n)} = \delta_{i,i'}\, \delta_{j,j'}\, \delta_{k,k'}, \quad i, i' \in I \text{ finite},\ j, j' \in \mathbb{Z}, k, k' \in \mathbb{Z}^n.$$

This additional flexibility allows to preserve compact support without major restrictions.

Based on the above considerations, this work is mainly motivated by the following two questions:

(Q1) What are minimal requirements such that a construction of multivariate pre-wavelet and pre-multiwavelet bases is still possible?

(Q2) How can we minimize the number of mother wavelets?

Question (Q1) is of theoretical interest. It aims to detect limitations of the existing theory. Question (Q2) is of theoretical and practical interest. Reducing the number of mother wavelets leads to less computations in applications. Moreover, we are mainly interested in constructing compactly supported pre-(multi)wavelets from compactly supported scaling functions in this work. The combination of compact support and a minimal number of mother wavelets is of special interest in signal and image analysis for instance. In this field high-pass and low-pass filters are used to obtain the wavelet decomposition of a signal. These filters are obtained from the coefficients of the refinement equation (0.8) and the functional equation (0.9), see [57]. For general dilation matrices these equations turn into

$$\varphi(x) = \sum_{k \in \mathbb{Z}^n} a_k \varphi(Mx - k)$$

and

$$\psi_i(x) = \sum_{k \in \mathbb{Z}^n} b_{i,k} \varphi(Mx - k), \quad 1 \leq i \leq \det M - 1.$$

Compact support of the generator usually reduces the length of the sequence $\{a_k\}_{k \in \mathbb{Z}^n}$ and as a result the length of the low-pass filter. Compact support of the wavelets usually reduces the length of the sequences $\{b_{i,k}\}_{k \in \mathbb{Z}^n}, 1 \leq i \leq \det M - 1$, and as a result the length of the high-pass filters. In addition, minimizing the number of mother wavelets, i.e., minimizing $\det M - 1$, leads to a minimization of the total number of high-pass filters.

In view of (Q1), we reduce the assumptions made in the definition of the multiresolution analysis. We only assume that the multivariate versions of conditions (M1), (M2) and (M3) hold. In this context we talk about a generalized multiresolution analysis. We remark that this definition does not coincide with the definition usually found in the literature, see [6, 59]. Here, a generalized multiresolution analysis of a separable Hilbert space $\mathcal{H}$ is a sequence of closed subspaces $\{S_j\}_{j\in\mathbb{Z}}$ of $\mathcal{H}$ such that

(1) $S_j \subset S_{j+1}$,

(2) $\overline{\bigcup_{j=-\infty}^{\infty} S_j} = \mathcal{H}$,

(3) $\bigcap_{j=-\infty}^{\infty} S_j = \{0\}$,

(4) $\delta(S_j) = S_{j+1}$,

(5) $S_0$ is invariant under the action of $\Upsilon$,

where $\Upsilon$ is a countable abelian group of unitary operators on $\mathcal{H}$ and $\delta$ is a unitary operator on $\mathcal{H}$ such that $\delta^{-1}\,\Upsilon\,\delta \subset \Upsilon$. In other words, the authors in [6] replaced translations and dilation by more general operators. Moreover, we observe that the main difference to our definition of a generalized multiresolution analysis is the existence of assumption (4). Therefore, we talk about a *stationary multiresolution analysis* while our definition can be classified as a *non-stationary multiresolution analysis*.

In view of (Q2), we incorporate a dilation matrix into our setting by assuming that the spaces $\{S_j\}_{j\in\mathbb{Z}}$ are $M^{-j}$-shift-invariant spaces. In summary, we obtain the subsequent definition.

**Definition 0.4.**
A sequence of closed, $M^{-j}$-shift-invariant subspaces $\{S_j\}_{j\in\mathbb{Z}}$, $S_j \subset L_2(\mathbb{R}^n)$, is called *generalized multiresolution analysis* if the following conditions are fulfilled:

(M1) $\ldots \subset S_{j-1} \subset S_j \subset S_{j+1} \subset \ldots$,

(M2) $\overline{\bigcup_{j=-\infty}^{\infty} S_j} = L_2(\mathbb{R}^n)$,

(M3) $\bigcap_{j=-\infty}^{\infty} S_j = \{0\}$.

This approach is inspired by the work of C. de Boor, R. A. DeVore and A. Ron. In [30] they constructed compactly supported and stable multivariate pre-wavelets. Their constructions fit into our setting for the special case $M = 2I$. Hence, we are eager to find generalized construction procedures for arbitrary dilation matrices which lead to pre-wavelets and pre-multiwavelets. Besides [30] there are more publications which evolved around pre-(multi)wavelets and dilation matrices. The main interest centered around stationary multiresolution analysis with one generator and $M = 2I$. In this setting Micchelli [63] as well as Chui and Wang [20] studied the existence of univariate compactly supported pre-wavelets in case $\varphi$ is refinable, compactly supported and has stable integer translates. In the multivariate setting Jia and Micchelli [49] studied the existence of stable pre-wavelets under the assumption that $\varphi$ is refinable with a mask $a \in \ell_1(\mathbb{Z}^n)$, $\varphi$ has stable integer translates and $\sum_{k\in\mathbb{Z}^n} |\varphi(\cdot - k)|$ is in $L_2(T^n)$. Moreover, compact support was obtained in case $\varphi$ is compactly supported. We remark that their investigations are based on the notion of extensibility of a finite set of Laurent polynomials and the Quillen-Suslin Theorem, see [68, 77]. In [54] M.-J. Lai extended the assumptions of Micchelli, Chui and Wang, and constructed compactly supported and stable pre-wavelets. Besides that, R. A. Zalik was concerned with pre-multiwavelets in case of a stationary multiresolution analysis and an arbitrary dilation matrix. In [82] he derived an explicit representation of the Fourier transform of stable pre-multiwavelets associated with such a multiresolution analysis.

In contrast to the aforementioned papers, we reduce the amount of initial assumptions by using the generalized multiresolution analysis defined in Definition 0.4. When it comes to the construction of pre-wavelets in Chapter 6, our main assumption is

$$\operatorname{supp} \hat{\varphi}_j = \mathbb{R}^n, \quad j \in \mathbb{Z},$$

where $\varphi_j$ is the generator of the space $S_j$ with Fourier transform $\hat{\varphi}_j$. This is not a very restrictive assumption. For practical applications one is interested in working with well-localized functions, i.e., functions with compact support. Due to the Paley-Wiener Theorem, such functions have a Fourier transform supported on $\mathbb{R}^n$. Besides that, we emphasize that we do not assume to have a refinement equation of the form

$$\varphi_j = \sum_{k\in\mathbb{Z}^n} a_k \, \varphi_{j+1}(\cdot + M^{-(j+1)}k), \tag{0.10}$$

where $a$ is a sequence in $\ell_2(\mathbb{Z}^n)$. Instead, we work with an equation of the form

$$\hat{\varphi}_j = A\, \hat{\varphi}_{j+1}, \tag{0.11}$$

where $A$ is $2\pi(M^T)^{j+1}\mathbb{Z}^n$-periodic. Formula (0.11) holds true in our setting due to the fact that we work with nested $M^{-j}$-shift-invariant spaces, see Chapter 4.

Notice that (0.10) is not equivalent to (0.11). To this end, suppose that $\varphi_{j+1}$ has $M^{-(j+1)}\mathbb{Z}^n$-stable shifts. Then (0.10) is equivalent to

$$\hat{\varphi}_j = A\hat{\varphi}_{j+1}, \quad A := \sum_{k\in\mathbb{Z}^n} a_k e^{i\left\langle \,\cdot\,, -M^{-(j+1)}k\right\rangle} \in L_2(\widetilde{C}_{j+1}),$$

where $\widetilde{C}_{j+1} := (M^T)^{j+1}[-\pi,\pi)^n$, see [35]. In contrast, (0.11) would allow us to choose $A$ such that it is not locally square-integrable on $\widetilde{C}_{j+1}$. When it comes to the construction of pre-multiwavelets, (0.11) turns into

$$\begin{pmatrix} \hat{\varphi}_1^j \\ \vdots \\ \hat{\varphi}_N^j \end{pmatrix} = \widetilde{A} \begin{pmatrix} \hat{\varphi}_1^{j+1} \\ \vdots \\ \hat{\varphi}_N^{j+1} \end{pmatrix}.$$

Here, $\widetilde{A}$ is a matrix with $2\pi(M^T)^{j+1}\mathbb{Z}^n$-periodic functions as entries and $\varphi_1^j, \ldots, \varphi_N^j$ are the generators for the space $S_j, j \in \mathbb{Z}$. Besides that we have two main assumptions. The first assumption states that special translates of the generators in $\Phi_j := \{\varphi_1^j, \ldots, \varphi_N^j\}$ provide a basis for the space $S_{j+1}(\Phi_{j+1})$. The second assumption is an equation of the form

$$\begin{pmatrix} \hat{\varphi}_1^{j+1} \\ \vdots \\ \hat{\varphi}_N^{j+1} \end{pmatrix} = \Gamma \begin{pmatrix} \hat{\varphi}_1^j \\ \vdots \\ \hat{\varphi}_N^j \end{pmatrix}, \quad j \in \mathbb{Z},$$

where $\Gamma$ is a quadratic matrix which is non-singular almost everywhere and has $2\pi(M^T)^{j+1}\mathbb{Z}^n$-periodic entries. For more details see Chapter 7.

This thesis is organized as follows. In Chapter 1 we introduce Lebesgue and Hilbert spaces as well as fundamental tools which we use throughout this thesis. In Chapter 2 we present the classical multivariate wavelet setting where one works with a single generator and the dilation matrix $M = 2I$. In view of (Q1) and (Q2), we then generalize this setting in Chapter 3. In particular, we introduce the concept of a generalized multiresolution analysis. The spaces $\{S_j\}_{j\in\mathbb{Z}}$ in this kind of multiresolution analysis are chosen to be $M^{-j}$-shift-invariant spaces where $M$ is an arbitrary dilation matrix. In case these spaces are generated by a single function we talk about *principal shift-invariant spaces* and in case they are generated by finitely many functions we talk about *finitely generated shift-invariant spaces*. In Chapter 4 we characterize both types of shift-invariant spaces and analyse under which assumptions they fulfill conditions (M2) and (M3). Next, we discuss in Chapter 5 orthogonal projections of functions in $L_2(\mathbb{R}^n)$ onto such spaces and how we can derive explicit representations of their Fourier transforms. This is important for the following two chapters which are dedicated to the actual construction of pre-wavelets and pre-multiwavelets. In Chapter 6 we are concerned with the construction of compactly supported pre-wavelets. We illustrate our construction procedure by an example where we choose exponential box splines as generators for the spaces $S_j, j \in \mathbb{Z}$. The advantage of exponential box splines in the context of a non-stationary multiresolution analysis is that they can be used to reproduce trigonometric functions, see [23]. Afterwards, we also discuss the construction of stable pre-wavelets. In Chapter 7 we generalize the construction procedure of compactly supported pre-wavelets in order to obtain compactly supported pre-multiwavelets. More precisely, we construct pre-multiwavelets in case the spaces $S_j, j \in \mathbb{Z}$, have finitely many generators and we construct compactly supported pre-multiwavelets in case each space $S_j$ has two or three generators. We observe that this increase in generators is accompanied by additional assumptions concerning the generators. Besides generalizing the construction procedure, we also generalize the example concerning exponential box splines from the previous chapter. Finally, we draw conclusions and discuss open problems and ideas for future research.

In summary, we can answer the questions (Q1) and (Q2) as follows:

(Q1) During the process of generalizing the results from [30] for a general dilation matrix and finitely many generators, we encounter several difficulties. One of the main tasks is to provide sufficient conditions such that the Gramian matrix corresponding to special translates of the generators of $S_0$ is nonzero, see Section 6.1.1 and Section 7.1.1. For a single generator this is possible under a mild assumption but for two and three generators more sophisticated estimates have to be fulfilled, see Corollary 6.1.9, (7.4) and (7.5). For finitely many generators the situation is even more complex, see (7.1). Hence, we have to add a rather strong assumption concerning the generator sets. Besides that, it is far from being trivial to generalize the construction procedure of a compactly supported basis such that it is applicable in the setting where each space $S_j$ has two or three generators, see Theorem 7.1.10. This indicates that it seems not possible to reduce the assumptions made in the definition of a generalized multiresolution analysis any further.

(Q2) We manage to incorporate a general dilation matrix into our construction procedures. Hence, we can minimize the number of required mother wavelets by choosing a dilation matrix with determinant $\pm 2$.

# Chapter 1

# Preliminaries

This chapter is dedicated to the repetition and introduction of mathematical concepts which are important for this thesis. It is divided into three sections which address Lebesgue and Hilbert spaces, the Fourier transform and the bracket product.

## 1.1 Lebesgue and Hilbert Spaces

In this section we define Lebesgue and Hilbert spaces. Moreover, we introduce the notion of orthogonal projections in Hilbert spaces. We close this section by summarizing some auxiliary results which we will need in Chapter 4.

**Definition 1.1.1.**
For $1 \leq p \leq \infty$, we define the space

$$\mathcal{L}_p(\mathbb{R}^n) := \left\{ f : \mathbb{R}^n \to \mathbb{C}\,\text{measurable} \;\middle|\; \|f\|_{\mathcal{L}_p(\mathbb{R}^n)} < \infty \right\},$$

where the (semi-)norm $\|\cdot\|_{\mathcal{L}_p(\mathbb{R}^n)}$ is given by

$$\|f\|_{\mathcal{L}_p(\mathbb{R}^n)} := \left( \int\limits_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x \right)^{1/p}, \quad p < \infty,$$

$$\|f\|_{\mathcal{L}_\infty(\mathbb{R}^n)} := \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} |f(x)| < \infty.$$

In addition, we define the following equivalence relation for functions $f, g \in \mathcal{L}_p(\mathbb{R}^n)$ :

$$f \sim g \quad :\Longleftrightarrow \quad \|f - g\|_{\mathcal{L}_p(\mathbb{R}^n)} = 0.$$

Then $L_p(\mathbb{R}^n) := \mathcal{L}_p(\mathbb{R}^n)/\sim$ with $\|\cdot\|_{L_p(\mathbb{R}^n)} := \|\cdot\|_{\mathcal{L}_p(\mathbb{R}^n)}$ is called *Lebesgue space*.

The Lebesgue spaces are Banach spaces and for $p = 2$ we have a Hilbert space.

**Definition 1.1.2.**
A Banach space $\mathcal{H}$ where the norm $\|\cdot\|_{\mathcal{H}}$ is induced by an inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}} : \mathcal{H}\times\mathcal{H}\to\mathbb{C}$, that is,

$$\|\cdot\|_{\mathcal{H}} = \sqrt{\langle\cdot,\cdot\rangle_{\mathcal{H}}}\,,$$

is called a *Hilbert space*.

For the most part of this thesis, we are going to work with the Hilbert space $L_2(\mathbb{R}^n)$ where the inner product is given by

$$\langle f,g\rangle_{L_2(\mathbb{R}^n)} = \int\limits_{\mathbb{R}^n} f(x)\overline{g(x)}\,\mathrm{d}x$$

for functions $f,g\in L_2(\mathbb{R}^n)$.

Based on the existence of an inner product in a Hilbert space $\mathcal{H}$, we can say that $f,g\in\mathcal{H}$ are orthogonal if $\langle f,g\rangle_{\mathcal{H}} = 0$. Moreover, we can work with the notion of an orthogonal projection.

**Theorem 1.1.3.** - [5, Chapter 3.2]
Let $S\subset\mathcal{H}$ be a closed subspace of a Hilbert space $\mathcal{H}$ and let $S^{\perp}$ be the orthogonal complement of $S$ in $\mathcal{H}$, i.e., $\mathcal{H} = S\oplus S^{\perp}$. Then there exists a unique mapping $\mathcal{P}_S:\mathcal{H}\to S$, called *orthogonal projection*, which satisfies

$$h-\mathcal{P}_S(h)\in S^{\perp},\quad h\in\mathcal{H}. \tag{1.1}$$

Formula (1.1) can be expressed equivalently as

$$\langle h-\mathcal{P}_S(h), s\rangle_{\mathcal{H}} = 0\quad\text{for all } s\in S.$$

**Proposition 1.1.4.** - [45, Theorem 8.3.6]
Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal sequence in a Hilbert space $\mathcal{H}$ and let $S$ be given by $S = \overline{\operatorname{span}\{e_n, n\in\mathbb{N}\}}$. Then the orthogonal projection of $h\in\mathcal{H}$ onto $S$ can be written as

$$\mathcal{P}_S(h) = \sum_{n\in\mathbb{N}}\langle h,e_n\rangle_{\mathcal{H}}\,e_n.$$

Finally, let us present some results which we will apply in Chapter 4. One of these results is the continuity of translations in $L_p(\mathbb{R}^n)$.

**Lemma 1.1.5.**
The translation in $L_p(\mathbb{R}^n), 1\le p<\infty$, is continuous. For every $r\in\mathbb{R}^n$ it holds

$$\lim_{|r|\to 0}\|u(\cdot+r)-u\|_{L_p(\mathbb{R}^n)} = 0,\quad u\in L_p(\mathbb{R}^n).$$

*Proof.*
We follow the proof of Theorem 4.21 in [36] where the lemma above was stated in a similar manner.

The space of continuous functions with compact support $C_0^0(\Omega)$ is dense in $L_p(\mathbb{R}^n)$. Moreover, a continuous function with compact support is uniformly continuous. Let $\phi$ be such a function. Then, for all $\varepsilon > 0$ there exists a $\delta = \delta(\phi, \varepsilon)$ such that

$$|\phi(\cdot + r) - \phi| < \varepsilon \quad \text{for } |r| < \delta.$$

For $\delta \to 0$, we obtain

$$\lim_{|r|\to 0} \|\phi(\cdot + r) - \phi\|_{L_\infty(\mathbb{R}^n)} = 0.$$

Since we consider functions with compact support, we have $L_q(\Omega) \subseteq L_p(\Omega)$ for $1 \le p \le q \le \infty$. Consequently,

$$\lim_{|r|\to 0} \|\phi(\cdot + r) - \phi\|_{L_p(\mathbb{R}^n)} = 0.$$

Finally, we consider the estimate

$$\begin{aligned} &\|u(\cdot + r) - u\|_{L_p(\mathbb{R}^n)} \\ &\quad \le \|u(\cdot + r) - \phi(\cdot + r)\|_{L_p(\mathbb{R}^n)} + \|\phi(\cdot + r) - \phi\|_{L_p(\mathbb{R}^n)} + \|\phi - u\|_{L_p(\mathbb{R}^n)}. \end{aligned}$$

For $|r| \to 0$, the right-hand side tends to $2\|\phi - u\|_{L_p(\mathbb{R}^n)}$. But since the continuous functions with compact support are dense in $L_p(\mathbb{R}^n)$, we can choose the function $\phi$ such that $\|\phi - u\|_{L_p(\mathbb{R}^n)} \le \varepsilon$. The proof is complete. □

Besides that we collect results concerning weak convergence in Hilbert spaces. A sequence $(x_n)$ in a Hilbert space $\mathcal{H}$ is *weakly convergent* to $x \in \mathcal{H}$, written $x_n \overset{w}{\to} x$, if

$$\langle x_n, y\rangle_{\mathcal{H}} \to \langle x, y\rangle_{\mathcal{H}} \quad \text{for all}\, y \in \mathcal{H}.$$

The weak limit $x$ is unique, see Lemma 3.6.3 in [44]. For the proof of Theorem 4.1.12 we will need the following results.

**Theorem 1.1.6.** - [34, Theorem 9.12]
Every bounded sequence in a Hilbert space has a weakly convergent subsequence.

**Proposition 1.1.7.** - [83, Proposition 21.23]
A bounded sequence $(x_n)$ in a Hilbert space $\mathcal{H}$ converges weakly to $x$ if every weakly convergent subsequence of $(x_n)$ has the same limit $x$.

**Theorem 1.1.8.** - [34, Theorem 9.10]
Let $(x_n)$ be a sequence in a Hilbert space $\mathcal{H}$. Then, $x_n \to x$ if and only if $x_n \overset{w}{\to} x$ and $\|x_n\| \to \|x\|$.

Moreover, the proof of Theorem 4.1.12 requires knowledge of weakly closed sets. A subset $K$ of a Hilbert space $\mathcal{H}$ is called *weakly closed* if $(x_n) \in K$ and $x_n \stackrel{w}{\to} x$ implies $x \in K$.

**Theorem 1.1.9.** - [34, Theorem 9.16]
A convex set is closed if and only if it is weakly closed.

Finally, we present a result which takes advantage of the fact that the notion of orthogonality exists in a Hilbert space.

**Theorem 1.1.10.** - [22, Theorem 2.6]
Let $S$ be a closed linear subspace of a Hilbert space $\mathcal{H}$. Moreover, let $h \in \mathcal{H}$ and let $f_0$ be the unique element of $S$ such that $\|h - f_0\| = \operatorname{dist}(h, S)$. Then $h - f_0 \perp S$. Conversely, if $f_0 \in S$ such that $h - f_0 \perp S$, then $\|h - f_0\| = \operatorname{dist}(h, S)$.

## 1.2 Fourier Transform

One of the most important tools in wavelet analysis is the Fourier transform. In this section we give the definition of the Fourier transform and the Fourier series and recall some basic properties, see [65], Chapter III in [76] and [80].

**Definition 1.2.1.**
Let $f$ be a function in $L_1(\mathbb{R}^n)$. Then the *Fourier transform* $F$ of the function $f$, denoted by $\hat{f}$, is defined by

$$\hat{f}(\xi) := Ff(x) = \int\limits_{\mathbb{R}^n} f(x)\, e_{-\xi}(x)\, \mathrm{d}x,$$

where $e_\xi(x) := e^{ix\cdot\xi}$ for $\xi \in \mathbb{R}^n$. Here, $x \cdot \xi$ denotes the Euclidean inner product of $x$ and $\xi$.

The following two results are well-known.

**Theorem 1.2.2.**
Let $f$ and $\hat{f}$ be functions in $L_1(\mathbb{R}^n)$. Then the *Fourier inversion formula*

$$f(x) = (\hat{f}(\xi))^\vee := F^{-1}\hat{f}(\xi) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \hat{f}(\xi) e_x(\xi)\, \mathrm{d}\xi$$

holds true for almost every $x \in \mathbb{R}^n$.

**Theorem 1.2.3.**
Let $f, g$ be functions in the space $L_1(\mathbb{R}^n)$. Then the Fourier transform of the convolution of $f$ and $g$ can be written as the product of the Fourier transforms of $f$ and $g$, that is,

$$\widehat{(f * g)} = \hat{f}\, \hat{g}.$$

Next, we introduce the Fourier coefficients and the Fourier series.

**Definition 1.2.4.**
Let $f \in L_1(T^n)$ where $T^n$ denotes the $n$-dimensional torus. Then the *Fourier coefficients* of $f$ are defined by

$$c_f(k) := \frac{1}{(2\pi)^n} \int\limits_{T^n} f(x) e_{-k}(x)\,\mathrm{d}x, \quad k \in \mathbb{Z}^n.$$

The *Fourier series* of $f$ is given by

$$\sum_{k\in\mathbb{Z}^n} c_f(k) e_x(k).$$

The following lemma shows that under a certain assumption on the Fourier coefficients, a function $f \in L_1(T^n)$ can be represented by a Fourier series.

**Lemma 1.2.5.** - [73, Chapter VII, Corollary 1.8]
Let $f \in L_1(T^n)$ and let $\{a_k\}_{k\in\mathbb{Z}^n}$ be the Fourier coefficients of $f$. Moreover, let $\sum_{k\in\mathbb{Z}^n} |a_k| < \infty$. Then it holds

$$f(x) = \sum_{k\in\mathbb{Z}^n} a_k\, e_k(x) \quad \text{almost everywhere.}$$

The definition of the Fourier transform can be extended such that it is also applicable to functions in the Hilbert space $L_2(\mathbb{R}^n)$.

**Theorem 1.2.6.**

i) The Fourier transform $F$ and the inverse Fourier transform $F^{-1}$ have unique extensions $\mathcal{F}, \mathcal{F}^{-1}$ on $L_2(\mathbb{R}^n)$ such that it holds

$$(2\pi)^n \langle f, g\rangle_{L_2(\mathbb{R}^n)} = \langle \mathcal{F}f, \mathcal{F}g\rangle_{L_2(\mathbb{R}^n)} = \left\langle \hat{f}, \hat{g}\right\rangle_{L_2(\mathbb{R}^n)}, \quad f, g \in L_2(\mathbb{R}^n). \tag{1.2}$$

ii) For a function $h \in L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^n)$, we have $Fh = \mathcal{F}h$ and $F^{-1}h = \mathcal{F}^{-1}h$.

iii) Let $f \in L_2(\mathbb{R}^n)$ and

$$f_\ell(\xi) := \int\limits_{|x|\leq\ell} f(x) e_{-\xi}(x)\,\mathrm{d}x, \quad f_\ell^*(x) := \frac{1}{(2\pi)^n} \int\limits_{|\xi|\leq\ell} f(\xi) e_x(\xi)\,\mathrm{d}\xi.$$

Then we have

$$\mathcal{F}f = \lim_{\ell\to\infty} f_\ell \quad \text{and} \quad \mathcal{F}^{-1}f = \lim_{\ell\to\infty} f_\ell^*.$$

**Lemma 1.2.7.**
Let $f, g \in L_2(\mathbb{R}^n)$. Then it holds

$$\begin{aligned} \mathcal{F}(af(x)+bg(x))(\xi) &= a\,\hat{f}(\xi)+b\,\hat{g}(\xi), \qquad && a,b\in\mathbb{C}, \xi\in\mathbb{R}^n,\\ \mathcal{F}(f(x-a))(\xi) &= e_{-a}(\xi)\hat{f}(\xi), && a,\xi\in\mathbb{R}^n. \end{aligned} \tag{1.3}$$

*Proof.*
The Fourier transform is linear because

$$\begin{aligned} \mathcal{F}(af(x)+bg(x))(\xi) &= \lim_{\ell\to\infty} \int\limits_{|x|\le\ell} (af(x)+bg(x))\, e_{-\xi}(x)\,\mathrm{d}x \\ &= a \lim_{\ell\to\infty} \int\limits_{|x|\le\ell} f(x)e_{-\xi}(x)\,\mathrm{d}x + b \lim_{\ell\to\infty} \int\limits_{|x|\le\ell} g(x)e_{-\xi}(x)\,\mathrm{d}x \\ &= a\,\hat{f}(\xi)+b\,\hat{g}(\xi). \end{aligned}$$

Next, we verify that a translation of a function results in a modulation of the corresponding Fourier transform. The definition of the Fourier transform yields

$$\mathcal{F}(f(x-a))(\xi) = \lim_{\ell\to\infty} \int\limits_{|x|\le\ell} f(x-a)e_{-\xi}(x)\,\mathrm{d}x.$$

We set $y = x - a$ and deduce

$$\begin{aligned} \lim_{\ell\to\infty} \int\limits_{|x|\le\ell} f(x-a)e_{-\xi}(x)\,\mathrm{d}x &= \lim_{\ell\to\infty} \int\limits_{|y+a|\le\ell} f(y)e_{-\xi}(y+a)\,\mathrm{d}y \\ &= e_{-\xi}(a) \lim_{\ell\to\infty} \int\limits_{|y|\le\ell} f(y)e_{-\xi}(y)\,\mathrm{d}y \\ &= e_{-\xi}(a)\hat{f}(\xi) \\ &= e_{-a}(\xi)\hat{f}(\xi). \end{aligned}$$

□

**Lemma 1.2.8.**
Let $M \in \mathbb{R}^{n\times n}$ be a non-singular matrix and $f \in L_2(\mathbb{R}^n)$. Then we have

$$\mathcal{F}(f(Mx))(\xi) = \frac{1}{|\det M|}\mathcal{F}(f)(M^{-T}\xi). \tag{1.4}$$

*Proof.*
First, we insert the definition of the $L_2$-Fourier transform into $\mathcal{F}(f(Mx))(\xi)$ to obtain

$$\begin{aligned}\mathcal{F}(f(Mx))(\xi) &= \lim_{\ell\to\infty} \int\limits_{|x|\le\ell} f(Mx)e_{-\xi}(x)\ \mathrm{d}x \\ &= \frac{1}{|\det M|}\lim_{\ell\to\infty} \int\limits_{|x|\le\ell} f(Mx)|\det M|e_{-\xi}(M^{-1}Mx)\ \mathrm{d}x.\end{aligned}$$

Then the transformation formula yields

$$\begin{aligned}&\frac{1}{|\det M|}\lim_{\ell\to\infty} \int\limits_{|x|\le\ell} f(Mx)|\det M|e_{-\xi}(M^{-1}Mx)\ \mathrm{d}x \\ &= \frac{1}{|\det M|}\lim_{\ell\to\infty} \int\limits_{|M^{-1}u|\le\ell} f(u)e_{-\xi}(M^{-1}u)\ \mathrm{d}u \\ &= \frac{1}{|\det M|}\lim_{\ell\to\infty} \int\limits_{|u|\le\ell} f(u)e_{-M^{-T}\xi}(u)\ \mathrm{d}u \\ &= \frac{1}{|\det M|}\mathcal{F}(f)(M^{-T}\xi).\end{aligned}$$

□

The following two theorems are versions of the so-called *Paley-Wiener Theorem.* They require knowledge about entire functions of exponential type. An entire function $g$ on $\mathbb{C}$ is of *exponential type* $\sigma$ if and only if for every $\varepsilon > 0$ there exists a constant $C_\varepsilon$ such that

$$|g(z)| \le C_\varepsilon\, e^{(\sigma+\epsilon)|z|} \quad \text{for all } z \in \mathbb{C}.$$

**Theorem 1.2.9.** - [73, Chapter III, Theorem 4.1]
Let $f, g \in L_2(\mathbb{R})$. Then $g = \mathcal{F}f$ with $\operatorname{supp} f \subset \left[-\frac{\sigma}{2\pi}, \frac{\sigma}{2\pi}\right]$ if and only if $g$ is the restriction of an entire function of exponential type $\sigma$ to the real axis.

Theorem 1.2.9 can be generalized for functions $f, g$ in $L_2(\mathbb{R}^n)$. Beforehand, let us introduce symmetric bodies and polar sets which are needed for this generalization. A compact and convex set in $\mathbb{R}^n$ which is symmetric about the origin and has non-empty interior is called *symmetric body.* Let $K$ denote such a symmetric body. Then the *polar set* $K^*$ of $K$ is defined by

$$K^* := \{\xi \in \mathbb{R}^n : x \cdot \xi \le 1 \quad \text{for all } x \in K\}.$$

$K^*$ is a symmetric body as well. Now, an entire function g on $\mathbb{C}^n$ is of exponential type $K^*$ if and only if for every $\varepsilon > 0$ there exists a constant $C_\varepsilon$ such that

$$|g(z)| \le C_\varepsilon e^{2\pi(1+\varepsilon)\|z\|^*} \quad \text{for all } z \in \mathbb{C}^n.$$

Here, the *dual norm* $\|\cdot\|^*$ is defined by $\|z\|^* := \sup_{y\in K} |y \cdot z|$.

**Theorem 1.2.10.** - [73, Chapter III, Theorem 4.9]
Let $f, g \in L_2(\mathbb{R}^n)$ and let $K$ be a symmetric body. Then $g = \mathcal{F}f$ is the Fourier transform of a function $f$ with $\operatorname{supp} f \subset K$ if and only if $g$ is the restriction to $\mathbb{R}^n$ of a function of exponential type $K^*$, where $K^*$ denotes the polar set of $K$.

## 1.3 Bracket Product

Another very important tool for our construction procedure of a pre-(multi)wavelet basis will be the so-called bracket product. In this section we summarize its most important properties.

**Definition 1.3.1.**
Let $f, g \in L_2(\mathbb{R}^n)$. Then the $2\pi(M^T)^j\mathbb{Z}^n$-periodization of $f\overline{g}$, i.e.,

$$[f, g]_j := \sum_{\beta \in 2\pi(M^T)^j\mathbb{Z}^n} f(\cdot + \beta)\, \overline{g(\cdot + \beta)}, \quad j \in \mathbb{Z}, \tag{1.5}$$

is called *bracket product*. We set $[\hat{f}, \hat{g}] := [\hat{f}, \hat{g}]_0$.

A standard procedure for evaluating integrals of bracket products is to partition the domain of integration into fundamental domains.

**Definition 1.3.2.** - [1, Definition 5.8]
Let $\mathcal{D}$ be either a lattice contained in $\mathbb{R}^n$ or $\mathbb{R}^n$ itself. Furthermore, let $\widetilde{C}$ be an arbitrary subset of $\mathcal{D}$ and let $\mathcal{L}$ be a lattice contained in $\mathcal{D}$. The set $\widetilde{C}$ is a *fundamental domain* of the lattice $\mathcal{L}$ in $\mathcal{D}$ if $\widetilde{C}$ intersects each coset of $\mathcal{D}/\mathcal{L}$ in exactly one point.

**Definition 1.3.3.**
Let $\mathcal{L}$ be a lattice. Then the *dual lattice* $\mathcal{L}^*$ is defined by

$$\mathcal{L}^* := \{\ell^* \in \mathbb{R}^n : \ell^* \cdot \ell \in 2\pi\mathbb{Z} \quad \text{for } \ell \in \mathcal{L}\}.$$

In this thesis we are interested in the lattice $\mathcal{L}_j = M^{-j}\mathbb{Z}^n$ and its dual lattice $\mathcal{L}_j^* = 2\pi(M^T)^j\mathbb{Z}^n, j \in \mathbb{Z}$. Therefore, we set $\mathcal{D} = \mathbb{R}^n$, $\widetilde{C}_j = (M^T)^j[-\pi, \pi)^n$ and thus, we can identify the space $L_2(\mathcal{D}/\mathcal{L}_j^*)$ with the space $L_2(\widetilde{C}_j)$. We mention that if we set $j = 0$, we obtain the $n$-dimensional torus given by $T^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$.

Next, we verify that the bracket product $[f, g]_j$ is an element of the space $L_1(\widetilde{C}_j)$ and hence, well-defined. Therefore, we need the subsequent lemma which was proven in [72, Lemma 1.1.5] for $\beta \in \mathbb{Z}^n$. The following version can be proven analogously.

**Lemma 1.3.4.**
Let $f, g \in L_2(\mathbb{R}^n)$. Then the series $\sum_{\beta \in 2\pi (M^T)^j \mathbb{Z}^n} |f(\cdot + \beta)\overline{g(\cdot + \beta)}|$, $j \in \mathbb{Z}$, converges almost everywhere.

*Proof.*
First, we remark that $f\overline{g}$ is measurable as a product of measurable functions. Now, let us consider the series

$$\sum_{\beta \in 2\pi (M^T)^j \mathbb{Z}^n} \int_{\widetilde{C}_j} \left| f(x+\beta)\overline{g(x+\beta)} \right| \mathrm{d}x = \sum_{\beta \in 2\pi (M^T)^j \mathbb{Z}^n} \int_{\widetilde{C}_j + \beta} \left| f(x)\overline{g(x)} \right| \mathrm{d}x$$
$$= \int_{\mathbb{R}^n} \left| f(x)\overline{g(x)} \right| \mathrm{d}x.$$

By Hölder's inequality, we deduce

$$\int_{\mathbb{R}^n} \left| f(x)\overline{g(x)} \right| \mathrm{d}x \leq \|f\|_{L_2(\mathbb{R}^n)} \|g\|_{L_2(\mathbb{R}^n)} < \infty. \tag{1.6}$$

We conclude by the Levi-Theorem, see [2, Theorem 11.18], that the series $\sum_{\beta \in 2\pi (M^T)^j \mathbb{Z}^n} |f(x+\beta)\overline{g(x+\beta)}|$ is integrable and that

$$\int_{\widetilde{C}_j} \sum_{\beta \in 2\pi (M^T)^j \mathbb{Z}^n} |f(x+\beta)\overline{g(x+\beta)}| \,\mathrm{d}x = \sum_{\beta \in 2\pi (M^T)^j \mathbb{Z}^n} \int_{\widetilde{C}_j} \left| f(x+\beta)\overline{g(x+\beta)} \right| \mathrm{d}x.$$

The assertion follows. □

**Lemma 1.3.5.**
Let $f, g \in L_2(\mathbb{R}^n)$. Then $[f, g]_j \in L_1(\widetilde{C}_j)$ for $j \in \mathbb{Z}$.

*Proof.*
Due to the Lemma 1.3.4, we know that the series in (1.5) converges almost everywhere. The proof of this Lemma yields

$$\int_{\widetilde{C}_j} |[f, g]_j(x)| \,\mathrm{d}x \leq \int_{\widetilde{C}_j} \sum_{\beta \in 2\pi (M^T)^j \mathbb{Z}^n} |f(x+\beta)\overline{g(x+\beta)}| \,\mathrm{d}x < \infty.$$

□

The following result is a special case of Theorem 3.1 in [49].

**Lemma 1.3.6.**
Let $f$ and $g$ be compactly supported functions in $L_2(\mathbb{R}^n)$. Then the sequence $c(f,g) := \{c(f,g)_k\}_{k\in\mathbb{Z}^n}$ with

$$c(f,g)_k := \int_{\mathbb{R}^n} f(x)\overline{g(x-k)}\,\mathrm{d}x$$

is an element of the sequence space $\ell_1(\mathbb{Z}^n)$.

The bracket product is related to the $L_2$-inner product, see, e.g., [30].

**Lemma 1.3.7.**
Let $f, g \in L_2(\mathbb{R}^n)$.

i) The inner product $\langle f(\cdot - k), g\rangle_{L_2(\mathbb{R}^n)}$, $k \in \mathbb{Z}^n$, is the $k$-th Fourier coefficient of the bracket product $[\hat{f}, \hat{g}]$, which means that

$$\langle f(\cdot - k), g\rangle_{L_2(\mathbb{R}^n)} = \frac{1}{(2\pi)^n} \int_{\widetilde{C}_0} e_{-k}(\xi)[\hat{f},\hat{g}](\xi)\,\mathrm{d}\xi. \tag{1.7}$$

ii) It holds $\langle f(\cdot - j), g(\cdot - k)\rangle_{L_2(\mathbb{R}^n)} = 0$ for $j, k \in \mathbb{Z}^n$, if and only if $[\hat{f}, \hat{g}] = 0$ almost everywhere.

iii) If $f, g$ are compactly supported functions, then the bracket product $[\hat{f}, \hat{g}]$ is a trigonometric polynomial.

*Proof.*
In order to prove part i), we use (1.2) and (1.3) first to deduce

$$\begin{aligned}
\langle f(\cdot - k), g\rangle_{L_2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} f(x-k)\overline{g(x)}\,\mathrm{d}x \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e_{-k}(\xi)\hat{f}(\xi)\overline{\hat{g}(\xi)}\,\mathrm{d}\xi \\
&= \frac{1}{(2\pi)^n}\sum_{\beta\in 2\pi\mathbb{Z}^n}\int_{\widetilde{C}_0} e_{-k}(\xi)\hat{f}(\xi+\beta)\overline{\hat{g}(\xi+\beta)}\,\mathrm{d}\xi.
\end{aligned} \tag{1.8}$$

Since $\hat{f}, \hat{g} \in L_2(\mathbb{R}^n)$, we can use (1.6) to obtain

$$\frac{1}{(2\pi)^n}\sum_{\beta\in 2\pi\mathbb{Z}^n}\left|\int\limits_{\widetilde{C}_0} e_{-k}(\xi)\hat{f}(\xi+\beta)\overline{\hat{g}(\xi+\beta)}\right| \mathrm{d}\xi$$
$$\le \frac{1}{(2\pi)^n}\sum_{\beta\in 2\pi\mathbb{Z}^n}\int\limits_{\widetilde{C}_0} |e_{-k}(\xi)||\hat{f}(\xi+\beta)\overline{\hat{g}(\xi+\beta)}| \,\mathrm{d}\xi$$
$$= \frac{1}{(2\pi)^n}\sum_{\beta\in 2\pi\mathbb{Z}^n}\int\limits_{\widetilde{C}_0} |\hat{f}(\xi+\beta)\overline{\hat{g}(\xi+\beta)}| \,\mathrm{d}\xi$$
$$= \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} |\hat{f}(\xi)\overline{\hat{g}(\xi)}| \,\mathrm{d}\xi < \infty.$$

Therefore, we can interchange the sum and the integral in (1.8) which yields

$$\langle f(\cdot - k), g\rangle_{L_2(\mathbb{R}^n)} = \frac{1}{(2\pi)^n}\int\limits_{\widetilde{C}_0}\sum_{\beta\in 2\pi\mathbb{Z}^n} e_{-k}(\xi)\hat{f}(\xi+\beta)\overline{\hat{g}(\xi+\beta)}\,\mathrm{d}\xi$$
$$= \frac{1}{(2\pi)^n}\int\limits_{\widetilde{C}_0} e_{-k}(\xi)[\hat{f},\hat{g}](\xi)\,\mathrm{d}\xi.$$

Now, let us prove part ii). We have already derived the relation

$$\langle f(\cdot - j), g(\cdot - k)\rangle_{L_2(\mathbb{R}^n)} = \frac{1}{(2\pi)^n}\int\limits_{\widetilde{C}_0} e_{k-j}(\xi)[\hat{f},\hat{g}](\xi)\,\mathrm{d}\xi, \quad j,k\in\mathbb{Z}^n. \tag{1.9}$$

If $[\hat{f},\hat{g}] = 0$ almost everywhere, it follows directly from (1.9) that the translates of the functions $f$ and $g$ are orthogonal to each other. Conversely, assume that the $L_2$-inner product vanishes for all $j,k\in\mathbb{Z}^n$. Then by (1.9), we have

$$0 = \sum_{j,k\in\mathbb{Z}^n} |\langle f(\cdot - j) g(\cdot - k)\rangle_{L_2(\mathbb{R}^n)}| = \sum_{j,k\in\mathbb{Z}^n}\left|\frac{1}{(2\pi)^n}\int\limits_{\widetilde{C}_0} e_{k-j}(\xi)[\hat{f},\hat{g}](\xi)\,\mathrm{d}\xi\right|. \tag{1.10}$$

Since $[\hat{f},\hat{g}]\in L_1(\widetilde{C}_0)$, see Lemma 1.3.5, and since the sequence of the Fourier coefficients of the bracket product is in the space $\ell_1(\mathbb{Z}^n)$, see (1.10), all conditions of Lemma 1.2.5 are fulfilled. Applying this lemma yields the desired result, i.e.,

$$[\hat{f},\hat{g}] = 0 \quad \text{almost everywhere.}$$

It remains to prove part iii). Again, we will use relation (1.7) and the fact that $[\hat{f},\hat{g}] \in L_1(\widetilde{C}_0)$. In addition, the sequence $c(\overline{g},\overline{f}) = \{c(\overline{g},\overline{f})_k\}_{k\in\mathbb{Z}^n}$ with

$$c(\overline{g},\overline{f})_k = \int\limits_{\mathbb{R}^n} f(x-k)\overline{g(x)}\,\mathrm{d}x$$

is an element of the sequence space $\ell_1(\mathbb{Z}^n)$, see Lemma 1.3.6. Thus, we can apply Lemma 1.2.5 to deduce that

$$\begin{aligned}[\hat{f},\hat{g}](x) &= \sum_{k\in\mathbb{Z}^n}\left(\frac{1}{(2\pi)^n}\int\limits_{\widetilde{C}_0} e_{-k}(\xi)[\hat{f},\hat{g}](\xi)\,\mathrm{d}\xi\right)e_k(x) \\ &= \sum_{k\in\mathbb{Z}^n}\langle f(\cdot-k),g\rangle_{L_2(\mathbb{R}^n)}\,e_k(x)\quad\text{almost everywhere.}\end{aligned} \tag{1.11}$$

Hence, we have equality in the $L_1$-sense. Moreover, the functions $f$ and $g$ are compactly supported. Consequently, there are finitely many non-vanishing coefficients. Thus, (1.11) is a trigonometric polynomial.

□

**Lemma 1.3.8.**
Let $f, g \in L_2(\mathbb{R}^n)$.

i) If $\tau$ is a $2\pi(M^T)^j\mathbb{Z}^n$-periodic function, we have $[\tau\hat{f},\hat{g}]_j = \tau[\hat{f},\hat{g}]_j = [\hat{f},\overline{\tau}\hat{g}]_j$ for $j \in \mathbb{Z}$.

ii) Let $f, g \in L_2(\mathbb{R}^n)$. Then we have the estimate

$$|[f,g]_j|^2 \le [f,f]_j[g,g]_j, \quad j \in \mathbb{Z}. \tag{1.12}$$

*Proof.*
For the proof of part i), we only need the $2\pi(M^T)^j\mathbb{Z}^n$-periodicity of $\tau$. We have

$$\begin{aligned}[\tau\hat{f},\hat{g}]_j &= \sum_{\beta\in 2\pi(M^T)^j\mathbb{Z}^n}(\tau\hat{f})(\cdot+\beta)\overline{\hat{g}(\cdot+\beta)} \\ &= \sum_{\beta\in 2\pi(M^T)^j\mathbb{Z}^n}\tau(\cdot+\beta)\hat{f}(\cdot+\beta)\overline{\hat{g}(\cdot+\beta)} \\ &= \tau\sum_{\beta\in 2\pi(M^T)^j\mathbb{Z}^n}\hat{f}(\cdot+\beta)\overline{\hat{g}(\cdot+\beta)} \\ &= \tau[\hat{f},\hat{g}]_j.\end{aligned} \tag{1.13}$$

Then the assumption follows by (1.13) because

$$[\hat{f}, \overline{\tau}\hat{g}]_j = \sum_{\beta \in 2\pi (M^T)^j \mathbb{Z}^n} \hat{f}(\cdot + \beta)\overline{(\overline{\tau}\hat{g})(\cdot + \beta)} = \sum_{\beta \in 2\pi (M^T)^j \mathbb{Z}^n} \hat{f}(\cdot + \beta)\tau(\cdot + \beta)\overline{\hat{g}(\cdot + \beta)}.$$

In order to prove part ii), we have to verify that the bracket product

$$[\cdot, \cdot]_j : L_2(\mathbb{R}^n) \times L_2(\mathbb{R}^n) \to \mathbb{C}$$

satisfies the properties of the inner product, see, e.g., [79, Chapter 1.2]. In particular, we have to check the following conditions:

(1) It holds $[f, f]_j \geq 0$ for all $f \in L_2(\mathbb{R}^n)$ and $[f, f]_j = 0$ if and only if $f = 0$.

(2) It holds $[f, g]_j = \overline{[g, f]_j}$ for all $f, g \in L_2(\mathbb{R}^n)$.

(3) It holds $[\lambda f, g]_j = \lambda [f, g]_j$ for all $f, g \in L_2(\mathbb{R}^n), \lambda \in \mathbb{C}$.

(4) It holds $[f + h, g]_j = [f, g]_j + [h, g]_j$ for all $f, g, h \in L_2(\mathbb{R}^n)$.

First, we prove condition (1). Since

$$[f, f]_j = \sum_{\beta \in 2\pi (M^T)^j \mathbb{Z}^n} |f(\cdot + \beta)|^2 \geq 0,$$

it follows directly that $[f, f]_j = 0$ if and only if $f = 0$. Next, we verify condition (2). By Lemma 1.3.4, we have

$$\overline{[g, f]_j} = \overline{\sum_{\beta \in 2\pi (M^T)^j \mathbb{Z}^n} g(\cdot + \beta)\overline{f(\cdot + \beta)}} = \sum_{\beta \in 2\pi (M^T)^j \mathbb{Z}^n} f(\cdot + \beta)\overline{g(\cdot + \beta)} = [f, g]_j.$$

Moreover, condition (3) is satisfied as well because

$$[\lambda f, g]_j = \sum_{\beta \in 2\pi (M^T)^j \mathbb{Z}^n} \lambda f(\cdot + \beta)\overline{g(\cdot + \beta)} = \lambda \sum_{\beta \in 2\pi (M^T)^j \mathbb{Z}^n} f(\cdot + \beta)\overline{g(\cdot + \beta)} = \lambda [f, g]_j.$$

Finally, condition (4) is fulfilled because

$$\begin{aligned}
[f + h, g]_j &= \sum_{\beta \in 2\pi (M^T)^j \mathbb{Z}^n} (f + h)(\cdot + \beta)\overline{g(\cdot + \beta)} \\
&= \sum_{\beta \in 2\pi (M^T)^j \mathbb{Z}^n} f(\cdot + \beta)\overline{g(\cdot + \beta)} + h(\cdot + \beta)\overline{g(\cdot + \beta)} \\
&= \sum_{\beta \in 2\pi (M^T)^j \mathbb{Z}^n} f(\cdot + \beta)\overline{g(\cdot + \beta)} + \sum_{\gamma \in 2\pi (M^T)^j \mathbb{Z}^n} h(\cdot + \gamma)\overline{g(\cdot + \gamma)} \\
&= [f, g]_j + [h, g]_j.
\end{aligned}$$

Consequently, the claim follows from the Cauchy-Schwarz inequality. □

# Chapter 2

# The Classical Setting

Before we begin with the construction of wavelet and multiwavelet bases with general dilation matrices, we recall the classical wavelet setting. This case has been intensively studied over the years and it is ideal to explain the basic concepts and motivate our approach.

A finite set of functions $\{\psi_i\}_{i\in I} \subset L_2(\mathbb{R}^n)$ is called a *wavelet basis* if the set

$$\{\psi_{i,j,k} := 2^{jn/2}\,\psi_i(2^j\cdot -k), i \in I \text{ finite},\ j\in\mathbb{Z}, k\in\mathbb{Z}^n\}$$

provides a basis for the function space $L_2(\mathbb{R}^n)$. The functions $\psi_i$ are also referred to as *mother wavelets*. If in addition the condition

$$\langle \psi_{i,j,k}, \psi_{i',j',k'}\rangle_{L_2(\mathbb{R}^n)} = \delta_{i,i'}\,\delta_{j,j'}\,\delta_{k,k'}, \quad i,i' \in I \text{ finite},\ j,j'\in\mathbb{Z}, k,k'\in\mathbb{Z}^n, \tag{2.1}$$

is satisfied, we obtain a so-called *orthonormal wavelet basis*. If (2.1) is weakened to

$$\langle \psi_{i,j,k}, \psi_{i',j',k'}\rangle_{L_2(\mathbb{R}^n)} = 0 \quad \text{for } j\neq j', \quad j,j'\in\mathbb{Z}, i,i'\in I \text{ finite}, k,k'\in\mathbb{Z}^n,$$

we call it a *pre-wavelet basis*. For the construction of such wavelet bases Y. Meyer and S. Mallat developed a systematic approach named multiresolution analysis, see [57, 62].

**Definition 2.1.**
A sequence of closed subspaces $\{S_j\}_{j\in\mathbb{Z}}$, $S_j \subset L_2(\mathbb{R}^n)$, is called *multiresolution analysis* if the following conditions are fulfilled:

(M1) $\ldots \subset S_{j-1} \subset S_j \subset S_{j+1} \subset \ldots,$

(M2) $\overline{\bigcup_{j=-\infty}^{\infty} S_j} = L_2(\mathbb{R}^n),$

(M3) $\bigcap_{j=-\infty}^{\infty} S_j = \{0\},$

(M4) $f \in S_j \Leftrightarrow f(2\,\cdot) \in S_{j+1}$,

(M5) there exists a generator $\varphi \in S_0$, also called *scaling function*, whose translates $\{\varphi(\cdot - k), k \in \mathbb{Z}^n\}$ provide a Riesz-basis for $S_0$.

In literature it is not uncommon to assume that $\{\varphi(\cdot - k), k \in \mathbb{Z}^n\}$ provides an orthonormal basis for $S_0$. We remark that this case is covered by condition (M5), see [4]. Moreover, the properties (M1), (M4) and (M5) ensure that the scaling function $\varphi$ is *refinable*. This means that $\varphi$ satisfies the *refinement equation*

$$\varphi(x) = \sum_{k \in \mathbb{Z}^n} a_k \varphi(2x - k), \tag{2.2}$$

with the *mask* $a = \{a_k\}_{k\in\mathbb{Z}^n} \in \ell_2(\mathbb{Z}^n)$. Applying the Fourier transform to (2.2) yields

$$\hat{\varphi}(\xi) = \frac{1}{2} a_s(e_{-1/2}(\xi))\, \hat{\varphi}(\xi/2),$$

where $a_s$ denotes the *symbol* defined by $a_s(z) = \sum_{k\in\mathbb{Z}} a_k z^k$ with

$$z \in \{(z_1, \dots, z_n) \in \mathbb{C}^n \,|\, |z_i| = 1, i = 1, \dots, n\} \quad \text{and} \quad z^k = (z_1^{k_1}, z_2^{k_2}, \dots, z_n^{k_n}).$$

By taking the concept of multiresolution analysis into consideration, we proceed as follows to obtain a wavelet basis of $L_2(\mathbb{R}^n)$. First, we define the functions $\psi_i, i \in I$ finite, whose integer translates span the nontrivial complement $W_0$ of $S_0$ in $S_1$, i.e., $S_1 = W_0 \oplus S_0$ or equivalently $W_0 = S_1 \ominus S_0$ with

$$W_0 := \overline{\text{span}\{\psi_i(\cdot - k), i \in I\,\text{finite}, k \in \mathbb{Z}^n\}}.$$

The spaces $W_j$ are given by

$$W_j := \{f \in L_2(\mathbb{R}^n) \mid f(2^{-j}\cdot) \in W_0\}, \quad j \in \mathbb{Z}. \tag{2.3}$$

We can easily check that $S_{j+1} = S_j \oplus W_j$ for $j \in \mathbb{Z}$. Let $f$ be a function of the space $S_{j+1}$. By (M4), we know that $f(2^{-j}\cdot) \in S_1$. A function in $S_1$ has a unique decomposition of the form

$$f(2^{-j}x) = s_0(x) + w_0(x), \quad s_0 \in S_0, w_0 \in W_0.$$

Set $y = 2^{-j}x$ to obtain $f(y) = s(y) + w(y)$ with $s := s_0(2^j\cdot)$ and $w := w_0(2^j\cdot)$. Property (M4) implies $s \in S_j$ and (2.3) implies $w \in W_j$.

Moreover, the subspaces $\{S_j\}_{j\in\mathbb{Z}}$ are dense in $L_2(\mathbb{R}^n)$ by definition and we observe

$$\overline{\bigcup_{j=-\infty}^{\infty} S_j} = S_0 \oplus \left( \bigoplus_{j=0}^{\infty} W_j \right) = L_2(\mathbb{R}^n).$$

The subspace $S_0$ has the equivalent representation

$$S_0 = \left( \bigcap_{j=-\infty}^{-1} S_j \right) \oplus \left( \bigoplus_{j=-\infty}^{-1} W_j \right).$$

By property (M3), the zero function is the only element in the intersection of the subspaces $\{S_j\}_{j\in\mathbb{Z}}$ and consequently, we obtain the following decomposition of the space $L_2(\mathbb{R}^n)$:

$$L_2(\mathbb{R}^n) = \bigoplus_{j=-\infty}^{\infty} W_j.$$

When using appropriately scaled, translated and dilated versions of the mother wavelets $\psi_i$, we finally obtain the wavelet basis

$$\Psi = \{\psi_{i,j,k} := 2^{jn/2}\, \psi_i(2^j \cdot -k), i \in I \,\text{finite}, j \in \mathbb{Z},\, k \in \mathbb{Z}^n\}.$$

If we assume further that $W_0$ is the orthogonal complement of $S_0$ in $S_1$, we obtain a pre-wavelet basis. This is due to property (M1) which yields that all spaces $\{W_j\}_{j\in\mathbb{Z}}$ are mutually orthogonal. In case (2.1) is fulfilled, we have an orthonormal wavelet basis.

Another possibility to construct a pre-wavelet basis is to define the space $W_0$ by projections. Let $\mathcal{P}_{S_0}$ be the orthogonal projector of $L_2(\mathbb{R}^n)$ onto $S_0$. Since $S_1$ is a closed subspace of $L_2(\mathbb{R}^n)$ with $S_1 = S_0 \oplus W_0$, we may define the space $W_0$ by

$$W_0 := \{s - \mathcal{P}_{S_0}\, s : s \in S_1\}. \tag{2.4}$$

Furthermore, by the nestedness of the spaces $S_j$, we have

$$\mathcal{P}_{S_j} \circ \mathcal{P}_{S_{j'}} = \mathcal{P}_{S_j} \quad \text{for all } j \le j',$$

where the projectors $\mathcal{P}_{S_j}$ are obtained from $\mathcal{P}_0$ by dilation. Thus, $\mathcal{P}_{S_j} - \mathcal{P}_{S_{j-1}}$ is an orthogonal projector of $L_2(\mathbb{R}^n)$ onto $W_{j-1}$. By properties (M2) and (M3), an arbitrary function $f \in L_2(\mathbb{R}^n)$ can be written as the telescopic sum

$$f = \sum_{j\in\mathbb{Z}} (\mathcal{P}_{S_j} f - \mathcal{P}_{S_{j-1}} f).$$

Hence,

$$L_2(\mathbb{R}^n) = \bigoplus_{j=-\infty}^{\infty} W_j.$$

It is also possible to increase the number of generators in (M5). Then the presented methods yield so-called *multiwavelet* and *pre-multiwavelet bases*.

# Chapter 3

# The General Setting

In this work we focus on the construction of pre-wavelet and pre-multiwavelet bases. Motivated by the questions

- How can we minimize the number of mother wavelets?
- What are minimal requirements such that a construction of pre-wavelet and pre-multiwavelet bases is still possible?

we generalize the concepts presented in Chapter 2. We note that if there is no risk of confusion, we will often not distinguish between mother wavelets, pre-(multi)wavelets and (multi)wavelets in the following.

The number of required mother wavelets depends on the so-called *dilation matrix* or *scaling matrix* denoted by $M$. In Chapter 2 the dilation matrix was given by $M = 2I$ where $I$ denotes the identity matrix. We assume throughout that $M$ is an integer $n \times n$ matrix which is expanding, that is, all its eigenvalues are greater than one in modulus. It follows directly that $M$ is invertible. Moreover, the matrix $M$ satisfies

$$\lim_{j\to\infty} \|M^{-j}\| = 0 \tag{3.1}$$

which is equivalent to

$$\lim_{j\to\infty} M^{-j} = 0 \quad \text{or} \quad \lim_{j\to\infty} M^{-j}x = 0 \quad \text{for all } x \in \mathbb{R}^n, \tag{3.2}$$

see [15, Theorem 7.17], [47, Theorem 4]. Moreover, (3.1) yields

$$\lim_{j\to\infty} |M^j x| = \infty \quad \text{for any } x \in \mathbb{R}^n \setminus \{0\}, \tag{3.3}$$

see [72, Chapter 2.1].

Such an arbitrary dilation matrix can be incorporated into Definition 2.1 by changing condition (M4) to

(M4*) $\quad f \in S_j \Leftrightarrow f(M\cdot) \in S_{j+1}$.

Then the multiresolution analysis approach presented in the previous chapter leads to

$$(m-1)N, \quad m := |\det M|, \tag{3.4}$$

mother wavelets where $N$ is the number of basis generators of $S_0$, see [61, 16, 82]. Hence, we can minimize the number of required mother wavelets by choosing a dilation matrix with determinant $\pm 2$.

Formula (3.4) does not remain valid for the tensor product approach with multiple generators.

**Example 3.1.**
Let $\{S_j\}_{j\in\mathbb{Z}}$ be a multiresolution analysis for the space $L_2(\mathbb{R})$ with the scaling function $\varphi$ and the scaling matrix $M = 2$. Assume further that the mother wavelet $\psi$ provides an orthonormal basis for $L_2(\mathbb{R})$. Then an $n$-dimensional multiresolution analysis can be obtained by considering the tensor product

$$V_j^n := \underbrace{V_j \otimes V_j \otimes \ldots \otimes V_j}_{\text{n-times}}.$$

For simplicity, we choose $n = 2$. Then the space $V_1^2$ has the decomposition

$$\begin{aligned} V_1^2 &= V_1 \otimes V_1 \\ &= (V_0 \oplus W_0) \otimes (V_0 \oplus W_0) \\ &= (V_0 \otimes V_0) \oplus (V_0 \otimes W_0) \oplus (W_0 \otimes V_0) \oplus (W_0 \otimes W_0). \end{aligned}$$

Hence, the wavelet space $W_0^2$ can be written as

$$W_0^2 = (V_0 \otimes W_0) \oplus (W_0 \otimes V_0) \oplus (W_0 \otimes W_0) \tag{3.5}$$

and the mother wavelets are given by

$$\psi^1(x,y) = \varphi(x)\psi(y), \quad \psi^2(x,y) = \psi(x)\varphi(y), \quad \psi^3(x,y) = \psi(x)\psi(y).$$

In [58, Chapter VII] it was proven that the set

$$\left\{\psi^i_{j,k,k'}(x,y) := \frac{1}{2^j}\psi^i\left(\frac{x-2^jk}{2^j}, \frac{y-2^jk'}{2^j}\right), i = 1,2,3, (k,k') \in \mathbb{Z}^2\right\}$$

provides an orthonormal basis for the space $L_2(\mathbb{R}^2)$. We observe that formula (3.4) remains valid. This is not true for $N > 1$. In this case the presented approach leads to $(2^n-1)N^2$ mother wavelets instead of $(2^n-1)N$. This can easily be seen by generalizing the above example for $N = 2$. Let $\varphi_1$, $\varphi_2$ be the scaling functions and assume that $\psi_1, \psi_2$ provide an orthonormal basis for $L_2(\mathbb{R})$. Then the decomposition (3.5) yields the following twelve mother wavelets:

$$\begin{aligned}
\psi^1(x,y) &= \varphi_1(x)\psi_1(y), & \psi^2(x,y) &= \varphi_2(x)\psi_1(y), & \psi^3(x,y) &= \varphi_1(x)\psi_2(y),\\
\psi^4(x,y) &= \varphi_2(x)\psi_2(y), & \psi^5(x,y) &= \psi_1(x)\varphi_1(y), & \psi^6(x,y) &= \psi_1(x)\varphi_2(y),\\
\psi^7(x,y) &= \psi_2(x)\varphi_1(y), & \psi^8(x,y) &= \psi_2(x)\varphi_2(y), & \psi^9(x,y) &= \psi_1(x)\psi_1(y),\\
\psi^{10}(x,y) &= \psi_1(x)\psi_2(y), & \psi^{11}(x,y) &= \psi_2(x)\psi_1(y), & \psi^{12}(x,y) &= \psi_2(x)\psi_2(y).
\end{aligned}$$

As this example illustrates, one should always weigh the simplicity of the tensor product approach against the number of required mother wavelets.

Another possibility to incorporate the dilation matrix into the definition of the multiresolution analysis is to specify the spaces $S_j$ as $h$-shift-invariant with $h := M^{-j}$.

**Definition 3.2.**
Let $\Phi := \{\varphi_1, \dots, \varphi_N\}$ with $\varphi_i \in L_2(\mathbb{R}^n)$ for $i = 1, \dots, N$. Then

$$S(\Phi) : = \overline{\operatorname{span}\{\varphi_1(\cdot - hk), \dots, \varphi_N(\cdot - hk), k \in \mathbb{Z}^n\}}$$

is called *h-shift-invariant space.* For $N = 1$ we call $S(\varphi)$ a *principal shift-invariant space* and for some finite $N$ we call $S(\Phi)$ a *finitely generated shift-invariant space.*

In Chapter 2, we illustrated that the assumptions made in the definition of a multiresolution analysis ensure that the construction of wavelet bases is possible. Since we are interested in identifying minimal requirements, we are going to omit conditions (M4), (M5) and analyse how this affects the construction procedure. Consequently, we are going to work with the following generalized multiresolution analysis which can be classified as non-stationary.

**Definition 3.3.**
A sequence of closed, $M^{-j}$-shift-invariant subspaces $\{S_j\}_{j\in\mathbb{Z}}$, $S_j \subset L_2(\mathbb{R}^n)$, is called *generalized multiresolution analysis (GMRA)* if the following conditions are fulfilled:

(M1) $\dots \subset S_{j-1} \subset S_j \subset S_{j+1} \subset \dots,$

(M2) $\overline{\bigcup_{j=-\infty}^{\infty} S_j} = L_2(\mathbb{R}^n),$

(M3) $\bigcap_{j=-\infty}^{\infty} S_j = \{0\}.$

# Chapter 4

# Shift-Invariant Spaces

Shift-invariant spaces are used not only in wavelet theory but also in approximation, frame and sampling theory. Their general definition can be found in Chapter 3, see Definition 3.2. In this chapter we will analyse their properties in detail.

The spaces $\{S_j\}_{j\in\mathbb{Z}}$ in the definition of the generalized multiresolution analysis are specified as $M^{-j}$-shift-invariant. Depending on the number of generators, we distinguish between principal and finitely generated shift-invariant spaces. In both cases we can describe $S_j$ by considering its Fourier transforms $\widehat{S_j} := \{\hat{f} : f \in S_j\}$ as Corollary 4.1.2 and Corollary 4.2.2 show. Many important results arise from this characterization which enable our construction procedures later on. We summarize these results in the Sections 4.1.1 and 4.2.1.

In view of the generalized multiresolution analysis, we also include an analysis of the union and the intersection of $M^{-j}$-shift-invariant spaces, see Sections 4.1.2 and 4.2.2.

## 4.1 Principal Shift-Invariant Spaces

### 4.1.1 Characterization

Principal shift-invariant spaces are generated by a single function. The subsequent characterization of $S_0(\varphi), \varphi \in L_2(\mathbb{R}^n)$, can be found in [31, Theorem 2.14] and [12].

**Theorem 4.1.1.**
Let $\varphi \in L_2(\mathbb{R}^n)$. A function $f$ is an element of the space $S_0(\varphi)$ if and only if $\hat{f} = \tau\hat{\varphi}$ for some $2\pi$-periodic $\tau$.

We generalize this result for the spaces $S_j, j \in \mathbb{Z}$. The method of proof we present will be used several times throughout this thesis.

**Corollary 4.1.2.**
Let $\varphi \in L_2(\mathbb{R}^n)$. A function $f$ is an element of the space $S_j(\varphi), j \in \mathbb{Z}$, if and only if $\hat{f} = \tau\hat{\varphi}$ for some $2\pi(M^T)^j\mathbb{Z}^n$-periodic $\tau$.

*Proof.*
Assume that $f \in S_j(\varphi), j \in \mathbb{Z}$. Then there exists a sequence $(f_\ell)_{\ell\in\mathbb{N}} \in S_j(\varphi)$ such that $f_\ell \to f$ in the $L_2$-sense. The definition of the space $S_j(\varphi)$ yields for all elements of the sequence the representation

$$\begin{aligned} f_\ell &= \sum_{k\in\mathbb{Z}^n} a_{\ell,k}\, \varphi(\cdot - M^{-j}k) \\ &= \sum_{k\in\mathbb{Z}^n} a_{\ell,k}\, \varphi(M^{-j}(M^j \cdot - k)) \\ &= \sum_{k\in\mathbb{Z}^n} a_{\ell,k}\, \varphi_{M^{-j}}(M^j \cdot - k) \end{aligned} \tag{4.1}$$

with $\varphi_{M^{-j}} := \varphi(M^{-j}\cdot)$. Next, we define the bijective, linear and continuous operator

$$\begin{aligned} J : L_2(\mathbb{R}^n) &\to L_2(\mathbb{R}^n), \\ f &\mapsto f(M^{-j}\cdot), \end{aligned} \tag{4.2}$$

and apply it to (4.1) which yields

$$Jf_\ell = \sum_{k\in\mathbb{Z}^n} a_{\ell,k}\, \varphi_{M^{-j}}(\cdot - k).$$

Hence, $Jf_\ell \in S_0(\varphi_{M^{-j}})$. With the continuity of the operator $J$, we obtain

$$\|Jf_\ell - Jf\|_{L_2(\mathbb{R}^n)} \to 0 \quad \text{for } \ell \to \infty.$$

It follows from the $L_2$-closure of the space $S_0(\varphi_{M^{-j}})$ that $Jf \in S_0(\varphi_{M^{-j}})$. By Theorem 4.1.1, there exists some $2\pi$-periodic function $\tau_\varphi$ such that the Fourier transform of $Jf$ can be written as

$$(\widehat{Jf})(\xi) = \tau_\varphi(\xi)\, \widehat{\varphi_{M^{-j}}}(\xi) \quad \text{almost everywhere.}$$

Due to (1.4), this is equivalent to

$$m^j \hat{f}((M^T)^j\xi) = m^j \tau_\varphi(\xi)\hat{\varphi}((M^T)^j\xi).$$

We set $\tilde{\xi} = (M^T)^j\xi$ and deduce

$$\hat{f}(\tilde{\xi}) = \tau_\varphi((M^T)^{-j}\tilde{\xi})\hat{\varphi}(\tilde{\xi}), \tag{4.3}$$

where $\tau_\varphi((M^T)^{-j}\widetilde{\xi})$ is $2\pi(M^T)^j\mathbb{Z}^n$-periodic. Since the Fourier transform and the operator $J^{-1}$ are continuous, we obtain

$$\|\hat{f}_\ell - \hat{f}\|_{L_2(\mathbb{R}^n)} \to 0 \quad \text{for } \ell \to \infty,$$

where $\hat{f}$ is given by (4.3).

Conversely, assume that $\hat{f}(\xi) = \tau(\xi)\hat{\varphi}(\xi)$ holds almost everywhere for a function $f \in L_2(\mathbb{R}^n)$ and some $2\pi(M^T)^j\mathbb{Z}^n$-periodic $\tau$. Set $\tilde{\xi} = (M^T)^{-j}\xi$ to obtain

$$\hat{f}((M^T)^j\tilde{\xi}) = \tau((M^T)^j\tilde{\xi})\,\hat{\varphi}((M^T)^j\tilde{\xi}) \quad \text{almost everywhere,}$$

where $\tau((M^T)^j\tilde{\xi})$ is $2\pi$-periodic. This is equivalent to

$$\widehat{(Jf)}(\tilde{\xi}) = \tau_{(M^T)^j}(\tilde{\xi})\,\widehat{\varphi_{M^{-j}}}(\tilde{\xi}) \quad \text{almost everywhere,}$$

with $\tau_{(M^T)^j} := \tau((M^T)^j\cdot)$. By Theorem 4.1.1, it follows that $Jf \in S_0(\varphi_{M^{-j}})$. By definition of this space, there exists a sequence of functions $(f_\ell)_{\ell\in\mathbb{N}} \in S_0(\varphi_{M^{-j}})$ which converges to $Jf$ in the $L_2$-sense, i.e.,

$$\|f_\ell - Jf\|_{L_2(\mathbb{R}^n)} \to 0 \quad \text{for } \ell \to \infty.$$

Every $f_\ell$ can be represented as

$$f_\ell = \sum_{k\in\mathbb{Z}^n} a_{\ell,k}\,\varphi_{M^{-j}}(\cdot - k).$$

Applying the operator $J^{-1}$ leads to

$$J^{-1}f_\ell = \sum_{k\in\mathbb{Z}^n} a_{\ell,k}\,\varphi_{M^{-j}}(M^j\cdot - k) = \sum_{k\in\mathbb{Z}^n} a_{\ell,k}\,\varphi(\cdot - M^{-j}k).$$

Consequently, $J^{-1}f_\ell \in S_j(\varphi)$ for all $\ell \in \mathbb{N}$. By the continuity of $J$, we have

$$\|J^{-1}f_\ell - f\|_{L_2(\mathbb{R}^n)} \to 0 \quad \text{for } \ell \to \infty.$$

Due to the $L_2$-closure of $S_j(\varphi)$, we conclude $f \in S_j(\varphi)$. □

Corollary 4.1.2 yields the characterization

$$\widehat{S_j(\varphi)} = \{\tau\hat{\varphi} \in L_2(\mathbb{R}^n) : \tau \text{ is } 2\pi(M^T)^j\mathbb{Z}^n\text{-periodic}\}. \tag{4.4}$$

**Remark 4.1.3.**
The function $\tau$ in (4.4) is measurable. This can be seen as follows: We have

$$\hat{f} = \tau\hat{\varphi}, \quad f \in L_2(\mathbb{R}^n), \tag{4.5}$$

where $\hat{f}$ and $\hat{\varphi}$ are Lebesgue measurable functions. Formula (4.5) implies

$$\operatorname{supp}\hat{f} \subset \operatorname{supp}\hat{\varphi}.$$

Hence, division by zero causes no problems when considering

$$\tau = \frac{\hat{f}}{\hat{\varphi}}.$$

If $\hat{\varphi} = 0$, then $\hat{f} = 0$ and $0/0 := 0$. Consequently, the measuarbility of $\tau$ follows from the well-known result that the set of Lebesgue measurable functions is closed under nonzero division, see [37, p. 518 in §20] or [10, Chapter 6.4].

A consequence of the characterization (4.4) is the following proposition which implies that if two functions generate the same principal shift-invariant space $S_j$, then the support of their Fourier transforms are equal.

**Proposition 4.1.4.**
Let $\varphi \in L_2(\mathbb{R}^n)$ and $f \in S_j(\varphi), j \in \mathbb{Z}$. Then the function $f$ generates $S_j(\varphi)$ if and only if $\operatorname{supp} \hat{\varphi} \subset \operatorname{supp} \hat{f}$.

*Proof.*
We follow the proof of [32, Corollary 2.4] where the result above was derived for $j = 0$.

Assume that $f$ generates the space $S_j(\varphi)$. Then the generator $\varphi$ is an element of the space $S_j(f)$ and Corollary 4.1.2 yields $\hat{\varphi} = \tau \hat{f}$ for some $2\pi(M^T)^j\mathbb{Z}^n$-periodic function $\tau$. Then the support of $\hat{\varphi}$ is given by

$$\operatorname{supp} \hat{\varphi} = \overline{\{x \in \mathbb{R}^n : (\tau \hat{f})(x) \neq 0\}} = \overline{\{x \in \mathbb{R}^n : \tau(x) \neq 0 \,\text{und}\, \hat{f}(x) \neq 0\}}.$$

As a consequence, $\operatorname{supp} \hat{\varphi} \subset \operatorname{supp} \hat{f}$.

Conversely, assume that it holds $\operatorname{supp} \hat{\varphi} \subset \operatorname{supp} \hat{f}$. We have to prove that $f$ and $\varphi$ generate the same space, i.e., $S_j(f) = S_j(\varphi)$. Since $f \in S_j(\varphi)$, we can use Corollary 4.1.2 again to obtain $\hat{f} = \tau \hat{\varphi}$ for some $2\pi(M^T)^j\mathbb{Z}^n$-periodic function $\tau$. Next, we define

$$\tau' := \begin{cases} \frac{1}{\tau}, & \text{on}\,\operatorname{supp}\tau, \\ 0, & \text{otherwise}. \end{cases} \tag{4.6}$$

Hence, we have $\hat{\varphi} = \tau' \hat{f}$ almost everywhere on $\operatorname{supp} \tau$. Since we know that

$$\operatorname{supp}\, \tau \supset \operatorname{supp}\, \hat{f} \supset \operatorname{supp}\, \hat{\varphi},$$

the equality holds everywhere. Then the claim follows with the help of Corollary 4.1.2 which tells us that $\varphi \in S_j(f)$. □

## 4.1.2 Density and Intersection

In the definition of the generalized multiresolution analysis the spaces $\{S_j\}_{j\in\mathbb{Z}}$ are required to satisfy

$$\text{(M2)} \ \overline{\bigcup_{j=-\infty}^{\infty} S_j(\varphi_j)} = L_2(\mathbb{R}^n) \quad \text{and} \quad \text{(M3)} \ \bigcap_{j=-\infty}^{\infty} S_j(\varphi_j) = \{0\}.$$

Here, we introduce the notation $\varphi_j$ for the generator of the space $S_j$ because we work with a non-stationary multiresolution analysis. Hence, every space $S_j$ might be generated by a different function.

In this section we investigate under which assumptions (M2) and (M3) are fulfilled. In [30] this was done for principal $2^{-j}$-shift-invariant spaces. We generalize these results for principal $M^{-j}$-shift-invariant spaces.

First, we have a closer look at condition (M2). To this end we need some preparations. The following lemma was proven within the proof of Theorem 2.3.5 in [72].

**Lemma 4.1.5.**
Any $t \in \mathbb{R}^n$ can be approximated by vectors of the form $M^{-j}k, k \in \mathbb{Z}^n, j \in \mathbb{N}$, for arbitrarily large $j$ and an $n \times n$ expanding integer matrix $M$.

In addition, we will use the so-called *Wiener's Theorem*, see [72, Theorem 2.3.4].

**Theorem 4.1.6.**
A closed subspace $X$ of the space $L_2(\mathbb{R}^n)$ is shift-invariant if and only if $\widehat{X} = L_2(\Omega)$ for some measurable set $\Omega \subset \mathbb{R}^n$. Here, $\Omega$ is uniquely determined up to a set of measure zero.

**Theorem 4.1.7.**
Let $(S_j := S_j(\varphi_j))_{j\in\mathbb{Z}}$ be a nested sequence of subspaces of the space $L_2(\mathbb{R}^n)$. Then $\overline{\bigcup_{j\in\mathbb{Z}} S_j(\varphi_j)} = L_2(\mathbb{R}^n)$ if and only if

$$\Omega_0 := \bigcup_{j\in\mathbb{Z}} \operatorname{supp} \hat{\varphi}_j = \mathbb{R}^n \text{ (modulo a null-set).} \tag{4.7}$$

*Proof.*
Let $X := \bigcup_{j\in\mathbb{Z}} S_j$. To begin with we verify that $\overline{X}$ is a closed shift-invariant subspace of $L_2(\mathbb{R}^n)$. Let $f \in X$. By assumption, the spaces $(S_j)_{j\in\mathbb{Z}}$ are nested and therefore, the function $f$ is an element of $S_j$ for all $j \geq j'$. Since $S_j$ is a $M^{-j}$-shift-invariant space, $f(\cdot + t)$ is in $X$ for $t = M^{-j}k, k \in \mathbb{Z}^n, j \in \mathbb{N}$. Lemma 1.1.5 states that the translation of functions in $L_2$ is continuous, that is, $\lim_{|r|\to 0} \|f(\cdot + r) - f\|_{L_2(\mathbb{R}^n)} = 0$

for $r \in \mathbb{R}^n$. Due to Lemma 4.1.5, it follows that $f(\cdot + t) \in \overline{X}$ for all $t \in \mathbb{R}^n$. Now, let $g \in \overline{X}$. Since the $L_2$-norm is invariant under shifts, we have

$$\|g(\cdot + t) - f(\cdot + t)\|_{L_2(\mathbb{R}^n)} = \|g - f\|_{L_2(\mathbb{R}^n)}.$$

Approximating $g$ by functions $f \in X$ yields that $g(\cdot + t) \in \overline{X}$.

With this result at hand, Theorem 4.1.6 tells us that $\widehat{\overline{X}} = L_2(\Omega)$, where $\Omega$ is a measurable subset of $\mathbb{R}^n$. Hence, $\overline{X} = L_2(\mathbb{R}^n)$ if and only if $\Omega = \mathbb{R}^n$ modulo a null-set. It remains to show that $\Omega = \Omega_0$.

Every function $\varphi_j$ is an element in $\overline{X}$ for $j \in \mathbb{Z}$. Consequently, $\operatorname{supp} \hat{\varphi}_j \subset \Omega$ modulo a null-set. Therefore, $\Omega_0 \subset \Omega$ modulo a null-set. Now, suppose that there exists a set $\Omega_1$ which is contained in $\Omega \setminus \Omega_0$ with positive measure. The Fourier transform of an element in $S_j, j \in \mathbb{Z}$, has the representation (4.4). We notice that the Fourier transform of such an element vanishes on $\Omega_1$ and thus, the Fourier transform of each element in $\bigcup_{j\in\mathbb{Z}} S_j$ vanishes on $\Omega_1$. Taking the closure, we observe that each element in $\overline{X}$ has a Fourier transform which vanishes on $\Omega_1$. This is a contradiction to the fact that $\widehat{\overline{X}}$ contains the space $L_2(\Omega_1)$. □

The subsequent corollary is a direct consequence of the preceding theorem.

**Corollary 4.1.8.**
Let $(S_j := S_j(\varphi_j))_{j\in\mathbb{Z}}$ be a nested sequence of subspaces of the space $L_2(\mathbb{R}^n)$ and $\bigcup_{j\in\mathbb{Z}} \operatorname{supp} \widehat{\varphi}_j = \mathbb{R}^n$. Then the orthogonal projectors $\mathcal{P}_{S_j}$ from $L_2(\mathbb{R}^n)$ onto $S_j$ satisfy

$$\lim_{j\to\infty} \mathcal{P}_{S_j} f = f \quad \text{for all } f \in L_2(\mathbb{R}^n).$$

*Proof.*
Due to the nestedness of the closed spaces $S_j$, we can apply Theorem 4.1.7 which yields that

$$\|f - \mathcal{P}_{S_j} f\|_{L_2(\mathbb{R}^n)} = \operatorname{dist}(f, S_j) \to 0 \quad \text{for } j \to \infty.$$

□

In case of a stationary multiresolution analysis, that is, condition (M4*) is satisfied, it is sufficient to assume that the generators of $S_0$ do not vanish in some neighborhood of the origin in order to obtain (4.7).

**Theorem 4.1.9.**
Let $(S_j := S_j(\varphi_j))_{j\in\mathbb{Z}}$ be a nested sequence of subspaces of the space $L_2(\mathbb{R}^n)$ and let $\varphi_j$ be the $M^j$-dilate of $\varphi_0$. If $\hat{\varphi}_0 \neq 0$ almost everywhere in some neighborhood of the origin, then $\overline{\bigcup_{j=-\infty}^{\infty} S_j} = L_2(\mathbb{R}^n)$.

*Proof.*
Due to (1.4), we have $\hat{\varphi}_j = c_j \hat{\varphi}_0((M^T)^{-j}\cdot)$ where $c_j$ is a constant. We know that $\lim_{j\to\infty}(M^T)^{-j}x = 0$ for all $x \in \mathbb{R}^n$, see (3.2). This means that for sufficiently large $j$ we are again in some neighborhood of the origin where $\hat{\varphi}_0 \neq 0$ holds almost everywhere. Consequently, $\bigcup_{j\in\mathbb{Z}} \operatorname{supp} \hat{\varphi}_j = \mathbb{R}^n$. Since $(S_j)_{j\in\mathbb{Z}}$ is a nested sequence our claim follows by Theorem 4.1.7. □

A possible candidate for $\varphi_j, j \in \mathbb{Z}$, in Theorem 4.1.7 and Theorem 4.1.9 would be a compactly supported $L_2(\mathbb{R}^n)$-function because such a function satisfies $\operatorname{supp} \hat{\varphi}_j = \mathbb{R}^n$, see Theorem 1.2.10.

We proceed with investigations concerning condition (M3). M. Bownik established the following relation between the number of generators and the dimension of the intersection of the spaces $(S_j)_{j\in\mathbb{Z}}$, see [13, Theorem 3.5].

**Theorem 4.1.10.**
Let $(\Phi_j)_{j\in Z}$ be a sequence of finite subsets of $L_2(\mathbb{R}^n)$ of cardinality $\leq L$, where $Z \subset \mathbb{Z}$ with $\inf_{j\in Z} j = -\infty$. Let $(S_j := S_j(\Phi_j))_{j\in Z}$ be a (not necessarily nested) sequence given by

$$S_j = \overline{\operatorname{span}\{\varphi(\cdot - M^{-j}k) : \varphi \in \Phi_j, k \in \mathbb{Z}^n\}}.$$

Then $Y := \bigcap_{j\in Z} S_j$ is a linear subspace of $L_2(\mathbb{R}^n)$ of dimension $\leq L$.

Setting $Z = \mathbb{Z}$ and $L = 1$ yields that $Y = \bigcap_{j\in\mathbb{Z}} S_j(\varphi_j)$ is a linear subspace of $L_2(\mathbb{R}^n)$ of dimension $\leq 1$. In case the dimension equals 1, we have one function $f$ which belongs to every spaces $S_j$. Hence, with Proposition 4.1.4 at hand, we can immediately deduce the next proposition.

**Proposition 4.1.11.**
Let $(S_j(\varphi_j))_{j\in\mathbb{Z}}$ be a sequence of subspaces of $L_2(\mathbb{R}^n)$ and let $f \in \bigcap_{j\in\mathbb{Z}} S_j(\varphi_j)$. Then $f$ is a generator for every space $S_j(\varphi_j)$ if and only if $\operatorname{supp} \hat{f} = \operatorname{supp} \hat{\varphi}_j$ for all $j \in \mathbb{Z}$. Moreover, the spaces $(S_j(\varphi_j))_{j\in\mathbb{Z}}$ are generated all by a single function if and only if $\operatorname{supp} \hat{\varphi}_j = \operatorname{supp} \hat{\varphi}_{j'}$ for all $j, j' \in \mathbb{Z}$.

In Corollary 4.1.8 we analysed $\lim_{j\to\infty} \mathcal{P}_{S_j} f$. Next, we analyse its counterpart, that is, $\lim_{j\to-\infty} \mathcal{P}_{S_j} f$.

**Theorem 4.1.12.**
Let $(S_j := S_j(\varphi_j))_{j\in\mathbb{Z}}$ be a nested sequence of subspaces of the space $L_2(\mathbb{R}^n)$ and let $Y = \bigcap_{j\in\mathbb{Z}} S_j$. Then $\lim_{j\to-\infty} \mathcal{P}_{S_j} f = \mathcal{P}_Y f$ for all $f \in L_2(\mathbb{R}^n)$.

*Proof.*
We follow the proof of Theorem 4.12 in [30]. First, we define $X_j := S_j$ and thus, $Y = \bigcap_{j\in\mathbb{Z}} X_j$. Moreover, we set $\mathcal{P}_j := \mathcal{P}_{X_j}$. The idea of the proof is to show

that $\mathcal{P}_j f \overset{w}{\to} g$ implies $g = \mathcal{P}_Y f$ and $g = \lim_{j\to-\infty} \mathcal{P}_j f$. This again implies that the sequence $(\mathcal{P}_j f)_j$ has limit points and that $\mathcal{P}_Y f$ is the only limit point. The existence of weak limit points follows from the boundedness of the sequence. Therefore, we initially derive the proof for a convergent subsequence $(\mathcal{P}_{j_k} f)_{j_k}$. With the help of this proof we deduce that every weak convergent subsequence converges to our desired limit and by Proposition 1.1.7, the whole sequence converges to our desired limit.

Let $g$ be the weak limit of the subsequence $(\mathcal{P}_{j_k} f)_{j_k}$. Every linear subspace $X_\ell$ with $\ell \in \mathbb{Z}$ is closed and convex and therefore, by Theorem 1.1.9, weakly closed. Since every $X_\ell$ contains every $\mathcal{P}_{j_k} f$ for $j_k \le \ell$, it follows that the weak limit $g$ is an element of every $X_\ell$. Thus, $g \in Y$. In addition, (1.1) yields that $x_{j_k} := f - \mathcal{P}_{j_k} f$ is perpendicular to $X_{j_k}$ and as a consequence, perpendicular to $Y$. Hence, the weak limit $x := f - g$ is perpendicular to $Y$ and $g = \mathcal{P}_Y f$. Since $Y$ is the intersection of the nested sequence $(X_j)_j$, we obtain with the help of Theorem 1.1.10

$$\lim_{j_k\to-\infty} \|x_{j_k}\|_{L_2(\mathbb{R}^n)} = \sup_{j_k} \operatorname{dist}(f, X_{j_k}) \le \operatorname{dist}(f, Y) = \|x\|_{L_2(\mathbb{R}^n)}. \tag{4.8}$$

From the definition of weak convergence it follows that $\langle x_{j_k}, x\rangle_{L_2(\mathbb{R}^n)} \to \langle x, x\rangle_{L_2(\mathbb{R}^n)}$. Moreover, we have

$$\begin{aligned}\|\mathcal{P}_{j_k} f - g\|^2_{L_2(\mathbb{R}^n)} &= \|x - x_{j_k}\|^2_{L_2(\mathbb{R}^n)} \\ &= \|x\|^2_{L_2(\mathbb{R}^n)} - 2\operatorname{Re}\langle x, x_{j_k}\rangle_{L_2(\mathbb{R}^n)} + \|x_{j_k}\|^2_{L_2(\mathbb{R}^n)}.\end{aligned} \tag{4.9}$$

Hence, it holds

$$\|\mathcal{P}_{j_k} f - g\|^2_{L_2(\mathbb{R}^n)} \to -\|x\|^2_{L_2(\mathbb{R}^n)} + \lim_{j_k} \|x_{j_k}\|^2_{L_2(\mathbb{R}^n)} \quad \text{for } j_k \to -\infty. \tag{4.10}$$

While (4.9) is non-negative, (4.8) yields that (4.10) is non-positive. We conclude that $\lim_{j_k} \mathcal{P}_{j_k} f = g = \mathcal{P}_Y f$. □

Corollary 4.1.8 and Theorem 4.1.12 ensure that even if $Y$ is nontrivial, we obtain an orthogonal decomposition of the space $L_2(\mathbb{R}^n)$. Consequently, a construction of wavelet bases via multiresolution analysis is possible regardless of the dimension of $Y$.

**Corollary 4.1.13.**
Let $(S_j := S_j(\varphi_j))_{j\in\mathbb{Z}}$ be a nested sequence of subspaces of the space $L_2(\mathbb{R}^n)$ and let

$$\Omega_0 = \bigcup_{j\in\mathbb{Z}} \operatorname{supp} \hat{\varphi}_j = \mathbb{R}^n \text{ (modulo a null-set).}$$

Moreover, let $Y$ denote the intersection of the spaces $S_j$, $j \in \mathbb{Z}$. Then we obtain the orthogonal decomposition

$$L_2(\mathbb{R}^n) = Y \oplus \bigoplus_{j\in\mathbb{Z}} W_j.$$

*Proof.*
Since $Y \subset S_j, j \in \mathbb{Z}$, it is orthogonal to each of the wavelet spaces $W_j := S_{j+1} \ominus S_j$. Our claim then follows by applying Corollary 4.1.8 and Theorem 4.1.12. □

In case (M4*) is satisfied, the intersection of the spaces $S_j, j \in \mathbb{Z}$, is always trivial.

**Corollary 4.1.14.**
Let $\varphi \in L_2(\mathbb{R}^n)$. We define $S_j := S_j(\varphi(M^j\cdot)), j \in \mathbb{Z}$. Then $\bigcap_{j\in\mathbb{Z}} S_j = \{0\}$.

*Proof.*
Suppose that $f$ is a nontrivial function in $\bigcap_{j\in\mathbb{Z}} S_j$. We are in the stationary case and therefore, we know that the space $S_\ell$ is the $M^\ell$-dilation of the space $S_0$. Thus, the intersection of the spaces $S_j$ is invariant under $M^\ell$-dilation. Moreover, Theorem 4.1.10 tells us that the intersection of the spaces is at most one-dimensional. With these considerations, we conclude that there exists a $\lambda$ such that

$$f(M^\ell\cdot) = \lambda f \quad \text{almost everywhere on } \mathbb{R}^n. \tag{4.11}$$

However, for a nontrivial function $f \in L_2(\mathbb{R}^n)$ equation (4.11) cannot be fulfilled. To this end consider the set

$$F_j := \{x \,|\, x \in D_j \setminus D_{j-1} \,\text{and}\, |f(x)| > C|\lambda|^j, C > 0\}, \;\; j \in \mathbb{Z}, \tag{4.12}$$

with $D_j := \{M^j x \,|\, x \in B_1(0)\}$ and $B_1(0) := \{x \in \mathbb{R}^n \,|\, |x| \leq 1\}$. Due to (3.3), $\mathbb{R}^n$ can be written as the disjoint union of the sets $F_j$. Furthermore, we use the transformation formula to obtain

$$F_j = MF_{j-1} \quad \text{and} \quad \text{meas}(F_j) = |\det M| \;\text{meas}(F_{j-1}), \quad j \in \mathbb{Z}. \tag{4.13}$$

The function $f$ is nontrivial and as a consequence the set

$$F_0 := \{x \,|\, x \in D_0 \setminus D_{-1} \,\text{and}\, |f(x)| > C, C > 0\}$$

has not measure 0. This can be proven as follows: Suppose $F_0$ has measure 0. Since the matrix $M$ is invertible, it holds $\det M \neq 0$. Then, by (4.13), every set $F_j$ has measure 0. By virtue of (4.12), we observe that $F_j, j \in \mathbb{Z}$, is of measure 0 for arbitrary $C > 0$. Consequently, $C$ can be chosen arbitrarily small such that the factor $|\lambda|^j$ gets compensated. It follows that $f$ vanishes almost everywhere on every $F_j$. This is a contradiction to our assumption $f \neq 0$. Hence, $F_0$ cannot have measure 0.

Furthermore, $M^j F_0 = F_j$ and therefore, for $x \in M^j F_0$ we have the estimate

$$|f(x)| \geq C|\lambda|^j, \tag{4.14}$$

see (4.13) and (4.11). In a last step we verify that $f$ cannot be in $L_2(\mathbb{R}^n)$. By (4.14) and (4.13), we obtain

$$\begin{aligned}\|f\|^2_{L_2(\mathbb{R}^n)} &= \int\limits_{\mathbb{R}^n} |f(x)|^2 \mathrm{d}x \\ &\geq \int\limits_{\bigcup_j F_j} |C|\lambda|^j|^2 \mathrm{d}x \\ &= C^2 \int\limits_{\bigcup_j F_j} (|\lambda|^2)^j \mathrm{d}x \\ &= C^2 \operatorname{meas}(F_0) \sum_{j\in\mathbb{Z}} (|\det M||\lambda|^2)^j. \end{aligned} \tag{4.15}$$

The series in (4.15) is divergent. As a consequence the norm of the function $f$ is not finite and thus, $f \notin L_2(\mathbb{R}^n)$. This is a contradiction. □

In summary, it is sufficient to assume in the stationary setting that $\varphi$ is compactly supported and that the spaces $S_j(\varphi(M^j\cdot)), j \in \mathbb{Z}$, are nested in order to ensure that (M2) and (M3) are satisfied.

# 4.2 Finitely Generated Shift-Invariant Spaces

## 4.2.1 Characterization

Finitely generated shift-invariant spaces are generated by multiple functions. An analogon of Theorem 4.1.1 for these spaces can be found in [32, Theorem 1.7].

**Theorem 4.2.1.**
Let $\Phi := \{\varphi_1, \ldots, \varphi_N\}$ be a subset of the space $L_2(\mathbb{R}^n)$. Then $f \in S_0(\Phi)$ if and only if

$$\hat{f} = \sum_{\varphi\in\Phi} \tau_\varphi \hat{\varphi}$$

for some $2\pi\mathbb{Z}^n$-periodic functions $\tau_\varphi$.

By adapting the periodicity of the functions $\tau_\varphi$, we obtain an analogue result for the spaces $S_j(\Phi), j \in \mathbb{Z}$.

**Corollary 4.2.2.**
Let $\Phi = \{\varphi_1, \ldots, \varphi_N\}$ be a subset of the space $L_2(\mathbb{R}^n)$. Then $f \in S_j(\Phi)$ if and only if

$$\hat{f} = \sum_{\varphi\in\Phi} \tau_\varphi \hat{\varphi} \tag{4.16}$$

for some $2\pi(M^T)^j\mathbb{Z}^n$-periodic functions $\tau_\varphi$.

*Proof.*
Assume $f \in S_j(\Phi)$. Then there exists a sequence $(f_\ell)_{\ell\in\mathbb{N}} \in S_j(\Phi)$ such that $f_\ell \to f$ in the $L_2$-sense. The definition of the space $S_j(\Phi)$ yields for all elements of the sequence the representation

$$f_\ell = \sum_{\varphi\in\Phi}\sum_{k\in\mathbb{Z}^n} a_{\ell,\varphi,k}\,\varphi(\cdot - M^{-j}k) = \sum_{\varphi\in\Phi}\sum_{k\in\mathbb{Z}^n} a_{\ell,\varphi,k}\,\varphi_{M^{-j}}(M^j\cdot - k) \tag{4.17}$$

with $\varphi_{M^{-j}} = \varphi(M^{-j}\cdot)$. Next, we apply the operator $J$ defined in (4.2) to (4.17) and we obtain

$$(Jf_\ell) = \sum_{\varphi\in\Phi}\sum_{k\in\mathbb{Z}^n} a_{\ell,\varphi,k}\,\varphi_{M^{-j}}(\cdot - k).$$

Hence, it holds $Jf_\ell \in S_0(\Phi_{M^{-j}})$ with $\Phi_{M^{-j}} := \{\varphi_{M^{-j}}, \varphi \in \Phi\}$. By the continuity of the operator $J$, we have

$$\|Jf_\ell - Jf\|_{L_2(\mathbb{R}^n)} \to 0 \quad \text{for } \ell \to \infty.$$

It follows from the $L_2$-closure of the space $S_0(\Phi_{M^{-j}})$ that $Jf \in S_0(\Phi_{M^{-j}})$. By Theorem 4.2.1, there exist $2\pi$-periodic functions $\tau_\varphi$ such that the Fourier transform of $Jf$ can be written as

$$\widehat{(Jf)}(\xi) = \sum_{\varphi\in\Phi} \tau_\varphi(\xi)\widehat{\varphi_{M^{-j}}}(\xi) \quad \text{almost everywhere.}$$

Since the Fourier transform and the operator $J^{-1}$ are continuous, we obtain

$$\|\hat{f}_\ell - \hat{f}\|_{L_2(\mathbb{R}^n)} \to 0 \quad \text{for } \ell \to \infty.$$

Moreover, the function $\hat{f}$ is given by

$$\hat{f} = \sum_{\varphi\in\Phi} \tau_\varphi((M^T)^{-j}\cdot)\hat{\varphi} \quad \text{almost everywhere.}$$

Here, the functions $\tau_\varphi((M^T)^{-j}\cdot)$ are $2\pi(M^T)^j\mathbb{Z}^n$-periodic.

Conversely, assume that $\hat{f}(\xi) = \sum_{\varphi\in\Phi}\tau_\varphi(\xi)\,\hat{\varphi}(\xi)$ holds almost everywhere for $f \in L_2(\mathbb{R}^n)$ and some $2\pi(M^T)^j\mathbb{Z}^n$-periodic functions $\tau_\varphi$. Set $\tilde{\xi} = (M^T)^{-j}\xi$. Then the functions $\tau_\varphi((M^T)^j\tilde{\xi})$ are $2\pi$-periodic and we obtain

$$\hat{f}((M^T)^j\tilde{\xi}) = \sum_{\varphi\in\Phi} \tau_\varphi((M^T)^j\tilde{\xi})\,\hat{\varphi}((M^T)^j\tilde{\xi}) \quad \text{almost everywhere.}$$

With $\tau_{\varphi,(M^T)^j} := \tau_\varphi((M^T)^j\cdot)$ this is equivalent to

$$\widehat{(Jf)}(\tilde{\xi}) = \sum_{\varphi\in\Phi} \tau_{\varphi,(M^T)^j}(\tilde{\xi})\,\widehat{\varphi_{M^{-j}}}(\tilde{\xi}) \quad \text{almost everywhere.}$$

It follows that $Jf$ is an element of the space $S_0(\Phi_{M^{-j}})$. Hence, there exists a sequence $(f_\ell)_{\ell\in\mathbb{N}} \in S_0(\Phi_{M^{-j}})$ with $f_\ell \to Jf$ in the $L_2$-sense. For each element of the sequence we have

$$f_\ell = \sum_{\varphi\in\Phi}\sum_{k\in\mathbb{Z}^n} a_{\ell,\varphi,k}\,\varphi_{M^{-j}}(\cdot - k).$$

Applying the operator $J^{-1}$ leads to

$$J^{-1}f_\ell = \sum_{\varphi\in\Phi}\sum_{k\in\mathbb{Z}^n} a_{\ell,\varphi,k}\,\varphi_{M^{-j}}(M^j\cdot - k) = \sum_{\varphi\in\Phi}\sum_{k\in\mathbb{Z}^n} a_{\ell,\varphi,k}\,\varphi(\cdot - M^{-j}k).$$

Consequently, $J^{-1}f_\ell \in S_j(\Phi)$. By the continuity of $J^{-1}$, we deduce further that

$$\|J^{-1}f_\ell - f\|_{L_2(\mathbb{R}^n)} \to 0 \quad \text{for } \ell \to \infty.$$

Due to the $L_2$-closure of the space $S_j(\Phi)$, $f$ is an element of the space $S_j(\Phi)$ as well. □

The characterization above yields

$$\widehat{S_j(\Phi)} = \left\{\sum_{\varphi\in\Phi} \tau_\varphi\hat{\varphi} \in L_2(\mathbb{R}^n) : \tau_\varphi \text{ is } 2\pi(M^T)^j\mathbb{Z}^n\text{-periodic, } \varphi\in\Phi\right\}. \tag{4.18}$$

Next, we deduce under which assumptions a finite set $F \subset S_j(\Phi)$ generates the space $S_j(\Phi)$. In the proof of Proposition 4.1.4 it was crucial that we could rearrange $\hat{f} = \tau\hat{\varphi}$ to $\frac{1}{\tau}\hat{f} = \hat{\varphi}$ on $\operatorname{supp}\tau$, see (4.6) for details. The following proposition generalizes this idea.

Hereinafter, we use the notation $\Phi$ for the set of functions $\{\varphi_1,\ldots,\varphi_N\}$ and for the vector consisting of the functions $\varphi_1,\ldots,\varphi_N$. It will always be clear from the context which interpretation is required.

**Proposition 4.2.3.**
Let $\Phi = \{\varphi_1,\ldots,\varphi_N\} \subset L_2(\mathbb{R}^n)$ and let $S_j(\Phi), j\in\mathbb{Z}$, be a finitely generated shift-invariant space. Moreover, let $F := \{f_1,\ldots,f_N\} \subset S_j(\Phi)$. Then the following equivalence holds: $F$ generates $S_j(\Phi)$ if and only if there exists a square matrix $\Gamma := (\tau_{f,\varphi})_{f\in F,\varphi\in\Phi}$ which is non-singular almost everywhere and has $2\pi(M^T)^j\mathbb{Z}^n$-periodic functions as entries such that $\widehat{\Phi} = \Gamma\,\widehat{F}$.

*Proof.*
The proof follows from Corollary 4.2.2. Every function $f_\ell, 1 \leq \ell \leq N$, in the set $F \subset S_j(\Phi)$ can be represented as

$$\hat{f}_\ell = \sum_{k=1}^{N} \tilde{\tau}_{\ell,k}\, \hat{\varphi}_k, \tag{4.19}$$

where the functions $\tilde{\tau}_{\ell,k}$ are $2\pi(M^T)^j\mathbb{Z}^n$-periodic functions. This is equivalent to $\widehat{F} = \widetilde{\Gamma}\widehat{\Phi}$ with $\widetilde{\Gamma} := (\tilde{\tau}_{\ell,k})_{\ell,k=1,\dots,N}$. Now, $F$ generates $S_j(\Phi)$ if and only if the Fourier transform of every generator $\varphi_i, 1 \leq i \leq N$, can be represented as

$$\hat{\varphi}_i = \sum_{\ell=1}^{N} \tau_{i,\ell}\, \hat{f}_\ell, \tag{4.20}$$

where the functions $\tau_{i,\ell}$ are $2\pi(M^T)^j\mathbb{Z}^n$-periodic. Condition (4.20) is equivalent to $\widehat{\Phi} = \Gamma\widehat{F}$ with $\Gamma := (\tau_{i,\ell})_{i,\ell=1,\dots,N}$. By combining (4.19) and (4.20), we obtain

$$\hat{\varphi}_i = \sum_{\ell=1}^{N} \tau_{i,\ell}\, \hat{f}_\ell = \sum_{\ell=1}^{N}\sum_{k=1}^{N} \tau_{i,\ell}\, \tilde{\tau}_{\ell,k}\, \hat{\varphi}_k.$$

Hence, $F$ generates $S_j(\Phi)$ if and only if

$$\sum_{\ell=1}^{N} \tau_{i,\ell}\, \tilde{\tau}_{\ell,k} = \delta_{i,k}.$$

In other words, the matrix $\widetilde{\Gamma}$ is the inverse of $\Gamma$. □

Following [32] and [30], we call $\Phi$ a *basis* for the space $S_j(\Phi)$ if the functions $\tau_\varphi$ in (4.16) are uniquely determined by $f$ or equivalently, if the determinant of the Gramian matrix is non-zero almost everywhere. We remark that in this case the functions $\tau_\varphi$ are proven to be measurable, see Corollary 3.11 in [32].

**Definition 4.2.4.**
Let $\Phi = \{\varphi_1, \dots, \varphi_N\} \subset S_j$ with $j \in \mathbb{Z}$. The *Gramian matrix* associated with $\Phi$ is defined by

$$G(\Phi) := \begin{pmatrix} [\hat{\varphi}_1, \hat{\varphi}_1]_j & [\hat{\varphi}_1, \hat{\varphi}_2]_j & \dots & [\hat{\varphi}_1, \hat{\varphi}_N]_j \\ [\hat{\varphi}_2, \hat{\varphi}_1]_j & [\hat{\varphi}_2, \hat{\varphi}_2]_j & \dots & [\hat{\varphi}_2, \hat{\varphi}_N]_j \\ \vdots & \vdots & \ddots & \vdots \\ [\hat{\varphi}_N, \hat{\varphi}_1]_j & [\hat{\varphi}_N, \hat{\varphi}_2]_j & \dots & [\hat{\varphi}_N, \hat{\varphi}_N]_j \end{pmatrix}.$$

The Gramian matrix is positive semidefinite and thus, all its eigenvalues are non-negative. Hence, it holds $\det G(\Phi) \geq 0$. Consequently, $\Phi$ is a basis if and only if the determinant of the Gramian matrix is non-zero almost everywhere.

For application purposes, one is interested in numerically stable algorithms. Therefore, it is desirable to work with $L_2$-stable bases.

**Definition 4.2.5.**
Let $\Phi = \{\varphi_1, \ldots, \varphi_N\} \subset L_2(\mathbb{R}^n)$. We call $\Phi$ an $L_2$-*stable basis* of $S_j(\Phi)$ if every element $f \in S_j(\Phi)$ has a unique representation of the form

$$f = \sum_{\varphi\in\Phi} \sum_{k\in\mathbb{Z}^n} c_{\varphi,k}(f)\varphi(\cdot - M^{-j}k) \tag{4.21}$$

and the coefficients satisfy

$$C_1 \sum_{\varphi\in\Phi} \sum_{k\in\mathbb{Z}^n} |c_{\varphi,k}(f)|^2 \leq \|f\|^2_{L_2(\mathbb{R}^n)} \leq C_2 \sum_{\varphi\in\Phi} \sum_{k\in\mathbb{Z}^n} |c_{\varphi,k}(f)|^2$$

with $0 < C_1 \leq C_2 < \infty$.

Hence, $L_2$-stability ensures that a small perturbation of the coefficient sequence has a controllable effect on the linear combination in (4.21).

Now, that we have established the terms basis and $L_2$-stability, we consider a space with an ($L_2$-stable) basis and analyse under which conditions a finite set of functions from the space provides an ($L_2$-stable) basis as well, see [30, Theorem 2.26].

**Theorem 4.2.6.**
Let the finite set of functions $\Phi = \{\varphi_1, \ldots, \varphi_N\} \subset L_2(\mathbb{R}^n)$ provide a basis for $S_0(\Phi)$. In addition, let $\Psi$ be any set of functions from $S_0(\Phi)$ and let $\Gamma := (\tau_{\psi,\varphi})_{\psi\in\Psi,\varphi\in\Phi}$ denote a square matrix with $2\pi\mathbb{Z}^n$-periodic measurable functions as entries. Then it holds:

i) $\Psi$ provides a basis for $S_0(\Phi)$ if and only if $\hat{\Psi} = \Gamma\hat{\Phi}$ for some $\Gamma$ which is non-singular almost everywhere.

ii) $\Psi$ provides a basis for $S_0(\Phi)$ if and only if $\Psi$ generates the space $S_0(\Phi)$ and $\#\Psi = \#\Phi$.

iii) $\Psi$ provides a basis for $S_0(\Phi)$ if and only if $\#\Psi = \#\Phi$ and $\det G(\Psi) \neq 0$ almost everywhere.

iv) $\Psi$ provides an $L_2$-stable basis for $S_0(\Phi)$ if $\Phi$ does and $\hat{\Psi} = \Gamma\hat{\Phi}$ with $\|\Gamma\|, \|\Gamma^{-1}\| \in L_\infty(\widetilde{C}_0), \widetilde{C}_0 = [-\pi, \pi)^n$.

In order to generalize the theorem above for spaces $S_j$, we will need the following lemma.

**Lemma 4.2.7.**
Let $\Phi = \{\varphi_1, \dots, \varphi_N\} \subset L_2(\mathbb{R}^n)$ generate the space $S_j(\Phi)$ and let $\Psi$ be any set from the space $S_j(\Phi), j \in \mathbb{Z}$. Then $S_j(\Phi) = S_j(\Psi)$ if and only if $S_0(\Phi_{M^{-j}}) = S_0(\Psi_{M^{-j}})$ with $\Phi_{M^{-j}} = \{\varphi(M^{-j}\cdot), \varphi \in \Phi\}$ and $\Psi_{M^{-j}} := \{\psi(M^{-j}\cdot), \psi \in \Psi\}$.

*Proof.*
Assume that $S_j(\Phi) = S_j(\Psi)$. Then for every generator $\varphi \in \Phi$ there exists a sequence $(f_\ell)_{\ell \in \mathbb{N}} \in S_j(\Phi)$ with

$$f_\ell = \sum_{\psi \in \Psi} \sum_{k \in \mathbb{Z}^n} a_{\ell,\psi,k} \psi_{M^{-j}}(M^j \cdot -k), \quad \psi_{M^{-j}} := \psi(M^{-j}\cdot),$$

such that $f_\ell \to \varphi$ in the $L_2$-sense. Consequently, $Jf_\ell$ is given by

$$Jf_\ell = \sum_{\psi \in \Psi} \sum_{k \in \mathbb{Z}^n} a_{\ell,\psi,k} \psi_{M^{-j}}(\cdot - k)$$

and therefore, $Jf_\ell \in S_0(\Psi_{M^{-j}})$. Due to the continuity of the operator $J$, we have

$$\|Jf_\ell - J\varphi\|_{L_2(\mathbb{R}^n)} = \|Jf_\ell - \varphi(M^{-j}\cdot)\|_{L_2(\mathbb{R}^n)} \to 0 \quad \text{for } \ell \to \infty.$$

Then it follows from the $L_2$-closure of the space $S_0$ that $\varphi(M^{-j}\cdot) \in S_0(\Psi_{M^{-j}})$ and thus, $S_0(\Phi_{M^{-j}}) \subseteq S_0(\Psi_{M^{-j}})$. It can be proven analogously that it holds $S_0(\Phi_{M^{-j}}) \supseteq S_0(\Psi_{M^{-j}})$ and thus, $S_0(\Phi_{M^{-j}}) = S_0(\Psi_{M^{-j}})$.

Conversely, assume that $S_0(\Phi_{M^{-j}}) = S_0(\Psi_{M^{-j}})$. Since $J$ is a bijection, we can use the same arguments as above to deduce that $S_j(\Phi) = S_j(\Psi)$. □

**Corollary 4.2.8.**
Let the finite set of functions $\Phi = \{\varphi_1, \dots, \varphi_N\} \subset L_2(\mathbb{R}^n)$ provide a basis for $S_j(\Phi)$. In addition, let $\Psi$ be any set of functions from $S_j(\Phi)$ and let $\Gamma = (\tau_{\psi,\varphi})_{\psi \in \Psi, \varphi \in \Phi}$ denote a square matrix with $2\pi(M^T)^j\mathbb{Z}^n$-periodic measurable functions as entries. Then it holds:

i) $\Psi$ provides a basis for $S_j(\Phi)$ if and only if $\hat{\Psi} = \Gamma\hat{\Phi}$ for some $\Gamma$ which is nonsingular almost everywhere.

ii) $\Psi$ provides a basis for $S_j(\Phi)$ if and only if $\Psi$ generates the space $S_j(\Phi)$ and $\#\Psi = \#\Phi$.

iii) $\Psi$ provides a basis for $S_j(\Phi)$ if and only if $\#\Psi = \#\Phi$ and $\det G(\Psi) \neq 0$ almost everywhere.

iv) $\Psi$ provides an $L_2$-stable basis for $S_j(\Phi)$ if $\Phi$ does and $\hat{\Psi} = \Gamma\hat{\Phi}$ with $\|\Gamma\|, \|\Gamma^{-1}\| \in L_\infty(\widetilde{C}_j), \widetilde{C}_j = (M^T)^j[-\pi,\pi)^n$.

*Proof.*
Let us start with the proof of part i). We assume that $\Phi$ and $\Psi$ provide a basis for $S_j(\Phi)$. This implies $S_j(\Phi) = S_j(\Psi)$ and by Lemma 4.2.7, $S_0(\Phi_{M^{-j}}) = S_0(\Psi_{M^{-j}})$. Next, we verify that $\Phi_{M^{-j}}$ and $\Psi_{M^{-j}}$ provide a basis for $S_0(\Phi_{M^{-j}})$. For every function $f \in S_j(\Phi)$ there exist sequences $(f_\ell^i)_{\ell\in\mathbb{N}} \in S_j(\Phi), i = 1,2$, with unique representations

$$f_\ell^1 = \sum_{\psi\in\Psi}\sum_{k\in\mathbb{Z}^n} a_{\ell,\psi,k}\psi(\cdot - M^{-j}k), \quad f_\ell^2 = \sum_{\varphi\in\Phi}\sum_{k\in\mathbb{Z}^n} b_{\ell,\varphi,k}\varphi(\cdot - M^{-j}k),$$

such that $f_\ell^i \to f$ in the $L_2$-sense. An application of $J$ yields

$$Jf_\ell^1 = \sum_{\psi\in\Psi}\sum_{k\in\mathbb{Z}^n} a_{\ell,\psi,k}\psi_{M^{-j}}(\cdot - k), \quad \psi_{M^{-j}} = \psi(M^{-j}\cdot),$$

and

$$Jf_\ell^2 = \sum_{\varphi\in\Phi}\sum_{k\in\mathbb{Z}^n} a_{\ell,\varphi,k}\varphi_{M^{-j}}(\cdot - k), \quad \varphi_{M^{-j}} = \varphi(M^{-j}\cdot).$$

We observe that $Jf_\ell^i \in S_0(\Phi_{M^{-j}}), i = 1,2$. Since $J$ is a bijection, the representations above are unique and since $J$ is continuous, we deduce

$$\|Jf_\ell^i - Jf\|_{L_2(\mathbb{R}^n)} \to 0 \quad \text{for } \ell \to \infty.$$

Due to the $L_2$-closure of the space $S_0$, it follows that $Jf \in S_0(\Phi_{M^{-j}})$. Now, part i) of Theorem 4.2.6 yields that

$$\widehat{\Psi_{M^{-j}}}(\xi) = \Gamma(\xi)\widehat{\Phi_{M^{-j}}}(\xi) \tag{4.22}$$

for some non-singular $\Gamma$ with $2\pi\mathbb{Z}^n$-periodic entries. We set $\tilde{\xi} = (M^T)^j\xi$ and replace $\xi$ by $(M^T)^{-j}\tilde{\xi}$ in (4.22) to obtain the desired result.

Conversely, we assume that $\hat{\Psi} = \Gamma\hat{\Phi}$ holds with $\Gamma$ being non-singular almost everywhere. Moreover, we assume that $\Gamma$ has $2\pi(M^T)^j\mathbb{Z}^n$-periodic entries. By analogue arguments as above, the equality $\hat{\Psi} = \Gamma\hat{\Phi}$ can be transformed to (4.22). Then part i) of Theorem 4.2.6 yields that $\Psi_{M^{-j}}$ provides a basis for $S_0(\Phi_{M^{-j}})$ and consequently, $\Psi$ provides a basis for $S_j(\Phi)$.

Next, we prove part ii). Assume that $\Psi$ provides a basis for the space $S_j(\Phi)$. It follows directly that $\Psi$ generates this space. Moreover, $\Psi_{M^{-j}}$ provides a basis for $S_0(\Phi_{M^{-j}})$, see proof of part i). By part ii) of Theorem 4.2.6, $\Psi_{M^{-j}}$ and $\Phi_{M^{-j}}$ are of the same cardinality. Applying the bijective operator $J^{-1}$ yields $\#\Psi = \#\Phi$.

Conversely, we assume that $\Psi$ generates the space $S_j(\Phi)$ and $\#\Psi = \#\Phi$. Since $J$ is a bijection, it follows directly that $\#\Phi_{M^{-j}} = \#\Psi_{M^{-j}}$. Moreover, $S_j(\Phi) = S_j(\Psi)$ and therefore, $S_0(\Phi_{M^{-j}}) = S_0(\Psi_{M^{-j}})$, see Lemma 4.2.7. Besides that we know that if $\Phi$ provides a basis for $S_j(\Phi)$, then $\Phi_{M^{-j}}$ provides a basis for $S_0(\Phi_{M^{-j}})$ as well, see proof of part i). By part ii) of Theorem 4.2.6, $\Psi_{M^{-j}}$ provides a basis for $S_0(\Phi_{M^{-j}})$ and hence, $\Psi$ provides a basis for $S_j(\Phi)$.

Now, we prove part iii). In the proof of part ii) we already deduced that every possible basis $\Psi$ of $S_j(\Phi)$ has the same number of elements as $\Phi$. Moreover, $\Psi$ provides a basis for $S_j(\Phi)$ if and only if every element in the space $S_j(\Phi)$ can be represented by a unique linear combination of the basis elements. This is the case if and only if $\det G(\Psi) \neq 0$ almost everywhere.

It remains to prove part iv). Let $\Phi$ provide an $L_2$-stable basis for $S_j(\Phi)$. Then every function $f \in S_j(\Phi)$ has a unique representation of the form

$$f = \sum_{\varphi\in\Phi} \sum_{k\in\mathbb{Z}^n} a_{\varphi,k}\varphi(\cdot - M^{-j}k) \tag{4.23}$$

with $(a_{\varphi,k})_{\varphi\in\Phi,k\in\mathbb{Z}^n} \in \ell_2(\mathbb{Z}^n)$. Moreover, the coefficients satisfy

$$C_1 \sum_{\varphi\in\Phi} \sum_{k\in\mathbb{Z}^n} |a_{\varphi,k}|^2 \leq \|f\|^2_{L_2(\mathbb{R}^n)} \leq C_2 \sum_{\varphi\in\Phi} \sum_{k\in\mathbb{Z}^n} |a_{\varphi,k}|^2 \tag{4.24}$$

with $0 < C_1 \leq C_2 < \infty$. Formula (4.23) is equivalent to

$$f = \sum_{\varphi\in\Phi} \sum_{k\in\mathbb{Z}^n} a_{\varphi,k}\, \varphi_{M^{-j}}(M^j \cdot - k).$$

Next, we apply the operator $J$ in order to obtain

$$Jf = \sum_{\varphi\in\Phi} \sum_{k\in\mathbb{Z}^n} a_{\varphi,k}\, \varphi_{M^{-j}}(\cdot - k).$$

The inequality (4.24) is still satisfied for the functions $Jf$. This follows from the transformation formula. Consequently, $\Phi_{M^{-j}}$ provides an $L_2$-stable basis for $S_0(\Phi_{M^{-j}})$. Moreover, let

$$\hat{\Psi} = \Gamma\hat{\Phi}$$

with $\|\Gamma\|, \|\Gamma^{-1}\| \in L_\infty(\widetilde{C}_j)$. Then it follows that

$$\widehat{\Psi_{M^{-j}}} = \Gamma((M^T)^j\cdot)\widehat{\Phi_{M^{-j}}}$$

with $\|\Gamma((M^T)^j\cdot)\|, \|\Gamma^{-1}((M^T)^j\cdot)\| \in L_\infty(\widetilde{C}_0)$. An application of part iv) of Theorem 4.2.6 yields that $\Psi_{M^{-j}}$ provides an $L_2$-stable basis for $S_0(\Phi_{M^{-j}})$ and thus, $\Psi$ provides an $L_2$-stable basis for $S_j(\Phi)$. □

The subsequent corollary states that a basis of a shift-invariant space $S_j$ can be orthonormalized. This result can be found in [30, Theorem 2.28] for $j = 0$. We obtain an extension of this result for spaces $S_j, j \in \mathbb{Z}$, by using the same arguments that we have already used several times in the preceding proofs.

**Corollary 4.2.9.**
Let the finite set of functions $\Phi = \{\varphi_1, \ldots, \varphi_N\} \subset L_2(\mathbb{R}^n)$ provide a basis for the space $S_j(\Phi), j \in \mathbb{Z}$.

i) Then there exists a set $\Phi^*$ of generators for $S_j(\Phi)$ that provides an orthonormal basis for $S_j(\Phi)$.

ii) If the functions in $\Phi$ have compact support, then there is another set $\Phi^* = \{\varphi_1^*, \ldots, \varphi_N^*\}$ of compactly supported functions which give the orthogonal decomposition of the space $S_j(\Phi)$:

$$S_j(\Phi) = S_j(\varphi_1^*) \oplus \ldots \oplus S_j(\varphi_N^*).$$

For the proof of Lemma 6.2.1 we generalize the following theorem which can be found in [49, Theorem 5.2].

**Theorem 4.2.10.**
Let the finite set of functions $\Phi = \{\varphi_1, \ldots, \varphi_N\} \subset L_2(\mathbb{R}^n)$ generate the space $S_0(\Phi)$. Moreover, let the functions $\varphi_i, i = 1, \ldots, N$, have compact support and linearly independent integer translates. Furthermore, let $a_1, \ldots, a_N$ be sequences on $\mathbb{Z}^n$. If $\sum_{i=1}^N \sum_{k \in \mathbb{Z}^n} a_{\varphi_i,k} \varphi_i(\cdot - k)$ is compactly supported, then all the sequences $a_1, \ldots, a_N$ are finitely supported.

**Corollary 4.2.11.**
Let the finite set of functions $\Phi = \{\varphi_1, \ldots, \varphi_N\} \subset L_2(\mathbb{R}^n)$ generate the space $S_j(\Phi), j \in \mathbb{Z}$. Moreover, let the functions $\varphi_i, i = 1, \ldots, N$, have compact support and linearly independent $M^{-j}\mathbb{Z}^n$ translates. Furthermore, let $a_1, \ldots, a_N$ be sequences on $\mathbb{Z}^n$. If $\sum_{i=1}^N \sum_{k \in \mathbb{Z}^n} a_{\varphi_i,k} \varphi_i(\cdot - M^{-j}k)$ is compactly supported, then all the sequences $a_1, \ldots, a_N$ are finitely supported.

*Proof.*
We set $\varphi_{M^{-j}}(y) = \varphi(M^{-j}y)$ and $x = M^j y$ for $x, y \in \mathbb{R}^n$. Then we have the equality $\varphi_{M^{-j}}(x - k) = \varphi(y - M^{-j}k), k \in \mathbb{Z}^n$. The function $\varphi_{M^{-j}}$ has linearly independent $\mathbb{Z}^n$-shifts whenever the function $\varphi$ has linearly independent $M^{-j}\mathbb{Z}^n$-shifts. Since

$$\sum_{\varphi \in \Phi} \sum_{k \in \mathbb{Z}^n} a_{\varphi,k}\, \varphi_{M^{-j}}(x - k) = \sum_{\varphi \in \Phi} \sum_{k \in \mathbb{Z}^n} a_{\varphi,k}\, \varphi(y - M^{-j}k),$$

we deduce further that the left-hand side is compactly supported whenever the right-hand side is compactly supported. By Theorem 4.2.10 our proof is complete. □

## 4.2.2 Density and Intersection

Conditions (M2) and (M3) in the definition of the generalized multiresolution analysis turn into

$$\text{(M2)} \ \overline{\bigcup_{j=-\infty}^{\infty} S_j(\Phi_j)} = L_2(\mathbb{R}^n) \quad \text{and} \quad \text{(M3)} \ \bigcap_{j=-\infty}^{\infty} S_j(\Phi_j) = \{0\}$$

if we work with finitely generated shift-invariant spaces. Here, $\Phi_j := \{\varphi_1^j, \dots, \varphi_N^j\}$ denotes the generator set of the space $S_j$ and $\varphi_i^j, 1 \le i \le N$, denotes a generator which belongs to the generator set $\Phi_j$. Hereinafter, we generalize the results from Section 4.1.2.

**Theorem 4.2.12.**
Let $(S_j := S_j(\Phi_j))_{j \in \mathbb{Z}}$ be a nested sequence of subspaces of the space $L_2(\mathbb{R}^n)$. Then $\overline{\bigcup_{j\in\mathbb{Z}} S_j} = L_2(\mathbb{R}^n)$ if and only if

$$\Omega_0 := \bigcup_{j\in\mathbb{Z}} \bigcup_{i\in\{1,\dots,N\}} \operatorname{supp} \hat{\varphi}_i^j = \mathbb{R}^n \text{ (modulo a null-set).} \tag{4.25}$$

*Proof.*
The proof of Theorem 4.1.7 yields that $\overline{X} = \overline{\bigcup_{j\in\mathbb{Z}} S_j}$ is a closed shift-invariant subspace of $L_2(\mathbb{R}^n)$. The number of generators of the spaces $S_j$ does not change the proof. With this result at hand, Theorem 4.1.6 tells us that

$$\widehat{\overline{X}} = L_2(\Omega),$$

where $\Omega$ is a measurable subset of $\mathbb{R}^n$. Hence, $\overline{X} = L_2(\mathbb{R}^n)$ if and only if $\Omega = \mathbb{R}^n$ modulo a null-set. It remains to show that $\Omega = \Omega_0$.

Every function $\varphi_i^j$ is an element in $\overline{X}$ for $j \in \mathbb{Z}$ and $i = 1, \dots, N$. Consequently, $\operatorname{supp} \hat{\varphi}_i^j \subset \Omega$ modulo a null-set. Therefore, $\Omega_0 \subset \Omega$ modulo a null-set. Now, suppose that there exists a set $\Omega_1$ which is contained in $\Omega \setminus \Omega_0$ with positive measure. The Fourier transform of elements in $S_j$ has the representation (4.18). We notice that the Fourier transform of these elements vanishes on $\Omega_1$ and thus, the Fourier transform of each element in $\bigcup_{j\in\mathbb{Z}} S_j$ vanishes on $\Omega_1$. Taking the closure, we observe that each element in $\overline{X}$ has a Fourier transform which vanishes on $\Omega_1$. This is a contradiction to the fact that $\widehat{\overline{X}}$ contains the space $L_2(\Omega_1)$. □

The subsequent corollary is a direct consequence of the preceding theorem.

**Corollary 4.2.13.**
Let $(S_j = S_j(\Phi_j))_{j\in\mathbb{Z}}$ be a nested sequence of subspaces of the space $L_2(\mathbb{R}^n)$ and $\bigcup_{j\in\mathbb{Z}} \bigcup_{i\in\{1,\dots,N\}} \operatorname{supp} \hat{\varphi}_i^j = \mathbb{R}^n$. Then the orthogonal projectors $\mathcal{P}_{S_j}$ from $L_2(\mathbb{R}^n)$ onto $S_j$ satisfy

$$\lim_{j\to\infty} \mathcal{P}_{S_j} f = f \quad \text{for all } f \in L_2(\mathbb{R}^n).$$

*Proof.*
Due to the nestedness of the closed spaces $S_j$, we can apply Theorem 4.2.12 which yields that

$$\|f - \mathcal{P}_{S_j} f\|_{L_2(\mathbb{R}^n)} = \operatorname{dist}(f, S_j) \to 0 \quad \text{for } j \to \infty.$$

$\square$

In case of a stationary multiresolution analysis it is sufficient to assume that the generators of $S_0$ do not vanish in some neighborhood of the origin in order to obtain (4.25).

**Theorem 4.2.14.**
Let $(S_j = S_j(\Phi_j))_{j\in\mathbb{Z}}$ be a nested sequence of subspaces of $L_2(\mathbb{R}^n)$. Moreover, assume that $S_j(\Phi_j)$ is the $M^j$-dilation of $S_0(\Phi_0)$, that is, $\Phi_j = \{\varphi_i^0(M^j\cdot), i = 1, \ldots, N\}$. If $\hat{\varphi}_i^0 \neq 0$ almost everywhere in some neighborhood of the origin for $i = 1, \ldots, N$, then it holds

$$\overline{\bigcup_{j=-\infty}^{\infty} S_j} = L_2(\mathbb{R}^n).$$

*Proof.*
Due to (1.4), we have $\hat{\varphi}_i^j = c_j\, \hat{\varphi}_i^0((M^T)^{-j}\cdot)$ where $c_j$ is a constant. We know that $\lim_{j\to\infty}(M^T)^{-j}x = 0$ for all $x \in \mathbb{R}^n$, see (3.2). This means that for sufficiently large $j$ we are again in some neighborhood of the origin where $\hat{\varphi}_i^0 \neq 0$ holds almost everywhere. Consequently, $\bigcup_{j\in\mathbb{Z}} \bigcup_{i\in\{1,\ldots,N\}} \operatorname{supp} \hat{\varphi}_i^j = \mathbb{R}^n$. Since $(S_j)_{j\in\mathbb{Z}}$ is a nested sequence our claim follows by Theorem 4.2.12. which yields $\overline{\bigcup_{j=-\infty}^{\infty} S_j} = L_2(\mathbb{R}^n)$. $\square$

Possible candidates for the generators of the finitely generated shift-invariant space in Theorem 4.2.12 and Theorem 4.2.14 are compactly supported functions in $L_2(\mathbb{R}^n)$ because the support of the Fourier transform of such functions equals $\mathbb{R}^n$, see Theorem 1.2.10.

Now, let us have a closer look at condition (M3). Theorem 4.1.10 in Section 4.1.2 yields that

$$Y = \bigcap_{j\in\mathbb{Z}} S_j(\Phi_j)$$

is a linear subspace of $L_2(\mathbb{R}^n)$ of dimension $\leq N$.

The following theorem states that the orthogonal projection of a function $f$ onto $Y$ is given by $\lim_{j\to-\infty} \mathcal{P}_{S_j} f$.

**Theorem 4.2.15.**
Let $(S_j = S_j(\Phi_j))_{j\in\mathbb{Z}}$ be a nested sequence of subspaces of the space $L_2(\mathbb{R}^n)$ and let $Y = \bigcap_{j\in\mathbb{Z}} S_j(\Phi_j)$. Then $\lim_{j\to-\infty} \mathcal{P}_{S_j} f = \mathcal{P}_Y f$ for all $f \in L_2(\mathbb{R}^n)$.

*Proof.*
The claim follows from the proof of Theorem 4.1.12 where the number of generators does not affect the proof. □

Taking Corollary 4.2.13 and Theorem 4.2.15 into consideration, we see that $L_2(\mathbb{R}^n)$ has an orthogonal decomposition. Therefore, we can construct wavelet bases via multiresolution analysis in case of finitely generated shift-invariant spaces as well.

**Corollary 4.2.16.**
Let $(S_j = S_j(\Phi_j))_{j\in\mathbb{Z}}$ be a nested sequence of subspaces of the space $L_2(\mathbb{R}^n)$ and let

$$\Omega_0 = \bigcup_{j\in\mathbb{Z}} \bigcup_{i\in\{1,\ldots,N\}} \operatorname{supp} \hat{\varphi}_i^j = \mathbb{R}^n \text{ (modulo a null-set).}$$

Moreover, let $Y$ denote the intersection of the spaces $S_j, j \in \mathbb{Z}$. Then we obtain the orthogonal decomposition

$$L_2(\mathbb{R}^n) = Y \oplus \bigoplus_{j\in\mathbb{Z}} W_j.$$

*Proof.*
Since $Y \subset S_j, j \in \mathbb{Z}$, it is orthogonal to each of the wavelet spaces $W_j := S_{j+1} \ominus S_j$. Our claim then follows by applying Corollary 4.2.13. and Theorem 4.2.15. □

Next, we want to prove that $Y$ is trivial in the stationary case. In [50], this was proven explicitly for $M = 2I$. In addition, the authors state in Remark 2.6 that the proof is analgoue for an arbitrary dilation matrix. In the following, we are going to adapt their method of proof to an arbitrary dilation matrix. The subsequent lemma is needed in this process, see [50, Lemma 2.3].

**Lemma 4.2.17.**
Let $S_0$ be generated by $\Phi_0 = \{\varphi_1^0, \ldots, \varphi_N^0\} \subset L_2(\mathbb{R}^n)$. Then there exists a set of functions $\Psi_0 = \{\psi_1^0, \ldots, \psi_N^0\} \subset L_2(\mathbb{R}^n)$ such that $S_0(\Phi_0) \subseteq S_0(\Psi_0)$ and the shifts of $\psi_1^0, \ldots, \psi_N^0$ are orthonormal.

**Theorem 4.2.18.**
Let $(S_j = S_j(\Phi_j))_{j\in\mathbb{Z}}$ be a nested sequence of subspaces of the space $L_2(\mathbb{R}^n)$ and let $S_j$ be the $M^j$-dilation of the space $S_0$, that is, $\Phi_j = \{\varphi_i^0(M^j\cdot), i = 1, \ldots, N\}$. Then it holds

$$\bigcap_{j\in\mathbb{Z}} S_j = \{0\}.$$

*Proof.*
Lemma 4.2.17 and the nestedness assumption $S_j \subset S_{j+1}$ imply that it is sufficient to prove the theorem for the case that the shifts of the generators of $S_0(\Phi_0)$ are orthonormal. In case the generators of $S_0(\Phi_0)$ are not orthonormal, we consider the space $S_0(\Psi_0)$ defined as in Lemma 4.2.17. Next, we set $S_j$ as the $M^j$-dilation of the space $S_0(\Psi_0)$. If the intersection of the spaces $S_j(\Psi_0(M^j\cdot))$ is trivial then it follows directly that the intersection of the spaces $S_j(\Phi_0(M^j\cdot)$ is trivial as well.

In the following we denote by $\mathcal{P}_{S_j}$ the orthogonal projector from the space $L_2(\mathbb{R}^n)$ onto $S_j$. Our claim follows if we are able to prove

$$\mathcal{P}_{S_j} f \to 0 \quad \text{as } j \to -\infty \text{ for every } f \in L_2(\mathbb{R}^n). \tag{4.26}$$

We note that

$$\|\mathcal{P}_{S_j}\| = \begin{cases} 1, & \text{if } S_j \neq \{0\}, \\ 0, & \text{if } S_j = \{0\}, \end{cases}$$

and that the set of continuous functions with compact support is dense in $L_2(\mathbb{R}^n)$. Hence, for each $f \in L_2(\mathbb{R}^n)$ there exists a continuous function with compact support $\tilde{f}$ with $\|\tilde{f} - f\|_{L_2(\mathbb{R}^n)} < \varepsilon$ for all $\varepsilon > 0$. Now, assume we have $\mathcal{P}_{S_j}\tilde{f} \to 0$ for $j \to -\infty$. Then, the estimate

$$\begin{aligned} \left|\|\mathcal{P}_{S_j}\tilde{f}\|_{L_2(\mathbb{R}^n)} - \|\mathcal{P}_{S_j} f\|_{L_2(\mathbb{R}^n)}\right| &\leq \|\mathcal{P}_{S_j}\tilde{f} - \mathcal{P}_{S_j} f\|_{L_2(\mathbb{R}^n)} \\ &= \|\mathcal{P}_{S_j}(\tilde{f} - f)\|_{L_2(\mathbb{R}^n)} \\ &\leq \|\tilde{f} - f\|_{L_2(\mathbb{R}^n)} \\ &< \varepsilon \end{aligned}$$

implies that $\mathcal{P}_{S_j} f \to 0$ for $j \to -\infty$. Hence, it suffices to prove (4.26) for any continuous function $f \in L_2(\mathbb{R}^n)$ with compact support. If $f$ is such a function the orthogonal projection of $f$ onto the space $S_j$ is given by

$$\mathcal{P}_{S_j} f = \sum_{k=1}^{N} \sum_{\alpha \in \mathbb{Z}^n} c_{j,k}(\alpha) m^{j/2} \varphi_k(M^j \cdot - \alpha), \quad \varphi_k := \varphi_k^0, \tag{4.27}$$

where $c_{j,k}$ are sequences in $\ell_2(\mathbb{Z}^n)$ for $k = 1, \dots, N$. The orthonormality of the shifts of $\varphi_1, \dots, \varphi_N$ and (4.27) yield

$$\|\mathcal{P}_{S_j} f\|^2_{L_2(\mathbb{R}^n)} = \sum_{k=1}^{N} \sum_{\alpha \in \mathbb{Z}^n} |c_{j,k}(\alpha)|^2 = \sum_{k=1}^{N} \|c_{j,k}\|^2_{\ell_2(\mathbb{Z}^n)}. \tag{4.28}$$

Moreover, with Proposition 1.1.4 we obtain

$$\mathcal{P}_{S_j} f = \sum_{k=1}^{N} \sum_{\alpha \in \mathbb{Z}^n} \left\langle f, m^{j/2} \varphi_k(M^j \cdot - \alpha) \right\rangle_{L_2(\mathbb{R}^n)} m^{j/2} \varphi_k(M^j \cdot - \alpha). \tag{4.29}$$

Therefore, we set $c_{j,k}(\alpha) := \left\langle f, m^{j/2}\varphi_k(M^j \cdot -\alpha)\right\rangle_{L_2(\mathbb{R}^n)}$. By (4.28) and (4.29), we further deduce that

$$\begin{aligned}\sum_{k=1}^{N}\sum_{\alpha\in\mathbb{Z}^n}|c_{j,k}(\alpha)|^2 &= \sum_{k=1}^{N}\sum_{\alpha\in\mathbb{Z}^n}\left|\left\langle f, m^{j/2}\varphi_k(M^j \cdot -\alpha)\right\rangle_{L_2(\mathbb{R}^n)}\right|^2\\ &= \sum_{k=1}^{N}\sum_{\alpha\in\mathbb{Z}^n}\left|\int_{\mathbb{R}^n} f(x)m^{j/2}\,\overline{\varphi_k(M^jx-\alpha)}\,\mathrm{d}x\right|^2.\end{aligned}$$

If we assume that $f$ is supported in the cube $[-R,R]^n$ for some $R>0$, it follows that

$$\begin{aligned}&\sum_{k=1}^{N}\sum_{\alpha\in\mathbb{Z}^n}\left|\int_{\mathbb{R}^n} f(x)m^{j/2}\,\overline{\varphi_k(M^jx-\alpha)}\,\mathrm{d}x\right|^2\\ &\quad= \sum_{k=1}^{N}\sum_{\alpha\in\mathbb{Z}^n}\left|\int_{M^j[-R,R]^n-\alpha} f(M^{-j}x+\alpha)\,m^{j/2}\,\overline{\varphi_k(x)}\,\mathrm{d}x\right|^2.\end{aligned}$$

Next, we apply the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}&\sum_{k=1}^{N}\sum_{\alpha\in\mathbb{Z}^n}\left|\int_{M^j[-R,R]^n-\alpha} f(M^{-j}x+\alpha)\,m^{j/2}\,\overline{\varphi_k(x)}\,\mathrm{d}x\right|^2\\ &\le \sum_{k=1}^{N}\sum_{\alpha\in\mathbb{Z}^n}\left(\int_{M^j[-R,R]^n-\alpha}|m^{j/2}f(M^{-j}x+\alpha)|^2\,\mathrm{d}x\right)\left(\int_{M^j[-R,R]^n-\alpha}|\varphi_k(x)|^2\,\mathrm{d}x\right).\end{aligned}$$

For $j<0, |j|$ sufficiently large, the intersection

$$\{M^j[-R,R]^n-\alpha\}\cap\{M^j[-R,R]^n-\alpha'\}$$

is trivial for $\alpha\neq\alpha'$, see (3.2), and the estimate

$$\begin{aligned}\|\mathcal{P}_{S_j}f\|^2_{L_2(\mathbb{R}^n)} &\le \|f\|^2_{L_2(\mathbb{R}^n)}\sum_{k=1}^{N}\left(\int_{E_j}|\varphi_k(x)|^2\,\mathrm{d}x\right)^2\\ &= \|f\|^2_{L_2(\mathbb{R}^n)}\sum_{k=1}^{N}\left(\int_{\mathbb{R}^n}\chi_{E_j}(x)\,|\varphi_k(x)|^2\,\mathrm{d}x\right)^2\end{aligned}$$

holds, where $E_j := \bigcup_{\alpha \in \mathbb{Z}^n} M^j[-R,R]^n - \alpha$. Notice that $\chi_{E_j}|\varphi_k|^2$ converges pointwise almost everywhere on $\mathbb{R}^n$ to 0 for $j \to -\infty$ and $|\chi_{E_j}\varphi_k|^2 \leq |\varphi_k|^2$. Then the dominated convergence theorem yields

$$\begin{aligned}\lim_{j\to-\infty} \|\mathcal{P}_j f\|^2_{L_2(\mathbb{R}^n)} &\leq \lim_{j\to-\infty} \|f\|^2_{L_2(\mathbb{R}^n)} \sum_{k=1}^{N} \left( \int_{\mathbb{R}^n} \chi_{E_j}(x)\,|\varphi_k(x)|^2 \,\mathrm{d}x \right)^2 \\ &= \|f\|^2_{L_2(\mathbb{R}^n)} \sum_{k=1}^{N} \left( \int_{\mathbb{R}^n} \lim_{j\to-\infty} \chi_{E_j}(x)\,|\varphi_k(x)|^2 \,\mathrm{d}x \right)^2 \to 0.\end{aligned}$$

Our proof is complete. □

We remark that in contrast to Corollary 4.1.14 we assume in the theorem above that the spaces $S_j$ are nested.

# Chapter 5

# Orthogonal Projection onto Shift-Invariant Spaces

In Chapter 3 and Chapter 4 we defined and analysed the setting of our construction procedures. Now, let us explain one basic construction idea. Since we are interested in a pre-(multi)wavelet basis, we define the space $W_j$ as the orthogonal complement of $S_j$ in $S_{j+1}$. Our aim is to determine a basis for every space $W_j, j \in \mathbb{Z}$. Considering the union of these bases then results in a pre-(multi)wavelet basis of the space $L_2(\mathbb{R}^n)$. But how do we obtain these bases? As stated in (2.4), the wavelet space $W_0$ can be defined by

$$W_0 := \{s - \mathcal{P}_{S_0} s : s \in S_1\}.$$

In the following chapters we will construct mother wavelets of the form $s - \mathcal{P}_{S_0} s$ such that they provide a basis for $W_0$. The analysis we are going to develop will be applicable to all wavelet spaces $W_j$ after a suitable dilation and hence, we are going to construct a basis for every wavelet space.

Before we start with this construction process, we dedicate this chapter to the derivation of an explicit representation of the Fourier transform of an orthogonal projection onto $S_0$. Later on this will be useful to derive an explicit representation of these mother wavelets.

For principal shift-invariant spaces Theorem 2.9 in [31] provides us with a representation of the Fourier transform of $\mathcal{P}_{S_0(\varphi)}$.

**Theorem 5.1.**
For every $f \in L_2(\mathbb{R}^n)$ the orthogonal projector $\mathcal{P}_{S_0(\varphi)}$ is given by $\widehat{\mathcal{P}_{S_0(\varphi)} f} = \tau_f \, \hat{\varphi}$ where $\tau_f$ denotes a $2\pi$-periodic function which is defined by

$$\tau_f := \begin{cases} [\hat{f}, \hat{\varphi}]/[\hat{\varphi}, \hat{\varphi}], & \text{on } \operatorname{supp}[\hat{\varphi}, \hat{\varphi}], \\ 0, & \text{otherwise.} \end{cases} \tag{5.1}$$

For finitely generated shift-invariant spaces $S_0(\Phi)$ there exist different representations depending on the assumptions on the generator set $\Phi$. If for example $\Phi$

provides a basis or a Riesz basis for the space $S_0(\Phi)$, see [32, Theorem 3.9] and [48, Theorem 4.2.5] for details. We are interested in a generalization of the representation given in Theorem 5.1 which can be found in [51]. Here, the generator set $\Phi$ is assumed to be minimal which means that for $1 \le i \le N$ it holds

$$\varphi_i \notin S_0(\Phi^{(i)}), \quad \Phi^{(i)} := \Phi \setminus \{\varphi_i\}.$$

Minimality is weaker in comparison to $L_2$-stability and a minimal generator set of $S_0$ is not automatically a basis.

**Lemma 5.2.**
Let $\Phi = \{\varphi_1, \dots, \varphi_N\} \subset L_2(\mathbb{R}^n)$ be a generator set for the space $S_0(\Phi)$. If $\Phi$ is $L_2$-stable, then $\Phi$ is minimal but not vice versa.

*Proof.*
We assume that $\Phi$ is $L_2$-stable. Now, suppose $\Phi$ is not minimal. Then there exists at least one generator $\varphi_i, 1 \le i \le N$, such that $\varphi_i \in S_0(\Phi^{(i)})$. Consequently, $\varphi_i$ has the following representation

$$\varphi_i = \sum_{\varphi \in \Phi^{(i)}} \sum_{k \in \mathbb{Z}^n} a_{\varphi,k}\, \varphi(\cdot - k)$$

and therefore,

$$\|\varphi_i - \sum_{\varphi \in \Phi^{(i)}} \sum_{k \in \mathbb{Z}^n} a_{\varphi,k}\, \varphi(\cdot - k)\|_{L_2(\mathbb{R}^n)} = 0.$$

By the definition of $L_2$-stability this is a contradiction to

$$\|\varphi_i - \sum_{\varphi \in \Phi^{(i)}} \sum_{k \in \mathbb{Z}^n} a_{\varphi,k}\, \varphi(\cdot - k)\|_{L_2(\mathbb{R}^n)} \gtrsim 1 + \left( \sum_{\varphi \in \Phi^{(i)}} \sum_{k \in \mathbb{Z}^n} |a_{\varphi,k}|^2 \right)^{1/2}.$$

Hence, $\Phi$ has to be minimal.

Next, we want to prove that $L_2$-stability does not follow from minimality. This can easily be seen with the help of an example for $N = 2$. As generators for $S_0$ we choose the Box Splines

$$\varphi_1(x) := B_{[2]}(x) = \frac{1}{2}\chi_{[0,2)} = \begin{cases} \frac{1}{2}, & x \in [0,2), \\ 0, & \text{elsewhere}, \end{cases}$$

$$\varphi_2(x) := B_{[1\,1]}(x) = \begin{cases} x, & x \in [0,1), \\ 2 - x, & x \in [1,2), \\ 0, & \text{elsewhere}. \end{cases}$$

To ensure that these two generators are minimal, it is sufficient to prove that

$$B_{[2]} \neq \sum_{k\in\mathbb{Z}} a_k \, B_{[1\,1]}(\cdot - k)$$

or equivalently

$$\chi_{[0,2)} \neq \sum_{k\in\mathbb{Z}} \tilde{a}_k \, B_{[1\,1]}(\cdot - k) \tag{5.2}$$

with $\tilde{a}_k := 2a_k$ on a set with positive measure. Then it follows directly that there exists no sequence $(f_n)_{n\in\mathbb{N}}$ with

$$f_n = \sum_{k\in\mathbb{Z}} a_{n,k} \, B_{[1\,1]}(\cdot - k)$$

such that $f_n \to B_{[2]}$ in the $L_2$-sense. The only possibility to obtain the value one on the interval $[0, 2)$ on the right-hand side in (5.2) is to set $\widetilde{a}_k := 1$ for $k \in \{-1, 0, 1\}$. Now, we consider the interval $[-1, 0)$ and the shifts of the Box Spline $B_{[1\,1]}$ which are supported on this interval, that is, $B_{[1\,1]}(\cdot + 1)$ and $B_{[1\,1]}(\cdot + 2)$. We obtain

$$\widetilde{a}_{-1}(x + 1) + \widetilde{a}_{-2}(2 - (x + 2)) = x + 1 - \widetilde{a}_{-2}x \quad \text{on } [-1, 0).$$

We observe that for all $\widetilde{a}_{-2} \in \mathbb{R}$ there exists a set of positive measure such that the right-hand side in (5.2) does not vanish on $[-1, 0)$. Hence, $\Phi$ is minimal. Next, we check if $\Phi$ is $L_2$-stable or not. Box Splines have $L_2$-stable shifts if and only if the matrix consisting of the direction vectors is unimodular, see [26]. Hence, the integer shifts of $\varphi_2$ are $L_2$-stable in contrast to the integer shifts of $\varphi_1$ and consequently, the generator set $\Phi = \{\varphi_1, \varphi_2\}$ cannot be $L_2$-stable. □

**Lemma 5.3.**
There exist minimal generator sets $\Phi = \{\varphi_1, \dots, \varphi_N\} \subset L_2(\mathbb{R}^n)$ of $S_0(\Phi)$ which do not provide a basis for $S_0(\Phi)$.

*Proof.*
Let $n = 1$. Moreover, let the Fourier transform of the generator $\varphi_1$ be compactly supported such that

$$\operatorname{supp} \hat{\varphi}_1 \subset [\delta, 2\pi - \delta) \quad \text{for } \delta \in [0, \pi). \tag{5.3}$$

Condition (5.3) ensures that $\operatorname{supp} \hat{\varphi}_1 \subset [0, 2\pi)$. Let us now consider two non-trivial functions

$$f_1 = \sum_{k\in\mathbb{Z}} a_{1,k}\varphi_1(\cdot - k), \quad f_2 = \sum_{k\in\mathbb{Z}} a_{2,k}\varphi_1(\cdot - k)$$

in $S_0(\Phi)$ with Fourier transform $\hat{f}_1 = \tau_1 \hat{\varphi}_1$and $\hat{f}_2 = \tau_2 \hat{\varphi}_1$. Here, $\tau_1$ and $\tau_2$ are assumed to be $2\pi\mathbb{Z}$-periodic functions in the space $L_2([0, 2\pi))$ with

$$\begin{aligned}\tau_1 = \tau_2 &\text{ on } \operatorname{supp} \hat{\varphi}_1,\\ \tau_1 \neq \tau_2 &\text{ on } E \subset [0, 2\pi) \setminus \operatorname{supp} \hat{\varphi}_1, |E| > 0.\end{aligned}$$

It follows that

$$\hat{f}_1 - \hat{f}_2 = (\tau_1 - \tau_2)\hat{\varphi}_1 = 0.$$

Hence, we have a non-trivial representation of zero of the form

$$\sum_{k\in\mathbb{Z}} (a_{1,k} - a_{2,k})\varphi_1(\cdot - k) = 0.$$

Consequently, the generator $\varphi_1$ does not provide a basis for its span. Now, it is possible to add finitely many compactly supported generators $\hat{\varphi}_i$ to the generator set with

$$\operatorname{supp} \hat{\varphi}_i \subset [0, 2\pi), \quad i = 2, \ldots, N,$$

and

$$\operatorname{supp} \hat{\varphi}_j \cap \operatorname{supp} \hat{\varphi}_k = \emptyset, \quad j \neq k, \quad j, k \in \{1, \ldots, N\}.$$

These conditions ensure that no function in the span of $\hat{\varphi}_i$ can be represented by the generators in $\widehat{\Phi^{(i)}}$. This means that we have a minimal generator set which does not provide a basis for $S_0(\Phi)$. □

In comparison to [51], we work with a different definition of the bracket product and of the Fourier transform. Therefore, we have to adapt the results in [51] which are needed for the derivation of the representation of $\widehat{\mathcal{P}_{S_0(\Phi)}}$. By combining [51] and [78], we are also going to include detailed proofs of the presented results.

In preparation for these results, we introduce the *weighted $L_2$-space*. Let $v \geq 0$ be a measurable function on a measurable set $\Omega \subseteq \mathbb{R}^n$. Then a function $\varphi$ is in $L_2(\Omega, v)$ if $\varphi : \Omega \to \mathbb{C}$ is measurable on the set $\Omega$ and the norm is given by

$$\|\varphi\|_{L_2(\Omega,v)} := \left( \int_\Omega |\varphi(x)|^2 v(x) \, \mathrm{d}x \right)^{1/2} < \infty.$$

Now, that we have established the term weighted $L_2$-space, we can derive the following variation of Corollary 4.1.2.

**Lemma 5.4.**
Let $\varphi \in L_2(\mathbb{R}^n)$. Then $f \in S_0(\varphi)$ if and only if there exists a measurable, $2\pi\mathbb{Z}^n$-periodic function $\tau$ in the space $L_2(\widetilde{C}_0, [\hat{\varphi}, \hat{\varphi}])$ such that

$$\hat{f} = \tau\hat{\varphi} \quad \text{almost everywhere on } \mathbb{R}^n,$$

and

$$\|f\|_{L_2(\mathbb{R}^n)} = \frac{1}{(2\pi)^n}\|\tau\|_{L_2(\widetilde{C}_0, [\hat{\varphi}, \hat{\varphi}])}. \tag{5.4}$$

*Proof.*
First, we prove formula (5.4). We observe that

$$\|f\|^2_{L_2(\mathbb{R}^n)} = \frac{1}{(2\pi)^n}\|\hat{f}\|^2_{L_2(\mathbb{R}^n)} = \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} |\hat{f}(x)|^2 \, \mathrm{d}x = \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} |\tau(x)|^2 |\hat{\varphi}(x)|^2 \, \mathrm{d}x.$$

Furthermore, the $2\pi\mathbb{Z}^n$-periodicity of $\tau$ yields

$$\begin{aligned}
\frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} |\tau(x)|^2 |\hat{\varphi}(x)|^2 \, \mathrm{d}x &= \frac{1}{(2\pi)^n}\sum_{\beta \in 2\pi\mathbb{Z}^n} \int\limits_{\widetilde{C}_0} |\tau(x)|^2 |\hat{\varphi}(x+\beta)|^2 \, \mathrm{d}x \\
&= \frac{1}{(2\pi)^n}\int\limits_{\widetilde{C}_0} |\tau(x)|^2 [\hat{\varphi}, \hat{\varphi}](x) \, \mathrm{d}x \\
&= \frac{1}{(2\pi)^n}\|\tau\|^2_{L_2(\widetilde{C}_0, [\hat{\varphi}, \hat{\varphi}])}.
\end{aligned}$$

We observe that $\tau \in L_2(\widetilde{C}_0, [\hat{\varphi}, \hat{\varphi}])$ if and only if $\tau\hat{\varphi} \in L_2(\mathbb{R}^n)$. Consequently, the claim follows directly by Corollary 4.1.2. □

Under the assumption that $\Phi$ is a set of orthogonal generators, we deduce a similar result as Lemma 5.4. Beforehand, we have to state the subsequent lemma which can be found in [48].

**Lemma 5.5.**
Let $\Phi = \{\varphi_1, \ldots, \varphi_N\} \subset L_2(\mathbb{R}^n)$. A function $f \in L_2(\mathbb{R}^n)$ is orthogonal to $S_0(\Phi)$ if and only if $[\hat{f}, \hat{\varphi}_i] = 0$ almost everywhere on $\widetilde{C}_0$ for $i = 1, \ldots, N$.

Hereinafter, we use the convention $\frac{0}{0} := 0$. In particular, this will be used for expressions of the form

$$\frac{[\,\cdot\,, \hat{\varphi}]}{[\hat{\varphi}, \hat{\varphi}]}.$$

In case the denominator vanishes, the function $\hat{\varphi}$ has $2\pi\mathbb{Z}^n$-periodic zeros. As a consequence, the nominator vanishes as well and we obtain $\frac{0}{0} = 0$.

**Proposition 5.6.**
Let $\Phi = \{\varphi_1, \ldots, \varphi_N\} \subset L_2(\mathbb{R}^n)$ be a finite set of orthogonal generators for $S_0(\Phi)$. For $i = 1, \ldots, N$ we define

$$m_i(f) := \frac{[\hat{f}, \hat{\varphi}_i]}{[\hat{\varphi}_i, \hat{\varphi}_i]}.$$

Then $f$ is an element of the space $S_0(\Phi)$ if and only if $m_i(f) \in L_2(\widetilde{C}_0, [\hat{\varphi}_i, \hat{\varphi}_i])$ and

$$\hat{f} = \sum_{i=1}^{N} m_i(f)\, \hat{\varphi}_i. \tag{5.5}$$

In addition, it holds

$$\|f\|^2_{L_2(\mathbb{R}^n)} = \frac{1}{(2\pi)^n} \sum_{i=1}^{N} \|m_i(f)\|^2_{L_2(\widetilde{C}_0, [\hat{\varphi}_i, \hat{\varphi}_i])}.$$

*Proof.*
If $N = 1$, the claim follows from Lemma 5.4. The definition of $m_1$ corresponds to the representation of the Fourier transform of the orthogonal projection onto $\widehat{S_0(\varphi_1)}$, see (5.1).

Now, let $N > 1$. Since our generators are orthogonal, the bracket product $[\hat{\varphi}_k, \hat{\varphi}_{k'}]$ vanishes almost everywhere on $\widetilde{C}_0$ for $k \neq k', 1 \leq k, k' \leq N$. By Lemma 5.5, it follows that $S_0(\varphi_k) \perp S_0(\varphi_{k'})$ for $k \neq k'$. Next, we verify that an arbitrary function $f \in S_0(\Phi)$ can be written as

$$f = \sum_{i=1}^{N} f_i, \quad f_i \in S_0(\varphi_i). \tag{5.6}$$

In order to prove (5.6), let $P_\ell = \sum_{i=1}^{N} f_{i,\ell}$ with $f_{i,\ell} \in S_0(\varphi_i)$ for all $\ell \in \mathbb{N}$ and $\lim_{\ell\to\infty} P_\ell = f$ in $L_2(\mathbb{R}^n)$. With the completeness of the Hilbert space $L_2$, the sequence $\{P_\ell\}_{\ell=1}^{\infty}$ is a Cauchy sequence. Consequently, for all $\varepsilon > 0$ there exists an $L > 0$ such that

$$\|P_\ell - P_{\ell'}\|_{L_2(\mathbb{R}^n)} < \varepsilon \quad \text{for all } \ell, \ell' > L.$$

By the orthogonality of the spaces $S_0(\varphi_i)$, we deduce

$$\begin{aligned}
\|P_\ell - P_{\ell'}\|^2_{L_2(\mathbb{R}^n)} &= \|\sum_{i=1}^N f_{i,\ell} - \sum_{i'=1}^N f_{i',\ell'}\|^2_{L_2(\mathbb{R}^n)} \\
&= \left\langle \sum_{i=1}^N f_{i,\ell} - \sum_{i'=1}^N f_{i',\ell'}, \sum_{i=1}^N f_{i,\ell} - \sum_{i'=1}^N f_{i',\ell'} \right\rangle_{L_2(\mathbb{R}^n)} \\
&= \sum_{i=1}^N \langle f_{i,\ell} - f_{i,\ell'}, f_{i,\ell} - f_{i,\ell'} \rangle_{L_2(\mathbb{R}^n)} \\
&= \sum_{i=1}^N \|f_{i,\ell} - f_{i,\ell'}\|^2_{L_2(\mathbb{R}^n)}.
\end{aligned}$$

Hence, $\{f_{i,\ell}\}_{\ell=1}^\infty$ is a Cauchy sequence in $S_0(\varphi_i)$ for $1 \le i \le N$. By the completeness of the spaces $S_0(\varphi_i)$, these Cauchy sequences converge in $L_2(\mathbb{R}^n)$. Thus, we have

$$f = \sum_{i=1}^N f_i \quad \text{with } f_i = \lim_{\ell\to\infty} f_{i,\ell} \in S_0(\varphi_i).$$

By Lemma 5.4, there exists a $2\pi\mathbb{Z}^n$-periodic function $\tau \in L_2(\widetilde{C}_0, [\hat{\varphi}_i, \hat{\varphi}_i])$ for every $f_i \in S_0(\varphi_i)$ such that

$$[\hat{f}_i, \hat{\varphi}_i] = [\tau\hat{\varphi}_i, \hat{\varphi}_i] = \tau[\hat{\varphi}_i, \hat{\varphi}_i].$$

Hence, $\tau = m_i(f_i)$ and we conclude

$$\|f\|^2_{L_2(\mathbb{R}^n)} = \sum_{i=1}^N \|f_i\|^2_{L_2(\mathbb{R}^n)} = \frac{1}{(2\pi)^n} \sum_{i=1}^N \|m_i(f_i)\|^2_{L_2(\widetilde{C}_0, [\hat{\varphi}_i, \hat{\varphi}_i])}.$$

Conversely, assume that (5.5) holds. Then $m_i(f)\hat{\varphi}_i \in \widehat{S_0(\varphi_i)}$ for all $1 \le i \le N$ and by Lemma 5.4, we obtain $f \in S_0(\Phi)$. □

The next proposition describes an orthogonalization procedure for a finite minimal set of generators. The minimality assumption is needed to avoid early termination of the algorithm.

**Proposition 5.7.**
Let $\Phi = \{\varphi_1, \ldots, \varphi_N\} \subset L_2(\mathbb{R}^n)$ be a finite minimal set of generators for the space $S_0(\Phi)$. In addition, let the functions $\{g_i\}_{i=1}^N$ be defined by $g_1 = \varphi_1$ and

$$\hat{g}_i = \hat{\varphi}_i - \sum_{k=1}^{i-1} b_k^{(i)} \hat{g}_k, \quad 1 < i \le N, \tag{5.7}$$

where

$$b_k^{(i)} := [\hat{\varphi}_i, \hat{g}_k][\hat{g}_k, \hat{g}_k]^{-1} \in L_2(\widetilde{C}_0, [\hat{g}_k, \hat{g}_k]), \quad 1 \leq k \leq i-1. \tag{5.8}$$

Then $g_i \in S_0(\Phi_i)$ with $\Phi_i := \{\varphi_k\}_{k=1}^i$ for all $1 \leq i \leq N$. Furthermore, for $j \neq k$ with $1 \leq j, k \leq N$ it holds

$$[\hat{g}_j, \hat{g}_k](x) = 0 \quad \text{for almost every } x \in \widetilde{C}_0. \tag{5.9}$$

*Proof.*
We proceed by induction.

Let $N = 2$. First, we verify that $[\hat{g}_1, \hat{g}_2] = 0$ almost everywhere on $\widetilde{C}_0$. If $b_1^{(2)}(x) \neq 0$ for $x \in \widetilde{C}_0$ we have

$$\begin{aligned}
[\hat{g}_1, \hat{g}_2](x) &= [\hat{\varphi}_1, \hat{\varphi}_2 - b_1^{(2)}\hat{\varphi}_1](x) \\
&= [\hat{\varphi}_1, \hat{\varphi}_2 - [\hat{\varphi}_2, \hat{\varphi}_1][\hat{\varphi}_1, \hat{\varphi}_1]^{-1}\hat{\varphi}_1](x) \\
&= [\hat{\varphi}_1, \hat{\varphi}_2](x) - [\hat{\varphi}_1, \hat{\varphi}_2](x)[\hat{\varphi}_1, \hat{\varphi}_1]^{-1}(x)[\hat{\varphi}_1, \hat{\varphi}_1](x) \\
&= 0.
\end{aligned}$$

If otherwise $b_1^{(2)}(x) = 0$ for $x \in \widetilde{C}_0$, then either $[\hat{\varphi}_2, \hat{\varphi}_1](x)$ or $[\hat{\varphi}_1, \hat{\varphi}_1](x)$ vanishes. In the first case the bracket product $\overline{[\hat{\varphi}_2, \hat{\varphi}_1]}(x)$ vanishes as well. Hence,

$$[\hat{g}_1, \hat{g}_2](x) = [\hat{\varphi}_1, \hat{\varphi}_2](x) = \overline{[\hat{\varphi}_2, \hat{\varphi}_1]}(x) = 0.$$

In the second case the function $\hat{\varphi}_1$ has $2\pi\mathbb{Z}^n$-periodic zeros. Consequently,

$$[\hat{g}_1, \hat{g}_2](x) = [\hat{\varphi}_1, \hat{\varphi}_2](x) = \sum_{\beta \in 2\pi\mathbb{Z}^n} \hat{\varphi}_1(x+\beta)\overline{\hat{\varphi}_2(x+\beta)} = 0.$$

Furthermore, it holds

$$\begin{aligned}
\|b_1^{(2)}\|_{L_2(\widetilde{C}_0, [\hat{g}_1, \hat{g}_1])} &= \int_{\widetilde{C}_0} |b_1^{(2)}(x)|^2 [\hat{\varphi}_1, \hat{\varphi}_1](x) \, \mathrm{d}x \\
&= \int_{\widetilde{C}_0} \left| \frac{[\hat{\varphi}_2, \hat{\varphi}_1](x)}{[\hat{\varphi}_1, \hat{\varphi}_1](x)} \right|^2 [\hat{\varphi}_1, \hat{\varphi}_1](x) \, \mathrm{d}x \\
&= \int_{\widetilde{C}_0} |[\hat{\varphi}_2, \hat{\varphi}_1](x)|^2 \, [\hat{\varphi}_1, \hat{\varphi}_1]^{-1}(x) \, \mathrm{d}x.
\end{aligned}$$

By (1.12), we obtain

$$\int_{\widetilde{C}_0} |[\hat{\varphi}_2, \hat{\varphi}_1](x)|^2 \, [\hat{\varphi}_1, \hat{\varphi}_1]^{-1}(x) \, \mathrm{d}x \leq \int_{\widetilde{C}_0} [\hat{\varphi}_2, \hat{\varphi}_2](x) \, \mathrm{d}x = \int_{\mathbb{R}^n} |\hat{\varphi}_2(x)|^2 \, \mathrm{d}x < \infty.$$

Hence, $b_1^{(2)} \in L_2(\widetilde{C}_0, [\hat{g}_1, \hat{g}_1])$. Moreover, the function $g_2$ is an element of the space $S_0(\Phi_2)$ because of Corollary 4.2.2.

For the induction step, assume that $[\hat{g}_k, \hat{g}_j] = 0$ for all distinct $1 \leq k, j \leq \ell - 1$ and some $2 \leq \ell \leq N - 1$. We observe that

$$[\hat{g}_\ell, \hat{g}_j] = [\hat{\varphi}_\ell - \sum_{k=1}^{\ell-1} b_k^{(\ell)} \hat{g}_k, \hat{g}_j] = [\hat{\varphi}_\ell, \hat{g}_j] - \sum_{k=1}^{\ell-1} b_k^{(\ell)} [\hat{g}_k, \hat{g}_j].$$

By induction hypothesis, $[\hat{g}_k, \hat{g}_j]$ vanishes for $k \neq j$ with $1 \leq k, j \leq \ell - 1$. Therefore, we obtain

$$\begin{aligned}[\hat{\varphi}_\ell, \hat{g}_j] - \sum_{k=1}^{\ell-1} b_k^{(\ell)} [\hat{g}_k, \hat{g}_j] &= [\hat{\varphi}_\ell, \hat{g}_j] - b_j^{(\ell)} [\hat{g}_j, \hat{g}_j] \\ &= [\hat{\varphi}_\ell, \hat{g}_j] - [\hat{\varphi}_\ell, \hat{g}_j] \\ &= 0 \quad \text{almost everywhere on } \mathbb{R}^n.\end{aligned}$$

Furthermore, we can prove $b_j^{(\ell)} \in L_2(\widetilde{C}_0, [\hat{g}_j, \hat{g}_j])$ for $1 \leq j \leq \ell - 1$ analogously to $b_1^{(2)} \in L_2(\widetilde{C}_0, [\hat{g}_1, \hat{g}_1])$ and by Corollary 4.2.2, we obtain $g_\ell \in S_0(\Phi_\ell)$. □

With this orthogonalization procedure we are able to represent a space generated by a finite minimal set of generators as the orthogonal sum of principal shift-invariant spaces.

**Proposition 5.8.**
Let $\Phi = \{\varphi_1, \ldots, \varphi_N\} \subset L_2(\mathbb{R}^n)$ be a minimal set of generators for the space $S_0(\Phi)$ and let the functions $\{g_i\}_{i=1}^N$ be defined by (5.7). Then it holds

$$S_0(\Phi) = \bigoplus_{i=1}^{N} S_0(g_i).$$

*Proof.*
The orthogonality of the spaces $S_0(g_i), i = 1, \ldots, N$, follows directly from Lemma 5.5. Next, we check that $\bigoplus_{i=1}^N S_0(g_i) \subseteq S_0(\Phi)$. Let $f \in \bigoplus_{i=1}^N S_0(g_i)$. Then $f$ can be represented as

$$f = \sum_{i=1}^{N} f_i, \quad f_i \in S_0(g_i).$$

By Lemma 5.4, we obtain $\hat{f}_i = \tau_i \, \hat{g}_i, \tau_i \in L_2(\widetilde{C}_0, [\hat{g}_i, \hat{g}_i])$. We further deduce that $\hat{f} = \sum_{i=1}^N \tau_i \, \hat{g}_i$ and consequently, $f \in S_0(\{g_i\}_{i=1}^N)$. According to Proposition 5.7, every function $g_i$ belongs to $S_0(\Phi_i)$ and thus, $f \in S_0(\Phi)$. It remains to prove that the embedding $S_0(\Phi) \subseteq \bigoplus_{i=1}^N S_0(g_i)$ holds. Let $f \in S_0(\Phi)$. By (5.7), the generators can be represented as $\hat{\varphi}_i = \hat{g}_i + \sum_{k=1}^{i-1} b_k^{(i)} \hat{g}_k$ with $b_k^{(i)} \in L_2(\widetilde{C}_0, [\hat{g}_k, \hat{g}_k])$ for $1 < i \leq N$. By (5.9), we deduce that on $\operatorname{supp}[\hat{g}_i, \hat{g}_i]$ we have

$$\begin{aligned} b_i^{(i)} &= \frac{[\hat{\varphi}_i, \hat{g}_i]}{[\hat{g}_i, \hat{g}_i]} \\ &= \frac{[\hat{g}_i + \sum_{k=1}^{i-1} b_k^{(i)} \hat{g}_k, \hat{g}_i]}{[\hat{g}_i, \hat{g}_i]} \\ &= \frac{[\hat{g}_i, \hat{g}_i] + \sum_{k=1}^{i-1} b_k^{(i)} [\hat{g}_k, \hat{g}_i]}{[\hat{g}_i, \hat{g}_i]} \\ &= \frac{[\hat{g}_i, \hat{g}_i]}{[\hat{g}_i, \hat{g}_i]} \\ &= 1, \end{aligned}$$

where $b_N^{(N)} := [\hat{\varphi}_N, \hat{g}_N][\hat{g}_N, \hat{g}_N]^{-1}$. Therefore, the representation

$$\hat{\varphi}_i = \hat{g}_i + \sum_{k=1}^{i-1} b_k^{(i)} \hat{g}_k$$

is equivalent to

$$\hat{\varphi}_i = b_i^{(i)} \hat{g}_i + \sum_{k=1}^{i-1} b_k^{(i)} \hat{g}_k = \sum_{k=1}^{i} b_k^{(i)} \hat{g}_k \quad \text{for } 1 \leq i \leq N.$$

Lemma 5.4 yields that $\hat{\varphi}_i \in \widehat{S_0(\{g_k\}_{k=1}^i)}$ for $1 \leq i \leq N$. By the orthogonality of the functions $\{g_i\}_{i=1}^N$, we obtain $f \in \bigoplus_{i=1}^N S_0(g_i)$. □

Finally, we obtain an explicit representation of the orthogonal projector in terms of $\widehat{P_{S_0(\Phi)}}$.

**Theorem 5.9.**
Let $\Phi = \{\varphi_1, \ldots, \varphi_N\} \subset L_2(\mathbb{R}^n)$ be a minimal set of generators for the space $S_0(\Phi)$. Then for any function $f \in L_2(\mathbb{R}^n)$ the orthogonal projection $\mathcal{P}_{S_0(\Phi)} f$ of $f$ onto the space $S_0(\Phi)$ is given by

$$\widehat{\mathcal{P}_{S_0(\Phi)} f} = \sum_{i=1}^N \frac{[\hat{f}, \hat{g}_i]}{[\hat{g}_i, \hat{g}_i]} \hat{g}_i, \tag{5.10}$$

where the functions $\{g_i\}_{i=1}^N$ are defined by (5.7) and (5.8).

*Proof.*
Let $f$ be a function in $L_2(\mathbb{R}^n)$ with

$$f = f_1 \oplus f_2, \quad f_1 \in S_0(\Phi), f_2 \in S_0(\Phi)^\perp.$$

By Proposition 5.8, we obtain

$$\mathcal{P}_{S_0(\Phi)} f = f_1 = \sum_{i=1}^{N} h_i, \quad h_i \in S_0(g_i). \tag{5.11}$$

Moreover, Proposition 5.6 yields

$$\hat{h}_i = m(h_i)\,\hat{g}_i = \frac{[\hat{h}_i, \hat{g}_i]}{[\hat{g}_i, \hat{g}_i]} \hat{g}_i, \quad 1 \le i \le N.$$

Hence, it holds

$$\widehat{\mathcal{P}_{S_0(\Phi)} f} = \sum_{i=1}^{N} \hat{h}_i = \sum_{i=1}^{N} \frac{[\hat{h}_i, \hat{g}_i]}{[\hat{g}_i, \hat{g}_i]} \hat{g}_i.$$

It remains to prove $[\hat{h}_i, \hat{g}_i] = [\hat{f}, \hat{g}_i]$. Since $g_i \in S_0(\Phi)$ and $f_2 \in S_0(\Phi)^\perp$, the bracket product of these two functions vanishes, see Lemma 5.5. We further deduce that

$$[\hat{f}, \hat{g}_i] = [\hat{f}_1 + \hat{f}_2, \hat{g}_i] = [\hat{f}_1, \hat{g}_i] + [\hat{f}_2, \hat{g}_i] = [\hat{f}_1, \hat{g}_i]. \tag{5.12}$$

Since the functions $g_i, 1 \le i \le N$, are orthogonal, inserting (5.11) into (5.12) yields the desired result:

$$[\hat{f}_1, \hat{g}_i] = \sum_{j=1}^{N} [\hat{h}_j, \hat{g}_i] = [h_i, \hat{g}_i].$$

□

We remark that (5.10) coincides with (5.1) for $N = 1$.

# Chapter 6

# Construction of Wavelets

With the help of the theory from the previous chapters, we now explicitly construct wavelet bases based on $M^{-j}$-principal shift-invariant spaces. In particular, we are interested in bases with desirable properties. In Section 6.1 we construct a compactly supported wavelet basis and in Section 6.2 we construct an $L_2$-stable wavelet basis. Both sections generalize construction procedures from [30] where the authors work with $2^{-j}$-principal shift-invariant spaces.

## 6.1 Compactly Supported Wavelet Bases

For application purposes one is interested in working with well-localized functions, that is, functions with compact support. Based on the construction idea presented in the beginning of Chapter 5, we develop a construction process which yields a compactly supported wavelet basis for $W_0$, see Section 6.1.1. Afterwards, we demonstrate in Section 6.1.2 how to adapt the presented analysis from Section 6.1.1 in order to obtain a compactly supported wavelet basis for every space $W_j, j \in \mathbb{Z}$. In Section 6.1.3 we give an example where we choose exponential box splines as our generators to demonstrate our construction procedure.

### 6.1.1 Compactly Supported Wavelet Bases for $\mathbf{W_0}$

Let $\varphi, \eta \in L_2(\mathbb{R}^n)$ and assume that

$$S_0(\varphi) \subset S_1(\eta),$$

where the space $S_0$ is $I$-shift-invariant and the space $S_1$ is $M^{-1}$-shift-invariant. Then the wavelet space $W_0$ is given by

$$W_0 := S_1(\eta) \ominus S_0(\varphi).$$

We proceed as follows. In a first step we show that there exist $m$ specific translates of the generator $\varphi$ which provide a basis for the space $S_1(\eta)$. Projecting $m-1$ of these basis elements onto the orthogonal complement of $S_0(\varphi)$ in $S_1(\eta)$ yields a basis for $W_0$. Then we modify this basis such that it is compactly supported.

In this process we work with a set of representatives of the disjoint cosets in $\mathbb{Z}^n/M\mathbb{Z}^n$ denoted by $R$ and a set of representatives of the disjoint cosets in $\mathbb{Z}^n/M^T\mathbb{Z}^n$ denoted by $R^T$. Let us collect some properties of $R$ and $R^T$ first.

**Lemma 6.1.1.** - [42, Lemma 2]
The cardinality of $R$ and $R^T$ is $m = |\det M| = |\det M^T|$.

**Lemma 6.1.2.**
Let $\{\tilde{\rho}_1, \dots, \tilde{\rho}_m\}$ be a full set of representatives of the disjoint cosets of $\mathbb{Z}^n/M^T\mathbb{Z}^n$. If we add an arbitrary element $d' \in R^T$ to each of the representatives, we obtain $m$ new representatives of the disjoint cosets of $\mathbb{Z}^n/M^T\mathbb{Z}^n$.

*Proof.*
Suppose $\tilde{\rho}_i + d'$ and $\tilde{\rho}_j + d'$ with $i \neq j, i, j \in \{1, \dots, m\}$, are representatives of the same coset. It holds

$$\mathbb{Z}^n = \bigcup_{\hat{\rho} \in R^T} (\hat{\rho} + M^T\mathbb{Z}^n), \tag{6.1}$$

see [42]. Consequently, there exist $\hat{\rho} \in R^T$ and $\ell_1, \ell_2 \in \mathbb{Z}^n$ such that

$$\begin{aligned} \tilde{\rho}_i + d' &= \hat{\rho} + M^T\ell_1, \\ \tilde{\rho}_j + d' &= \hat{\rho} + M^T\ell_2. \end{aligned}$$

It follows that

$$\tilde{\rho}_i - \tilde{\rho}_j = M^T(\ell_1 - \ell_2).$$

But since $\tilde{\rho}_i$ and $\tilde{\rho}_j$ are representatives of disjoint cosets, we know that

$$\tilde{\rho}_i - \tilde{\rho}_j \notin M^T\mathbb{Z}^n.$$

This is a contradiction. □

Let $\{\rho_1, \dots, \rho_m\}$ be a full set of representatives of the disjoint cosets in $\mathbb{Z}^n/M\mathbb{Z}^n$ and let $d \in R$. Then, by the same arguments as above, $\{\rho_1 + d, \dots, \rho_m + d\}$ is a set of representatives of the disjoint cosets in $\mathbb{Z}^n/M\mathbb{Z}^n$ as well.

**Lemma 6.1.3.** - [19, Lemma 2.3]
Let $\rho_j$ and $\hat{\rho}_j$, $j = 0, \dots, m-1$, be the full representatives of $\mathbb{Z}^n/M\mathbb{Z}^n$ and $\mathbb{Z}^n/M^T\mathbb{Z}^n$, respectively. Then it holds

$$\frac{1}{m} \sum_{k=0}^{m-1} e_{M^{-1}\rho_k}(2\pi\hat{\rho}_\ell) = \delta_{\ell,0}, \quad 0 \leq \ell \leq m-1,$$

and

$$\frac{1}{m}\sum_{\ell=0}^{m-1} e_{M^{-1}\rho_k}(-2\pi\hat{\rho}_\ell) = \delta_{k,0}, \quad 0 \le k \le m-1.$$

Based on the set of representatives $R$, we can determine certain translates of the function $\varphi$ which generate the space $S_1(\eta)$.

**Theorem 6.1.4.**
Let $S_0(\varphi) \subset S_1(\eta)$ and $\operatorname{supp}\hat{\varphi} = \operatorname{supp}\hat{\eta}$. Moreover, let $0 \in R$. Then it holds

$$S_1(\eta) = S_1(\varphi) = S_0(\phi), \quad \phi := (\varphi(\cdot + M^{-1}d))_{d\in R}.$$

*Proof.*
Let us start by showing that $S_1(\varphi) = S_0(\phi)$. The space $S_1(\varphi)$ is generated by the $M^{-1}\mathbb{Z}^n$-shifts of $\varphi$. Besides that $\mathbb{Z}^n$ can be written as

$$\bigcup_{d\in R}(d + M\mathbb{Z}^n). \tag{6.2}$$

Therefore, we obtain

$$M^{-1}\mathbb{Z}^n = \bigcup_{d\in R}(M^{-1}d + \mathbb{Z}^n).$$

This yields

$$S_1(\varphi) = S_0((\varphi(\cdot + M^{-1}d))_{d\in R}) = S_0(\phi).$$

It remains to prove that $S_1(\eta) = S_1(\varphi)$. Set $g := \eta(M^{-1}\cdot)$ and $f := \varphi(M^{-1}\cdot)$. By assumption, $\varphi \in S_1(\eta)$ and as a consequence $f \in S_0(g)$. Furthermore, it holds $\operatorname{supp}\hat{\varphi} = \operatorname{supp}\hat{\eta}$ and thus, $\operatorname{supp}\hat{f} = \operatorname{supp}\hat{g}$. By Proposition 4.1.4, we deduce that $S_0(f) = S_0(g)$. Since $S_1(\varphi)$ is the $M$-dilation of $S_0(f)$ and $S_1(\eta)$ is the $M$-dilation of $S_0(g)$, the claim follows. □

Next, we will prove that $\phi$ provides a basis for $S_0(\phi)$. We proceed by verifying that the product of the eigenvalues of the Gramian matrix $G(\phi)$ is greater than zero almost everywhere. In order to determine these eigenvalues, we introduce the operator $Q$. For a $2\pi M^T\mathbb{Z}^n$-periodic function $f$ the operator $Q_d$ is defined by

$$Q_d(f)(\xi) := \sum_{\tilde{d}\in R^T} e_{M^{-1}d}(\xi + 2\pi\tilde{d}) f(\xi + 2\pi\tilde{d}), \quad d \in R,\ \xi \in \mathbb{R}^n. \tag{6.3}$$

**Lemma 6.1.5.**
The operator $Q_d$ in (6.3) is $2\pi\mathbb{Z}^n$-periodic.

*Proof.*
For $\ell \in \mathbb{Z}^n$ and $\xi \in \mathbb{R}^n$, we have

$$Q_d(f)(\xi + 2\pi\ell) = \sum_{\tilde{d}\in R^T} e_{M^{-1}d}(\xi + 2\pi\ell + 2\pi\tilde{d}) f(\xi + 2\pi\ell + 2\pi\tilde{d}), \quad d \in R.$$

Since $\mathbb{Z}^n = \bigcup_{d'\in R^T}(d' + M^T\mathbb{Z}^n)$, we set $\ell = d' + M^T n$ with $n \in \mathbb{Z}^n$. Furthermore, we use the $2\pi$-periodicity of the exponential function as well as the $2\pi M^T\mathbb{Z}^n$-periodicity of the function $f$ to deduce

$$\begin{aligned}
&\sum_{\tilde{d}\in R^T} e_{M^{-1}d}(\xi + 2\pi\ell + 2\pi\tilde{d}) f(\xi + 2\pi\ell + 2\pi\tilde{d}) \\
&\quad = \sum_{\tilde{d}\in R^T} e_{M^{-1}d}(\xi + 2\pi d' + 2\pi M^T n + 2\pi\tilde{d}) f(\xi + 2\pi d' + 2\pi M^T n + 2\pi\tilde{d}) \\
&\quad = \sum_{\tilde{d}\in R^T} e_{M^{-1}d}(\xi + 2\pi(d' + \tilde{d})) f(\xi + 2\pi(d' + \tilde{d})).
\end{aligned}$$

By Lemma 6.1.2, we conclude

$$\begin{aligned}
\sum_{\tilde{d}\in R^T} e_{M^{-1}d}(\xi + 2\pi(d' + \tilde{d})) f(\xi + 2\pi(d' + \tilde{d})) &= \sum_{\hat{d}\in R^T} e_{M^{-1}d}(\xi + 2\pi\hat{d}) f(\xi + 2\pi\hat{d}) \\
&= Q_d(f)(\xi).
\end{aligned}$$

□

From now on, we always assume that $0 \in R$ and $0 \in R^T$.

**Lemma 6.1.6.**
A $2\pi M^T\mathbb{Z}^n$-periodic function $f$ can be decomposed in its $2\pi\mathbb{Z}^n$-periodic components by

$$f = \frac{1}{m} \sum_{d^*\in R} e_{-M^{-1}d^*}\, Q_{d^*}(f). \tag{6.4}$$

*Proof.*
First of all, we insert the definition of the operator $Q_{d^*}$ into (6.4). We obtain

$$\begin{aligned}
\frac{1}{m} \sum_{d^*\in R} e_{-M^{-1}d^*}\, Q_{d^*}(f) &= \frac{1}{m} \sum_{d^*\in R} e_{-M^{-1}d^*} \sum_{\tilde{d}\in R^T} e_{M^{-1}d^*}(\cdot + 2\pi\tilde{d}) f(\cdot + 2\pi\tilde{d}) \\
&= \frac{1}{m} \sum_{d^*\in R} \left( \sum_{\tilde{d}\in R^T} e_{M^{-1}d^*}(2\pi\tilde{d}) \right) f(\cdot + 2\pi\tilde{d}).
\end{aligned}$$

Then, we use Lemma 6.1.3 to deduce

$$\begin{aligned}\frac{1}{m}\sum_{d^*\in R}\left(\sum_{\tilde d\in R^T} e_{M^{-1}d^*}(2\pi\tilde d)\right) f(\cdot+2\pi\tilde d) &= \frac{1}{m}\sum_{\tilde d\in R^T}\left(\sum_{d^*\in R} e_{M^{-1}d^*}(2\pi\tilde d)\right) f(\cdot+2\pi\tilde d)\\ &= \frac{1}{m}\, m\,\delta_{\tilde d,0}\, f(\cdot+2\pi\tilde d)\\ &= f.\end{aligned}$$

□

**Lemma 6.1.7.**
Let $\phi = (\varphi(\cdot + M^{-1}d))_{d\in R}$. For all $d, d^* \in R$ the corresponding entry of the Gramian matrix $G(\phi)$ is given by

$$[e_{M^{-1}d}\,\hat\varphi, e_{M^{-1}d^*}\,\hat\varphi] = Q_{d-d^*}([\hat\varphi,\hat\varphi]_1). \tag{6.5}$$

*Proof.*
By the definition of the bracket product and (6.1), we obtain

$$\begin{aligned}[e_{M^{-1}d}\,\hat\varphi, e_{M^{-1}d^*}\,\hat\varphi] &= \sum_{\alpha\in\mathbb{Z}^n} e_{M^{-1}d}(\cdot+2\pi\alpha)e_{-M^{-1}d^*}(\cdot+2\pi\alpha)\,|\hat\varphi(\cdot+2\pi\alpha)|^2\\ &= \sum_{\alpha\in\mathbb{Z}^n} e_{M^{-1}(d-d^*)}(\cdot+2\pi\alpha)\,|\hat\varphi(\cdot+2\pi\alpha)|^2\\ &= \sum_{\tilde d\in R^T}\sum_{\alpha\in\mathbb{Z}^n} e_{M^{-1}(d-d^*)}(\cdot+2\pi(\tilde d+M^T\alpha))\,|\hat\varphi(\cdot+2\pi(\tilde d+M^T\alpha))|^2.\end{aligned}$$

Since the exponential function ist $2\pi$-periodic, we deduce

$$\begin{aligned}&\sum_{\tilde d\in R^T}\sum_{\alpha\in\mathbb{Z}^n} e_{M^{-1}(d-d^*)}(\cdot+2\pi(\tilde d+M^T\alpha))\,|\hat\varphi(\cdot+2\pi(\tilde d+M^T\alpha))|^2\\ &\quad= \sum_{\tilde d\in R^T}\sum_{\alpha\in\mathbb{Z}^n} e_{M^{-1}(d-d^*)}(\cdot+2\pi\tilde d)\,|\hat\varphi(\cdot+2\pi\tilde d+2\pi M^T\alpha)|^2\\ &\quad= \sum_{\tilde d\in R^T} e_{M^{-1}(d-d^*)}(\cdot+2\pi\tilde d)\sum_{\alpha\in\mathbb{Z}^n}|\hat\varphi(\cdot+2\pi\tilde d+2\pi M^T\alpha)|^2\\ &\quad= \sum_{\tilde d\in R^T} e_{M^{-1}(d-d^*)}(\cdot+2\pi\tilde d)\,[\hat\varphi,\hat\varphi]_1(\cdot+2\pi\tilde d)\\ &\quad= Q_{d-d^*}([\hat\varphi,\hat\varphi]_1).\end{aligned}$$

□

**Lemma 6.1.8.**
For every $d^\circ \in R$ and $x \in \widetilde{C}_0$ the number $m[\hat\varphi,\hat\varphi]_1(x+2\pi d^\circ)$ is an eigenvalue of $G(\phi)(x)$ corresponding to the eigenvector $a_{d^\circ} := (e_{M^{-1}d}(x+2\pi d^\circ))_{d\in R}$.

*Proof.*
We have to prove that

$$G(\phi)\, a_{d^\circ} = m\, [\hat{\varphi}, \hat{\varphi}]_1(x + 2\pi d^\circ)\, a_{d^\circ}.$$

Therefore, we calculate the $d$-th entry of $G(\phi)\, a_{d^\circ}$. By (6.5) and the $2\pi\mathbb{Z}^n$-periodicity of the operator $Q$, we obtain

$$\begin{aligned} G(\phi)\, a_{d^\circ} &= \sum_{d^* \in R} Q_{d-d^*}([\hat{\varphi}, \hat{\varphi}]_1)\, e_{M^{-1}d^*}(\cdot + 2\pi d^\circ) \\ &= \sum_{d^* \in R} Q_{d-d^*}([\hat{\varphi}, \hat{\varphi}]_1)(\cdot + 2\pi d^\circ)\, e_{M^{-1}d^*}(\cdot + 2\pi d^\circ)\, e_{M^{-1}d}(\cdot + 2\pi d^\circ)\, e_{-M^{-1}d}(\cdot + 2\pi d^\circ) \\ &= e_{M^{-1}d}(\cdot + 2\pi d^\circ) \sum_{d^* \in R} Q_{d-d^*}([\hat{\varphi}, \hat{\varphi}]_1)(\cdot + 2\pi d^\circ)\, e_{M^{-1}(d^*-d)}(\cdot + 2\pi d^\circ). \end{aligned}$$

Applying Lemma 6.1.2 and Lemma 6.1.6 yields

$$\begin{aligned} & e_{M^{-1}d}(\cdot + 2\pi d^\circ) \sum_{d^* \in R} Q_{d-d^*}([\hat{\varphi}, \hat{\varphi}]_1)(\cdot + 2\pi d^\circ)\, e_{M^{-1}(d^*-d)}(\cdot + 2\pi d^\circ) \\ &\quad = e_{M^{-1}d}(\cdot + 2\pi d^\circ) \sum_{\hat{d} \in R} Q_{\hat{d}}([\hat{\varphi}, \hat{\varphi}]_1)(\cdot + 2\pi d^\circ)\, e_{-M^{-1}\hat{d}}(\cdot + 2\pi d^\circ) \\ &\quad = e_{M^{-1}d}(\cdot + 2\pi d^\circ)\, m\, [\hat{\varphi}, \hat{\varphi}]_1(\cdot + 2\pi d^\circ). \end{aligned}$$

□

**Corollary 6.1.9.**
The set $\phi = (\varphi(\cdot + M^{-1}d))_{d \in R}$ provides a basis for $S_0(\phi)$ if $\operatorname{supp} \hat{\varphi} = \mathbb{R}^n$.

*Proof.*
Lemma 6.1.8 yields that the determinant of the Gramian matrix $G(\phi)$ is given by

$$\det G(\phi) = C \prod_{d^\circ \in R} [\hat{\varphi}, \hat{\varphi}]_1(\cdot + 2\pi d^\circ),$$

where $C$ is a strictly positive constant. Since $\operatorname{supp} \hat{\varphi} = \mathbb{R}^n$, we have $\operatorname{supp} [\hat{\varphi}, \hat{\varphi}]_1 = \mathbb{R}^n$. Consequently, $\det G(\phi) > 0$ almost everywhere and thus, $\phi$ provides a basis for the space $S_0(\phi)$. □

Combining Theorem 6.1.4 and Corollary 6.1.9 yields that under the assumption

$$\operatorname{supp} \hat{\varphi} = \operatorname{supp} \hat{\eta} = \mathbb{R}^n \tag{6.6}$$

we obtain $S_1(\eta) = S_0(\phi)$, where $\phi$ provides a basis for $S_0(\phi)$. The Paley-Wiener Theorem 1.2.10 tells us that the support of the Fourier transform of every compactly supported function equals $\mathbb{R}^n$. Therefore, we notice that (6.6) is not a very restrictive assumption.

With the basis $\phi$ at hand, we want to derive a basis for the orthogonal complement $W_0$ of $S_0(\varphi)$ in $S_0(\phi)$. The set $\phi$ consists of $m = |\det M|$ elements denoted by $\varphi_d := \varphi(\cdot + M^{-1}d), d \in R$. In case $d = 0$, we obtain the generator of the space $S_0(\varphi)$. Hence, the idea is to project the $m-1$ functions $(\varphi(\cdot + M^{-1}d))_{d\in R'}, R' := R \setminus \{0\}$, onto the orthogonal complement of $S_0(\varphi)$. This can be done with the help of Theorem 1.1.3.

**Theorem 6.1.10.**
Let $S_0(\varphi) \subset S_1(\eta)$ and $\operatorname{supp}\hat{\varphi} = \operatorname{supp}\hat{\eta}$. Moreover, let $w_d, d \in R'$, be defined by

$$w_d := \varphi_d - \mathcal{P}_{S_0(\varphi)}\varphi_d.$$

Then the space $W_0 = S_1 \ominus S_0$ is a finitely generated shift-invariant space generated by the set $\mathscr{W} := (w_d)_{d\in R'}$, that is,

$$W_0 = S_0(\mathscr{W}).$$

The set $\mathscr{W}$ provides a basis for $W_0$ if $\operatorname{supp}\hat{\varphi} = \mathbb{R}^n$ holds.

*Proof.*
First, we prove that $S_0(\varphi) \oplus S_0(\mathscr{W}) = S_1(\eta)$ which means that $\{\varphi\} \cup \mathscr{W}$ generates $S_1(\eta)$ and therefore, $W_0 = S_0(\mathscr{W})$. For every function $f \in S_1(\eta)$ there exists a sequence $(f_\ell)_{\ell\in\mathbb{N}}$ in the space $S_1(\eta)$ such that $f_\ell \to f$ in the $L_2$-sense. Due to Theorem 6.1.4, every element of this sequence has a representation of the form

$$\begin{aligned} f_\ell &= \sum_{k\in\mathbb{Z}^n}\sum_{d\in R} a_{\ell,k,d}\varphi_d(\cdot - k) \\ &= \sum_{k\in\mathbb{Z}^n} a_{\ell,k,0}\varphi(\cdot - k) + \sum_{k\in\mathbb{Z}^n}\sum_{d\in R'} a_{\ell,k,d}\varphi_d(\cdot - k) && (6.7)\\ &= \sum_{k\in\mathbb{Z}^n} a_{\ell,k,0}\varphi(\cdot - k) + \sum_{k\in\mathbb{Z}^n}\sum_{d\in R'} a_{\ell,k,d}\mathcal{P}_{S_0(\varphi)}\varphi_d(\cdot - k) && (6.8)\\ &\quad + \sum_{k\in\mathbb{Z}^n}\sum_{d\in R'} a_{\ell,k,d}\left(\varphi_d(\cdot - k) - \mathcal{P}_{S_0(\varphi)}\varphi_d(\cdot - k)\right). && (6.9)\end{aligned}$$

Since (6.8) lies in the space $S_0(\phi)$ and (6.9) lies in the space $S_0(\mathscr{W})$, we have shown that $S_1(\eta) \subset S_0(\varphi) \oplus S_0(\mathscr{W})$. Conversely, for every function $g \in S_0(\varphi) \oplus S_0(\mathscr{W})$ there exists a sequence $(g_\ell)_{\ell\in\mathbb{N}}$ in the space $S_0(\varphi) \oplus S_0(\mathscr{W})$ such that $g_\ell \to g$ in the

$L_2$-sense. Every element of this sequence can be represented as

$$g_\ell = \sum_{k\in\mathbb{Z}^n} b_{\ell,k,0}\varphi(\cdot - k) + \sum_{k\in\mathbb{Z}^n}\sum_{d\in R'} b_{\ell,k,d}\left(\varphi_d - \mathcal{P}_{S_0(\varphi)}\varphi_d\right)(\cdot - k)$$

$$= \sum_{k\in\mathbb{Z}^n} b_{\ell,k,0}\varphi(\cdot - k) - \sum_{k\in\mathbb{Z}^n}\sum_{d\in R'} b_{\ell,k,d}\mathcal{P}_{S_0(\varphi)}\varphi_d(\cdot - k) \tag{6.10}$$

$$+ \sum_{k\in\mathbb{Z}^n}\sum_{d\in R'} b_{\ell,k,d}\varphi_d(\cdot - k). \tag{6.11}$$

Comparing (6.7) with (6.10) and (6.11) yields $S_0(\varphi) \oplus S_0(\mathscr{W}) \subset S_1(\eta)$.

Now, assume that $\operatorname{supp}\hat{\varphi} = \mathbb{R}^n$ holds. It remains to prove that $\mathscr{W}$ provides a basis for $S_0(\mathscr{W})$. Corollary 6.1.9 yields that $\phi = (\varphi(\cdot + M^{-1}d))_{d\in R}$ provides a basis for $S_0(\phi)$. Besides that Theorem 6.1.4 yields $S_1(\eta) = S_0(\phi)$. We have already shown that the integer translates of $\phi_* := \{\varphi\} \cup \mathscr{W}$ generate the space $S_1(\eta)$ and therefore, also the space $S_0(\phi)$. Moreover, the sets $\phi$ and $\phi_*$ have the same number of elements. In this case part ii) of Corollary 4.2.8 states that $\phi_*$ provides a basis for $S_0(\phi)$. As a consequence, we have $\det G(\phi_*) \neq 0$ almost everywhere. Due to the orthogonality between $W_0 = S_0(\mathscr{W})$ and $S_0(\varphi)$, the Gramian matrix $G(\phi_*)$ has the form

$$G(\phi_*) = \begin{pmatrix} G(\mathscr{W}) & 0 \\ 0 & [\hat{\varphi}, \hat{\varphi}] \end{pmatrix}.$$

Therefore, an application of the Laplace formula results in

$$\det G(\phi_*) = [\hat{\varphi}, \hat{\varphi}] \det G(\mathscr{W}).$$

Hence, $\det G(\mathscr{W}) \neq 0$ almost everywhere. Consequently, $\mathscr{W}$ provides a basis for $W_0$. □

Theorem 5.1 provides us with an explicit representation of the Fourier transform of the functions $w_d, d \in R'$. It is given by

$$\widehat{w_d} = \widehat{\varphi_d} - \widehat{\mathcal{P}_{S_0(\varphi)}\varphi_d} = \widehat{\varphi_d} - \hat{\varphi}\frac{[\widehat{\varphi_d}, \hat{\varphi}]}{[\hat{\varphi}, \hat{\varphi}]}, \tag{6.12}$$

where $[\widehat{\varphi_d}, \hat{\varphi}][\hat{\varphi}, \hat{\varphi}]^{-1} = 0$ if $[\hat{\varphi}, \hat{\varphi}] = 0$.

The next theorem shows that if we modify the representation (6.12) by multiplication with $[\hat{\varphi}, \hat{\varphi}]$, we obtain a compactly supported basis of the space $W_0$.

**Theorem 6.1.11.**
Let $S_0(\varphi) \subset S_1(\eta)$ and $\operatorname{supp}\hat{\varphi} = \operatorname{supp}\hat{\eta} = \mathbb{R}^n$. Moreover, let $[\hat{\varphi}, \hat{\varphi}]$ be bounded. Then the functions

$$\mathscr{W}_c := (([\hat{\varphi}, \hat{\varphi}]\widehat{\varphi_d} - [\widehat{\varphi_d}, \hat{\varphi}]\hat{\varphi})^\vee)_{d\in R'}$$

provide a basis for the space $W_0$. If in addition $\varphi$ is compactly supported, then the functions in $\mathscr{W}_c$ also have compact support.

*Proof.*
We obtain $\widehat{\mathscr{W}_c}$ by multiplying $\widehat{\mathscr{W}}$ with the $2\pi$-periodic diagonal matrix $\Gamma := [\hat{\varphi}, \hat{\varphi}]I$, that is, $\widehat{\mathscr{W}_c} = \Gamma\widehat{\mathscr{W}}$. By assumption, the matrix $\Gamma$ only consists of bounded entries and thus, we deduce $\widehat{\mathscr{W}_c} \subset L_2(\mathbb{R}^n)$. Since $\operatorname{supp}\hat{\varphi} = \mathbb{R}^n$, we know that $\mathscr{W}$ provides a basis for $W_0 = S_0(\mathscr{W})$, see Theorem 6.1.10. By taking part i) of Corollary 4.2.8 into consideration, we conclude that $\mathscr{W}_c$ provides a basis for the space $W_0$.

To complete the proof we have to show that $\mathscr{W}_c$ is compactly supported whenever $\varphi$ is compactly supported. Part iii) of Lemma 1.3.7 tells us that if $\varphi$ has compact support, then $[\widehat{\varphi_d}, \hat{\varphi}]$ is a trigonometric polynomial. As a consequence, the inverse Fourier transform of $[\widehat{\varphi_d}, \hat{\varphi}]\hat{\varphi}$ is a finite linear combination of the shifts of $\varphi$ and compactly supported because $\varphi$ is. The same reasoning shows that the inverse Fourier transform of $[\hat{\varphi}, \hat{\varphi}]\widehat{\varphi_d}$ is compactly supported. Therefore, the functions in $\mathscr{W}_c$ are also compactly supported. □

Moreover, we can prove the existence of an orthogonal basis of $W_0$ consisting of compactly supported functions.

**Theorem 6.1.12.**
Let $S_0(\varphi) \subset S_1(\eta)$ and $\operatorname{supp}\hat{\varphi} = \operatorname{supp}\hat{\eta} = \mathbb{R}^n$.

i) There exists a set of mother wavelets which provides an orthonormal basis for $W_0$.

ii) If the function $\varphi$ is compactly supported there exists a subset $\Psi$ of compactly supported functions from $W_0$ which provides a basis for $W_0$. Moreover, there exists a set $\Psi^* := (\psi_d^*)_{d\in R'}$ consisting of compactly supported functions which provides a basis for $W_0$ and fulfills

$$S_0(\psi_d^*) \perp S_0(\psi_{\tilde{d}}^*), \quad d \neq \tilde{d}.$$

*Proof.*
We start with the proof of part i). The conditions of Theorem 6.1.10 are satisfied and therefore, $\mathscr{W}$ provides a basis for $W_0$. Our claim then follows from part i) of Corollary 4.2.9.

It remains to prove part ii). Here, we assume that $\varphi$ is compactly supported. Theorem 6.1.11 yields that the set of compactly supported functions $\mathscr{W}_c$ provides a basis for $W_0$. By part ii) of Corollary 4.2.9, the proof is complete. □

The next result answers the question under which conditions a set of compactly supported functions $(w(\cdot + M^{-1}d))_{d\in R'}, w \in L_2(\mathbb{R}^n)$, provides a basis for $W_0$.

**Theorem 6.1.13.**
Let $S_0(\varphi) \subset S_1(\eta)$ and $\operatorname{supp}\hat{\varphi} = \operatorname{supp}\hat{\eta} = \mathbb{R}^n$. In addition, let $\varphi$ be compactly supported.

i) If $w$ is a compactly supported function contained in the space $S_1(\eta)$, then the functions $w(\cdot + M^{-1}d), d \in R'$, are in $W_0$ if and only if $[\hat{w}, \hat{\varphi}]_1$ is $2\pi$-periodic.

ii) If $w$ is a compactly supported generator for the space $S_1(\eta)$ and $[\hat{w}, \hat{\varphi}]_1$ is $2\pi$-periodic, then the functions $(w(\cdot + M^{-1}d))_{d\in R'}$ provide a basis for $W_0$.

*Proof.*
We start with the proof of part i). The functions $w(\cdot + M^{-1}d), d \in R'$, are in $W_0$ if and only if $\langle w(\cdot + M^{-1}d), \varphi(\cdot - k)\rangle_{L_2(\mathbb{R}^n)} = 0$ for all $d \in R', k \in \mathbb{Z}^n$. We proceed as in the proof of part i) of Lemma 1.3.7 to obtain

$$\begin{aligned}
0 &= \left\langle w(\cdot + M^{-1}d), \varphi(\cdot - k)\right\rangle_{L_2(\mathbb{R}^n)} \\
&= \frac{1}{(2\pi)^n} \left\langle \widehat{w(\cdot + M^{-1}d)}, \widehat{\varphi(\cdot - k)}\right\rangle_{L_2(\mathbb{R}^n)} \\
&= \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} e_{M^{-1}d}(\xi)\, e_k(\xi)\, \hat{w}(\xi)\, \overline{\hat{\varphi}(\xi)}\, \mathrm{d}\xi \\
&= \frac{1}{(2\pi)^n} \int\limits_{\widetilde{C}_1} \sum_{\ell \in 2\pi M^T \mathbb{Z}^n} e_{M^{-1}d}(\xi + \ell)\, e_k(\xi + \ell)\, \hat{w}(\xi + \ell)\, \overline{\hat{\varphi}(\xi + \ell)}\, \mathrm{d}\xi \\
&= \frac{1}{(2\pi)^n} \int\limits_{\widetilde{C}_1} e_{M^{-1}d+k}(\xi)\, [\hat{w}, \hat{\varphi}]_1(\xi)\, \mathrm{d}\xi.
\end{aligned}$$

It follows that

$$\frac{1}{(2\pi)^n} \frac{1}{|\det M^T|} \int\limits_{\widetilde{C}_1} e_{M^{-1}d+k}(\xi)\, [\hat{w}, \hat{\varphi}]_1(\xi)\, \mathrm{d}\xi = 0. \tag{6.13}$$

Since we know that $\{(2\pi)^{-n/2} e_{-k}(\xi)\}_{k\in\mathbb{Z}^n}$ is an orthonormal basis of $L_2([-\pi,\pi)^n)$, we can show that $\{(2\pi)^{-n/2} |\det M^T|^{-1/2} e_{-M^{-1}k}(\xi)\}_{k\in\mathbb{Z}^n}$ is an orthonormal system in $L_2(\widetilde{C}_1)$. Let $\ell, m \in \mathbb{Z}^n, \ell \neq m$. Then, we have

$$\begin{aligned}
0 &= \frac{1}{(2\pi)^n} \int\limits_{[-\pi,\pi)^n} e_{-\ell}(\xi)\, e_m(\xi)\, \mathrm{d}\xi \\
&= \frac{1}{(2\pi)^n} \int\limits_{[-\pi,\pi)^n} e_{m-\ell}(\xi)\, \mathrm{d}\xi \\
&= \frac{1}{(2\pi)^n} \frac{1}{|\det M^T|} \int\limits_{[-\pi,\pi)^n} e_{m-\ell}(M^{-T} M^T \xi)\, |\det M^T|\, \mathrm{d}\xi.
\end{aligned}$$

We set $u = M^T\xi$ and use the transformation formula to deduce

$$\begin{aligned}
&\frac{1}{(2\pi)^n}\frac{1}{|\det M^T|}\int\limits_{[-\pi,\pi)^n} e_{m-\ell}(M^{-T}M^T\xi)\,|\det M^T|\,\mathrm{d}\xi \\
&= \frac{1}{(2\pi)^n}\frac{1}{|\det M^T|}\int\limits_{M^T[-\pi,\pi)^n} e_{m-\ell}(M^{-T}u)\,\mathrm{d}u \\
&= \frac{1}{(2\pi)^n}\frac{1}{|\det M^T|}\int\limits_{\widetilde{C}_1} e_{-M^{-1}(\ell-m)}(u)\,\mathrm{d}u.
\end{aligned}$$

As a consequence,

$$\{(2\pi)^{-n/2}|\det M^T|^{-1/2}e_{-M^{-1}k}(\xi)\}_{k\in\mathbb{Z}^n}$$

is an orthonormal system in $L_2(\widetilde{C}_1)$. Besides that, we recall

$$M^{-1}\mathbb{Z}^n = \bigcup_{d\in R}(M^{-1}d + \mathbb{Z}^n).$$

With these considerations we conclude that only for $d = 0$ the Fourier coefficients in (6.13) do not vanish. In this case, we have

$$\frac{1}{|\det M^T|}\langle w, \varphi(\cdot - k)\rangle = \frac{1}{(2\pi)^n}\frac{1}{|\det M^T|}\int\limits_{\widetilde{C}_1} e_k(\xi)\,[\hat{w},\hat{\varphi}]_1(\xi)\,\mathrm{d}\xi. \tag{6.14}$$

Our aim is to show with the help of Lemma 1.2.5 that $[\hat{w},\hat{\varphi}]_1$ can be represented as a $2\pi$-periodic Fourier series where the Fourier coefficients are given by (6.14). In order to apply Lemma 1.2.5, we have to prove that the Fourier coefficients are in $\ell_1(\mathbb{Z}^n)$. To this end we define the function

$$\begin{aligned}
g : \widetilde{C}_0 &\to \mathbb{C}, \\
x &\mapsto [\hat{w},\hat{\varphi}](M^T x).
\end{aligned}$$

Since $[\hat{w},\hat{\varphi}]_1 \in L_1(\widetilde{C}_1)$, we obtain

$$\int\limits_{\widetilde{C}_0} |g(x)|\,dx = \int\limits_{\widetilde{C}_0} |[\hat{w},\hat{\varphi}]_1(M^T x)|\,\mathrm{d}x = \frac{1}{|\det M^T|}\int\limits_{\widetilde{C}_1} |[\hat{w},\hat{\varphi}]_1(u)|\,\mathrm{d}u < \infty.$$

Furthermore, the Fourier coefficients of the function $g$ for $k, \ell \in \mathbb{Z}^n, d \in R$, are

$$\begin{aligned}\frac{1}{(2\pi)^n}\int\limits_{\widetilde{C}_0} e_k(\xi)g(\xi)\,\mathrm{d}\xi &= \frac{1}{(2\pi)^n}\int\limits_{\widetilde{C}_0} e_k(\xi)[\hat{w},\hat{\varphi}]_1(M^T\xi)\,\mathrm{d}\xi \\ &= \frac{1}{(2\pi)^n}\frac{1}{|\det M^T|}\int\limits_{\widetilde{C}_1} e_{M^{-1}k}(u)[\hat{w},\hat{\varphi}]_1(u)\,\mathrm{d}u \\ &= \frac{1}{(2\pi)^n}\frac{1}{|\det M^T|}\int\limits_{\widetilde{C}_1} e_{M^{-1}d+\ell}(u)[\hat{w},\hat{\varphi}]_1(u)\,\mathrm{d}u \\ &= \frac{1}{|\det M^T|}\left\langle w(\cdot + M^{-1}d), \varphi(\cdot - \ell)\right\rangle_{L_2(\mathbb{R}^n)}.\end{aligned}$$

We already observed that these Fourier coefficients are nonzero for $d = 0$. By Lemma 1.3.6, the sequence $\{c(w,\varphi)_\ell\}_{\ell\in\mathbb{Z}^n}$ with $c(w,\varphi)_\ell := |\det M^T|^{-1}\langle w, \varphi(\cdot - \ell)\rangle_{L_2(\mathbb{R}^n)}$ is an element of the space $\ell_1(\mathbb{Z}^n)$. Consequently, the conditions of Lemma 1.2.5 are satisfied and with $y = M^T x, x \in \mathbb{R}^n$, and $e_{-k}(x) = e_{-k}(M^{-T}y) = e_{-M^{-1}k}(y)$ we obtain

$$\begin{aligned}[\hat{w},\hat{\varphi}]_1(y) &= g(x) \\ &= \sum_{k\in\mathbb{Z}^n} a_k\, e_{-k}(x) \\ &= \sum_{k\in\mathbb{Z}^n}\left(\frac{1}{(2\pi)^n}\frac{1}{|\det M^T|}\int\limits_{\widetilde{C}_1} e_{M^{-1}k}(\xi)[\hat{w},\hat{\varphi}]_1(\xi)\mathrm{d}\xi\right) e_{-M^{-1}k}(y).\end{aligned}$$

We further deduce

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^n}\left(\frac{1}{(2\pi)^n}\frac{1}{|\det M^T|}\int\limits_{\widetilde{C}_1} e_{M^{-1}k}(\xi)[\hat{w},\hat{\varphi}]_1(\xi)\mathrm{d}\xi\right) e_{-M^{-1}k}(y) \\ &= \sum_{d\in R}\sum_{\ell\in\mathbb{Z}^n}\left(\frac{1}{(2\pi)^n}\frac{1}{|\det M^T|}\int\limits_{\widetilde{C}_1} e_{M^{-1}d+\ell}(\xi)[\hat{w},\hat{\varphi}]_1(\xi)\mathrm{d}\xi\right) e_{-(M^{-1}d+\ell)}(y) \\ &= \sum_{\ell\in\mathbb{Z}^n}\left(\frac{1}{(2\pi)^n}\frac{1}{|\det M^T|}\int\limits_{\widetilde{C}_1} e_{\ell}(\xi)[\hat{w},\hat{\varphi}]_1(\xi)\mathrm{d}\xi\right) e_{-\ell}(y).\end{aligned}$$

In other words, $[\hat{w},\hat{\varphi}]_1$ is $2\pi$-periodic.

Now, let us prove part ii). Since $w$ is compactly supported, it follows directly that $\operatorname{supp}\hat{w} = \mathbb{R}^n$. By Corollary 6.1.9, $\mathscr{W}_* := (w(\cdot + M^{-1}d))_{d\in R}$ provides a basis for $S_1(\eta)$. This is equivalent to $\det G(\mathscr{W}_*) \neq 0$ almost everywhere on $\widetilde{C}_0$. Thus, the determinant of the Gramian matrix corresponding to $\mathscr{W}_* \setminus \{w\}$ is nonzero almost everywhere. Since we assume that $[\hat{w}, \hat{\varphi}]_1$ is $2\pi$-periodic, we can apply part i) to deduce that $\mathscr{W}_* \setminus \{w\}$ is in $W_0$. Hence, we obtain $m-1$ functions whose Gramian matrix is nonzero almost everywhere. Since Theorem 6.1.10 ensures that $W_0$ contains a basis of cardinality $m-1$, we conclude by part iii) of Corollary 4.2.8 that $\mathscr{W}_* \setminus \{w\}$ is a basis for $W_0$. □

**Remark 6.1.14.**
While verifying part i) of Theorem 6.1.13, we proved the following: Let $f, g$ be compactly supported functions in $L_2(\mathbb{R}^n)$. Then we have

$$[\hat{f}, \hat{g}]_1(y) = \sum_{d\in R}\sum_{k\in\mathbb{Z}^n} a_{d,k}\, e_{-(M^{-1}d+k)}(y) \quad \text{almost everywhere,} \tag{6.15}$$

where the coefficients $a_{d,k}$ are given by

$$\begin{aligned} a_{d,k} &= \frac{1}{(2\pi)^n}\frac{1}{|\det M^T|}\int\limits_{\widetilde{C}_1} e_{M^{-1}d+k}(\xi)\,[\hat{f}, \hat{g}]_1(\xi)\,\mathrm{d}\xi \\ &= \frac{1}{|\det M^T|}\left\langle f(\cdot + M^{-1}d), g(\cdot - k)\right\rangle_{L_2(\mathbb{R}^n)}. \end{aligned}$$

Theorem 6.1.13 raises the question how such a function $w \in S_1(\eta)$ might look like. On the one hand, it is necessary that

$$[\hat{w}, \hat{\varphi}]_1 = [\tau\hat{\eta}, \hat{\varphi}]_1 = \tau[\hat{\eta}, \hat{\varphi}]_1, \quad \tau\ 2\pi M^T\mathbb{Z}^n\text{-periodic,}$$

takes the value 0 or 1 because we need orthogonality between the functions $w$ and $\varphi$, see (1.7). As a result, the bracket product $[\hat{w}, \hat{\varphi}]_1$ is $2\pi$-periodic. On the other hand, we demand $\operatorname{supp}\hat{w} = \operatorname{supp}\hat{\eta} = \mathbb{R}^n$ because this implies that $w$ is a generator for the space $S_1(\eta)$. Both requirements can be met with the choice $\tau = [\hat{\eta}, \hat{\varphi}]_1^{-1}$, i.e., we obtain

$$\hat{w}_0 := \frac{\hat{\eta}}{[\hat{\eta}, \hat{\varphi}]_1} = \frac{\frac{1}{\tilde{\tau}}\hat{\varphi}}{[\frac{1}{\tilde{\tau}}\hat{\varphi}, \hat{\varphi}]_1} = \frac{\hat{\varphi}}{[\hat{\varphi}, \hat{\varphi}]_1}, \tag{6.16}$$

where $\tilde{\tau}$ is $2\pi M^T\mathbb{Z}^n$-periodic. This representation has the disadvantage that the division by $[\hat{\varphi}, \hat{\varphi}]_1$ might cause that $w_0$ is not an $L_2(\mathbb{R}^n)$-function. Moreover, we need to ensure that $w_0$ is compactly supported. Hence, we have to modify this representation such that there is no denominator in (6.16).

**Corollary 6.1.15.**
Let $S_0(\varphi) \subset S_1(\eta)$ and $\operatorname{supp}\hat{\varphi} = \operatorname{supp}\hat{\eta} = \mathbb{R}^n$. Moreover, let $\varphi$ be compactly supported. Then the function $w$ defined by

$$\hat{w} := \frac{\hat{\varphi}}{[\hat{\varphi},\hat{\varphi}]_1}\prod_{d\in R}[\hat{\varphi},\hat{\varphi}]_1(\cdot + 2\pi d) = \hat{\varphi}\prod_{d\in R'}[\hat{\varphi},\hat{\varphi}]_1(\cdot + 2\pi d) \tag{6.17}$$

is an element of the space $L_2(\mathbb{R}^n)$ and the set $\{w(\cdot + M^{-1}d) : d \in R'\}$ provides a basis for $W_0$. Furthermore, the function $w$ is compactly supported.

*Proof.*
First of all, $[\hat{\varphi},\hat{\varphi}]_1$ is bounded because it is a trigonometric polynomial, see part iii) of Lemma 1.3.7. It follows that $\hat{w}$ and thus, $w$ are functions in $L_2(\mathbb{R}^n)$. Moreover, $w \in S_1(\eta)$ because we obtain (6.17) by multiplying (6.16) with a $2\pi M^T\mathbb{Z}^n$-periodic product.

Since $\operatorname{supp}\hat{\varphi} = \mathbb{R}^n$, it follows that $[\hat{\varphi},\hat{\varphi}]_1 > 0$ almost everywhere. Hence, we have $\operatorname{supp}\hat{w} = \mathbb{R}^n$ and Proposition 4.1.4 yields that $w$ is another generator for the space $S_1(\eta)$.

Finally, we want to apply part ii) of Theorem 6.1.13 to show that the set of functions $\{w(\cdot + M^{-1}d) : d \in R'\}$ provides a basis for $W_0$. Beforehand, we have to verify that $[\hat{w},\hat{\varphi}]_1$ is $2\pi$-periodic and that $w$ is compactly supported. We constructed $\hat{w}_0$ such that $[\hat{w}_0,\hat{\varphi}]_1$ is $2\pi$-periodic. Hence, it suffices to show that the product $\prod_{d\in R}[\hat{\varphi},\hat{\varphi}]_1(\cdot + 2\pi d)$ is $2\pi$-periodic in order to deduce the $2\pi$-periodicity of $[\hat{w},\hat{\varphi}]_1$. For this purpose we consider

$$\prod_{d\in R}[\hat{\varphi},\hat{\varphi}]_1(\cdot + 2\pi d + 2\pi\mathbb{1}) = \prod_{d\in R}[\hat{\varphi},\hat{\varphi}]_1(\cdot + 2\pi(d+\mathbb{1})),$$

where $\mathbb{1}$ denotes the vector of ones. Due to (6.2), there exists a $\tilde{d} \in R$ and an $\ell \in \mathbb{Z}^n$ such that $\mathbb{1}$ can be written as $\mathbb{1} = \tilde{d} + M\ell$. As we can see in the proof of Lemma 6.1.2, the summation $d + \mathbb{1}, d \in R$, provides a new set of representatives of the distinct cosets of $\mathbb{Z}^n/M\mathbb{Z}^n$. Consequently, we obtain

$$\prod_{d\in R}[\hat{\varphi},\hat{\varphi}]_1(\cdot + 2\pi(d+\mathbb{1})) = \prod_{d^*\in R}[\hat{\varphi},\hat{\varphi}]_1(\cdot + 2\pi d^*).$$

It remains to show that the function $w$ is compactly supported. Since $\varphi$ is compactly supported, we know due to the Paley-Wiener Theorem 1.2.10 that $\hat{\varphi}$ is an entire function of exponential type and $\operatorname{supp}\hat{\varphi} = \mathbb{R}^n$. Finite sums and finite products of entire functions are entire functions. Therefore, $\hat{w}$ is an entire function of exponential type with $\operatorname{supp}\hat{w} = \mathbb{R}^n$. Our claim follows by using the Paley-Wiener Theorem 1.2.10. again. Now, part ii) of Theorem 6.1.13 yields the desired result. □

### 6.1.2 Compactly Supported Wavelet Basis for $\mathbf{W_j}$

Our construction procedure for a basis of $W_0$ can be applied to all spaces $W_j$ after a suitable dilation. We illustrate this for the case $j = 1$. This choice allows us to clearly observe how the role of $S_1(\eta)$ is changing when compared with our investigations in the previous section.

Let $\eta, p \in L_2(\mathbb{R}^n)$ with

$$\operatorname{supp}\hat{\eta} = \operatorname{supp}\hat{p} = \mathbb{R}^n. \tag{6.18}$$

Further assume that

$$S_1(\eta) \subset S_2(p), \tag{6.19}$$

where the space $S_1$ is $M^{-1}$-shift-invariant and the space $S_2$ is $M^{-2}$-shift-invariant. The subsequent theorem tells us that the properties (6.18) and (6.19) of the generators $\eta$ and $p$ are preserved under dilation.

**Theorem 6.1.16.**
Let $\eta_{M^{-1}} := \eta(M^{-1}\cdot)$ and $p_{M^{-1}} := p(M^{-1}\cdot)$. Under the assumptions (6.18) and (6.19) we have

$$S_0(\eta_{M^{-1}}) \subset S_1(p_{M^{-1}}), \tag{6.20}$$

where $\operatorname{supp}\widehat{\eta_{M^{-1}}} = \operatorname{supp}\widehat{p_{M^{-1}}} = \mathbb{R}^n$.

*Proof.*
First, we prove (6.20). Let $f$ be a function in $S_0(\eta_{M^{-1}})$. Then there exists a sequence $(f_\ell)_{\ell\in\mathbb{N}} \in S_0(\eta_{M^{-1}})$ such that $f_\ell \to f$ in the $L_2$-sense. The definition of the space $S_0(\eta_{M^{-1}})$ yields for all elements of the sequence the representation

$$\begin{aligned} f_\ell &= \sum_{k\in\mathbb{Z}^n} a_{\ell,k}\, \eta_{M^{-1}}(\cdot - k) \\ &= \sum_{k\in\mathbb{Z}^n} a_{\ell,k}\, \eta(M^{-1}\cdot - M^{-1}k). \end{aligned} \tag{6.21}$$

Analogue to Chapter 4, we define the following bijective, linear and continuous operator

$$\begin{aligned} \widetilde{J} : L_2(\mathbb{R}^n) &\to L_2(\mathbb{R}^n) \\ f &\mapsto f(M^{-1}\cdot) \end{aligned}$$

and apply $\widetilde{J}^{-1}$ to (6.21). We obtain

$$\widetilde{J}^{-1} f_\ell = \sum_{k\in\mathbb{Z}^n} a_{\ell,k}\, \eta(\cdot - M^{-1}k).$$

Consequently, $\widetilde{J}^{-1} f_\ell \in S_1(\eta)$ and by (6.19), $\widetilde{J}^{-1} f_\ell \in S_2(p)$. Hence, there exists a sequence $(g_r)_{r\in\mathbb{N}} \in S_2(p)$ such that $g_r \to \widetilde{J}^{-1} f_\ell$ in the $L_2$-sense. The function $g_r$ can be represented as

$$g_r = \sum_{k\in\mathbb{Z}^n} b_{r,k}\, p(\cdot - M^{-2}k) = \sum_{k\in\mathbb{Z}^n} b_{r,k}\, p_{M^{-1}}(M\cdot - M^{-1}k).$$

Applying the operator $\widetilde{J}$ yields

$$\widetilde{J} g_r = \sum_{k\in\mathbb{Z}^n} b_{r,k}\, p_{M^{-1}}(\cdot - M^{-1}k).$$

Hence, we have a sequence $(\widetilde{J} g_r)_{r\in\mathbb{N}} \in S_1(p_{M^{-1}})$ which converges to $f_\ell$ in the $L_2$-sense. The $L_2$-closure of $S_1(p_{M^{-1}})$ yields $f_\ell \in S_1(p_{M^{-1}})$. Therefore, it holds $(f_\ell)_{\ell\in\mathbb{N}} \in S_1(p_{M^{-1}})$ and analogously $f \in S_1(p_{M^{-1}})$. Besides that $\operatorname{supp}\widehat{\eta_{M^{-1}}} = \operatorname{supp}\widehat{p_{M^{-1}}} = \mathbb{R}^n$, see (1.4). □

The theorem above ensures that the conditions of Theorem 6.1.4 are fulfilled. Hence, we obtain

$$S_1(p_{M^{-1}}) = S_1(\eta_{M^{-1}}) = S_0(H_{M^{-1}}),$$

where $H_{M^{-1}} := \{\eta_{M^{-1},d} := \eta_{M^{-1}}(\cdot + M^{-1}d), d \in R\}$. Moreover, Corollary 6.1.9 yields that $H_{M^{-1}}$ provides a basis for $S_0(H_{M^{-1}})$. Consequently, we know by Theorem 6.1.10 that the set of functions

$$\left(\eta_{M^{-1},d} - \mathcal{P}_{S_0(\eta_{M^{-1}})}\eta_{M^{-1},d}\right)_{d\in R'}$$

provides a basis for

$$\widetilde{W}_0 := S_1(p_{M^{-1}}) \ominus S_0(\eta_{M^{-1}}).$$

If we further assume that $\eta$ is compactly supported, then $\eta_{M^{-1}}$ is compactly supported as well. This implies that $[\widehat{\eta_{M^{-1}}}, \widehat{\eta_{M^{-1}}}]$ is a trigonometric polynomial and thus, bounded. Consequently, Theorem 6.1.11 provides us with a compactly supported basis for $\widetilde{W}_0$ where the basis elements are defined by

$$\widehat{w_d} := [\widehat{\eta_{M^{-1}}}, \widehat{\eta_{M^{-1}}}]\widehat{\eta_{M^{-1},d}} - [\widehat{\eta_{M^{-1},d}}, \widehat{\eta_{M^{-1}}}]\widehat{\eta_{M^{-1}}}, \quad d \in R'.$$

Analogue to the proof of part i) of Corollary 4.2.8 and the fact that dilation preserves orthogonality, we obtain that the compactly supported functions

$$w_{M,d} := w_d(M\cdot), \quad d \in R',$$

provide a basis for $W_1 := S_2(p) \ominus S_1(\eta)$. In case $(w_d)_{d\in R'}$ is an $L_2$-stable basis for $W_0$, then $(w_{M,d})_{d\in R'}$ is an $L_2$-stable basis for $W_1$, see proof of part iv) of Corollary 4.2.8.

### 6.1.3 Example: Exponential Box Splines

In this section we apply our construction methods for general dilation matrices to exponential box splines contained in $\mathbb{R}^2$. In the following we present two examples where we reduce the number of required mother wavelets for $W_0$ from three in the case $M = 2I, n = 2$, to only one. Moreover, we prove within these examples that there exist non-stationary refinement equations for exponential box splines with dilation matrices other than $M = cI$ with $|c| \geq 2, c \in \mathbb{Z}$. To the best of the author's knowledge, this is a completely new result.

Exponential box splines are a generalization of polynomial box splines. The following definition of polynomial box splines as well as the properties listed below this definition can be found in [33, Chapter I].

**Definition 6.1.17.**
Let $n, p \in \mathbb{N}$ with $n \leq p$. Moreover, let $x^j = (x_1^j, \ldots, x_n^j)^T \in \mathbb{R}^n \setminus \{0\}$ with $j = 1, \ldots, p$, be the columns of the matrix $X_p = (x^1, \ldots, x^p)$. Then the *polynomial box spline* $B_{X_p}$ associated with the matrix $X_p$ is inductively defined by

$$\begin{aligned} B_{X_p} &: \mathbb{R}^n \to \mathbb{R}, \\ &u \mapsto \int_0^1 B_{X_{p-1}}(u - tx^p)\, \mathrm{d}t. \end{aligned}$$

In case $X_n$ is invertible it holds

$$B_{X_n}(u) = \begin{cases} \frac{1}{|\det X_n|}, & \text{for } u \in Z_{X_n} := \left\{ \sum_{j=1}^n t_j x^j : 0 \leq t_j < 1 \right\}, \\ 0, & \text{elsewhere.} \end{cases}$$

The polynomial box spline $B_{X_p}$ is non-negative and compactly supported on

$$\overline{Z_{X_p}} := \left\{ \sum_{j=1}^p t_j x^j : 0 \leq t_j \leq 1 \right\}.$$

In addition, it is a piecewise polynomial function of degree $p - n$. Moreover, the Fourier transform of $B_{X_p}$ is given by

$$\widehat{B_{X_p}}(\xi) = \prod_{j=1}^p \frac{1 - e^{-i\xi \cdot x^j}}{i\xi \cdot x^j} = \prod_{j=1}^p \frac{e^{-i\xi \cdot x^j} - 1}{-i\xi \cdot x^j}.$$

In [25] it was shown for certain polynomial box splines in $\mathbb{R}^2$ that $M^2 = 2I$ is a necessary condition in order to obtain $2I$-refinability. Such a matrix can be constructed in the following way, see [25, Corollary 4.1]:

**Lemma 6.1.18.**
Let $M \in \mathbb{Z}^{2\times 2}$ be a dilation matrix with $\det M < 0$. Then $M$ satisfies $M^2 = 2I$ if and only if trace $M = 0$ and $\det M = -2$.

Our aim is to choose appropriate exponential box splines as generators $\hat{\varphi}_j$ for the spaces $S_j, j \in \mathbb{Z}$, such that

$$\hat{\varphi}_j = A\,\hat{\varphi}_{j+1}, \quad A\,2\pi(M^T)^{j+1}\mathbb{Z}^2\text{-periodic}, \tag{6.22}$$

holds in the setting of the generalized multiresolution analysis. We refer to (6.22) as a *non-stationary refinement equation.* Since the lemma above leads to refinability in the special case of polynomial box splines and a stationary multiresolution analysis, we take it as a starting point. Hence, we choose the dilation matrix as

$$M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{6.23}$$

It can easily be checked that the trace of $M$ is 0. Besides that, the eigenvalues are $\sqrt{2}$ and $-\sqrt{2}$. Thus, we have an expanding matrix with integer entries. Furthermore, it holds $|\det M| = 2$. Consequently, we have to determine one mother wavelet in order to obtain a basis for $W_0$. In contrast, for $M = 2I, n = 2$, we would need three mother wavelets. Hence, already in the two-dimensional case, working with a different dilation matrix leads to a significant reduction of the number of mother wavelets. The inverse of $M$ is given by

$$M^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}.$$

Moreover, we observe that $M^2 = 2I$ and therefore, we obtain for all $\ell \in \mathbb{N}$:

$$\begin{aligned} M^{2\ell} &= 2^\ell I, & M^{2\ell+1} &= 2^\ell M, \\ (M^{-1})^{2\ell} &= \frac{1}{2^\ell} I, & (M^{-1})^{2\ell+1} &= \frac{1}{2^\ell} M^{-1}. \end{aligned}$$

Besides choosing an appropriate matrix, we have to define the spaces $S_j, j \in \mathbb{Z}$, to obtain a generalized multiresolution analysis. We choose exponential box splines as generators for the spaces $S_j$. The following definition and classification of exponential box splines can be found in [30].

**Definition 6.1.19.**
Let $\Gamma$ be a finite index set consisting of pairs $\gamma = (x^\gamma, \lambda_\gamma)$ with $x^\gamma \in \mathbb{R}^n \setminus \{0\}, \lambda_\gamma \in \mathbb{C}$. Moreover, let $\lambda := (\lambda_\gamma)_{\gamma\in\Gamma}$. Then the *exponential box spline* $C_\lambda$ is defined by its Fourier transform as

$$\widehat{C}_\lambda(\xi) := \prod_{\gamma\in\Gamma} \frac{e^{\lambda_\gamma - i\xi\cdot x^\gamma} - 1}{\lambda_\gamma - i\xi \cdot x^\gamma}, \quad \xi \in \mathbb{R}^n. \tag{6.24}$$

We refer to $(x^\gamma)_{\gamma\in\Gamma}$ as *directions*.

**Lemma 6.1.20.**
The exponential box spline $C_\lambda$ has the following properties:

i) If $\operatorname{span}(x_\gamma)_{\gamma\in\Gamma} = \mathbb{R}^n$, the exponential box spline $C_\lambda$ is a compactly supported piecewise-exponential-polynomial function supported on

$$\overline{Z_\Gamma} = \left\{ \sum_{\gamma\in\Gamma} t_\gamma x^\gamma : 0 \leq t_\gamma \leq 1 \right\}.$$

ii) If $\lambda = 0$, then $C_\lambda$ is a polynomial box spline.

iii) If $n = 1$ and $x^\gamma = 1$ for all $\gamma \in \Gamma$, then $C_\lambda$ is an exponential B-spline.

iv) If all the directions $(x^\gamma)_{\gamma\in\Gamma}$ are standard unit vectors, then $C_\lambda$ is a tensor spline.

v) If $\lambda \in \mathbb{R}$, we obtain $C_\lambda \geq 0$ in the interior of $\overline{Z_\Gamma}$.

For more detailed information on exponential box splines we refer to [69], [70] and [27].

Hereinafter, we assume that $(x^\gamma)_{\gamma\in\Gamma}$ consists of $2p$ vectors, $p \in \mathbb{N} \setminus \{0\}$, and that these vectors split up into two vectors, each appearing $p$ times. These two vectors are chosen to be

$$x^{\gamma_1} := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad x^{\gamma_2} := M \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{6.25}$$

Hence, $\operatorname{span}(x^\gamma)_{\gamma\in\Gamma} = \mathbb{R}^2$. Next, we state the following important observation.

**Lemma 6.1.21.**
Let $M$ be defined as in (6.23) and let $x^{\gamma_1}$ and $x^{\gamma_2}$ be defined as in (6.25). Moreover, let the directions be given by $p$-times the vector $x^{\gamma_1}$ and $p$-times the vector $x^{\gamma_2}$ with $p \in \mathbb{N} \setminus \{0\}$. Furthermore, for $j \in \mathbb{Z}$ we define

$$\lambda_j := (\lambda_{j,\gamma_1}, \lambda_{j,\gamma_2}), \quad \lambda_{j-1} = (\lambda_{j-1,\gamma_1}, \lambda_{j-1,\gamma_2}) := (\lambda_{j,\gamma_2}/2, \lambda_{j,\gamma_1}),$$

where $\lambda_{j,\gamma_i} \in \mathbb{C}$ for $i = 1, 2$. Then the exponential box spline $C_{\lambda_j}$ given by

$$\widehat{C}_{\lambda_j}(\xi) = \left( \frac{e^{\lambda_{j,\gamma_1} - i\xi\cdot x^{\gamma_1}} - 1}{\lambda_{j,\gamma_1} - i\xi \cdot x^{\gamma_1}} \right)^p \left( \frac{e^{\lambda_{j,\gamma_2} - i\xi\cdot x^{\gamma_2}} - 1}{\lambda_{j,\gamma_2} - i\xi \cdot x^{\gamma_2}} \right)^p$$

satisfies

$$\widehat{C}_{\lambda_j}(M^j\xi) = A_{\lambda_j,\lambda_{j-1}}(M^{j-1}\xi)\widehat{C}_{\lambda_{j-1}}(M^{j-1}\xi)$$

with

$$A_{\lambda_j,\lambda_{j-1}}(\xi) := \left(\frac{1}{2}\frac{e^{\lambda_{j,\gamma_2}-iM\xi\cdot x^{\gamma_2}}-1}{e^{\lambda_{j-1,\gamma_1}-i\xi\cdot x^{\gamma_1}}-1}\right)^p.$$

Furthermore, $A_{\lambda_j,\lambda_{j-1}}$ is a $2\pi$-periodic trigonometric polynomial.

*Proof.*
Let $\xi = (\xi_1,\xi_2)^T \in \mathbb{R}^2$ and let $j \in \mathbb{Z}$, $\ell \in \mathbb{N}$. Furthermore, we set $\lambda_{\gamma_i} := \lambda_{j,\gamma_i}$ and $\tilde{\lambda}_{\gamma_i} := \lambda_{j-1,\gamma_i}$ for $i = 1,2$. First, we insert the definition of $\widehat{C}_{\lambda_j}$ into $\widehat{C}_{\lambda_j}(M^j\xi)$. We obtain

$$\widehat{C}_{\lambda_j}(M^j\xi) = \begin{cases} \left(\frac{e^{\lambda_{\gamma_1}-i\xi_1}-1}{\lambda_{\gamma_1}-i\xi_1}\right)^p\left(\frac{e^{\lambda_{\gamma_2}-i(\xi_1+\xi_2)}-1}{\lambda_{\gamma_2}-i(\xi_1+\xi_2)}\right)^p, & j=0,\\ \left(\frac{e^{\lambda_{\gamma_1}-i2^\ell\xi_1}-1}{\lambda_{\gamma_1}-i2^\ell\xi_1}\right)^p\left(\frac{e^{\lambda_{\gamma_2}-i2^\ell(\xi_1+\xi_2)}-1}{\lambda_{\gamma_2}-i2^\ell(\xi_1+\xi_2)}\right)^p, & j=2\ell>0,\\ \left(\frac{e^{\lambda_{\gamma_1}-i2^{-\ell}\xi_1}-1}{\lambda_{\gamma_1}-i2^{-\ell}\xi_1}\right)^p\left(\frac{e^{\lambda_{\gamma_2}-i2^{-\ell}(\xi_1+\xi_2)}-1}{\lambda_{\gamma_2}-i2^{-\ell}(\xi_1+\xi_2)}\right)^p, & j=-2\ell<0,\\ \left(\frac{e^{\lambda_{\gamma_1}-i2^\ell(\xi_1+\xi_2)}-1}{\lambda_{\gamma_1}-i2^\ell(\xi_1+\xi_2)}\right)^p\left(\frac{e^{\lambda_{\gamma_2}-i2^{\ell+1}\xi_1}-1}{\lambda_{\gamma_2}-i2^{\ell+1}\xi_1}\right)^p, & j=2\ell+1,\\ \left(\frac{e^{\lambda_{\gamma_1}-i2^{-\ell-1}(\xi_1+\xi_2)}-1}{\lambda_{\gamma_1}-i2^{-\ell-1}(\xi_1+\xi_2)}\right)^p\left(\frac{e^{\lambda_{\gamma_2}-i2^{-\ell}\xi_1}-1}{\lambda_{\gamma_2}-i2^{-\ell}\xi_1}\right)^p, & j=-2\ell-1. \end{cases}$$

For $\widehat{C}_{\lambda_{j-1}}(M^{j-1}\xi)$ we have

$$\widehat{C}_{\lambda_{j-1}}(M^{j-1}\xi) = \begin{cases} \left(\frac{e^{\tilde{\lambda}_{\gamma_1}-i/2(\xi_1+\xi_2)}-1}{\tilde{\lambda}_{\gamma_1}-i/2(\xi_1+\xi_2)}\right)^p\left(\frac{e^{\tilde{\lambda}_{\gamma_2}-i\xi_1}-1}{\tilde{\lambda}_{\gamma_2}-i\xi_1}\right)^p, & j=0,\\ \left(\frac{e^{\tilde{\lambda}_{\gamma_1}-i2^{\ell-1}(\xi_1+\xi_2)}-1}{\tilde{\lambda}_{\gamma_1}-i2^{\ell-1}(\xi_1+\xi_2)}\right)^p\left(\frac{e^{\tilde{\lambda}_{\gamma_2}-i2^\ell\xi_1}-1}{\tilde{\lambda}_{\gamma_2}-i2^\ell\xi_1}\right)^p, & j=2\ell>0,\\ \left(\frac{e^{\tilde{\lambda}_{\gamma_1}-i2^{-\ell-1}(\xi_1+\xi_2)}-1}{\tilde{\lambda}_{\gamma_1}-i2^{-\ell-1}(\xi_1+\xi_2)}\right)^p\left(\frac{e^{\tilde{\lambda}_{\gamma_2}-i2^{-\ell}\xi_1}-1}{\tilde{\lambda}_{\gamma_2}-i2^{-\ell}\xi_1}\right)^p, & j=-2\ell<0,\\ \left(\frac{e^{\tilde{\lambda}_{\gamma_1}-i2^\ell\xi_1}-1}{\tilde{\lambda}_{\gamma_1}-i2^\ell\xi_1}\right)^p\left(\frac{e^{\tilde{\lambda}_{\gamma_2}-i2^\ell(\xi_1+\xi_2)}-1}{\tilde{\lambda}_{\gamma_2}-i2^\ell(\xi_1+\xi_2)}\right)^p, & j=2\ell+1\\ \left(\frac{e^{\tilde{\lambda}_{\gamma_1}-i2^{-\ell-1}\xi_1}-1}{\tilde{\lambda}_{\gamma_1}-i2^{-\ell-1}\xi_1}\right)^p\left(\frac{e^{\tilde{\lambda}_{\gamma_2}-i2^{-\ell-1}(\xi_1+\xi_2)}-1}{\tilde{\lambda}_{\gamma_2}-i2^{-\ell-1}(\xi_1+\xi_2)}\right)^p, & j=-2\ell-1. \end{cases}$$

Since $\tilde{\lambda}_{\gamma_1} = \lambda_{\gamma_2}/2$ and $\tilde{\lambda}_{\gamma_2} = \lambda_{\gamma_1}$, the quotient of $\widehat{C}_{\lambda_j}(M^j\xi)$ and $\widehat{C}_{\lambda_{j-1}}(M^{j-1}\xi)$ has the form

$$\frac{\widehat{C}_{\lambda_j}(M^j\xi)}{\widehat{C}_{\lambda_{j-1}}(M^{j-1}\xi)} = \begin{cases} \left(\frac{1}{2}\frac{e^{\lambda_{\gamma_2}-i(\xi_1+\xi_2)}-1}{e^{\tilde{\lambda}_{\gamma_1}-i/2(\xi_1+\xi_2)}-1}\right)^p, & j=0,\\ \left(\frac{1}{2}\frac{e^{\lambda_{\gamma_2}-i2^\ell(\xi_1+\xi_2)}-1}{e^{\tilde{\lambda}_{\gamma_1}-i2^{\ell-1}(\xi_1+\xi_2)}-1}\right)^p, & j=2\ell>0,\\ \left(\frac{1}{2}\frac{e^{\lambda_{\gamma_2}-i2^{-\ell}(\xi_1+\xi_2)}-1}{e^{\tilde{\lambda}_{\gamma_1}-i2^{-\ell-1}(\xi_1+\xi_2)}-1}\right)^p, & j=-2\ell<0,\\ \left(\frac{1}{2}\frac{e^{\lambda_{\gamma_2}-i2^{\ell+1}\xi_1}-1}{e^{\tilde{\lambda}_{\gamma_1}-i2^\ell\xi_1}-1}\right)^p, & j=2\ell+1,\\ \left(\frac{1}{2}\frac{e^{\lambda_{\gamma_2}-i2^{-\ell}\xi_1}-1}{e^{\tilde{\lambda}_{\gamma_1}-i2^{-\ell-1}\xi_1}-1}\right)^p, & j=-2\ell-1. \end{cases}$$

Then $A$ is given by

$$\begin{aligned} A_{\lambda_j,\lambda_{j-1}}(\xi) &= \left(\frac{1}{2}\frac{e^{\lambda_{\gamma_2}-iM\xi\cdot x^{\gamma_2}}-1}{e^{\tilde{\lambda}_{\gamma_1}-i\xi\cdot x^{\gamma_1}}-1}\right)^p \\ &= \left(\frac{1}{2}\frac{e^{\lambda_{\gamma_2}-i2\xi_1}-1}{e^{\tilde{\lambda}_{\gamma_1}-i\xi_1}-1}\right)^p \\ &= \frac{1}{2^p}\left(\frac{e^{2(\lambda_{\gamma_2}/2-i\xi_1)}-1}{e^{\lambda_{\gamma_2}/2-i\xi_1}-1}\right)^p\left(\frac{e^{\lambda_{\gamma_2}/2-i\xi_1}+1}{e^{\lambda_{\gamma_2}/2-i\xi_1}+1}\right)^p \\ &= \frac{1}{2^p}\left(e^{\lambda_{\gamma_2}/2-i\xi_1}+1\right)^p. \end{aligned}$$

Since

$$A_{\lambda_j,\lambda_{j-1}}(\xi+2\pi\mathbb{1}) = \frac{1}{2^p}\left(e^{\lambda_{\gamma_2}/2-i(\xi_1+2\pi)}+1\right)^p = \frac{1}{2^p}\left(e^{\lambda_{\gamma_2}/2-i\xi_1}+1\right)^p = A_{\lambda_j,\lambda_{j-1}}(\xi),$$

the function $A_{\lambda_j,\lambda_{j-1}}$ is a $2\pi$-periodic trigonometric polynomial. □

Based on the lemma above, we define the generators of the spaces $S_j$, $j \in \mathbb{Z}$, by

$$\varphi_j := C_{\lambda_{-j}}(M^j\cdot) \tag{6.26}$$

to obtain a non-stationary refinement equation. This can be seen as follows: Formula (1.4) and the symmetry of the matrix $M$ yield

$$\hat{\varphi}_j = \frac{1}{m^j}\widehat{C}_{\lambda_{-j}}(M^{-j}\cdot).$$

Therefore, it holds

$$\hat{\varphi}_j = \frac{1}{m^j}A_{\lambda_{-j},\lambda_{-j-1}}(M^{-j-1}\cdot)\widehat{C}_{\lambda_{-j-1}}(M^{-j-1}\cdot) = mA_{\lambda_{-j},\lambda_{-j-1}}(M^{-j-1}\cdot)\hat{\varphi}_{j+1}.$$

Furthermore, we fix values for $\lambda_0$ for our construction, i.e.,

$$\lambda_1^* := \lambda_{0,\gamma_1}, \quad \lambda_2^* := \lambda_{0,\gamma_2}.$$

According to Lemma 6.1.21, we obtain

$$\begin{aligned} &\vdots \\ \lambda_2 &= (2\lambda_1^*, 2\lambda_2^*), \\ \lambda_1 &= (\lambda_2^*, 2\lambda_1^*), \\ \lambda_0 &= (\lambda_1^*, \lambda_2^*), \\ \lambda_{-1} &= (\lambda_2^*/2, \lambda_1^*), \\ \lambda_{-2} &= (\lambda_1^*/2, \lambda_2^*/2), \\ &\vdots \end{aligned}$$

or more general

$$\lambda_j = \begin{cases} (2^{j/2}\lambda_1^*, 2^{j/2}\lambda_2^*), & j \in 2\mathbb{Z}, \\ (2^{(j-1)/2}\lambda_2^*, 2^{(j+1)/2}\lambda_1^*), & j \in 2\mathbb{Z}+1. \end{cases}$$

Next, we investigate if the conditions (M1)-(M3) of a generalized multiresolution analysis are fulfilled for the spaces defined above. Due to our non-stationary refinement equation the corresponding spaces are nested. Furthermore, the compact support of the generators yields $\overline{\bigcup_j S_j} = L_2(\mathbb{R}^2)$, see Theorem 4.1.7. Moreover, we know by Theorem 4.1.10 that the intersection of the spaces $S_j$ in the non-stationary case is of dimension 0 or 1. Besides that, we obtain a stationary multiresolution analysis if and only if $\lambda_0 = (0,0)$.

In summary, we have shown so far that the spaces defined above fit into the setting of our construction procedure developed in Section 6.1. For this construction procedure we have to choose a set of representatives of $\mathbb{Z}^2/M\mathbb{Z}^2$. Since $m = 2$, such a set consists of two elements. Furthermore, $(0,0)^T$ has to be one of the representatives. We set

$$R := \left\{ d_0 := \begin{pmatrix} 0 \\ 0 \end{pmatrix}, d_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}, \quad R' := \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

Moreover, let $\varphi := \varphi_0$ and $\eta := \varphi_1$. Then Theorem 6.1.4 states that $S_1(\eta) = S_0(\phi)$ with

$$\phi := \{\varphi_{d_i} := \varphi(\cdot + M^{-1}d_i), i = 0,1\}.$$

Since the function $\varphi$ is compactly supported, we have $\operatorname{supp}\hat{\varphi} = \mathbb{R}^2$. Therefore, $\phi$ provides a basis for $S_0(\phi)$, see Corollary 6.1.9. Since $m-1 = 1$, we define a single mother wavelet for the space $W_0$ by

$$\widehat{w_{d_1}} := \begin{cases} \widehat{\varphi_{d_1}} - \widehat{P_\varphi \varphi_{d_1}} = \widehat{\varphi_{d_1}} - \hat{\varphi}[\widehat{\varphi_{d_1}}, \hat{\varphi}]/[\hat{\varphi}, \hat{\varphi}], & [\hat{\varphi}, \hat{\varphi}] \neq 0, \\ \hat{\varphi}_{d_1}, & [\hat{\varphi}, \hat{\varphi}] = 0. \end{cases}$$

By Theorem 6.1.10, the function $w_{d_1}$ and its integer translates provide a basis for $W_0 = S_0(w_{d_1})$. If we want the basis of $W_0$ to be compactly supported, we can modify $\widehat{w_{d_1}}$ by multiplying it with $[\hat{\varphi}, \hat{\varphi}]$. This bracket product is a trigonometric polynomial because of the compact support of $\varphi$. Consequently, $[\hat{\varphi}, \hat{\varphi}]$ is bounded. Theorem 6.1.11 yields that

$$w_c := ([\hat{\varphi}, \hat{\varphi}]\widehat{\varphi_{d_1}} - [\widehat{\varphi_{d_1}}, \hat{\varphi}]\hat{\varphi})^\vee \tag{6.27}$$

provides a compactly supported basis for $W_0 = S_0(w_c)$.

Next, we want to address stability. $L_2$-stability of the integer translates of $\varphi := C_{\lambda_0}$ can be deduced from their linear independence, see [49]. In [23] we find the following important result:

**Lemma 6.1.22.**
Let $C_\lambda$ be an exponential box spline defined as in (6.24). Moreover, let the directions $(x^\gamma)_{\gamma\in\Gamma}$ be the columns of the matrix $X$. Then the integer translates of $C_\lambda$ are linearly independent if and only if $X$ is unimodular and one has $\lambda_\gamma - \lambda_{\gamma'} \notin 2\pi i\mathbb{Z}\setminus\{0\}$ for all $\lambda_\gamma, \lambda_{\gamma'} \in \lambda$.

In our case the matix $X$ in Lemma 6.1.22 has the form

$$\begin{pmatrix} 1 & \dots & 1 & 1 & \dots & 1 \\ 0 & \dots & 0 & 1 & \dots & 1 \end{pmatrix} \in \mathbb{Z}^{2\times 2p}.$$

This is an unimodular matrix since every $2\times 2$ submatrix has determinant $-1, 0$ or 1. Hence, by assuming that

$$\lambda_1^* - \lambda_2^* \notin 2\pi i\mathbb{Z}\setminus\{0\}, \tag{6.28}$$

we ensure that $\varphi = C_{\lambda_0}, \lambda_0 = (\lambda_1^*, \lambda_2^*)$, has $L_2$-stable integer translates. We remark, that if $\lambda_1^*$ and $\lambda_2^*$ are real-valued, (6.28) is always satisfied. Next, we check if (6.27) preserves the $L_2$-stability. Therefore, we need the following result which is a special case of Theorem 3.3 in [49].

**Lemma 6.1.23.**
Let $\varphi \in L_2(\mathbb{R}^n)$ be compactly supported. Then the integer translates of $\varphi$ are $L_2$-stable if and only if

$$\sum_{\alpha\in\mathbb{Z}^n} |\hat{\varphi}(\xi + 2\pi\alpha)|^2 > 0 \quad \text{for all } \xi \in \mathbb{R}^n.$$

Hence, in order to deduce $L_2$-stability of the integer translates of $w_c$ we have to prove that

$$\sum_{\alpha\in\mathbb{Z}^2} |\hat{w}_c(\xi + 2\pi\alpha)|^2 > 0 \quad \text{for all } \xi \in \mathbb{R}^2.$$

By inserting the definiton of $\hat{w}_c$, we obtain

$$\begin{aligned}&\sum_{\alpha\in\mathbb{Z}^2} |\hat{w}_c(\xi + 2\pi\alpha)|^2 \\ &\quad= \sum_{\alpha\in\mathbb{Z}^2} |[\hat{\varphi},\hat{\varphi}](\xi + 2\pi\alpha)\widehat{\varphi_{d_1}}(\xi + 2\pi\alpha) - [\widehat{\varphi_{d_1}},\hat{\varphi}](\xi + 2\pi\alpha)\hat{\varphi}(\xi + 2\pi\alpha)|^2.\end{aligned}$$

Due to the $2\pi\mathbb{Z}^2$-periodicity of the bracket product and (1.3), we obtain

$$\begin{aligned}&\sum_{\alpha\in\mathbb{Z}^2} |\hat{w}_c(\xi + 2\pi\alpha)|^2 \\ &= \sum_{\alpha\in\mathbb{Z}^2} |[\hat{\varphi},\hat{\varphi}](\xi)e_{M^{-1}d_1}(\xi + 2\pi\alpha)\hat{\varphi}(\xi + 2\pi\alpha) - [\widehat{\varphi_{d_1}},\hat{\varphi}](\xi)\hat{\varphi}(\xi + 2\pi\alpha)|^2 \\ &= \sum_{\alpha\in\mathbb{Z}^2} |\hat{\varphi}(\xi + 2\pi\alpha)|^2 |[\hat{\varphi},\hat{\varphi}](\xi)e_{M^{-1}d_1}(\xi + 2\pi\alpha) - [\widehat{\varphi_{d_1}},\hat{\varphi}](\xi)|^2.\end{aligned}$$

Since

$$\begin{aligned}|[\hat\varphi,\hat\varphi](\xi)e_{M^{-1}d_1}(\xi+2\pi\alpha)-[\widehat{\varphi_{d_1}},\hat\varphi](\xi)|^2\\ =|\sum_{\beta\in\mathbb{Z}^2}|\hat\varphi(\xi+2\pi\beta)|^2e_{M^{-1}d_1}(\xi+2\pi\alpha)-|\hat\varphi(\xi+2\pi\beta)|^2e_{M^{-1}d_1}(\xi+2\pi\beta)|^2\end{aligned}$$

we further deduce

$$\begin{aligned}\sum_{\alpha\in\mathbb{Z}^2}|\hat w_c(\xi+2\pi\alpha)|^2=&\sum_{\alpha\in\mathbb{Z}^2}|\hat\varphi(\xi+2\pi\alpha)e_{M^{-1}d_1}(\xi)|^2\\ &\cdot|\sum_{\beta\in\mathbb{Z}^2}|\hat\varphi(\xi+2\pi\beta)|^2e_{M^{-1}d_1}(2\pi\alpha)-|\hat\varphi(\xi+2\pi\beta)|^2e_{M^{-1}d_1}(2\pi\beta)|^2.\end{aligned}$$

Every summand is non-negative. Hence, this sum is greater than zero for every $\xi\in\mathbb{R}^2$, if we can find at least one summand greater than zero for every $\xi\in\mathbb{R}^2$. Moreover, the integer translates of the generator $\varphi$ are $L_2$-stable and therefore, $\hat\varphi$ cannot have $2\pi\mathbb{Z}^2$-periodic zeros. It follows that it is sufficient to consider a summand where $\hat\varphi(\xi+2\pi\tilde\alpha)\neq 0,\tilde\alpha\in\mathbb{Z}^2$, holds. Let

$$|\hat\varphi(\xi+2\pi\tilde\alpha)e_{M^{-1}d_1}(\xi)|^2|\sum_{\beta\in\mathbb{Z}^2}|\hat\varphi(\xi+2\pi\beta)|^2e_{M^{-1}d_1}(2\pi\tilde\alpha)-|\hat\varphi(\xi+2\pi\beta)|^2e_{M^{-1}d_1}(2\pi\beta)|^2$$

be such a summand. Since $|\hat\varphi(\xi+2\pi\tilde\alpha)e_{M^{-1}d_1}(\xi)|^2$ is strictly positive, we have to verify that the second factor is strictly positive as well. Since $M^{-1}d_1=(1/2,1/2)^T$, it follows that

$$e_{M^{-1}d_1}(2\pi\tilde\alpha)=e^{i\pi(\tilde\alpha_1+\tilde\alpha_2)}=\begin{cases}1, & \tilde\alpha_1+\tilde\alpha_2\in 2\mathbb{Z},\\ -1, & \tilde\alpha_1+\tilde\alpha_2\in 2\mathbb{Z}+1,\end{cases}\tag{6.29}$$

for $\tilde\alpha=(\tilde\alpha_1,\tilde\alpha_2)^T\in\mathbb{Z}^2$.

**Case A:** If $\tilde\alpha_1+\tilde\alpha_2\in 2\mathbb{Z}$, the second factor has the form

$$|\sum_{\beta\in\mathbb{Z}^2}|\hat\varphi(\xi+2\pi\beta)|^2-|\hat\varphi(\xi+2\pi\beta)|^2e^{i\pi(\beta_1+\beta_2)}|^2\tag{6.30}$$

for $\beta=(\beta_1,\beta_2)^T$. Since $e^{i\pi(\beta_1+\beta_2)}=\pm1$ every summand is nonnegative. Again, we only have to determine one strictly positive summand. Therefore, we have a closer look at

$$\hat\varphi(\xi)=\left(\frac{e^{\lambda_1^*-i\xi_1}-1}{\lambda_1^*-i\xi_1}\right)^p\left(\frac{e^{\lambda_2^*-i(\xi_1+\xi_2)}-1}{\lambda_2^*-i(\xi_1+\xi_2)}\right)^p.\tag{6.31}$$

In case $\lambda_0$ is real-valued, the first factor vanishes if $\lambda_1^* = 0$ and $\xi_1 = 2\pi k, k \in \mathbb{Z}\setminus\{0\}$. For $\lambda_1^* = 0$ and $\xi_1 \to 0$, we have

$$\begin{aligned}\lim_{\xi_1\to 0}\left(\frac{e^{-i\xi_1}-1}{-i\xi_1}\right)^p &= \left(\lim_{\xi_1\to 0}\frac{\cos(\xi_1)-i\sin(\xi_1)-1}{-i\xi_1}\right)^p \\ &= \left(\lim_{\xi_1\to 0}\frac{\cos(\xi_1)-1}{-i\xi_1}+\frac{\sin(\xi_1)}{\xi_1}\right)^p \\ &= 1.\end{aligned}$$

The second factor in (6.31) vanishes if $\lambda_2^* = 0$ and $\xi_1 + \xi_2 = 2\pi\ell, \ell \in \mathbb{Z} \setminus \{0\}$. For $\lambda_2^* = 0$ and $\xi_1 + \xi_2 \to 0$, we have

$$\lim_{\xi_1+\xi_2\to 0}\left(\frac{e^{-i(\xi_1+\xi_2)}-1}{-i(\xi_1+\xi_2)}\right)^p = 1.$$

Hence, there are three different cases which lead to $\hat{\varphi}(\xi) = 0$:

(1) $\lambda_1^* = 0, \lambda_2^* = 0, \quad \xi_1 = 2\pi k, \quad \xi_2 = 2\pi(\ell - k), \quad \ell, k \in \mathbb{Z} \setminus \{0\}$,

(2) $\lambda_1^* \neq 0, \lambda_2^* = 0, \quad \xi_1 + \xi_2 = 2\pi\ell, \quad \ell \in \mathbb{Z} \setminus \{0\}$,

(3) $\lambda_1^* = 0, \lambda_2^* \neq 0, \quad \xi_1 = 2\pi k, \quad k \in \mathbb{Z} \setminus \{0\}$.

In (6.30) we consider $\hat{\varphi}(\xi + 2\pi\beta)$. Hence, we investigate for the cases above which choice of $\beta = (\beta_1, \beta_2)^T \in \mathbb{Z}^2$ yields $\hat{\varphi}(\xi + 2\pi\beta) \neq 0$. In the first case the choice $\beta_1 = -k, \beta_2 = -(\ell - k)$, in the second case the choice $\beta_1 + \beta_2 = -\ell$ and in the third case the choice $\beta_1 = -k, \beta_2 = k + m, m \in 2\mathbb{Z} + 1$, leads to $\hat{\varphi}(\xi + 2\pi\beta) \neq 0$. We observe that the sum of $\beta_1$ and $\beta_2$ in the first and second case is not always in $2\mathbb{Z} + 1$. Hence, there exist $\xi \in \mathbb{R}^2$ such that $e^{i\pi(\beta_1+\beta_2)} = 1$ in (6.30). Consequently, we obtain $L_2$-stability only in case (3) and in case $\lambda_1^*, \lambda_2^* \in \mathbb{R} \setminus \{0\}$.

In case $\lambda_0$ is complex-valued, we consider the following two cases:

(1) $\lambda_0 = (\lambda_1^*, \lambda_2^*) = (iy, iz), \quad y, z \in \mathbb{R} \setminus \{0\}$,

(2) $\lambda_0 = (\lambda_1^*, \lambda_2^*) = (a + iy, b + iz), \quad a, b, y, z \in \mathbb{R} \setminus \{0\}$.

First, let us have a closer look at the case (1). We observe that

$$\lim_{\xi_1\to y}\left(\frac{e^{iy-i\xi_1}-1}{iy-i\xi_1}\right)^p = \left(\lim_{\xi_1\to y} e^{iy-i\xi_1}\right)^p = 1$$

and that

$$\lim_{\xi_1+\xi_2\to z}\left(\frac{e^{iz-i(\xi_1+\xi_2)}-1}{iz-i(\xi_1+\xi_2)}\right)^p = 1.$$

Hence, the numerator of the first factor in (6.31) vanishes if $\xi_1 = y + 2\pi k, k \in \mathbb{Z}$, and the numerator of the second factor in (6.31) vanishes if $\xi_1 + \xi_2 = z + 2\pi\ell, \ell \in \mathbb{Z}$. Hence, $\hat{\varphi}(\xi + 2\pi\beta)$ with $\xi_1 = y + 2\pi k$ and $\xi_2 = z - y + 2\pi(\ell - k)$ is nonzero if and only if $\beta_1 = -k$ and $\beta_2 = -(\ell - k)$. Again, the sum of $\beta_1$ and $\beta_2$ is not always in $2\mathbb{Z} + 1$ and therefore, we do not obtain $L_2$-stability. This changes if we consider the second case where

$$\begin{aligned}&\hat{\varphi}(\xi + 2\pi\beta)\\ &= \left(\frac{e^{a+iy-i\xi_1} - 1}{a + iy - i(\xi_1 + 2\pi\beta_1)}\right)^p \left(\frac{e^{b+iz-i(\xi_1+\xi_2)} - 1}{b + iz - i(\xi_1 + 2\pi\beta_1 + \xi_2 + 2\pi\beta_2)}\right)^p. \end{aligned} \tag{6.32}$$

Since $a$ and $b$ are nonzero, the denominators cannot vanish. Next, we check if

$$e^{a+iy-i\xi_1} = e^a(\cos(y - \xi_1) + i\sin(y - \xi_1)) \tag{6.33}$$

can take the value 1. Suppose (6.33) takes the value 1. Then $\sin(y - \xi_1)$ has to vanish. This is the case if and only if $y - \xi_1 = \pi k, k \in \mathbb{Z}$. Consequently, (6.33) reduces to $e^a$ or $-e^a$. Since $a \in \mathbb{R} \setminus \{0\}$, we obtain a contradiction in both cases. By the same arguments, we can deduce that the numerator of the second factor in (6.32) cannot take the value 0. Hence, $\hat{\varphi}(\xi + 2\pi\beta)$ is nonzero for every $\xi \in \mathbb{R}^2$ and thus, $L_2$-stability is preserved.

**Case B:** If $\tilde{\alpha}_1 + \tilde{\alpha}_2 \in 2\mathbb{Z} + 1$ in (6.29), we obtain

$$|(-1)\sum_{\beta\in\mathbb{Z}^2} |\hat{\varphi}(\xi + 2\pi\beta)|^2 + |\hat{\varphi}(\xi + 2\pi\beta)|^2 e^{i\pi(\beta_1+\beta_2)}|^2.$$

Therefore, it is possible to obtain a strictly positive summand only if $\beta_1 + \beta_2 \in 2\mathbb{Z}$. Due to analogue arguments as above, we obtain $L_2$-stability in the same cases as before.

In summary, the integer translates of $w_c$ are $L_2$-stable if

- $\lambda_1^* - \lambda_2^* \notin 2\pi i\mathbb{Z} \setminus \{0\}$,

and if one of the following conditions is satisfied:

- $\lambda_1^*, \lambda_2^* \in \mathbb{R} \setminus \{0\}$,
- $\lambda_1^* = 0, \lambda_2^* \in \mathbb{R} \setminus \{0\}$,
- $\lambda_1^* = a + iy, \lambda_2^* = b + iz,\ a, b, y, z \in \mathbb{R} \setminus \{0\}$.

This list can be further extended by considering combinations of all our cases considered so far:

- $\lambda_1^* = a + iy, \lambda_2^* = b,\ a, b, y \in \mathbb{R} \setminus \{0\}$,

- $\lambda_1^* = a, \lambda_2^* = b + iz,\ a, b, z \in \mathbb{R} \setminus \{0\}$,
- $\lambda_1^* = iy, \lambda_2^* = b + iz,\ b, y, z \in \mathbb{R} \setminus \{0\}$,
- $\lambda_1^* = 0, \lambda_2^* = b + iz,\ b, z \in \mathbb{R} \setminus \{0\}$,
- $\lambda_1^* = iy, \lambda_2^* = b,\ b, y \in \mathbb{R} \setminus \{0\}$.

This whole construction procedure to obtain a wavelet basis with certain properties can be done for every $W_j, j \in \mathbb{Z}$. In order to obtain a wavelet basis for the whole space $L_2(\mathbb{R}^2)$, we have to consider the intersection of all spaces $S_j$, denoted by Y, as well because

$$L_2(\mathbb{R}^2) = Y \oplus \bigoplus_{j \in \mathbb{Z}} W_j.$$

In the stationary case $Y$ is trivial. However, in the non-stationary case it can be zero- or one-dimensional. The first case has the advantage that we obtain a wavelet basis for $L_2(\mathbb{R}^2)$, if we have constructed a wavelet basis for all spaces $W_j$. Therefore, in Theorem 6.1.25 we are going to deduce another condition for $Y$ to be trivial. For the proof of this theorem we will need the following lemma, see [30, Lemma 4.6]:

**Lemma 6.1.24.**
If $\Omega$ is a measurable subset of $\mathbb{R}^n$ and $\alpha \neq 0$ is a fixed real constant such that for each dyadic $t \in \mathbb{R}^n$, we have $\Omega + \alpha\, t = \Omega$ modulo a null-set, then $\Omega = \mathbb{R}^n$ or $\Omega = \emptyset$ modulo a null-set. Furthermore, if $f$ is a measurable function on $\mathbb{R}^n$ with $f(\cdot + \alpha\, t) = f$ almost everywhere for each dyadic $t$, then $f$ is constant almost everywhere.

Following the method of proof of [30, Theorem 8.4] we obtain a trivial intersection if $\operatorname{Re} \lambda_1^* = 0$ or $\operatorname{Re} \lambda_2^* = 0$.

**Theorem 6.1.25.**
Let $\{S_j\}_{j \in \mathbb{Z}}$ be defined as in (6.26) with

$$M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and let $(x^\gamma)_{\gamma \in \Gamma}$ consist of the vectors

$$x^{\gamma_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad x^{\gamma_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

each appearing $p$ times, $p \in \mathbb{N} \setminus \{0\}$. In addition, set

$$\lambda_j = \begin{cases} (2^{j/2}\lambda_1^*, 2^{j/2}\lambda_2^*), & j \in 2\mathbb{Z}, \\ (2^{(j-1)/2}\lambda_2^*, 2^{(j+1)/2}\lambda_1^*), & j \in 2\mathbb{Z}+1. \end{cases}$$

Then $Y = \bigcap_{j \in \mathbb{Z}} S_j$ is trivial if $\operatorname{Re} \lambda_1^* = 0$ or $\operatorname{Re} \lambda_2^* = 0$.

*Proof.*
Let $f \in Y, f \neq 0$. It is sufficient to prove

$$\bigcap_{j \in 2\mathbb{Z}} S_j = \{0\}.$$

Therefore, we assume throughout this proof that $j \in 2\mathbb{Z}$. Next, we define the function $G$ by

$$\widehat{G}(\xi) := \left(\frac{1}{\lambda_1^* - i\xi \cdot x^{\gamma_1}}\right)^p \left(\frac{1}{\lambda_2^* - i\xi \cdot x^{\gamma_2}}\right)^p, \quad \xi \in \mathbb{R}^2.$$

Since $f \in S_{-j}$, it can be written as a linear combination of the $M^j\mathbb{Z}^2$-translates of the generator $\varphi_{-j} = C_{\lambda_j}(M^{-j}\cdot)$. Next, we verify that the quotient $2^{j(p-1)}\hat{\varphi}_{-j}/\widehat{G}$ is $2\pi M^{-j}\mathbb{Z}^2$-periodic. For $k = (k_1, k_2)^T \in \mathbb{Z}^2$ the nominator $2^{j(p-1)}\hat{\varphi}_{-j}(\cdot + 2\pi M^{-j}k)$ has the form

$$\begin{aligned}
&2^{j(p-1)}\hat{\varphi}_{-j}(\xi + 2\pi M^{-j}k)\\
&= 2^{j(p-1)}|-2|^j\widehat{C_{\lambda_j}}(M^j(\xi + 2\pi M^{-j}k))\\
&= 2^{pj/2}2^{pj/2}\left(\frac{e^{2^{j/2}\lambda_1^* - i(2^{j/2}\xi_1 + 2\pi k_1)} - 1}{2^{j/2}\lambda_1^* - i(2^{j/2}\xi_1 + 2\pi k_1)}\right)^p\left(\frac{e^{2^{j/2}\lambda_2^* - i(2^{j/2}(\xi_1+\xi_2) + 2\pi(k_1+k_2))} - 1}{2^{j/2}\lambda_2^* - i(2^{j/2}(\xi_1 + \xi_2) + 2\pi(k_1 + k_2))}\right)^p
\end{aligned}$$

and the denominator $\widehat{G}(\cdot + 2\pi M^{-j}k)$ has the form

$$\begin{aligned}
&\widehat{G}(\xi + 2\pi M^{-j}k)\\
&\quad = \left(\lambda_1^* - i(\xi_1 + 2\pi 2^{-j/2}k_1)\right)^{-p}\left(\lambda_2^* - i(\xi_1 + \xi_2 + 2\pi 2^{-j/2}(k_1 + k_2))\right)^{-p}.
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
\frac{2^{j(p-1)}\hat{\varphi}_{-j}(\xi + 2\pi M^{-j}k)}{\widehat{G}(\xi + 2\pi M^{-j}k)} &= \left(e^{2^{j/2}\lambda_1^* - i2^{j/2}\xi_1} - 1\right)^p \left(e^{2^{j/2}\lambda_2^* - i(2^{j/2}(\xi_1+\xi_2))} - 1\right)^p\\
&= \frac{2^{j(p-1)}|-2|^j\widehat{C_{\lambda_j}}(M^j\xi)}{\widehat{G}(\xi)}\\
&= \frac{2^{j(p-1)}\hat{\varphi}_{-j}(\xi)}{\hat{G}(\xi)}.
\end{aligned}$$

We observe that $2^{j(p-1)}\hat{\varphi}_{-j}/\widehat{G}$ is a trigonometric polynomial. Since the spaces $S_{-j}$ are linear subspaces we also know that $g \in S_{-j}$ if and only if $\tilde{g} := 2^{j(p-1)}g \in S_{-j}$. Then Corollary 4.1.2 yields for every function $\tilde{g} \in S_{-j}, j \in 2\mathbb{Z}$, the representation

$$\hat{\tilde{g}} = 2^{j(p-1)}\hat{g} = 2^{j(p-1)}\tau_{-j}\hat{\varphi}_{-j} = 2^{j(p-1)}\tau_{-j}\frac{\hat{\varphi}_{-j}}{\widehat{G}}\widehat{G} = \tilde{\tau}_{-j}\widehat{G},$$

where $\tau_{-j}$ and $\tilde{\tau}_{-j} := (2^{j(p-1)}\tau_{-j}\hat{\varphi}_{-j})/\widehat{G}$ are $2\pi 2^{-j/2}\mathbb{Z}^2$-periodic. Due to the nestedness assumption, we can also write

$$\hat{\tilde{g}} = \tilde{\tau}_{-j}\widehat{G} = \tilde{\tau}_{-j+2}\widehat{G}.$$

This is equivalent to

$$(\tilde{\tau}_{-j+2} - \tilde{\tau}_{-j})\,\widehat{G} = 0.$$

Since $\widehat{G}(\xi) \neq 0$ for every $\xi \in \mathbb{R}^2$, it follows that all $\tilde{\tau}_{-j}$ agree almost everywhere with one measurable function $\tau$ and this function is invariant under all $2\pi 2^{-j/2}\mathbb{Z}^2$-shifts for $j \in 2\mathbb{Z}$. We observe that $2^{-j/2}\mathbb{Z}^2$ contains the dyadic points which are dense in $\mathbb{R}^2$. By Lemma 6.1.24 and the choice $\alpha = 2\pi$, the function $\tau$ is constant almost everywhere. Hence, the Fourier transform of every function in $Y$ can be represented by a scalar multiple of $\widehat{G}$. Therefore, $Y$ is trivial if and only if $G \notin Y$. Due to the $2\pi 2^{-j/2}\mathbb{Z}^2$-periodicity of $\tilde{\tau}_{-j}$, Corollary 4.1.2 yields that $G \in S_{-j}$ if and only if $G \in L_2(\mathbb{R}^2)$. Hence, in the following we are going to prove that if $\operatorname{Re}\lambda_1^* = 0$ or $\operatorname{Re}\lambda_2^* = 0$, $G$ cannot be a function in $L_2(\mathbb{R}^2)$. To this end we calculate the $L_2$-norm of $\widehat{G}$. We obtain

$$\begin{aligned}
&\|\widehat{G}\|^2_{L_2(\mathbb{R}^2)} \\
&\quad = \int\limits_{\mathbb{R}^2} \Big| \Big(\frac{1}{\lambda_1^* - i\xi\cdot x^{\gamma_1}}\Big)^p \Big(\frac{1}{\lambda_2^* - i\xi\cdot x^{\gamma_2}}\Big)^p \Big|^2 \,\mathrm{d}\xi \\
&\quad = \int\limits_{\mathbb{R}^2} \Big(\Big|\frac{1}{\lambda_1^* - i\xi\cdot x^{\gamma_1}}\Big|^2\Big)^p \Big(\Big|\frac{1}{\lambda_2^* - i\xi\cdot x^{\gamma_2}}\Big|^2\Big)^p \,\mathrm{d}\xi \\
&\quad = \int\limits_{\mathbb{R}^2} \Big(\frac{1}{(\operatorname{Re}(\lambda_1^*))^2 + (\operatorname{Im}(\lambda_1^*) - \xi\cdot x^{\gamma_1})^2}\Big)^p \Big(\frac{1}{(\operatorname{Re}(\lambda_2^*))^2 + (\operatorname{Im}(\lambda_2^*) - \xi\cdot x^{\gamma_2})^2}\Big)^p \mathrm{d}\xi.
\end{aligned}$$

In the following we denote the matrix with the columns $x^{\gamma^1}$ and $x^{\gamma^2}$ by

$$Y_\Gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and we denote the unit vectors by $e_{(1)} := (1,0)^T, e_{(2)} := (0,1)^T$. Then we further deduce

$$\begin{aligned}
\Big(\frac{1}{(\operatorname{Re}(\lambda_1^*))^2 + (\operatorname{Im}(\lambda_1^*) - \xi\cdot x^{\gamma_1})^2}\Big)^p &= \Big(\frac{1}{(\operatorname{Re}(\lambda_1^*))^2 + (\operatorname{Im}(\lambda_1^*) - \xi\cdot (Y_\Gamma e_{(1)}))^2}\Big)^p \\
&= \Big(\frac{1}{(\operatorname{Re}(\lambda_1^*))^2 + (\operatorname{Im}(\lambda_1^*) - (Y_\Gamma^T\xi)\cdot e_{(1)})^2}\Big)^p
\end{aligned}$$

and

$$\left(\frac{1}{(\mathrm{Re}(\lambda_2^*))^2 + (\mathrm{Im}(\lambda_2^*) - \xi \cdot x^{\gamma_2})^2}\right)^p = \left(\frac{1}{(\mathrm{Re}(\lambda_2^*))^2 + (\mathrm{Im}(\lambda_2^*) - \xi \cdot (Y_\Gamma e_{(2)}))^2}\right)^p$$
$$= \left(\frac{1}{(\mathrm{Re}(\lambda_2^*))^2 + (\mathrm{Im}(\lambda_2^*) - (Y_\Gamma^T \xi) \cdot e_{(2)})^2}\right)^p.$$

Since $\det(Y_\Gamma^T)^{-1} = 1$, the transformation formula yields

$$\|\widehat{G}\|^2_{L_2(\mathbb{R}^2)}$$
$$= \int_{\mathbb{R}^2} \left(\frac{1}{(\mathrm{Re}(\lambda_1^*))^2 + (\mathrm{Im}(\lambda_1^*) - \xi \cdot e_{(1)})^2}\right)^p \left(\frac{1}{(\mathrm{Re}(\lambda_2^*))^2 + (\mathrm{Im}(\lambda_2^*) - \xi \cdot e_{(2)})^2}\right)^p \mathrm{d}\xi$$
$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\frac{1}{(\mathrm{Re}(\lambda_1^*))^2 + (\mathrm{Im}(\lambda_1^*) - \xi_1)^2}\right)^p \left(\frac{1}{(\mathrm{Re}(\lambda_2^*))^2 + (\mathrm{Im}(\lambda_2^*) - \xi_2)^2}\right)^p \mathrm{d}\xi_1\,\mathrm{d}\xi_2$$
$$= \int_{\mathbb{R}} \left(\frac{1}{(\mathrm{Re}(\lambda_2^*))^2 + (\mathrm{Im}(\lambda_2^*) - \xi_2)^2}\right)^p \int_{\mathbb{R}} \left(\frac{1}{(\mathrm{Re}(\lambda_1^*))^2 + (\mathrm{Im}(\lambda_1^*) - \xi_1)^2}\right)^p \mathrm{d}\xi_1\,\mathrm{d}\xi_2.$$

If $\mathrm{Re}(\lambda_1^*) = 0$, we obtain

$$\int_{\mathbb{R}} \left(\frac{1}{(\mathrm{Re}(\lambda_2^*))^2 + (\mathrm{Im}(\lambda_2^*) - \xi_2)^2}\right)^p \int_{\mathbb{R}} \left(\frac{1}{(\mathrm{Re}(\lambda_1^*))^2 + (\mathrm{Im}(\lambda_1^*) - \xi_1)^2}\right)^p \mathrm{d}\xi_1\,\mathrm{d}\xi_2$$
$$= \int_{\mathbb{R}} \left(\frac{1}{(\mathrm{Re}(\lambda_2^*))^2 + (\mathrm{Im}(\lambda_2^*) - \xi_2)^2}\right)^p \int_{\mathbb{R}} \frac{1}{(\mathrm{Im}(\lambda_1^*) - \xi_1)^{2p}} \mathrm{d}\xi_1 \mathrm{d}\xi_2.$$

To determine the integral

$$\int_{\mathbb{R}} \frac{1}{(\mathrm{Im}(\lambda_1^*) - \xi_1)^{2p}} \mathrm{d}\xi_1$$

we calculate

$$\lim_{b \to -\infty} \lim_{a \to \mathrm{Im}(\lambda_1^*)^+} \int_b^a \frac{1}{(\mathrm{Im}(\lambda_1^*) - \xi_1)^{2p}} \mathrm{d}\xi_1 + \lim_{b \to \infty} \lim_{a \to \mathrm{Im}(\lambda_1^*)^-} \int_a^b \frac{1}{(\mathrm{Im}(\lambda_1^*) - \xi_1)^{2p}} \mathrm{d}\xi_1.$$

Since $p \geq 1$, we see that

$$\lim_{a \to \mathrm{Im}(\lambda_1^*)^+} \frac{(\mathrm{Im}(\lambda_1^*) - a)^{1-2p}}{2p-1} \to -\infty, \qquad \lim_{b \to -\infty} \frac{(\mathrm{Im}(\lambda_1^*) - b)^{1-2p}}{2p-1} \to 0,$$
$$\lim_{a \to \mathrm{Im}(\lambda_1^*)^-} \frac{(\mathrm{Im}(\lambda_1^*) - a)^{1-2p}}{2p-1} \to \infty, \qquad \lim_{b \to \infty} \frac{(\mathrm{Im}(\lambda_1^*) - b)^{1-2p}}{2p-1} \to 0.$$

Hence, the second integral does not converge and therefore, $\widehat{G} \notin L_2(\mathbb{R}^2)$. The case $\mathrm{Re}(\lambda_2^*) = 0$ can be treated analogue. □

Now, let us summarize the most important results concerning our first example. We have constructed a compactly supported wavelet basis for $W_0$ in the non-stationary setting. As stated in the previous section, our analysis can be applied to all spaces $W_j, j \in \mathbb{Z}$, after a suitable dilation and hence, we obtain a compactly supported basis for all spaces $W_j$. Under the assumption

(I) The real part of at least one entry of $\lambda_0$ is zero.

the intersection of the spaces $S_j$ is trivial. Therefore, the union of our compactly supported wavelet bases is a compactly supported wavelet basis for the whole space $L_2(\mathbb{R}^2)$.

**Remark 6.1.26.**
But one question still remains open: How can we construct a compactly supported and $L_2$-stable wavelet basis of $L_2(\mathbb{R}^2)$? If we assume

(II) The entries of $\lambda_0$ satisfy one of the following conditions:

- $\lambda_1^*, \lambda_2^* \in \mathbb{R} \setminus \{0\}$,
- $\lambda_1^* = 0, \lambda_2^* \in \mathbb{R} \setminus \{0\}$,
- $\lambda_1^* = a + iy, \lambda_2^* = b + iz,\ a, b, y, z \in \mathbb{R} \setminus \{0\}$,
- $\lambda_1^* = a + iy, \lambda_2^* = b,\ a, b, y \in \mathbb{R} \setminus \{0\}$,
- $\lambda_1^* = a, \lambda_2^* = b + iz,\ a, b, z \in \mathbb{R} \setminus \{0\}$,
- $\lambda_1^* = iy, \lambda_2^* = b + iz,\ b, y, z \in \mathbb{R} \setminus \{0\}$,
- $\lambda_1^* = 0, \lambda_2^* = b + iz,\ b, z \in \mathbb{R} \setminus \{0\}$,
- $\lambda_1^* = iy, \lambda_2^* = b,\ b, y \in \mathbb{R} \setminus \{0\}$.

and

(III) The difference of $\lambda_1^*$ and $\lambda_2^*$ satisfies $\lambda_1^* - \lambda_2^* \notin 2\pi i\mathbb{Z} \setminus \{0\}$.

we obtain compactly supported and $L_2$-stable wavelet bases for $W_j, j \in \mathbb{Z}$. The space $L_2(\mathbb{R}^2)$ has the orthogonal decomposition

$$L_2(\mathbb{R}^2) = Y \oplus \bigoplus_{j \in \mathbb{Z}} W_j.$$

Even if choose the entries of $\lambda_0$ such that $Y$ is trivial, this decomposition does not imply that the union of $L_2$-stable wavelet bases is $L_2$-stable as well. For $L_2$-stability we need to ensure that the stability constants can be chosen independently of $j$. In the stationary case this is always possible since dilation doesn't change the stability constants. In the non-stationary case the answer to this question is far from being trivial. Therefore, this problem will be the subject of future research.

As a second example we consider the matrix

$$M = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}. \tag{6.34}$$

The absolute value of the determinant is 2 and $M^2 = 2I$. Moreover, the inverse of $M$ is given by

$$M^{-1} = \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}.$$

With the matrix (6.34) at hand, we want to prove an analogon of Lemma 6.1.21. Let $(x_\gamma)_{\gamma\in\Gamma}$ consist of the vectors

$$x^{\gamma_1} := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } x^{\gamma_2} := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{6.35}$$

each appearing $p$ times, $p \in \mathbb{N} \setminus \{0\}$. According to part iv) of Lemma 6.1.20, the corresponding exponential box spline can be classified as a tensor spline. Besides that, $\operatorname{span}(x_\gamma)_{\gamma\in\Gamma} = \mathbb{R}^2$ and the matrix

$$\begin{pmatrix} 0 & \dots & 0 & 1 & \dots & 1 \\ 1 & \dots & 1 & 0 & \dots & 0 \end{pmatrix} \in \mathbb{Z}^{2\times 2p}$$

is unimodular. Furthermore, there is a slight change in the Fourier transform of our generators $\varphi_j$ because $M$ is not symmetric anymore. We have

$$\hat{\varphi}_j = \frac{1}{m^j} C_{\lambda_{-j}}((M^T)^{-j}\cdot).$$

The matrix $M^T$ still satisfies $(M^T)^2 = 2I$.

**Lemma 6.1.27.**
Let $M$ be defined as in (6.34) and let $x^{\gamma_1}$ and $x^{\gamma_2}$ be defined as in (6.35). Moreover, let the directions be given by $p$-times the vector $x^{\gamma_1}$ and $p$-times the vector $x^{\gamma_2}$ with $p \in \mathbb{N} \setminus \{0\}$. Furthermore, for $j \in \mathbb{Z}$ we define

$$\lambda_j := (\lambda_{j,\gamma_1}, \lambda_{j,\gamma_2}), \quad \lambda_{j-1} = (\lambda_{j-1,\gamma_1}, \lambda_{j-1,\gamma_2}) := (\lambda_{j,\gamma_2}/2, \lambda_{j,\gamma_1}),$$

where $\lambda_{j,\gamma_i} \in \mathbb{C}$ for $i = 1, 2$. Then the exponential box spline $C_{\lambda_j}$ given by

$$\widehat{C}_{\lambda_j}(\xi) = \left(\frac{e^{\lambda_{j,\gamma_1} - i\xi\cdot x^{\gamma_1}} - 1}{\lambda_{j,\gamma_1} - i\xi \cdot x^{\gamma_1}}\right)^p \left(\frac{e^{\lambda_{j,\gamma_2} - i\xi\cdot x^{\gamma_2}} - 1}{\lambda_{j,\gamma_2} - i\xi \cdot x^{\gamma_2}}\right)^p$$

satisfies

$$\widehat{C}_{\lambda_j}((M^T)^j\xi) = A_{\lambda_j,\lambda_{j-1}}((M^T)^{j-1}\xi)\widehat{C}_{\lambda_{j-1}}((M^T)^{j-1}\xi)$$

with

$$A_{\lambda_j,\lambda_{j-1}}(\xi) := \left(\frac{1}{2}\frac{e^{\lambda_{j,\gamma_2}-iM^T\xi\cdot x^{\gamma_2}}-1}{e^{\lambda_{j-1,\gamma_1}-i\xi\cdot x^{\gamma_1}}-1}\right)^p.$$

Furthermore, $A_{\lambda_j,\lambda_{j-1}}$ is a $2\pi$-periodic trigonometric polynomial.

*Proof.*
Let $\xi = (\xi_1,\xi_2) \in \mathbb{R}^2$ and let $j \in \mathbb{Z}$ and $\ell \in \mathbb{N}$. Furthermore we set $\lambda_{\gamma_i} := \lambda_{j,\gamma_i}$ and $\tilde\lambda_{\gamma_i} := \lambda_{j-1,\gamma_i}$ for $i = 1,2$. First, we insert the definition of $\widehat{C}_{\lambda_j}$ into $\widehat{C}_{\lambda_j}((M^T)^j\xi)$. We obtain

$$\widehat{C}_{\lambda_j}((M^T)^j\xi) = \begin{cases} \left(\frac{e^{\lambda_{\gamma_1}-i\xi_2}-1}{\lambda_{\gamma_1}-i\xi_2}\right)^p\left(\frac{e^{\lambda_{\gamma_2}-i\xi_1}-1}{\lambda_{\gamma_2}-i\xi_1}\right)^p, & j=0,\\ \left(\frac{e^{\lambda_{\gamma_1}-i2^\ell\xi_2}-1}{\lambda_{\gamma_1}-i2^\ell\xi_2}\right)^p\left(\frac{e^{\lambda_{\gamma_2}-i2^\ell\xi_1}-1}{\lambda_{\gamma_2}-i2^\ell\xi_1}\right)^p, & j=2\ell>0,\\ \left(\frac{e^{\lambda_{\gamma_1}-i2^{-\ell}\xi_2}-1}{\lambda_{\gamma_1}-i2^{-\ell}\xi_2}\right)^p\left(\frac{e^{\lambda_{\gamma_2}-i2^{-\ell}\xi_1}-1}{\lambda_{\gamma_2}-i2^{-\ell}\xi_1}\right)^p, & j=-2\ell<0,\\ \left(\frac{e^{\lambda_{\gamma_1}-i2^\ell\xi_1}-1}{\lambda_{\gamma_1}-i2^\ell\xi_1}\right)^p\left(\frac{e^{\lambda_{\gamma_2}-i2^{\ell+1}\xi_2}-1}{\lambda_{\gamma_2}-i2^{\ell+1}\xi_2}\right)^p, & j=2\ell+1,\\ \left(\frac{e^{\lambda_{\gamma_1}-i2^{-\ell-1}\xi_1}-1}{\lambda_{\gamma_1}-i2^{-\ell-1}\xi_1}\right)^p\left(\frac{e^{\lambda_{\gamma_2}-i2^{-\ell}\xi_2}-1}{\lambda_{\gamma_2}-i2^{-\ell}\xi_2}\right)^p, & j=-2\ell-1. \end{cases}$$

For $\widehat{C}_{\lambda_{j-1}}((M^T)^{j-1}\xi)$ we have

$$\widehat{C}_{\lambda_{j-1}}((M^T)^{j-1}\xi) = \begin{cases} \left(\frac{e^{\tilde\lambda_{\gamma_1}-i/2\xi_1}-1}{\tilde\lambda_{\gamma_1}-i/2\xi_1}\right)^p\left(\frac{e^{\tilde\lambda_{\gamma_2}-i\xi_2}-1}{\tilde\lambda_{\gamma_2}-i\xi_2}\right)^p, & j=0,\\ \left(\frac{e^{\tilde\lambda_{\gamma_1}-i2^{\ell-1}\xi_1}-1}{\tilde\lambda_{\gamma_1}-i2^{\ell-1}\xi_1}\right)^p\left(\frac{e^{\tilde\lambda_{\gamma_2}-i2^\ell\xi_2}-1}{\tilde\lambda_{\gamma_2}-i2^\ell\xi_2}\right)^p, & j=2\ell>0,\\ \left(\frac{e^{\tilde\lambda_{\gamma_1}-i2^{-\ell-1}\xi_1}-1}{\tilde\lambda_{\gamma_1}-i2^{-\ell-1}\xi_1}\right)^p\left(\frac{e^{\tilde\lambda_{\gamma_2}-i2^{-\ell}\xi_2}-1}{\tilde\lambda_{\gamma_2}-i2^{-\ell}\xi_2}\right)^p, & j=-2\ell<0,\\ \left(\frac{e^{\tilde\lambda_{\gamma_1}-i2^\ell\xi_2}-1}{\tilde\lambda_{\gamma_1}-i2^\ell\xi_2}\right)^p\left(\frac{e^{\tilde\lambda_{\gamma_2}-i2^\ell\xi_1}-1}{\tilde\lambda_{\gamma_2}-i2^\ell\xi_1}\right)^p, & j=2\ell+1\\ \left(\frac{e^{\tilde\lambda_{\gamma_1}-i2^{-\ell-1}\xi_2}-1}{\tilde\lambda_{\gamma_1}-i2^{-\ell-1}\xi_2}\right)^p\left(\frac{e^{\tilde\lambda_{\gamma_2}-i2^{-\ell-1}\xi_1}-1}{\tilde\lambda_{\gamma_2}-i2^{-\ell-1}\xi_1}\right)^p, & j=-2\ell-1. \end{cases}$$

Since $\tilde\lambda_{\gamma_1} = \lambda_{\gamma_2}/2$ and $\tilde\lambda_{\gamma_2} = \lambda_{\gamma_1}$, the quotient of $\widehat{C}_{\lambda_j}(M^j\xi)$ and $\widehat{C}_{\lambda_{j-1}}(M^{j-1}\xi)$ has the form

$$\frac{\widehat{C}_{\lambda_j}((M^T)^j\xi)}{\widehat{C}_{\lambda_{j-1}}((M^T)^{j-1}\xi)} = \begin{cases} \left(\frac{1}{2}\frac{e^{\lambda_{\gamma_2}-i\xi_1}-1}{e^{\tilde\lambda_{\gamma_1}-i/2\xi_1}-1}\right)^p, & j=0,\\ \left(\frac{1}{2}\frac{e^{\lambda_{\gamma_2}-i2^\ell\xi_1}-1}{e^{\tilde\lambda_{\gamma_1}-i2^{\ell-1}\xi_1}-1}\right)^p, & j=2\ell>0,\\ \left(\frac{1}{2}\frac{e^{\lambda_{\gamma_2}-i2^{-\ell}\xi_1}-1}{e^{\tilde\lambda_{\gamma_1}-i2^{-\ell-1}\xi_1}-1}\right)^p, & j=-2\ell<0,\\ \left(\frac{1}{2}\frac{e^{\lambda_{\gamma_2}-i2^{\ell+1}\xi_2}-1}{e^{\tilde\lambda_{\gamma_1}-i2^\ell\xi_2}-1}\right)^p, & j=2\ell+1,\\ \left(\frac{1}{2}\frac{e^{\lambda_{\gamma_2}-i2^{-\ell}\xi_2}-1}{e^{\tilde\lambda_{\gamma_1}-i2^{-\ell-1}\xi_2}-1}\right)^p, & j=-2\ell-1. \end{cases}$$

Then $A$ is given by

$$\begin{aligned} A_{\lambda_j,\lambda_{j-1}}(\xi) &= \left(\frac{1}{2}\frac{e^{\lambda_{\gamma_2}-i(M^T)\xi\cdot x^{\gamma_2}}-1}{e^{\tilde{\lambda}_{\gamma_1}-i\xi\cdot x^{\gamma_1}}-1}\right)^p \\ &= \left(\frac{1}{2}\frac{e^{\lambda_{\gamma_2}-i2\xi_2}-1}{e^{\tilde{\lambda}_{\gamma_1}-i\xi_2}-1}\right)^p \\ &= \frac{1}{2^p}\left(\frac{e^{2(\lambda_{\gamma_2}/2-i\xi_2)}-1}{e^{\lambda_{\gamma_2}/2-i\xi_2}-1}\right)^p\left(\frac{e^{\lambda_{\gamma_2}/2-i\xi_2}+1}{e^{\lambda_{\gamma_2}/2-i\xi_2}+1}\right)^p \\ &= \frac{1}{2^p}(e^{\lambda_{\gamma_2}/2-i\xi_2}+1)^p. \end{aligned}$$

Since

$$A_{\lambda_j,\lambda_{j-1}}(\xi+2\pi\mathbb{1}) = \frac{1}{2^p}(e^{\lambda_{\gamma_2}/2-i(\xi_2+2\pi)}+1)^p = \frac{1}{2^p}(e^{\lambda_{\gamma_2}/2-i\xi_2}+1)^p = A_{\lambda_j,\lambda_{j-1}}(\xi),$$

the function $A_{\lambda_j,\lambda_{j-1}}$ is a $2\pi$-periodic trigonometric polynomial. □

Hence, the theory developed for the first example remains valid. In the following, we point out the differences which occur when working with this new matrix. First, the vector $d_1 = (1,0)^T$ in $R$ is substituted by $\tilde{d}_1 := (0,1)^T$. As a result, (6.29) has the form

$$e_{M^{-1}\tilde{d}_1}(2\pi\tilde{\alpha}) = e^{i\pi\tilde{\alpha}_1} = \begin{cases} 1, & \tilde{\alpha}_1 \in 2\mathbb{Z}, \\ -1, & \tilde{\alpha}_1 \in 2\mathbb{Z}+1. \end{cases}$$

Therefore, if $\tilde{\alpha}_1 \in 2\mathbb{Z}$ we consider

$$|\sum_{\beta\in\mathbb{Z}^2} |\hat{\varphi}(\xi+2\pi\beta)|^2 - |\hat{\varphi}(\xi+2\pi\beta)|^2 e^{i\pi\beta_1}|^2$$

and if $\tilde{\alpha}_1 \in 2\mathbb{Z}+1$ we consider

$$|(-1)\sum_{\beta\in\mathbb{Z}^2} |\hat{\varphi}(\xi+2\pi\beta)|^2 + |\hat{\varphi}(\xi+2\pi\beta)|^2 e^{i\pi\beta_1}|^2.$$

Since the Fourier transform of $\hat{\varphi}$ is given by

$$\hat{\varphi}(\xi) = \left(\frac{e^{\lambda_1^*-i\xi_2}-1}{\lambda_1^*-i\xi_2}\right)^p\left(\frac{e^{\lambda_2^*-i\xi_1}-1}{\lambda_2^*-i\xi_1}\right)^p,$$

we can use similar arguments as in the first example to deduce $L_2$-stability under the assumptions

(II') The entries of $\lambda_0$ satisfy one of the following conditions:

- $\lambda_1^*, \lambda_2^* \in \mathbb{R} \setminus \{0\}$,
- $\lambda_1^* \in \mathbb{R} \setminus \{0\}, \lambda_2^* = 0$,
- $\lambda_1^* = 0, \lambda_2^* \in \mathbb{R} \setminus \{0\}$,
- $\lambda_1^* = a + iy, \lambda_2^* = b + iz,\ a, b, y, z \in \mathbb{R} \setminus \{0\}$,
- $\lambda_1^* = a + iy, \lambda_2^* = b,\ a, b, y \in \mathbb{R} \setminus \{0\}$,
- $\lambda_1^* = a + iy, \lambda_2^* = iz, a, y, z \in \mathbb{R} \setminus \{0\}$,
- $\lambda_1^* = a + iy, \lambda_2^* = 0, a, y \in \mathbb{R} \setminus \{0\}$,
- $\lambda_1^* = a, \lambda_2^* = iz, a, z \in \mathbb{R} \setminus \{0\}$,
- $\lambda_1^* = a, \lambda_2^* = b + iz,\ a, b, z \in \mathbb{R} \setminus \{0\}$,
- $\lambda_1^* = iy, \lambda_2^* = b + iz,\ b, y, z \in \mathbb{R} \setminus \{0\}$,
- $\lambda_1^* = 0, \lambda_2^* = b + iz,\ b, z \in \mathbb{R} \setminus \{0\}$,
- $\lambda_1^* = iy, \lambda_2^* = b,\ b, y \in \mathbb{R} \setminus \{0\}$.

and (III). Hence, our construction procedure yields a compactly supported and $L_2$-stable wavelet basis for $W_j, j \in \mathbb{Z}$. Moreover, the quotient $2^{j(p-1)}\hat{\varphi}_{-j}/\widehat{G}$ in Theorem 6.1.25 changes to

$$\frac{2^{j(p-1)}|-2|^j\widehat{C_{\lambda_j}}((M^T)^j(\xi + 2\pi(M^T)^{-j}k))}{\widehat{G}(\xi + 2\pi(M^T)^{-j}k)}$$

because $M$ is not symmetric anymore. Since $(M^T)^j = 2^{j/2}I = M^j$ for $j \in 2\mathbb{Z}$, the proof remains valid and Theorem 6.1.25 holds for our new matrix.

## 6.2 Stable Wavelet Bases

In this section we present a second construction procedure which yields an $L_2$-stable basis for every space $W_j, j \in \mathbb{Z}$. As before we consider two $L_2(\mathbb{R}^n)$-functions $\varphi$ and $\eta$ such that

$$S_0(\varphi) \subset S_1(\eta).$$

In contrast to Section 6.1, this construction process will not be based on an orthogonal projection. Indeed, a main component will be the bracket product

$$B := [\hat{\eta}, \hat{\varphi}]_1 = \overline{A}[\hat{\eta}, \hat{\eta}]_1, \tag{6.36}$$

where $\hat{\varphi} = A\hat{\eta}$ and $A$ is $2\pi M^T\mathbb{Z}^n$-periodic.

**Lemma 6.2.1.**
Let $S_0(\varphi) \subset S_1(\eta)$ and $\operatorname{supp}\hat{\varphi} = \operatorname{supp}\hat{\eta} = \mathbb{R}^n$. Moreover, let the operator $Q_0$ be defined as in (6.3). A necessary and sufficient condition for a function $w \in L_2(\mathbb{R}^n)$ to be an element of the space $W_0$ is that there exists a $2\pi M^T\mathbb{Z}^n$-periodic function $\tau$ such that

$$\hat{w} = \tau\hat{\eta} \tag{6.37}$$

and

$$Q_0(\tau B) = \sum_{\tilde{d}\in R^T} (\tau B)(\cdot + 2\pi\tilde{d}) = 0. \tag{6.38}$$

If $\eta$ is compactly supported, then a sufficient condition for the the function $w$ to have compact support is that $\tau$ is a $2\pi M^T\mathbb{Z}^n$-periodic trigonometric polynomial. Whenever the function $\eta$ has linearly independent $M^{-1}\mathbb{Z}^n$-shifts, the last property characterizes all compactly supported functions of the space $W_0$.

*Proof.*
First of all, we prove (6.38). Since $\tau B$ is $2\pi M^T\mathbb{Z}^n$-periodic, we can insert it into the definition (6.3) of the operator $Q_0$ and obtain

$$Q_0(\tau B) = \sum_{\tilde{d}\in R^T} e_0(\cdot + 2\pi\tilde{d})(\tau B)(\cdot + 2\pi\tilde{d}) = \sum_{\tilde{d}\in R^T} (\tau B)(\cdot + 2\pi\tilde{d}).$$

Moreover, $W_0 \subset S_1(\eta)$ and by Corollary 4.1.2, every Fourier transform of a function in $W_0$ has a representation of the form (6.37). Besides that, we can use (6.1) to deduce for two arbitrary functions $f, g \in L_2(\mathbb{R}^n)$

$$\begin{aligned} Q_0([f,g]_1) &= \sum_{\tilde{d}\in R^T} \left( \sum_{\gamma^*\in 2\pi M^T\mathbb{Z}^n} f(\cdot + 2\pi\tilde{d} + \gamma^*)\overline{g(\cdot + 2\pi\tilde{d} + \gamma^*)} \right) \\ &= \sum_{\beta\in 2\pi\mathbb{Z}^n} f(\cdot + \beta)\overline{g(\cdot + \beta)} \\ &= [f,g]. \end{aligned}$$

Together with (6.37) and (6.36) this yields

$$[\hat{w}, \hat{\varphi}] = Q_0([\tau\hat{\eta}, \hat{\varphi}]_1) = Q_0(\tau[\hat{\eta}, \hat{\varphi}]_1) = Q_0(\tau B).$$

By Lemma 5.5, the function $w \in S_1(\eta)$ belongs to $W_0$ if and only if the bracket product $[\hat{w}, \hat{\varphi}]$ vanishes almost everywhere on $\widetilde{C}_0$. Thus, (6.38) holds.

In case $\tau$ is a trigonometric polynomial and $\eta$ is a compactly supported function, then $\hat{w} = \tau\hat{\eta}$ is the Fourier transform of a compactly supported function $w$. Furthermore, if $f \in S_1(\eta)$ is compactly supported and in addition, $\eta$ has linearly independent $M^{-1}\mathbb{Z}^n$-shifts, Corollary 4.2.11 implies that $\hat{f} = \tau\hat{\eta}$ holds with $\tau$ being a trigonometric polynomial. □

In view of Lemma 6.2.1, we want to find $m-1$ functions which provide an $L_2$-stable basis for $W_0$. Let $R_0$ be a subset of $R$ with cardinality $m-1$. Then it is our aim to define functions $\tau_d, d \in R_0$ such that (6.37), (6.38) are satisfied and we obtain an $L_2$-stable basis.

In preparation for this process, we recall Lemma 6.1.6 and apply it to the function $B = [\hat{\eta}, \hat{\varphi}]_1$. We obtain

$$B = \sum_{d \in R} e_{-M^{-1}d} B_d, \quad B_d := \frac{Q_d(B)}{m}. \tag{6.39}$$

In case the functions $\varphi$ and $\eta$ are compactly supported, we have already seen in (6.15) that $B = [\hat{\eta}, \hat{\varphi}]_1$ is a $2\pi M^T \mathbb{Z}^n$-periodic trigonometric polynomial. Consequently, by the definition of $Q_d, d \in R$, the functions $B_d$ are $2\pi\mathbb{Z}^n$-periodic trigonometric polynomials.

**Theorem 6.2.2.**
Let the functions $\varphi$ and $\eta$ fulfill $S_0(\varphi) \subset S_1(\eta)$ and $\operatorname{supp} \hat{\varphi} = \operatorname{supp} \hat{\eta} = \mathbb{R}^n$. Furthermore, let $B_d$ be defined as in (6.39) and let $d_0 \in R, R_0 := R \backslash \{d_0\}$. Then the $m-1$ functions

$$\tau_d := e_{M^{-1}d_0} B_d - e_{M^{-1}d} B_{d_0}, \quad d \in R_0,$$

satisfy

$$Q_0(\tau_d B) = 0.$$

Moreover, if $\hat{w}_d \in L_2(\mathbb{R}^n)$, then the functions $w_d, d \in R_0$, with the Fourier transforms $\hat{w}_d := \tau_d \hat{\eta}$, are elements of the space $W_0$. If $\varphi$ and $\eta$ are compactly supported, then the functions $w_d, d \in R_0$, are compactly supported. If the function $\eta$ has $L_2$-stable $M^{-1}\mathbb{Z}^n$-shifts and if $B$ and $1/B_{d_0}$ are in $L_\infty(\widetilde{C}_0)$, then $(w_d)_{d \in R_0}$ provides an $L_2$-stable basis for $W_0$.

*Proof.*
Let $d, d^* \in R$. By the $2\pi\mathbb{Z}^n$-periodicity of $Q_0$ and $B_d$, (6.3) and (6.39), we obtain

$$Q_0(e_{M^{-1}d^*} B_d B) = B_d Q_0(e_{M^{-1}d^*} B) = B_d Q_{d^*}(B) = B_d B_{d^*} m. \tag{6.40}$$

In view of (6.40), we consider two different cases. For $d = d, d^* = d_0$, we have

$$Q_0(e_{M^{-1}d_0} B_d B) = B_d B_{d_0} m$$

and for $d = d_0, d^* = d$, we obtain

$$Q_0(e_{M^{-1}d} B_{d_0} B) = B_{d_0} B_d m.$$

It follows that

$$\begin{aligned}
0 &= B_d B_{d_0} - B_{d_0} B_d \\
&= m^{-1}(Q_0(e_{M^{-1}d_0} B_d B) - Q_0(e_{M^{-1}d} B_{d_0} B)) \\
&= m^{-1}Q_0(e_{M^{-1}d_0} B_d B - e_{M^{-1}d} B_{d_0} B) \\
&= m^{-1}Q_0((e_{M^{-1}d_0} B_d - e_{M^{-1}d} B_{d_0}) B) \\
&= m^{-1}Q_0(\tau_d B).
\end{aligned}$$

Therefore, (6.38) is satisfied and $w_d \in W_0$ for all $d \in R_0$ if $\hat{w}_d \in L_2(\mathbb{R}^n)$ holds.

We already mentioned that if the functions $\eta$ and $\varphi$ are compactly supported, then $B = [\hat{\eta}, \hat{\varphi}]_1$ is a trigonometric polynomial. Hence, $B_d$ and every function $\tau_d$, $d \in R_0$, are trigonometric polynomials. This implies that the $m-1$ functions $w_d$ are compactly supported.

It remains to show the $L_2$-stability of $(w_d)_{d \in R_0}$. Let us consider the matrix

$$\Gamma = (\tau_{d,d^*})_{d,d^* \in R} := \begin{pmatrix} -B_{d_0} & 0 & \dots & \dots & \dots & 0 \\ B_{d_1} & -B_{d_0} & 0 & \dots & \dots & 0 \\ B_{d_2} & 0 & -B_{d_0} & 0 & \dots & 0 \\ \vdots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ B_{d_{m-1}} & 0 & 0 & \dots & 0 & -B_{d_0} \end{pmatrix}.$$

The product $\Gamma(e_{M^{-1}d}\hat{\eta})_{d \in R}$ can be written as

$$\begin{pmatrix} -B_{d_0} e_{M^{-1}d_0} \hat{\eta} \\ (B_{d_1} e_{M^{-1}d_0} - B_{d_0} e_{M^{-1}d_1})\hat{\eta} \\ (B_{d_2} e_{M^{-1}d_0} - B_{d_0} e_{M^{-1}d_2})\hat{\eta} \\ \vdots \\ \vdots \\ (B_{d_{m-1}} e_{M^{-1}d_0} - B_{d_0} e_{M^{-1}d_{m-1}})\hat{\eta} \end{pmatrix} = \begin{pmatrix} -B_{d_0} e_{M^{-1}d_0} \hat{\eta} \\ \tau_{d_1}\hat{\eta} \\ \tau_{d_2}\hat{\eta} \\ \vdots \\ \vdots \\ \tau_{d_{m-1}}\hat{\eta} \end{pmatrix} = \begin{pmatrix} -B_{d_0} e_{M^{-1}d_0} \hat{\eta} \\ \hat{w}_{d_1} \\ \hat{w}_{d_2} \\ \vdots \\ \vdots \\ \hat{w}_{d_{m-1}} \end{pmatrix}. \quad (6.41)$$

We observe that $\Gamma(e_{M^{-1}d}\hat{\eta})_{d \in R}$ coincides with $(\hat{w}_d)_{d \in R_0}$ in the $R_0$-entries. As a result, it is sufficient to prove that the shifts of the inverse transforms of $\Gamma(e_{M^{-1}d}\hat{\eta})_{d \in R}$ are $L_2$-stable. Since we assume $\operatorname{supp} \hat{\eta} = \mathbb{R}^n$ and the $L_2$-stability of the $M^{-1}\mathbb{Z}^n$-shifts of the function $\eta$, or equivalently of the $\mathbb{Z}^n$ shifts of $(\eta(\cdot + M^{-1}d))_{d \in R}$, we know that $(\eta(\cdot + M^{-1}d))_{d \in R}$ provides an $L_2$-stable basis for $S_1(\eta)$, see Theorem 6.1.4 and Corollary 6.1.9. According to part iv) of Corollary 4.2.8, proving that $\|\Gamma\|$ and $\|\Gamma^{-1}\|$ are essentially bounded on $\widetilde{C}_0$ yields our claim. Because of the boundedness of $B$, every component of $B_d$ is bounded and therefore, also all the entries of the matrix $\Gamma$. Applying the Laplace formula yields $|\det \Gamma| = |B_{d_0}|^m$ which is by assumption bounded away from 0. We deduce that $\|\Gamma\|$ and $\|\Gamma^{-1}\|$ are bounded almost everywhere. Hence, (6.41) provides an $L_2$-stable basis for $S_1(\eta)$. Since we know that the functions $w_d, d \in R_0$, are $m-1$ linearly independent elements in $W_0$, we finally obtain that $(w_d)_{d \in R_0}$ provides an $L_2$-stable basis for $W_0$. □

We remark that if $[\hat{\varphi}, \hat{\varphi}]_1$ and $[\hat{\eta}, \hat{\eta}]_1$ are bounded, then

$$|B|^2 = [\hat{\eta}, \hat{\varphi}]_1 [\hat{\varphi}, \hat{\eta}]_1 = \overline{A} [\hat{\eta}, \hat{\eta}]_1 A [\hat{\eta}, \hat{\eta}]_1 = [\hat{\varphi}, \hat{\varphi}]_1 [\hat{\eta}, \hat{\eta}]_1$$

yields the boundedness of $B$.

**Example 6.2.3.**
Let us have a closer look at Theorem 6.2.2 with $\varphi$ being the cardinal B-spline of order 1, i.e., $\varphi := \chi_{[0,1)}$ and we define $\eta$ as the 2-dilate of $\varphi$, that is, $\eta := \varphi(2\cdot)$. It follows directly that $S_0(\varphi) \subset S_1(\eta)$ and $\operatorname{supp} \hat{\varphi} = \operatorname{supp} \hat{\eta} = \mathbb{R}$. Moreover, let $M = 2$, $R = \{0, 1\}$ and $d_0 = 1$. Due to (6.15), we have

$$\begin{aligned} B(y) &= [\hat{\eta}, \hat{\varphi}]_1(y) \\ &= \sum_{d \in R} \sum_{k \in \mathbb{Z}} \frac{1}{2} \langle \varphi(2 \cdot + d), \varphi(\cdot - k) \rangle_{L_2(\mathbb{R})} e_{-(d/2+k)}(y) \\ &= \frac{1}{2} \langle \varphi(2\cdot), \varphi \rangle_{L_2(\mathbb{R})} + \frac{1}{2} \langle \varphi(2 \cdot + 1), \varphi(\cdot + 1) \rangle_{L_2(\mathbb{R}^n)} e_{1/2}(y) \\ &= \frac{1}{4} + \frac{1}{4} e_{1/2}(y). \end{aligned}$$

By (6.39), we further deduce that

$$\begin{aligned} B_0 &= \frac{Q_0(B)}{2} \\ &= \frac{1}{2} B + \frac{1}{2} B(\cdot + 2\pi) \\ &= \frac{1}{2} \left( \frac{1}{4} + \frac{1}{4} e_{1/2} \right) + \frac{1}{2} \left( \frac{1}{4} + \frac{1}{4} e_{1/2}(\cdot + 2\pi) \right) \\ &= \frac{1}{4}. \end{aligned}$$

Similarly we can show that

$$\begin{aligned} B_1 &= \frac{Q_1(B)}{2} \\ &= \frac{1}{2} e_{1/2} B + \frac{1}{2} e_{1/2}(\cdot + 2\pi) B(\cdot + 2\pi) \\ &= \frac{1}{2} e_{1/2} \left( \frac{1}{4} + \frac{1}{4} e_{1/2} \right) + \frac{1}{2} e_{1/2}(\cdot + 2\pi) \left( \frac{1}{4} + \frac{1}{4} e_{1/2}(\cdot + 2\pi) \right) \\ &= \frac{1}{4} e_1. \end{aligned}$$

Then the function

$$\tau_0 := e_{1/2}B_0 - B_1 = e_{1/2}\frac{1}{4} - \frac{1}{4}e_1$$

satisfies

$$\begin{aligned}
Q_0(\tau_0 B) &= \sum_{\tilde{d}\in R^T} \tau_0(\cdot + 2\pi\tilde{d})B(\cdot + 2\pi\tilde{d}) \\
&= \left(\frac{1}{4}e_{1/2} - \frac{1}{4}e_1\right)\left(\frac{1}{4} + \frac{1}{4}e_{1/2}\right) + \left(-\frac{1}{4}e_{1/2} - \frac{1}{4}e_1\right)\left(\frac{1}{4} - \frac{1}{4}e_{1/2}\right) \\
&= 0.
\end{aligned}$$

Next, we define $\hat{w}_0 := \tau_0\hat{\eta}$. We calculate

$$\begin{aligned}
\|\hat{w}_0\|^2_{L_2(\mathbb{R})} &= \int_{\mathbb{R}} |\tau_0(x)\hat{\eta}(x)|^2 \,\mathrm{d}x \\
&= \int_{\mathbb{R}} \left|\left(\frac{1}{4}e_{1/2}(x) - \frac{1}{4}e_1(x)\right)\frac{\mathrm{i}e_{-1/2}(x) - \mathrm{i}}{x}\right|^2 \mathrm{d}x \\
&= \int_{\mathbb{R}} \frac{\sin^4\left(\frac{x}{4}\right)}{x^2}\,\mathrm{d}x \\
&= \frac{\pi}{8} < \infty.
\end{aligned}$$

Therefore, $w_0 \in L_2(\mathbb{R})$ and thus, $w_0 \in W_0$. Moreover, Theorem 6.2.2 yields that $w_0$ is compactly supported. Finally, we verify that $w_0$ provides an $L_2$-stable basis for $W_0$. The generator $\varphi$ has $L_2$-stable integer shifts, see [26]. Since the stability constants do not change under dilation, $\eta$ has $L_2$-stable half-shifts. Furthermore, it holds

$$\|B\|_{L_\infty(\widetilde{C}_0)} = \operatorname*{ess\,sup}_{x\in\widetilde{C}_0}\left|\frac{1}{4} + \frac{1}{4}e_{1/2}(x)\right| \leq \frac{1}{2} < \infty.$$

For the inverse of $B_1$ we obtain

$$\left\|\frac{1}{B_1}\right\|_{L_\infty(\widetilde{C}_0)} = \operatorname*{ess\,sup}_{x\in\widetilde{C}_0} |4e_{-1}(x)| = 4 < \infty.$$

Consequently, Theorem 6.2.2 yields the desired result.

Next, we want to present an application of Theorem 6.2.2. For this application, we need the subsequent theorem which can be found in [64, Theorem 3.4.12]. For a more general version see also [49, Theorem 3.3].

**Theorem 6.2.4.**
Let the function $f$ be compactly supported. Then the following statements are equivalent:

i) The function $f$ is $L_2$-stable.

ii) The Fourier transform $\hat{f}$ of the function $f$ has no real periodic zeros.

**Corollary 6.2.5.**
Assume that the functions $\varphi$ and $\eta$ are compactly supported and that they fulfill $S_0(\varphi) \subset S_1(\eta)$ and $\operatorname{supp}\hat{\varphi} = \operatorname{supp}\hat{\eta} = \mathbb{R}^n$. In addition, let the function $\varphi$ have $L_2$-stable shifts and the function $\eta$ have $L_2$-stable $M^{-1}\mathbb{Z}^n$-shifts. Moreover, let $A$ in (6.36) be a trigonometric polynomial. In addition, we define

$$\varphi_1 := (\varphi * \overline{\varphi_-}), \quad \varphi_- := \varphi(-\cdot),$$

and

$$\eta_1 := (\eta * \overline{\eta_-}), \quad \eta_- := \eta(-\cdot).$$

Then the sequence $(w_d)_{d\in R'}$ of compactly supported functions defined in Theorem 6.2.2 with respect to $\varphi_1$ and $\eta_1$ provides an $L_2$-stable basis for the space $W_0$ if $\hat{w}_d \in L_2(\mathbb{R}^n)$ for all $d \in R'$.

*Proof.*
To prove this corollary, we have to verify that the conditions of Theorem 6.2.2 are fulfilled for $d_0 = 0$.

We start by noticing that the functions $\varphi$ and $\eta$ are elements of the space $L_1(\mathbb{R}^n)$ because it holds

$$\|\varphi\, \chi_{\operatorname{supp}\varphi}\|_{L_1(\mathbb{R}^n)} \le \|\varphi\|_{L_2(\mathbb{R}^n)} \|\chi_{\operatorname{supp}\varphi}\|_{L_2(\mathbb{R}^n)} < \infty,$$

and

$$\|\eta\, \chi_{\operatorname{supp}\eta}\|_{L_1(\mathbb{R}^n)} \le \|\eta\|_{L_2(\mathbb{R}^n)} \|\chi_{\operatorname{supp}\eta}\|_{L_2(\mathbb{R}^n)} < \infty.$$

Therefore, we can apply Theorem 1.2.3 and we obtain

$$\hat{\eta}_1(\xi) = \widehat{(\eta * \overline{\eta_-})}(\xi) = \hat{\eta}(\xi)\widehat{\overline{\eta_-}}(\xi) = \hat{\eta}(\xi)\overline{\hat{\eta}(\xi)} = |\hat{\eta}(\xi)|^2.$$

Similarly we observe that $\hat{\varphi}_1 = |\hat{\varphi}|^2$. It follows $\hat{\varphi}_1 = |A|^2\, \hat{\eta}_1$ and thus, it holds $S_0(\varphi_1) \subset S_1(\eta_1)$. Moreover, the compact support transfers from $\varphi$ and $\eta$ to $\varphi_1$ and $\eta_1$, respectively. This implies $\operatorname{supp}\hat{\varphi}_1 = \operatorname{supp}\hat{\eta}_1 = \mathbb{R}^n$. Next, we use (6.15) and the fact that $A$ is a trigonometric polynomial to deduce that the non-negative function

$$B = [\hat{\eta}_1, \hat{\varphi}_1]_1 = |A|^2\, [\hat{\eta}_1, \hat{\eta}_1]_1$$

is a trigonometric polynomial as well. The boundedness of $B$ follows directly. Furthermore, the $M^{-1}\mathbb{Z}^n$-shifts of $\eta$ are $L_2$-stable and as a consequence, Theorem 6.2.4 tells us that neither $\hat{\eta}$ nor $\hat{\eta}_1 = |\hat{\eta}|^2$ have $2\pi M^T\mathbb{Z}^n$- periodic zeros. Hence, $[\hat{\eta}_1, \hat{\eta}_1]_1$ is strictly positive. With the same arguments (with respect to the function $\varphi$), the function $\widetilde{A} := |A|^2 = \hat{\varphi}_1/\hat{\eta}_1 = |\hat{\varphi}|^2/|\hat{\eta}|^2$ cannot have $2\pi\mathbb{Z}^n$-periodic zeros. Consequently, $B$ is a non-negative $2\pi M^T\mathbb{Z}^n$-periodic function which has no $2\pi\mathbb{Z}^n$-periodic zeros. It follows that $B_0 = m^{-1}\sum_{\tilde{d}\in R^T} B(\cdot + 2\pi\tilde{d})$ is a strictly positive $2\pi\mathbb{Z}^n$-periodic trigonometric polynomial. In summary, $B$ and $1/B_0$ are essentially bounded on $\widetilde{C}_0$ and by Theorem 6.2.2, our claim follows. □

Next, we derive a general construction procedure for functions $\tau$ which fulfill (6.38). To this end we have a closer look at the choice of representatives of $\mathbb{Z}^n/M^T\mathbb{Z}^n$ and $\mathbb{Z}^n/M^{-T}\mathbb{Z}^n$. We follow the notation and definitions of [55]. First of all, we state that our matrix $M^T$ defines the pattern

$$P(M^T) := \{y \in \mathbb{R}^n : M^T y \in \mathbb{Z}^n\} = M^{-T}\mathbb{Z}^n.$$

The congruence classes $[d'] \in \mathbb{Z}^n/M^T\mathbb{Z}^n$ can be identified by their representatives $d' \in \mathbb{Z}^n$. Thus, we obtain

$$M^{-T}[d'] = \{y \in \mathbb{R}^n : y \equiv M^{-T}d' \mod I\}.$$

The congruence relation $x \equiv y \mod I$ for $x, y \in \mathbb{R}^n$ holds if and only if there exists a vector $z \in \mathbb{Z}^n$ with

$$x = y + Iz = y + z.$$

Hence, every unit cube of $\mathbb{R}^n$ contains exactly one element of $M^{-T}[d']$. With

$$P_I(M^T) := P(M^T) \cap [0,1)^n$$

it follows that an appropriate choice of representatives $d' \in \mathbb{Z}^n/M^T\mathbb{Z}^n$ is given by

$$y = M^{-T}d' \in P_I(M^T).$$

**Lemma 6.2.6.**
$P_I(M^T)$ equipped with the addition modulo the unit matrix $I$ is a group.

*Proof.*
The addition modulo the unit matrix $I$ is defined for all $x, y \in P_I(M^T)$ by

$$x_i +_I y_i := \begin{cases} x_i + y_i, & 0 \le x_i + y_i < 1, \\ x_i + y_i - 1, & x_i + y_i \ge 1, \end{cases} \qquad \text{for all } i \in \{1, \dots, n\}.$$

Since $M^T x + M^T y \in \mathbb{Z}^n$ and $x +_I y \in [0,1)^n$, $P_I(M^T)$ is closed under the addition modulo $I$. Furthermore, we can deduce associativity from the associativity of the componentwise addition. Finally, the identity element is 0 and the inverse element of $x \in P_I(M^T)$ is $1 - x \in P_I(M^T)$. □

**Lemma 6.2.7.**
The quotient group $\mathbb{Z}^n/M^T\mathbb{Z}^n$ is isomorphic to $G_I := (P_I(M^T), +_I)$.

*Proof.*
We have to prove that there exists a bijective group homomorphism between the quotient group $\mathbb{Z}^n/M^T\mathbb{Z}^n$ and $G_I$. Therefore, we define the function

$$\begin{aligned} h : \mathbb{Z}^n/M^T\mathbb{Z}^n &\to G_I \\ [d'] &\mapsto M^{-T}[d'] \cap [0,1)^n. \end{aligned} \tag{6.42}$$

As mentioned before, the set $M^{-T}[d']$ is equivalent to $\{h \in \mathbb{R}^n : h \equiv M^{-T}d' \mod I\}$. As a result it holds

$$|M^{-T}[d'] \cap [0,1)^n| = 1. \tag{6.43}$$

Consequently, for arbitrary elements $[d'_1], [d'_2] \in \mathbb{Z}^n/M^T\mathbb{Z}^n$ and $y_i \in G_I, i \in \{1,2,3\}$, we have

$$h([d'_1]) +_I h([d'_2]) = M^{-T}[d'_1] \cap [0,1)^n +_I M^{-T}[d'_2] \cap [0,1)^n = y_1 +_I y_2$$

and

$$h([d'_1] + [d'_2]) = h([d'_1 + d'_2]) = M^{-T}[d'_1 + d'_2] \cap [0,1)^n = y_3.$$

Since $y_1 + y_2 \in M^{-T}[d'_1 + d'_2]$, we deduce that $y_1 +_I y_2 \in M^{-T}[d'_1 + d'_2] \cap [0,1)^n$. Hence, it holds $y_1 +_I y_2 = y_3$.

Since $|G_I| = |\mathbb{Z}^n/M^T\mathbb{Z}^n| = m$, it is sufficient to show the injectivity of $h$ in order to show the bijectivity of $h$. For this purpose, let $[d'_1], [d'_2] \in \mathbb{Z}^n/M^T\mathbb{Z}^n$ such that $h([d'_1]) = h([d'_2])$. By (6.43), this implies $M^{-T}[d'_1] \cap [0,1)^n = M^{-T}[d'_2] \cap [0,1)^n = y$ with $y \in G_I$. Since equivalence classes are either equal or disjoint, we conclude that $[d'_1] = [d'_2]$ and our claim follows. □

Moreover, we introduce the notion of a character of a group, see [56, Definition 3.1.1.].

**Definition 6.2.8.**
A *character* of a group $(G, *)$ is a homomorphism from $G$ into the multiplicative group of nonzero complex numbers. That is, a character of $G$ is a function $\chi : G \to \mathbb{C}^*$ that satisfies $\chi(a * b) = \chi(a)\chi(b)$ for all $a, b \in G$. The group of characters of $G$ is denoted by $\widehat{G}$.

We will also need the following theorem which can be found in [56], see Theorem 3.2.1.

**Theorem 6.2.9.**
Let $G$ be a finite abelian group and let $\widetilde{U}$ be a subgroup of $G$. Moreover, let $\chi \in \widehat{G}$ and let $\widehat{G}_{\widetilde{U}}$ be the subgroup of $\widehat{G}$ formed by characters of $G$ which are identically 1 on $\widetilde{U}$. Then it holds

$$\sum_{\tilde{u} \in \widetilde{U}} \chi(\tilde{u}) = \begin{cases} |\widetilde{U}|, & \chi \in \widehat{G}_{\widetilde{U}}, \\ 0, & \text{otherwise.} \end{cases}$$

An application of Theorem 6.2.9 yields the subsequent lemma.

**Lemma 6.2.10.**
Let $U$ be a subgroup of $G_I$ and let $e_d, d \in R$, be an exponential which is not constant on $2\pi U$. Then it holds

$$\sum_{u \in U} e_d(2\pi u) = 0.$$

*Proof.*
Let us start by proving that

$$\begin{aligned} \chi_d : G_I &\to \mathbb{C}^* \\ g &\mapsto e_d(2\pi g) \end{aligned}$$

defines a character. Therefore, we have to verify that $\chi_d(g_1 +_I g_2) = \chi_d(g_1)\chi_d(g_2)$ holds for arbitrary elements $g_1, g_2 \in G_I$. Since $d \in \mathbb{Z}^n$ and $\lfloor g_1 + g_2 \rfloor \in \mathbb{Z}^n$, we obtain

$$\begin{aligned} \chi_d(g_1 +_I g_2) &= e_d(2\pi(g_1 + g_2 - \lfloor g_1 + g_2 \rfloor)) \\ &= e_d(2\pi g_1)\, e_d(2\pi g_2)\, e_d(-2\pi \lfloor g_1 + g_2 \rfloor) \\ &= e_d(2\pi g_1)\, e_d(2\pi g_2). \end{aligned}$$

By Theorem 6.2.9 our claim follows. □

As a last step of preparation we generalize Corollary 1 in Chapter II, §19, of [9].

**Corollary 6.2.11.**
Let $(X, \mathscr{E})$, $(Y, \mathscr{F})$ be two measurable spaces and let $\mu$ be a measure on $(X, \mathscr{E})$. Moreover, let $F : X \to Y$ be a $(\mathscr{E}, \mathscr{F})$-measurable mapping and let $\varphi : Y \to \mathbb{C}$ be a $\mathscr{F}$-measurable function. In addition, set $F_{\#}\mu(I) := \mu(F^{-1}(I))$, $I \in \mathscr{F}$. Then the $F_{\#}\mu$-integrability of $\varphi$ entails the $\mu$-integrability of $\varphi \circ F$. In case of integrability, we have

$$\int_X \varphi(F(x))\, \mathrm{d}\mu(x) = \int_Y \varphi(y)\, \mathrm{d}F_{\#}\mu(y).$$

*Proof.*
In case $\varphi : Y \to \mathbb{R}$ the claim follows by [9], see Corollary 1 in Chapter II, §19. Hence, we split $\varphi$ into real and imaginary parts to obtain

$$\int_X \mathrm{Re}(\varphi(F(x)))\,\mathrm{d}\mu(x) = \int_X \mathrm{Re}(\varphi)(F(x))\,\mathrm{d}\mu(x) = \int_Y \mathrm{Re}(\varphi)(y)\,\mathrm{d}F_{\#}\mu(y),$$

and

$$i\int_X \mathrm{Im}(\varphi(F(x)))\,\mathrm{d}\mu(x) = i\int_X \mathrm{Im}(\varphi)(F(x))\,\mathrm{d}\mu(x) = i\int_Y \mathrm{Im}(\varphi)(y)\,\mathrm{d}F_{\#}\mu(y).$$

Our proof is complete. □

Based on the previous considerations, we are going to present a general method to construct functions $\tau$ which fulfill (6.38).

**Remark 6.2.12.**
Let $U$ be a subgroup of $G_I$. Then the disjoint cosets $g +_I U, g \in G_I$, form a partition of $G_I$. We let $J \subset G_I$ be a set of representatives of these distinct cosets. By partitioning $J$ into disjoint sets $J = J_0 \cup J_1$ , we obtain the sets $K_i := \bigcup_{g \in J_i}(g +_I U), i = 0, 1,$ which form a partition of $G_I$.

**Theorem 6.2.13.**
Let $S_0(\varphi) \subset S_1(\eta)$ and let $\varphi$ and $\eta$ be compactly supported. In addition, let $B = [\hat{\varphi}, \hat{\eta}]_1$. Furthermore, let $U$ be a subgroup of $G_I$ and let $d$ be an arbitrary element of $R$ for which $e_d$ is not constant on $2\pi U$. Moreover, let $K$ be any union of cosets in $G_I$ of $U$ which contains 0. Then the function $w_{d,K}$ defined by the Fourier transform

$$\hat{w}_{d,K} := e_{M^{-1}d}\hat{\eta} \prod_{[\alpha] \in K \setminus \{0\}} B(\cdot + 2\pi M^T \alpha)$$

is a compactly supported function of the space $W_0$, if $w_{d,K} \in L_2(\mathbb{R}^n)$.

*Proof.*
Since the functions $\varphi$ and $\eta$ are compactly supported, we know that $B = [\hat{\eta}, \hat{\varphi}]_1$ is a trigonometric polynomial. By [71, Lemma 12.7.], the product of two trigonometric polynomials is a trigonometric polynomial and consequently, the product $\prod_{[\alpha] \in K \setminus \{0\}} B(\cdot + 2\pi M^T \alpha)$ is one as well. Therefore, $w_{d,K}$ is a well-defined $L_2(\mathbb{R}^n)$-function. Moreover, $\operatorname{supp} \hat{w}_{d,K} = \mathbb{R}^n$ and by applying the Paley-Wiener Theorem 1.2.10, we obtain that $w_{d,K}$ is compactly supported.

Now, Lemma 6.2.1 states that our claim follows if

$$\widetilde{\tau} := \prod_{[\alpha] \in K} B(\cdot + 2\pi M^T \alpha)$$

satisfies

$$\sum_{\tilde{d}\in R^T} e_{M^{-1}d}(\cdot + 2\pi\tilde{d})\widetilde{\tau}(\cdot + 2\pi\tilde{d}) = 0. \tag{6.44}$$

To verify (6.44), we will use Corollary 6.2.11. To apply this corollary, we set $(X, \mathscr{E}) = (G_I, \mathcal{P}(G_I))$, $(Y, \mathscr{F}) = (\mathbb{Z}^n/M^T\mathbb{Z}^n, \mathcal{P}(\mathbb{Z}^n/M^T\mathbb{Z}^n))$ and $F = h^{-1}$, see (6.42). In addition, let $\mu$ be the counting measure. It follows that $F_{\#}\mu(I) = \mu(F^{-1}(I)), I \in \mathscr{F}$, is the counting measure as well. Finally, we define the function $\varphi$ by

$$\begin{aligned}\varphi : \mathbb{Z}^n/M^T\mathbb{Z}^n &\to \mathbb{C}\\ [\tilde{d}] &\mapsto e_{M^{-1}d}(\cdot + 2\pi\tilde{d})\,\widetilde{\tau}(\cdot + 2\pi\tilde{d}).\end{aligned}$$

Since $\widetilde{\tau}$ is a trigonometric polynomial, the composition $\varphi \circ F$ is $\mu$-integrable. Now, we apply Corollary 6.2.11 and use that $e_{M^{-1}d}$ and $\widetilde{\tau}$ are $2\pi M^T\mathbb{Z}^n$-periodic. We obtain

$$\begin{aligned}\int_Y \varphi(y)\,\mathrm{d}F_{\#}\mu(y) &= \sum_{\tilde{d}\in\widetilde{R}^T} e_{M^{-1}d}(\cdot + 2\pi\tilde{d})\widetilde{\tau}(\cdot + 2\pi\tilde{d})\\ &= \int_X \varphi(F(x))\,\mathrm{d}\mu(x)\\ &= \sum_{g\in G_I} e_{M^{-1}d}(\cdot + 2\pi h^{-1}(g))\,\widetilde{\tau}(\cdot + 2\pi h^{-1}(g)),\end{aligned} \tag{6.45}$$

where $h^{-1}g$ is a representative of the equivalence class $[h^{-1}g]$. Next, we split up the sum in (6.45) with the help of Remark 6.2.12. We get

$$\begin{aligned}&\sum_{g\in G_I} e_{M^{-1}d}(\cdot + 2\pi h^{-1}(g))\,\widetilde{\tau}(\cdot + 2\pi h^{-1}(g))\\ &\quad= \sum_{j\in J}\sum_{u\in U} e_{M^{-1}d}(\cdot + 2\pi h^{-1}(j +_I u))\,\widetilde{\tau}(\cdot + 2\pi h^{-1}(j +_I u))\\ &\quad= \sum_{j\in J}\sum_{u\in U} e_{M^{-1}d}(\cdot + 2\pi h^{-1}(j) + 2\pi h^{-1}(u))\,\widetilde{\tau}(\cdot + 2\pi h^{-1}(j) + 2\pi h^{-1}(u))\\ &\quad= \sum_{j\in J}\sum_{u\in U} e_{M^{-1}d}(\cdot + 2\pi M^T j + 2\pi M^T u)\,\widetilde{\tau}(\cdot + 2\pi M^T j + 2\pi M^T u).\end{aligned}$$

Moreover, for an arbitrary element $M^T u \in R^T$ with $u \in U$ we deduce

$$\begin{aligned}\widetilde{\tau}(\cdot + 2\pi M^T u) &= \prod_{\alpha \in K} B(\cdot + 2\pi(M^T\alpha + M^T u)) \\ &= \prod_{\alpha \in K} B(\cdot + 2\pi(h^{-1}(\alpha +_I u)) \\ &= \prod_{\alpha \in K} B(\cdot + 2\pi(h^{-1}(\alpha)) \\ &= \prod_{\alpha \in K} B(\cdot + 2\pi M^T\alpha).\end{aligned}$$

We conclude

$$\begin{aligned}&\sum_{j\in J}\sum_{u\in U} e_{M^{-1}d}(\cdot + 2\pi M^T j + 2\pi M^T u)\,\widetilde{\tau}(\cdot + 2\pi M^T j + 2\pi M^T u) \\ &\quad = \sum_{j\in J} e_{M^{-1}d}(\cdot + 2\pi M^T j)\widetilde{\tau}(\cdot + 2\pi M^T j)\sum_{u\in U} e_{M^{-1}d}(2\pi M^T u) \\ &\quad = \sum_{j\in J} e_{M^{-1}d}(\cdot + 2\pi M^T j)\widetilde{\tau}(\cdot + 2\pi M^T j)\sum_{u\in U} e_d(2\pi u).\end{aligned}$$

By assumption, $e_d$ is not constant on $2\pi U$ and the assertion follows from Lemma 6.2.10. □

Theorem 6.2.13 yields the following corollary.

**Corollary 6.2.14.**
Let $S_0(\varphi) \subset S_1(\eta)$. Moreover, let $\varphi$ and $\eta$ be compactly supported. In addition, let $B = [\hat{\varphi}, \hat{\eta}]_1$ and $G'_I := G_I \setminus \{0\}$ . If $|\det M| \in 2\mathbb{N}$ and if $e_d(2\pi\alpha) = -1$ for a $d \in R'$ and $\alpha \in G'_I$, then the function $w$ with Fourier transform

$$\hat{w} = e_{M^{-1}d} B(\cdot + 2\pi M^T\alpha)\hat{\eta}$$

is a compactly supported function of the space $W_0$ in case $w \in L_2(\mathbb{R}^n)$.

*Proof.*
This corollary describes a special case of Theorem 6.2.13. Let $U$ be a group of order 2, i.e., $U = \{0, \alpha\}$. Since $e_0(2\pi u) = e_d(2\pi 0) = 1$ and $e_d(2\pi u) = -1$ for $d \in R', u \in U' = \{\alpha\}$, the function $e_d$ is not constant on $2\pi U$. By Lagrange's Theorem, for $g_i \in G_I$ with $i = 1, \ldots, m/2 - 1, m = |\det M|$, we obtain

$$G_I = U \cup g_1 +_I U \cup \ldots \cup g_{m/2-1} +_I U.$$

We observe that the subgroup $U$ itself is a coset. Hence, we can set $K = U = \{0, \alpha\}$. Consequently, we obtain

$$\hat{w} = e_{M^{-1}d} B(\cdot + 2\pi M^T\alpha)\hat{\eta}.$$

Finally, Theorem 6.2.13 yields our claim. □

**Example 6.2.15.**
With the help of Corollary 6.2.14, we extend Example 6.2.3. Let $G_I = \left\{0, \frac{1}{2}\right\}$. Since $\det M = 2$ and $e_d(2\pi\alpha) = -1$ for $d = 1$ and $\alpha = \frac{1}{2}$, we obtain

$$\hat{w} = e_{1/2} B(\cdot + 2\pi)\hat{\eta} = \left(\frac{1}{4} e_{1/2} - \frac{1}{4} e_1\right)\hat{\eta}.$$

We observe that $\left(\frac{1}{4}e_{1/2} - \frac{1}{4}e_1\right) = \tau_0$. Hence, $w$ is a function in $L_2(\mathbb{R}^n)$ and provides an $L_2$-stable basis for $W_0$.

The construction method we presented can also be applied to find $L_2$-stable bases for the spaces $W_j, j \in \mathbb{Z}$. This can be done analogously to Section 6.1.2 where we considered the case $S_1(\eta) \subset S_2(p)$. Let $\widetilde{B} := [\hat{p}, \hat{\eta}]_2$ and $B^* := [\widehat{p_{M^{-1}}}, \widehat{\eta_{M^{-1}}}]_1$. We remark that $1/\widetilde{B}_{d_0} \in L_\infty(\widetilde{C}_1)$ implies $1/B^*_{d_0} \in L_\infty(\widetilde{C}_0)$.

# Chapter 7

# Construction of Multiwavelets

In this chapter we generalize the construction procedure from Section 6.1 in order to obtain compactly supported multiwavelets. More precisely, we construct multiwavelet bases in case the spaces $S_j$ have finitely many generators and we construct compactly supported multiwavelet bases in case each space $S_j$ has two or three generators.

## 7.1 Compactly Supported Multiwavelet Bases

Throughout this chapter we assume that every space $S_j, j \in \mathbb{Z}$, has finitely many generators and that the generator sets may vary from space to space. In particular, we assume that

$$S_0(\Phi) \subset S_1(H),$$

where $\Phi = \{\varphi_1, \ldots, \varphi_N\}$ and $H = \{\eta_1, \ldots, \eta_N\}$ are generator sets consisting of $L_2(\mathbb{R}^n)$-functions. Then the wavelet space $W_0$ is defined as the orthogonal complement of $S_0(\Phi)$ in $S_1(H)$. In Section 7.1.1 we derive a basis for $W_0$ in case $N < \infty$ and we derive a compactly supported basis for $W_0$ in case $N \leq 3$. In contrast to Section 6.1.1, it is no longer sufficient for the generators to satisfy

$$\operatorname{supp} \hat{\varphi}_i = \operatorname{supp} \hat{\eta}_i = \mathbb{R}^n \quad \text{for } i = 1, \ldots N,$$

in order to obtain a basis. Indeed, we have to assume two conditions. The first condition is that the integer translates of

$$\widetilde{\Phi} := \{\varphi_i(\cdot + M^{-1}d), i = 1, \ldots, N, d \in R\}$$

provide a basis for $S_1(H)$. Secondly, we assume that

$$\widehat{H} = \Gamma \widehat{\Phi}$$

holds, where $\Gamma$ is a quadratic matrix which is non-singular almost everywhere and has $2\pi M^T \mathbb{Z}^n$-periodic entries. In Section 7.1.2 we demonstrate that all results concerning the space $W_0$ can be applied to the wavelet spaces $W_j = S_{j+1} \ominus S_j, j \in \mathbb{Z}$, after a suitable dilation. Afterwards, we illustrate this construction procedure in Section 7.1.3 by extending the example in Section 6.1.3 from $N = 1$ to $N = 2$.

### 7.1.1 Compactly Supported Multiwavelet Bases for $W_0$

We follow the same construction idea as in Section 6.1.1. We show that under certain assumptions the integer translates of $\widetilde{\Phi} = \{\varphi_i(\cdot + M^{-1}d), i = 1, \ldots, N, d \in R\}$ provide a basis for $S_1(H)$. Projecting the $(m-1)N$ functions $\varphi_i(\cdot + M^{-1}d)$ with $i = 1, \ldots, N, d \in R'$, onto the orthogonal complement of $S_0(\Phi)$ in $S_1(H)$ yields a basis for $W_0$. This basis then can be modified such that it is compactly supported provided that $N \leq 3$.

First, we verify the equality of the spaces $S_1(H)$ and $S_0(\widetilde{\Phi})$. We recall that we assume $0 \in R$.

**Theorem 7.1.1.**
Let $S_0(\Phi) \subset S_1(H)$ and let $\widetilde{\Phi} = \{\varphi_i(\cdot + M^{-1}d), i = 1, \ldots, N, d \in R\}$. Moreover, assume that $\widehat{H} = \Gamma\widehat{\Phi}$ where $\Gamma$ is a matrix with $2\pi M^T\mathbb{Z}^n$-periodic entries which is non-singular almost everywhere. Then it holds

$$S_1(H) = S_1(\Phi) = S_0(\widetilde{\Phi}).$$

*Proof.*
Let us start with the proof of $S_1(\Phi) = S_0(\widetilde{\Phi})$. The space $S_1(\Phi)$ is generated by the $M^{-1}\mathbb{Z}^n$-shifts of $\varphi_i, i = 1, \ldots, N$. Besides that, we have

$$M^{-1}\mathbb{Z}^n = \bigcup_{d \in R} (M^{-1}d + \mathbb{Z}^n).$$

This yields

$$S_1(\Phi) = S_0\left( (\varphi_i(\cdot + M^{-1}d))_{\substack{i=1,\ldots,N,\\ d \in R}} \right) = S_0(\widetilde{\Phi}).$$

With Proposition 4.2.3 we obtain $S_1(H) = S_1(\Phi)$ and the proof is complete. □

Next, we are going to show that the determinant of the Gramian matrix $G(\widetilde{\Phi})$ is strictly positive almost everywhere in order to deduce that $\widetilde{\Phi}$ provides a basis for $S_0(\widetilde{\Phi})$. In preparation for this, we derive a representation of the entries of the Gramian matrix $G(\widetilde{\Phi})$.

**Lemma 7.1.2.**
Let $\widetilde{\Phi} = \{\varphi_i(\cdot + M^{-1}d), i = 1, \ldots, N, d \in R\}$. For all $d, d^* \in R$ the corresponding entry of the Gramian matrix $G(\widetilde{\Phi})$ is given by

$$[e_{M^{-1}d}\,\hat{\varphi}_i, e_{M^{-1}d^*}\,\hat{\varphi}_k] = Q_{d-d^*}([\hat{\varphi}_i, \hat{\varphi}_k]_1).$$

*Proof.*
By inserting the definition of the bracket product and (6.1), we obtain

$$
\begin{aligned}
&[e_{M^{-1}d}\,\hat{\varphi}_i, e_{M^{-1}d^*}\,\hat{\varphi}_k] \\
&= \sum_{\alpha\in\mathbb{Z}^n} e_{M^{-1}d}(\cdot+2\pi\alpha)e_{-M^{-1}d^*}(\cdot+2\pi\alpha)\,\hat{\varphi}_i(\cdot+2\pi\alpha)\overline{\hat{\varphi}_k(\cdot+2\pi\alpha)} \\
&= \sum_{\alpha\in\mathbb{Z}^n} e_{M^{-1}(d-d^*)}(\cdot+2\pi\alpha)\,\hat{\varphi}_i(\cdot+2\pi\alpha)\overline{\hat{\varphi}_k(\cdot+2\pi\alpha)} \\
&= \sum_{\tilde{d}\in R^T}\sum_{\alpha\in\mathbb{Z}^n} e_{M^{-1}(d-d^*)}(\cdot+2\pi(\tilde{d}+M^T\alpha))\hat{\varphi}_i(\cdot+2\pi(\tilde{d}+M^T\alpha))\overline{\hat{\varphi}_k(\cdot+2\pi(\tilde{d}+M^T\alpha))}.
\end{aligned}
$$

Since the exponential function is $2\pi$-periodic, we deduce

$$
\begin{aligned}
&\sum_{\tilde{d}\in R^T}\sum_{\alpha\in\mathbb{Z}^n} e_{M^{-1}(d-d^*)}(\cdot+2\pi(\tilde{d}+M^T\alpha))\,\hat{\varphi}_i(\cdot+2\pi(\tilde{d}+M^T\alpha))\overline{\hat{\varphi}_k(\cdot+2\pi(\tilde{d}+M^T\alpha))} \\
&= \sum_{\tilde{d}\in R^T}\sum_{\alpha\in\mathbb{Z}^n} e_{M^{-1}(d-d^*)}(\cdot+2\pi\tilde{d})\,\hat{\varphi}_i(\cdot+2\pi\tilde{d}+2\pi M^T\alpha)\overline{\hat{\varphi}_k(\cdot+2\pi\tilde{d}+2\pi M^T\alpha)} \\
&= \sum_{\tilde{d}\in R^T} e_{M^{-1}(d-d^*)}(\cdot+2\pi\tilde{d})\sum_{\alpha\in\mathbb{Z}^n}\hat{\varphi}_i(\cdot+2\pi\tilde{d}+2\pi M^T\alpha)\overline{\hat{\varphi}_k(\cdot+2\pi\tilde{d}+2\pi M^T\alpha)} \\
&= \sum_{\tilde{d}\in R^T} e_{M^{-1}(d-d^*)}(\cdot+2\pi\tilde{d})\,[\hat{\varphi}_i,\hat{\varphi}_k]_1(\cdot+2\pi\tilde{d}) \\
&= Q_{d-d^*}([\hat{\varphi}_i,\hat{\varphi}_k]_1).
\end{aligned}
$$

□

Since the determinant is the product of the eigenvalues, it is our aim to ensure that each eigenvalue of the Gramian matrix $G(\widetilde{\Phi})$ is strictly positive. Although we do not know the eigenvalues of the Gramian matrix for $N > 1$, we can derive a lower bound for each of them by applying the following lemma, see [11, Theorem 3.2.8].

**Lemma 7.1.3.**
Let $A, B \in \mathbb{C}^{n\times n}$ be Hermitian matrices with eigenvalues

$$\alpha_1 \geq \ldots \geq \alpha_r \quad \text{and} \quad \beta_1 \geq \ldots \geq \beta_r.$$

Then the eigenvalues $\gamma_i$ of $C = A + B$ satisfy

$$\alpha_i + \beta_r \leq \gamma_i \leq \alpha_i + \beta_1, \quad i = 1, .., r.$$

**Theorem 7.1.4.**
The set $\widetilde{\Phi} = \{\varphi_i(\cdot + M^{-1}d), i = 1, \ldots, N, d \in R\}$ provides a basis for $S_0(\widetilde{\Phi})$, if for almost every $x \in \widetilde{C}_0$ it holds

$$0 < \min_{i=1,\ldots,N}\min_{d\in R}[\hat{\varphi}_i,\hat{\varphi}_i]_1(x+2\pi d) - \sum_{r=1}^{N-1}\sum_{s=r+1}^{N}\max_{d^\circ\in R}|[\hat{\varphi}_r,\hat{\varphi}_s]_1(x+2\pi d^\circ)|. \qquad (7.1)$$

*Proof.*
First, we partition the Gramian matrix into $N^2$ blocks of size $m \times m$ and then we split it up into a sum of $1 + (N^2 - N)/2$ matrices. We obtain

$$G(\widetilde{\Phi}) = \left[\begin{array}{c|c|c|c|c} A_1 & B_1 & B_2 & \dots & B_{N-1} \\ \hline B_1^* & A_2 & B_N & \dots & B_{N+N-3} \\ \hline B_2^* & B_N^* & \ddots & \ddots & \vdots \\ \hline \vdots & \vdots & \ddots & \ddots & B_{(N^2-N)/2} \\ \hline B_{N-1}^* & B_{N+N-3}^* & \dots & B_{(N^2-N)/2}^* & A_N \end{array}\right] = \widetilde{A} + \sum_{i=1}^{(N^2-N)/2} \widetilde{B}_i,$$

where

$$\widetilde{A} := \left[\begin{array}{c|c|c|c|c} A_1 & 0 & 0 & 0 & 0 \\ \hline 0 & A_2 & 0 & 0 & 0 \\ \hline 0 & 0 & \ddots & 0 & 0 \\ \hline 0 & 0 & 0 & \ddots & 0 \\ \hline 0 & 0 & 0 & 0 & A_N \end{array}\right]$$

and

$$\widetilde{B}_1 := \left[\begin{array}{c|c|c|c|c} 0 & B_1 & 0 & \dots & 0 \\ \hline B_1^* & 0 & 0 & \dots & 0 \\ \hline 0 & 0 & \ddots & \ddots & 0 \\ \hline \vdots & \vdots & \ddots & \ddots & \vdots \\ \hline 0 & 0 & \dots & 0 & 0 \end{array}\right],$$

$$\vdots$$

$$\vdots$$

$$\widetilde{B}_{(N^2-N)/2} := \left[\begin{array}{c|c|c|c|c} 0 & 0 & 0 & \dots & 0 \\ \hline 0 & 0 & \ddots & \ddots & \vdots \\ \hline 0 & \ddots & \ddots & 0 & 0 \\ \hline \vdots & \ddots & 0 & 0 & B_{(N^2-N)/2} \\ \hline 0 & \dots & 0 & B_{(N^2-N)/2}^* & 0 \end{array}\right].$$

The matrices $\widetilde{A}$ and $\widetilde{B}_i, i = 1, \dots, (N^2 + N)/2$, are Hermitian matrices. Therefore, we can apply Lemma 7.1.3 to derive a lower bound for the eigenvalues of the Gramian matrix. Beforehand, we have to determine the eigenvalues of the matrices $\widetilde{A}$ and $\widetilde{B}_i$. Due to Lemma 7.1.2, we can show analogue to the proof of Lemma 6.1.8 that the eigenvalues of $\widetilde{A}$ are given by

$$\left\{ m[\hat{\varphi}_i, \hat{\varphi}_i]_1(x + 2\pi d^\circ), i = 1, \dots, N, x \in \widetilde{C}_0, d^\circ \in R \right\}.$$

The eigenvalues of $\widetilde{B}_i$ can be determined by computing the singular values $\sigma_i$ of $B_i$ and $B_i^*$. Let $v$ be a right singular vector and $u$ a left singular vector of $\widetilde{B}_1$. Then it holds

$$\widetilde{B}_1 \begin{pmatrix} v \\ u \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \left[\begin{array}{c|c|c|c|c} 0 & B_1 & 0 & \dots & 0 \\ \hline B_1^* & 0 & 0 & \dots & 0 \\ \hline 0 & 0 & \ddots & \ddots & 0 \\ \hline \vdots & \vdots & \ddots & \ddots & \vdots \\ \hline 0 & 0 & \dots & 0 & 0 \end{array}\right] \begin{pmatrix} v \\ u \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \sigma_1 \begin{pmatrix} v \\ u \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and

$$\widetilde{B}_1 \begin{pmatrix} -v \\ u \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \left[\begin{array}{c|c|c|c|c} 0 & B_1 & 0 & \dots & 0 \\ \hline B_1^* & 0 & 0 & \dots & 0 \\ \hline 0 & 0 & \ddots & \ddots & 0 \\ \hline \vdots & \vdots & \ddots & \ddots & \vdots \\ \hline 0 & 0 & \dots & 0 & 0 \end{array}\right] \begin{pmatrix} -v \\ u \\ 0 \\ \vdots \\ 0 \end{pmatrix} = -\sigma_1 \begin{pmatrix} -v \\ u \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since $B_1$ and $B_1^*$ are square matrices with the same eigenvectors $a_{d^\circ}$ defined in Lemma 6.1.8, we obtain

$$B_1^* B_1 a_{d^\circ} = m^2 |[\hat{\varphi}_1, \hat{\varphi}_2]_1(\cdot + 2\pi d^\circ)|^2 a_{d^\circ}, \quad d^\circ \in R.$$

Consequently, the eigenvalues are

$$\{0, m|[\hat{\varphi}_1, \hat{\varphi}_2]_1(\cdot + 2\pi d^\circ)|, -m|[\hat{\varphi}_1, \hat{\varphi}_2]_1(\cdot + 2\pi d^\circ)|, d^\circ \in R\}. \tag{7.2}$$

With the same procedure we obtain the eigenvalues of $\widetilde{B}_i, i = 2, \dots, (N^2 - N)/2$. The only difference to (7.2) is that the generators in the bracket product vary from matrix to matrix. In a next step we sort the eigenvalues of every matrix, i.e., we let the eigenvalues of the matrix $\widetilde{A}$ denoted by $\{\alpha_\ell\}_{\ell=1,\dots,Nm}$ and the eigenvalues of the matrices $\widetilde{B}_i$ denoted by $\{\beta_{i,\ell}\}_{\ell=1,\dots,Nm}$ satisfy

$$\alpha_1 \geq \dots \geq \alpha_{Nm} \quad \text{and} \quad \beta_{i,1} \geq \dots \geq \beta_{i,Nm}.$$

Next, we successively apply Lemma 7.1.3. Let $\{\gamma_{1,\ell}\}_{\ell=1,\dots,Nm}$ be the eigenvalues of $\widetilde{A} + \widetilde{B}_1$. Then we have

$$\alpha_\ell + \beta_{1,Nm} \leq \gamma_{1,\ell}, \quad \ell = 1, \dots, Nm.$$

Next, let $\{\gamma_{2,\ell}\}_{\ell=1,\dots,Nm}$ be the eigenvalues of $\widetilde{A} + \widetilde{B}_1 + \widetilde{B}_2$. Then we obtain

$$\gamma_{1,\ell} + \beta_{2,Nm} \leq \gamma_{2,\ell}, \quad \ell = 1, \dots, Nm,$$

and thus,

$$\alpha_\ell + \beta_{1,Nm} + \beta_{2,Nm} \le \gamma_{2,\ell}, \quad \ell = 1, \ldots, Nm.$$

We repeat this procedure until we have a lower bound for the eigenvalues of $G(\widetilde{\Phi}) = \widetilde{A} + \sum_{i=1}^{(N^2-N)/2} \widetilde{B}_i$:

$$\alpha_\ell + \sum_{i=1}^{(N^2-N)/2} \beta_{i,Nm} \le \gamma_{(N^2-N)/2,\ell}, \quad \ell = 1, \ldots, Nm.$$

Inserting the calculated eigenvalues yields (7.1). □

Generator sets which satisfy (7.1) necessarily fulfill

$$\operatorname{supp}[\hat{\varphi}_i, \hat{\varphi}_i]_1 = \mathbb{R}^n \text{ for } i = 1, \ldots, N. \tag{7.3}$$

Condition (7.3) is not a completely new assumption. For a single generator $\operatorname{supp} \hat{\varphi} = \mathbb{R}^n$ was a sufficient condition to obtain a basis for $S_0(\phi)$ and (7.3) is a direct consequence of this assumption. Moreover, having a closer look at the proof of Corollary 6.1.9, one can observe that the weaker assumption (7.3) would have led to the same result.

For $N = 2$ and $N = 3$, we can also derive an alternative to (7.1).

**Theorem 7.1.5.**

i) Let $N = 2$ and let $\operatorname{supp}[\hat{\varphi}_i, \hat{\varphi}_i]_1 = \mathbb{R}^n$ for $i = 1, 2$. If for all $d^\circ \in R$ it holds

$$0 < [\hat{\varphi}_2, \hat{\varphi}_2]_1(\cdot + 2\pi d^\circ) - \frac{|[\hat{\varphi}_1, \hat{\varphi}_2]_1(\cdot + 2\pi d^\circ)|^2}{[\hat{\varphi}_1, \hat{\varphi}_1]_1(\cdot + 2\pi d^\circ)} \tag{7.4}$$

almost everywhere, then the generator set $\widetilde{\Phi} = \{\varphi_i(\cdot + M^{-1}d), i = 1, 2, d \in R\}$ provides a basis for $S_0(\widetilde{\Phi})$.

ii) Let $N = 3$ and suppose that $\operatorname{supp}[\hat{\varphi}_i, \hat{\varphi}_i]_1 = \mathbb{R}^n$ for $i = 1, 2, 3$. Moreover, for $d^\circ \in R$ let

$$\mathscr{A}_{1,d^\circ} := m[\hat{\varphi}_3, \hat{\varphi}_3]_1(\cdot + 2\pi d^\circ) - m\frac{|[\hat{\varphi}_1, \hat{\varphi}_3]_1(\cdot + 2\pi d^\circ)|^2}{[\hat{\varphi}_1, \hat{\varphi}_1]_1(\cdot + 2\pi d^\circ)},$$

$$\mathscr{A}_{2,d^\circ} := \left| m[\hat{\varphi}_3, \hat{\varphi}_2]_1(\cdot + 2\pi d^\circ) - m\frac{[\hat{\varphi}_3, \hat{\varphi}_1]_1(\cdot + 2\pi d^\circ)[\hat{\varphi}_1, \hat{\varphi}_2]_1(\cdot + 2\pi d^\circ)}{[\hat{\varphi}_1, \hat{\varphi}_1]_1(\cdot + 2\pi d^\circ)} \right|^2,$$

and

$$\mathscr{A}_{3,d^\circ} := \left( m[\hat{\varphi}_2, \hat{\varphi}_2]_1(\cdot + 2\pi d^\circ) - m\frac{|[\hat{\varphi}_1, \hat{\varphi}_2]_1(\cdot + 2\pi d^\circ)|^2}{[\hat{\varphi}_1, \hat{\varphi}_1]_1(\cdot + 2\pi d^\circ)} \right)^{-1}.$$

If (7.4) is satisfied and if

$$0 < \mathscr{A}_{1,d^\circ} - \mathscr{A}_{2,d^\circ}\mathscr{A}_{3,d^\circ} \tag{7.5}$$

almost everywhere for all $d^\circ \in R$, then $\widetilde{\Phi} = \{\varphi_i(\cdot + M^{-1}d), i = 1,2,3, d \in R\}$ provides a basis for $S_0(\widetilde{\Phi})$.

*Proof.*
Let us start with part i). As in the proof of Theorem 7.1.4 we split up the Gramian matrix into four blocks of size $m \times m$, i.e.,

$$G(\widetilde{\Phi}) = \left[\begin{array}{c|c} A_1 & B \\ \hline B^* & A_2 \end{array}\right].$$

The determinant of $A_1$ is given by

$$\det(A_1) = C \prod_{d^\circ \in R} [\hat{\varphi}_1, \hat{\varphi}_1]_1(\cdot + 2\pi d^\circ)$$

and it is greater than zero almost everywhere, see Corollary 6.1.9. Consequently, $A_1$ is invertible and the determinant of the Gramian matrix can be calculated as

$$\det(G(\widetilde{\Phi})) = \det(A_1)\det(A_2 - B^* A_1^{-1} B).$$

Our aim is to prove that $\det G(\widetilde{\Phi}) > 0$ almost everywhere. We observe that this is the case if and only if $\det(A_2 - B^* A_1^{-1} B) > 0$ almost everywhere or equivalently if the product of the eigenvalues of $A_2 - B^* A_1^{-1} B$ is greater than zero almost everywhere. Since the matrices $A_2, B^*, A_1^{-1}$ and $B$ have the same eigenvectors, the eigenvalues of $A_2 - B^* A_1^{-1} B$ are given by

$$m[\hat{\varphi}_2, \hat{\varphi}_2]_1(\cdot + 2\pi d^\circ) - m\,\frac{|[\hat{\varphi}_1, \hat{\varphi}_2]_1(\cdot + 2\pi d^\circ)|^2}{[\hat{\varphi}_1, \hat{\varphi}_1]_1(\cdot + 2\pi d^\circ)}, \quad d^\circ \in R.$$

By (7.4), our claim follows.

Part ii) can be proven analogously. Here, we work with a Gramian matrix of size $3m \times 3m$ which is partitioned into four blocks, i.e.,

$$G(\widetilde{\Phi}) = \left[\begin{array}{c|cc} A_1 & B_1 & B_2 \\ \hline B_1^* & A_2 & B_3 \\ B_2^* & B_3^* & A_3 \end{array}\right].$$

We already know, that $A_1$ is invertible. Therefore, we can calculate the determinant of the Gramian matrix as

$$\begin{aligned}
\det G(\widetilde{\Phi}) &= \det(A_1)\det\left(\begin{pmatrix} A_2 & B_3 \\ B_3^* & A_3 \end{pmatrix} - \begin{pmatrix} B_1^* \\ B_2^* \end{pmatrix} A_1^{-1} \begin{pmatrix} B_1 & B_2 \end{pmatrix}\right) \\
&= \det(A_1)\det\left(\begin{pmatrix} A_2 & B_3 \\ B_3^* & A_3 \end{pmatrix} - \begin{pmatrix} B_1^* A_1^{-1} B_1 & B_1^* A_1^{-1} B_2 \\ B_2^* A_1^{-1} B_1 & B_2^* A_1^{-1} B_2 \end{pmatrix}\right) \\
&= \det(A_1)\det\begin{pmatrix} A_2 - B_1^* A_1^{-1} B_1 & B_3 - B_1^* A_1^{-1} B_2 \\ B_3^* - B_2^* A_1^{-1} B_1 & A_3 - B_2^* A_1^{-1} B_2 \end{pmatrix}.
\end{aligned}$$

Since we assume that (7.4) is satisfied, $A_2 - B_1^* A_1^{-1} B_1$ is invertible. Hence, we can apply the formula for the determinant of block matrices again and we obtain

$$\begin{aligned}
&\det(A_1)\det\begin{pmatrix} A_2 - B_1^* A_1^{-1} B_1 & B_3 - B_1^* A_1^{-1} B_2 \\ B_3^* - B_2^* A_1^{-1} B_1 & A_3 - B_2^* A_1^{-1} B_2 \end{pmatrix} \\
&= \det(A_1)\det(A_2 - B_1^* A_1^{-1} B_1) \\
&\quad \cdot \det((A_3 - B_2^* A_1^{-1} B_2) - (B_3^* - B_2^* A_1^{-1} B_1)(A_2 - B_1^* A_1^{-1} B_1)^{-1}(B_3 - B_1^* A_1^{-1} B_2)) \\
&= \det(A_1)\det(A_2 - B_1^* A_1^{-1} B_1) \\
&\quad \cdot \det((A_3 - B_2^* A_1^{-1} B_2) - (B_3^* - B_2^* A_1^{-1} B_1)(A_2 - B_1^* A_1^{-1} B_1)^{-1}(B_3^* - B_2^* A_1^{-1} B_1)^*).
\end{aligned}$$

By our assumptions, each factor is greater than zero almost everywhere and the proof is complete. □

In case of finitely many generators, the method of proof above is also applicable. However, an increase of $N$ results in significantly more computations. Moreover, we observe that this method does not involve any kind of estimate. Hence, (7.4) and (7.5) are sharp estimates that the generators have to satisfy in order to guarantee the strict positivity almost everywhere of the determinant of the Gramian matrix. In contrast, Theorem 7.1.4 gives us a general estimate for finitely many generators which is not sharp. Depending on the choice of generators, this inaccuracy can cause that (7.4) and (7.5) are satisfied while (7.1) is not. Let us give an example.

**Example 7.1.6.**
Let $N = 2$ and let $M = 2$. Moreover, let $S_0(\Phi_2), \Phi_2 = \{\varphi_1, \varphi_2\}$, with

$$\varphi_1(x) := x\chi_{[0,0.5)}(x), \quad \varphi_2(x) := x^2\chi_{[0,0.5)}(x).$$

In view of Theorem 7.1.4 and Theorem 7.1.5, we want to check if the set

$$\widetilde{\Phi}_2 = \{\varphi_1(\cdot + \frac{1}{2}d), \varphi_2(\cdot + \frac{1}{2}d), d = 0, 1\}$$

provides a basis for the space $S_0(\widetilde{\Phi}_2)$. Formula (6.15) yields

$$\begin{aligned}
[\hat{\varphi}_1, \hat{\varphi}_1]_1(\xi) &= \sum_{d\in R}\sum_{k\in\mathbb{Z}} \frac{1}{|\det M^T|} \left\langle \varphi_1(\cdot + M^{-1}d), \varphi_1(\cdot - k)\right\rangle_{L_2(\mathbb{R})} e_{-(M^{-1}d+k)}(\xi) \\
&= \frac{1}{2}\sum_{d=0}^{1}\sum_{k\in\mathbb{Z}} \langle \varphi_1(\cdot + d/2), \varphi_1(\cdot - k)\rangle_{L_2(\mathbb{R})} e_{-(d/2+k)}(\xi) \\
&= \frac{1}{2}\langle \varphi_1, \varphi_1\rangle_{L_2(\mathbb{R})}.
\end{aligned}$$

Hence, we obtain

$$\frac{1}{2}\langle \varphi_1, \varphi_1\rangle_{L_2(\mathbb{R})} = \frac{1}{2}\int_{[0,0.5)} x^2\,\mathrm{d}x = \frac{1}{48}.$$

Moreover, we calculate

$$[\hat{\varphi}_2, \hat{\varphi}_2]_1(\xi) = \frac{1}{2} \langle \varphi_2, \varphi_2 \rangle_{L_2(\mathbb{R})} = \frac{1}{2} \int\limits_{[0,0.5)} x^4 \,\mathrm{d}x = \frac{1}{320},$$

$$[\hat{\varphi}_1, \hat{\varphi}_2]_1(\xi) = \frac{1}{2} \langle \varphi_1, \varphi_2 \rangle_{L_2(\mathbb{R})} = \frac{1}{2} \int\limits_{[0,0.5)} x^3 \,\mathrm{d}x = \frac{1}{128}.$$

Inserting these values into (7.1) and (7.4) yields

$$\frac{1}{320} - \frac{1}{128} = -\frac{3}{640} = -0.0046875 \not> 0$$

and

$$\frac{1}{320} - 48\left(\frac{1}{128}\right)^2 = \frac{1}{5120} = 0.0001953125 > 0.$$

Since one of the estimates is fulfilled, $\widetilde{\Phi}_2$ provides a basis for $S_0(\widetilde{\Phi}_2)$.

In a next step, we add the generator $\varphi_3(x) := x^3 \chi_{[0,0.5)}(x)$ to the generator set $\widetilde{\Phi}_2$ and we want to verify that

$$\widetilde{\Phi}_3 = \{\varphi_i(\cdot + \frac{1}{2}d), i = 1, 2, 3, d = 0, 1\}$$

provides a basis for the space $S_0(\widetilde{\Phi}_3)$. In order to calculate (7.1) and (7.5), we have to determine the values of the following bracket products first:

$$[\hat{\varphi}_3, \hat{\varphi}_3]_1(\xi) = \frac{1}{2} \langle \varphi_3, \varphi_3 \rangle = \frac{1}{2} \int\limits_{[0,0.5)} x^6 \,\mathrm{d}x = \frac{1}{1792},$$

$$[\hat{\varphi}_1, \hat{\varphi}_3]_1(\xi) = \frac{1}{2} \langle \varphi_1, \varphi_3 \rangle = \frac{1}{2} \int\limits_{[0,0.5)} x^4 \,\mathrm{d}x = \frac{1}{320},$$

$$[\hat{\varphi}_2, \hat{\varphi}_3]_1(\xi) = \frac{1}{2} \langle \varphi_1, \varphi_3 \rangle = \frac{1}{2} \int\limits_{[0,0.5)} x^5 \,\mathrm{d}x = \frac{1}{768}.$$

Hence, formula (7.1) yields

$$\frac{1}{1792} - \frac{1}{128} - \frac{1}{320} - \frac{1}{768} = -\frac{157}{13440} \approx -0.01168155 \not> 0.$$

In preparation for (7.5), we calculate

$$\mathscr{A}_{1,d^\circ} = \frac{2}{1792} - 96\left(\frac{1}{320}\right)^2 = \frac{1}{5600},$$
$$\mathscr{A}_{2,d^\circ} = \left(\frac{2}{768} - \frac{96}{320\cdot 128}\right)^2 = \frac{1}{14\,745\,600},$$
$$\mathscr{A}_{3,d^\circ} = \left(\frac{2}{320} - \frac{96}{16384}\right)^{-1} = 2560.$$

Hence, we obtain

$$\mathscr{A}_{1,d^\circ} - \mathscr{A}_{2,d^\circ}\mathscr{A}_{3,d^\circ} = \frac{1}{5600} - \frac{2560}{14\,745\,600} = \frac{1}{201\,600} \approx 4.96\cdot 10^{-6} > 0.$$

Moreover, we have already verified that (7.4) is satisfied. Hence, part ii) of Theorem 7.1.5 yields that $\widetilde{\Phi}_3$ provides a basis.

For finitely many generators condition (7.1) is a sufficient but not necessary condition for $\widetilde{\Phi}$ to provide a basis for $S_0(\widetilde{\Phi})$. Therefore, we are going to assume in the following that $\widetilde{\Phi}$ possesses the *basis property.*

Next, we want to combine Theorem 1.1.3 and Theorem 7.1.1. Since $W_0$ is the orthogonal complement of $S_0$ in $S_1$ and since $S_1$ can be generated by the functions $(\varphi_{i,d} := \varphi_i(\cdot + M^{-1}d))_{\substack{i=1,\dots,N,\\ d\in R'}}$, the set

$$\mathscr{W} := (w_{i,d} := \varphi_{i,d} - \mathcal{P}_{S_0(\Phi)}\varphi_{i,d})_{\substack{i=1,\dots,N,\\ d\in R'}}, \quad R' = R\setminus\{0\},$$

is a subset of $W_0$. Moreover, we will show that this set provides a basis for $W_0$. Besides that Theorem 5.9 gives us an explicit representation of the functions $\hat{w}_{i,d}$ if $\Phi$ is a minimal generator set for $S_0(\Phi)$. We obtain

$$\hat{w}_{i,d} = \hat{\varphi}_{i,d} - \widehat{\mathcal{P}_{S_0(\Phi)}\varphi_{i,d}} = \hat{\varphi}_{i,d} - \sum_{j=1}^{N}[\hat{\varphi}_{i,d},\hat{g}_j]/[\hat{g}_j,\hat{g}_j]\hat{g}_j, \tag{7.6}$$

with $[\hat{\varphi}_{i,d},\hat{g}_j][\hat{g}_j,\hat{g}_j]^{-1} = 0$ if $[\hat{g}_j,\hat{g}_j] = 0$. For a definition of the functions $\hat{g}_j$ see (5.7).

**Theorem 7.1.7.**
Let $S_0(\Phi) \subset S_1(H)$ with $\Phi = \{\varphi_1,\dots,\varphi_N\}$ and $H = \{\eta_1,\dots,\eta_N\}$. Moreover, assume that $\widehat{H} = \Gamma\widehat{\Phi}$ where $\Gamma$ is a matrix with $2\pi M^T\mathbb{Z}^n$-periodic entries which is non-singular almost everywhere. In addition, let $\widetilde{\Phi}$ possess the basis property and let

$$\mathscr{W} = (w_{i,d})_{\substack{i=1,\dots,N,\\ d\in R'}}$$

be defined as in (7.6). Then the space $W_0 = S_1 \ominus S_0$ is a finitely generated shift-invariant space and $\mathscr{W}$ is a generating set for $W_0$, that is, $W_0 = S_0(\mathscr{W})$. Moreover, the set $\mathscr{W}$ provides a basis for $W_0$.

*Proof.*
First, we prove that $S_0(\Phi) \oplus S_0(\mathscr{W}) = S_1(H)$ which means that $\{\Phi\} \cup \mathscr{W}$ generates $S_1(H)$ and therefore, $W_0 = S_0(\mathscr{W})$. For every function $f \in S_1(H)$ there exists a sequence $(f_\ell)_{\ell\in\mathbb{N}}$ with $f_\ell \to f$ in the $L_2$-sense. Due to Theorem 7.1.1, every element of this sequence has a representation of the form

$$
\begin{aligned}
f_\ell &= \sum_{k\in\mathbb{Z}^n}\sum_{i=1}^{N}\sum_{d\in R} a_{\ell,k,i,d}\varphi_{i,d}(\cdot - k) \\
&= \sum_{k\in\mathbb{Z}^n}\sum_{i=1}^{N} a_{\ell,k,i,0}\varphi_i(\cdot - k) + \sum_{k\in\mathbb{Z}^n}\sum_{i=1}^{N}\sum_{d\in R'} a_{\ell,k,i,d}\varphi_{i,d}(\cdot - k) \qquad (7.7)\\
&= \sum_{k\in\mathbb{Z}^n}\sum_{i=1}^{N} a_{\ell,k,i,0}\varphi_i(\cdot - k) + \sum_{k\in\mathbb{Z}^n}\sum_{i=1}^{N}\sum_{d\in R'} a_{\ell,k,i,d}\mathcal{P}_{S_0(\Phi)}\varphi_{i,d}(\cdot - k) \qquad (7.8)\\
&\quad + \sum_{k\in\mathbb{Z}^n}\sum_{i=1}^{N}\sum_{d\in R'} a_{\ell,k,i,d}\left(\varphi_{i,d}(\cdot - k) - \mathcal{P}_{S_0(\Phi)}\varphi_{i,d}(\cdot - k)\right). \qquad (7.9)
\end{aligned}
$$

Since (7.8) lies in the space $S_0(\Phi)$ and (7.9) lies in the space $S_0(\mathscr{W})$, we have shown that $S_1(H) \subset S_0(\Phi) \oplus S_0(\mathscr{W})$. Conversely, for every function $g \in S_0(\Phi) \oplus S_0(\mathscr{W})$ there exists a sequence $(g_\ell)_{\ell\in\mathbb{N}}$ in the space $S_0(\Phi) \oplus S_0(\mathscr{W})$ with $g_\ell \to g$ in the $L_2$-sense. Every element of this sequence can be represented by

$$
\begin{aligned}
g_\ell &= \sum_{k\in\mathbb{Z}^n}\sum_{i=1}^{N} b_{\ell,k,i,0}\varphi_i(\cdot - k) + \sum_{k\in\mathbb{Z}^n}\sum_{i=1}^{N}\sum_{d\in R'} b_{\ell,k,i,d}\left(\varphi_{i,d} - \mathcal{P}_{S_0(\Phi)}\varphi_{i,d}\right)(\cdot - k) \\
&= \sum_{k\in\mathbb{Z}^n}\sum_{i=1}^{N} b_{\ell,k,i,0}\varphi_i(\cdot - k) - \sum_{k\in\mathbb{Z}^n}\sum_{i=1}^{N}\sum_{d\in R'} b_{\ell,k,i,d}\mathcal{P}_{S_0(\Phi)}\varphi_{i,d}(\cdot - k) \qquad (7.10)\\
&\quad + \sum_{k\in\mathbb{Z}^n}\sum_{i=1}^{N}\sum_{d\in R'} b_{\ell,k,i,d}\varphi_{i,d}(\cdot - k). \qquad (7.11)
\end{aligned}
$$

Comparing (7.7) with (7.10) and (7.11) yields $S_0(\Phi) \oplus S_0(\mathscr{W}) \subset S_1(H)$.

It remains to prove that $\mathscr{W}$ provides a basis for $S_0(\mathscr{W})$. We assume that the set $\widetilde{\Phi} := (\varphi_i(\cdot + M^{-1}d))_{i=1,\ldots,N,d\in R}$ provides a basis for $S_0(\widetilde{\Phi})$. In addition, we know by Theorem 7.1.1 that in the case at hand it holds $S_1(H) = S_0(\widetilde{\Phi})$. We have already shown that $\Phi_* := \{\Phi\} \cup \mathscr{W}$ generates the space $S_1(H)$ and because of the equality of the spaces $S_0(\widetilde{\Phi})$ as well. Moreover, the sets $\widetilde{\Phi}$ and $\Phi_*$ have the same number of elements. In this case part ii) of Corollary 4.2.8 states that $\Phi_*$ provides a basis for $S_0(\widetilde{\Phi})$. As a consequence, we have $\det G(\Phi_*) \neq 0$ almost everywhere. Due to the orthogonality between $W_0 = S_0(\mathscr{W})$ and $S_0(\Phi)$, the Gramian matrix has the form

$$G(\Phi_*) = \begin{pmatrix} G(\mathscr{W}) & 0 \\ 0 & G(\Phi) \end{pmatrix}.$$

Since $G(\Phi_*)$ is a block diagonal matrix, we obtain

$$\det G(\Phi_*) = \det G(\mathscr{W}) \det G(\Phi).$$

Hence, $\det G(\mathscr{W}) \neq 0$ almost everywhere and consequently, $\mathscr{W}$ provides a basis for $W_0$. □

**Remark 7.1.8.**
Within the proof of the theorem above, we have shown that if $\widetilde{\Phi}$ possesses the basis property, then the generator set $\Phi$ is minimal.

In the following we are going to modify the mother wavelets obtained from Theorem 7.1.7 such that they provide a compactly supported basis for $W_0$. Beforehand, we derive an alternative representation for the bracket product $[\hat{g}_\ell, \hat{g}_\ell], 1 < \ell \leq N$.

**Lemma 7.1.9.**
Let $\Phi = \{\varphi_1, \dots, \varphi_N\} \subset L_2(\mathbb{R}^n)$ be a finite minimal set of generators for the space $S_0(\Phi)$. Moreover, let the functions $g_i, i = 1, \dots N$ be defined as in (5.7). Then for $1 < \ell \leq N$ it holds

$$[\hat{g}_\ell, \hat{g}_\ell] = [\hat{\varphi}_\ell, \hat{\varphi}_\ell] - \sum_{k=1}^{\ell-1} \frac{|[\hat{\varphi}_\ell, \hat{g}_k]|^2}{[\hat{g}_k, \hat{g}_k]}. \tag{7.12}$$

*Proof.*
Due to (5.9), we can deduce

$$\begin{aligned}
&[\hat{g}_\ell, \hat{g}_\ell] \\
&= [\hat{\varphi}_\ell, \hat{\varphi}_\ell] - [\hat{\varphi}_\ell, \sum_{r=1}^{\ell-1} \frac{[\hat{\varphi}_\ell, \hat{g}_r]}{[\hat{g}_r, \hat{g}_r]} \hat{g}_r] - [\sum_{k=1}^{\ell-1} \frac{[\hat{\varphi}_\ell, \hat{g}_k]}{[\hat{g}_k, \hat{g}_k]} \hat{g}_k, \hat{\varphi}_\ell] + [\sum_{k=1}^{\ell-1} \frac{[\hat{\varphi}_\ell, \hat{g}_k]}{[\hat{g}_k, \hat{g}_k]} \hat{g}_k, \sum_{r=1}^{\ell-1} \frac{[\hat{\varphi}_\ell, \hat{g}_r]}{[\hat{g}_r, \hat{g}_r]} \hat{g}_r] \\
&= [\hat{\varphi}_\ell, \hat{\varphi}_\ell] - \sum_{r=1}^{\ell-1} \frac{|[\hat{\varphi}_\ell, \hat{g}_r]|^2}{[\hat{g}_r, \hat{g}_r]} - \sum_{k=1}^{\ell-1} \frac{|[\hat{\varphi}_\ell, \hat{g}_k]|^2}{[\hat{g}_k, \hat{g}_k]} + \sum_{k=1}^{\ell-1} \frac{|[\hat{\varphi}_\ell, \hat{g}_k]|^2}{[\hat{g}_k, \hat{g}_k]} \\
&= [\hat{\varphi}_\ell, \hat{\varphi}_\ell] - \sum_{r=1}^{\ell-1} \frac{|[\hat{\varphi}_\ell, \hat{g}_r]|^2}{[\hat{g}_r, \hat{g}_r]}.
\end{aligned}$$

□

**Theorem 7.1.10.**
Let $S_0(\Phi) \subset S_1(H)$ with $\Phi = \{\varphi_1, \dots \varphi_N\}$, $H = \{\eta_1, \dots, \eta_N\}$ and $1 \leq N \leq 3$. Moreover, let $\Phi$ be a generator set consisting of compactly supported functions and let $\widetilde{\Phi}$ possess the basis property. In addition, assume that $\widehat{H} = \Gamma\widehat{\Phi}$ where $\Gamma$ is a matrix with $2\pi M^T\mathbb{Z}^n$-periodic entries which is non-singular almost everywhere. Then the $(m-1)N$ functions

$$\mathscr{W}_c := \left(\left(\left(\hat{\varphi}_{i,d} - \sum_{k=1}^{N} \frac{[\hat{\varphi}_{i,d}, \hat{g}_k]}{[\hat{g}_k, \hat{g}_k]}\hat{g}_k\right)\prod_{r=1}^{N}[\hat{g}_r, \hat{g}_r]^{2^{N-r}}\right)^{\vee}\right)_{\substack{i=1,..,N,\\ d\in R'}} \tag{7.13}$$

provide a compactly supported basis for the space $W_0$.

*Proof.*
If $N = 1$ the claim follows from Theorem 6.1.11.

Let $N = 2$. With the help of (7.6), we obtain $\widehat{\mathscr{W}_c}$ by multiplying $\widehat{\mathscr{W}}$ with the $2\pi$-periodic diagonal matrix $\Gamma^* := \prod_{r=1}^{2}[\hat{g}_r, \hat{g}_r]^{2^{2-r}}I = [\hat{g}_1, \hat{g}_1]^2[\hat{g}_2, \hat{g}_2]I$, that is, $\widehat{\mathscr{W}_c} = \Gamma^*\widehat{\mathscr{W}}$. From (5.7) and (7.12) we know that the diagonal entries of $\Gamma^*$ are

$$\begin{aligned}
[\hat{g}_1, \hat{g}_1]^2[\hat{g}_2, \hat{g}_2] &= [\hat{\varphi}_1, \hat{\varphi}_1]^2\left([\hat{\varphi}_2, \hat{\varphi}_2] - \frac{|[\hat{\varphi}_2, \hat{\varphi}_1]|^2}{[\hat{\varphi}_1, \hat{\varphi}_1]}\right)\\
&= [\hat{\varphi}_1, \hat{\varphi}_1]^2[\hat{\varphi}_2, \hat{\varphi}_2] - [\hat{\varphi}_1, \hat{\varphi}_1]|[\hat{\varphi}_2, \hat{\varphi}_1]|^2.
\end{aligned} \tag{7.14}$$

Since our generators $\varphi_1$ and $\varphi_2$ are compactly supported, (7.14) is a trigonometric polynomial as a finite linear combination of trigonometric polynomials, see part iii) of Lemma 1.3.7. Hence, the matrix $\Gamma^*$ consists only of bounded entries and thus, we deduce $\widehat{\mathscr{W}_c} \subset L_2(\mathbb{R}^n)$. By Theorem 7.1.7, $\mathscr{W}$ provides a basis for $W_0 = S_0(\mathscr{W})$. By taking part i) of Corollary 4.2.8 into consideration, it remains to prove that $\Gamma^*$ is non-singular almost everywhere in order to deduce that $\mathscr{W}_c$ provides a basis for $W_0$ as well. We have already shown that (7.14) is a trigonometric polynomial. Trigonometric polynomials are analytic functions and the zeros of analytic functions are isolated. Consequently, the entries of the diagonal matrix $\Gamma^*$ are non-zero almost everywhere which yields that the matrix is non-singular almost everywhere. To complete the proof for $N = 2$ we have to show that the functions in $\mathscr{W}_c$ are compactly supported. Therefore, we have to evaluate

$$\left(\hat{\varphi}_{i,d} - \frac{[\hat{\varphi}_{i,d}, \hat{g}_1]}{[\hat{g}_1, \hat{g}_1]}\hat{g}_1 - \frac{[\hat{\varphi}_{i,d}, \hat{g}_2]}{[\hat{g}_2, \hat{g}_2]}\hat{g}_2\right)[\hat{g}_1, \hat{g}_1]^2[\hat{g}_2, \hat{g}_2]. \tag{7.15}$$

For the first component of the functions in $\mathscr{W}_c$ we obtain

$$[\hat{g}_1, \hat{g}_1]^2[\hat{g}_2, \hat{g}_2]\hat{\varphi}_{i,d} = ([\hat{\varphi}_1, \hat{\varphi}_1]^2[\hat{\varphi}_2, \hat{\varphi}_2] - [\hat{\varphi}_1, \hat{\varphi}_1]|[\hat{\varphi}_2, \hat{\varphi}_1]|^2)\hat{\varphi}_{i,d}.$$

For the second component we calculate

$$-[\hat{\varphi}_{i,d},\hat{g}_1][\hat{g}_1,\hat{g}_1][\hat{g}_2,\hat{g}_2]\hat{g}_1 = -([\hat{\varphi}_1,\hat{\varphi}_1][\hat{\varphi}_2,\hat{\varphi}_2] - |[\hat{\varphi}_2,\hat{\varphi}_1]|^2)[\hat{\varphi}_{i,d},\hat{\varphi}_1]\hat{\varphi}_1.$$

Finally, for the third component we observe

$$\begin{aligned}
-[\hat{\varphi}_{i,d},\hat{g}_2][\hat{g}_1,\hat{g}_1]^2\hat{g}_2 &= -[\hat{\varphi}_{i,d},\hat{\varphi}_2 - \frac{[\hat{\varphi}_2,\hat{\varphi}_1]}{[\hat{\varphi}_1,\hat{\varphi}_1]}\hat{\varphi}_1][\hat{\varphi}_1,\hat{\varphi}_1]^2\left(\hat{\varphi}_2 - \frac{[\hat{\varphi}_2,\hat{\varphi}_1]}{[\hat{\varphi}_1,\hat{\varphi}_1]}\hat{\varphi}_1\right)\\
&= -\left([\hat{\varphi}_{i,d},\hat{\varphi}_2] - \frac{[\hat{\varphi}_1,\hat{\varphi}_2]}{[\hat{\varphi}_1,\hat{\varphi}_1]}[\hat{\varphi}_{i,d},\hat{\varphi}_1]\right)[\hat{\varphi}_1,\hat{\varphi}_1]^2\left(\hat{\varphi}_2 - \frac{[\hat{\varphi}_2,\hat{\varphi}_1]}{[\hat{\varphi}_1,\hat{\varphi}_1]}\hat{\varphi}_1\right)\\
&= -([\hat{\varphi}_{i,d},\hat{\varphi}_2][\hat{\varphi}_1,\hat{\varphi}_1] - [\hat{\varphi}_1,\hat{\varphi}_2][\hat{\varphi}_{i,d},\hat{\varphi}_1])(\hat{\varphi}_2[\hat{\varphi}_1,\hat{\varphi}_1] - [\hat{\varphi}_2,\hat{\varphi}_1]\hat{\varphi}_1).
\end{aligned}$$

Since we assume that $\varphi_1$ and $\varphi_2$ are compactly supported, the functions $\varphi_{i,d}$, $i = 1,2$, $d \in R'$, are compactly supported as well. Now, we consider the first component $[\hat{\varphi}_1,\hat{\varphi}_1]^2[\hat{\varphi}_2,\hat{\varphi}_2]\hat{\varphi}_{i,d}$. The bracket products $[\hat{\varphi}_1,\hat{\varphi}_1]$ and $[\hat{\varphi}_2,\hat{\varphi}_2]$ are trigonometric polynomials. As a consequence, the inverse Fourier transform of $[\hat{\varphi}_1,\hat{\varphi}_1]^2[\hat{\varphi}_2,\hat{\varphi}_2]\hat{\varphi}_{i,d}$ is a finite linear combination of the shifts of $\varphi_{i,d}$ and compactly supported because $\varphi_{i,d}$ is. The same reasoning shows that the inverse Fourier transform of all other components of the functions in $\mathscr{W}_c$ is compactly supported. Consequently, the functions in $\mathscr{W}_c$ are compactly supported.

For $N = 3$ the proof is analogue. First, we calculate

$$\begin{aligned}
\prod_{r=1}^{3}[\hat{g}_r,\hat{g}_r]^{2^{3-r}} &= [\hat{g}_1,\hat{g}_1]^4[\hat{g}_2,\hat{g}_2]^2[\hat{g}_3,\hat{g}_3]\\
&= \left([\hat{g}_1,\hat{g}_1]^2[\hat{g}_2,\hat{g}_2]\right)\left([\hat{g}_1,\hat{g}_1]^2[\hat{g}_2,\hat{g}_2][\hat{g}_3,\hat{g}_3]\right)\\
&= \left([\hat{\varphi}_1,\hat{\varphi}_1]^2[\hat{\varphi}_2,\hat{\varphi}_2] - [\hat{\varphi}_1,\hat{\varphi}_1]|[\hat{\varphi}_2,\hat{\varphi}_1]|^2\right)\left([\hat{g}_1,\hat{g}_1]^2[\hat{g}_2,\hat{g}_2][\hat{g}_3,\hat{g}_3]\right).
\end{aligned}$$

Moreover, we have

$$[\hat{g}_3,\hat{g}_3] = [\hat{\varphi}_3,\hat{\varphi}_3] - \frac{|[\hat{\varphi}_3,\hat{\varphi}_1]|^2}{[\hat{\varphi}_1,\hat{\varphi}_1]} - \frac{|[\hat{\varphi}_3,\hat{g}_2]|^2}{[\hat{g}_2,\hat{g}_2]},$$

where

$$\begin{aligned}
&|[\hat{\varphi}_3,\hat{g}_2]|^2\\
&=\left(|[\hat{\varphi}_2,\hat{\varphi}_3]|^2 - \frac{[\hat{\varphi}_3,\hat{\varphi}_2][\hat{\varphi}_2,\hat{\varphi}_1][\hat{\varphi}_1,\hat{\varphi}_3]}{[\hat{\varphi}_1,\hat{\varphi}_1]} - \frac{[\hat{\varphi}_1,\hat{\varphi}_2][\hat{\varphi}_3,\hat{\varphi}_1][\hat{\varphi}_2,\hat{\varphi}_3]}{[\hat{\varphi}_1,\hat{\varphi}_1]} + \frac{|[\hat{\varphi}_1,\hat{\varphi}_2]|^2|[\hat{\varphi}_1,\hat{\varphi}_3]|^2}{[\hat{\varphi}_1,\hat{\varphi}_1]^2}\right).
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
[\hat{g}_1,\hat{g}_1]^2[\hat{g}_2,\hat{g}_2][\hat{g}_3,\hat{g}_3] &= \left([\hat{\varphi}_1,\hat{\varphi}_1]^2[\hat{\varphi}_2,\hat{\varphi}_2] - [\hat{\varphi}_1,\hat{\varphi}_1]|[\hat{\varphi}_2,\hat{\varphi}_1]|^2\right)[\hat{\varphi}_3,\hat{\varphi}_3]\\
&\quad - \left([\hat{\varphi}_1,\hat{\varphi}_1][\hat{\varphi}_2,\hat{\varphi}_2] - |[\hat{\varphi}_2,\hat{\varphi}_1]|^2\right)|[\hat{\varphi}_3,\hat{\varphi}_1]|^2\\
&\quad - [\hat{\varphi}_1,\hat{\varphi}_1]^2|[\hat{\varphi}_2,\hat{\varphi}_3]|^2 + [\hat{\varphi}_3,\hat{\varphi}_2][\hat{\varphi}_2,\hat{\varphi}_1][\hat{\varphi}_1,\hat{\varphi}_3][\hat{\varphi}_1,\hat{\varphi}_1]\\
&\quad + [\hat{\varphi}_1,\hat{\varphi}_2][\hat{\varphi}_3,\hat{\varphi}_1][\hat{\varphi}_2,\hat{\varphi}_3][\hat{\varphi}_1,\hat{\varphi}_1] - |[\hat{\varphi}_1,\hat{\varphi}_2]|^2|[\hat{\varphi}_1,\hat{\varphi}_3]|^2.
\end{aligned}$$

Hence, we verified that $\prod_{r=1}^{3}[\hat{g}_r, \hat{g}_r]^{2^{3-r}}$ is a finite linear combination of trigonometric polynomials. Following the proof above for $N = 2$, we have to prove that

$$\left(\hat{\varphi}_{i,d} - \frac{[\hat{\varphi}_{i,d}, \hat{g}_1]}{[\hat{g}_1, \hat{g}_1]}\hat{g}_1 - \frac{[\hat{\varphi}_{i,d}, \hat{g}_2]}{[\hat{g}_2, \hat{g}_2]}\hat{g}_2 - \frac{[\hat{\varphi}_{i,d}, \hat{g}_3]}{[\hat{g}_3, \hat{g}_3]}\hat{g}_3\right)[\hat{g}_1, \hat{g}_1]^4[\hat{g}_2, \hat{g}_2]^2[\hat{g}_3, \hat{g}_3]$$

has the same structure as (7.15). Based on our previous investigations, we observe that this is true for

$$\begin{aligned}&\left(\hat{\varphi}_{i,d} - \frac{[\hat{\varphi}_{i,d}, \hat{g}_1]}{[\hat{g}_1, \hat{g}_1]}\hat{g}_1 - \frac{[\hat{\varphi}_{i,d}, \hat{g}_2]}{[\hat{g}_2, \hat{g}_2]}\hat{g}_2\right)[\hat{g}_1, \hat{g}_1]^4[\hat{g}_2, \hat{g}_2]^2[\hat{g}_3, \hat{g}_3]\\ &\quad = \left(\left(\hat{\varphi}_{i,d} - \frac{[\hat{\varphi}_{i,d}, \hat{g}_1]}{[\hat{g}_1, \hat{g}_1]}\hat{g}_1 - \frac{[\hat{\varphi}_{i,d}, \hat{g}_2]}{[\hat{g}_2, \hat{g}_2]}\hat{g}_2\right)[\hat{g}_1, \hat{g}_1]^2[\hat{g}_2, \hat{g}_2]\right)\left([\hat{g}_1, \hat{g}_1]^2[\hat{g}_2, \hat{g}_2][\hat{g}_3, \hat{g}_3]\right).\end{aligned}$$

It remains to prove that

$$\frac{[\hat{\varphi}_{i,d}, \hat{g}_3]}{[\hat{g}_3, \hat{g}_3]}\hat{g}_3[\hat{g}_1, \hat{g}_1]^4[\hat{g}_2, \hat{g}_2]^2[\hat{g}_3, \hat{g}_3]$$

has the required structure. We calculate

$$\begin{aligned}&\frac{[\hat{\varphi}_{i,d}, \hat{g}_3]}{[\hat{g}_3, \hat{g}_3]}\hat{g}_3[\hat{g}_1, \hat{g}_1]^4[\hat{g}_2, \hat{g}_2]^2[\hat{g}_3, \hat{g}_3]\\ &= [\hat{\varphi}_{i,d}, \hat{g}_3]\hat{g}_3[\hat{g}_1, \hat{g}_1]^4[\hat{g}_2, \hat{g}_2]^2\\ &= \left[\hat{\varphi}_{i,d}, \hat{\varphi}_3 - \frac{[\hat{\varphi}_3, \hat{\varphi}_1]}{[\hat{\varphi}_1, \hat{\varphi}_1]}\hat{\varphi}_1 - \frac{[\hat{\varphi}_3, \hat{g}_2]}{[\hat{g}_2, \hat{g}_2]}\hat{g}_2\right]\left(\hat{\varphi}_3 - \frac{[\hat{\varphi}_3, \hat{\varphi}_1]}{[\hat{\varphi}_1, \hat{\varphi}_1]}\hat{\varphi}_1 - \frac{[\hat{\varphi}_3, \hat{g}_2]}{[\hat{g}_2, \hat{g}_2]}\hat{g}_2\right)[\hat{g}_1, \hat{g}_1]^4[\hat{g}_2, \hat{g}_2]^2.\end{aligned}$$

Let us investigate the expression

$$\mathscr{B}_1 := \left[\hat{\varphi}_{i,d}, \hat{\varphi}_3 - \frac{[\hat{\varphi}_3, \hat{\varphi}_1]}{[\hat{\varphi}_1, \hat{\varphi}_1]}\hat{\varphi}_1 - \frac{[\hat{\varphi}_3, \hat{g}_2]}{[\hat{g}_2, \hat{g}_2]}\hat{g}_2\right][\hat{g}_1, \hat{g}_1]^2[\hat{g}_2, \hat{g}_2].$$

We observe

$$\begin{aligned}\mathscr{B}_1 = {}&[\hat{\varphi}_{i,d}, \hat{\varphi}_3]\left([\hat{\varphi}_1, \hat{\varphi}_1]^2[\hat{\varphi}_2, \hat{\varphi}_2] - [\hat{\varphi}_1, \hat{\varphi}_1]|[\hat{\varphi}_2, \hat{\varphi}_1]|^2\right)\\ &- [\hat{\varphi}_{i,d}, \hat{\varphi}_1][\hat{\varphi}_1, \hat{\varphi}_3]\left([\hat{\varphi}_1, \hat{\varphi}_1][\hat{\varphi}_2, \hat{\varphi}_2] - |[\hat{\varphi}_2, \hat{\varphi}_1]|^2\right)\\ &- [\hat{\varphi}_{i,d}, \hat{g}_2][\hat{g}_2, \hat{\varphi}_3][\hat{\varphi}_1, \hat{\varphi}_1]^2.\end{aligned}$$

This can be rewritten as

$$\begin{aligned}\mathscr{B}_1 = {}&[\hat{\varphi}_{i,d}, \hat{\varphi}_3]\left([\hat{\varphi}_1, \hat{\varphi}_1]^2[\hat{\varphi}_2, \hat{\varphi}_2] - [\hat{\varphi}_1, \hat{\varphi}_1]|[\hat{\varphi}_2, \hat{\varphi}_1]|^2\right)\\ &- [\hat{\varphi}_{i,d}, \hat{\varphi}_1][\hat{\varphi}_1, \hat{\varphi}_3]\left([\hat{\varphi}_1, \hat{\varphi}_1][\hat{\varphi}_2, \hat{\varphi}_2] - |[\hat{\varphi}_2, \hat{\varphi}_1]|^2\right)\\ &- [\hat{\varphi}_{i,d}, \hat{\varphi}_2 - \frac{[\hat{\varphi}_2, \hat{\varphi}_1]}{[\hat{\varphi}_1, \hat{\varphi}_1]}\hat{\varphi}_1][\hat{\varphi}_2 - \frac{[\hat{\varphi}_2, \hat{\varphi}_1]}{[\hat{\varphi}_1, \hat{\varphi}_1]}\hat{\varphi}_1, \hat{\varphi}_3][\hat{\varphi}_1, \hat{\varphi}_1]^2.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\mathscr{B}_1 = &[\hat\varphi_{i,d},\hat\varphi_3]\left([\hat\varphi_1,\hat\varphi_1]^2[\hat\varphi_2,\hat\varphi_2]-[\hat\varphi_1,\hat\varphi_1]|[\hat\varphi_2,\hat\varphi_1]|^2\right)\\ &-[\hat\varphi_{i,d},\hat\varphi_1][\hat\varphi_1,\hat\varphi_3]\left([\hat\varphi_1,\hat\varphi_1][\hat\varphi_2,\hat\varphi_2]-|[\hat\varphi_2,\hat\varphi_1]|^2\right)\\ &-\left([\hat\varphi_{i,d},\hat\varphi_2][\hat\varphi_1,\hat\varphi_1]-[\hat\varphi_{i,d},\hat\varphi_1][\hat\varphi_1,\hat\varphi_2]\right)\left([\hat\varphi_2,\hat\varphi_3][\hat\varphi_1,\hat\varphi_1]-[\hat\varphi_2,\hat\varphi_1][\hat\varphi_1,\hat\varphi_3]\right).\end{aligned}$$

Furthermore, let us have a closer look at

$$\mathscr{B}_2 := \left(\hat\varphi_3-\frac{[\hat\varphi_3,\hat\varphi_1]}{[\hat\varphi_1,\hat\varphi_1]}\hat\varphi_1-\frac{[\hat\varphi_3,\hat g_2]}{[\hat g_2,\hat g_2]}\hat g_2\right)[\hat g_1,\hat g_1]^2[\hat g_2,\hat g_2].$$

This can be written as

$$\begin{aligned}\mathscr{B}_2 = &\hat\varphi_3\left([\hat\varphi_1,\hat\varphi_1]^2[\hat\varphi_2,\hat\varphi_2]-[\hat\varphi_1,\hat\varphi_1]|[\hat\varphi_2,\hat\varphi_1]|^2\right)\\ &-[\hat\varphi_3,\hat\varphi_1]\left([\hat\varphi_1,\hat\varphi_1][\hat\varphi_2,\hat\varphi_2]-|[\hat\varphi_2,\hat\varphi_1]|^2\right)\hat\varphi_1\\ &-[\hat\varphi_3,\hat g_2]\hat g_2[\hat\varphi_1,\hat\varphi_1]^2.\end{aligned}$$

Finally, we can write $\mathscr{B}_2$ as

$$\begin{aligned}\mathscr{B}_2 = &\left([\hat\varphi_1,\hat\varphi_1]^2[\hat\varphi_2,\hat\varphi_2]-[\hat\varphi_1,\hat\varphi_1]|[\hat\varphi_2,\hat\varphi_1]|^2\right)\hat\varphi_3\\ &-[\hat\varphi_3,\hat\varphi_1]\left([\hat\varphi_1,\hat\varphi_1][\hat\varphi_2,\hat\varphi_2]-|[\hat\varphi_2,\hat\varphi_1]|^2\right)\hat\varphi_1\\ &-[\hat\varphi_3,\hat\varphi_2][\hat\varphi_1,\hat\varphi_1]^2\hat\varphi_2+[\hat\varphi_3,\hat\varphi_2][\hat\varphi_2,\hat\varphi_1][\hat\varphi_1,\hat\varphi_1]\hat\varphi_1\\ &+[\hat\varphi_1,\hat\varphi_2][\hat\varphi_3,\hat\varphi_1][\hat\varphi_1,\hat\varphi_1]\hat\varphi_2-[\hat\varphi_1,\hat\varphi_2][\hat\varphi_3,\hat\varphi_1][\hat\varphi_2,\hat\varphi_1]\hat\varphi_1.\end{aligned}$$

Then the claim follows with the same arguments as above.

□

Furthermore, we can ensure the existence of a compactly supported and orthogonal basis for $W_0$ in our setting.

**Theorem 7.1.11.**
Let $S_0(\Phi)\subset S_1(H)$ with $\Phi=\{\varphi_1,\dots,\varphi_N\}, H=\{\eta_1,\dots,\eta_N\}$, and let $\widetilde\Phi$ possess the basis property. In addition, assume that $\widehat H=\Gamma\widehat\Phi$ where $\Gamma$ is a matrix with $2\pi M^T\mathbb{Z}^n$-periodic entries which is non-singular almost everywhere.

i) There exists a set of mother wavelets $\Psi$ for $W_0$ which provides an orthonormal basis for $W_0$.

ii) If $N\le 3$ and if the functions $\varphi_i, i=1,\dots,N$, are compactly supported, then there exists a subset $\Psi=(\psi_i)_{i=1,\dots,(m-1)N}$ of compactly supported functions from $W_0$ which provides a basis for $W_0$. Moreover, there exists a set $\Psi^*:=(\psi_r^*)_{r=1,\dots,(m-1)N}$ consisting of compactly supported functions which provides a basis for $W_0$ and fulfills

$$S_0(\psi_r^*)\perp S_0(\psi_{r'}^*),\quad r\neq r'.$$

*Proof.*
Let us prove part i). The conditions of Theorem 7.1.7 are satisfied and therefore, $\mathscr{W}$ is a basis for $W_0$. Our claim then follows from part i) of Corollary 4.2.9.

For the proof of part ii) we need Theorem 7.1.10 which tells us that we can choose the set of compactly supported functions $\mathscr{W}_c$ for $\Psi$ in order to obain a compactly supported basis for $W_0$. By part ii) of Corollary 4.2.9 the proof is complete. □

Finally, we deduce under which assumptions the set

$$\{w_i(\cdot + M^{-1}d), i = 1, \ldots, N, d \in R'\} \subset S_1(H)$$

is in $W_0$ and provides a basis for $W_0$.

**Theorem 7.1.12.**
Let $S_0(\Phi) \subset S_1(H)$ with $\Phi = \{\varphi_1, \ldots, \varphi_N\}$ and $H = \{\eta_1, \ldots, \eta_N\}$. Moreover, let $\Phi$ be a generator set consisting of compactly supported functions and let $\widetilde{\Phi}$ possess the basis property. In addition, assume that $\widehat{H} = \Gamma\widehat{\Phi}$ where $\Gamma$ is a matrix with $2\pi M^T\mathbb{Z}^n$-periodic entries which is non-singular almost everywhere.

i) If the functions $w_i, i = 1, \ldots, N$, are in $S_1(H)$ and compactly supported, then the functions $w_i(\cdot + M^{-1}d)$ with $i = 1, \ldots, N, d \in R'$, are in $W_0$ if and only if the bracket products $[\hat{w}_i, \hat{\varphi}_j]_1$ are $2\pi$-periodic for all $i, j = 1, \ldots, N$.

ii) Assume that the integer translates of $\{w_i(\cdot + M^{-1}d), i = 1, \ldots, N, d \in R\}$ provide a compactly supported basis for $S_1(H)$ and let $[\hat{w}_i, \hat{\varphi}_j]_1$ be $2\pi$-periodic for $i, j = 1, \ldots, N$. Then the integer translates of

$$\{w_i(\cdot + M^{-1}d), i = 1, \ldots, N, d \in R'\}$$

provide a basis for $W_0$.

*Proof.*
We begin with the proof of part i). The functions $w_i(\cdot + M^{-1}d)$ are in $W_0$ if and only if

$$\left\langle w_i(\cdot + M^{-1}d), \varphi_j(\cdot - k)\right\rangle_{L_2(\mathbb{R}^n)} = 0, \quad \text{for all } i, j = 1, \ldots, N, d \in R', k \in \mathbb{Z}^n.$$

By the proof of the analogue result for one generator, see Theorem 6.1.13, we can directly deduce that this is equivalent to the $2\pi$-periodicity of $[\hat{w}_i, \hat{\varphi}_j]_1, i, j = 1, \ldots, N$.

In part ii) we assume that the integer translates of

$$\mathscr{W}_* := (w_i(\cdot + M^{-1}d))_{\substack{i=1,\ldots,N,\\ d\in R}}$$

provide a basis for $S_1(H)$. This is equivalent to $\det G(\mathscr{W}_*) \neq 0$ almost everywhere on $\widetilde{C}_0$. Thus, the determinant of the Gramian matrix corresponding to

$\mathscr{W}_* \setminus \{w_1, \ldots, w_N\}$ is non-zero almost everywhere. Since we assume that $[\hat{w}_i, \hat{\varphi}_j]_1$ is $2\pi$-periodic, we can apply i) and consequently, $\mathscr{W}_* \setminus \{w_1, \ldots, w_N\}$ is in $W_0$. Hence, we obtain $(m-1)N$ functions whose Gramian matrix is nonzero almost everywhere. Since Theorem 7.1.7 ensures that $W_0$ contains a basis of cardinality $(m-1)N$, we conclude by part iii) of Corollary 4.2.8 that $\mathscr{W}_* \setminus \{w_1, \ldots, w_N\}$ is a basis for $W_0$. □

### 7.1.2 Compactly Supported Multiwavelet Basis for $\mathbf{W_j}$

Our procedure for constructing a basis of $W_0$ can be applied to all spaces $W_j$ after a suitable dilation. We illustrate this for the case $j = 1$. This choice allows us to clearly observe how the role of $S_1(H)$ is changing when compared with our investigations in the previous chapters.

Let $H = \{\eta_1, \ldots, \eta_N\}$ and $P := \{p_1, \ldots, p_N\}$ such that

$$S_1(H) \subset S_2(P). \tag{7.16}$$

We assume further that

$$\widehat{P} = \widetilde{\Gamma}\, \widehat{H}, \tag{7.17}$$

where $\widetilde{\Gamma}$ is a matrix with $2\pi(M^T)^2\mathbb{Z}^n$-periodic entries which is non-singular almost everywhere. Moreover, let

$$\widetilde{H} := \{\eta_i(\cdot + M^{-2}d), i = 1, \ldots, N, d \in R\} \tag{7.18}$$

provide a basis for the space $S_1(\widetilde{H})$. It is crucial that all the properties (7.16), (7.17) and (7.18) of the generator sets $P$ and $H$ are preserved under dilation.

**Theorem 7.1.13.**
Let $H_{M^{-1}} := \{\eta_{1,M^{-1}}, \ldots, \eta_{N,M^{-1}}\}$ and $P_{M^{-1}} := \{p_{1,M^{-1}}, \ldots, p_{N,M^{-1}}\}$ with

$$\eta_{i,M^{-1}} := \eta_i(M^{-1}\cdot), \quad p_{i,M^{-1}} := p_i(M^{-1}\cdot), \quad i = 1, \ldots, N.$$

Under the assumptions (7.16), (7.17) and (7.18) we have

$$S_0(H_{M^{-1}}) \subset S_1(P_{M^{-1}}) \tag{7.19}$$

and

$$\widehat{P_{M^{-1}}} = \Gamma\, \widehat{H_{M^{-1}}}.$$

Here, $\Gamma$ is a matrix with $2\pi M^T\mathbb{Z}^n$-periodic entries which is non-singular almost everywhere. Moreover,

$$\widetilde{H}_{M^{-1}} := \{\eta_{i,M^{-1}}(\cdot + M^{-1}d), i = 1, \ldots, N, d \in R\}$$

provides a basis for the space $S_0(\widetilde{H}_{M^{-1}})$.

*Proof.*
First, we prove (7.19). Let $f$ be a function in $S_0(H_{M^{-1}})$. Then there exists a sequence $(f_\ell)_{\ell\in\mathbb{N}} \in S_0(H_{M^{-1}})$ such that $f_\ell \to f$ in the $L_2$-sense. The definition of the space $S_0(H_{M^{-1}})$ yields for all elements of the sequence the representation

$$f_\ell = \sum_{i=1}^{N}\sum_{k\in\mathbb{Z}^n} a_{\ell,i,k}\,\eta_{i,M^{-1}}(\cdot - k) = \sum_{i=1}^{N}\sum_{k\in\mathbb{Z}^n} a_{\ell,i,k}\,\eta_i(M^{-1}\cdot - M^{-1}k). \tag{7.20}$$

Recall that the bijective, linear and continuous operator $\widetilde{J}$ from Section 6.1.2 is given by

$$\begin{aligned} \widetilde{J} : L_2(\mathbb{R}^n) &\to L_2(\mathbb{R}^n) \\ f &\mapsto f(M^{-1}\cdot). \end{aligned}$$

We apply $\widetilde{J}^{-1}$ to (7.20) and obtain

$$\widetilde{J}^{-1} f_\ell = \sum_{i=1}^{N}\sum_{k\in\mathbb{Z}^n} a_{\ell,i,k}\,\eta_i(\cdot - M^{-1}k).$$

Consequently, $\widetilde{J}^{-1} f_\ell \in S_1(H)$ and by (7.16), we have $\widetilde{J}^{-1} f_\ell \in S_2(P)$. Hence, for every function $\widetilde{J}^{-1} f_\ell$, $\ell \in \mathbb{N}$, there exists a sequence of functions $(g_r)_{r\in\mathbb{N}} \in S_2(P)$ such that $g_r \to \widetilde{J}^{-1} f_\ell$ in the $L_2$-sense. The function $g_r$ can be represented as

$$g_r = \sum_{i=1}^{N}\sum_{k\in\mathbb{Z}^n} b_{r,i,k}\,p_i(\cdot - M^{-2}k) = \sum_{i=1}^{N}\sum_{k\in\mathbb{Z}^n} b_{r,i,k}\,p_{i,M^{-1}}(M\cdot - M^{-1}k).$$

Applying the operator $\widetilde{J}$ yields

$$\widetilde{J} g_r = \sum_{i=1}^{N}\sum_{k\in\mathbb{Z}^n} b_{r,i,k}\,p_{i,M^{-1}}(\cdot - M^{-1}k).$$

Hence, it holds $(\widetilde{J} g_r)_{r\in\mathbb{N}} \in S_1(P_{M^{-1}})$. From the $L_2$-closure of the space $S_1(P_{M^{-1}})$ it follows that $f_\ell \in S_1(P_{M^{-1}})$. Consequently, $(f_\ell)_{\ell\in\mathbb{N}} \in S_1(P_{M^{-1}})$ and due to the same argument, we obtain $f \in S_1(P_{M^{-1}})$.

Besides that it holds

$$\widehat{P}(M^T\cdot) = \widetilde{\Gamma}(M^T\cdot)\,\widehat{H}(M^T\cdot).$$

By (1.4), this is equivalent to

$$\widehat{P_{M^{-1}}} = \widetilde{\Gamma}(M^T\cdot)\,\widehat{H_{M^{-1}}} = \Gamma\,\widehat{H_{M^{-1}}},$$

where $\Gamma := \widetilde{\Gamma}(M^T\cdot)$ has $2\pi M^T\mathbb{Z}^n$-periodic entries. By assumption, the matrix $\widetilde{\Gamma}$ is non-singular almost everywhere. Hence, the matrix $\Gamma$ is non-singular almost everywhere as well.

Finally, we have to prove that $\widetilde{H}_{M^{-1}}$ provides a basis for $S_0(\widetilde{H}_{M^{-1}})$. Let $f \in S_1(\widetilde{H})$. Then there exits a sequence $(f_\ell)_{\ell\in\mathbb{N}} \in S_1(\widetilde{H})$ such that $f_\ell \to f$ in the $L_2$-sense. By assumption, $\widetilde{H}$ provides a basis for $S_1(\widetilde{H})$. Hence, every function $f_\ell$ has a unique representation

$$f_\ell = \sum_{i=1}^{N} \sum_{d\in R} \sum_{k\in\mathbb{Z}^n} a_{\ell,i,d,k}\eta_i(\cdot + M^{-2}d - M^{-1}k).$$

Applying the operator $\widetilde{J}$ yields

$$\begin{aligned}\widetilde{J}f_\ell &= \sum_{i=1}^{N} \sum_{d\in R} \sum_{k\in\mathbb{Z}^n} a_{\ell,i,d,k}\eta_i(M^{-1}\cdot + M^{-2}d - M^{-1}k)\\ &= \sum_{i=1}^{N} \sum_{d\in R} \sum_{k\in\mathbb{Z}^n} a_{\ell,i,d,k}\eta_{i,M^{-1}}(\cdot + M^{-1}d - k).\end{aligned} \tag{7.21}$$

Since $\widetilde{J}$ is bijective, the representation (7.21) is unique as well and the proof is complete. □

Now, we can apply our construction procedure from Section 7.1.1 to find a wavelet basis for

$$\widetilde{W}_0 := S_1(P_{M^{-1}}) \ominus S_0(H_{M^{-1}}).$$

By (7.6), the set of functions

$$\widetilde{\mathscr{W}} := (\widetilde{w}_{i,d} := \eta_{i,M^{-1},d} - \mathcal{P}_{S_0(H_{M^{-1}})}\eta_{i,M^{-1},d})_{\substack{i=1,\dots,N,\\ d\in R'}}, \quad R' = R\setminus\{0\},$$

with $\eta_{i,M^{-1},d} := \eta_{i,M^{-1}}(\cdot + M^{-1}d)$ is a possible candidate for a wavelet basis. According to Remark 7.1.8, $H_{M^{-1}}$ is minimal and thus, Theorem 5.9 provides us with an explicit representation of the functions $\widehat{\widetilde{w}}_{i,d}$. We obtain

$$\widehat{\widetilde{w}}_{i,d} = \widehat{\eta_{i,M^{-1},d}} - \widehat{\mathcal{P}_{S_0(H_{M^{-1}})}\eta_{i,M^{-1},d}} = \widehat{\eta_{i,M^{-1},d}} - \sum_{j=1}^{N} \frac{[\widehat{\eta_{i,M^{-1},d}}, \hat{g}_j]}{[\hat{g}_j, \hat{g}_j]}\hat{g}_j$$

with $g_1 = \eta_{1,M^{-1}}$ and $[\widehat{\eta_{i,M^{-1},d}}, \hat{g}_j][\hat{g}_j, \hat{g}_j]^{-1} = 0$ if $[\hat{g}_j, \hat{g}_j] = 0$.

Let $\widetilde{w}_{i,d,M} := \widetilde{w}_{i,d}(M\cdot)$. With the same arguments as in the proofs of Theorem 7.1.13 and Corollary 4.2.8 and the fact that dilation preserves orthogonality, we obtain that

$$\widetilde{\mathscr{W}}_M := (\widetilde{w}_{i,d,M})_{\substack{i=1,\dots,N,\\ d\in R'}}$$

provides a wavelet basis for the space

$$W_1 = S_1(\widetilde{\mathscr{W}_M}) = S_2(P) \ominus S_1(H).$$

In case $\widetilde{\mathscr{W}}$ is an $L_2$-stable basis for $\widetilde{W_0} = S_0(\widetilde{\mathscr{W}})$, the same holds true for $\widetilde{\mathscr{W}_M}$ concerning the space $W_1 = S_1(\widetilde{\mathscr{W}_M})$, see proof of part iv) of Corollary 4.2.8. Moreover, compact support is preserved under dilation, see (1.4). Thus, Theorem 7.1.10 can be applied whenever the functions in $H$ are compactly supported.

### 7.1.3 Example: Exponential Box Splines

In the past sections we presented a way to construct a compactly supported multiwavelet basis for the space $L_2(\mathbb{R}^n)$. In this section we want to illustrate this construction process. In particular, we present an application for Theorem 7.1.10 by extending the example presented in Section 6.1.3 from $N = 1$ generator to $N = 2$ generators.

As in Section 6.1.3 we choose the symmetric dilation matrix

$$M := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and the directions

$$x^{\gamma_1} := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad x^{\gamma_2} := M \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

with cardinality $p \in \mathbb{N} \setminus \{0\}$ each. Furthermore, we set

$$\lambda_j = (\lambda_{j,\gamma_1}, \lambda_{j,\gamma_2}), \quad \lambda_{j-1} = (\lambda_{j-1,\gamma_1}, \lambda_{j-1,\gamma_2}) = (\lambda_{j,\gamma_2}/2, \lambda_{j,\gamma_1}).$$

Then the exponential box spline corresponding to these directions is defined by

$$\widehat{C}_{\lambda_j,p}(\xi) := \left( \frac{e^{\lambda_{j,\gamma_1} - i\xi \cdot x^{\gamma_1}} - 1}{\lambda_{j,\gamma_1} - i\xi \cdot x^{\gamma_1}} \right)^p \left( \frac{e^{\lambda_{j,\gamma_2} - i\xi \cdot x^{\gamma_2}} - 1}{\lambda_{j,\gamma_2} - i\xi \cdot x^{\gamma_2}} \right)^p, \quad j \in \mathbb{Z}, p \in \mathbb{N} \setminus \{0\},$$

and satisfies the equation

$$\widehat{C}_{\lambda_j,p}(M^j \xi) = A_{\lambda_j,\lambda_{j-1},p}(M^{j-1}\xi) \widehat{C}_{\lambda_{j-1},p}(M^{j-1}\xi),$$

where $A_{\lambda_j,\lambda_{j-1},p}$ is a $2\pi$-periodic trigonometric polynomial given by

$$A_{\lambda_j,\lambda_{j-1},p}(\xi) := \left( \frac{1}{2} \frac{e^{\lambda_{j,\gamma_2} - iM\xi \cdot x^{\gamma_2}} - 1}{e^{\lambda_{j-1,\gamma_1} - i\xi \cdot x^{\gamma_1}} - 1} \right)^p.$$

Again, we fix the values for $\lambda_0$ by setting

$$\lambda_1^* = \lambda_{0,\gamma_1}, \quad \lambda_2^* = \lambda_{0,\gamma_2}.$$

Hence, the values for $\lambda_j$ are defined as

$$\lambda_j = (\lambda_{j,\gamma_1}, \lambda_{j,\gamma_2}) = \begin{cases} (2^{j/2}\lambda_1^*, 2^{j/2}\lambda_2^*), & j \in 2\mathbb{Z}, \\ (2^{(j-1)/2}\lambda_2^*, 2^{(j+1)/2}\lambda_1^*), & j \in 2\mathbb{Z}+1. \end{cases}$$

Based on this knowledge, we define the generators $\Phi_j := \{\varphi_1^j, \varphi_2^j\}$ for the spaces $S_j, j \in \mathbb{Z}$, as

$$\hat{\varphi}_1^j := \frac{1}{m^j}\widehat{C}_{\lambda_{-j},p}(M^{-j}\cdot), \quad \hat{\varphi}_2^j := \frac{1}{m^j}\widehat{C}_{\lambda_{-j},p+1}(M^{-j}\cdot). \tag{7.22}$$

Then we obtain

$$\begin{pmatrix} \hat{\varphi}_1^j \\ \hat{\varphi}_2^j \end{pmatrix} = \Gamma \begin{pmatrix} \hat{\varphi}_1^{j+1} \\ \hat{\varphi}_2^{j+1} \end{pmatrix}, \tag{7.23}$$

where $\Gamma$ is the $2\pi(M^T)^{j+1}\mathbb{Z}^2$-periodic matrix given by

$$\Gamma := \begin{pmatrix} mA_{\lambda_{-j},\lambda_{-j-1},p}(M^{-j-1}\cdot) & 0 \\ 0 & mA_{\lambda_{-j},\lambda_{-j-1},p+1}(M^{-j-1}\cdot) \end{pmatrix}.$$

It is important for our construction procedure with two generators that the matrix $\Gamma$ is non-singular almost everywhere and that $\Gamma^{-1}$ has $2\pi(M^T)^{j+1}\mathbb{Z}^2$-periodic entries. This is the case if and only if

$$A_{\lambda_{-j},\lambda_{-j-1},p}(\xi) = \frac{1}{2^p}\left(e^{\lambda_{-j,\gamma_2}/2-i\xi_1}+1\right)^p \neq 0 \quad \text{almost everywhere.}$$

An equivalent condition is given by

$$e^{\lambda_{-j,\gamma_2}/2-i\xi_1} \neq -1 \quad \text{almost everywhere.} \tag{7.24}$$

To ensure (7.24), we set $\lambda_1^* = 0$ and $\lambda_2^* \in \mathbb{R}\setminus\{0\}$. In addition, our non-stationary refinement equation (7.23) ensures that the corresponding spaces are nested and due the compact support of the generators, we have $\overline{\bigcup_j S_j} = L_2(\mathbb{R}^2)$, see Theorem 4.2.12. Finally, we verify that our generator sets possess the basis property which implies that the generator sets $\Phi_j$ of the spaces $S_j$ are minimal for $j \in \mathbb{Z}$, see Remark 7.1.8.

**Lemma 7.1.14.**
Let $\varphi_1^j$ and $\varphi_2^j$ be defined as in (7.22). Then the set

$$\widetilde{\Phi}_j := \{\hat{\varphi}_i^j(\cdot + M^{-(j+1)}d), i = 1,2, d \in R\}$$

provides a basis for $S_j(\widetilde{\Phi}_j)$.

*Proof.*
Suppose that $\widetilde{\Phi}_j$ does not provide a basis for $S_j(\widetilde{\Phi}_j)$. Then there exists a nontrivial representation of 0 given by

$$\sum_{d\in R}\sum_{k\in\mathbb{Z}^2} c_k\,\varphi_1^j(x+M^{-(j+1)}d-M^{-j}k)-\sum_{d\in R}\sum_{\ell\in\mathbb{Z}^2} d_\ell\,\varphi_2^j(x+M^{-(j+1)}d-M^{-j}\ell)=0.$$

Applying the Fourier transform yields

$$c_s(z)\hat{\varphi}_1^j(\xi)=d_s(z)\hat{\varphi}_2^j(\xi), \tag{7.25}$$

where $c_s(z)$ and $d_s(z)$ are the corresponding symbols, see Chapter 2. Next, we insert the definition of $\hat{\varphi}_i^j, i=1,2$, into (7.25). We obtain

$$c_s(z)=d_s(z)\left(\frac{e^{\lambda_{-j,\gamma_1}-iM^{-j}\xi\cdot x^{\gamma_1}}-1}{\lambda_{-j,\gamma_1}-iM^{-j}\xi\cdot x^{\gamma_1}}\right)\left(\frac{e^{\lambda_{-j,\gamma_2}-iM^{-j}\xi\cdot x^{\gamma_2}}-1}{\lambda_{-j,\gamma_2}-iM^{-j}\xi\cdot x^{\gamma_2}}\right).$$

In case $j\in 2\mathbb{Z}$, we have $M^{-j}=2^{-j/2}I$. Consequently, we deduce

$$c_s(z)=d_s(z)\left(\frac{e^{\lambda_{-j,\gamma_1}-i2^{-j/2}\xi_1}-1}{\lambda_{-j,\gamma_1}-i2^{-j/2}\xi_1}\right)\left(\frac{e^{\lambda_{-j,\gamma_2}-i2^{-j/2}(\xi_1+\xi_2)}-1}{\lambda_{-j,\gamma_2}-i2^{-j/2}(\xi_1+\xi_2)}\right). \tag{7.26}$$

In case $j\in 2\mathbb{N}+1$, the matrix $M^{-j}$ equals $2^{(-j+1)/2}M^{-1}$ and thus, we have

$$c_s(z)=d_s(z)\left(\frac{e^{\lambda_{-j,\gamma_1}-i2^{(-j-1)/2}(\xi_1+\xi_2)}-1}{\lambda_{-j,\gamma_1}-i2^{(-j-1)/2}(\xi_1+\xi_2)}\right)\left(\frac{e^{\lambda_{-j,\gamma_2}-i2^{(-j+1)/2}\xi_1}-1}{\lambda_{-j,\gamma_2}-i2^{(-j+1)/2}\xi_1}\right). \tag{7.27}$$

In case $j\in -2\mathbb{N}-1$, the matrix $M^{-j}$ equals $2^{(j-1)/2}M$ and hence, we obtain

$$c_s(z)=d_s(z)\left(\frac{e^{\lambda_{-j,\gamma_1}-i2^{(j-1)/2}(\xi_1+\xi_2)}-1}{\lambda_{-j,\gamma_1}-i2^{(j-1)/2}(\xi_1+\xi_2)}\right)\left(\frac{e^{\lambda_{-j,\gamma_2}-i2^{(j+1)/2}\xi_1}-1}{\lambda_{-j,\gamma_2}-i2^{(j+1)/2}\xi_1}\right). \tag{7.28}$$

Since it holds $\mathcal{F}(\varphi_i^j(x+M^{-j-1}d-M^{-j}k))(\xi)=e_{M^{-j-1}d-M^{-j}k}(\xi)\hat{\varphi}_i^j(\xi)$, each symbol $c_s(z),d_s(z)$ is $2\pi(M^T)^{j+1}\mathbb{Z}^2$-periodic. We conclude that the left-hand sides in (7.26), (7.27) and (7.28) are periodic in contrast to the right-hand sides. This is a contradiction. □

In summary, we have shown that every space $S_j(\Phi_j)$, $j\in\mathbb{Z}$, has the required properties such that we can apply our construction procedure. Hence, it causes no problems if we focus on the spaces $S_0(\Phi_0)$ and $S_1(\Phi_1)$. Theorem 7.1.10 yields that

$$\mathscr{W}_c:=\left(\left([\hat{g}_1,\hat{g}_1]^2[\hat{g}_2,\hat{g}_2]\hat{\varphi}_{i,d}^0-[\hat{\varphi}_{i,d}^0,\hat{g}_1][\hat{g}_1,\hat{g}_1][\hat{g}_2,\hat{g}_2]\hat{g}_1-[\hat{\varphi}_{i,d}^0,\hat{g}_2][\hat{g}_1,\hat{g}_1]^2\hat{g}_2\right)^\vee\right)_{\substack{i=1,2\\ d\in R'}}$$

provides a compactly supported basis for $W_0$. Finally, we have to check whether the intersection of the spaces $S_j$ is trivial for $\lambda_1^*=0$ and $\lambda_2^*\in\mathbb{R}\setminus\{0\}$.

**Theorem 7.1.15.**
Let $\{S_j\}_{j\in\mathbb{Z}}$ be defined as in (7.22) with

$$M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and let $(x^\gamma)_{\gamma\in\Gamma}$ consist of the vectors

$$x^{\gamma_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad x^{\gamma_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

each appearing $p$ times, $p \in \mathbb{N} \setminus \{0\}$. In addition, set

$$\lambda_j = \begin{cases} (2^{j/2}\lambda_1^*, 2^{j/2}\lambda_2^*), & j \in 2\mathbb{Z}, \\ (2^{(j-1)/2}\lambda_2^*, 2^{(j+1)/2}\lambda_1^*), & j \in 2\mathbb{Z}+1. \end{cases}$$

Then $Y = \bigcap_{j\in\mathbb{Z}} S_j$ is trivial if $\operatorname{Re}\lambda_1^* = 0$ or $\operatorname{Re}\lambda_2^* = 0$.

*Proof.*
Let $f \in Y, f \neq 0$. It is sufficient to prove

$$\bigcap_{j\in 2\mathbb{Z}} S_j = \{0\}.$$

Therefore, we assume throughout this proof that $j \in 2\mathbb{Z}$. Furthermore, we define the function $G_p$ by

$$\widehat{G}_p(\xi) := \left(\frac{1}{\lambda_1^* - i\xi\cdot x^{\gamma_1}}\right)^p \left(\frac{1}{\lambda_2^* - i\xi\cdot x^{\gamma_2}}\right)^p, \quad p \in \mathbb{N}\setminus\{0\}, \xi \in \mathbb{R}^2.$$

Since $f \in S_{-j}$, it can be written as a linear combination of the $M^j\mathbb{Z}^2$-translates of the generators $\varphi_1^{-j} = C_{\lambda_j,p}(M^{-j}\cdot)$ and $\varphi_2^{-j} = C_{\lambda_j,p+1}(M^{-j}\cdot)$. Next, we verify that the quotient $2^{j(p-1)}\hat{\varphi}_1^{-j}/\widehat{G}_p$ is $2\pi M^{-j}\mathbb{Z}^2$-periodic. For $(k_1,k_2)^T \in \mathbb{Z}^2$ the nominator has the form

$$\begin{aligned} &2^{j(p-1)}\hat{\varphi}_1^{-j}(\xi + 2\pi M^{-j}k) \\ &= 2^{j(p-1)}|-2|^j\widehat{C_{\lambda_j,p}}(M^j(\xi + 2\pi M^{-j}k)) \\ &= 2^{pj/2}2^{pj/2}\left(\frac{e^{2^{j/2}\lambda_1^* - i(2^{j/2}\xi_1 + 2\pi k_1)} - 1}{2^{j/2}\lambda_1^* - i(2^{j/2}\xi_1 + 2\pi k_1)}\right)^p \left(\frac{e^{2^{j/2}\lambda_2^* - i(2^{j/2}(\xi_1+\xi_2) + 2\pi(k_1+k_2))} - 1}{2^{j/2}\lambda_2^* - i(2^{j/2}(\xi_1+\xi_2) + 2\pi(k_1+k_2))}\right)^p \end{aligned}$$

and the denominator has the form

$$\begin{aligned} &\widehat{G}_p(\xi + 2\pi M^{-j}k) \\ &\quad = \left(\lambda_1^* - i(\xi_1 + 2\pi 2^{-j/2}k_1)\right)^{-p}\left(\lambda_2^* - i(\xi_1 + \xi_2 + 2\pi 2^{-j/2}(k_1+k_2))\right)^{-p}. \end{aligned}$$

Consequently, we obtain

$$\begin{aligned}\frac{2^{j(p-1)}\hat{\varphi}_1^{-j}(\xi+2\pi M^{-j}k)}{\widehat{G}_p(\xi+2\pi M^{-j}k)} &= \left(e^{2^{j/2}\lambda_1^*-i2^{j/2}\xi_1}-1\right)^p\left(e^{2^{j/2}\lambda_2^*-i(2^{j/2}(\xi_1+\xi_2))}-1\right)^p\\ &= \frac{2^{j(p-1)}|-2|^j\widehat{C_{\lambda_j,p}}(M^j\xi)}{\widehat{G}_p(\xi)}\\ &= \frac{2^{j(p-1)}\hat{\varphi}_1^{-j}(\xi)}{\widehat{G}_p(\xi)}.\end{aligned}$$

The same holds true for the quotient $2^{jp}\hat{\varphi}_2^{-j}/\widehat{G}_{p+1}$. We observe that $2^{j(p-1)}\hat{\varphi}_1^{-j}/\widehat{G}_p$ and $2^{jp}\hat{\varphi}_2^{-j}/\widehat{G}_{p+1}$ are trigonometric polynomials. Since the spaces $S_{-j}$ are linear subspaces we also know that $g \in S_{-j}$ if and only if $\tilde{g} := 2^{j(p-1)}2^{jp}g$ is in $S_{-j}$. Then (4.18) yields for every function $\tilde{g} \in S_{-j}$ with $j \in 2\mathbb{Z}$ the representation

$$\begin{aligned}\hat{\tilde{g}} &= 2^{j(p-1)}2^{jp}\hat{g}\\ &= 2^{j(p-1)}2^{jp}\sum_{i=1}^{2}\tau_i^{-j}\hat{\varphi}_i^{-j}\\ &= 2^{j(p-1)}2^{jp}\tau_1^{-j}\frac{\hat{\varphi}_1^{-j}}{\widehat{G}_p}\widehat{G}_p + 2^{j(p-1)}2^{jp}\tau_2^{-j}\frac{\hat{\varphi}_2^{-j}}{\widehat{G}_{p+1}}\widehat{G}_{p+1}\\ &= \tilde{\tau}_1^{-j}\widehat{G}_p + \tilde{\tau}_2^{-j}\widehat{G}_{p+1},\end{aligned}$$

where $\tau_1^{-j}, \tau_2^{-j}$ and hence,

$$\tilde{\tau}_1^{-j} := (2^{j(p-1)}2^{jp}\tau_1^{-j}\hat{\varphi}_1^{-j})/\widehat{G}_p, \quad \tilde{\tau}_2^{-j} := (2^{j(p-1)}2^{jp}\tau_2^{-j}\hat{\varphi}_2^{-j})/\widehat{G}_{p+1}$$

are $2\pi 2^{-j/2}\mathbb{Z}^2$-periodic. Due to the nestedness of the spaces $S_j$, we further deduce that

$$\hat{\tilde{g}} = \tilde{\tau}_1^{-j}\widehat{G}_p + \tilde{\tau}_2^{-j}\widehat{G}_{p+1} = \widehat{G}_p\left(\tilde{\tau}_1^{-j} + \tilde{\tau}_2^{-j}\widehat{G}_1\right) = \widehat{G}_p\left(\tilde{\tau}_1^{-j+2} + \tau_2^{-j+2}\widehat{G}_1\right),$$

where $\widehat{G}_p \neq 0$ almost everywhere. Hence, we obtain

$$\left(\tilde{\tau}_1^{-j+2} - \tilde{\tau}_1^{-j}\right) + \left(\tilde{\tau}_2^{-j+2} - \tilde{\tau}_2^{-j}\right)\widehat{G}_1 = 0.$$

This is equivalent to

$$\left(\tilde{\tau}_1^{-j+2} - \tilde{\tau}_1^{-j}\right) = -\left(\tilde{\tau}_2^{-j+2} - \tilde{\tau}_2^{-j}\right)\widehat{G}_1. \tag{7.29}$$

The difference $\tilde{\tau}_i^{-j+2} - \tilde{\tau}_i^{-j}$ is $4\pi 2^{-j/2}\mathbb{Z}^n$-periodic for $i = 1, 2$ and $\widehat{G}_1$ is not periodic. Consequently, the left hand side in (7.29) is periodic in contrast to the right hand side. It follows that

$$\tilde{\tau}_1^{-j+2} - \tilde{\tau}_1^{-j} = 0$$

and

$$\tilde{\tau}_2^{-j+2} - \tilde{\tau}_2^{-j} = 0.$$

This implies that all $\tilde{\tau}_1^{-j}$ agree almost everywhere with one measurable function $\tau_1$ and that all $\tilde{\tau}_2^{-j}$ agree almost everywhere with one measurable function $\tau_2$. Moreover, we deduce that these two functions $\tau_1, \tau_2$ are invariant under all $2\pi 2^{-j/2}\mathbb{Z}^2$-shifts for $j = 2\mathbb{Z}$. We observe that $2^{-j/2}\mathbb{Z}^2$ contains the dyadic points which are dense in $\mathbb{R}^2$. By Lemma 6.1.24 and the choice $\alpha = 2\pi$, the functions $\tau_1$ and $\tau_2$ are constant almost everywhere. Hence, the Fourier transform of every function in $Y$ can be represented by a linear combination of $\widehat{G}_p$ and $\widehat{G}_{p+1}$. Therefore, $Y$ is trivial if and only if $G_p \notin Y$ and $G_{p+1} \notin Y$ for $p \in \mathbb{N} \setminus \{0\}$. With the $2\pi 2^{-j/2}\mathbb{Z}^2$-periodicity of $\tilde{\tau}_1^{-j}$ and $\tilde{\tau}_2^{-j}$, Corollary 4.2.2 yields that $G_p$ and $G_{p+1}$ are elements of $S_{-j}$ if and only if $G_p$ and $G_{p+1}$ are elements of $L_2(\mathbb{R}^2)$. Due to the proof of Theorem 6.1.25, we already know that if $\operatorname{Re}\lambda_1^* = 0$ or $\operatorname{Re}\lambda_2^* = 0$, the function $G_p, p \in \mathbb{N} \setminus \{0\}$, cannot be a function in $L_2(\mathbb{R}^2)$. Hence, the proof is complete. □

Theorem 7.1.15 yields that we have found a compactly supported basis for

$$L_2(\mathbb{R}^2) = \bigoplus_{j=-\infty}^{\infty} W_j. \tag{7.30}$$

If we consider the matrix

$$M := \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

and the directions

$$x^{\gamma_1} := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad x^{\gamma_2} := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

with cardinality $p \in \mathbb{N}\setminus\{0\}$ each, we can use similar arguments to obtain a compactly supported basis for (7.30). For more details we refer the reader to Section 6.1.3.

# Conclusion and Outlook

In this thesis we studied the construction of multivariate pre-wavelets and pre-multiwavelets with focus on compact support and stability. Our investigations were motivated by the questions:

(Q1) What are minimal requirements such that a construction of multivariate pre-wavelet and pre-multiwavelet bases is still possible?

(Q2) How can we minimize the number of mother wavelets?

In this final chapter we discuss the results presented in this thesis as well as ideas for future research.

Our main results concerning the construction of compactly supported pre-wavelets and pre-multiwavelets can be found in Chapter 6 and Chapter 7. In Section 6.1 we constructed a compactly supported basis for the space $L_2(\mathbb{R}^n)$ under the assumption that every space $S_j, j \in \mathbb{Z}$, is generated by a single function, see Theorem 6.1.11. This process consisted of four steps:

(S1) Proving that it holds $S_0(\phi) = S_1(\eta)$ with $\phi = \{\varphi(\cdot + M^{-1}d), d \in R\}$.

(S2) Proving that the Gramian matrix $G(\phi)$ is regular in order to show that $\phi$ provides a basis for $S_0(\phi)$.

(S3) Finding an explicit representation of $\mathcal{P}_{S_0(\varphi)}$ in order to obtain an explicit representation of the pre-wavelets.

(S4) Modifying the representation found in step (S3) such that we obtain compactly supported pre-wavelets.

We faced the biggest challenge in step (S2) where we had to find an appropriate generalization of the operator $Q_v$ from [30]. In Section 7.1 we generalized the process above for finitely many generators. Again, we encountered one of the main difficulties in step (S2). In case of a single generator, we computed the determinant of the Gramian matrix by calculating the eigenvalues. It is not clear how we can calculate the eigenvalues of the Gramian matrix $G(\widetilde{\Phi})$ with

$$\widetilde{\Phi} = \{\varphi_i(\cdot + M^{-1}d), d \in R, i = 1, \ldots, N\}$$

of size $mN \times mN, N > 1$. Therefore, we derived in Theorem 7.1.4 a lower bound for each eigenvalue which led to condition (7.1). This condition ensures that $G(\widetilde{\Phi})$ is regular. An alternative approach is presented in Theorem 7.1.5 for $N = 2$ and $N = 3$ generators. Here, we managed to calculate $\det G(\widetilde{\Phi})$ and thus, we derived two estimates which have to be fulfilled in order to ensure $\det G(\widetilde{\Phi}) > 0$ almost everywhere. Depending on the choice of generators both approaches yield conditions which may be difficult to check. Moreover, condition (7.1) is a sufficient but not necessary condition. Thus, we added the assumption that $\widetilde{\Phi}$ provides a basis for $S_0(\widetilde{\Phi})$. Consequently, there is space for improvement. More sophisticated approaches to calculate the exact eigenvalues of $G(\widetilde{\Phi})$ might lead to less restrictive assumptions. Besides that step (S4) was far from being trivial. In Theorem 7.1.10 we obtained a compactly supported basis in case each space $S_j$ has one, two or three generators. We reckon that formula (7.13) remains valid in case each space $S_j$ has finitely many generators. However, it is an open problem to verify this conjecture.

Our main results concerning the construction of stable pre-wavelets can be found in Section 6.2. In particular, we were concerned with the construction of an $L_2$-stable basis for every space $W_j, j \in \mathbb{Z}$, see Theorem 6.2.2. If the intersection of the spaces $\{S_j\}_{j\in\mathbb{Z}}$ is trivial, then the union of these bases yields a wavelet basis for the space $L_2(\mathbb{R}^n)$. This union is $L_2$-stable as well if the stability constants can be chosen independently of $j$. This is always possible if we consider a stationary multiresolution analysis because dilation doesn't change the stability constants. In contrast, this is a non-trivial problem when working with a non-stationary multiresolution analysis. Therefore, we suggest to make this a subject for future research. The work of C. de Boor, R. A. DeVore and A. Ron could serve as a starting point. In [30] they discussed stability with respect to exponential box splines with $M = 2$ and $N = 1$. Their investigations are based on a wavelet $\psi$ in $W_0$ whose stability constants, i.e., the positive constants $C_1, C_2$ in $C_1(\psi) \leq [\hat{\psi}, \hat{\psi}] \leq C_2(\psi)$ almost everywhere, are related to the stability constants of the generators. Hence, the first step to adapt their results to our setting would be to prove that $[\hat{w}_d, \hat{w}_d]$ can be represented with the help of the bracket products $[\hat{\eta}, \hat{\eta}]_1$ and $[\hat{\varphi}, \hat{\varphi}]$.

Taking the discussion above into consideration, we draw the following conclusions:

(Q1) It seems not possible to reduce the initial assumptions made in the definition of a generalized multiresolution analysis any further.

(Q2) We managed to incorporate a general dilation matrix into our construction procedures. Hence, we can minimize the number of required mother wavelets by choosing a dilation matrix with determinant $\pm 2$.

Finally, let us discuss two more suggestions for future research. In the introduction of this thesis we have stated that a function $f \in L_2(\mathbb{R}^n)$ has a representation of the form

$$f(x) = \sum_{i \in I} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} c_{i,j,k} \psi_{i,j,k}(x).$$

One of the most important applications of such wavelet expansions is that they can be used to characterize function spaces. Besov spaces which can be classified as smoothness spaces are of special interest. It has been shown that weighted sequence norms of the wavelet expansion coefficients $\{c_{i,j,k}\}_{i \in I, j \in \mathbb{Z}, k \in \mathbb{Z}^n}$ are equivalent to Besov norms. Hence, these coefficients can be used to verify if a function $f$ belongs to a certain Besov space or not. For more information see, e.g., Section 2 in [24]. In literature, we find characterizations of Besov spaces if $N = 1$ and if $M$ is a diagonal, anisotropic dilation matrix or an arbitrary dilation matrix with real entries, see [39, 38, 14]. An open problem is how Besov spaces can be characterized in case of multigenerators and an arbitrary dilation matrix.

Another promising field of research are frames, see, e.g., [17]. If a sequence of functions $\{f_j\}_{j=1}^{\infty}$ is a frame for a Hilbert space $\mathcal{H}$, then $\overline{\operatorname{span}}\{f_j\}_{j=1}^{\infty} = \mathcal{H}$. In contrast to wavelets, frames do not provide a basis for $\mathcal{H}$. Hence, there is redundancy which leads to more flexibility in the construction process. A special class of frames are wavelet frames which can be constructed via frame multiresolution analysis. This kind of multiresolution analysis is a natural generalization of the multiresolution analysis defined in Definition 0.3. Indeed, the only difference to Definition 0.3 is that the integer translates of the scaling function in condition (M5) are assumed to provide a frame for $S_0$. If there is more than one scaling function, we obtain multiwavelet frames. Consequently, the following generalization of question (Q1) arises naturally: What are minimal requirements such that a construction of wavelet frames and multiwavelet frames is still possible? Up to the author's knowledge, this question has not been answered yet.

# Bibliography


[1] M. D. Adams, *Multiresolution Signal and Geometry Processing: Filter Banks, Wavelets and Subdivision*, University of Victoria, Victoria, 2013.

[2] C. D. Aliprantis and K. C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, 2nd ed., Springer, Berlin, Heidelberg, 1999.

[3] B. K. Alpert, *Wavelets and Other Bases for Fast Numerical Linear Algebra*, vol. 2, pp. 181–216, In: Wavelets: A Tutorial in Theory and Applications (C. K. Chui ed.), Academic Press, 1992.

[4] R. Ashino and M. Kametani, *A Lemma on Matrices and a Construction of Multi-Wavelets*, Math. Japan **45** (1997), no. 2, 267–287.

[5] G. Bachman, L. Narici, and E. Beckenstein, *Fourier and Wavelet Analysis*, Springer, New York, 2000.

[6] L. W. Baggett, H. A. Medina, and K. D. Merrill, *Generalized multi-resolution analyses and a construction procedure for all wavelet sets in* $\mathbb{R}^n$, J. Fourier Anal. Appl. **5** (1999), no. 6, 563–573.

[7] G. Battle, *A block spin construction of ondelettes. Part I: Lemarié functions*, Comm. Math. Phys. **110** (1987), 601–615.

[8] ———, *Cardinal Spline Interpolation and the Block Spin Construction of Wavelets*, In: Wavelets: A Tutorial in Theory and Applications (C. K. Chui, ed.), vol. 2, Academic Press, 1992, pp. 73–90.

[9] H. Bauer, *Measure and Integration Theory*, De Gruyter, Berlin, 2001.

[10] S. K. Berberian, *Fundamentals of Real Analysis*, Springer, New York, 1999.

[11] Å. Björck, *Numerical Methods in Matrix Computations*, Springer, Cham, 2015.

[12] M. Bownik, *The Structure of Shift-Invariant Subspaces of* $L_2(\mathbb{R}^n)$, J. Funct. Anal. **177** (2000), no. 2, 282–309.

[13] ———, *Riesz wavelets and generalized multiresolution analyses*, Appl. Comput. Harmon. Anal. **14** (2003), no. 3, 181–194.

[14] ———, *Atomic and molecular decompositions of anisotropic Besov spaces*, Math. Z. **250** (2005), 539–571.

[15] R. L. Burden and J. D. Faires, *Numerical Analysis*, 9th ed., Brooks/Coles, Boston, 2011.

[16] C. A. Cabrelli and M. L. Gordillo, *Existence of multiwavelets in* $\mathbb{R}^n$, Proc. Amer. Math. Soc. **130** (2002), no. 5, 1413–1424.

[17] O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser, 2016.

[18] C. K. Chui, *An Introduction to Wavelets*, Academic Press, 1992.

[19] C. K. Chui and C. Li, *A General Framework of Multivariate Wavelets with Duals*, Appl. Comput. Harmon. Anal. **1** (1994), no. 4, 368–390.

[20] C. K. Chui and J.-Z. Wang, *A general framework of compactly supported splines and wavelets*, J. Approx. Theory **71** (1992), no. 3, 263–304.

[21] A. Cohen and I. Daubechies, *Non-separable bidimensional wavelet bases*, Rev. Mat. Iberoam. **9** (1993), no. 1, 51–137.

[22] J. B. Conway, *A Course in Functional Analysis*, 2nd ed., Graduate Texts in Mathematics, vol. 96, Springer, New York, 2007.

[23] S. Dahlke, W. Dahmen, I. Weinreich, and E. Schmitt, *Multiresolution and Wavelets on* $S^2$ *and* $S^3$, Numer. Funct. Anal. Optim. **16** (1995), no. 1-2, 19–41.

[24] S. Dahlke and R. A. DeVore, *Besov Regularity for Elliptic Boundary Value Problems*, Comm. Partial Differential Equations **22** (1997), no. 1-2, 1–16.

[25] S. Dahlke, V. Latour, and M. Neeb, *Generalized Cardinal B-Splines: Stability, Linear Independence, and Appropriate Scaling Matrices*, Constr. Approx. **13** (1997), 29–56.

[26] W. Dahmen and C. A. Micchelli, *Translates of Multivariate Splines*, Linear Algebra Appl. **52/53** (1983), 217–234.

[27] ———, *On multivariate E-Splines*, Adv. Math. **76** (1989), no. 1, 33–93.

[28] I. Daubechies, *Orthonormal bases of wavelets with compact support*, Comm. Pure Appl. Math. **41** (1987), 909–996.

[29] ———, *Ten Lectures on Wavelets*, CBMS-NSF Regional Conference Series in Applied Math., vol. 61, SIAM, Philadelphia, 1992.

[30] C. de Boor, R. A. DeVore, and A. Ron, *On the Construction of Multivariate (Pre)Wavelets*, Constr. Approx. **9** (1993), 123–166.

[31] ———, *Approximation from shift-invariant subspaces of* $L_2(\mathbb{R}^d)$, Trans. Amer. Math. Soc. **341** (1994), no. 2, 787–806.

[32] ———, *The Structure of Finitely Generated Shift-Invariant Spaces in* $L_2(\mathbb{R}^d)$, J. Funct. Anal. **119** (1994), no. 1, 37–78.

[33] C. de Boor, K. Höllig, and S. Riemenschneider, *Box Splines*, Appl. Math. Sci., vol. 98, Springer, New York, 1993.

[34] F. Deutsch, *Best Approximation in Inner Product Spaces*, CMS Books in Mathematics, Springer, New York, 2001.

[35] R. A. DeVore and B. J. Lucier, *Wavelets*, Acta Numer. **1** (1992), 1–56.

[36] M. Dobrowolski, *Angewandte Funktionalanalysis: Funktionalanalysis, Sobolev-Räume und elliptische Differentialgleichungen*, 2nd ed., Springer, Berlin, Heidelberg, 2010.

[37] H. Fischer and H. Kaul, *Mathematik für Physiker Band 2: Gewöhnliche und partielle Differentialgleichungen, mathematische Grundlagen der Quantenmechanik*, 4th ed., Springer Spektrum, Wiesbaden, 2014.

[38] G. Garrigós, R. Hochmuth, and A. Tabacco, *Wavelet Characterizations for Anisotropic Besov Spaces with* $0 < p < 1$, Proc. Edinb. Math. Soc. (2) **47** (2004), no. 3, 573–595.

[39] G. Garrigós and A. Tabacco, *Wavelet Decompositions of Anisotropic Besov Spaces*, Math. Nachr. **239-240** (2002), no. 1, 80–102.

[40] J. S. Geronimo, D. P. Hardin, and P. R. Massopust, *Fractal Functions and Wavelet Expansions Based on Several Scaling Functions*, J. Approx. Theory **78** (1994), no. 3, 373–401.

[41] T. N. T. Goodman, S. L. Lee, and W. S. Tang, *Wavelets in wandering subspaces*, Trans. Amer. Math. Soc. **338** (1993), no. 2, 639–654.

[42] K. Gröchenig and W. R. Madych, *Multiresolution Analysis, Haar Bases, and Self-Similar Tilings of* $R^n$, IEEE Trans. Inform. Theory **38** (1992), no. 2, 556–568.

[43] A. Haar, *Zur Theorie der orthogonalen Funktionensysteme*, Math. Ann. **69** (1910), 331–371.

[44] V. L. Hansen, *Functional Analysis: Entering Hilbert Space*, World Scientific, Singapore, 2006.

[45] C. Heil, *Introduction to Real Analysis*, Grad. Texts in Math., vol. 280, Springer, Cham, 2019.

[46] L. Hervé, *Multi-Resolution Analysis of Multiplicity d: Applications to Dyadic Interpolation*, Appl. Comput. Harmon. Anal. **1** (1994), no. 4, 299–315.

[47] E. Isaacson and H. B. Keller, *Analysis of Numerical Methods*, John Wiley & Sons, New York, 1966.

[48] K. Jetter and G. Plonka, *A survey on $L_2$-approximation order from shift-invariant spaces*, pp. 73–111, In: Multivariate Approximation and Applications (N. Dyn, D. Leviatan, D. Levin, A. Pinkus, ed.), 2001.

[49] R.-Q. Jia and C. A. Micchelli, *Using the Refinement Equations for the Construction of Pre-Wavelets II: Powers of Two*, Curves and Surfaces (L. L. Schumaker P.-J. Laurent, A. Le Méhauté, ed.), Academic Press, 1991, pp. 209–246.

[50] R. Q. Jia and Z. Shen, *Multiresolution and Wavelets*, Proc. Edinb. Math. Soc. (2) **37** (1994), no. 2, 271–300.

[51] K. S. Kazarian, *On the structure of finitely generated shift-invariant subspaces*, arXiv: 1605.02456 (2016).

[52] F. Keinert, *Wavelets and Multiwavelets*, Chapman & Hall/CRC, 2003.

[53] K. Koch, *Interpolating Scaling Vectors and Multiwavelets in $\mathbb{R}^d$*, Ph.D. thesis, Philipps-Universität Marburg, 2006.

[54] M.-J. Lai, *Construction of multivariate compactly supported prewavelets in $L_2$ space and pre-Riesz bases in Sobolev spaces*, J. Approx. Theory **142** (2006), no. 2, 83–115.

[55] D. Langemann and J. Prestin, *Multivariate periodic wavelet analysis*, Appl. Comput. Harmon. Anal. **28** (2010), no. 1, 46–66.

[56] B. Luong, *Fourier Analysis on Finite Abelian Groups*, Birkhäuser, Boston, 2009.

[57] S. Mallat, *Multiresolution approximations and wavelet orthonormal bases of $L_2(\mathbb{R})$*, Trans. Amer. Math. Soc. **315** (1989), no. 1, 69–87.

[58] ———, *A wavelet tour of signal processing*, Academic Press, 1998.

[59] K. D. Merrill, *Generalized Multiresolution Analyses*, Birkhäuser, Cham, 2018.

[60] Y. Meyer, *Principe d'incertitude, bases hilbertiennes et algèbres d'opérateurs*, Séminaire Bourbaki **662** (1985/86), 209–223.

[61] ——, *Ondelettes, fonctions splines et analyses graduées*, Rend. Semin. Mat. Univ. Politec. Torino **45** (1987), 1–42.

[62] ——, *Ondelettes et opérateurs I: Ondelettes*, Hermann, Paris, 1990.

[63] C. A. Micchelli, *Using the refinement equation for the construction of pre-wavelets*, Numer. Algorithms **1** (1991), 75–116.

[64] I. Ya. Novikov, V. Yu. Protasov, and M. A. Skopina, *Wavelet Theory*, vol. 239, American Mathematical Society, Providence, 2011.

[65] G. Plonka, D. Potts, G. Steidl, and M. Tasche, *Numerical Fourier Analysis*, Appl. Numer. Harmon. Anal., Birkhäuser, Cham, 2018.

[66] G. Plonka and V. Strela, *Construction of Multiscaling Functions with Approximation and Symmetry*, SIAM J. Math. Anal. **29** (1998), no. 2, 481–510.

[67] ——, *From Wavelets to Multiwavelets*, pp. 1–25, In: Mathematical Methods for Curves and Surfaces II (M. Daehlen, T. Lyche, L. L. Schumaker, eds.), 1998.

[68] D. Quillen, *Projective Modules over Polynomial Rings*, Invent. Math. **36** (1976), 167–171.

[69] A. Ron, *Exponential Box Splines*, Constr. Approx. **4** (1988), 357–378.

[70] ——, *Linear independence of the translates of an exponential box spline*, Rocky Mountain J. Math. **22** (1992), no. 1, 331–351.

[71] R. Schaback and H. Wendland, *Numerische Mathematik*, 5th ed., Springer, Berlin, Heidelberg, 2005.

[72] M. Skopina, A. Krivoshein, and V. Protasov, *Multivariate Wavelet Frames*, Springer, Singapore, 2016.

[73] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton, New Jersey, 1971.

[74] G. Strang and T. Nguyen, *Wavelets and Filter Banks*, Wellesley-Cambridge Press, Wellesley, 1997.

[75] J.-O. Strömberg, *A modified Franklin system and higher-order spline systems on* $\mathbb{R}^n$ *as unconditional bases for Hardy spaces*, Conference on Harmonic Analysis in Honor of A. Zygmund, 1981, pp. 475–494.

[76] M. Sugiura, *Unitary Representations and Harmonic Analysis - An Introduction*, Kodansha Scientific Books, Tokyo, 1975.

[77] A. A. Suslin, *Projective modules over a polynomial ring are free*, Sov. Math. Dokl. **17** (1976), 1160–1164.

[78] S. M. Tabatabaie, K. Kazarian, R. A. K. Gol, and S. Jokar, *The Structure of Finitely Generated Shift-Invariant Subspaces on Locally Compact Abelian Groups*, Mediterr. J. Math. **18** (2021), no. 1, 27.

[79] J. Weidmann, *Lineare Operatoren in Hilberträumen - Teil I Grundlagen*, B. G. Teubner, Wiesbaden, 2000.

[80] F. Weisz, *Lebesgue Points and Summability of Higher Dimensional Fourier Series*, Birkhäuser, Cham, 2021.

[81] P. Wojtaszczyk, *A Mathematical Introduction to Wavelets*, Cambridge University Press, Cambridge, 1997.

[82] R. A. Zalik, *Bases of translates and multiresolution analyses*, Appl. Comput. Harmon. Anal. **24** (2008), no. 1, 41–57.

[83] E. Zeidler, *Nonlinear Functional Analysis and its Applications II/A: Linear Monotone Operators*, Springer, New York, 1990.