A novel approach to the cohomology of symplectic quotients

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Abstract

We develop a novel approach to the topology of singular symplectic quotients by extending Sjamaar's complex of differential forms to the complex of resolution differential forms. The motivation for this is to extend Sjamaar's complex in a way which makes the definition of a Kirwan map possible. In his theory this is not possible due to the singularities of any connection form at the fixed points of the action. Thus, the idea is to resolve the group action by blow-ups. Doing this using real blow-ups results in a locally free action on the whole manifold but also in difficult exceptional bundles. Carrying out the construction using symplectic blow-ups the exceptional bundles turn out to be more controllable. This approach then allows to define a Kirwan map, whose surjectivity we study in case that the fixed point set has vanishing cohomology in odd degrees. It turns out that this map is surjective in even degrees while it is not surjective in odd degrees.

Zusammenfassung

Wir entwickeln einen neuartigen Zugang zur Topologie singulärer symplektischer Quotienten, indem wir Sjamaars Komplex von Differentialformen zu dem Komplex der Auflösungsformen erweitern. Die Motivation hierfür ist, dass wir Sjamaars Komplex in einer solchen Weise erweitern wollen, dass die Definition einer Kirwan-Abbildung möglich wird. In seiner Theorie ist dies nicht möglich aufgrund von Singularitäten einer jeden Zusammenhangsform in Fixpunkten der Wirkung. Deshalb ist die Idee, die Gruppenwirkung mit Hilfe von Aufblasungen aufzulösen. Der Versuch mittels reeller Aufblasungen resultiert zwar in einer lokal freien Wirkung auf der Mannigfaltigkeit, allerdings auch in schwierigen exzeptionellen Bündeln. Symplektische Aufblasungen führen hingegen zu exzeptionellen Bündeln, die besser zu kontrollieren sind. Dieser Zugang erlaubt uns dann auch die Definition einer Kirwan-Abbildung, deren Surjektivität wir im Falle untersuchen, dass die Komponenten der Fixpunktmenge verschwindende ungerade Kohomologie haben. Es zeigt sich, dass diese Abbildung surjektiv in geraden Graden, allerdings im allgemeinen nicht surjektiv in ungeraden Graden ist.

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Contents

Chapter 1

Introduction

Since Kirwan's seminal thesis [37] was published in 1984, a branch of symplectic geometry focused on studying the so called Kirwan map. This map relates the equivariant cohomology of a Hamiltonian G-manifold to the de Rham cohomology of the corresponding symplectic quotient and is defined as follows. Let (M, σ) be a compact connected symplectic manifold which carries a Hamiltonian symmetry of a compact Lie group G with momentum map

$$\mathrm{J}\colon M \longrightarrow \mathfrak{g}^*,$$

which means that for all X in the Lie algebra of G we have the equality $i_{\overline{X}}\sigma = d\mathrm{J}^X$, where $\overline{X}$ is the fundamental vector field associated to X, and J is a smooth map which is equivariant with respect to the coadjoint action of G on $\mathfrak{g}^*$. It is the equation $i_{\overline{X}}\sigma = d\mathrm{J}^X$ which ties Hamiltonian actions to equivariant cohomology as explained in [3, Section 6] and [14, Section 5], a tie which is moreover strengthened by the Kirwan map. To introduce it, we assume that $0 \in \mathfrak{g}^*$ is a regular value of the momentum map, so that $\mathrm{J}^{-1}(0)$ is a closed submanifold of M invariant under the S^1-action and we form the symplectic quotient $(\mathscr{M}_0 := \mathrm{J}^{-1}(0)/S^1, \sigma_0)$ which is a symplectic orbifold according to a classical theorem by Marsden-Weinstein and Meyer, see [42, Lemma 3.9]. Important to this theorem is that the regularity assumption on the reduction parameter $0 \in \mathfrak{g}^*$ assures finiteness of all stabilizers of the G-action on $\mathrm{J}^{-1}(0)$, which means that the action restricted to the zero level set is locally free. We may therefore look at the canonical projection

$$\pi\colon \mathrm{J}^{-1}(0) \longrightarrow \mathscr{M}_0$$

from the zero level set to the symplectic quotient $\mathscr{M}_0$ which is an orbifold and the induced isomorphism

$$\pi^*\colon \Omega(\mathscr{M}_0) \longrightarrow \Omega_{\mathrm{bas}G}(\mathrm{J}^{-1}(0)),$$

where $\Omega(\mathscr{M}_0)$ is the complex of differential forms on $\mathscr{M}_0$ and $\Omega_{\mathrm{bas}\,G}(\mathrm{J}^{-1}(0))$ is the complex of G-basic differential forms on $\mathrm{J}^{-1}(0)$ where a differential form is called G-basic if it is both G-invariant and G-horizontal. Moreover, since the action of G on $\mathrm{J}^{-1}(0)$ is locally free, the natural map from basic differential forms $\Omega_{\mathrm{bas}\,G}(\mathrm{J}^{-1}(0))$ on $\mathrm{J}^{-1}(0)$ to equivariant differential forms $C_G(\mathrm{J}^{-1}(0))$ on $\mathrm{J}^{-1}(0)$ given as

$$\begin{aligned}\Omega_{\mathrm{bas}\,G}(\mathrm{J}^{-1}(0)) &\longrightarrow C_G(\mathrm{J}^{-1}(0)) = \left(S(\mathfrak{g}^*) \otimes \Omega\left(\mathrm{J}^{-1}(0)\right)\right)^G \\ \omega &\longmapsto 1 \otimes \omega\end{aligned}$$

induces an isomorphism in cohomology, whose homotopy inverse can be made explicit after choosing a connection form $\alpha \in \Omega^1(\mathrm{J}^{-1}(0), \mathfrak{g})$ with curvature $\Omega := d\alpha + \frac{1}{2}[\alpha, \alpha] \in \Omega^2(\mathrm{J}^{-1}(0), \mathfrak{g})$ by the so-called Cartan map. This map is defined as

$$\begin{aligned}\mathrm{Car}\colon C_G(\mathrm{J}^{-1}(0)) &\longrightarrow \Omega_{\mathrm{bas}\,G}(\mathrm{J}^{-1}(0))\\ \omega &\longmapsto P_{\mathrm{hor}}(\omega(\Omega))\end{aligned}$$

on the level of differential forms, where the curvature components are plugged in ω for the Lie algebra variables and P_{hor} denotes horizontal projection as in [48, Theorem 5.17]. If we now let $\iota\colon \mathrm{J}^{-1}(0) \to M$ denote the inclusion, we may form the composition

$$\kappa\colon H^*_G(M) \xrightarrow{\iota^*} H^*_G(\mathrm{J}^{-1}(0)) \xrightarrow{\mathrm{Car}} H^*_{\mathrm{bas}\,G}(\mathrm{J}^{-1}(0)) \xrightarrow{(\pi^*)^{-1}} H^*(\mathscr{M}_0)$$

which is known as *Kirwan map*. Here Car and $(\pi^*)^{-1}$ are isomorphisms and it was Kirwan's major discovery that ι^* is surjective, which she proved by using Morse-theoretic arguments which do not rely on the regularity of $0 \in \mathbb{R}$. This regularity assumption is only needed to build a bridge from the basic cohomology of the zero level set to the cohomology of the symplectic quotient.

Starting with the landmark paper of Lerman-Sjamaar [56] an interest in singular symplectic spaces, namely stratified symplectic spaces, emerged. Lerman-Sjamaar showed that the stratification of a symplectic manifold M acted upon in a Hamiltonian way by a Lie group G into orbit types induces a stratification of the symplectic quotient $\mathscr{M}_0$ and that this is the case even if $0 \in \mathfrak{g}^*$ is not a regular value. Furthermore, they proved that those induced strata are themselves symplectic manifolds again. Since the topology of singular spaces is difficult to handle due to an absence of for example Poincaré duality, the focus in the exploration of the topology of singular symplectic quotients shifted to studying the intersection cohomology of singular symplectic reductions. Intersection cohomology was introduced by Goresky-McPherson in 1971 and overcomes the difficulties of the usual singular cohomology of singular spaces. Even though a lot of work has been done on the intersection cohomology of singular symplectic quotients, see Chapter 6, the (non-intersection) cohomology of singular symplectic quotients remains somewhat mysterious. Nevertheless, one work focussing on the singular cohomology with real coefficients of singular symplectic quotients was carried out by Sjamaar in 2005 [57], giving a de Rham model for the real cohomology of singular symplectic quotients which works as follows: Consider G and set $\mathrm{J}^{-1}(0)^\top$ as the points in the zero level set whose orbit type is just the identity subgroup $\{1\} \subset G$, called the top stratum of $\mathrm{J}^{-1}(0)$. Assume that this set is open and dense in the zero level set. Moreover, define the top stratum of the symplectic quotient as

$$\mathscr{M}_0^\top := \frac{\mathrm{J}^{-1}(0)^\top}{G}.$$

The top stratum $\mathscr{M}_0^\top$ is a manifold and sits inside the diagram

$$\begin{tikzcd} M & \mathrm{J}^{-1}(0)^\top \arrow[l, "\iota_\top"'] \arrow[d, "\pi_\top"] \\ & \mathscr{M}_0^\top. \end{tikzcd}$$

Sjamaar defines the complex of differential forms on the symplectic quotient $\mathscr{M}_0$ as

$$\Omega(\mathscr{M}_0) := \left\{\omega \in \Omega(\mathscr{M}_0^\top) \,\middle|\, \exists\, \eta \in \Omega(M)\colon \pi_\top^*\omega = \iota_\top^*\eta\right\}$$

and proves a de Rham theorem which states that the cohomology of the complex of differential forms, where the differential is the usual exterior derivative of differential forms, computes the cohomology of $\mathscr{M}_0$ with real coefficients, i.e.

$$H^*(\mathscr{M}_0;\mathbb{R}) \cong H^*(\Omega(\mathscr{M}_0), d).$$

It was this de Rham model of the real cohomology of the symplectic quotient $\mathscr{M}_0$ which inspired us to define a singular Kirwan map using differential forms. In case that $G = S^1$, a first naive attempt would be to simply define the singular Kirwan map as follows. Take an equivariant differential form $\sum_I \omega_I \cdot x^I$ on M, restrict it to the top stratum of the zero level set $\mathrm{J}^{-1}(0)^\top$, apply the Cartan map with respect to a connection form on the top stratum of the zero level set, where the action of S^1 is free and obtain a basic differential form on the top stratum of the zero level set of the form

$$\sum_I \omega_I|_{\mathrm{J}^{-1}(0)^\top} \wedge \Omega^I - \alpha \wedge \sum_I (i_{\overline{X}} \omega_I|_{\mathrm{J}^{-1}(0)^\top}) \wedge \Omega^I,$$

which induces a differential form on the top stratum of the symplectic quotient $\mathscr{M}_0^\top$. While this seems to be a nice expression for a singular Kirwan map it is no differential form on $\mathscr{M}_0$ in Sjamaar's sense, the reason beeing that the presence of fixed points in $\mathrm{J}^{-1}(0)$, in which the fundamental vector fields of the S^1 action vanish, forces the connection form α, which is dual to the fundamental vector fields, to be unbounded near the fixed points. Hence, there cannot be a global 1-form on M extending α and our above expression cannot be a differential form on $\mathscr{M}_0$ in Sjamaar's sense. To overcome this obstacle, the idea is to find a space related to M on which the pull-back of the connection form of $\mathrm{J}^{-1}(0)^\top$ admits an extension to the whole space. Our first idea was to consider a resolution of the group action on M by performing real blow-ups of M along the isotropy components F in M as in [13, Section 2.9]. This procedure leads to a G-space $\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M)$ on which the G-action is locally free and a smooth G-equivariant map

$$\beta^{\mathbb{R}} \colon \mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M) \longrightarrow M,$$

which is a diffeomorphism away from the isotropy components in M. Since the action on $\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M)$ is now locally free, there is a connection form on $\mathrm{J}^{-1}(0)^\top$ whose pullback under $\beta^{\mathbb{R}}|_{(\beta^{\mathbb{R}})^{-1}(\mathrm{J}^{-1}(0)^\top)}$ extends to the blown-up space $\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M)$ and thus makes this space attractive for studying a singular Kirwan map. We therefore extend Sjamaar's notion of differential forms on $\mathscr{M}_0$ by considering the diagram

$$\begin{tikzcd}
\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M) \arrow[d, "\beta^{\mathbb{R}}"] & (\beta^{\mathbb{R}})^{-1}\left(\mathrm{J}^{-1}(0)^\top\right) \arrow[l, hook', "\iota^{\mathbb{R}}_\top"'] \arrow[d, "\beta^{\mathbb{R}}_\top"] \\
M & \mathrm{J}^{-1}(0)^\top \arrow[l, hook', "\iota_\top"] \arrow[d, "\pi_\top"] \\
& \mathscr{M}_0^\top,
\end{tikzcd}$$

and introducing the space

$$\widehat{\Omega}(\mathscr{M}_0) := \left\{ \omega_0 \in \Omega(\mathscr{M}_0^\top) \;\middle|\; \exists \widetilde{\eta} \in \Omega(\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M)) \colon (\beta^{\mathbb{R}}_\top)^* \pi_\top^* \omega_0 = (\iota^{\mathbb{R}}_\top)^* \widetilde{\eta} \right\}$$

of real resolution differential forms on $\mathscr{M}_0$. Since every differential form in Sjamaar's sense is a real resolution differential form, we may form the short exact sequence

$$0 \longrightarrow \Omega(\mathscr{M}_0) \longrightarrow \widehat{\Omega}(\mathscr{M}_0) \longrightarrow C(\mathscr{M}_0) \longrightarrow 0,$$

where $C(\mathscr{M}_0)$ denotes the cokernel complex of the inclusion $\Omega(\mathscr{M}_0) \to \widehat{\Omega}(\mathscr{M}_0)$. This short exact sequence of complexes induces a long exact cohomology sequence in a standard way of the form

$$\begin{tikzcd}
0 \arrow[r] & H^0(\Omega(\mathscr{M}_0),d) \arrow[r] & H^0(\widehat{\Omega}(\mathscr{M}_0),d) \arrow[r] & H^0(C(\mathscr{M}_0),d) \arrow[dll, "\delta"] & \\
& H^1(\Omega(\mathscr{M}_0),d) \arrow[r] & H^1(\widehat{\Omega}(\mathscr{M}_0),d) \arrow[r] & H^1(C(\mathscr{M}_0),d) \arrow[dll, "\delta"] & \\
& H^2(\Omega(\mathscr{M}_0),d) \arrow[r] & H^2(\widehat{\Omega}(\mathscr{M}_0),d) \arrow[r] & H^2(C(\mathscr{M}_0),d) \arrow[r] & \ldots,
\end{tikzcd}$$

where δ denotes the boundary operator. When dealing with the blow-up $\beta^{\mathbb{R}}$ we may form the strict transform of $\mathrm{J}^{-1}(0)$ defined as $\widehat{C}_{\mathbb{R}} := \overline{(\beta^{\mathbb{R}})^{-1}\mathrm{J}^{-1}(0)^{\top}}$, its quotient $\widehat{\mathscr{M}_0} := \widehat{C}_{\mathbb{R}}/G$ by the induced G-action, and consider the induced map

$$\beta_0^{\mathbb{R}} \colon \widehat{\mathscr{M}_0} \longrightarrow \mathscr{M}_0.$$

Since this is again a diffeomorphism away from the exceptional loci $\widehat{F} := (\beta_0^{\mathbb{R}})^{-1}(F)$, it seems plausible that the cohomology of the cokernel complex $C(\mathscr{M}_0)$ is connected to the cohomology of these exceptional loci. We therefore study these spaces using a Gysin sequence for locally free circle actions and determine the cohomology of the fibres of the bundles $\widehat{F} \to F$. Since we cannot apply Leray-Hirsch's theorem to the bundles

$$\widehat{F} \longrightarrow F$$

they seem to be not suitable to work with, forcing us to end our investigation of the real blow-up method at this point.

We therefore consider symplectic blow-ups to resolve the S^1 action instead. These blow-ups seem on the one hand less attractive since they introduce invariant complex projective spaces sitting inside the blown-up space $\mathrm{Bl}_{\mathrm{G}}^{\mathbb{C}}(M)$, which make it impossible for the blown-up action to be locally free. On the other hand they seem more natural in the context of Hamiltonian group actions since the blown-up space is again Hamiltonian with respect to the lifted action with momentum map $\widetilde{\mathrm{J}}$. We proceed analougously to the real situation by performing symplectic blow-ups along all fixed point components to obtain a Hamiltonian manifold $(\mathrm{Bl}_{\mathrm{G}}^{\mathbb{C}}(M), \widetilde{\sigma})$ for which zero is now a regular value of the momentum map $\widetilde{\mathrm{J}}$ and its zero level is the strict transform of $\mathrm{J}^{-1}(0)$. Now, we consider the map

$$\beta_0^{\mathbb{C}} \colon \widetilde{\mathscr{M}_0} := \frac{\widetilde{\mathrm{J}}^{-1}(0)}{G} \longrightarrow \mathscr{M}_0$$

where the space $\widetilde{\mathscr{M}_0}$ is called the partial desingularization of $\mathscr{M}_0$. This space was already considerd by Kirwan [38] in the GIT context and was later brought to the general Hamiltonian setting by Meinrenken-Sjamaar [46]. We find that the fibres of the exceptional bundles of the partial desingularization $\beta_0^{\mathbb{C}}$ are themselves regular symplectic reductions of complex projective spaces and their cohomology was determined by Kalkman [32, Section 5]. Now the situation is much more well-behaved than in the real analogue due to the existence of the symplectic class of $\widetilde{\mathscr{M}_0}$ and the easier topology of the exceptional bundles; in fact Leray-Hirsch's theorem now applies to those bundles. Again, we define a complex of resolution forms, but now with respect to the symplectically blown-up space by considering the diagram

$$
\begin{tikzcd}
\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M) \arrow[d, "\beta^{\mathbb{C}}"] & (\beta^{\mathbb{C}})^{-1}\left(\mathrm{J}^{-1}(0)^{\top}\right) \arrow[l, hook', "\iota^{\mathbb{C}}_{\top}"'] \arrow[d, "\beta^{\mathbb{C}}_{\top}"] \\
M & \mathrm{J}^{-1}(0)^{\top} \arrow[l, hook', "\iota_{\top}"] \arrow[d, "\pi_{\top}"] \\
 & \mathscr{M}_0^{\top}
\end{tikzcd}
$$

and making the definition

$$\widetilde{\Omega}(\mathscr{M}_0) := \left\{ \omega_0 \in \Omega(\mathscr{M}_0^{\top}) \;\middle|\; \exists \widetilde{\eta} \in \Omega(\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M))\colon (\beta^{\mathbb{C}}_{\top})^* \pi^*_{\top} \omega_0 = (\iota^{\mathbb{C}}_{\top})^* \widetilde{\eta} \right\}.$$

We then consider the resulting short exact sequence of cochain complexes

$$0 \longrightarrow \Omega(\mathscr{M}_0) \longrightarrow \widetilde{\Omega}(\mathscr{M}_0) \longrightarrow C(\mathscr{M}_0) \longrightarrow 0,$$

and the induced long exact sequence in cohomology. This sequence is of the form

$$
\begin{tikzcd}
0 \arrow[r] & H^0(\Omega(\mathscr{M}_0), d) \arrow[r] & H^0(\widetilde{\Omega}(\mathscr{M}_0), d) \arrow[r] & H^0(C(\mathscr{M}_0), d) \arrow[dll, out=-10, in=170, "\delta"'] & \\
 & H^1(\Omega(\mathscr{M}_0), d) \arrow[r] & H^1(\widetilde{\Omega}(\mathscr{M}_0), d) \arrow[r] & H^1(C(\mathscr{M}_0), d) \arrow[dll, out=-10, in=170, "\delta"'] & \\
 & H^2(\Omega(\mathscr{M}_0), d) \arrow[r] & H^2(\widetilde{\Omega}(\mathscr{M}_0), d) \arrow[r] & H^2(C(\mathscr{M}_0), d) \arrow[r] & \ldots,
\end{tikzcd}
$$

which enables us to interpret all occuring terms geometrically. In fact, Sjamaar's de Rham theorem shows that

$$H^*(\Omega(\mathscr{M}_0), d) \cong H^*(\mathscr{M}_0; \mathbb{R}),$$

we prove the isomorphism

$$H^*(\widetilde{\Omega}(\mathscr{M}_0), d) \cong H^*(\widetilde{\mathscr{M}_0})$$

and determine the cohomology of the cokernel complex as

$$H^*(C(\mathscr{M}_0), d) \cong \bigoplus_{F \in \mathcal{F}_0} \operatorname{coker}(H^*(F) \to H^*(\widetilde{F})) \cong \bigoplus_{F \in \mathcal{F}_0} H^*(F) \otimes \frac{\mathbb{R}[\widetilde{\sigma}_0|_{\widetilde{F}}, \widetilde{\Omega}|_{\widetilde{F}}]_{\geq 1}}{I_F},$$

where $\widetilde{\sigma}_0|_{\widetilde{F}}$ and $\widetilde{\Omega}|_{\widetilde{F}}$ are generators of the cohomology of the fibre of the exceptional bundle $\widetilde{F} \to F$ of degree two, I_F is a certain ideal of relations between these generators and $\mathbb{R}[\widetilde{\sigma}_0|_{\widetilde{F}}, \widetilde{\Omega}|_{\widetilde{F}}]_{\geq 1}$ is the set of polynomials in these two generators of degree at least one. $\mathcal{F}_0$ denotes the set of fixed point components contained in the zero level set. This reveals the long exact sequence as being of the form

$$\ldots \longrightarrow H^k(\mathscr{M}_0; \mathbb{R}) \longrightarrow H^k(\widetilde{\mathscr{M}_0}) \longrightarrow \bigoplus_{F \in \mathcal{F}_0} \operatorname{coker}(H^k(F) \to H^k(\widetilde{F})) \longrightarrow H^{k+1}(\mathscr{M}_0; \mathbb{R}) \longrightarrow \ldots.$$

This sequence allows us to detect the case of Hamiltonian circle actions, where the components of the fixed point set have vanishing cohomology in odd degrees. In this case the long exact sequence above splits into short exact sequences and there is a (non-canonical) splitting

$$H^{\mathrm{ev}}\left(\widetilde{\mathscr{M}_0}\right) \cong H^{\mathrm{ev}}\left(\mathscr{M}_0\right) \oplus V$$

of even-degree cohomology groups for some vector space V. Thus there is a projection map $H^{\mathrm{ev}}(\widetilde{\mathscr{M}_0}) \to H^{\mathrm{ev}}(\mathscr{M}_0;\mathbb{R})$. Note that while the cohomology of resolution differential forms of $\mathscr{M}_0$ simply computes the cohomology of the partial desingularization $\widetilde{\mathscr{M}_0}$ the main merit of the concept of resolution differential forms is that it allows a comparison of $H^*\left(\widetilde{\mathscr{M}_0}\right)$ and $H^*\left(\mathscr{M}_0\right)$ and provides the specific framework of differential forms in which one can carry out this comparison. Now, in order to define the singular Kirwan map we introduce the $\mathfrak{g}$-differential graded algebras $\Omega(\mathrm{J}^{-1}(0))$ and $\widetilde{\Omega}(\mathrm{J}^{-1}(0))$ by pulling back the differential forms and resolution differential forms of $\mathscr{M}_0$ to the top stratum of the zero level set $\mathrm{J}^{-1}(0)^{\top}$. This allows us to consider the corresponding basic complexes $\Omega^*\left(\mathrm{J}^{-1}(0)\right)_{\mathrm{bas}\,\mathfrak{g}}$ and $\widetilde{\Omega}^*\left(\mathrm{J}^{-1}(0)\right)_{\mathrm{bas}\,\mathfrak{g}}$ as well as the associated Cartan complexes $C_G(\mathrm{J}^{-1}(0)) := C_G\big(\Omega^*\left(\mathrm{J}^{-1}(0)\right)\big)$ and $C_G\big(\widetilde{\Omega}^*\left(\mathrm{J}^{-1}(0)\right)\big)$ and their cohomologies. Ultimately, we define the singular Kirwan map $\mathcal{K}$, called the *resolution Kirwan map*, as the composition

$$\begin{tikzcd}
H^*_G(M) \arrow[r, "\iota^*_{\top}"] \arrow[drrr, "\mathcal{K}"'] & H^*_G\left(\mathrm{J}^{-1}(0)\right) \arrow[r, "\mathrm{inc}"] & H^*\left(C_G\big(\widetilde{\Omega}^*\left(\mathrm{J}^{-1}(0)\right)\big), d_G\right) \arrow[r, "\mathrm{Car}"] & H^*\left(\widetilde{\Omega}^*\left(\mathrm{J}^{-1}(0)\right)_{\mathrm{bas}\,\mathfrak{g}}, d\right) \arrow[d, "(\pi^*_{\top})^{-1}"] \\
 & & & H^*(\widetilde{\Omega}^*(\mathscr{M}_0), d),
\end{tikzcd}$$

where $H^*_G\left(\mathrm{J}^{-1}(0)\right) = H^*(C_G(\mathrm{J}^{-1}(0)), d_G)$ and inc denotes the map induced by the natural inclusion $C_G(\mathrm{J}^{-1}(0)) \to C_G\big(\widetilde{\Omega}^*\left(\mathrm{J}^{-1}(0)\right)\big)$. We prove that the image of the resolution Kirwan map $\mathcal{K}\colon H^*_G(M) \to H^*(\widetilde{\Omega}^*(\mathscr{M}_0), d)$ contains the image of the natural map $H^*(\mathscr{M}_0;\mathbb{R}) \to H^*(\widetilde{\Omega}^*(\mathscr{M}_0), d)$, which can be seen as a weak form of resolution Kirwan surjectivity. In particular, this implies that in the case where the components of the fixed point set have vanishing cohomology in odd degrees we obtain a surjection

$$H^{\mathrm{ev}}_G(M) \longrightarrow H^{\mathrm{ev}}(\mathscr{M}_0;\mathbb{R}).$$

Notice that this surjectivity is interesting for many applications as the cohomology classes of even degree are particularly interesting for problems concerning integration on the even dimensional spaces M or $\mathscr{M}_0$, such as residue formulas. We furthermore find an example which shows that it is in general not true that there is a surjection

$$H^*_G(M) \longrightarrow H^*(\mathscr{M}_0;\mathbb{R}).$$

We then apply our results to a class of examples, namely to Abelian polygon spaces, where we consider the example of the standard diagonal rotational action of S^1 on the product $S^2\times\ldots\times S^2$ of 2-spheres. In the case of four spheres, we show that the singular symplectic quotient $\mathscr{M}_0$ has non-vanishing third cohomology.

The outline of the thesis is as follows: In Chapter 2, we recall the relevant aspects of Hamiltonian group actions and equivariant cohomology. Chapter 3 deals with the real blow-up construction. Its implications and limitations for studying resolution cohomology are dealt with in Chapter 4. In the following Chapter 5 we develop the partial desingularization via symplectic blow-ups. Our main results are contained in Chapter 6, where we define resolution differential forms and prove 'weak' resolution Kirwan surjectivity. The chapter concludes with a discussion of an important family of examples, the so-called Abelian polygon spaces. Chapter 7 discusses other approaches to the topology of singular symplectic quotients and singular Kirwan maps and some connections to our work. Finally, in Chapter 8 we give a brief outlook on what could be done on this topic in the future.

Throughout the thesis cohomology is considered with real coefficients and we consider effective left actions of compact Lie groups G on compact connected manifolds. Often, we consider a Hamiltonian S^1-action on (M, σ). Throughout we fix an invariant almost complex structure on M and an identification $\mathrm{Lie}(S^1) \cong \mathbb{R}$ where a fixed generator $X \in \mathrm{Lie}(S^1)$ corresponds to $1 \in \mathbb{R}$.

Chapter 2

Symplectic geometry and the Kirwan map

2.1 Hamiltonian group actions and symplectic reduction

In order to introduce our setup and fix notation, let's briefly recapitulate the basic framework of symplectic geometry and Hamiltonian group actions. We omit the proofs since these are excellently covered in the literature. The reader unfamiliar with the material is encouraged to consult [9], [4] and [45].

2.1.1 Symplectic geometry and Hamiltonian group actions

Definition 2.1.1. A differential form $\sigma \in \Omega^2(M)$ on a $2n$-dimensional manifold M is called *symplectic* if

- σ is closed and
- σ is non-degenerate, which means σ^n is nowhere vanishing.

The pair (M, σ) is called a *symplectic manifold*.

Definition 2.1.2. Two symplectic manifolds (M_1, σ_1) and (M_2, σ_2) are called *symplectomorphic* if there is a diffeomorphism $\varphi\colon M_1 \to M_2$ with $\varphi^*\sigma_2 = \sigma_1$.

Example 2.1.3. The wide list of examples of symplectic manifolds contains orientable surfaces, Kähler manifolds, coadjoint orbits of compact Lie groups, cotangent bundles and the Euclidean spaces $\left(\mathbb{R}^{2n}, \sigma = \sum\limits_{k=1}^{n} dx_k \wedge dy_k\right)$, where $x_1, y_1, \ldots, x_n, y_n$ are coordinates on $\mathbb{R}^{2n}$.

Remark 2.1.4. One can produce further examples by taking products of the aforementioned according to the following simple principle: Let (M_1, σ_1) and (M_2, σ_2) be symplectic manifolds. Then their product $(M_1 \times M_2, \mathrm{pr}_1^*\sigma_1 + \mathrm{pr}_2^*\sigma_2)$ is again a symplectic manifold.

A fundamental feature of symplectic geometry is that there are no local invariants of symplectic manifolds. This statement is made precise in the following

Theorem 2.1.5 (Darboux)**.** *Let (M, σ) be a $2n$-dimensional symplectic manifold. Then (M, σ) is locally symplectomorphic to $\left(\mathbb{R}^{2n}, \sum\limits_{k=1}^{n} dx_k \wedge dy_k\right)$.*

We thus have to study symplectic manifolds by global means. One possibility is to impose symmetries on (M, σ) and explore their nature. Let therefore G be a compact connected Lie group acting on a compact connected symplectic manifold (M, σ).

Definition 2.1.6. The action of G on (M, σ) is called *symplectic* if $g^*\sigma = \sigma$ for any $g \in G$, so when the symplectic form σ is G-invariant.

Given such an action, any element $X \in \mathfrak{g}$ induces a vector field $\overline{X} \in \mathcal{X}(M)$, defined as

$$\overline{X}_p := \frac{d}{dt}\Big|_{t=0} \exp(tX) \cdot p,$$

where $p \in M$ and the action is defined by the map $G \times M \to M$ where $(g, p) \mapsto g \cdot p$. Since we assumed the action to be symplectic σ will be invariant under the flow of $\overline{X}$ for all $X \in \mathfrak{g}$ and thus

$$L_{\overline{X}}\sigma = \frac{d}{dt}\Big|_{t=0} \exp(tX)^*\sigma = 0,$$

where L denotes the Lie derivative. By Cartan's formula and the closedness of σ we have

$$0 = L_{\overline{X}}\sigma = d(i_{\overline{X}}\sigma) + i_{\overline{X}}d\sigma = d(i_{\overline{X}}\sigma),$$

so the differential form $i_{\overline{X}}\sigma$ is closed for all $X \in \mathfrak{g}$. As turns out, we want ask for even more, namely that the differential form $i_{\overline{X}}\sigma$ is exact for all $X \in \mathfrak{g}$. This directly leads to

Definition 2.1.7. An action of a Lie group G on a compact symplectic manifold (M, σ) is called *Hamiltonian* if there exists a smooth G-equivariant map $\mathrm{J}\colon M \to \mathfrak{g}^*$, such that for every $X \in \mathfrak{g}$

$$d\mathrm{J}^X = i_{\overline{X}}\sigma,$$

where $\mathrm{J}^X\colon M \to \mathbb{R}$ is defined by $\mathrm{J}^X(p) := \mathrm{J}(p)(X)$ and G acts on $\mathfrak{g}^*$ by the coadjoint action. Such a map J is called *momentum map*. For Abelian Lie groups G an additive constant may be added to J and one still has a momentum map.

Example 2.1.8. The 2-sphere is a sympletic manifold where the symplectic form is the standard volume form. The standard S^1-action given by rotation around the z-axis is Hamiltonian with momentum map

$$\begin{aligned} \mathrm{J}\colon S^2 &\longrightarrow \mathbb{R} \\ (x, y, z) &\longmapsto z. \end{aligned}$$

Example 2.1.9. The standard diagonal action of S^1 on $\mathbb{C}^n$ given as $e^{it} \cdot (z_k) := (e^{it} \cdot z_k)$ is Hamiltonian when we equip $\mathbb{C}^n \cong \mathbb{R}^{2n}$ with the standard symplectic form. A momentum map is

$$\begin{aligned} \mathrm{J}\colon \mathbb{C}^n &\longrightarrow \mathbb{R} \\ (z_k)_k &\longmapsto -\frac{1}{2}\sum_{k=1}^{n} |z_k|^2. \end{aligned}$$

If more generally a compact n-torus $T^n = S^1 \times \ldots \times S^1$ acts on $\mathbb{C}^n$ by $(e^{it_k}) \cdot (z_k) = (e^{it_k} \cdot z_k)$ this action is again Hamiltonian, a momentum map is

$$\begin{aligned} \mathrm{J}\colon \mathbb{C}^n &\longrightarrow \mathbb{R}^n \\ (z_k)_k &\longmapsto -\frac{1}{2}(|z_k|^2)_k. \end{aligned}$$

Example 2.1.10. When we endow complex projective space $\mathbb{CP}^n$ with the Fubini-Study form σ_{FS} the T^n-action $(e^{it_1}, \dots, e^{it_n}) \cdot [z_1 : \dots : z_{n+1}] := [e^{it_1} \cdot z_1 : \dots : e^{it_n} \cdot z_n : z_{n+1}]$ is Hamiltonian with momentum map

$$
\begin{aligned}
\mathrm{J}\colon \mathbb{CP}^n &\longrightarrow \mathbb{R}^n \\
[z_1 : \dots : z_{n+1}] &\longmapsto -\frac{1}{2}\left(\frac{|z_1|^2}{\sum_{i=1}^{n+1} |z_i|^2}, \dots, \frac{|z_n|^2}{\sum_{i=1}^{n+1} |z_i|^2} \right).
\end{aligned}
$$

And for the circle action $e^{it} \cdot [z_1 : \dots : z_{n+1}] := [e^{it} \cdot z_1 : \dots : e^{it} \cdot z_n : z_{n+1}]$ we have the momentum map

$$
\begin{aligned}
\mathrm{J}\colon \mathbb{CP}^n &\longrightarrow \mathbb{R} \\
[z_1 : \dots : z_{n+1}] &\longmapsto -\frac{1}{2} \frac{\sum_{i=1}^{n} |z_i|^2}{\sum_{i=1}^{n+1} |z_i|^2}.
\end{aligned}
$$

Their rich structure on the one hand and local simplicity on the other have made momentum maps a driving force not only in symplectic geometry but also in combinatorics, algebraic geometry and theoretical mechanics where the origins of momentum maps lie. Since we will be mostly concerned with circle actions later on, we will now state the local and convexity properties of momentum maps in restricted generality only for $G = S^1$ or Abelian G. Denote by $\mathcal{F}$ the set of fixed point components of the fixed point set M^G and by $\mathcal{F}_0$ the set of those fixed point components contained in the zero level set. When M is compact both $\mathcal{F}$ and $\mathcal{F}_0$ are finite as explained in [51, Section 2.4.14 (i)]. Full local understanding of an effective Hamiltonian action of S^1 on a compact connected symplectic manifold (M, σ) near a fixed point component $F \in \mathcal{F}$ is provided by the local normal form theorem, [40, Lemma 3.1], [51, Theorem 7.5.5] and [10, Prop. 3.2]:

Lemma 2.1.11. *For each fixed point component $F \in \mathcal{F}$, there exist numbers $\ell_F^+, \ell_F^- \in \mathbb{N}$ such that there is*

1. *a faithful unitary representation $\varrho\colon S^1 \to (S^1)^{\ell_F^+ + \ell_F^-} \subset \mathrm{U}(\ell_F^+ + \ell_F^-)$ with positive weights $\lambda_1, \dots, \lambda_{\ell_F^+}$ and negative weights $\lambda_{\ell_F^+ + 1}, \dots, \lambda_{\ell_F^+ + \ell_F^-}$,*

2. *a principal K_F-bundle P_F over F, where K_F is a subgroup of $\mathrm{U}(\ell_F^+) \times \mathrm{U}(\ell_F^-)$ that commutes with $\varrho(S^1)$,*

such that there is a symplectomorphism Φ from a neighbourhood U of F in M to a neighbourhood U_0 of the zero section in the associated normal bundle $\Sigma_F \cong P_F \times_{K_F} \mathbb{C}^{\ell_F^+ + \ell_F^-} \to F$. This symplectomorphism is equivariant with respect to the circle action on $P_F \times_{K_F} \mathbb{C}^{\ell_F^+ + \ell_F^-}$ which is given by $z \cdot [p, z_1, \dots, z_{\ell_F^+ + \ell_F^-}] := [z \cdot p, z^{\lambda_1} z_1, \dots, z^{\lambda_{\ell_F^+ + \ell_F^-}} z_{\ell_F^+ + \ell_F^-}]$. Moreover Φ pulls back the momentum map J to the map $\mu\colon P_F \times_{K_F} \mathbb{C}^{\ell_F^+ + \ell_F^-} \to \mathbb{R}$ given below; that is $\mathrm{J} \circ \Phi = \mu$, where

$$
\mu\left([p, z_1, \dots, z_{\ell_F^+ + \ell_F^-}]\right) = \frac{1}{2} \sum_{i=1}^{\ell_F^+ + \ell_F^-} \lambda_i |z_i|^2 + J(F).
$$

The numbers $2\ell_F^-$ and $2\ell_F^+$ are called the index and the co-index of F. The tuple $(2\ell_F^-, 2\ell_F^+)$ is called the signature of F.

Remark 2.1.12. From the local normal form one can guess that it is worth it to study momentum maps within Morse-Bott theory. In fact, for a Hamiltonian action of a torus T^n on a compact connected symplectic manifold (M, σ) with momentum map

$$\mathrm{J}\colon M \longrightarrow \mathfrak{t}^*$$

and $X \in \mathfrak{t}$ the component function $\mathrm{J}^X : M \to \mathbb{R}$ is a Morse-Bott function. For further information on this see [50, Theorem 3.52].

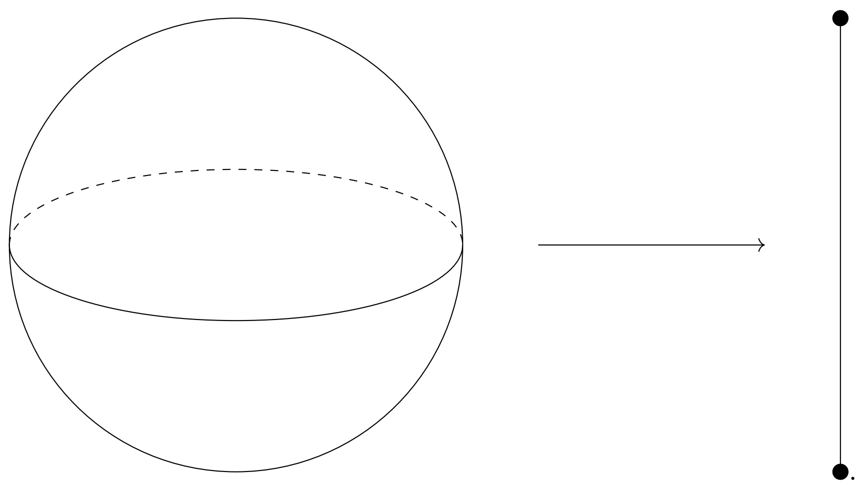
One striking result of the study of momentum maps is the convexity theorem from Atiyah and Guillemin-Sternberg, see [9, Theorem 27.1]. This theorem tells us that the momentum image of a compact connected symplectic manifold (M, σ) is a convex polytope in $\mathfrak{g}^*$. We have therefore now overcome the struggle of the absence of local invariants of symplectic manifolds and found an invariant of Hamiltonian manifolds, the momentum polytope.

Theorem 2.1.13 (Atiyah, Guillemin-Sternberg)**.** *Let T^k act on a compact connected symplectic manifold (M, σ) in a Hamiltonian way with momentum map $\mathrm{J}\colon M \to \mathfrak{t}^*$. Then*

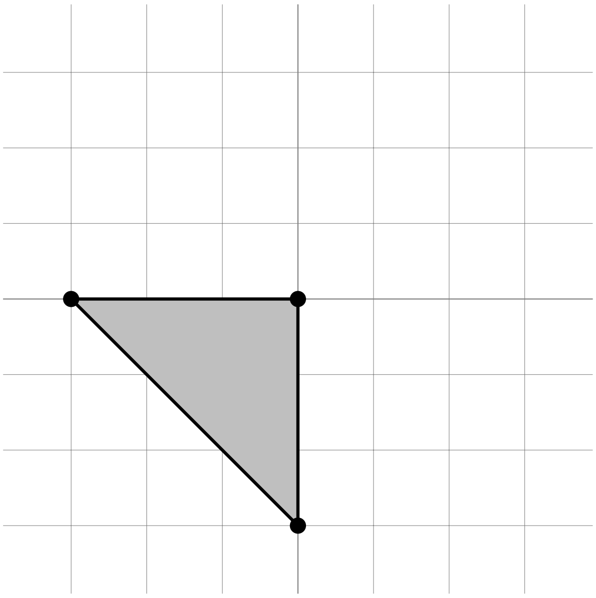
1. *The level sets of J are connected.*
2. *$\mathrm{J}(M)$ is a closed convex polytope in $\mathfrak{t}^*$.*
3. *The image of J is the convex hull of the images of the fixed points under J.*

Example 2.1.14.

1. Let us apply the convexity theorem in Example 2.1.8. The fixed point set of the rotational circle action consists of the north- and southpole. Hence the momentum image is the convex hull of $\mathrm{J}(0,0,1) = 1$ and $\mathrm{J}(0,0,-1) = -1$ inside $\mathbb{R}$, i.e. the closed intervall $[-1, 1]$. We therefore might picture the momentum map as



2. If we consider the T^2 action on $\mathbb{C}\mathrm{P}^2$ from Example 2.1.10, we obtain the fixed point set $\{[1:0:0], [0:1:0], [0:0:1]\}$ and the momentum polytope is the convex hull of $\left\{\left(-\frac{1}{2}, 0\right), \left(0, -\frac{1}{2}\right), (0,0)\right\}$. This convex hull is the following triangle



The relation between Hamiltonian torus actions $T^k \curvearrowright (M, \sigma)$ with momentum map $\mathrm{J} : M \to \mathbb{R}^k$ and their momentum polytopes is particularly fruitful in the case that the dimension of T^k is half the dimension of M, i.e. $k = n$. We call $(M, \sigma, T^n, \mathrm{J})$ a toric symplectic manifold in this case. Only special polytopes arise as momentum polytopes of toric symplectic manifolds, called *Delzant polytopes*, see [9, Definition 28.1], and Delzant proved that toric symplectic manifolds are classified by their momentum polytope, see see [4, Theorem IV.4.20 and Section VII.2], [21, Chapter 1] and [9, Theorem 28.2]. In fact, he proved

Theorem 2.1.15 (Delzant)**.** *The momentum map constitutes a bijective correspondence*

$$\begin{aligned} \{\textit{symplectic toric manifolds}\} &\longrightarrow \{\textit{Delzant polytopes}\} \\ (M, \sigma, T^n, \mathrm{J}) &\longmapsto \mathrm{J}(M). \end{aligned}$$

In particular, two toric symplectic manifolds, whose momentum polytopes coincide, are equivariantly symplectomorphic.

2.1.2 Symplectic reduction

To form a new symplectic manifold from a Hamiltonian action, a cornerstone result of Marsden-Weinstein and Meyer is contained in

Theorem 2.1.16. *Let G act on (M, σ) in a Hamiltonian fashion with momentum map $\mathrm{J}\colon M \to \mathfrak{g}^*$ and suppose that 0 is a regular value of J and G acts freely on $\mathrm{J}^{-1}(0)$. Then the quotient space $\mathscr{M}_0 := \mathrm{J}^{-1}(0)/G$ carries a unique symplectic form $\sigma_0 \in \Omega^2(\mathscr{M}_0)$ satisfying $\pi^*\sigma_0 = \iota^*\sigma$, where π and ι are the natural projection and inclusion sitting in*

$$\begin{tikzcd} \mathrm{J}^{-1}(0) \arrow[r, hook, "\iota"] \arrow[d, "\pi"] & M \\ \mathscr{M}_0. & \end{tikzcd}$$

Definition 2.1.17. In the situation of the aforementioned theorem the symplectic manifold $(\mathscr{M}_0, \sigma_0)$ is called the *symplectic quotient* or *symplectic reduction* of the Hamiltonian action $G \curvearrowright (M, \sigma)$ with momentum map J.

Remark 2.1.18. The assumption on the freeness of the action on the zero level set in the Marsden-Weinstein-Meyer theorem is not too necessary in the sense that even if G does not act freely on $\mathrm{J}^{-1}(0)$ it is guaranteed by the regularity assumption that G acts locally freely on $\mathrm{J}^{-1}(0)$, which means that all stabilizer groups are finite. Thus for 0 a regular value of J the symplectic quotient $\mathscr{M}_0$ is still a reasonable smooth space, namely a symplectic orbifold, see [4, Propositon III.2.20] (For more information on (symplectic) orbifolds we may recommend [42, Part 1], [8, Chapter 4], [11, Section 14.1] and [1]). When we restrict ourselves to $G = S^1$ there is nothing special about the level set $\mathrm{J}^{-1}(0)$ because for any $\varepsilon \in \mathbb{R} \cong \mathfrak{g}^*$ we may form the map $\mathrm{J} - \varepsilon$ which is again a momentum map of the S^1-action on M.

As locally free actions are central to our endeavour, we note the following definition.

Definition 2.1.19. An action of a Lie group G on a manifold M is called *locally free* if the stabilizer G_p is finite for every point $p \in M$.

If we drop the regularity assumption the reduced space $\mathscr{M}_0 := \mathrm{J}^{-1}(0)/G$ will have serious singularities. Nevertheless, Lerman-Sjamaar [56] managed to find a nice symplectic structure on $\mathscr{M}_0$: It is a symplectic stratified space. For the general definitions and properties of stratified spaces see [56, Section 1, Section 6] or [13, Section 2.7]. We can summarize the relevant part of Lerman-Sjamaar's work for us in the following theorem, which is an amalgamation of [56, Theorem 2.1] and [56, Theorem 5.9]

Theorem 2.1.20 (Lerman-Sjamaar)**.** *The stratification of* $M = \coprod_{H<G} M_{(H)}$ *by orbit types induces a stratification*

$$\mathscr{M}_0 = \coprod_{H<G} \frac{\mathrm{J}^{-1}(0) \cap M_{(H)}}{G} =: \coprod_{H<G} (\mathscr{M}_0)_{(H)}$$

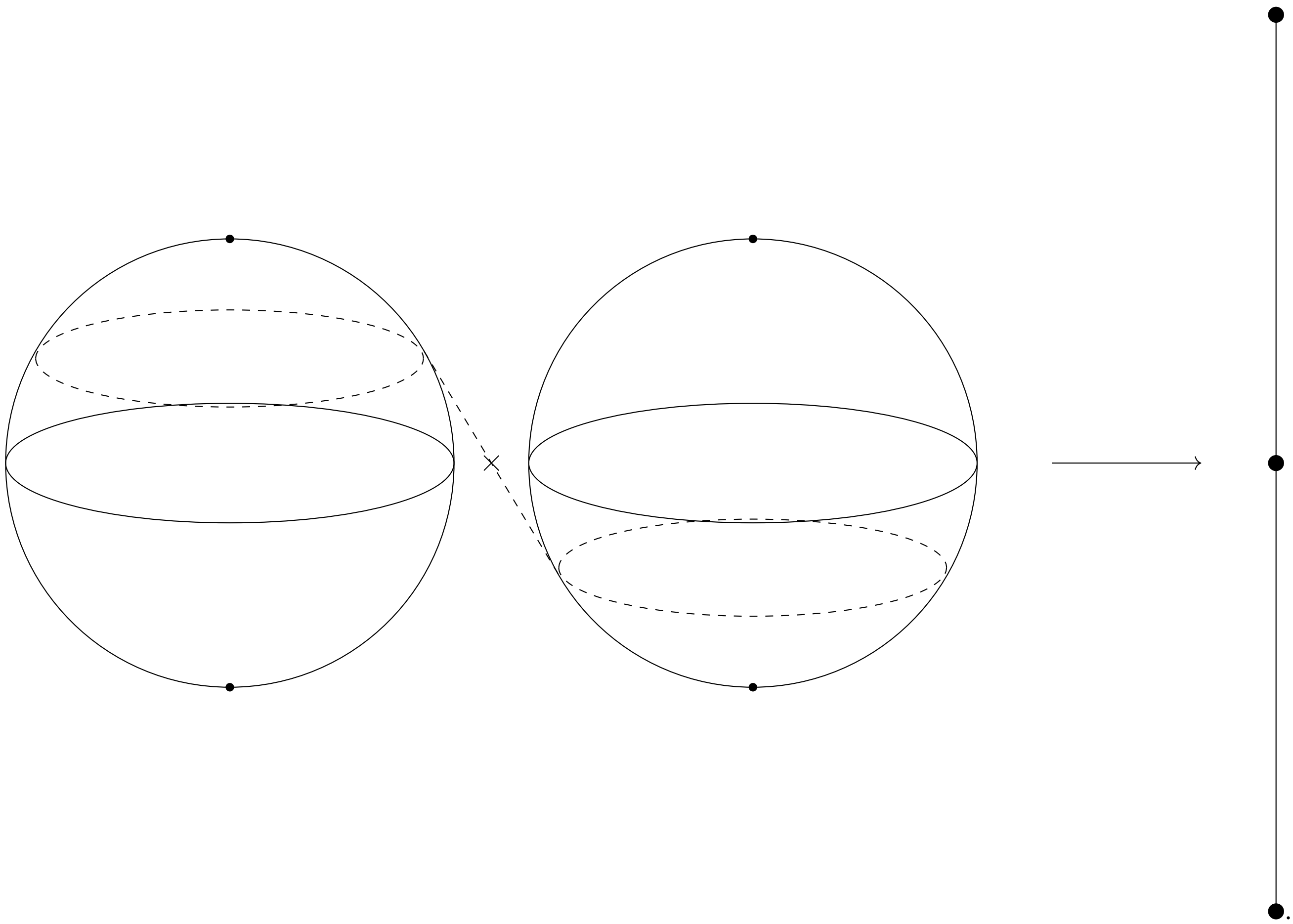
of the symplectic quotient. Each stratum $(\mathscr{M}_0)_{(H)}$ *is a symplectic manifold, where the symplectic form is induced by the restriction of* σ *to* $\mathrm{J}^{-1}(0) \cap M_{(H)}$*. Furthermore there is a unique open, connected and dense stratum* $\mathscr{M}_0^{\top}$*, which we call the regular stratum or top stratum.*

Remark 2.1.21. We could also partition M by infinitesimal orbit types $M_{(\mathfrak{h})}$ which would result in a stratification of $\mathscr{M}_0$ by symplectic orbifolds.

Example 2.1.22. Let us examine one example of this stratification. Consider the product $S^2 \times S^2$ of two spheres. This is a symplectic manifold with symplectic form $\sigma = \mathrm{pr}_1^*(\mathrm{vol}_{S^2}) + \mathrm{pr}_2^*(\mathrm{vol}_{S^2})$ and the diagonal S^2-action induced by rotation around the z-axis is Hamiltonian with momentum map

$$\begin{aligned} \mathrm{J}\colon S^2 \times S^2 &\longrightarrow \mathbb{R} \\ ((x_1, y_1, z_1)(x_2, y_2, z_2)) &\longmapsto z_1 + z_2 \end{aligned}$$

as depicted in



Zero is not a regular value of this as the zero level set $\mathrm{J}^{-1}(0)$ contains the fixed points $((0,0,1),(0,0,-1))$ and $((0,0,-1),(0,0,1))$. Apart from these fixed points the circle action is free on the zero level set and $\mathrm{J}^{-1}(0)$ is homeomorphic to a suspended 2-torus which is indicated partly by the dashed portion of the above picture. Quotienting this by S^1 gives us as symplectic quotient $\mathscr{M}_0$ a suspension of the circle where the singular stratum consists of the two cone points.

In general, the regular stratum $\mathscr{M}_0^\top$ is of the form $\frac{\mathrm{J}^{-1}(0)\cap M_{(H)}}{G} =: \frac{\mathrm{J}^{-1}(0)^\top}{G}$ for some orbit type $M_{(H)}$ and its symplectic form $\sigma_\top$ satisfies $\pi_\top^*\sigma_\top = \iota_\top^*\sigma$, where $\pi_\top$ and $\iota_\top$ are the natural projection and inclusion as in

$$\begin{array}{ccc} \mathrm{J}^{-1}(0)^\top & \xhookrightarrow{\iota_\top} & M \\ \big\downarrow{\scriptstyle \pi_\top} & & \\ \mathscr{M}_0^\top . & & \end{array}$$

This observation can be seen as a first instance of Sjamaar's de Rham theory of symplectic quotients which he developed in [57] and we will review now. Let

$$\Omega^*(M)^G := \left\{ \omega \in \Omega^*(M) \;\middle|\; g^*\omega = \omega \quad \forall g \in G \right\}$$

denote the complex of G-invariant differential forms on M.

Definition 2.1.23. The complex of *differential forms* on $\mathscr{M}_0$ is defined as

$$\Omega(\mathscr{M}_0) := \left\{ \omega_0 \in \Omega(\mathscr{M}_0^\top) \;\middle|\; \exists \eta \in \Omega(M) \colon \pi_\top^*\omega_0 = \iota_\top^*\eta \right\}.$$

By averaging over G, we may replace $\Omega(M)$ by $\Omega(M)^G$ in this definition as mentioned in [57, p. 155]. It is straight forward to relate this complex to a subcomplex of the G-invariant differential forms on M in the following way. Consider the complex

$$\Omega_{\mathrm{J}}(M) := \left\{\omega \in \Omega(M)^G \;\middle|\; \iota_{\top}^*\omega = \omega|_{J^{-1}(0)^{\top}} \text{ horizontal}\right\}$$

and the ideal

$$I_{\mathrm{J}}(M) := \left\{\eta \in \Omega(M)^G \;\middle|\; \iota_{\top}^*\eta = 0\right\}.$$

Proposition 2.1.24. *We have an isomorphism of complexes*

$$\Omega(\mathscr{M}_0) \cong \frac{\Omega_{\mathrm{J}}(M)}{I_{\mathrm{J}}(M)}.$$

Sjamaar further noticed that any differential form on $\mathscr{M}_0$ induces forms on the lower dimensional strata. In [57, Lemma 3.3] he proves that there is a natural restriction map

$$\Omega(\mathscr{M}_0) \longrightarrow \Omega((\mathscr{M}_0)_{(H)}),$$

for any lower dimensional stratum $(\mathscr{M}_0)_{(H)}$ of the reduced space $\mathscr{M}_0$. The significance of the complex of differential forms on $\mathscr{M}_0$ and the main result of [57] is a de Rham theorem for $\Omega(\mathscr{M}_0)$. In [57, Theorem 5.5] he proves

Theorem 2.1.25 (Sjamaar)**.** *The cohomology of the complex of differential forms on $\mathscr{M}_0$ is isomorphic to the singular cohomology of $\mathscr{M}_0$ with real coefficients, that is*

$$H^*(\Omega(\mathscr{M}_0), d) \cong H^*(\mathscr{M}_0; \mathbb{R}).$$

He proved his result for cohomology with values in the locally constant sheaf $\underline{\mathbb{R}}$, but you can also think of singular cohomology with real coefficients since $\mathscr{M}_0$ is (para-)compact and locally contractible.

A very interesting question combining symplectic reduction and the convexity theorem is how the symplectic quotients $\left(\mathscr{M}_\varepsilon := \frac{\mathrm{J}^{-1}(\varepsilon)}{G}, \sigma_\varepsilon\right)$ vary as one varies the reduction parameter $\varepsilon \in \mathfrak{g}^*$. Let us restrict to the case $G = S^1$. Then the momentum polytope is in fact a closed interval in $\mathbb{R} \cong \mathfrak{g}^*$ which we can decompose into the connected components of the subset of regular values inside $\mathrm{J}(M)$. It is common to refer to these connected components of the momentum polytope as *chambers* which are seperated by critical values referred to as *walls*. As long as the reduction parameter ε varies inside one chamber the classical theorem of Duistermaat-Heckman, see [12, Theorem 1.1] or [21, Theorem 2.3] for details, reads as follows.

Theorem 2.1.26 (Duistermaat-Heckman)**.** *If ε and ε' are in the same connected component of the set of regular values of J, then the symplectic quotients $\mathscr{M}_\varepsilon$ and $\mathscr{M}_{\varepsilon'}$ are diffeomorphic and their symplectic classes are related by*

$$[\sigma_\varepsilon] = [\sigma_{\varepsilon'}] + (\varepsilon - \varepsilon') \cdot [c] \in H^2(\mathscr{M}_\varepsilon),$$

where $c \in H^2(\mathscr{M}_\varepsilon)$ is the (common) curvature class of the $\mathscr{M}_\varepsilon$.

Now, if ε crosses a critical value, a so called wall-crossing occurs and the change of the symplectic quotient is controlled by masterly work of Guillemin-Sternberg, see [21, Section 2.3] and [20] for details. In fact the part interesting for us of what they proved is

Theorem 2.1.27 (Guillemin-Sternberg). *As ε crosses a critical value, the diffeomorphism type of the $\mathscr{M}_\varepsilon$ undergoes the change of a blow-up followed by a blow-down.*

Moreover they had the insight, see [20, p. 511 and p. 499] and [21, p. 35 f.], that whenever a critical level set only contains fixed point components of index or co-index 2 the reduction at this critical value is well-behaved in the following sense

Proposition 2.1.28. *If 0 is a critical value of J and every fixed point component $F \subset \mathrm{J}^{-1}(0) \cap M^G$ has signature $(2, 2p)$ or $(2q, 2)$ and S^1 acts semi-freely on M (this means that the isotropy subgroups are either $\{1\}$ or S^1), then the reduced space $\mathscr{M}_0$ is in fact a smooth symplectic manifold, whose symplectic form is induced by the symplectic form of M.*

2.2 Equivariant cohomology and $\mathfrak{g}$-differential graded algebras

Hamiltonian group actions and equivariant cohomology are tightly connected, one such connection being the *Kirwan map*, which is one of the driving forces of this dissertation. In this section we will recall the necessary background from equivariant cohomology, again without proofs, and refer the reader to [18], [23] and [60].

2.2.1 Algebraic topology of group actions

It is very convenient to study the topology of manifolds by de Rham's theorem, see [61, Theorem 5.45].

Theorem 2.2.1 (de Rham). *There is an isomorphism*

$$\Phi_{\mathrm{dR}} \colon H^*(M) \cong H^*(M; \mathbb{R})$$

of graded $\mathbb{R}$-algebras between the (de Rham) cohomology of the complex of differential forms on M and the singular cohomology of M with real coefficients.

When concerned with compact Lie group actions on smooth compact manifolds, one might wonder whether the topology of such actions could be studied by investigating cohomological properties of some suitable complex of differential forms tied to the action. Natural candidates might me the following two complexes

Definition 2.2.2. Let a compact Lie group G act on a smooth compact manifold M. Then the complex of *G-invariant* differential forms is

$$\Omega^*(M)^G := \left\{ \omega \in \Omega^*(M) \;\middle|\; g^*\omega = \omega \quad \forall g \in G \right\}.$$

The complex of *G-basic* differential forms is

$$\Omega^*_{\operatorname{bas} G}(M) := \left\{ \omega \in \Omega^*(M)^G \;\middle|\; i_{\overline{X}}\omega = 0 \quad \forall X \in \mathfrak{g} \right\},$$

so a G-basic differential form is a differential form which is G-invariant and *G-horizontal.*

From these complexes we obtain the G-invariant cohomology algebra $H^*(M)^G$ and the G-basic cohomology algebra $H^*_{\operatorname{bas} G}(M)$. Concerning the invariant cohomology, a straightforward averaging argument, [18, Theorem 2.2], shows that for compact and connected G the inclusion $\big(\Omega^*(M)^G, d\big) \to (\Omega^*(M), d)$ of complexes induces an isomorphism

$$H^*(M)^G \longrightarrow H^*(M).$$

Thus, the invariant cohomology does not contain information about the group action. The G-basic cohomology on the other hand is a bit more interesting. If G acts freely/locally freely on M the quotient space M/G is a manifold/orbifold and the projection

$$\pi\colon M \longrightarrow M/G$$

is a smooth principal G-bundle/G-orbibundle which induces an isomorphism of complexes

$$\pi^*\colon \Omega^*(M/G) \longrightarrow \Omega^*_{\mathrm{bas}\,G}(M).$$

In particular, it defines an isomorphism in cohomology $H^*(M/G) \cong H^*_{\mathrm{bas}\,G}(M)$, see [18, Proposition 2.5], [22, Corollary B.30], [22, Corollary B.31], [22, Corollary B.36], [31, Proposition 3.5.4], [11, Section 16.2], [14, Section 3] and [47, p. 380].
As noted in [18, Remark 2.8] the G-basic cohomology $H^*_{\mathrm{bas}\,G}(M)$ is isomorphic to $H^*(M/G;\mathbb{R})$, the singular cohomology of the quotient space, even if the G-action is singular and the quotient is not a manifold. Since there are many interesting non-trivial group actions with contractible quotient space the basic cohomology is not appropriate to study those. The 'right' definition of equivariant cohomology was found by topologists and we will now shortly describe their construction, also called *Borel construction*. You can find further information on this in [23, Chapter I] and [60, Part I].
For each compact Lie group G there is a principal G-bundle

$$\mathrm{E}G \longrightarrow \mathrm{B}G,$$

called the *universal G-bundle*, which is unique up to homotopy equivalence. The space $\mathrm{E}G$ is contractible and G acts freely on $\mathrm{E}G$. The base space of the universal bundle $\mathrm{B}G$ is called *classifying space*. Now, we can form the product

$$M \times \mathrm{E}G.$$

This product is homotopy-equivalent to M because $\mathrm{E}G$ is contractible and the diagonal G-action on $M \times \mathrm{E}G$ is free since it is free on the second factor. We arrive at

Definition 2.2.3. The space $M_G := M \times_G \mathrm{E}G = (M \times \mathrm{E}G)/G$ is called the *homotopy quotient* of the action of G on M. Its singular cohomology is called the *singular equivariant cohomology* of the G action on M and we set

$$H^*_G(M;\mathbb{R}) := H^*(M_G;\mathbb{R}).$$

Now, this definition, while tremendously useful to topologists, is of limited help in our search for a de Rham model for the study of smooth actions, because $\mathrm{E}G$ is no longer a manifold, but an infinite-dimensional space. It was the seminal work of Cartan in the 1950s that translated the topological simplicity of the homotopy quotient into the algebraic simplicity of a suitable de Rham model.
The action of G on M induces an action of G on $\Omega(M)$ by pull-back. On the other hand G acts on the symmetric algebra $S(\mathfrak{g}^*)$, which consists of polynomials $P\colon \mathfrak{g} \to \mathbb{R}$, by the coadjoint representation, which means that for $\omega \in \Omega(M)$, $X \in \mathfrak{g}$ and $P \in S(\mathfrak{g}^*)$ we have the actions

$$g \cdot \omega := \left(g^{-1}\right)^* \omega \qquad \text{and} \qquad (g \cdot P)(X) := P(\mathrm{Ad}_{g^{-1}}(X)).$$

This defines an action of G on $\Omega(M) \otimes S(\mathfrak{g}^*)$ and we set

Definition 2.2.4. The *Cartan complex* of the G action on M is defined as

$$C_G(M) := \left(\Omega(M) \otimes S(\mathfrak{g}^*)\right)^G.$$

Elements of $C_G(M)$ can be regarded as polynomial maps $\omega\colon \mathfrak{g} \longrightarrow \Omega(M)$ which are G-equivariant in the sense that for $\omega \in C_G(M)$:

$$\left(g^{-1}\right)^* (\omega(X)) = \omega\left(\mathrm{Ad}_g X\right).$$

Now, we define the map $d_G\colon C_G(M) \to C_G(M)$ by

$$d_G(\omega)(X) := d(\omega(X)) - i_{\overline{X}}\omega(X)$$

and give $C_G(M)$ the grading

$$C_G^n(M) := \bigoplus_{k,l\colon k+2l=n} \left(\Omega^k(M) \otimes S^l(\mathfrak{g}^*)\right)^G.$$

This turns d_G into a differential of degree 1.

Remark 2.2.5. When we fix a basis $\{X_i\}_{i=1}^k$ of $\mathfrak{g}$ with dual basis $\{x_i\}$ we can write an element $\omega \in C_G(M)$ as

$$\omega = \sum_I \omega_I \cdot x_I$$

for multi-indices I and differential forms $\omega_I \in \Omega(M)$. In such a basis the differential looks like

$$d_G\omega = d_G\left(\sum_I \omega_I \cdot x_I\right) = \sum_I \left(d\omega_I - \sum_{i=1}^k i_{\overline{X_i}}\omega_I \cdot x_i\right) \cdot x_I.$$

Definition 2.2.6. The *equivariant de Rham cohomology* of the action of G on M is

$$H_G^*(M) := H^*(C_G(M), d_G).$$

That we have reached the goal of our search for a suitable model guarantees the following equivariant version of the de Rham theorem [60, Theorem 21.6 and Appenidx A]:

Theorem 2.2.7 (Cartan)**.** *Let G be a compact connected Lie group acting on a manifold M. Then there is a graded $S(\mathfrak{g}^*)^G$-algebra isomorphism between the singular equivariant cohomology of M and the equivariant de Rham cohomology of M:*

$$\Phi_{dR}^G\colon H_G^*(M;\mathbb{R}) \longrightarrow H_G^*(M).$$

2.2.2 Locally free actions and the Cartan map

Let G act on M. Then for each $X \in \mathfrak{g}$, the Lie derivative with respect to the fundamental vector field $\overline{X}$ is a derivation of degree 0, while the contraction with the fundamental vector field is an anti-derivation of $\Omega(M)$ of degree -1:

$$L_X := L_{\overline{X}}\colon \Omega^k(M) \to \Omega^k(M) \qquad \text{and} \qquad i_X := i_{\overline{X}}\colon \Omega^k(M) \to \Omega^{k-1}(M).$$

These two operators are related to the exterior differential d by Cartan's homotopy formula

$$L_X = i_X \circ d + d \circ i_X.$$

Generalizing this one arrives at the following definition, see [17, Definition 3.1], [15, Definition 6] and [48, Sections 5.2 and 5.6].

Definition 2.2.8. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $A = \bigoplus A_k$ a $\mathbb{Z}$-graded algebra. We call A a $\mathfrak{g}$-*differential graded algebra ($\mathfrak{g}$-dga)* if there is a derivation $d\colon A \to A$ of degree 1, together with derivations $i_X\colon A \to A$ of degree -1 and $L_X\colon A \to A$ of degree 0 for all $X \in \mathfrak{g}$, i_X and L_X depending linearly on X, such that

1. $d^2 = 0$,
2. $i_X^2 = 0$,
3. $[L_X, L_Y] = L_{[X,Y]}$,
4. $[L_X, i_Y] = i_{[X,Y]}$,
5. $[d, L_X] = 0$,
6. $L_X = di_X + i_X d$.

At present, our main example is the de Rham complex $\Omega(M)$ of a manifold M on which a Lie group G with Lie algebra $\mathfrak{g}$ acts. For connected G a differential form $\omega \in \Omega(M)$ is G-invariant if and only if $L_{\overline{X}}\omega = 0$ for all $X \in \mathfrak{g}$. Thus we may state the following definition, which translates the former concepts of invariant, basic and equivariant differential forms to the language of $\mathfrak{g}$-differential graded algebras.

Definition 2.2.9. Let A be a $\mathfrak{g}$-differential graded algebra. From A we may form

- the *invariant* subcomplex $A^G := \{\omega \in A \mid \forall X \in \mathfrak{g} : L_X\omega = 0\}$,
- the *basic* subcomplex $A_{\mathrm{bas}\,\mathfrak{g}} := \{\omega \in A^G \mid \forall X \in \mathfrak{g} : i_X\omega = 0\}$, whose cohomology $H_{\mathrm{bas}\,\mathfrak{g}}(A) := H^*(A_{\mathrm{bas}\,\mathfrak{g}}, d)$ is the *basic cohomology* of A,
- the *Cartan* complex $C_{\mathfrak{g}}(A) := (A \otimes S(\mathfrak{g}^*))^{\mathfrak{g}}$, where $\omega\colon \mathfrak{g} \to A$ is an invariant element if $L_X(\omega) = 0$ for all $X \in \mathfrak{g}$ and the action of $\mathfrak{g}$ on $A \otimes S(\mathfrak{g}^*)$ is defined similarly to Definition 2.2.4. The differential of $C_{\mathfrak{g}}(A)$ is
$$d_{\mathfrak{g}}\omega(X) := d(\omega(X)) - i_X(\omega(X)),$$
the cohomology of $(C_{\mathfrak{g}}(A), d_{\mathfrak{g}})$, denoted by $H_{\mathfrak{g}}(A)$, is the *equivariant cohomology* of A.

For a compact connected Lie group G acting on a manifold M the notions $H_{\mathrm{bas}\,\mathfrak{g}}(A)$ and $H_{\mathrm{bas}\,G}(A)$ as well as $H_{\mathfrak{g}}(A)$ and $H_G(A)$ coincide when A is some subcomplex of the de Rham complex of M. A situation in which basic and equivariant cohomology of A relate very well by the natural map

$$H_{\mathrm{bas}\,\mathfrak{g}}(A) \longrightarrow H_{\mathfrak{g}}(A)$$

induced by sending a basic element $\omega \in A_{\mathrm{bas}\,\mathfrak{g}}$ to $\omega \otimes 1 \in C_{\mathfrak{g}}(A)$ occurs for locally free $\mathfrak{g}$-differential graded algebras. To reformulate the geometric concept of locally free group actions from Definition 2.1.19 algebraically, we have to speak about connection forms of Lie group actions.

Definition 2.2.10. A $\mathfrak{g}$-valued 1-form $\theta \in \Omega^1(M) \otimes \mathfrak{g} =: \Omega^1(M, \mathfrak{g})$ is called *connection form*, if

- for every $g \in G$: $(g^{-1})^*\, \theta = \mathrm{Ad}_{\,g} \circ \theta$,
- for every $X \in \mathfrak{g}$: $i_{\overline{X}}\theta = X$.

The *curvature form* $F^\theta \in \Omega^2(M) \otimes \mathfrak{g}$ of a connection θ is defined by

$$F^\theta(X, Y) = d\theta(X, Y) + \frac{1}{2}\theta([X, Y]),$$

where X, Y are vector fields on M. Sometimes we will denote a connection form by α and its curvature form by Ω.

As explained in [11, Section 16.2] and [23, Section 2.3.4] we have the following characterizations of local freeness.

Proposition 2.2.11. *For the action of G on M the following are equivalent:*

- *The action is locally free.*
- *The fundamental vector field $\overline{X}$ is nowhere vanishing for all $X \in \mathfrak{g}$.*
- *There exists a connection form $\theta \in \Omega^1(M) \otimes \mathfrak{g}$.*

Remark 2.2.12. Let $\{X_i\}_{i=1}^k$ be a basis of $\mathfrak{g}$ with dual basis $\{x_i\}$ of $\mathfrak{g}^*$ and $\theta \in \Omega^1(M) \otimes \mathfrak{g}$ a connection form. Then we may define the real-valued 1-forms

$$\theta_i := x_i \circ \theta,$$

which are simply the components of θ in our chosen basis and we call them *connection elements* in accordance with [23, Definition 2.3.4].
Fix a basis $\{X_1, \dots, X_k\}$ of $\mathfrak{g}$. Then, as in [23, Definition 2.3.4]

Definition 2.2.13. A $\mathfrak{g}$-differential graded algebra A is called locally free if there are invariant elements $\theta_l \in A_1$, $1 \le l \le k$ satisfying

$$i_{X_m}\theta_n = \delta_{m,n} \qquad \text{for all}\, 1 \le m, n \le k.$$

A is moreover called of Type (C) if the θ_l can be chosen such that their span inside A_1 is $\mathfrak{g}$-invariant.

Remark 2.2.14. As mentioned in [23, p. 24] or [17, p. 8] the condition of span$(\{\theta_k\})$ being $\mathfrak{g}$-invariant inside A_1 is automatically fulfilled for actions induced by compact connected Lie groups.

Now, let A be a locally free $\mathfrak{g}$-differential graded algebra with connection $\theta \in A_1 \otimes \mathfrak{g}$. In our usual basis $\{x_i\}$ of $\mathfrak{g}^*$ dual to $\{X_i\} \subset \mathfrak{g}$ we can decompose θ into its components $\theta_i \in A_1$. These connection elements θ_i allow us to define the horizontal projection

$$\begin{aligned} P^\theta_{\text{hor}}\colon A &\to A_{\text{hor}} := \{\omega \in A \mid \forall X \in \mathfrak{g} : i_X\omega = 0\} \\ \omega &\mapsto \prod_i (\omega - \theta_i \cdot i_{X_i}\omega). \end{aligned}$$

The importance of this notion lies in the following theorem, see [48, Theorem 5.17], [23, Chapter 5] and [18, Theorem 5.2]:

Theorem 2.2.15. *Let A be a locally free $\mathfrak{g}$-differential graded algebra with connection $\theta \in A_1 \otimes \mathfrak{g}$. Then the* Cartan map

$$\begin{aligned} Car^\theta\colon C_{\mathfrak{g}}(A) &\longrightarrow A_{\operatorname{bas}\mathfrak{g}} \\ \alpha &\longmapsto P^\theta_{\text{hor}}\left(\alpha(F^\theta)\right) \end{aligned}$$

induces an isomorphism in cohomology, which is inverse to the map $H_{\operatorname{bas}\mathfrak{g}}(A) \to H_{\mathfrak{g}}(A)$ induced by sending a basic element $\omega \in A_{\operatorname{bas}\mathfrak{g}}$ to $\omega \otimes 1 \in C_{\mathfrak{g}}(A)$.

Remark 2.2.16. When we fix a basis $\{X_i\}$ of $\mathfrak{g}$ with dual basis $\{x_i\}$, we can write the curvature as

$$F^\theta = \sum_{i=1}^k F_i^\theta \cdot X_i.$$

An element $\alpha \in C_{\mathfrak{g}}(A)$ is again of the form

$$\alpha = \sum_I \alpha_I \cdot x^I,$$

where $I = (i_1, \dots, i_l)$ is some multi-index and $x^I = x_{i_1} \cdot \dots \cdot x_{i_l}$. Plugging the curvature into α is then

$$\alpha(F^\theta) = \sum_I \alpha_I \cdot \left(F^\theta\right)^I = \sum_{I=(i_1,\dots,i_l)} \alpha_I \cdot F^\theta_{i_1} \cdot \dots \cdot F^\theta_{i_l}.$$

Remark 2.2.17. Since we are mostly interested in S^1 actions we may summarize the aforementioned theory as follows, c.f. [32, p. 43] and [14, Section 3]. Let A be a locally free $\mathrm{Lie}(S^1)$-differential graded algebra. Then the map

$$\begin{aligned} A_{\mathrm{bas}\,\mathrm{Lie}(S^1)} &\longrightarrow C_{\mathrm{Lie}(S^1)}(A) \\ \omega &\longmapsto \omega \otimes 1 \end{aligned}$$

induces an isomorphism in cohomology whose inverse is, after the choice of an connection element α with curvature $\Omega := d\alpha$ and a generator $X \in \mathrm{Lie}(S^1)$, induced by the Cartan map

$$\begin{aligned} \mathrm{Car}\colon C_{\mathrm{Lie}(S^1)}(A) &\longrightarrow A_{\mathrm{bas}\,\mathrm{Lie}(S^1)} \\ \sum_I \omega_I \cdot x^I &\longmapsto \sum_I \omega_I \cdot \Omega^I - \alpha \cdot \sum_I (i_{\overline{X}}\omega_I) \cdot \Omega^I. \end{aligned}$$

2.2.3 The Kirwan map

A major link between Hamiltonian actions and equivariant cohomology is the Kirwan map. Suppose that G acts on a compact symplectic manifold (M, σ) in a Hamiltonian fashion with momentum map

$$\mathrm{J}\colon M \longrightarrow \mathfrak{g}^*.$$

Assume furthermore, that 0 is a regular value of the momentum map. Then the level set $\mathrm{J}^{-1}(0)$ is a closed G-invariant submanifold of M and the symplectic quotient $\mathscr{M}_0$ is naturally a symplectic orbifold. After considering the inclusion $\iota\colon \mathrm{J}^{-1}(0) \to M$, we can define the Kirwan map as the composition

$$\kappa\colon H^*_G(M) \xrightarrow{\iota^*} H^*_G(\mathrm{J}^{-1}(0)) \xrightarrow{\mathrm{Car}} H^*_{\mathrm{bas}\,G}(\mathrm{J}^{-1}(0)) \xrightarrow{(\pi^*)^{-1}} H^*(\mathscr{M}_0). \tag{2.2.1}$$

By the previous considerations we have that Car and π^* are isomorphisms. Generalizing techniques from Morse-Bott theory, Kirwan proved that the inclusion ι induces a surjectiction in equivariant cohomology, proving surjectivity of the Kirwan map

$$\kappa\colon H^*_G(M) \longrightarrow H^*(\mathscr{M}_0).$$

We may summarize Kirwan's reasoning roughly for an actual Morse-Bott function f in the following way, for details see [16, Theorem 7.1], [58, Section 3], [38, Section 5], [7, Theorem 1] and [27]: Consider a G-invariant Morse-Bott function $f\colon M \longrightarrow \mathbb{R}$ and let $\nu_0 \in \mathbb{R}$ be critical value of f. Moreover, let $\{N_i\}$ be the critical components of the critical level set $f^{-1}(\nu_0)$. Denote the negative Disk-/ Sphere-bundle of the components N_i by D^-N_i/S^-N_i, where D^-N_i has rank λ_i. When we now look at a small $\varepsilon > 0$, such that ν_0 is the only critical value of f in the interval $(\nu_0 - \varepsilon, \nu_0 + \varepsilon)$ we consider the sublevel sets $M^{\nu_0 \pm \varepsilon} := f^{-1}((-\infty, \nu_0 \pm \varepsilon])$. The equivariant long exact sequence of the pair $(M^{\nu_0+\varepsilon}, M^{\nu_0-\varepsilon})$ turns out, using excision and the Thom isomorphism on the left side, as

$$\begin{tikzcd}
\dots \arrow[r] & H^*_G(M^{\nu_0+\varepsilon}, M^{\nu_0-\varepsilon}) \arrow[r] \arrow[d] & H^*_G(M^{\nu_0+\varepsilon}) \arrow[r] \arrow[d] & H^*_G(M^{\nu_0-\varepsilon}) \arrow[r] & \dots \\
& \bigoplus_i H^*_G(D^-N_i, S^-N_i) \arrow[r] \arrow[d] & \bigoplus_i H^*_G(D^-N_i) \arrow[d] & & \\
& \bigoplus_i H^{*-\lambda_i}_G(N_i) \arrow[r] & \bigoplus_i H^*_G(N_i) & &
\end{tikzcd}$$

Now the downmost horizontal map $H_G^{*-\lambda_i}(N_i) \to H_G^*(N_i)$ is multiplication with the equivariant Euler class and therefore injective since the Euler class is not a zero divisor. But this means, that the long exact sequence of the pair $(M^{\nu_0+\varepsilon}, M^{\nu_0-\varepsilon})$ breaks down into short exact sequences of the form

$$0 \longrightarrow H_G^*(M^{\nu_0+\varepsilon}, M^{\nu_0-\varepsilon}) \longrightarrow H_G^*(M^{\nu_0+\varepsilon}) \longrightarrow H_G^*(M^{\nu_0-\varepsilon}) \longrightarrow 0.$$

If we now argue like this, starting from the top (remember that M is compact) critical value, the sublevel set at the top will be M itself, while at the bottom we arrive at the minimal level set $f^{-1}(\nu_{\min})$ and obtain a surjection $H_G^*(M) \to H_G^*(f^{-1}(\nu_{\min}))$. Kirwan showed that one can argue in a similar manner for the function $\|\mathrm{J}\|^2$, which is not a Morse-Bott function by [27, Remark 7.2 (4)], but realizes the zero level set $\mathrm{J}^{-1}(0)$ as its minimal level set. Moreover, it turns out that the regularity assumption we made at the beginning of the subsection – 0 should be a regular value of the momentum map – is not necessary to prove that $\iota^*\colon H_G^*(M;\mathbb{R}) \to H_G^*(\mathrm{J}^{-1}(0);\mathbb{R})$ is surjective, but only necessary to connect $H_G^*(\mathrm{J}^{-1}(0))$ and $H^*(\mathscr{M}_0)$. In fact, as in [27, Theorem 8.1], one has

Theorem 2.2.18 (Kirwan)**.** *The inclusion $\iota\colon \mathrm{J}^{-1}(0) \to M$ induces a surjection*

$$\iota^*\colon H_G^*(M;\mathbb{R}) \longrightarrow H_G^*(\mathrm{J}^{-1}(0);\mathbb{R})$$

in equivariant cohomology.

The purpose of this thesis is to overcome this gap between $H_G^*(\mathrm{J}^{-1}(0);\mathbb{R})$ and $H^*(\mathscr{M}_0)$ for circle actions.

Remark 2.2.19. As noted in [18, Example 7.9] a similar argument as above implies that the G-action on M is equivariantly formal for Abelian G. This means, that there is an isomorphism of graded $S(\mathfrak{g}^*)^G$-modules

$$H_G^*(M) \cong S(\mathfrak{g}^*)^G \otimes H^*(M).$$

If we combine this with the Borel localization theorem, [18, Theorem 8.1], we obtain an injection

$$S(\mathfrak{g}^*)^G \otimes H^*(M) \cong H_G^*(M) \longrightarrow H_G^*(M^G) = \bigoplus_{F\in\mathcal{F}} S(\mathfrak{g}^*)^G \otimes H^*(F).$$

An easy consequence of this is, that if we have a Hamiltonian circle action where the components of the fixed point set have vanishing odd cohomology, then the odd cohomology of M is also zero, because all polynomial parts have even degree.

Ansatz: We would like to define the Kirwan map in the singular setting analougously to (2.2.1) by restricting to $\mathrm{J}^{-1}(0)^\top$ and $\mathscr{M}_0^\top$ on the level of differential forms and then use Sjamaar's de Rham theory. This is not so easily possible as the incorporation of the Cartan map, and therefore the incorporation of multiplication with the connection form, introduces singularities of the differential forms on $\mathrm{J}^{-1}(0)^\top$ at the fixed point strata. Our main idea is now, to desingularize the group action in order to make multiplication with the connection form well-defined and generalize Sjamaar's de Rham complex in a suitable way using equivariant blow-ups.

Chapter 3

Equivariant real blow-up and partial desingularization of group actions

Summary: Let (M, σ) be a compact connected symplectic manifold with a Hamiltonian action of $G = S^1$ with momentum map $\mathrm{J}\colon M \to \mathbb{R}$. In this chapter we follow the common approach to desingularizing group actions by equivariant real blow-ups as in [13, Section 2.9] to obtain a manifold $\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M)$, which admits a locally free G-action and a smooth G-equivariant map

$$\beta^{\mathbb{R}}\colon \mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M) \longrightarrow M,$$

such that $\beta^{\mathbb{R}}\colon \mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M) \setminus (\beta^{\mathbb{R}})^{-1}(M^G) \to M \setminus M^G$ is an equivariant diffeomorphism where we focus on the case $G = S^1$.

3.1 The basic construction

In this section we describe a blow-up procedure to define a manifold $\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M)$, which admits a locally free G-action and a smooth G-equivariant map

$$\beta^{\mathbb{R}}\colon \mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M) \longrightarrow M,$$

such that $\beta^{\mathbb{R}}\colon \mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M) \setminus (\beta^{\mathbb{R}})^{-1}(M^G) \to M \setminus M^G$ is an equivariant diffeomorphism. We will spend some time reviewing the procedure and various angles from where to see it. We focus on the case $G = S^1$ even though this procedure works for any smooth action of a compact Lie group G on a smooth manifold as explained in [13, Section 2.9].

3.1.1 Glueing

Let M be an $2n$-dimensional smooth manifold and $N \subset M$ a smooth submanifold of codimension $2k$ whose normal bundle

$$\nu\colon Q \longrightarrow N, \qquad Q_p := T_pM/T_pN \text{ for all } p \in M$$

has structure group $\mathrm{U}(k)$. Projectivize this bundle fibrewise and denote the resulting bundle by $\mathbb{R}\mathrm{P}(Q) \to N$. Next, consider the space

$$\mathrm{l}_Q := \{(l, q) \in \mathbb{R}\mathrm{P}(Q) \times Q \mid q \in l\},$$

where $q \in l$ in particular means that l is a real line in $Q_{\nu(q)}$, together with the commutative diagram

$$\begin{tikzcd} \mathrm{l}_Q \arrow[r, "\varphi"] \arrow[d, "\lambda"] & Q \arrow[d, "\nu"] \\ \mathbb{R}\mathrm{P}(Q) \arrow[r] & N, \end{tikzcd} \tag{3.1.1}$$

where the map φ sends (l, q) to q and the map λ sends (l, q) to l.

In order to describe the structure of l_Q it is convenient to first introduce the universal line bundle

$$\mathrm{l} := \{(l, w) \in \mathbb{R}\mathrm{P}^{2k-1} \times \mathbb{R}^{2k} \mid w \in l\} \longrightarrow \mathbb{R}\mathrm{P}^{2k-1}. \tag{3.1.2}$$

In particular, l represents the real blow-up of $\mathbb{R}^{2k}$ along the origin. With this notation, we see that l_Q is a smooth fibre bundle over N with bundle projection $\nu \circ \varphi$ and fibre l. Furthermore, l_Q is a smooth line bundle over $\mathbb{R}\mathrm{P}(Q)$ with bundle projection given by the map λ in (3.1.1).

Let further $V \subset Q$ be a closed disc bundle, diffeomorphic to a closed tubular neighbourhood $W \subset M$ of N, put $\widetilde{V} := \varphi^{-1}(V)$, and identify V and W, which allows us to consider $\varphi|_{\widetilde{V}}$ as a map $\varphi\colon \widetilde{V} \to W$. In particular, since φ is clearly a diffeomorphism outside the zero section of Q, we get that $\varphi|_{\partial\widetilde{V}} : \partial\widetilde{V} \to \partial W$ is a diffeomorphism. With these preparations, we make the following

Definition 3.1.1. The *real blow-up of M along N* is the smooth manifold

$$\mathrm{Bl}_N^{\mathbb{R}}(M) := \overline{M \setminus W} \cup_{\varphi|_{\partial\widetilde{V}}} \widetilde{V}$$

obtained by glueing the manifolds with boundary $\overline{M \setminus W}$ and $\widetilde{V}$ with the gluing map $\varphi|_{\partial\widetilde{V}} : \partial\widetilde{V} \to \partial W$.

The map $\beta^{\mathbb{R}}\colon \mathrm{Bl}_N^{\mathbb{R}}(M) \to M$ defined by

$$\beta^{\mathbb{R}} := \begin{cases} \mathrm{id} & \text{on } \overline{M \setminus W} \\ \varphi & \text{on } \widetilde{V} \end{cases}$$

is called the *blow-down map*.

The set $(\beta^{\mathbb{R}})^{-1}(N) \subset \widetilde{V}$ is called the *exceptional divisor* of $\mathrm{Bl}_N^{\mathbb{R}}(M)$. It is the zero section of l_Q, regarded as a line bundle over $\mathbb{R}\mathrm{P}(Q)$, and can thus be identified with $\mathbb{R}\mathrm{P}(Q)$.

Remark 3.1.2. The various choices involved in the construction of the blow-up do not cause problems since they lead to equivalent results in the sense that two different choices lead to blow-down maps $\beta^{\mathbb{R}}\colon \mathrm{Bl}_N^{\mathbb{R}}(M) \to M$ and $(\beta^{\mathbb{R}})'\colon \mathrm{Bl}_N^{\mathbb{R}}(M)' \to M$ for which there is a diffeomorphism $f\colon \mathrm{Bl}_N^{\mathbb{R}}(M) \to \mathrm{Bl}_N^{\mathbb{R}}(M)'$ such that $(\beta^{\mathbb{R}})' \circ f = \beta^{\mathbb{R}}$.

We immediately see that the blow-down map $\beta^{\mathbb{R}}$ is a diffeomorphism outside the exceptional divisor.

3.1.2 Principal bundles

We can also see the construction in the light of principal bundles as in [44] and [59, Chapter 4]. By assumption, there is a reduction of the normal frame bundle to $\mathrm{U}(k)$. Therefore there is a principal $\mathrm{U}(k)$-bundle $P \to N$ such that

$$Q \cong P \times_{\mathrm{U}(k)} \mathbb{C}^k \cong P \times_{\mathrm{U}(k)} \mathbb{R}^{2k}.$$

From $\mathbb{R}^{2k}$ we may form the projective space $\mathbb{R}P^{2k-1}$ and the universal line bundle l as in (3.1.2) to obtain

$$
\begin{tikzcd}
P \times_{\mathrm{U}(k)} \mathrm{l} \arrow[r, "\varphi"] \arrow[d] & P \times_{\mathrm{U}(k)} \mathbb{R}^{2k} \cong Q \arrow[d, "\nu"] \\
\mathbb{R}P(Q) \cong P \times_{\mathrm{U}(k)} \mathbb{R}P^{2k-1} \arrow[r] & N,
\end{tikzcd}
$$

where again $\varphi([A,(l,p)]) = [A,p]$. From here we proceed as above to define the blow-up of M along N. This description will allow us, by the local normal form theorem, to desingularize the group action as sketched in the beginning of this chapter.

Remark 3.1.3. We can also regard subgroups of $H \subset \mathrm{U}(k)$, when reducing the structure group of the normal frame bundle, the blow-up is then defined analogously.

We can also describe l in two other fashions:

Lemma 3.1.4. $\mathrm{l} = \left\{([t_i],(p_j)) \in \mathbb{R}P^{2k-1} \times \mathbb{R}^{2k} \,\middle|\, t_i p_j = t_j p_i \quad \forall 1 \leq i,j \leq 2k\right\}$
$\cong (S^{2k-1} \times \mathbb{R})/\mathbb{Z}_2.$

In particular, l *is equal to the real blow-up of* $\mathbb{R}^{2k}$ *along the origin.*

Proof. Let us begin with clarifying the first equality. So let $l = [t_1 : \ldots : t_{2k}] \in \mathbb{R}P^{2k-1}$ be homogeneous coordinates and $p = (p_j) \in \mathbb{R}^{2k}$ such that $p \in l$. Then there is a real number $\lambda \in \mathbb{R}$ such that $p_i = \lambda t_i$ for all $1 \leq i \leq 2k$ and $t_i p_j = \lambda t_i t_j = p_i t_j$. On the other hand, if $l = [t_i] \in \mathbb{R}P^{2k-1}$ and $p = (p_j) \in \mathbb{R}^{2k}$ are such that $t_i p_j = t_j p_i$ for all $1 \leq i,j \leq 2k$, there is some $t_j \neq 0$. It follows that for all $1 \leq i \leq 2k$:

$$p_i = \frac{p_j}{t_j} t_i.$$

So, by setting $\lambda := \frac{p_j}{t_j}$, we have $p_i = \lambda t_i$ for all i, meaning that and $p \in l$.
In order to understand the second isomorphism, we first have to say how $\mathbb{Z}_2$ acts on $S^{2k-1} \times \mathbb{R}$. The action is given by

$$(-1)(w,r) := (-w,-r).$$

Now, we consider

$$
\begin{aligned}
f\colon (S^{2k-1} \times \mathbb{R})/\mathbb{Z}_2 &\to \left\{(l,p) \in \mathbb{R}P^{2k-1} \times \mathbb{R}^{2k} \mid p \in l\right\} \\
[w,r] &\mapsto ([w], rw).
\end{aligned}
$$

The action of $\mathbb{Z}_2$ on the sphere is exactly the antipodal action defining real projective space, $\mathbb{R}P^{2k-1} \cong S^{2k-1}/\mathbb{Z}_2$, and the inverse of f is given by

$$
\begin{aligned}
f^{-1}\colon \left\{(l,p) \in S^{2k-1}/\mathbb{Z}_2 \times \mathbb{R}^{2k} \mid p \in l\right\} &\to (S^{2k-1} \times \mathbb{R})/\mathbb{Z}_2 \\
([w], \lambda w) &\mapsto ([w], \lambda),
\end{aligned}
$$

where one easily checks that all occuring maps are well-defined. □

3.1.3 Coordinates

For later use it will be convenient to have explicit coordinates on $\mathrm{Bl}_N^{\mathbb{R}}(M)$ at our disposal. Since N is a codimension $2k$ submanifold of M, there is an atlas $\{(U,\varphi_U)\}$ of M with coordinate maps $\varphi_U(p) = w = (w_1,\ldots,w_{2n})$ such that if $U \cap N \neq \emptyset$,

$$\varphi_U(U \cap N) = \{(w_1,\ldots,w_{2n}) \in \varphi_U(U) \mid (w_1,\ldots,w_{2k}) = 0\}.$$

By shrinking the sets U, we assume w.l.o.g. that every U with $U \cap N \neq \emptyset$ is contained in the interior of W. Then, recalling the identification $W \cong V$, the atlas $\{(U,\varphi_U)\}$ induces an open cover $\left\{U^{\#} := (\beta^{\mathbb{R}})^{-1}(U)\right\}$ of $\mathrm{Bl}_N^{\mathbb{R}}(M)$ such that

- if $U^{\#} \cap (\beta^{\mathbb{R}})^{-1}(N) = \emptyset$, $\beta^{\mathbb{R}} : U^{\#} \to U$ is a diffeomorphism;
- if $U^{\#} \cap (\beta^{\mathbb{R}})^{-1}(N) \neq \emptyset$, we have

$$U^{\#} \cong \{(l,p) \in \mathbb{R}\mathrm{P}^{2k-1} \times U \mid (l,(w_1,\dots,w_{2k})) \in \mathrm{l}\},$$

  where $\varphi_U(U) \subset \mathbb{R}^{2k} \times \mathbb{R}^{2(n-k)}$.

In the first case, we get coordinates $\varphi_{U^{\#}} := \varphi_U \circ \beta^{\mathbb{R}}$ on $U^{\#}$; in the following, it will sometimes be convenient to identify $U \cong U^{\#}$ and just write φ_U instead of $\varphi_{U^{\#}}$.
To obtain coordinate maps in the second case, consider the standard open cover $\{V_i\}_{1 \leq i \leq 2k}$ of $\mathbb{R}\mathrm{P}^{2k-1}$, where $V_i := \{l = [t_1 : \ldots : t_{2k}] \mid t_i \neq 0\}$. Introduce coordinates on V_i by setting

$$(v_1,\dots,v_{2k-1}) := \left(\frac{t_1}{t_i},\dots,\frac{t_{i-1}}{t_i},\frac{t_{i+1}}{t_i},\dots,\frac{t_{2k}}{t_i}\right). \tag{3.1.3}$$

The V_i induces a cover of $U^{\#}$ by sets $U_i^{\#} \cong \{(l,p) \in V_i \times U \mid (l,(w_1,\dots,w_{2k})) \in \mathrm{l}\}$, which by Lemma 3.1.4 are given in terms of the equations

$$w_j = w_i v_j, \quad 1 \leq j < i \quad \text{and} \quad w_j = w_i v_{j-1}, \quad i < j \leq 2k,$$

yielding the coordinate maps

$$\begin{aligned} \varphi_{U_i^{\#}} : U_i^{\#} &\longrightarrow \varphi_{U_i^{\#}}(U_i^{\#}) \subset \mathbb{R}^{2k-1} \times W_i \\ (l,p) &\longmapsto (v_1,\dots,v_{2k-1},w_i,w_{2k+1},\dots,w_{2n}), \end{aligned}$$

where W_i is the image of $\varphi_U(U)$ in $\mathbb{R} \times \mathbb{R}^{2(n-k)}$ under the projection

$$\mathbb{R}^{2n} \longrightarrow \mathbb{R} \times \mathbb{R}^{2(n-k)},\ (w_1,\dots,w_n) \longmapsto (w_i,w_{2k+1},\dots,w_{2n}).$$

A simple computation then shows that the charts $\{(U^{\#},\varphi_{U^{\#}})\}$ and $\{(U_i^{\#},\varphi_{U_i^{\#}})\}$ indeed constitute a smooth atlas for $\mathrm{Bl}_N^{\mathbb{R}}(M)$, compare [28, Section 3.1].

With respect to the charts introduced above, the blow-down map $\beta^{\mathbb{R}}$ is given near the exceptional divisor as follows: If $U^{\#} \cap (\beta^{\mathbb{R}})^{-1}(N) \neq \emptyset$, then $\beta|_{U^{\#}}$ amounts to mapping (l,p) to p. In the latter case, if $p \notin N$, one has $w_i \neq 0$ for some $1 \leq i \leq 2k$ and $(\beta^{\mathbb{R}})|_{U^{\#}}^{-1}(\{p\}) \cong \{(l,p) \mid l = [w_1 : \ldots : w_{2k}]\}$; on the contrary, if $p \in N$, then $(\beta^{\mathbb{R}})|_{U^{\#}}^{-1}(\{p\}) \cong \mathbb{R}\mathrm{P}^{2k-1} \times \{p\}$. More precisely, in the coordinates provided by $\varphi_{U_i^{\#}}$ and φ_U,

- the map $\beta^{\mathbb{R}}$ is given by the monoidal transformation

$$\begin{aligned} &\varphi_U \circ (\beta^{\mathbb{R}})|_{U_i^{\#}} \circ \varphi_{U_i^{\#}}^{-1} : \\ &(v_1,\dots,v_{2k-1},w_i,w_{2k+1},\dots,w_{2n}) \longmapsto (w_i(v_1,\dots,1,\dots,v_{2k-1}),w_{2k+1},\dots,w_{2n}) \end{aligned}$$

  with 1 at the i-th position,
- $(\beta^{\mathbb{R}})^{-1}(N) \cap U_i^{\#}$ corresponds to the set of points $\{(l,p) \mid w_i = 0\}$.

In particular, the second statement means that the exceptional divisor $(\beta^{\mathbb{R}})^{-1}(N) \subset \mathrm{Bl}_N^{\mathbb{R}}(M)$ is a smooth submanifold of real codimension one. Now, let (M,σ) be a compact connected symplectic manifold with a Hamiltonian action of $G = S^1$ with momentum map $\mathrm{J}\colon M \to \mathbb{R}$.

Definition 3.1.5. We denote by $\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M)$ the *real blow-up* of M, which we define as the smooth manifold that results from successively blowing up M according to Definition 5.1.1 along all $F \in \mathcal{F}$ with respect to Lemma 2.1.11, and by $\beta^{\mathbb{R}} : \mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M) \to M$ the composition of all the blow-down maps. Recall that $\mathcal{F}$ is a finite set due to the compactness of M.

We call $E^{\mathbb{R}} := (\beta^{\mathbb{R}})^{-1}(M^G) \subset \mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M)$ the *exceptional divisor* and for $F \in \mathcal{F}$ we call $E^{\mathbb{R}}_F := (\beta^{\mathbb{R}})^{-1}(F)$ the *exceptional locus* associated with F.

Furthermore, the *strict transform* is the closure

$$\widehat{C}_{\mathbb{R}} := \overline{(\beta^{\mathbb{R}})^{-1}(\mathrm{J}^{-1}(0)^{\top})} \subset \mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M).$$

Note that the exceptional loci are the connected components of the exceptional divisor.

Proposition 3.1.6. *The strict transform $\widehat{C}_{\mathbb{R}}$ and the exceptional divisor $E_{\mathbb{R}}$ are smooth submanifolds of $\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M)$ of real codimension 1, with simple normal crossings. More precisely, there is an atlas $\{(\mathscr{U}, \varphi_{\mathscr{U}})\}$ of $\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M)$ such that for each $\mathscr{U}$ satisfying $\mathscr{U} \cap \widehat{C}_{\mathbb{R}} \cap E^{\mathbb{R}}_F \neq 0$ for some $F \in \mathcal{F}$, the coordinate map $\varphi_{\mathscr{U}}(p) = (w_1, \dots, w_{2n}) = (w_1, \dots, w_{2k}, w_{2n-\dim F+1}, \dots, w_{2n})$ fulfills*

- *$w_{2n-\dim F+1}, \dots, w_{2n}$ are local coordinates on F;*
- *$E^{\mathbb{R}}_F \cap \mathscr{U} = \{w_i = 0\}$ for some $1 \leq i \leq 2n - \dim F$;*
- *$\widehat{C}_{\mathbb{R}} \cap \mathscr{U} = \{w_j = 0\}$ for some $1 \leq j \leq 2n - \dim F$ with $i \neq j$;*
- *$\beta^{\mathbb{R}}(w_1, \dots, w_{2n}) = (w_i(w_1, \dots, 1, \dots, w_{2k}), w_{2n-\dim F+1}, \dots, w_{2n})$ with 1 at the i-th position, where on the right-hand side we use local coordinates of M in the open set $\beta^{\mathbb{R}}(\mathscr{U})$.*

Proof. Since the exceptional loci $E^{\mathbb{R}}_F$ are the connected components of $E^{\mathbb{R}}$ and Section 5.1 shows that each $E^{\mathbb{R}}_F$ is a smooth codimension 1-submanifold of $\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M)$, we see that the same holds for $E^{\mathbb{R}}$.

If an $F \in \mathcal{F}$ satisfies $\widehat{C}_{\mathbb{R}} \cap E^{\mathbb{R}}_F \neq 0$, then $F \subset \mathrm{J}^{-1}(0)$. For such an F, recall the description of the symplectic normal bundle $\Sigma_F \cong P_F \times_{K_F} \mathbb{C}^{\ell^+_F + \ell^-_F}$ with $\operatorname{codim} F = 2n - \dim F = 2(\ell^+_F + \ell^-_F) = 2k$ and the local normal form of the momentum map near F given in Proposition 2.1.11, by which the zero level set is locally described by the relation

$$\Phi_F(\mathrm{J}^{-1}(0) \cap U_F) = \left\{ [\wp, z_1, \dots, z_{\ell^+_F + \ell^-_F}] \in V_F \subset P_F \times_{K_F} \mathbb{C}^{\ell^+_F + \ell^-_F} \;\middle|\; \sum \lambda^F_i |z_i|^2 = 0 \right\}.$$

Construct an atlas $\{(\mathscr{U}, \varphi_{\mathscr{U}})\}$ of $\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M)$ following the procedure described in Section 5.1, where the atlas $\{(U, \varphi_U)\}$ of M with $\varphi_U(p) = w = (w_1, \dots, w_{2n})$ underlying this construction is given in terms of the local trivializations of the fibre bundle $E^{\mathbb{R}}_F$ over F and the diffeomorphisms $\Phi_F : U_F \to V_F$ with the identification $\mathbb{C}^{\ell^+_F + \ell^-_F} \ni z = (z_1, \dots, z_{\ell^+_F + \ell^-_F}) \equiv (w_1, \dots, w_{2\ell^+_F + 2\ell^-_F}) \in \mathbb{R}^{2\ell^+_F + 2\ell^-_F}$. Each $\mathscr{U}$ with $\mathscr{U} \cap E^{\mathbb{R}}_F \neq \emptyset$ is then mapped by $\varphi_{\mathscr{U}}$ onto a set of the shape

$$\left\{ (l, p) \in V_i \times U \;\middle|\; (l, (w_1, \dots, w_{2\ell^+_F + 2\ell^-_F})) \in \mathrm{l} \right\}$$

for some suitable U and $1 \leq i \leq 2k$, with $w_{2\ell^+_F + 2\ell^-_F + 1}, \dots, w_{2n}$ local coordinates of F and $\varphi_{\mathscr{U}}(\mathscr{U} \cap E^{\mathbb{R}}_F) = \{w_i = 0\}$. Since the group $K_F \subset U(\ell^+_F) \times U(\ell^-_F)$ leaves the quadratic form $\sum \lambda^F_i |z_i|^2$ invariant, we get

$$\begin{aligned} \varphi_{\mathscr{U}}\big(\mathscr{U} \cap (\beta^{\mathbb{R}})^{-1}(\mathrm{J}^{-1}(0)^{\top})\big) &= \left\{ (l, p) \in V_i \times (\mathrm{J}^{-1}(0)^{\top} \cap U) \;\middle|\; (l, (w_1, \dots, w_{2\ell^+_F + 2\ell^-_F})) \in \mathrm{l} \right\} \\ &= \Bigg\{ (l, p) \in V_i \times U \;\Bigg|\; (l, (w_1, \dots, w_{2\ell^+_F + 2\ell^-_F})) \in \mathrm{l}, \\ &\qquad \sum_{k=1}^{\ell^+_F + \ell^-_F} \lambda^F_k (w_k^2 + w^2_{\ell^+_F + \ell^-_F + k}) = 0,\ (w_1, \dots, w_{2\ell^+_F + 2\ell^-_F}) \neq 0 \Bigg\}. \end{aligned}$$

Thus,

$$\varphi_{\mathscr{U}}(\mathscr{U} \cap \widehat{C}_{\mathbb{R}}) = \left\{ (l,p) \in V_i \times U \;\middle|\; ([t_1 : \ldots : t_k], (w_1, \ldots, w_{2\ell_F^+ + 2\ell_F^-})) \in \mathrm{l},\ \sum_{k=1}^{\ell_F^+ + \ell_F^-} \lambda_k^F |t_k|^2 = 0 \right\} \tag{3.1.4}$$

since the standard coordinates $(v_1, \ldots, v_{2n-\dim F-1})$ from (3.1.3) for the elements $l \in V_i$ reveal that if $(l, (w_1, \ldots, w_{2\ell_F^+ + 2\ell_F^-})) \in \mathrm{l}$, then

$$\sum_{k=1}^{\ell_F^+ + \ell_F^-} \lambda_k^F (w_k^2 + w_{\ell_F^+ + \ell_F^- + k}^2) = 0 \iff \sum_{k=1}^{i-1} \lambda_{\iota(k)}^F v_k^2 + \lambda_i^F + \sum_{k=i+1}^{2n-\dim F} \lambda_{\iota(k)}^F v_k^2 = 0$$

for suitable indices $\iota(k)$. Moreover, (3.1.4) is actually a local description of $\widehat{C}_{\mathbb{R}}$ as a non-singular quadric (recall that $\lambda_i^F \neq 0$ for all i), which reveals that $\widehat{C}_{\mathbb{R}}$ is a smooth submanifold of $\mathrm{Bl}_{\mathrm{G}}^{\mathbb{R}}(M)$. Further, since $\varphi_{\mathscr{U}}(\mathscr{U} \cap E_F^{\mathbb{R}}) = \{w_i = 0\}$, $\widehat{C}_{\mathbb{R}}$ is transversal to $E_F^{\mathbb{R}}$. Taking $\sum_{k=1}^{i-1} \lambda_{\iota(k)}^F v_k^2 + \lambda_i^F + \sum_{k=i+1}^{2n-\dim F} \lambda_{\iota(k)}^F v_k^2$ as a coordinate and relabling the others proves the third claim. Finally, the statement about the blow-down map in coordinates follows from the description of $\beta^{\mathbb{R}}$ in coordinates given in Section 3.1.3. □

3.2 Lifting symmetries

We want to define a canonical locally free G-action on $\mathrm{Bl}_{\mathrm{G}}^{\mathbb{R}}(M)$, making $\beta^{\mathbb{R}} \colon \mathrm{Bl}_{\mathrm{G}}^{\mathbb{R}}(M) \to M$ equivariant. Denote by $\mathcal{F}$ the set of components of M^G, and recall that for any component $F \in \mathcal{F}$ the local normal form theorem, see Lemma 2.1.11, yielded a diffeomorphism Φ of a neighbourhood of F in M to a neighbourhood U_0 of the zero section in the associated bundle $P_F \times_{K_F} \mathbb{C}^{\ell_F^+ + \ell_F^-} \to F$.

As was explained in Section 3.1.2, we can blow-up M along all components F of the fixed point set using for each $F \in \mathcal{F}$ the diagram

$$\begin{tikzcd}
P_F \times_{K_F} \mathrm{l} \arrow[r, "\varphi"] \arrow[d] & P_F \times_{K_F} \mathbb{R}^{2\ell_F^+ + 2\ell_F^-} \arrow[d, "\nu"] \\
P_F \times_{K_F} \mathbb{R}P^{2\ell_F^+ + 2\ell_F^- - 1} \arrow[r] & F.
\end{tikzcd}$$

After identifying $\mathbb{C}^{\ell_F^+ + \ell_F^-}$ with $\mathbb{R}^{2\ell_F^+ + 2\ell_F^-}$, the action on $\mathbb{C}^{\ell_F^+ + \ell_F^-}$ reads

$$\begin{aligned}
z \cdot (z_i) &= \left(z^{\lambda_i^F} \cdot z_i \right) \\
&= \left(e^{i\lambda_i^F t} \cdot z_i \right) \\
&= \left((\cos(\lambda_i^F t) + i \sin(\lambda_i^F t)) \cdot (x_i + i y_i) \right) \\
&= \begin{pmatrix} \cos(\lambda_i^F t) & -\sin(\lambda_i^F t) \\ \sin(\lambda_i^F t) & \cos(\lambda_i^F t) \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix} \\
&= (\cos(\lambda_i^F t) x_i - \sin(\lambda_i^F t) y_i, \cos(\lambda_i^F t) y_i + \sin(\lambda_i^F t) x_i).
\end{aligned}$$

The sphere $S^{2k+2l-1} \subset \mathbb{R}^{2k+2l}$ is invariant under this S^1 action and the action furthermore commutes with the antipodal $\mathbb{Z}_2$-action on the sphere. Therefore S^1 also acts on $P \times_{K_F} \mathbb{R}P^{2\ell_F^+ + 2\ell_F^- - 1}$ by

$$e^{it} \cdot [p, [x_n : y_n]_n] := [e^{it} \cdot p, [\cos(\lambda_n^F t) x_n - \sin(\lambda_n^F t) y_n : \cos(\lambda_n^F t) y_n + \sin(\lambda_n^F t) x_n]_n].$$

Now we take the diagonal action $S^1 \curvearrowright \mathbb{R}P^{2k-1} \times \mathbb{R}^{2k}$, for l is invariant under this action and $\varphi\colon P \times_{K_F} \mathrm{l} \to P \times_{K_F} \mathbb{R}^{2k}$ becomes equivariant. This equips $\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M)$ with a natural circle action. Moreover, one hase the following

Lemma 3.2.1. *The circle action on $\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M)$ is locally free and $\beta^{\mathbb{R}}$ is equivariant with respect to this action.*

Proof. The action is locally free away from the exceptional locus, since it coincides with the original action there. On the exceptional divisor the action is equal to the isotropy action on the odd-dimensional real projective space, coming from the circle action on the sphere with no zero weights. Thus, the action on the sphere is locally free and the action on the exceptional divisor stays locally free after dividing by $\mathbb{Z}_2$. The equivariance of $\beta^{\mathbb{R}}$ is immediate from the construction. □

Corollary 3.2.2. *There is a connection form $\alpha \in \Omega^1(\mathrm{J}^{-1}(0)^{\top})$ for the G-action on the regular part of $\mathrm{J}^{-1}(0)$ such that there is a connection form $\hat{\alpha} \in \Omega^1(\widehat{C}_{\mathbb{R}})$ for the G-action on $\widehat{C}_{\mathbb{R}}$ which extends α in the sense that $\hat{\alpha}\big|_{(\beta^{\mathbb{R}})^{-1}(\mathrm{J}^{-1}(0)^{\top})} = (\beta^{\mathbb{R}})^*\alpha$.*

Thus, we have resolved the G-action on M by going over to the locally free G-action on $\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M)$, see also [13, Section 2.9].

Remark 3.2.3. Let us understand in more detail, how the action of S^1 on the exceptional fibres l behaves under the isomorphism $\mathrm{l} \cong (S^{2\ell_F^+ + 2\ell_F^- - 1} \times \mathbb{R})/\mathbb{Z}_2$ from Lemma 3.1.4. For some $(l, p) \in \mathbb{R}P^{2\ell_F^+ + 2\ell_F^- - 1} \times \mathbb{R}^{2\ell_F^+ + 2\ell_F^-}$ with $p \in l$, the action is simply

$$z \cdot (l, p) := (z \cdot l, z \cdot p).$$

Now we represent some line $l \in \mathbb{R}P^{2k-1}$ by $l = [w] \in S^{2k-1}/\mathbb{Z}_2$. Then there is a scalar $\lambda \in \mathbb{R}$ such that $p = \lambda w$ and the circle acts as

$$z \cdot ([w], \lambda w) = ([z \cdot w], \lambda(z \cdot w)).$$

To make the isomorphism $\mathrm{l} \cong (S^{2k-1} \times \mathbb{R})/\mathbb{Z}_2$ equivariant, we therefore define the circle action on $(S^{2k-1} \times \mathbb{R})/\mathbb{Z}_2$ to be given by

$$z \cdot [w, r] := [z \cdot w, r].$$

Chapter 4

Real resolution differential forms and the real blow-up

Summary: Using the notation as in the previous chapter, we now consider the diagram

$$\begin{tikzcd}
\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M) \arrow[d, "\beta^{\mathbb{R}}"] & (\beta^{\mathbb{R}})^{-1}\left(\mathrm{J}^{-1}(0)^{\top}\right) \arrow[l, hook', "\iota^{\mathbb{R}}_{\top}"'] \arrow[d, "\beta^{\mathbb{R}}_{\top}"] \\
M & \mathrm{J}^{-1}(0)^{\top} \arrow[l, hook', "\iota_{\top}"] \arrow[d, "\pi_{\top}"] \\
 & \mathscr{M}_0^{\top}
\end{tikzcd}$$

and define the complex of real resolution differential forms on $\mathscr{M}_0$ as

$$\widehat{\Omega}(\mathscr{M}_0) := \left\{\omega_0 \in \Omega(\mathscr{M}_0^{\top}) \;\middle|\; \exists \widetilde{\eta} \in \Omega(\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M))\colon (\beta^{\mathbb{R}}_{\top})^{*}\pi^{*}_{\top}\omega_0 = (\iota^{\mathbb{R}}_{\top})^{*}\widetilde{\eta}\right\}$$

in which we can embed Sjamaar's complex of differential forms on $\mathscr{M}_0$. As the S^1-action is locally free on $\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M)$ we find a global connection form on $\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M)$ which is what we wanted as explained in our ansatz. We then continue to relate the cohomology of $\widehat{\Omega}(\mathscr{M}_0)$ to Sjamaar's complex of differential forms by studying the short exact sequence

$$0 \longrightarrow \Omega(\mathscr{M}_0) \longrightarrow \widehat{\Omega}(\mathscr{M}_0) \longrightarrow C(\mathscr{M}_0) \longrightarrow 0,$$

where $C(\mathscr{M}_0)$ denotes the cokernel complex. As the cohomology of the cokernel complex seems impalpable, we end our investigation of real resolution forms at this point.

To begin with, the blow-up $\beta^{\mathbb{R}}\colon \mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M) \to M$ sits in the following diagram

$$\begin{tikzcd}
\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M) \arrow[d, "\beta^{\mathbb{R}}"] & (\beta^{\mathbb{R}})^{-1}\left(\mathrm{J}^{-1}(0)^{\top}\right) \arrow[l, hook', "\iota^{\mathbb{R}}_{\top}"'] \arrow[d, "\beta^{\mathbb{R}}_{\top}"] \\
M & \mathrm{J}^{-1}(0)^{\top} \arrow[l, hook', "\iota_{\top}"] \arrow[d, "\pi_{\top}"] \\
 & \mathscr{M}_0^{\top}.
\end{tikzcd}$$

We use this to make the following

Definition 4.0.1. The complex of *real resolution forms* on $\mathscr{M}_0$ is defined as

$$\widehat{\Omega}(\mathscr{M}_0) := \left\{\omega_0 \in \Omega(\mathscr{M}_0^\top) \,\middle|\, \exists \widetilde{\eta} \in \Omega(\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M))\colon (\beta^{\mathbb{R}}_\top)^* \pi_\top^* \omega_0 = (\iota^{\mathbb{R}}_\top)^* \widetilde{\eta}\right\}.$$

Again, by averaging over G we can replace $\Omega(\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M))$ by $\Omega(\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M))^G$ in this definition. We may characterize these differential forms by considering

$$\Omega_{\mathrm{J}}(\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M)) := \left\{\omega \in \Omega(\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M))^G \,\middle|\, \omega|_{(\beta^{\mathbb{R}})^{-1}\left(J^{-1}(0)^\top\right)} \text{ is horizontal}\right\}$$

and

$$I_{\mathrm{J}}(\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M)) := \left\{\omega \in \Omega(\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M))^G \,\middle|\, \omega|_{(\beta^{\mathbb{R}})^{-1}\left(J^{-1}(0)^\top\right)} = 0\right\}.$$

Proposition 4.0.2. *There is an isomorphism of cochain complexes*

$$\widehat{\Omega}(\mathscr{M}_0) \cong \frac{\Omega_{\mathrm{J}}(\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M))}{I_{\mathrm{J}}(\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M))}.$$

Proof. We have the natural surjection

$$\Omega_{\mathrm{J}}(\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M)) \xrightarrow{(\iota^{\mathbb{R}}_\top)^*} \Omega_{\mathrm{bas}\,G}\left((\beta^{\mathbb{R}})^{-1}\left(\mathrm{J}^{-1}(0)^\top\right)\right) \xrightarrow{\left((\beta^{\mathbb{R}}_\top)^{-1}\right)^*} \Omega_{\mathrm{bas}\,G}\left(\mathrm{J}^{-1}(0)^\top\right) \xrightarrow{\left(\pi^*_\top\right)^{-1}} \widehat{\Omega}(\mathscr{M}_0),$$

whose kernel is precisely $I_{\mathrm{J}}(\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M))$. □

The relation between real resolution forms and the exceptional loci is described in

Proposition 4.0.3. *Let $E_F^{\mathbb{R}} \in \mathcal{E}$ be a component of the exceptional divisor $(\beta^{\mathbb{R}})^{-1}\left(M^G\right)$ intersecting $\widehat{C}_{\mathbb{R}}$, where $\mathcal{E}$ is the set of all components of the exceptional divisor. Then*

1. *For $\omega \in \Omega_{\mathrm{J}}(\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M))$ the restriction $\omega|_{E_F^{\mathbb{R}} \cap \widehat{C}_{\mathbb{R}}}$ is horizontal.*

2. *For $\eta \in I_{\mathrm{J}}(\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M))$ the restriction $\eta|_{E_F^{\mathbb{R}} \cap \widehat{C}_{\mathbb{R}}}$ is zero.*

3. *There is a well-defined surjective restriction map $\widehat{\Omega}(\mathscr{M}_0) \to \Omega((E_F^{\mathbb{R}} \cap \widehat{C}_{\mathbb{R}})/G)$.*

Proof. Let $z \in E_F^{\mathbb{R}} \cap \widehat{C}_{\mathbb{R}}$ and $(z_n) \subset (\beta^{\mathbb{R}})^{-1}(\mathrm{J}^{-1}(0)^\top)$ be a sequence converging to z. Then for $\omega \in \Omega_{\mathrm{J}}(\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M))$, $\eta \in I_{\mathrm{J}}(\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M))$ and $X \in \mathfrak{g}$ one has

$$i_{\overline{X}_z}\omega_z = \lim_{n\to\infty} i_{\overline{X}_{z_n}}\omega_{z_n} = 0 \quad \text{and} \quad \eta_z = \lim_{n\to\infty} \eta_{z_n} = 0,$$

showing *(1)* and *(2)*. Furthermore, this implies that, in view of Proposition 4.0.2, each real resolution form gives a well-defined form on the quotient $(E_F^{\mathbb{R}} \cap \widehat{C}_{\mathbb{R}})/G$, yielding a map

$$\widehat{\Omega}(\mathscr{M}_0) \longrightarrow \Omega((E_F^{\mathbb{R}} \cap \widehat{C}_{\mathbb{R}})/G).$$

To show its surjectivity take $\omega \in \Omega_{\mathrm{bas}\,G}(E_F^{\mathbb{R}} \cap \widehat{C}_{\mathbb{R}})$. Since $E_F^{\mathbb{R}} \cap \widehat{C}_{\mathbb{R}}$ is a transverse intersection of two closed submanifolds of $\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M)$ it is itself again a closed submanifold of $\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M)$. Now, let U be a G-invariant tubular neighbourhood of $E_F^{\mathbb{R}} \cap \widehat{C}_{\mathbb{R}}$ in $\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M)$ diffeomorphic to the normal bundle $\nu(E_F^{\mathbb{R}} \cap \widehat{C}_{\mathbb{R}})$ of $E_F^{\mathbb{R}} \cap \widehat{C}_{\mathbb{R}}$. Denote the natural equivariant retraction by $\pi\colon U \to E_F^{\mathbb{R}} \cap \widehat{C}_{\mathbb{R}}$. We then define

$$\omega' := \varrho_U \cdot \pi^*\omega,$$

where $\varrho_U \in \mathcal{C}^\infty_c(U)$ is a smooth G-invariant cut-off function on U with $\varrho_U = 1$ near $E_F^{\mathbb{R}} \cap \widehat{C}_{\mathbb{R}}$. Then

- $g^*w' = (\varrho_U \circ g) \cdot g^*\pi^*\omega = \varrho_U \cdot \pi^* g^*\omega = \varrho_U \cdot \pi^*\omega = \omega'$, where $g \in G$
- $i_{\overline{X}}\omega' = \varrho_U \cdot \pi^* i_{\overline{X}}\omega = 0$ for $X \in \mathfrak{g}$.

This proves that $\omega' \in \Omega_{\mathrm{J}}(\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M))$ and $\omega'\big|_{E^{\mathbb{R}}_F \cap \widehat{C}_{\mathbb{R}}} = \omega$, yielding *(3)*. □

Alternatively, we can describe real resolution forms by introducing the real partial desingularization of $\mathscr{M}_0$.

Definition 4.0.4. The *real partial desingularization* of $\mathscr{M}_0$ is the orbifold $\widehat{\mathscr{M}_0} := \widehat{C}_{\mathbb{R}}/G$ and comes together with the continuous map $\beta^{\mathbb{R}}_0$ which is a diffeomorphism away from the exceptional set defined by the following diagram:

$$\begin{array}{ccc} \widehat{\mathscr{M}_0} & \xrightarrow{\beta^{\mathbb{R}}_0} & \mathscr{M}_0 \\ {\scriptstyle \pi^{\mathbb{R}}}\uparrow & & \uparrow{\scriptstyle \pi} \\ \widehat{C}_{\mathbb{R}} & \xrightarrow{\beta^{\mathbb{R}}} & \mathrm{J}^{-1}(0). \end{array}$$

We denote the exceptional fibre bundles of $\beta^{\mathbb{R}}_0$ by

$$\beta^{\mathbb{R},F}_0 \colon \widehat{F} := (\beta^{\mathbb{R}}_0)^{-1}(F) \longrightarrow F$$

and the restriction of $\beta^{\mathbb{R}}_0$ to the top stratum as

$$(\beta^{\mathbb{R}}_0)_{\top} \colon \widehat{\mathscr{M}_0}^{\top} := (\beta^{\mathbb{R}}_0)^{-1}\left(\mathscr{M}_0^{\top}\right) \longrightarrow \mathscr{M}_0^{\top}.$$

Proposition 4.0.5. *There is an isomorphism of cochain complexes*

$$\widehat{\Omega}(\mathscr{M}_0) \cong \Omega(\widehat{\mathscr{M}_0}).$$

Proof. Let ω be a real resolution form on $\mathscr{M}_0$. Then there is $\eta \in \Omega(\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M))$ such that $(\beta^{\mathbb{R}}_{\top})^*\pi^*_{\top}\omega = (\iota^{\mathbb{R}}_{\top})^*\eta$. By restricting this form η to $\widehat{C}_{\mathbb{R}}$ we obtain a G-basic form which does not depend on the extension η of ω since $(\beta^{\mathbb{R}})^{-1}(\mathrm{J}^{-1}(0)^{\top})$ is dense in $\widehat{C}_{\mathbb{R}}$ and $(\beta^{\mathbb{R}}_0)_{\top}$ is a diffeomorphism. Thus we have a natural map

$$\widehat{\Omega}(\mathscr{M}_0) \longrightarrow \Omega(\widehat{\mathscr{M}_0}).$$

On the other hand, $\widehat{C}_{\mathbb{R}}$ is a closed submanifold of $\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M)$ and therefore every differential form on $\widehat{C}_{\mathbb{R}}$, and in particular every G-basic form, admits an extension to $\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M)$ and therefore gives us a real resolution form on $\mathscr{M}_0$ and a map

$$\Omega(\widehat{\mathscr{M}_0}) \longrightarrow \widehat{\Omega}(\mathscr{M}_0).$$

These maps are inverse to each other, commute with the differentials and the isomorphism is proved. □

The advantage of studying the complex $\widehat{\Omega}(\mathscr{M}_0)$ is that there is a natural inclusion

$$\begin{aligned} \Omega(\mathscr{M}_0) &\hookrightarrow \widehat{\Omega}(\mathscr{M}_0) \\ \omega &\mapsto \omega. \end{aligned}$$

Indeed, let $\omega \in \Omega(\mathscr{M}_0)$ and $\eta \in \Omega(M)$ be such that $\pi^*_{\top}\omega = \iota^*_{\top}\eta$. Pulling back this equation to $\mathrm{Bl}^{\mathbb{R}}_{\mathrm{G}}(M)$ gives $(\beta^{\mathbb{R}}_{\top})^*\pi^*_{\top}\omega = (\beta^{\mathbb{R}}_{\top})^*\iota^*_{\top}\eta = (\iota^{\mathbb{R}}_{\top})^*(\beta^{\mathbb{R}})^*\eta$. Hence ω is a real resolution form with smooth extension $(\beta^{\mathbb{R}})^*\eta$.

Definition 4.0.6. We set $C(\mathscr{M}_0)$ to be the cokernel complex of the inclusion $\Omega(\mathscr{M}_0) \hookrightarrow \widehat{\Omega}(\mathscr{M}_0)$:

$$C(\mathscr{M}_0) := \frac{\widehat{\Omega}(\mathscr{M}_0)}{\Omega(\mathscr{M}_0)}.$$

The three differential complexes $\Omega(\mathscr{M}_0), \widehat{\Omega}(\mathscr{M}_0)$ and $C(\mathscr{M}_0)$ are naturally related by the short exact sequence of complexes

$$0 \longrightarrow \Omega(\mathscr{M}_0) \longrightarrow \widehat{\Omega}(\mathscr{M}_0) \longrightarrow C(\mathscr{M}_0) \longrightarrow 0.$$

This induces a long exact sequence in cohomology, see [6, p. 17],

$$\begin{tikzcd}
0 \arrow[r] & H^0(\Omega(\mathscr{M}_0), d) \arrow[r] & H^0(\widehat{\Omega}(\mathscr{M}_0), d) \arrow[r] & H^0(C(\mathscr{M}_0), d) \arrow[dll, "\delta"] & \\
 & H^1(\Omega(\mathscr{M}_0), d) \arrow[r] & H^1(\widehat{\Omega}(\mathscr{M}_0), d) \arrow[r] & H^1(C(\mathscr{M}_0), d) \arrow[dll, "\delta"] & \\
 & H^2(\Omega(\mathscr{M}_0), d) \arrow[r] & H^2(\widehat{\Omega}(\mathscr{M}_0), d) \arrow[r] & H^2(C(\mathscr{M}_0), d) \arrow[r] & \dots,
\end{tikzcd}$$

where δ is the so-called boundary operator. We shall try to understand the individual terms of this sequence.

Since the blow-down map $\beta_0^{\mathbb{R}} \colon \widehat{\mathscr{M}_0} \to \mathscr{M}_0$ is a diffeomorphism away from the exceptional set, we expect the cokernel $C(\mathscr{M}_0)$ to be related to $\bigoplus_{F \in \mathcal{F}_0} \operatorname{coker}\left(\Omega(F) \hookrightarrow \Omega\left(\frac{E_F^{\mathbb{R}} \cap \widehat{C}_{\mathbb{R}}}{G}\right)\right)$ where $\mathcal{F}_0$ denotes the set of fixed point components contained in the zero level set. So let us examine the exceptional divisors $\frac{E_F^{\mathbb{R}} \cap \widehat{C}_{\mathbb{R}}}{G}$ of $\beta_0^{\mathbb{R}}$ more closely.

Proposition 4.0.7. *For each $F \in \mathcal{F}_0$, $\frac{E_F^{\mathbb{R}} \cap \widehat{C}_{\mathbb{R}}}{G}$ is the total space of a fibre bundle*

$$((S^{2\ell_F^+ - 1} \times S^{2\ell_F^- - 1})/\mathbb{Z}_2)/G \longrightarrow \frac{E_F^{\mathbb{R}} \cap \widehat{C}_{\mathbb{R}}}{G} \longrightarrow F.$$

Proof. From the construction of the blow-up, it is clear that the projection

$$\frac{E_F^{\mathbb{R}} \cap \widehat{C}_{\mathbb{R}}}{G} \longrightarrow F,$$

with $E_F^{\mathbb{R}} = (\beta^{\mathbb{R}})^{-1}(F)$, defines a fibre bundle. From the local normal form theorem, see Lemma 2.1.11, we know that in a neighbourhood U of F the zero level of the momentum map is diffeomorphic to

$$\begin{aligned}
U \cap \mathrm{J}^{-1}(0) &\cong \left\{ [p, z_1, \dots, z_{\ell_F^+ + \ell_F^-}] \in P_F \times_{K_F} \mathbb{C}^{\ell_F^+ + \ell_F^-} \;\Big|\; \sum_{i=1}^{\ell_F^+} \lambda_i^F \left(x_i^2 + y_i^2\right) = - \sum_{i=\ell_F^+ + 1}^{\ell_F^+ + \ell_F^-} \lambda_i^F \left(x_i^2 + y_i^2\right) \right\} \\
&\cong P_F \times_{K_F} \left(\frac{[0, \infty) \times S^{2\ell_F^+ - 1} \times S^{2\ell_F^- - 1}}{\{0\} \times S^{2\ell_F^+ - 1} \times S^{2\ell_F^- - 1}} \right).
\end{aligned}$$

Therefore in a neighbourhood $\widetilde{U}$ of $E_F^{\mathbb{R}}$ we can describe $\widehat{C}_{\mathbb{R}}$ as

$$\begin{aligned}
\widetilde{U} \cap \widehat{C}_{\mathbb{R}} \cong \Big\{ & [p, (t_s), [x_r : y_r]_r] \in P_F \times_{K_F} (\mathbb{R}^{2\ell_F^+ + 2\ell_F^-} \times \mathbb{R}P^{2\ell_F^+ + 2\ell_F^- - 1}) \\
& \Big| \sum_{i=1}^{\ell_F^+} \lambda_i^F \left(x_i^2 + y_i^2\right) = - \sum_{i=\ell_F^+ + 1}^{\ell_F^+ + \ell_F^-} \lambda_i^F \left(x_i^2 + y_i^2\right), (t_s) \in [x_r : y_r]_r \Big\}
\end{aligned}$$

Intersecting with $E_F^{\mathbb{R}}$ is now simply looking at $(t_s) = 0$, so

$$\widetilde{U} \cap E_F^{\mathbb{R}} \cap \widehat{C}_{\mathbb{R}} \cong \left\{ [p, [x_r : y_r]_r] \in P_F \times_{K_F} \mathbb{R}P^{2\ell_F^+ + 2\ell_F^- - 1} \;\middle|\; \sum_{i=1}^{\ell_F^+} \lambda_i^F \left(x_i^2 + y_i^2\right) = - \sum_{i=\ell_F^+ + 1}^{\ell_F^+ + \ell_F^-} \lambda_i^F \left(x_i^2 + y_i^2\right) \right\}.$$

Now $\widetilde{U} \cap E_F^{\mathbb{R}} \cap \widehat{C}_{\mathbb{R}}$ is diffeomorphic to $P_F \times_{K_F} V$, where $V = \frac{S^{2\ell_F^+ - 1} \times S^{2\ell_F^- - 1}}{\mathbb{Z}_2}$. The diffeomorphism is given by

$$\begin{aligned} \widetilde{U} \cap E_F^{\mathbb{R}} \cap \widehat{C}_{\mathbb{R}} &\longrightarrow P_F \times_{K_F} \frac{S^{2\ell_F^+ - 1} \times S^{2\ell_F^- - 1}}{\mathbb{Z}_2} \\ [p, [x_r : y_r]_r] &\longmapsto \left[p, \left[\sqrt{\frac{|\lambda_r^F|}{C}} \cdot x_r, \sqrt{\frac{|\lambda_r^F|}{C}} \cdot y_r \right]_r \right] \\ \left[p, \left[\frac{1}{\sqrt{|\lambda_r^F|}} \cdot x_r : \frac{1}{\sqrt{|\lambda_r^F|}} \cdot y_i \right]_r \right] &\longleftarrow [p, [x_r : y_r]_r], \end{aligned}$$

where $C = \sum_{i=1}^{\ell_F^+} \lambda_i^F \left(x_i^2 + y_i^2\right) = - \sum_{i=\ell_F^+ + 1}^{\ell_F^+ + \ell_F^-} \lambda_i^F \left(x_i^2 + y_i^2\right)$. One easily checks that these maps are well-defined and inverse to each other. □

We continue, by examining the cohomology of that fibre $((S^{2\ell_F^+ - 1} \times S^{2\ell_F^- - 1})/\mathbb{Z}_2)/G$. For this we need a certain long exact sequence, called the Gysin sequence for locally free S^1-actions whose construction is inspired by [54, Section 3.7]. Let S^1 act locally freely on a compact manifold M. Let $X \in \mathfrak{g}$ be the infinitesimal generator of the Lie algebra of S^1 and consider the sequence

$$0 \longrightarrow \Omega^*_{\mathrm{bas}\, G}(M) \longrightarrow \Omega^*(M)^G \xrightarrow{i_{\overline{X}}} \Omega^{*-1}_{\mathrm{bas}\, G}(M) \longrightarrow 0.$$

By definition, this sequence is exact in the first and second place. We should therefore check, that the contraction with $\overline{X}$ gives a surjective map

$$i_{\overline{X}} \colon \Omega^*(M)^G \longrightarrow \Omega^{*-1}_{\mathrm{bas}\, G}(M).$$

Note that for any invariant form $\omega \in \Omega^*(M)^G$ the contraction $i_{\overline{X}}\omega$ is basic. In order to show surjectivity, consider some G-basic form $\eta \in \Omega^*_{\mathrm{bas}\, G}(M)$. Since the action was locally free, there is a connection form $\alpha \in \Omega^1(M)^G$ such that $\alpha(\overline{X}) = 1$. Thus

$$i_{\overline{X}}(\alpha \wedge \eta) = i_{\overline{X}}\alpha \wedge \eta - \alpha \wedge i_{\overline{X}}\eta = \eta,$$

and the above sequence is exact. Therefore, it induces a long exact sequence in cohomology

$$\ldots \longrightarrow H^k_{\mathrm{bas}\, G}(M) \longrightarrow H^k(M)^G \longrightarrow H^{k-1}_{\mathrm{bas}\, G}(M) \xrightarrow{\delta} H^{k+1}_{\mathrm{bas}\, G}(M) \longrightarrow \ldots$$

After using the canonical isomorphisms $H^*(M) \cong H^*(M)^G$ and $\pi^* \colon H^*(M/G) \to H^*_{\mathrm{bas}\, G}(M)$ we obtain

$$\ldots \longrightarrow H^k(M/G) \xrightarrow{\pi^*} H^k(M) \xrightarrow{i_{\overline{X}}} H^{k-1}(M/G) \xrightarrow{\delta} H^{k+1}(M/G) \longrightarrow \ldots$$

While $i_{\overline{X}}$ can be identified with the fibre integration π_* of the S^1-orbibundle $\pi \colon M \to M/G$, it remains to specify the connecting homomorphism

$$\delta \colon H^{k-1}(M/G) \longrightarrow H^{k+1}(M/G).$$

From the definition of δ, see [6, p. 17] again, we have

$$\delta([\eta]) = [d(\alpha \wedge \eta)] = [d\alpha][\eta].$$

Thus, δ is multiplication with the curvature class $\Omega = [d\alpha]$ and summarizing, we obtain the Gysin sequence as

Proposition 4.0.8. *Let $G = S^1$ act locally freely on M and let $\alpha \in \Omega^1(M)$ be a connection form of the S^1-orbibundle $\pi\colon M \to M/G$ with curvature form $\Omega = d\alpha \in \Omega^2_{\mathrm{bas}\,G}(M)$. Then there is a long exact sequence*

$$\ldots \longrightarrow H^k(M/G) \xrightarrow{\pi^*} H^k(M) \xrightarrow{\pi_*} H^{k-1}(M/G) \xrightarrow{\cdot[\Omega]} H^{k+1}(M/G) \longrightarrow \ldots,$$

which we call the Gysin sequence for locally free circle actions.

As an application of this sequence, we compute the cohomology of $((S^{2m-1} \times S^{2l-1})/\mathbb{Z}_2)/G$ where $m := \ell_F^+$ and $l := \ell_F^-$. Note that, since the spheres are odd-dimensional we know that

$$\begin{aligned} H^*((S^{2m-1} \times S^{2l-1})/\mathbb{Z}_2) &\cong H^*_{\mathrm{bas}\,\mathbb{Z}_2}(S^{2m-1} \times S^{2l-1}) \\ &= H^*(S^{2m-1} \times S^{2l-1})^{\mathbb{Z}_2} \\ &= H^*(S^{2m-1} \times S^{2l-1}) \\ &\cong \frac{\mathbb{R}[x,y]}{(x^2,y^2)}, \end{aligned}$$

where x has degree $2m-1$ and y has degree $2l-1$. Thus

$$H^k((S^{2m-1} \times S^{2l-1})/\mathbb{Z}_2) = \begin{cases} \mathbb{R}\,, & k = 0,\ 2m-1,\ 2l-1,\ 2m+2l-2 \\ 0\,, & \text{else.} \end{cases}$$

If we plug this into the Gysin sequence, we get similar results as [52, Lemma 1.1], where the authors computed the integral cohomology ring of the quotient space of a free S^1 action on a space with the same cohomology as $S^{2m-1} \times S^{2l-1}$, while their proof only points to a standard calculation with the Gysin sequence, we want to give some details.

Lemma 4.0.9. *When $G = S^1$ acts locally freely on $M = S^{2m-1} \times S^{2l-1}$ the real cohomology ring of the resulting quotient orbifold M/G is isomorphic to*

1. $\frac{\mathbb{R}[\Omega,x]}{(\Omega^m,x^2)}$, *where* $\deg(\Omega) = 2$ *and* $\deg(x) = 2l-1$ *or*
2. $\frac{\mathbb{R}[\Omega,y]}{(\Omega^l,y^2)}$, *where* $\deg(\Omega) = 2$ *and* $\deg(y) = 2m-1$.

Here Ω denotes the curvature class associated to a connection form $\alpha \in \Omega^1(M)$.

Proof. Without loss of generality we can assume $l \leq m$. From the cohomology of $S^{2m-1} \times S^{2l-1}$ we find that the interesting portions of the Gysin sequence of the locally free action $S^1 \curvearrowright M$ look like

$$\begin{tikzcd}
0 \arrow[r] & H^0(M/G) \arrow[r] & H^0(M) \arrow[r] & H^{-1}(M/G) \arrow[dll] & \\
& H^1(M/G) \arrow[r] & H^1(M) \arrow[r] & H^0(M/G) \arrow[dll] & \\
& H^2(M/G) \arrow[r] & H^2(M) \arrow[r] & H^1(M/G) \arrow[dll] & \\
& H^3(M/G) \arrow[r] & H^3(M) \arrow[r] & H^2(M/G) \arrow[r] & \dots
\end{tikzcd}$$

$$\vdots$$

$$\begin{tikzcd}
\dots \arrow[r] & H^{2l-4}(M/G) \arrow[r] & H^{2l-4}(M) \arrow[r] & H^{2l-5}(M/G) \arrow[dll] & \\
& H^{2l-3}(M/G) \arrow[r] & H^{2l-3}(M) \arrow[r] & H^{2l-4}(M/G) \arrow[dll] & \\
& H^{2l-2}(M/G) \arrow[r] & H^{2l-2}(M) \arrow[r] & H^{2l-3}(M/G) \arrow[dll] & \\
& H^{2l-1}(M/G) \arrow[r] & H^{2l-1}(M) \arrow[r] & H^{2l-2}(M/G) \arrow[dll] & \\
& H^{2l}(M/G) \arrow[r] & H^{2l}(M) \arrow[r] & H^{2l-1}(M/G) \arrow[dll] & \\
& H^{2l+1}(M/G) \arrow[r] & H^{2l+1}(M) \arrow[r] & H^{2l}(M/G) \arrow[r] & \dots
\end{tikzcd}$$

$$\vdots$$

$$\begin{tikzcd}
\dots \arrow[r] & H^{2m-4}(M/G) \arrow[r] & H^{2m-4}(M) \arrow[r] & H^{2m-5}(M/G) \arrow[dll] & \\
& H^{2m-3}(M/G) \arrow[r] & H^{2m-3}(M) \arrow[r] & H^{2m-4}(M/G) \arrow[dll] & \\
& H^{2m-2}(M/G) \arrow[r] & H^{2m-2}(M) \arrow[r] & H^{2m-3}(M/G) \arrow[dll] & \\
& H^{2m-1}(M/G) \arrow[r] & H^{2m-1}(M) \arrow[r] & H^{2m-2}(M/G) \arrow[dll] & \\
& H^{2m}(M/G) \arrow[r] & H^{2m}(M) \arrow[r] & H^{2m-1}(M/G) \arrow[dll] & \\
& H^{2m+1}(M/G) \arrow[r] & H^{2m+1}(M) \arrow[r] & H^{2m}(M/G) \arrow[r] & \dots
\end{tikzcd}$$

$$\vdots$$

$$\begin{tikzcd}
\dots \arrow[r] & H^{2l+2m-4}(M/G) \arrow[r] & H^{2l+2m-4}(M) \arrow[r] & H^{2l+2m-5}(M/G) \arrow[dll] & \\
& H^{2l+2m-3}(M/G) \arrow[r] & H^{2l+2m-3}(M) \arrow[r] & H^{2l+2m-4}(M/G) \arrow[dll] & \\
& H^{2l+2m-2}(M/G) \arrow[r] & H^{2l+2m-2}(M) \arrow[r] & H^{2l+2m-3}(M/G) \arrow[dll] & \\
& H^{2l+2m-1}(M/G) \arrow[r] & H^{2l+2m-1}(M) \arrow[r] & H^{2l+2m-2}(M/G) \arrow[r] & 0
\end{tikzcd}$$

And after plugging in $H^*(S^{2m-1}\times S^{2l-1})$ we have

$$
\begin{tikzcd}
0 \arrow[r] & H^0(M/G) \arrow[r] & \mathbb{R} \arrow[r] & 0 \arrow[dll] & \\
 & H^1(M/G) \arrow[r] & 0 \arrow[r] & H^0(M/G) \arrow[dll] & \\
 & H^2(M/G) \arrow[r] & 0 \arrow[r] & H^1(M/G) \arrow[dll] & \\
 & H^3(M/G) \arrow[r] & 0 \arrow[r] & H^2(M/G) \arrow[r] & \dots
\end{tikzcd}
$$

$$\vdots$$

$$
\begin{tikzcd}
\dots \arrow[r] & H^{2l-4}(M/G) \arrow[r] & 0 \arrow[r] & H^{2l-5}(M/G) \arrow[dll] & \\
 & H^{2l-3}(M/G) \arrow[r] & 0 \arrow[r] & H^{2l-4}(M/G) \arrow[dll] & \\
 & H^{2l-2}(M/G) \arrow[r] & 0 \arrow[r] & H^{2l-3}(M/G) \arrow[dll] & \\
 & H^{2l-1}(M/G) \arrow[r] & \mathbb{R} \arrow[r] & H^{2l-2}(M/G) \arrow[dll] & \\
 & H^{2l}(M/G) \arrow[r] & 0 \arrow[r] & H^{2l-1}(M/G) \arrow[dll] & \\
 & H^{2l+1}(M/G) \arrow[r] & 0 \arrow[r] & H^{2l}(M/G) \arrow[r] & \dots
\end{tikzcd}
$$

$$\vdots$$

$$
\begin{tikzcd}
\dots \arrow[r] & H^{2m-4}(M/G) \arrow[r] & 0 \arrow[r] & H^{2m-5}(M/G) \arrow[dll] & \\
 & H^{2m-3}(M/G) \arrow[r] & 0 \arrow[r] & H^{2m-4}(M/G) \arrow[dll] & \\
 & H^{2m-2}(M/G) \arrow[r] & 0 \arrow[r] & H^{2m-3}(M/G) \arrow[dll] & \\
 & H^{2m-1}(M/G) \arrow[r] & \mathbb{R} \arrow[r] & H^{2m-2}(M/G) \arrow[dll] & \\
 & H^{2m}(M/G) \arrow[r] & 0 \arrow[r] & H^{2m-1}(M/G) \arrow[dll] & \\
 & H^{2m+1}(M/G) \arrow[r] & 0 \arrow[r] & H^{2m}(M/G) \arrow[r] & \dots
\end{tikzcd}
$$

$$\vdots$$

$$
\begin{tikzcd}
\dots \arrow[r] & H^{2l+2m-4}(M/G) \arrow[r] & 0 \arrow[r] & H^{2l+2m-5}(M/G) \arrow[dll] & \\
 & H^{2l+2m-3}(M/G) \arrow[r] & 0 \arrow[r] & H^{2l+2m-4}(M/G) \arrow[dll] & \\
 & H^{2l+2m-2}(M/G) \arrow[r] & \mathbb{R} \arrow[r] & H^{2l+2m-3}(M/G) \arrow[dll] & \\
 & H^{2l+2m-1}(M/G) \arrow[r] & 0 \arrow[r] & H^{2l+2m-2}(M/G) \arrow[r] & 0
\end{tikzcd}
$$

Chasing this diagram, we see that for all even degrees $2n \leq 2l-2$ we have $H^{2n}(M/G) = \mathbb{R} = \langle [\Omega^n] \rangle$ while for all odd degrees $2n-1 \leq 2l-3$ the cohomology group $H^{2n-1}(M/G)$ vanishes. Now we have to distinguish the cases where

$$\begin{tikzcd} H^{2l-1}(M) \arrow[r] \arrow[d, "\cong"] & H^{2l-2}(M/G) \arrow[d, "\cong"] \\ \mathbb{R} & \mathbb{R} \end{tikzcd}$$

is either zero or an isomorphism.

Case 1: Let this map be the zero map. Then $H^{2l-1}(M/G) = \mathbb{R} = \langle x \rangle$ for some element $x \in H^{2l-1}(M/G)$ such that $\pi^* x$ is the generator of $H^{2l-1}(S^{2k-1} \times S^{2l-1})$. Furthermore, we find that for all degrees $2n \leq 2m-2 : H^{2n}(M/G) = \mathbb{R} = \langle [\Omega^n] \rangle$ and for all odd degrees $2n-1 \leq 2m-3$ the cohomology group $H^{2n-1}(M/G) \cong \mathbb{R}$. Since M/G, as an orbifold, fulfills Poincaré duality, we have $H^{2m}(M/G) \cong H^{2m+2l-3-2k}(M/G) = H^{2l-3}(M/G) = 0$ and $[\Omega]^m = 0$. Moreover, this implies that $H^{2m-1}(M/G) \cong \mathbb{R}$ and that fibre integration $\pi_* \colon H^{2m-1}(M) \to H^{2m-2}(M/G)$ is an isomorphism which sends a generator $y \in H^{2m-1}(S^{2m-1} \times S^{2l-1})$ to some multiple of $[\Omega^{m-1}]$. As an orbifold M/G has dimension $2m+2l-3$ implying that $H^n(M/G) = 0$ for all $n \geq 2m+2l-2$ and $H^{2l+2m-3}(M/G) \cong \mathbb{R}$. We arrive at

$$H^*(M/G) \cong \frac{\mathbb{R}[\Omega, x]}{(\Omega^m, x^2)}, \qquad \deg(x) = 2l-1,\ \deg(\Omega) = 2.$$

Case 2: Now, let $H^{2l-1}(M) \to H^{2l-2}(M/G)$ be an isomorphism. Then $H^{2l-1}(M/G) = 0$, $H^{2l}(M/G) = 0$ and all even and odd cohomologies of M/G vanish until degree $2m-1$. Here we obtain $H^{2m}(M/G) = 0$ and an isomorphism $H^{2m-1}(M/G) \cong \mathbb{R}$. Therefore, all even degree cohomologies above this degree vanish, while the odd cohomologies are isomorphic to $\mathbb{R}$. Finally $H^{2l+2m-3}(M/G) \cong \mathbb{R}$ and the cohomologies vanish above. □

With this, we are now able to proof

Proposition 4.0.10. *In the situation above, the cohomology of the exceptional fibre is given as*

$$H^*\left(((S^{2\ell_F^+-1} \times S^{2\ell_F^- -1})/\mathbb{Z}_2)/G\right) \cong \frac{\mathbb{R}[\Omega, x]}{(\Omega^m, x^2)},$$

where $m \in \{\ell_F^+, \ell_F^-\}$ *and* $\deg(x) \in \{2\ell_F^- - 1, 2\ell_F^+ - 1\}$.

Proof. The locally free action of G on $(S^{2\ell_F^+-1} \times S^{2\ell_F^- -1})/\mathbb{Z}_2$ is given by

$$z \cdot [(x_i, y_i), (x_j, y_j)] := [(z^{\lambda_i} x_i, z^{\lambda_i} y_i), (z^{\lambda_j} x_j, z^{\lambda_j} y_j)].$$

Let us initially ignore the $\mathbb{Z}_2$-action and consider the S^1-orbibundle

$$S^1 \longrightarrow S^{2\ell_F^+-1} \times S^{2\ell_F^- -1} \longrightarrow (S^{2\ell_F^+-1} \times S^{2\ell_F^- -1})/S^1.$$

With the help of the Gysin sequence, see Proposition 4.0.8, one calculates

$$H^*\left((S^{2\ell_F^+-1} \times S^{2\ell_F^- -1})/G\right) \cong \frac{\mathbb{R}[\Omega, x]}{(\Omega^m, x^2)},$$

as in Lemma 4.0.9, where Ω is the curvature of the action and x is the a generator of $H^{2m-1}(S^{2m-1})$ and m is either ℓ_F^+ or ℓ_F^-. But now Ω and x are invariant under the antipodal action, the G-action and the $\mathbb{Z}_2$-action commute, and therefore

$$H^*(((S^{2\ell_F^+-1} \times S^{2\ell_F^- -1})/\mathbb{Z}_2)/G) \cong H^*((S^{2\ell_F^+-1} \times S^{2\ell_F^- -1})/G)^{\mathbb{Z}_2} = \frac{\mathbb{R}[\Omega, x]}{(\Omega^m, x^2)},$$

by [5, III, Corollary 2.3]. □

Since we are not able to find a global cohomology class on the exceptional divisor $\widehat{F}$ restricting to the generator x in the cohomology of the fibre we cannot apply Leray-Hirsch's theorem. We will see later why this is a major flaw of the real blow-up procedure that ends our investigation of real resolution cohomology associated to the real blow-up at this point.

We want to conclude this section by regarding our complex of real resolution differential forms in the light of $\mathfrak{g}$-differential graded algebras as in Section 2.2.2 and define the $\mathfrak{g}$-differential graded algebras

$$\Omega^*(\mathrm{J}^{-1}(0)) := \left\{\omega \in \Omega(\mathrm{J}^{-1}(0)^\top) \;\middle|\; \exists \eta \in \Omega(M) \colon \iota_\top^* \eta = \omega\right\}$$

$$\widehat{\Omega}^*(\mathrm{J}^{-1}(0)) := \left\{\omega \in \Omega(\mathrm{J}^{-1}(0)^\top) \;\middle|\; \exists \eta \in \Omega(\mathrm{Bl}_{\mathrm{G}}^{\mathbb{R}}(M)) \colon (\iota_\top^{\mathbb{R}})^* \eta = (\beta_\top^{\mathbb{R}})^* \omega\right\}.$$

Our complexes of resolution and differential forms on $\mathscr{M}_0$ are isomorphic to the basic subcomplexes of these $\mathfrak{g}$-differential graded algebras via the pullback associated to the quotient map $\pi_\top \colon \mathrm{J}^{-1}(0)^\top \to \mathscr{M}_0^\top$. Now, the big difference between $\Omega^*(\mathrm{J}^{-1}(0))$ and $\widehat{\Omega}^*(\mathrm{J}^{-1}(0))$ and driving force of our investigations is that $\Omega^*(\mathrm{J}^{-1}(0))$ is not invariant under multiplication with a connection form, while $\widehat{\Omega}^*(\mathrm{J}^{-1}(0))$ is when considering a connection form as in Corollary 3.2.2. This makes $\widehat{\Omega}^*(\mathrm{J}^{-1}(0))$ a W^*-module, and even a $\mathfrak{g}$-differential graded algebra of type (C) in the sense of [23, Definition 3.4.1] and [23, Definition 2.3.4]. This is a powerful property, because for such a $\mathfrak{g}$-differential graded algebra A of type (C), the map

$$\begin{aligned} A_{\mathrm{bas}\,\mathfrak{g}} &\longrightarrow C_G(A) \\ \omega &\longmapsto 1 \otimes \omega \end{aligned}$$

induces an isomorphism in cohomology with homotopy inverse $\mathrm{Car} \colon C_G(A) \to A_{\mathrm{bas}\,\mathfrak{g}}$ as in Theorem 2.2.15.

Chapter 5

Equivariant symplectic blow-up and partial desingularization of symplectic quotients

Summary: Instead of desingularizing the whole group action as in Chapter 3, we focus on desingularizing the group action on the zero level set in this chapter as in [38, Remark 6.10] and [46, Section 4] in order to obtain a symplectic manifold $\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M)$ as a result of an iteration of symplectic blow-ups, which admits a Hamiltonian G-action and a smooth G-equivariant map

$$\beta^{\mathbb{C}} \colon \mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M) \longrightarrow M,$$

such that $\beta^{\mathbb{C}} \colon \mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M) \setminus (\beta^{\mathbb{C}})^{-1}(M^G) \to M \setminus M^G$ is an equivariant diffeomorphism and $0 \in \mathbb{R}$ is now a regular value of the momentum map $\widetilde{\mathrm{J}} \colon \mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M) \to \mathbb{R}$. This leads to a desingularization of the symplectic quotient

$$\beta^{\mathbb{C}}_0 \colon \widetilde{\mathscr{M}_0} \longrightarrow \mathscr{M}_0$$

whose exceptional loci we investigate closely. It turns out that they are much more well-behaved than their analogues in the former real desingularization of Chapter 3. Again, we work with $G = S^1$ here.

5.1 Complex blow-up

Let M be a $2n$-dimensional smooth symplectic manifold and $N \subset M$ a smooth submanifold of codimension $2k$ whose normal bundle

$$\nu \colon Q \longrightarrow N, \qquad Q_p := T_pM/T_pN \text{ for all } p \in M$$

has structure group $\mathrm{U}(k)$. Projectivize this bundle fibrewise where the fibres are now considered as complex vector spaces invoking the almost complex structure of M, and denote the resulting bundle by $\mathbb{C}\mathrm{P}(Q) \to N$. Next, consider the space

$$L_Q := \{(l, q) \in \mathbb{C}\mathrm{P}(Q) \times Q \mid q \in l\},$$

where $q \in l$ in particular means that l is a line in $Q_{\nu(q)}$, together with the commutative diagram

$$\begin{array}{ccc} L_Q & \xrightarrow{\varphi} & Q \\ \big\downarrow{\scriptstyle\lambda} & & \big\downarrow{\scriptstyle\nu} \\ \mathbb{C}\mathrm{P}(Q) & \longrightarrow & N, \end{array} \tag{5.1.1}$$

where the map φ sends (l, q) to q and the map λ sends (l, q) to l.

In order to describe the structure of L_Q it is convenient to first introduce the universal line bundle

$$L := \{(l, z) \in \mathbb{CP}^{k-1} \times \mathbb{C}^k \mid z \in l\} \longrightarrow \mathbb{CP}^{k-1},$$

whose total space can also be described as

$$L = \{(l, z) \in \mathbb{CP}^{k-1} \times \mathbb{C}^k \mid l_i z_j = l_j z_i \quad \forall\, 1 \leq i < j \leq k\}, \tag{5.1.2}$$

where $l = [l_1 : \cdots : l_k]$ denote homogeneous coordinates. In particular, L represents the complex blow-up of $\mathbb{C}^k$ along the origin. With this notation, we see that L_Q is a smooth fibre bundle over N with bundle projection $\nu \circ \varphi$ and fibre L. Furthermore, L_Q is a smooth line bundle over $\mathbb{CP}(Q)$ with bundle projection given by the map λ in (5.1.1).

Let further $V \subset Q$ be a closed disc bundle, diffeomorphic to a closed tubular neighbourhood $W \subset M$ of N, put $\widetilde{V} := \varphi^{-1}(V)$, and identify V and W, which allows us to consider $\varphi|_{\widetilde{V}}$ as a map $\varphi\colon \widetilde{V} \to W$. In particular, since φ is clearly a diffeomorphism outside the zero section of Q, we get that $\varphi|_{\partial\widetilde{V}} : \partial\widetilde{V} \to \partial W$ is a diffeomorphism. With these preparations, we make the following

Definition 5.1.1. The *complex blow-up of M along N* is the smooth manifold

$$\mathrm{Bl}_N^{\mathbb{C}}(M) := \overline{M \setminus W} \cup_{\varphi|_{\partial\widetilde{V}}} \widetilde{V}$$

obtained by glueing the manifolds with boundary $\overline{M \setminus W}$ and $\widetilde{V}$ with the gluing map $\varphi|_{\partial\widetilde{V}} : \partial\widetilde{V} \to \partial W$.

The map $\beta^{\mathbb{C}}\colon \mathrm{Bl}_N^{\mathbb{C}}(M) \to M$ defined by

$$\beta^{\mathbb{C}} := \begin{cases} \mathrm{id} & \text{on } \overline{M \setminus W} \\ \varphi & \text{on } \widetilde{V} \end{cases}$$

is called the *blow-down map*.

The set $(\beta^{\mathbb{C}})^{-1}(N) \subset \widetilde{V}$ is called the *exceptional divisor* of $\mathrm{Bl}_N^{\mathbb{C}}(M)$. It is the zero section of L_Q, regarded as a line bundle over $\mathbb{CP}(Q)$, and can thus be identified with $\mathbb{CP}(Q)$.

Remark 5.1.2. The various choices involved in the construction of the blow-up do not cause problems since they lead to equivalent results in the sense that two different choices lead to blow-down maps $\beta^{\mathbb{C}}\colon \mathrm{Bl}_N^{\mathbb{C}}(M) \to M$ and $(\beta^{\mathbb{C}})'\colon \mathrm{Bl}_N^{\mathbb{C}}(M)' \to M$ for which there is a diffeomorphism $f\colon \mathrm{Bl}_N^{\mathbb{C}}(M) \to \mathrm{Bl}_N^{\mathbb{C}}(M)'$ such that $(\beta^{\mathbb{C}})' \circ f = \beta^{\mathbb{C}}$. Furthermore, we can also regard subgroups $H \subset \mathrm{U}(k)$ when reducing the structure group of the normal frame bundle of N, the blow-up is then defined analogously.

We immediately see that the blow-down map $\beta^{\mathbb{C}}$ is a diffeomorphism outside the exceptional divisor.

Blow-up in coordinates

For later use it will be convenient to have explicit coordinates on $\mathrm{Bl}_N^{\mathbb{C}}(M)$ at our disposal. Since N is a codimension $2k$ submanifold of M, there is an atlas $\{(U, \varphi_U)\}$ of M with coordinate maps $\varphi_U(p) = w = (w_1, \ldots, w_{2n})$ such that if $U \cap N \neq \emptyset$,

$$\varphi_U(U \cap N) = \{(w_1, \ldots, w_{2n}) \in \varphi_U(U) \mid (w_1, \ldots, w_{2k}) = 0\}.$$

By shrinking the sets U, we assume w.l.o.g. that every U with $U \cap N \neq \emptyset$ is contained in the interior of W. Then, recalling the identification $W \cong V$, the atlas $\{(U, \varphi_U)\}$ induces an open cover $\{U^{\#} := (\beta^{\mathbb{C}})^{-1}(U)\}$ of $\mathrm{Bl}_N^{\mathbb{C}}(M)$ such that

- if $U^\# \cap (\beta^{\mathbb{C}})^{-1}(N) = \emptyset$, $\beta^{\mathbb{C}} : U^\# \to U$ is a diffeomorphism;
- if $U^\# \cap (\beta^{\mathbb{C}})^{-1}(N) \neq \emptyset$, we have

$$U^\# \cong \{(l,p) \in \mathbb{C}\mathrm{P}^{k-1} \times U \mid (l,(z_1,\dots,z_k)) \in L\},$$

  where we identified $\varphi_U(U) \subset \mathbb{R}^{2k} \times \mathbb{R}^{2(n-k)} \cong \mathbb{C}^k \times \mathbb{R}^{2(n-k)}$, introducing the complex coordinates $(z_1,\dots,z_k) := (w_1,\dots,w_{2k})$, where $z_d := w_d + i \cdot w_{k+d}$.

In the first case, we get coordinates $\varphi_{U^\#} := \varphi_U \circ \beta^{\mathbb{C}}$ on $U^\#$; in the following, it will sometimes be convenient to identify $U \cong U^\#$ and just write φ_U instead of $\varphi_{U^\#}$.

To obtain coordinate maps in the second case, consider the standard open cover $\{V_i\}_{1\leq i\leq k}$ of $\mathbb{C}\mathrm{P}^{k-1}$, where $V_i := \{l = [l_1 : \dots : l_k] \mid l_i \neq 0\}$. Introduce coordinates on V_i by setting

$$(u_1,\dots,u_{k-1}) := \left(\frac{l_1}{l_i},\dots,\frac{l_{i-1}}{l_i},\frac{l_{i+1}}{l_i},\dots,\frac{l_k}{l_i}\right). \tag{5.1.3}$$

The V_i induce a cover of $U^\#$ by sets $U_i^\# \cong \{(l,p) \in V_i \times U \mid (l,(z_1,\dots,z_k)) \in L\}$, which by (5.1.2) are given in terms of the equations

$$z_j = z_i u_j, \quad 1 \leq j < i \quad \text{and} \quad z_j = z_i u_{j-1}, \quad i < j \leq k,$$

yielding the coordinate maps

$$\begin{aligned} \varphi_{U_i^\#} : U_i^\# &\longrightarrow \varphi_{U_i^\#}(U_i^\#) \subset \mathbb{C}^{k-1} \times W_i \\ (l,p) &\longmapsto (u_1,\dots,u_{k-1},z_i,w_{2k+1},\dots,w_{2n}), \end{aligned}$$

where W_i is the image of $\varphi_U(U)$ in $\mathbb{C} \times \mathbb{R}^{2(n-k)}$ under the projection

$$\mathbb{R}^{2n} \longrightarrow \mathbb{C} \times \mathbb{R}^{2(n-k)},\ (w_1,\dots,w_n) \longmapsto (z_i,w_{2k+1},\dots,w_{2n}).$$

A simple computation then shows that the charts $\{(U^\#,\varphi_{U^\#})\}$ and $\{(U_i^\#,\varphi_{U_i^\#})\}$ indeed constitute a smooth atlas for $\mathrm{Bl}_N^{\mathbb{C}}(M)$, compare [28, Section 3.1].

With respect to the charts introduced above, the blow-down map $\beta^{\mathbb{C}}$ is given near the exceptional divisor as follows: If $U^\# \cap (\beta^{\mathbb{C}})^{-1}(N) \neq \emptyset$, then $\beta^{\mathbb{C}}|_{U^\#}$ amounts to mapping (l,p) to p. In the latter case, if $p \notin N$, one has $z_i \neq 0$ for some $1 \leq i \leq k$ and $(\beta^{\mathbb{C}})|_{U^\#}^{-1}(\{p\}) \cong \{(l,p) \mid l = [z_1 : \dots : z_k]\}$ as p defines a unique line; on the contrary, if $p \in N$, then $(\beta^{\mathbb{C}})|_{U^\#}^{-1}(\{p\}) \cong \mathbb{C}\mathrm{P}^{k-1} \times \{p\}$. More precisely, in the coordinates provided by $\varphi_{U_i^\#}$ and φ_U,

- the map $\beta^{\mathbb{C}}$ is given by the monoidal transformation

$$\begin{aligned} &\varphi_U \circ (\beta^{\mathbb{C}})|_{U_i^\#} \circ \varphi_{U_i^\#}^{-1} : \\ &(u_1,\dots,u_{k-1},z_i,w_{2k+1},\dots,w_{2n}) \longmapsto (z_i(u_1,\dots,1,\dots,u_{k-1}),w_{2k+1},\dots,w_{2n}) \end{aligned}$$

  with 1 at the i-th position,
- $(\beta^{\mathbb{C}})^{-1}(N) \cap U_i^\#$ corresponds to the set of points $\{(l,p) \mid z_i = 0\}$.

In particular, the second statement means that the exceptional divisor $(\beta^{\mathbb{C}})^{-1}(N) \subset \mathrm{Bl}_N^{\mathbb{C}}(M)$ is a smooth submanifold of real codimension two.

5.2 Partial desingularization of a Hamiltonian circle action

The case relevant to us is when M is a $2n$-dimensional symplectic manifold and $N \subset M$ is a connected symplectic submanifold of codimension $2k$, which we will assume from now on. In this case, the normal bundle $\nu\colon Q \to N$ can be identified with the symplectic normal bundle and carries an almost complex structure. As a consequence, there is a reduction of the normal frame bundle to $\mathrm{U}(k)$, i.e., we get a principal $\mathrm{U}(k)$-bundle $P \to N$ with

$$Q \cong P \times_{\mathrm{U}(k)} \mathbb{C}^k.$$

Thus, the diagram (5.1.1) from the above construction can be written as

$$\begin{tikzcd} L_Q \cong P \times_{\mathrm{U}(k)} L \arrow[r, "\varphi"] \arrow[d, "\lambda"] & P \times_{\mathrm{U}(k)} \mathbb{C}^k \cong Q \arrow[d, "\nu"] \\ \mathbb{CP}(Q) \cong P \times_{\mathrm{U}(k)} \mathbb{CP}^{k-1} \arrow[r] & N \end{tikzcd}$$

with the map φ given by $\varphi([\wp,(l,z)]) = [\wp, z]$. Compare [44] and [59, Chapter 4].

From now on, M carries a Hamiltonian action of $G = S^1$ with momentum map $\mathrm{J}\colon M \to \mathbb{R}$. Recall the notation from Chapter 2; in particular, $\mathcal{F}$ denotes the set of connected components of M^G. Given Lemma 2.1.11 we make the following

Definition 5.2.1. We denote by $\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M)$ the *complex blow-up* of M, which we define as the smooth manifold that results from successively blowing up M according to Definition 5.1.1 along all $F \in \mathcal{F}$, and by $\beta^{\mathbb{C}} : \mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M) \to M$ the composition of all the blow-down maps. Recall that $\mathcal{F}$ is a finite set due to the compactness of M.

We call $E^{\mathbb{C}} := (\beta^{\mathbb{C}})^{-1}(M^G) \subset \mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M)$ the *exceptional divisor* and for $F \in \mathcal{F}$ we call $E^{\mathbb{C}}_F := (\beta^{\mathbb{C}})^{-1}(F)$ the *exceptional locus* associated with F.

Furthermore, the *strict transform* is the closure

$$\widetilde{C}_{\mathbb{C}} := \overline{(\beta^{\mathbb{C}})^{-1}(\mathrm{J}^{-1}(0)^{\top})} \subset \mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M).$$

Note that the exceptional loci are the connected components of the exceptional divisor.

Proposition 5.2.2. *The strict transform $\widetilde{C}_{\mathbb{C}}$ and the exceptional divisor $E_{\mathbb{C}}$ are smooth submanifolds of $\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M)$ of real codimension 1 and 2, respectively, with simple normal crossings. More precisely, there is an atlas $\{(\mathscr{U}, \varphi_{\mathscr{U}})\}$ of $\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M)$ such that for each $\mathscr{U}$ satisfying $\mathscr{U} \cap \widetilde{C}_{\mathbb{C}} \cap E^{\mathbb{C}}_F \neq 0$ for some $F \in \mathcal{F}$, the coordinate map $\varphi_{\mathscr{U}}(p) = (w_1, \dots, w_{2n}) = (z_1, \dots, z_k, w_{2n-\dim F+1}, \dots, w_{2n})$ fulfills*

- *$w_{2n-\dim F+1}, \dots, w_{2n}$ are local coordinates on F;*
- *$E^{\mathbb{C}}_F \cap \mathscr{U} = \{w_i = w_{i+k} = 0\} = \{z_i = 0\}$ for some $1 \le i \le k$;*
- *$\widetilde{C}_{\mathbb{C}} \cap \mathscr{U} = \{w_j = 0\}$ for some $1 \le j \le 2n - \dim F$ with $j \neq i$, $j \neq i+k$;*
- *$\beta^{\mathbb{C}}(w_1, \dots, w_{2n}) = (z_i(z_1, \dots, 1, \dots, z_k), w_{2n-\dim F+1}, \dots, w_{2n})$ with 1 at the i-th position, where on the right-hand side we use local coordinates of M in the open set $\beta^{\mathbb{C}}(\mathscr{U})$.*

Proof. Since the exceptional loci $E^{\mathbb{C}}_F$ are the connected components of $E^{\mathbb{C}}$ and Section 5.1 shows that each $E^{\mathbb{C}}_F$ is a smooth codimension 2-submanifold of $\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M)$, we see that the same holds for $E^{\mathbb{C}}$.

If an $F \in \mathcal{F}$ satisfies $\widetilde{C}_{\mathbb{C}} \cap E_F^{\mathbb{C}} \neq 0$, then $F \subset \mathrm{J}^{-1}(0)$. For such an F, recall the description of the symplectic normal bundle $\Sigma_F \cong P_F \times_{K_F} \mathbb{C}^{\ell_F^+ + \ell_F^-}$ with $\operatorname{codim} F = 2n - \dim F = 2(\ell_F^+ + \ell_F^-) = 2k$ and the local normal form of the momentum map near F given in Proposition 2.1.11, by which the zero level set is locally described by the relation

$$\Phi_F(\mathrm{J}^{-1}(0) \cap U_F) = \left\{ [\wp, z_1, \dots, z_{\ell_F^+ + \ell_F^-}] \in V_F \subset P_F \times_{K_F} \mathbb{C}^{\ell_F^+ + \ell_F^-} \;\Big|\; \sum \lambda_i^F |z_i|^2 = 0 \right\}.$$

Construct an atlas $\{(\mathscr{U}, \varphi_{\mathscr{U}})\}$ of $\mathrm{Bl}_{\mathrm{G}}^{\mathbb{C}}(M)$ following the procedure described in Section 5.1, where the atlas $\{(U, \varphi_U)\}$ of M with $\varphi_U(p) = w = (w_1, \dots, w_{2n})$ underlying this construction is given in terms of the local trivializations of the fibre bundle $E_F^{\mathbb{C}}$ over F and the diffeomorphisms $\Phi_F : U_F \to V_F$ with the identification $\mathbb{C}^{\ell_F^+ + \ell_F^-} \ni z = (z_1, \dots, z_{\ell_F^+ + \ell_F^-}) \equiv (w_1, \dots, w_{2\ell_F^+ + 2\ell_F^-}) \in \mathbb{R}^{2\ell_F^+ + 2\ell_F^-}$. Each $\mathscr{U}$ with $\mathscr{U} \cap E_F^{\mathbb{C}} \neq \emptyset$ is then mapped by $\varphi_{\mathscr{U}}$ onto a set of the shape

$$\left\{ (l, p) \in V_i \times U \;\Big|\; (l, (w_1, \dots, w_{2\ell_F^+ + 2\ell_F^-})) \in L \right\}$$

for some suitable U and $1 \leq i \leq k$, with $w_{2\ell_F^+ + 2\ell_F^- + 1}, \dots, w_{2n}$ local coordinates of F and $\varphi_{\mathscr{U}}(\mathscr{U} \cap E_F^{\mathbb{C}}) = \{z_i = 0\}$. Since the group $K_F \subset U(\ell_F^+) \times U(\ell_F^-)$ leaves the quadratic form $\sum \lambda_i^F |z_i|^2$ invariant, we get

$$\begin{aligned}
\varphi_{\mathscr{U}}\big(\mathscr{U} \cap (\beta^{\mathbb{C}})^{-1}(\mathrm{J}^{-1}(0)^{\top})\big) &= \left\{ (l, p) \in V_i \times (\mathrm{J}^{-1}(0)^{\top} \cap U) \;\Big|\; (l, (w_1, \dots, w_{2\ell_F^+ + 2\ell_F^-})) \in L \right\} \\
&= \Bigg\{ (l, p) \in V_i \times U \;\Big|\; (l, (w_1, \dots, w_{2\ell_F^+ + 2\ell_F^-})) \in L, \\
&\qquad \sum_{k=1}^{\ell_F^+ + \ell_F^-} \lambda_k^F (w_k^2 + w_{\ell_F^+ + \ell_F^- + k}^2) = 0, \ (w_1, \dots, w_{2\ell_F^+ + 2\ell_F^-}) \neq 0 \Bigg\}.
\end{aligned}$$

Thus,

$$\varphi_{\mathscr{U}}(\mathscr{U} \cap \widetilde{C}_{\mathbb{C}}) = \left\{ (l, p) \in V_i \times U \;\Big|\; ([l_1 : \ldots : l_k], (w_1, \dots, w_{2\ell_F^+ + 2\ell_F^-})) \in L, \ \sum_{k=1}^{\ell_F^+ + \ell_F^-} \lambda_k^F |l_k|^2 = 0 \right\} \tag{5.2.1}$$

since the standard coordinates $(u_1, \dots, u_{2n - \dim F - 1})$ from (5.1.3) for the elements $l \in V_i$ reveal that if $(l, (w_1, \dots, w_{2\ell_F^+ + 2\ell_F^-})) \in L$, then

$$\sum_{k=1}^{\ell_F^+ + \ell_F^-} \lambda_k^F (w_k^2 + w_{\ell_F^+ + \ell_F^- + k}^2) = 0 \iff \sum_{k=1}^{i-1} \lambda_{\iota(k)}^F u_k^2 + \lambda_i^F + \sum_{k=i+1}^{2n - \dim F} \lambda_{\iota(k)}^F u_k^2 = 0$$

for suitable indices $\iota(k)$. Moreover, (5.2.1) is actually a local description of $\widetilde{C}_{\mathbb{C}}$ as a non-singular quadric (recall that $\lambda_i^F \neq 0$ for all i), which reveals that $\widetilde{C}_{\mathbb{C}}$ is a smooth submanifold of $\mathrm{Bl}_{\mathrm{G}}^{\mathbb{C}}(M)$. Further, since $\varphi_{\mathscr{U}}(\mathscr{U} \cap E_F^{\mathbb{C}}) = \{z_i = 0\}$, $\widetilde{C}_{\mathbb{C}}$ is transversal to $E_F^{\mathbb{C}}$. Taking $\sum_{k=1}^{i-1} \lambda_{\iota(k)}^F u_k^2 + \lambda_i^F + \sum_{k=i+1}^{2n - \dim F} \lambda_{\iota(k)}^F u_k^2$ as a coordinate and relabling the others proves the third claim. Finally, the statement about the blow-down map in coordinates follows from the description of $\beta^{\mathbb{C}}$ in coordinates given in Section 5.1. □

Corollary 5.2.3. *For any component F of the fixed point set the restriction of $\beta^{\mathbb{C}}$ to the intersection of the exceptional locus $E_F^{\mathbb{C}}$ associated with F and the strict transform $\widetilde{C}_{\mathbb{C}}$ defines a fibre bundle over F with fibre*

$$\left\{ l = [l_1 : \ldots : l_k] \in \mathbb{CP}^{k-1} \;\Big|\; \frac{1}{2} \sum_{i=1}^{k} \lambda_i^F |l_i|^2 = 0 \right\}. \tag{5.2.2}$$

Proof. Let's compute the strict transform $\widetilde{C}_{\mathbb{C}}$ more explicitely first. In a neighbourhood of F the zero level set is diffeomorphic to

$$\left\{[p,z] \in V_F \subset P_F \times_{K_F} \mathbb{C}^k \,\middle|\, \frac{1}{2}\sum_{i=1}^{k} \lambda_i^F |z_i|^2 = 0\right\}.$$

Thus, the inverse image under the blow-down map of the non-singular part in this neighbourhood is

$$\left\{[p,z,l] \in P_F \times_{K_F} \mathbb{C}^k \times \mathbb{C}\mathrm{P}^{k-1} \,\middle|\, [p,z] \in V_F,\ \frac{1}{2}\sum_{i=1}^{k} \lambda_i^F |z_i|^2 = 0,\ z \in l, z \neq 0\right\}.$$

Using homogeneous coordinates, this is isomorphic to

$$\Bigg\{[p,(z_1,\dots,z_k),[l_1:\dots:l_k]] \in P_F \times_{K_F} \mathbb{C}^k \times \mathbb{C}\mathrm{P}^{k-1}\colon$$
$$[p,z] \in V_F,\ \frac{1}{2}\sum_{i=1}^{k} \lambda_i^F |z_i|^2 = 0,\ z_i l_j = z_j l_i, z \neq 0\Bigg\}.$$

To proceed, we study this set in a standard open subset $U_j := \{[l_1 : \dots : l_k] \in \mathbb{C}\mathrm{P}^k \mid l_j \neq 0\} \cong \mathbb{C}^{k-1}$ of complex projective space and obtain after normalizing $l_j = 1$:

$$\Bigg\{[p,(z_1,\dots,z_k),[l_1:\dots:1:\dots:l_k]] \in P_F \times_{K_F} \mathbb{C}^k \times U_i\colon$$
$$[p,z] \in V_F,\ \frac{1}{2}\sum_{i=1}^{k} \lambda_i^F |z_i|^2 = 0,\ z_i = z_j l_i, z \neq 0\Bigg\}.$$

Using the relation $z_i = z_j l_i$ we get

$$\Bigg\{[p,(z_j l_1,\dots,z_j l_k, l_1,\dots,l_k)] \in P_F \times_{K_F} \mathbb{C}^k \times \mathbb{C}^{k-1}\colon$$
$$[p,z] \in V_F,\ \frac{1}{2}|z_j|^2\sum_{i=1}^{k} \lambda_i^F |l_i|^2 = 0, (l_1,\dots,z_j,\dots,l_k) \neq 0\Bigg\}.$$

Since $z_j \neq 0$ in this affine piece, we can divide the crucial equation by $|z_j|^2$. This reveals

$$\Bigg\{[p,(z_j l_1,\dots,z_j l_k, l_1,\dots,l_k)] \in P_F \times_{K_F} \mathbb{C}^k \times \mathbb{C}^{k-1}\colon$$
$$[p,z] \in V_F,\ \frac{1}{2}\sum_{i=1}^{k} \lambda_i^F |l_i|^2 = 0, (l_1,\dots,z_j,\dots,l_k) \neq 0\Bigg\}.$$

The closure of this is

$$\left\{[p,(z_j l_1,\dots,z_j l_k, l_1,\dots,l_k)] \in P_F \times_{K_F} \mathbb{C}^k \times \mathbb{C}^{k-1} \,\middle|\, [p,z] \in V_F,\ \frac{1}{2}\sum_{i=1}^{k} \lambda_i^F |l_i|^2 = 0\right\}.$$

In total, we obtain that in a neighbourhood of $E_F^{\mathbb{C}}$ the strict transform $\widetilde{C}_{\mathbb{C}}$ is diffeomorphic to

$$\left\{[p,z,l] \in P_F \times_{K_F} \mathbb{C}^k \times \mathbb{C}\mathrm{P}^{k-1} \,\middle|\, [p,z] \in V_F,\ \frac{1}{2}\sum_{i=1}^{k} \lambda_i^F |l_i|^2 = 0,\ z \in l\right\}$$

which proves our claim. $\square$

Remark 5.2.4. Notice that the fibre (5.2.2) is the zero level set $\mathrm{J}^{-1}_{\mathbb{CP}^{k-1},\lambda}(0)$ of the Hamiltonian circle action

$$S^1 \times \mathbb{CP}^{k-1} \longrightarrow \mathbb{CP}^{k-1}, z \cdot [l_1 : \ldots : l_k] := [z^{\lambda_1^F} l_1 : \ldots : z^{\lambda_k^F} l_k]$$

with momentum map

$$\mathrm{J}_{\mathbb{CP}^{k-1},\lambda} \colon \mathbb{CP}^{k-1} \longrightarrow \mathbb{R}, \qquad [l_1 : \ldots : l_k] \longmapsto \frac{\sum_{i=1}^{k} \lambda_i^F \cdot |l_i|^2}{\sum_{i=1}^{k} |l_i|^2}$$

and 0 is a regular value in this case, see [32, Section 5.1]. As a consequence, S^1 acts locally freely on $\mathrm{J}^{-1}_{\mathbb{CP}^{k-1},\lambda}(0)$. We will denote this complex projective space with the Hamiltonian circle action induced by the isotropy representation of F by $\mathbb{CP}^{k-1}_{\lambda,F}$.

The *resolution* or *desingularization* of the G-action on M is the content of the following

Proposition 5.2.5. $\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M)$ *carries a unique G-action making $\beta^{\mathbb{C}} : \mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M) \to M$ equivariant. Furthermore when restricted to $\widetilde{C}_{\mathbb{C}}$ the G-action is locally free.*

Proof. Let $F \in \mathcal{F}$ and recall the description of the symplectic normal bundle $\Sigma_F \cong P_F \times_{K_F} \mathbb{C}^{\ell_F^+ + \ell_F^-}$ of F given in Proposition 2.1.11. Blowing up M along F leads to the diagram

$$\begin{tikzcd}
P_F \times_{K_F} L \arrow[r, "\varphi"] \arrow[d] & P_F \times_{K_F} \mathbb{C}^{\ell_F^+ + \ell_F^-} \cong \Sigma_F \arrow[d, "\nu"] \\
P_F \times_{K_F} \mathbb{CP}^{\ell_F^+ + \ell_F^- - 1} \arrow[r] & F
\end{tikzcd}$$

where L is defined using $k = \ell_F^+ + \ell_F^-$ in the notation above. The sphere $S^{2\ell_F^+ + 2\ell_F^- - 1} \subset \mathbb{C}^{\ell_F^+ + \ell_F^-}$ is invariant under the S^1-action on $\mathbb{C}^{\ell_F^+ + \ell_F^-}$ from Proposition 2.1.11. Since the latter also commutes with the diagonal action on the sphere, it induces an S^1-action on $\mathbb{CP}^{\ell_F^+ + \ell_F^- - 1}$. Thus, S^1 acts on the line bundle L by means of the product action $S^1 \curvearrowright \mathbb{CP}^{\ell_F^+ + \ell_F^- - 1} \times \mathbb{C}^{\ell_F^+ + \ell_F^-}$, and since this action commutes with the K_F-action by Proposition 2.1.11, we obtain an S^1-action on $L_Q \cong P_F \times_{K_F} L$ induced by the product action on $P_F \times L$. The map $\varphi \colon P_F \times_{K_F} L \to P_F \times_{K_F} \mathbb{C}^{\ell_F^+ + \ell_F^-}$ is S^1-equivariant, which means that the S^1-actions on L_Q and M glue together to an S^1-action on $\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M)$.

This action is locally free away from the exceptional divisor, since there it coincides with the original G-action on M, while on the exceptional divisor the action is equal to the isotropy action on the complex projective space. When we restrict this action to the strict transform $\widetilde{C}_{\mathbb{C}}$ it is locally free by Remark 5.2.4.

The equivariance of $\beta^{\mathbb{C}}$ is immediate from the construction. Finally, the uniqueness of the action follows from the fact that the complement of the exceptional divisor, on which the action is uniquely determined by the G-action on M, is dense in $\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M)$. □

As a consequence we obtain

Corollary 5.2.6. *There is a connection form $\alpha \in \Omega^1(\mathrm{J}^{-1}(0)^{\top})$ for the G-action on the regular part of $\mathrm{J}^{-1}(0)$ such that there is a connection form $\widetilde{\alpha} \in \Omega^1(\widetilde{C}_{\mathbb{C}})$ for the G-action on $\widetilde{C}_{\mathbb{C}}$ which extends α in the sense that $\widetilde{\alpha}\big|_{(\beta^{\mathbb{C}})^{-1}(\mathrm{J}^{-1}(0)^{\top})} = (\beta^{\mathbb{C}}_{\top})^* \alpha$ where $\beta^{\mathbb{C}}_{\top}$ is the restriction of $\beta^{\mathbb{C}}$ to $(\beta^{\mathbb{C}})^{-1}(\mathrm{J}^{-1}(0)^{\top})$.*

From now on, fix such a connection α.

Remark 5.2.7. Note that one could have hoped that the action of S^1 on the whole of $\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M)$ was locally free, but in order for the S^1-action to be (locally) free on the blow-up, it has in particular to be (locally) free on the exceptional divisor. But for a compact manifold X acted on by a torus T, the Euler characteristic satisfies $\chi(X) = \chi(X^T)$ by [18, Theorem 9.3]. The Euler characteristic of $\mathbb{C}P^k$ is $k+1$, so there cannot exist a (locally) free S^1-action on $\mathbb{C}P^k$ because the Euler characteristic of the empty set is $\chi(\emptyset) = 0$.

Definition 5.2.8. The *partial desingularization* of $\mathscr{M}_0$ is the orbifold $\widetilde{\mathscr{M}_0} := \widetilde{C}_{\mathbb{C}}/G$ and comes together with the continuous map $\beta_0^{\mathbb{C}}$ which is a diffeomorphism away from the exceptional set defined by the following diagram:

$$\begin{tikzcd} \widetilde{\mathscr{M}_0} \arrow[r, "\beta_0^{\mathbb{C}}"] & \mathscr{M}_0 \\ \widetilde{C}_{\mathbb{C}} \arrow[u, "\pi^{\mathbb{C}}"] \arrow[r, "\beta^{\mathbb{C}}"] & \mathrm{J}^{-1}(0). \arrow[u, "\pi"'] \end{tikzcd}$$

We denote the exceptional fibre bundles of $\beta_0^{\mathbb{C}}$ by

$$\beta_0^{\mathbb{C},F} \colon \widetilde{F} := (\beta_0^{\mathbb{C}})^{-1}(F) \longrightarrow F$$

and the restriction of $\beta_0^{\mathbb{C}}$ to the top stratum as

$$(\beta_0^{\mathbb{C}})_{\top} \colon \widetilde{\mathscr{M}_0}^{\top} := (\beta_0^{\mathbb{C}})^{-1}\left(\mathscr{M}_0^{\top}\right) \longrightarrow \mathscr{M}_0^{\top}.$$

According to Remark 5.2.4, the exceptional bundles are of the form

$$\mathbb{C}\mathrm{P}^{k-1}_{\lambda,F} /\!\!/ S^1 \longrightarrow \widetilde{F} \longrightarrow F,$$

where $2k$ is the codimension of F in M.

The curvature form $d\widetilde{\alpha} \in \Omega^2(\widetilde{C}_{\mathbb{C}})$ is basic and therefore descends to a form $\widetilde{\Omega} \in \Omega^2(\widetilde{\mathscr{M}_0})$ still called curvature form. Let S^1 act on $\mathbb{C}^k$ as induced by the isotropy representation of a fixed point component F. We can decompose $\mathbb{C}^k = \mathbb{C}^{\ell_F^+} \times \mathbb{C}^{\ell_F^-}$ into positive and negative weight spaces and define

$$S_F^{\pm} := \left\{ (z_1, \dots, z_{\ell_F^{\pm}}) \in \mathbb{C}^{\ell_F^{\pm}} \;\middle|\; \sum_{i=1}^{\ell_F^{\pm}} \pm\lambda_i^F |z_i|^2 = 1 \right\}.$$

Now the isotropy representation induces an S^1-action on these ellipsoids as

$$z \cdot (z_1, \dots, z_{\ell_F^{\pm}}) := (z^{\lambda_1} \cdot z_1, \dots, z^{\lambda_{\ell_F^{\pm}}} \cdot z_{\ell_F^{\pm}}).$$

Proposition 5.2.9. *If all weights of the S^1-action on $\mathbb{C}^k$ have the same absolute value, there is a diffeomorphism*

$$\mathbb{C}\mathrm{P}^{k-1}_{\lambda,F} /\!\!/ S^1 \longrightarrow (S_F^+/S^1) \times (S_F^-/S^1).$$

Proof. Let $[l_1 : \ldots : l_k] \in \mathbb{C}\mathrm{P}^{k-1}$ be such that $\sum_{i=1}^{k} \lambda_i^F |l_i|^2 = 0$. Define

$$C := \sum_{i=1}^{\ell_F^+} \lambda_i^F |l_i|^2 = \sum_{i=1}^{\ell_F^-} -\lambda_{\ell_F^+ + i}^F |l_{\ell_F^+ + i}|^2.$$

Then the smooth map

$$\begin{aligned} H\colon \mathbb{CP}^{k-1}_{\lambda,F}/\!\!/S^1 &\longrightarrow (S_F^+/S^1)\times(S_F^-/S^1) \\ S^1\cdot[l_1:\ldots:l_k] &\longmapsto \left(S^1\cdot\frac{1}{\sqrt{C}}(l_1,\ldots,l_{\ell_F^+}), S^1\cdot\frac{1}{\sqrt{C}}(l_{\ell_F^++1},\ldots,l_{\ell_F^++\ell_F^-})\right) \end{aligned}$$

is well-defined with smooth inverse

$$\begin{aligned} G:(S_F^+/S^1)\times(S_F^-/S^1) &\longrightarrow \mathbb{CP}^{k-1}_{\lambda,F}/\!\!/S^1 \\ \left(S^1\cdot(l_1,\ldots,l_{\ell_F^+}), S^1\cdot(l_{\ell_F^++1},\ldots,l_{\ell_F^++\ell_F^-})\right) &\longmapsto S^1\cdot[l_1:\ldots:l_k]. \end{aligned}$$

Indeed, denote by $\tilde{H}$ the map of which we claim that it induces H by passing to the S^1-quotient in the domain, i.e., $\tilde{H}$ is defined like H but without the "$S^1\cdot$" on the left-hand side. Consider firstly $z\in S^1$. Then $\tilde{H}$ maps

$$\begin{aligned} [z^{\lambda_1}\cdot l_1:\ldots:z^{\lambda_k}\cdot l_k] \mapsto &\Bigg(S^1\cdot\frac{1}{\sqrt{C}}(z^{\lambda_1}\cdot l_1,\ldots,z^{\lambda_{\ell_F^+}}\cdot l_{\ell_F^+}), \\ &\quad S^1\cdot\frac{1}{\sqrt{C}}(z^{\lambda_{\ell_F^++1}}\cdot l_{\ell_F^++1},\ldots,z^{\lambda_{\ell_F^++\ell_F^-}}\cdot l_{\ell_F^++\ell_F^-})\Bigg) \\ &= \left(S^1\cdot\frac{1}{\sqrt{C}}(l_1,\ldots,l_{\ell_F^+}), S^1\cdot\frac{1}{\sqrt{C}}(l_{\ell_F^++1},\ldots,l_{\ell_F^++\ell_F^-})\right). \end{aligned}$$

Secondly, consider $w\in\mathbb{C}\setminus\{0\}$. Then

$$\sum_{i=1}^{\ell_F^+}\lambda_i^F|w\cdot l_i|^2 = |w|^2\cdot C$$

and $\tilde{H}$ maps

$$\begin{aligned} [w\cdot l_1:\ldots:w\cdot l_k] \mapsto &\Bigg(S^1\cdot\frac{1}{|w|\cdot\sqrt{C}}(w\cdot l_1,\ldots,w\cdot l_{\ell_F^+}), \\ &\quad S^1\cdot\frac{1}{|w|\cdot\sqrt{C}}(w\cdot l_{\ell_F^++1},\ldots,w\cdot l_{\ell_F^++\ell_F^-})\Bigg) \\ = &\left(S^1\cdot\frac{1}{\sqrt{C}}(\frac{w}{|w|}\cdot l_1,\ldots,\frac{w}{|w|}\cdot l_{\ell_F^+}), S^1\cdot\frac{1}{\sqrt{C}}(\frac{w}{|w|}\cdot l_{\ell_F^++1},\ldots,\frac{w}{|w|}\cdot l_{\ell_F^++\ell_F^-})\right). \end{aligned}$$

Now there are $z_1,z_2\in S^1$ such that $z_1^\lambda = z_2^{-\lambda} = \frac{w}{|w|}$, where $\lambda_1^F=\ldots=\lambda_{\ell_F^+}^F =: \lambda$ and $\lambda_{\ell_F^++1}^F=\ldots=\lambda_{\ell_F^++\ell_F^-}^F = -\lambda$ by assumption, so that

$$\begin{aligned} &\left(S^1\cdot\frac{1}{\sqrt{C}}(\frac{w}{|w|}\cdot l_1,\ldots,\frac{w}{|w|}\cdot l_{\ell_F^+}), S^1\cdot\frac{1}{\sqrt{C}}(\frac{w}{|w|}\cdot l_{\ell_F^++1},\ldots,\frac{w}{|w|}\cdot l_k)\right) \\ &= \left(S^1\cdot\frac{1}{\sqrt{C}}(z_1^\lambda\cdot l_1,\ldots,z_1^\lambda\cdot l_{\ell_F^+}), S^1\cdot\frac{1}{\sqrt{C}}(z_2^{-\lambda}\cdot l_{\ell_F^++1},\ldots,z_2^{-\lambda}\cdot l_k)\right) \\ &= \left(S^1\cdot\frac{1}{\sqrt{C}}(l_1,\ldots,l_{\ell_F^+}), S^1\cdot\frac{1}{\sqrt{C}}(l_{\ell_F^++1},\ldots,l_k)\right). \end{aligned}$$

Thus, H is well-defined. On the other hand, take $z_1, z_2 \in S^1$ and let $z_1', z_2' \in S^1$ be such that $(z_i')^2 = z_i$. Then

$$\begin{aligned}[z_1^\lambda \cdot l_1 : \ldots : z_2^{-\lambda} \cdot l_k] &= [(z_1' z_2')^\lambda \cdot \frac{(z_1')^\lambda}{(z_2')^\lambda} \cdot l_1 : \ldots : (z_1' z_2')^{-\lambda} \cdot \frac{(z_1')^\lambda}{(z_2')^\lambda} \cdot l_k] \\ &= [(z_1' z_2')^\lambda \cdot l_1 : \ldots : (z_1' z_2')^{-\lambda} \cdot l_k] \\ &= (z_1' z_2') \cdot [l_1 : \ldots : l_k]\end{aligned}$$

implying that G is well-defined. □

5.3 Symplectic blow-ups, symplectic cuts and partial desingularization

Based on ideas of Gromov, McDuff [44] showed that the blow-up construction described above is compatible with the symplectic structure in the following sense

Proposition 5.3.1. *There is a symplectic form $\widetilde{\sigma} \in \Omega^2(\mathrm{Bl}_{\mathrm{G}}^{\mathbb{C}}(M))$ such that*

(i) $\widetilde{\sigma} = (\beta^{\mathbb{C}})^* \sigma$ outside a neighbourhood of all exceptional loci $E_F^{\mathbb{C}}$,

(ii) When restricted to a fibre of some exceptional bundle $E_F^{\mathbb{C}} \to F$, $\widetilde{\sigma}$ is equal to $\varepsilon \cdot \sigma_{\mathrm{FS}}$ for some $\varepsilon > 0$, where $\sigma_{\mathrm{FS}} \in \Omega^2(\mathbb{C}\mathrm{P}^{k-1})$ is the Fubini-Study form.

This was then utilized by Kirwan [38, Remark 6.10] to give a first idea of a partial desingularization of a symplectic quotient by a succession of symplectic blow-ups along certain isotropy strata followed by a symplectic reduction. This was made rigorous by Meinrenken-Sjamaar [46, Section 4] who used the symplectic cut technique by Lerman [41]. Let us recall symplectic cuts briefly first.
Suppose S^1 acts on a symplectic manifold (M, σ) in a Hamiltonian fashion with momentum map

$$\mathrm{J} \colon M \longrightarrow \mathbb{R}.$$

Now consider the S^1-action on the product $(M \times \mathbb{C}, \sigma \oplus dx \wedge dy)$ defined by $z \cdot (p, z') := (z \cdot p, z^{-1} \cdot z')$. This is again a Hamiltonian action with momentum map

$$\begin{aligned}\overline{\mathrm{J}} \colon M \times \mathbb{C} &\longrightarrow \mathbb{R} \\ (p, z') &\longmapsto J(p) - \frac{1}{2}|z'|^2.\end{aligned}$$

Then if ε is a regular value of J it is also a regular value of $\overline{\mathrm{J}}$ and

$$\begin{aligned}\overline{\mathrm{J}}^{-1}(\varepsilon) &= \left\{ (p, z') \;\middle|\; \mathrm{J}(p) > \varepsilon, z' = e^{i\varphi} \cdot \sqrt{2(\mathrm{J}(p) - \varepsilon)} \right\} \cup \{(p, 0) \mid \mathrm{J}(p) = \varepsilon\} \\ &\cong \left(\mathrm{J}^{-1}((\varepsilon, \infty) \times S^1)\right) \cup \mathrm{J}^{-1}(\varepsilon).\end{aligned}$$

The *symplectic cut* of (M, σ) at ε is defined as the symplectic reduction

$$(\overline{M}_{\mathrm{J} \geq \varepsilon}, \overline{\sigma}_\varepsilon) := (M \times \mathbb{C}) /\!\!/_\varepsilon S^1 = \overline{\mathrm{J}}^{-1}(\varepsilon)/S^1 \cong \mathrm{J}^{-1}((\varepsilon, \infty)) \cup M /\!\!/_\varepsilon S^1.$$

If furthermore another Lie group G acts on (M, σ) with momentum map $\mathrm{J}_G \colon M \to \mathfrak{g}^*$ and the G-action commutes with the S^1-action, then G also acts on $(\overline{M}_{\mathrm{J} \geq \varepsilon}, \overline{\sigma}_\varepsilon)$ in a Hamiltonian way with momentum map

$$\begin{aligned}\overline{\mathrm{J}_G} \colon \overline{M}_{\mathrm{J} \geq \varepsilon} &\longrightarrow \mathfrak{g}^* \\ p &\longmapsto \begin{cases} \mathrm{J}_G(p), & \text{if } p \in \mathrm{J}^{-1}((\varepsilon, \infty)), \\ \mathrm{J}_G(q), & \text{if } p = [q] \in M /\!\!/_\varepsilon S^1. \end{cases}\end{aligned} \tag{5.3.1}$$

As one easily sees, the symplectic blow-up along a symplectic submanifold is a special case of the symplectic cut where one considers a tubular neighbourhood of the symplectic submanifold and the S^1-action of fibrewise scalar multiplication as explained in [46, Section 4.1.1] and one obtains the Hamiltonian S^1-manifold $(\mathrm{Bl}_{\mathrm{G}}^{\mathbb{C}}(M), \widetilde{\sigma})$ with momentum map $\widetilde{J}\colon \mathrm{Bl}_{\mathrm{G}}^{\mathbb{C}}(M) \to \mathbb{R}$. Meinrenken-Sjamaar define the partial desingularization of $\mathscr{M}_0$ as the (now regular) symplectic quotient $\mathrm{Bl}_{\mathrm{G}}^{\mathbb{C}}(M) /\!\!/ S^1$ of $\mathrm{Bl}_{\mathrm{G}}^{\mathbb{C}}(M)$ at 0, see [46, Section 4.1.2].

By above expression (5.3.1) of the momentum map of the blow-up, the local normal form theorem and Remark 5.2.4 our strict transform $\widetilde{C}_{\mathbb{C}}$ coincides with the regular zero level set $\widetilde{J}^{-1}(0)$ and we conclude

Corollary 5.3.2. *Our partial desingularization* $\widetilde{\mathscr{M}_0} = \widetilde{C}_{\mathbb{C}}/G$ *is diffeomorphic to Kirwan-Meinrenken-Sjamaars's partial desingularization* $\mathrm{Bl}_{\mathrm{G}}^{\mathbb{C}}(M) /\!\!/ S^1$.

This equips the partial desingularization $\widetilde{\mathscr{M}_0}$ with a symplectic form $\widetilde{\sigma}_0$ which allows the identification of Remark 5.2.4 to be a symplectic one after rescaling the Fubiny-Study form on $\mathbb{C}\mathrm{P}^{k-1}_{\lambda,F}$ and with which we can understand the topology of the exceptional fibre bundles

$$\mathbb{C}\mathrm{P}^{k-1}_{\lambda,F} /\!\!/ S^1 \longrightarrow \widetilde{F} \longrightarrow F$$

more deeply:

Proposition 5.3.3. *Let*

$$[\widetilde{\sigma}_0|_{\widetilde{F}}], [\widetilde{\Omega}|_{\widetilde{F}}] \in H^*(\widetilde{F})$$

be the restriction of the symplectic class and the curvature class of $\widetilde{\mathscr{M}_0}$ *to* $(\beta_0^{\mathbb{C}})^{-1}(F)$. *Then*

$$H^*(\widetilde{F}) \cong H^*(F) \otimes \frac{\mathbb{R}[\widetilde{\sigma}_0|_{\widetilde{F}}, \widetilde{\Omega}|_{\widetilde{F}}]}{I_F},$$

where I_F *is an ideal encoding relations between* $[\widetilde{\sigma}_0|_{\widetilde{F}}]$ *and* $[\widetilde{\Omega}|_{\widetilde{F}}]$.

Proof. By a theorem of Kalkman [32, Theorem 5.2 and Remark 5.3], the cohomology of the fibre is isomorphic to

$$H^*(\mathbb{C}\mathrm{P}^{k-1}_{\lambda,F} /\!\!/ S^1) \cong \frac{\mathbb{R}[\phi, \eta]}{I},$$

where ϕ and η are the cohomology classes represented by the symplectic form of $\mathbb{C}\mathrm{P}^{k-1}_{\lambda,F} /\!\!/ S^1$ and the curvature class of the S^1-orbibundle $\mathrm{J}^{-1}_{\lambda,F}(0) \to \mathbb{C}\mathrm{P}^{k-1}_{\lambda,F} /\!\!/ S^1$, respectively. Moreover, I is an ideal of relations between these generators which is specified in [32, loc. cit.]. Since both of the generators are restrictions of classes on $\widetilde{F}$, namely the restriction of the symplectic class $[\widetilde{\sigma}_0]$ and the curvature class $[\widetilde{\Omega}]$ of $\widetilde{\mathscr{M}_0}$ to $\widetilde{F}$, we may apply Leray-Hirsch's theorem, see [6, Theorem 5.11], which finishes the proof. $\square$

Remark 5.3.4. This proposition manifests a striking difference between real and complex partial desingularization because Leray-Hirsch's theorem applies to the exceptional bundles in the complex case while this is not clear in case of real partial desingularization!

Let S^1 act on $\mathbb{C}^k$ as induced by the isotropy representation of a fixed point component F. We can define connection forms on $S^{\pm}_F$ by

$$\Xi_{\pm} := \pm \sum_{i=1}^{\ell^{\pm}_F} x_i dy_i - y_i dx_i.$$

The basic forms $d\Xi_{\pm}$ descend to forms on $S^{\pm}_F/S^1$ denoted by the same name.

Proposition 5.3.5. *If all weights of the S^1-action on $\mathbb{C}^k$ have the same absolute value, the diffeomorphism*

$$\mathbb{C}\mathrm{P}^{k-1}_{\lambda,F} /\!\!/ S^1 \longrightarrow (S_F^+/S^1) \times (S_F^-/S^1),$$

which exists by Proposition 5.2.9, pulls back the form $\frac{1}{2}(d\Xi_+ - d\Xi_-)$ to the symplectic form induced by the Fubini-Study form σ_{FS} on $\mathbb{C}\mathrm{P}^{k-1}$ and $d\Xi_+ + d\Xi_-$ to the curvature form of $\mathbb{C}\mathrm{P}^{k-1}_{\lambda,F} /\!\!/ S^1$.

Proof. The Fubini-Study form of $\mathbb{C}\mathrm{P}^{k-1}_{\lambda,F} /\!\!/ S^1$ is induced by restricting the standard symplectic form $\sum_{i=1}^{k} dx_i \wedge dy_i$ of $\mathbb{C}^k$ to S^{2k-1}, so it is equal to $\frac{1}{2}(d\Xi_+ - d\Xi_-)$. The same reasoning applies to the curvature form. $\square$

Chapter 6

Resolution cohomology and the Kirwan map

Summary: We consider the diagram

$$\begin{tikzcd}
\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M) \arrow[d, "\beta^{\mathbb{C}}"] & & (\beta^{\mathbb{C}})^{-1}\left(\mathrm{J}^{-1}(0)^{\top}\right) \arrow[ll, hook', "\iota^{\mathbb{C}}_{\top}"'] \arrow[d, "\beta^{\mathbb{C}}_{\top}"] \arrow[dd, bend left, "\pi'_{\top}"] \\
M & & \mathrm{J}^{-1}(0)^{\top} \arrow[ll, hook', "\iota_{\top}"] \arrow[d, "\pi_{\top}"] \\
& & \mathscr{M}_0^{\top},
\end{tikzcd}$$

and define the complex of resolution differential forms

$$\widetilde{\Omega}(\mathscr{M}_0) := \left\{ \omega_0 \in \Omega(\mathscr{M}_0^{\top}) \;\middle|\; \exists\, \tilde{\varrho} \in \Omega(\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M)) \colon (\pi'_{\top})^{*}\omega_0 = (\iota^{\mathbb{C}}_{\top})^{*}\tilde{\varrho} \right\},$$

which we use to obtain information on the singular cohomology of $\mathscr{M}_0$ by embedding Sjamaar's complex of differential forms in $\widetilde{\Omega}(\mathscr{M}_0)$, forming the short exact sequence

$$0 \longrightarrow \Omega(\mathscr{M}_0) \longrightarrow \widetilde{\Omega}(\mathscr{M}_0) \longrightarrow C(\mathscr{M}_0) \longrightarrow 0,$$

and studying the induced long exact cohomology sequence. We give geometrical meaning to each occuring cohomological term and identify some special cases in which the long exact cohomology sequence is particularly nice. We then move on to define the resolution Kirwan map

$$\mathcal{K} \colon H^{*}_{G}(M) \longrightarrow H^{*}(\widetilde{\Omega}^{*}(\mathscr{M}_0), d)$$

and investigate its surjectivity properties. To shed some light on our results some examples are studied.

6.1 Resolution differential forms and their underlying $\mathfrak{g}$-differential graded algebra

In this section we introduce a new topological invariant for symplectic quotients of S^1-actions called *resolution cohomology*, and study its basic properties. We use the notation from the previous sections. In order to define resolution forms consider the following diagram

$$
\begin{tikzcd}
\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M) \arrow[d, "\beta^{\mathbb{C}}"] & (\beta^{\mathbb{C}})^{-1}\left(\mathrm{J}^{-1}(0)^{\top}\right) \arrow[l, hook', "\iota^{\mathbb{C}}_{\top}"'] \arrow[d, "\beta^{\mathbb{C}}_{\top}"] \arrow[dd, bend left=60, "\pi'_{\top}"] \\
M & \mathrm{J}^{-1}(0)^{\top} \arrow[l, hook', "\iota_{\top}"] \arrow[d, "\pi_{\top}"] \\
 & \mathscr{M}_0^{\top}.
\end{tikzcd}
\tag{6.1.1}
$$

Definition 6.1.1. With the notation of the diagram (6.1.1) we define the *cochain complex of resolution forms* on $\mathscr{M}_0$ as

$$\widetilde{\Omega}(\mathscr{M}_0) := \left\{\omega_0 \in \Omega(\mathscr{M}_0^{\top}) \;\middle|\; \exists\, \tilde{\varrho} \in \Omega(\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M))\colon (\pi'_{\top})^*\omega_0 = (\iota^{\mathbb{C}}_{\top})^*\tilde{\varrho}\right\},$$

with the exterior derivative d as the differential.

Again, by averaging over G we can replace $\Omega(\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M))$ by $\Omega(\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M))^G$ in this definition.

Remark 6.1.2. The first thing to notice about this definition is that it does not depend on the choices involved in the definition of the blow-up $\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M)$ by Remark 5.1.2.
Another useful observation is that, even though for a given form $\omega_0 \in \Omega(\mathscr{M}_0^{\top})$ the existence of a form $\widetilde{\varrho} \in \Omega(\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M))$ satisfying

$$(\beta^{\mathbb{C}}_{\top})^*\pi^*_{\top}\omega_0 = (\pi'_{\top})^*\omega_0 = (\iota^{\mathbb{C}}_{\top})^*\widetilde{\varrho}$$

might seem quite restrictive because the exceptional divisor $E^{\mathbb{C}} \subset \mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M)$ is a "larger space" than the fixed point set M^G, it is actually less restrictive than being a differential form on $\mathscr{M}_0$ because for any $p \in (\beta^{\mathbb{C}})^{-1}(\mathrm{J}^{-1}(0)^{\top})$ one has

$$((\beta^{\mathbb{C}}_{\top})^*\pi^*_{\top}\omega_0)_p(v_1, \dots, v_{\deg\omega_0}) = (\pi^*_{\top}\omega_0)_{\beta^{\mathbb{C}}(p)}((\beta^{\mathbb{C}}_{\top})_*v_1, \dots, (\beta^{\mathbb{C}}_{\top})_*v_{\deg\omega_0}),$$

and the kernel of $\beta^{\mathbb{C}}_*$ at any point $p_0 \in E^{\mathbb{C}}$ contains the tangent space $T(E^{\mathbb{C}}_F)_{\beta^{\mathbb{C}}(p_0)}$ of the fibre of $E^{\mathbb{C}}_F$ over $\beta^{\mathbb{C}}(p_0)$, where $F \in \mathcal{F}$ is such that $p_0 \in E^{\mathbb{C}}_F$. This makes extending $(\beta^{\mathbb{C}}_{\top})^*\pi^*_{\top}\omega_0$ to $E^{\mathbb{C}}_F$ potentially easier than extending $\pi^*_{\top}\omega_0$ to F – informally, one could describe this as the phenomenon that there is "more room" for finding an extension in $\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M)$ than there is in M.

Let

$$\Omega_{\mathrm{J}}(\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M)) := \left\{\widetilde{\omega} \in \Omega(\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M))^G \;\middle|\; \widetilde{\omega}|_{(\beta^{\mathbb{C}})^{-1}\left(\mathrm{J}^{-1}(0)^{\top}\right)} \text{ is } G\text{-horizontal}\right\}$$

and

$$I_{\mathrm{J}}(\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M)) := \left\{\widetilde{\omega} \in \Omega(\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M))^G \;\middle|\; \widetilde{\omega}|_{(\beta^{\mathbb{C}})^{-1}\left(\mathrm{J}^{-1}(0)^{\top}\right)} = 0\right\}.$$

Proposition 6.1.3. *We have an isomorphism of cochain complexes*

$$\widetilde{\Omega}(\mathscr{M}_0) \cong \frac{\Omega_{\mathrm{J}}(\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M))}{I_{\mathrm{J}}(\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M))}.$$

In particular, considering an element $\omega \in \widetilde{\Omega}(\mathscr{M}_0)$ as a coset on the right-hand side, it is only specified by the restriction of any of its representatives to $(\beta^{\mathbb{C}})^{-1}\left(\mathrm{J}^{-1}(0)^{\top}\right)$.

Proof. We have the natural surjection

$$\Omega_{\mathrm{J}}(\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M)) \xrightarrow{(\iota^{\mathbb{C}}_{\top})^*} \Omega_{\mathrm{bas}\,G}\left((\beta^{\mathbb{C}})^{-1}\left(\mathrm{J}^{-1}(0)^{\top}\right)\right) \xrightarrow{\left((\beta^{\mathbb{C}}_{\top})^{-1}\right)^*} \Omega_{\mathrm{bas}\,G}\left(\mathrm{J}^{-1}(0)^{\top}\right) \xrightarrow{\left(\pi^*_{\top}\right)^{-1}} \widetilde{\Omega}(\mathscr{M}_0),$$

whose kernel is precisely $I_{\mathrm{J}}(\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M))$. Here $\Omega_{\mathrm{bas}\,G}\left(\mathrm{J}^{-1}(0)^{\top}\right)$ and $\Omega_{\mathrm{bas}\,G}((\beta^{\mathbb{C}})^{-1}(\mathrm{J}^{-1}(0)^{\top}))$ are the complexes of basic differential forms on $\mathrm{J}^{-1}(0)^{\top}$ and $(\beta^{\mathbb{C}})^{-1}(\mathrm{J}^{-1}(0)^{\top})$, respectively. □

The relation between resolution forms and the exceptional loci is described in

Proposition 6.1.4. *Let $F \in \mathcal{F}$ be such that $F \subset \mathrm{J}^{-1}(0)$, i.e. $F \in \mathcal{F}_0$. Then*

1. *For $\omega \in \Omega_{\mathrm{J}}(\mathrm{Bl}_{\mathrm{G}}^{\mathbb{C}}(M))$ the restriction $\omega|_{E_F^{\mathbb{C}} \cap \widetilde{C}_{\mathbb{C}}}$ is horizontal.*
2. *For $\eta \in I_{\mathrm{J}}(\mathrm{Bl}_{\mathrm{G}}^{\mathbb{C}}(M))$ the restriction $\eta|_{E_F^{\mathbb{C}} \cap \widetilde{C}_{\mathbb{C}}}$ is zero.*
3. *There is a well-defined surjective restriction map $\widetilde{\Omega}(\mathscr{M}_0) \to \Omega((E_F^{\mathbb{C}} \cap \widetilde{C}_{\mathbb{C}})/G) = \Omega(\widetilde{F})$.*

Proof. First we note that $F \subset \mathrm{J}^{-1}(0)$ implies $E_F^{\mathbb{C}} \cap \widetilde{C}_{\mathbb{C}} \neq \emptyset$. Let $z \in E_F^{\mathbb{C}} \cap \widetilde{C}_{\mathbb{C}}$ and $(z_n) \subset (\beta^{\mathbb{C}})^{-1}(\mathrm{J}^{-1}(0)^{\top})$ be a sequence converging to z. Then for $\omega \in \Omega_{\mathrm{J}}(\mathrm{Bl}_{\mathrm{G}}^{\mathbb{C}}(M))$, $\eta \in I_{\mathrm{J}}(\mathrm{Bl}_{\mathrm{G}}^{\mathbb{C}}(M))$ and $X \in \mathfrak{g}$ one has

$$i_{\overline{X}_z}\omega_z = \lim_{n\to\infty} i_{\overline{X}_{z_n}}\omega_{z_n} = 0 \quad \text{and} \quad \eta_z = \lim_{n\to\infty} \eta_{z_n} = 0,$$

showing *(1)* and *(2)*. Furthermore, this implies that, in view of Proposition 6.1.3, each resolution form gives a well-defined form on the quotient $(E_F^{\mathbb{C}} \cap \widetilde{C}_{\mathbb{C}})/G$, yielding a map $\widetilde{\Omega}(\mathscr{M}_0) \to \Omega((E_F^{\mathbb{C}} \cap \widetilde{C}_{\mathbb{C}})/G)$. To show its surjectivity take $\omega \in \Omega_{\mathrm{bas}\, G}(E_F^{\mathbb{C}} \cap \widetilde{C}_{\mathbb{C}})$. Since $E_F^{\mathbb{C}} \cap \widetilde{C}_{\mathbb{C}}$ is a transverse intersection of two closed submanifolds of $\mathrm{Bl}_{\mathrm{G}}^{\mathbb{C}}(M)$ it is itself again a closed submanifold of $\mathrm{Bl}_{\mathrm{G}}^{\mathbb{C}}(M)$. Now, let U be a G-invariant tubular neighbourhood of $E_F^{\mathbb{C}} \cap \widetilde{C}_{\mathbb{C}}$ in $\mathrm{Bl}_{\mathrm{G}}^{\mathbb{C}}(M)$ diffeomorphic to the normal bundle $\nu(E_F^{\mathbb{C}} \cap \widetilde{C}_{\mathbb{C}})$ of $E_F^{\mathbb{C}} \cap \widetilde{C}_{\mathbb{C}}$. Denote the natural equivariant retraction by $\pi \colon U \to E_F^{\mathbb{C}} \cap \widetilde{C}_{\mathbb{C}}$. We then define

$$\omega' := \varrho_U \cdot \pi^*\omega,$$

where $\varrho_U \in \mathcal{C}_c^{\infty}(U)$ is a smooth G-invariant cut-off function on U with $\varrho_U = 1$ near $E_F^{\mathbb{C}} \cap \widetilde{C}_{\mathbb{C}}$. Then

- $g^*w' = (\varrho_U \circ g) \cdot g^*\pi^*\omega = \varrho_U \cdot \pi^* g^*\omega = \varrho_U \cdot \pi^*\omega = \omega'$, where $g \in G$
- $i_{\overline{X}}\omega' = \varrho_U \cdot \pi^* i_{\overline{X}}\omega = 0$ for $X \in \mathfrak{g}$.

This proves that $\omega' \in \Omega_{\mathrm{J}}(\mathrm{Bl}_{\mathrm{G}}^{\mathbb{C}}(M))$ and $\omega'|_{E_F^{\mathbb{C}} \cap \widetilde{C}_{\mathbb{C}}} = \omega$, yielding *(3)*. □

Next, observe that there is a natural inclusion

$$\Omega(\mathscr{M}_0) \hookrightarrow \widetilde{\Omega}(\mathscr{M}_0), \qquad \omega \mapsto \omega.$$

Indeed, let $\omega \in \Omega(\mathscr{M}_0)$ and $\eta \in \Omega(M)$ be such that $\pi_{\top}^*\omega = \iota_{\top}^*\eta$. Pulling back this equation to $\mathrm{Bl}_{\mathrm{G}}^{\mathbb{C}}(M)$ gives $(\beta_{\top}^{\mathbb{C}})^*\pi_{\top}^*\omega = (\beta_{\top}^{\mathbb{C}})^*\iota_{\top}^*\eta = (\iota_{\top}^{\mathbb{C}})^*(\beta^{\mathbb{C}})^*\eta$. Hence ω is a resolution form with smooth extension $(\beta^{\mathbb{C}})^*\eta$. This leads to the following

Definition 6.1.5. Denote by $C(\mathscr{M}_0)$ the cokernel complex of the natural inclusion $\Omega(\mathscr{M}_0) \hookrightarrow \widetilde{\Omega}(\mathscr{M}_0)$, that is,

$$C(\mathscr{M}_0) := \frac{\widetilde{\Omega}(\mathscr{M}_0)}{\Omega(\mathscr{M}_0)}.$$

The three differential complexes $\Omega(\mathscr{M}_0)$, $\widetilde{\Omega}(\mathscr{M}_0)$ and $C(\mathscr{M}_0)$ are naturally related by the short exact sequence of complexes

$$0 \longrightarrow \Omega(\mathscr{M}_0) \longrightarrow \widetilde{\Omega}(\mathscr{M}_0) \longrightarrow C(\mathscr{M}_0) \longrightarrow 0.$$

This induces a long exact sequence in cohomology

$$
\begin{tikzcd}
0 \arrow[r] & H^0(\Omega(\mathscr{M}_0), d) \arrow[r] & H^0(\widetilde{\Omega}(\mathscr{M}_0), d) \arrow[r] & H^0(C(\mathscr{M}_0), d) \arrow[dll, "\delta"] \\
& H^1(\Omega(\mathscr{M}_0), d) \arrow[r] & H^1(\widetilde{\Omega}(\mathscr{M}_0), d) \arrow[r] & H^1(C(\mathscr{M}_0), d) \arrow[dll, "\delta"] \\
& H^2(\Omega(\mathscr{M}_0), d) \arrow[r] & H^2(\widetilde{\Omega}(\mathscr{M}_0), d) \arrow[r] & H^2(C(\mathscr{M}_0), d) \arrow[r] & \dots .
\end{tikzcd}
\tag{6.1.2}
$$

We want to understand this sequence by interpeting each occuring cohomology geometrically. By Sjamaar's theorem [57, Theorem 5.5] $H^*\left(\Omega(\mathscr{M}_0)\right) \cong H^*(\mathscr{M}_0; \mathbb{R})$, which interprets one third of the occuring spaces. Moreover,

Proposition 6.1.6. *There is an isomorphism of cochain complexes*

$$\widetilde{\Omega}(\mathscr{M}_0) \cong \Omega(\widetilde{\mathscr{M}_0}).$$

Proof. Let ω be a resolution form on $\mathscr{M}_0$. Then there is $\eta \in \Omega(\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M))$ such that $(\beta^{\mathbb{C}}_{\top})^* \pi^*_{\top}\omega = (\iota^{\mathbb{C}}_{\top})^*\eta$. By restricting this form η to $\widetilde{C}_{\mathbb{C}}$ we obtain a G-basic form which does not depend on the extension η of ω since $(\beta^{\mathbb{C}})^{-1}(\mathrm{J}^{-1}(0)^{\top})$ is dense in $\widetilde{C}_{\mathbb{C}}$ and $(\beta^{\mathbb{C}}_0)_{\top}$ is a diffeomorphism. Thus we have a natural map

$$\widetilde{\Omega}(\mathscr{M}_0) \longrightarrow \Omega(\widetilde{\mathscr{M}_0}).$$

On the other hand, $\widetilde{C}_{\mathbb{C}}$ is a closed G-invariant submanifold of $\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M)$ and therefore every G-invariant differential form on $\widetilde{C}_{\mathbb{C}}$, and in particular every G-basic form, admits a G-invariant extension to $\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M)$ and therefore (since $\widetilde{C}$ contains $(\beta^{\mathbb{C}})^{-1}(\mathrm{J}^{-1}(0)^{\top})$) gives us a resolution form on $\mathscr{M}_0$ and a map

$$\Omega(\widetilde{\mathscr{M}_0}) \longrightarrow \widetilde{\Omega}(\mathscr{M}_0).$$

These maps are inverse to each other, commute with the differentials and the isomorphism is proved. ☐

Thus, the natural map

$$H^*(\mathscr{M}_0; \mathbb{R}) \cong H^*(\Omega(\mathscr{M}_0), d) \longrightarrow H^*(\widetilde{\Omega}(\mathscr{M}_0), d) \cong H^*(\widetilde{\mathscr{M}_0})$$

is in fact induced by the pullback of forms on $\mathscr{M}_0$ to forms on $\widetilde{\mathscr{M}_0}$ which we denote by

$$(\beta^{\mathbb{C}}_0)^* \colon \Omega(\mathscr{M}_0) \longrightarrow \Omega(\widetilde{\mathscr{M}_0}).$$

Corollary 6.1.7. *The complex of resolution forms can equivalently be defined as*

$$
\begin{aligned}
\widetilde{\Omega}(\mathscr{M}_0) &= \left\{ \omega_0 \in \Omega(\mathscr{M}_0^{\top}) \,\middle|\, \exists\, \widetilde{\varrho} \in \Omega_{\mathrm{bas}\, G}(\widetilde{C}_{\mathbb{C}}) \colon (\pi'_{\top})^*\omega_0 = (\widetilde{\iota}_{\top})^*\widetilde{\varrho} \right\} \\
&= \left\{ \omega_0 \in \Omega(\mathscr{M}_0^{\top}) \,\middle|\, \exists\, \varrho' \in \Omega(\widetilde{\mathscr{M}_0}) \colon ((\beta^{\mathbb{C}}_0)_{\top})^*\omega_0 = (\iota^{\top}_0)^*\varrho' \right\},
\end{aligned}
$$

where $\widetilde{\iota}_{\top}$ denotes the inclusion $(\beta^{\mathbb{C}})^{-1}(\mathrm{J}^{-1}(0)^{\top}) \to \widetilde{C}_{\mathbb{C}}$ and $\iota^{\top}_0$ is the inclusion $\widetilde{\mathscr{M}_0}^{\top} \to \widetilde{\mathscr{M}_0}$ of $\widetilde{\mathscr{M}_0}^{\top} := \pi^{\mathbb{C}}\left((\beta^{\mathbb{C}})^{-1}(\mathrm{J}^{-1}(0)^{\top})\right) = (\beta^{\mathbb{C}}_0)^{-1}\left(\mathrm{J}^{-1}(0)^{\top}\right)$ and $(\beta^{\mathbb{C}}_0)_{\top} \colon \widetilde{\mathscr{M}_0}^{\top} \to \mathscr{M}_0^{\top}$ is the restriction of $\beta^{\mathbb{C}}_0$ to the top stratum of $\mathscr{M}_0$.

$\square$

So it remains to understand $H^*(C(\mathscr{M}_0))$, the cohomology of the cokernel. We start by looking at the restriction maps

$$\Omega(\mathscr{M}_0) \longrightarrow \Omega(F) \quad \text{and} \quad \Omega(\widetilde{\mathscr{M}_0}) \longrightarrow \Omega(\widetilde{F})$$

from [57, Lemma 3.3] and the collection $\mathcal{F}_0 := \left\{ F \subset \mathrm{J}^{-1}(0) \cap M^{S^1} \right\}$ consisting of all components of the fixed point set contained in the zero level. We obtain the maps

$$\begin{aligned} r\colon \Omega(\mathscr{M}_0) &\longrightarrow \bigoplus_{F \in \mathcal{F}_0} \Omega(F), \\ \omega &\longmapsto (\omega|_F)_{F \in \mathcal{F}_0} =: (r_F(\omega))_{F \in \mathcal{F}_0} \end{aligned}$$

and

$$\begin{aligned} \widetilde{r}\colon \Omega(\widetilde{\mathscr{M}_0}) &\longrightarrow \bigoplus_{F \in \mathcal{F}_0} \Omega(\widetilde{F}), \\ \omega &\longmapsto (\omega|_{\widetilde{F}})_{F \in \mathcal{F}_0}. \end{aligned}$$

Now, consider the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker(r) \arrow[r] \arrow[d, "(\beta_0^{\mathbb{C}})^*_r"] & \Omega(\mathscr{M}_0) \arrow[r, "r"] \arrow[d, "(\beta_0^{\mathbb{C}})^*"] & \bigoplus_{F \in \mathcal{F}_0} \Omega(F) \arrow[r] \arrow[d, "\bigoplus_{F \in \mathcal{F}_0} (\beta_0^{\mathbb{C},F})^*"] & 0 \\ 0 \arrow[r] & \ker(\widetilde{r}) \arrow[r] & \Omega\left(\widetilde{\mathscr{M}_0}\right) \arrow[r, "\widetilde{r}"] & \bigoplus_{F \in \mathcal{F}_0} \Omega(\widetilde{F}) \arrow[r] & 0, \end{tikzcd}$$

where $(\beta_0^{\mathbb{C}})^*_r\colon \ker(r) \to \ker(\widetilde{r})$ is the restriction of $(\beta_0^{\mathbb{C}})^*$ to $\ker(r)$. This diagram is of major interest to us because

$$C(\mathscr{M}_0) \cong \operatorname{coker}((\beta_0^{\mathbb{C}})^*)$$

as a consequence of Proposition 6.1.6. By the Snake lemma, there is an exact sequence of the form

$$\begin{tikzcd} 0 \arrow[r] & \ker((\beta_0^{\mathbb{C}})^*_r) \arrow[r] & \ker((\beta_0^{\mathbb{C}})^*) \arrow[r] & \ker(\bigoplus_{F \in \mathcal{F}_0} (\beta_0^{\mathbb{C},F})^*) \arrow[dll, out=-10, in=170] \\ & \operatorname{coker}((\beta_0^{\mathbb{C}})^*_r) \arrow[r] & \operatorname{coker}((\beta_0^{\mathbb{C}})^*) \arrow[r] & \operatorname{coker}(\bigoplus_{F \in \mathcal{F}_0} (\beta_0^{\mathbb{C},F})^*) \arrow[r] & 0. \end{tikzcd}$$

But $\ker((\beta_0^{\mathbb{C},F})^*) = 0$ because $\beta_0^{\mathbb{C},F}$ is a surjective orbifold submersion. Thus, we have the exact sequence

$$0 \longrightarrow \operatorname{coker}((\beta_0^{\mathbb{C}})^*_r) \longrightarrow \operatorname{coker}((\beta_0^{\mathbb{C}})^*) \longrightarrow \operatorname{coker}(\bigoplus_{F \in \mathcal{F}_0} (\beta_0^{\mathbb{C},F})^*) \longrightarrow 0. \tag{6.1.3}$$

We now claim that

$$H^*(\operatorname{coker}((\beta_0^{\mathbb{C}})^*_r)) = 0, \tag{6.1.4}$$

which by the long exact cohomology sequence associated with the short exact sequence (6.1.3) immediately implies that there is an isomorphism

$$H^*(C(\mathscr{M}_0)) = H^*(\mathrm{coker}((\beta_0^{\mathbb{C}})^*)) \cong H^*(\mathrm{coker}(\bigoplus_{F\in\mathcal{F}_0}(\beta_0^{\mathbb{C},F})^*)) = \bigoplus_{F\in\mathcal{F}_0} H^*(\mathrm{coker}((\beta_0^{\mathbb{C},F})^*)). \tag{6.1.5}$$

To prove our claim, we first observe that since $\ker((\beta_0^{\mathbb{C}})^*_r) = 0$, we have the short exact sequence

$$0 \longrightarrow \ker(r) \longrightarrow \ker(\widetilde{r}) \longrightarrow \mathrm{coker}((\beta_0^{\mathbb{C}})^*_r) \longrightarrow 0 \tag{6.1.6}$$

to compute $H^*(\mathrm{coker}((\beta_0^{\mathbb{C}})^*_r))$ again by using the induced long exact sequence in cohomology.

By [23, Theorem 11.1.1] we have an isomorphism

$$H^*(\ker(\widetilde{r})) \cong H^*_c\left(\widetilde{\mathscr{M}_0} \setminus \left(\bigcup_{F\in\mathcal{F}_0}\widetilde{F}\right)\right),$$

where the right-hand side denotes the cohomology of the complex $\Omega_c(\widetilde{\mathscr{M}_0} \setminus (\cup_{F\in\mathcal{F}_0}\widetilde{F}))$ formed by the differential forms on $\widetilde{\mathscr{M}_0} \setminus (\cup_{F\in\mathcal{F}_0}\widetilde{F}))$ with compact support, and the map $\Omega_c(\widetilde{\mathscr{M}_0} \setminus (\cup_{F\in\mathcal{F}_0}\widetilde{F})) \to \Omega(\widetilde{\mathscr{M}_0})$ inducing the above isomorphism is extension by zero. This implies (by the long exact sequence associated with (6.1.6)) that our claim (6.1.4) follows from the following Proposition, taking into account that $\beta_0^{\mathbb{C}}$ is a diffeomorphism away from the exceptional loci:

Proposition 6.1.8. *Extension by zero induces an isomorphism*

$$\mathrm{ext}\colon H^*_c\left(\mathscr{M}_0 \setminus \left(\bigcup_{F\in\mathcal{F}_0} F\right)\right) \longrightarrow H^*(\ker(r)).$$

For the proof we need the following basic lemma:

Lemma 6.1.9. *Let $F \subset M$ be a component of the fixed point set, $\pi\colon U \to F$ be an invariant tubular neighbourhood and $i\colon F \to U$ be the inclusion. If $\omega \in \Omega^k_{\mathrm{bas}\,G}(U)$ is closed and $i^*\omega = 0$, there is $\nu \in \Omega^{k-1}_{\mathrm{bas}\,G}(U)$ with $i^*\nu = 0$ and $d\nu = \omega$.*

Proof of Lemma 6.1.9. Since U is an equivariant deformation retraction, we have isomorphisms

$$\pi^*\colon H^*_{\mathrm{bas}\,G}(F) \longrightarrow H^*_{\mathrm{bas}\,G}(U) \qquad i^*\colon H^*_{\mathrm{bas}\,G}(U) \longrightarrow H^*_{\mathrm{bas}\,G}(F).$$

From $i^*\omega = 0$ it follows that $[i^*\omega] = 0 \in H^*_{\mathrm{bas}\,G}(F)$. Since i^* is an isomorphism, we find that $[\omega] = 0 \in H^*_{\mathrm{bas}\,G}(U)$ and there is $\nu' \in \Omega^{k-1}_{\mathrm{bas}\,G}(U)$ with $d\nu' = \omega$. Moreover, $0 = i^*\omega = i^*d\nu' = di^*\nu'$ and $i^*\nu'$ is closed. Then

$$\nu := \nu' - \pi^*i^*\nu'$$

is such that

- $d\nu = d(\nu' - \pi^*i^*\nu') = d\nu' - \pi^*(di^*\nu') = d\nu' = \omega,$
- $i^*\nu = i^*(\nu' - \pi^*i^*\nu') = i^*\nu' - i^*\pi^*i^*\nu' = i^*\nu' - (\underbrace{\pi \circ i}_{=\mathrm{id}})^*i^*\nu' = 0.$

□

Proof of Proposition 6.1.8. 1. **Well-definedness:** At first, we have to make sure that the extension map is well-defined with the specified target. To this end, we use the following identification from Section 2.1.2:

$$\Omega(\mathscr{M}_0) \cong \frac{\Omega_{\mathrm{J}}(M)}{I_{\mathrm{J}}(M)},$$

where

$$\Omega_{\mathrm{J}}(M) := \left\{\omega \in \Omega(M)^G \,\middle|\, \omega|_{\mathrm{J}^{-1}(0)^\top} \text{ basic}\right\}$$

and

$$I_{\mathrm{J}}(M) := \left\{\omega \in \Omega(M)^G \,\middle|\, \omega|_{\mathrm{J}^{-1}(0)^\top} = 0\right\}.$$

Then

$$\ker(r) \cong \frac{\left\{\omega \in \Omega(M)^G \,\middle|\, \omega|_{\mathrm{J}^{-1}(0)^\top} \text{ basic, } \omega|_F = 0 \text{ for all } F \in \mathcal{F}_0\right\}}{\left\{\omega \in \Omega(M)^G \,\middle|\, \omega|_{\mathrm{J}^{-1}(0)^\top} = 0,\ \omega|_F = 0 \text{ for all } F \in \mathcal{F}_0\right\}}.$$

Thus, extension by zero induces a well-defined map

$$\operatorname{ext}\colon \Omega_{\mathrm{bas}\ G}(\mathrm{J}^{-1}(0)^\top)_c \longrightarrow \ker(r).$$

2. **Surjectivity:** Let $\omega \in \Omega^k(M)^G$ be such that

   - $\omega|_{\mathrm{J}^{-1}(0)^\top}$ is basic,
   - $\omega|_F = 0$ for all $F \in \mathcal{F}_0$,
   - $d\omega = 0$.

   For $F \in \mathcal{F}_0$, let $\pi\colon U \to F$ be an invariant tubular neighbourhood and $i\colon F \to U$ the inclusion. By the above lemma, we find $\nu_F \in \Omega^{k-1}_{\mathrm{bas}\,G}(U)$ such that

   $$\omega = d\nu_F \text{ on } U \qquad \text{and} \qquad (\nu_F)|_F = 0.$$

   Now, let $\varrho_F \in \mathcal{C}^\infty_c(U)^G$ be a G-invariant compactly supported function, which is equal to 1 on a neighbourhood of F. Then

   $$\widetilde{\omega} := \left(\omega - \sum_{F \in \mathcal{F}_0} d(\varrho_F \cdot \nu_F)\right)\Bigg|_{\mathrm{J}^{-1}(0)^\top} \in \Omega^k_{\mathrm{bas}\ G}(\mathrm{J}^{-1}(0)^\top)_c$$

   is well-defined and such that

   - $d\widetilde{\omega} = 0$,
   - $\widetilde{\omega}$ is basic,

   which shows that $\operatorname{ext}(\widetilde{\omega}) \in \ker(r)$, and by construction we have $[\operatorname{ext}(\widetilde{\omega})] = [\omega] \in H^*(\ker(r))$.

3. **Injectivity:** Let $\omega \in \Omega^k_{\mathrm{bas}\,G}(\mathrm{J}^{-1}(0)^\top)_c$ and $\nu' \in \Omega^{k-1}(M)^G$ be such that

   $$\operatorname{ext}(\omega) = d\nu', \qquad \nu'|_{\mathrm{J}^{-1}(0)^\top} \text{ is basic}, \qquad \text{and} \qquad \nu'|_F = 0 \text{ for all } F \in \mathcal{F}_0.$$

   Then, for each $F \in \mathcal{F}_0$, there is an invariant tubular neighbourhood U of F on which $\operatorname{ext}(\omega)|_U = 0$, and consequently $\nu'|_U$ is closed. By the above lemma there exists $\alpha_F \in \Omega^{k-2}_{\mathrm{bas}\,G}(U)$ such that

   $$\nu'|_U = d\alpha_F \qquad \text{and} \qquad (\alpha_F)|_F = 0.$$

Now, let $\varrho_F \in \mathcal{C}_c^\infty(U)^G$ be a G-invariant compactly supported function, which is equal to 1 on a neighbourhood of F. Then

$$\nu := \left(\nu' - \sum_{F \in \mathcal{F}_0} d(\varrho_F \cdot \alpha_F) \right) \Bigg|_{\mathrm{J}^{-1}(0)^\top}$$

is an element of $\Omega_{\mathrm{bas}\, G}(\mathrm{J}^{-1}(0)^\top)_c$ and $\omega = d\nu$.

□

As stated above, this proves the following

Corollary 6.1.10. *The cohomology of the cokernel complex* $\mathrm{coker}((\beta_0^{\mathbb{C}})_r^*)$ *is*

$$H^*(\mathrm{coker}((\beta_0^{\mathbb{C}})_r^*)) = 0.$$

□

Using Corollary 6.1.10, we deduce from Proposition 5.3.3 that

$$\begin{aligned} H^*(C(\mathscr{M}_0)) &\cong \bigoplus_{F \in \mathcal{F}_0} H^*(\mathrm{coker}((\beta_0^{\mathbb{C},F})^*)) = \bigoplus_{F \in \mathcal{F}_0} \mathrm{coker}\left(H^*(F) \overset{(\beta_0^{\mathbb{C},F})^*}{\longrightarrow} H^*(\widetilde{F}) \right) \\ &= \bigoplus_{F \in \mathcal{F}_0} \mathrm{coker} \left(H^*(F) \to H^*(F) \otimes \frac{\mathbb{R}[\widetilde{\sigma}_0|_{\widetilde{F}}, \widetilde{\Omega}|_{\widetilde{F}}]}{I_F} \right) \\ &\cong \bigoplus_{F \in \mathcal{F}_0} H^*(F) \otimes \frac{\mathbb{R}[\widetilde{\sigma}_0|_{\widetilde{F}}, \widetilde{\Omega}|_{\widetilde{F}}]_{\geq 1}}{I_F}, \end{aligned}$$

where $\mathbb{R}[\widetilde{\sigma}_0|_{\widetilde{F}}, \widetilde{\Omega}|_{\widetilde{F}}]_{\geq 1}$ denotes the ideal in the polynomial ring $\mathbb{R}[\widetilde{\sigma}_0|_{\widetilde{F}}, \widetilde{\Omega}|_{\widetilde{F}}]$ given by all polynomials of degree ≥ 1. In the first line we used that $\ker((\beta_0^F)^*) = 0$ on forms since $\beta_0^{\mathbb{C},F}$ is a surjective orbifold submersion, and in the last line that $(\beta_0^{\mathbb{C},F})^*$ embeds $H^*(F)$ into $H^*(F) \otimes \frac{\mathbb{R}[\widetilde{\sigma}_0|_{\widetilde{F}}, \widetilde{\Omega}|_{\widetilde{F}}]}{I_F}$ as the degree zero subspace.

This finally gives a full understanding of the long exact sequence (6.1.2), yielding:

Theorem 6.1.11. *There is a long exact sequence of the form*

$$\ldots \longrightarrow H^k(\mathscr{M}_0) \overset{(\beta_0^{\mathbb{C}})^*}{\longrightarrow} H^k(\widetilde{\mathscr{M}_0}) \overset{\widetilde{R}}{\longrightarrow} \bigoplus_{F \in \mathcal{F}_0} H^*(F) \otimes \frac{\mathbb{R}[\widetilde{\sigma}_0|_{\widetilde{F}}, \widetilde{\Omega}|_{\widetilde{F}}]_{\geq 1}}{I_F} \overset{\delta}{\longrightarrow} H^{k+1}(\widetilde{\mathscr{M}_0}) \longrightarrow \ldots,$$

where $\widetilde{R}\colon H^*(\widetilde{\mathscr{M}_0}) \to \bigoplus_{F \in \mathcal{F}_0} \mathrm{coker}(H^*(F) \to H^*(\widetilde{F}))$ *is the natural map*

$$\widetilde{R}\colon H^*(\widetilde{\mathscr{M}_0}) \longrightarrow \bigoplus_{F \in \mathcal{F}_0} H^*(\widetilde{F}) \longrightarrow \bigoplus_{F \in \mathcal{F}_0} \mathrm{coker}(H^*(F) \to H^*(\widetilde{F})),$$

induced by restriction to the exceptional components and projection to the according cokernels.

□

As a special case we get

Corollary 6.1.12. *If the fixed point set* $\mathrm{J}^{-1}(0) \cap M^{S^1}$ *consists only of isolated fixed points and one has* $H^{2k+1}(\mathscr{M}_0) = 0$ *for all* k*, then the natural map* $H^*(\mathscr{M}_0) \to H^*(\widetilde{\mathscr{M}_0})$ *is injective and there is a (non-canonical) isomorphism*

$$H^*(\widetilde{\mathscr{M}_0}) \cong H^*(\mathscr{M}_0) \oplus \bigoplus_{F \in \mathcal{F}_0} \frac{\mathbb{R}[\widetilde{\sigma}_0|_{\widetilde{F}}, \widetilde{\Omega}|_{\widetilde{F}}]_{\geq 1}}{I_F}.$$

Remark 6.1.13. This corollary gives us a way to identify those classes in $H^*(\widetilde{\mathscr{M}_0})$ which come from classes in $H^*(\mathscr{M}_0)$. In fact, we have

$$H^*(\mathscr{M}_0) \cong \left\{ [\omega] \in H^*(\widetilde{\mathscr{M}_0}) \,\middle|\, \widetilde{R}([\omega]) = 0 \right\}.$$

Proof. Since $H^*(F) = \mathbb{R}$ when F is a point and the generators $\widetilde{\sigma}_0|_{\widetilde{F}}, \widetilde{\Omega}|_{\widetilde{F}}$ are of degree 2, and in particular even, the odd degree components of $\bigoplus_{F\in\mathcal{F}_0} H^*(F) \otimes \frac{\mathbb{R}[\widetilde{\sigma}_0|_{\widetilde{F}},\widetilde{\Omega}|_{\widetilde{F}}]_{\geq 1}}{I_F} = \bigoplus_{F\in\mathcal{F}_0} \frac{\mathbb{R}[\widetilde{\sigma}_0|_{\widetilde{F}},\widetilde{\Omega}|_{\widetilde{F}}]_{\geq 1}}{I_F}$ vanish and the long exact sequence from Theorem 6.1.11 breaks down into sequences

$$0 \to H^{2k}\left(\mathscr{M}_0\right) \to H^{2k}(\widetilde{\mathscr{M}_0}) \to \left(\bigoplus_{F\in\mathcal{F}_0} \frac{\mathbb{R}[\widetilde{\sigma}_0|_{\widetilde{F}},\widetilde{\Omega}|_{\widetilde{F}}]_{\geq 1}}{I_F} \right)_{2k} \to H^{2k+1}\left(\mathscr{M}_0\right) \to H^{2k+1}(\widetilde{\mathscr{M}_0}) \to 0,$$

where we denote by the subindex $2k$ the subspace of degree $2k$ elements. Now the vanishing of the odd cohomology of $\mathscr{M}_0$ implies that also the odd cohomology of $\widetilde{\mathscr{M}_0}$ vanishes and the above sequence further reduces to

$$0 \longrightarrow H^{k}\left(\mathscr{M}_0\right) \longrightarrow H^{k}(\widetilde{\mathscr{M}_0}) \longrightarrow \left(\bigoplus_{F\in\mathcal{F}_0} \frac{\mathbb{R}[\widetilde{\sigma}_0|_{\widetilde{F}},\widetilde{\Omega}|_{\widetilde{F}}]_{\geq 1}}{I_F} \right)_{k} \longrightarrow 0,$$

where in case of odd k all the vector spaces in the sequence are 0. But every short exact sequence of vector spaces is split by [43, Proposition 4.3] and the corollary follows. $\square$

More generally we can also obtain

Theorem 6.1.14. *Suppose that for each component $F \in \mathcal{F}$ we have $H^{2k+1}(F) = 0$ for all $k \in \mathbb{N}$. Then, the natural map $H^{2k}(\mathscr{M}_0) \to H^{2k}(\widetilde{\mathscr{M}_0})$ of even cohomology groups is injective for every $k \in \mathbb{N}$ and there is a (non-canonical) isomorphism*

$$H^{2k}(\widetilde{\mathscr{M}_0}) \cong H^{2k}(\mathscr{M}_0) \oplus V,$$

where $V := \ker\left(\bigoplus_{F\in\mathcal{F}_0} \operatorname{coker}\left(H^{2k}(F) \to H^{2k}(\widetilde{F})\right) \xrightarrow{\delta} H^{2k+1}\left(\mathscr{M}_0\right) \right).$

Proof. Again, since $H^{2k+1}(F) = 0$ our long exact sequence from Theorem 6.1.11 breaks down into sequences of the form

$$0 \to H^{2k}\left(\mathscr{M}_0\right) \to H^{2k}\left(\widetilde{\mathscr{M}_0}\right) \to \bigoplus_{F\in\mathcal{F}_0} \operatorname{coker}\left(H^{2k}(F) \to H^{2k}(\widetilde{F})\right) \to H^{2k+1}\left(\mathscr{M}_0\right) \to H^{2k+1}\left(\widetilde{\mathscr{M}_0}\right) \to 0.$$

Moreover, by [18, Example 7.9] the S^1-action on M is equivariantly formal so on the one hand, [18, Corollary 8.7], there is an injection

$$H^*_{S^1}(M) \longrightarrow H^*_{S^1}(M^{S^1}) = \bigoplus_{F\subset M^{S^1}} \mathbb{R}[x] \otimes H^*(F)$$

while on the other hand, [18, Theorem 7.3], there is an isomorphism $H^*_{S^1}(M) \cong \mathbb{R}[x] \otimes H^*(M)$. But now the degree of x being two and the odd cohomology of the F vanishing implies that $H^*(M)$ vanishes in odd degrees. By [44, Proposition 2.4] we also have $H^{2k+1}(\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M)) = 0$ and $H^{2k+1}_{S^1}(\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M)) = 0$ as $H^*_{S^1}(\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M)) \cong \mathbb{R}[x] \otimes H^*(\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M))$ again. Since $\widetilde{\mathscr{M}_0}$ is a regular symplectic reduction of $\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M)$ the Kirwan map

$$\kappa\colon H^*_{S^1}(\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M)) \longrightarrow H^*(\widetilde{\mathscr{M}_0})$$

is surjective and degree-preserving which implies that $H^*(\widetilde{\mathscr{M}_0})$ vanishes in odd degrees. We have thus shortened our sequence of interest to

$$0 \to H^{2k}(\mathscr{M}_0) \xrightarrow{A} H^{2k}\left(\widetilde{\mathscr{M}_0}\right) \xrightarrow{B} \bigoplus_{F\in\mathcal{F}_0} \operatorname{coker}\left(H^{2k}(F) \to H^{2k}(\widetilde{F})\right) \xrightarrow{C} H^{2k+1}(\mathscr{M}_0) \to 0,$$

where $A := (\beta_0^{\mathbb{C}})^*$, $B := \widetilde{R}$ and C is the connecting homomorphism δ. Now, setting $V := \ker(C)$ we may look at the sequence

$$0 \longrightarrow H^{2k}(\mathscr{M}_0) \xrightarrow{A} H^{2k}\left(\widetilde{\mathscr{M}_0}\right) \xrightarrow{B'} \ker(C) \longrightarrow 0,$$

where B is equal to B' and we only changed the codomain of B from $\bigoplus_F \operatorname{coker}\left(H^{2k}(F) \to H^{2k}(\widetilde{F})\right)$ to $\ker(C)$ which was allowed since $\ker(C) = \operatorname{Im}(B)$. This sequence is still exact since $\ker(B) = \ker(B')$ and since every short exact sequence of real vector spaces splits, we obtain the desired

$$H^{2k}(\widetilde{\mathscr{M}_0}) \cong H^{2k}(\mathscr{M}_0) \oplus V.$$

□

Remark 6.1.15. In the situation of Theorem 6.1.14, the injectivity does not have to hold in odd degrees, even if $H^k(F) = 0$ for all $k \in \mathbb{N}$, $k \geq 1$ and $F \in \mathcal{F}$, as we will see later in Example 6.3.4. Note that also from the general theory of resolution of singularities we cannot expect that the natural map $H^*(\mathscr{M}_0) \longrightarrow H^*(\widetilde{\mathscr{M}_0})$ is always injective, since resolutions do not always induce injective maps in cohomology as the example of the nodal curve shows, see [24, Example I.4.9.1]. In fact, the nodal curve has non trivial cohomology in positive degrees while the resolution is contractible.

Remark 6.1.16. The isomorphism $H^*\left(\widetilde{\mathscr{M}_0}\right) \cong H^*(\mathscr{M}_0) \oplus \bigoplus_{F\in\mathcal{F}_0} \frac{\mathbb{R}[\widetilde{\sigma}_0|_{\widetilde{F}}, \widetilde{\Omega}|_{\widetilde{F}}]_{\geq 1}}{I_F}$ of the previous Corollary 6.1.12 can be made explicit by recalling how short exact sequences of vector spaces split, compare [43, Propositions 4.2 and 4.3]. More precisely, let A, B, C be real vector spaces and

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

a short exact sequence of these. Now take a basis $\{c_i\}_{i\in I}$ of C. By surjectivity of g, there are elements $\{b_i\}_{i\in I}$ in B which map to that basis. We can now specify a map

$$h\colon C \longrightarrow B$$

by sending c_i to b_i because $\{c_i\}_{i\in I}$ is a basis. Since h is a right inverse of g, the sequence is split and

$$A \oplus C \longrightarrow B, \qquad (a, c) \longmapsto f(a) + h(c)$$

is the splitting isomorphism. By means of the map h one can also find a projection $p\colon B \to A$ which is left inverse to f. For this one considers the mapping

$$B \longrightarrow \ker(g) \oplus \operatorname{Im}(h), \qquad b \longmapsto (b - h(g(b))) + h(g(b)).$$

Since $\operatorname{Im}(h) \cong C$ and $\ker(g) = \operatorname{Im}(f) \cong A$, f being injective,

$$B \longrightarrow \ker(g) \cong A \qquad b \longmapsto (b - h(g(b))),$$

is the desired projection. Applying this to our present situation, we see that the splitting isomorphism from Corollary 6.1.12 and a projection

$$H^*(\widetilde{\mathscr{M}_0}) \longrightarrow H^*(\mathscr{M}_0)$$

can be made explicit by fixing a basis of $\bigoplus_{F\in\mathcal{F}_0} \frac{\mathbb{R}[\widetilde{\sigma}_0|_{\widetilde{F}},\widetilde{\Omega}|_{\widetilde{F}}]_{\geq 1}}{I_F}$ and finding preimages of the basis vectors inside $H^*\left(\widetilde{\mathscr{M}_0}\right)$, or in other words by fixing a right-inverse

$$h\colon \bigoplus_{F\in\mathcal{F}_0} \frac{\mathbb{R}[\widetilde{\sigma}_0|_{\widetilde{F}},\widetilde{\Omega}|_{\widetilde{F}}]_{\geq 1}}{I_F} \longrightarrow H^*\left(\widetilde{\mathscr{M}_0}\right)$$

to the map $\widetilde{R}$ from Theorem 6.1.11.

Example 6.1.17. As an example consider the product $S^2 \times S^2$ with its product symplectic form where we equipped each sphere with its standard volume form. Then the S^1 action defined by diagonally rotating around the z-axis with speed 1 is a Hamiltonian action with momentum map

$$\begin{aligned} \mathrm{J}\colon S^2\times S^2 &\longrightarrow \mathbb{R} \\ ((x_i,y_i,z_i))_{i=1,2} &\longmapsto z_1+z_2. \end{aligned}$$

Now, zero is not a regular value of J as the zero level contains the fixed points $((0,0,1),(0,0,-1))$ and $((0,0,-1),(0,0,1))$. In fact, the zero level set is a suspended 2-torus and the symplectic quotient $\mathscr{M}_0$ is a suspension of S^1 and in particular homeomorphic to S^2. From Proposition 5.2.9 we see that in this case all the occuring exceptional divisors of the partial desingularization are in fact points, as the occuring spheres $S_F^{\pm}$ in the weight spaces are one-dimensional. Thus, for all fixed points F in the zero level set we have $F=\widetilde{F}$ and the long exact sequence from Theorem 6.1.11 shows that

$$\beta^*\colon H^k(\mathscr{M}_0)\cong H^k(\widetilde{\mathscr{M}_0}) \qquad \forall k\in\mathbb{N}.$$

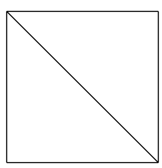
Consequently, resolution cohomology and ordinary cohomology of $\mathscr{M}_0$ are isomorphic. Moreover, using techniques from toric geometry, one shows that the curvature class of the partial desingularization $\widetilde{\mathscr{M}_0}$ vanishes in this case. In fact, $M=S^2\times S^2$ is a toric symplectic manifold acted upon by the two-torus $T^2=S^1\times S^1$ where each circle rotates one of the spheres around the z-axis. A momentum map for this action is given by

$$\begin{aligned} \mathrm{J}'\colon M &\longrightarrow \mathbb{R}^2 \\ ((x_i,y_i,z_i))_{i=1,2} &\longmapsto (z_1,z_2), \end{aligned}$$

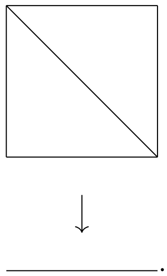
where we identified the dual of the Lie algebra of T^2 with $\mathbb{R}^2$. Our original momentum map J arises from J' by composing it with the map

$$\begin{aligned} \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x,y) &\longmapsto x+y \end{aligned}$$

as we consider the diagonal S^1-action. Now, the momentum image of $\mathrm{J}'(M)\subset\mathbb{R}^2$ is the convex polytope

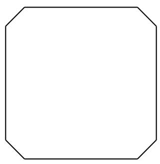


where the drawn anti-diagonal depicts $\mathrm{J}'(\mathrm{J}^{-1}(0))$. From Proposition 2.1.28 we already know, that $\mathscr{M}_0$ will be indeed a symplectic manifold. Then, the complementary $S^1 \subset T^2, z \mapsto (z,1)$ acts in a Hamiltonian way with momentum map J'' whose momentum image we obtain by the projection of the anti-diagonal as

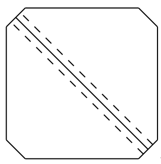


.

Now, $\mathscr{M}_0$ is a toric symplectic two-manifold whose momentum polytope associated to the S^1-action is an interval. By the classification of symplectic toric manifolds due to Delzant, see [4, Theorem IV.4.20 and Section VII.2], [21, Chapter 1] and [9, Theorem 28.2] $\mathscr{M}_0$ has to be S^2. A similar reasoning applies to the blow-up $\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}$ whose momentum polytope is

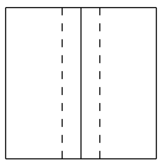


by [41, Remark 1.5]. Now, we may depict the strict transform as the anti-diagonal in



.

The dashed part represents a neighbourhood of $\widetilde{\mathrm{J}}^{-1}(0)$ in $\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M)$ and the image tells us that it is T^2-equivariantly symplectomorphic to a neighbourhood of $S^2 \times S^1$ in $S^2 \times S^2$ as depicted in

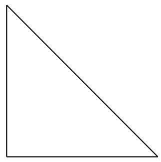


and the bundle $\widetilde{\mathrm{J}}^{-1}(0) \to \widetilde{\mathscr{M}_0}$ is trivial.

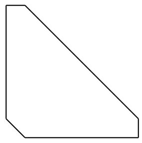
Example 6.1.18. Moving on from the previous example to the case of the standard diagonal S^1-action on $M := S^2 \times S^2 \times S^2$ with momentum map

$$\begin{aligned} \mathrm{J}\colon S^2 \times S^2 \times S^2 &\longrightarrow \mathbb{R} \\ ((x_i, y_i, z_i))_{i=1,2,3} &\longmapsto z_1 + z_2 + z_3 \end{aligned}$$

we see that the critical values are $-3, -1, 1$ and 3. There are 3 isolated fixed points in the -1-level set and the isotropy representation at each of those has one positive and two negative weights. Again, using toric geometry and the fact that M is indeed a toric symplectic manifold, one proves that $\mathscr{M}_0$ is $\mathbb{C}\mathrm{P}^2$ and $\widetilde{\mathscr{M}_0}$ is the blow-up of $\mathscr{M}_0$ in three points. Moreover, the exceptional bundles $\widetilde{F}$ are equal to $\mathbb{C}\mathrm{P}^1$s in this case and the curvature of $\mathscr{M}_0$ does not vanish. Thus, in this example all terms in the long exact sequence from Theorem 6.1.11 are excplicitely known. In fact, we consider M as a toric Hamiltonian T^3-manifold, whose momentum polytope is a cube. By arguments similar to those in Example 6.1.17, one obtains that the momentum image of a complementary T^2-action on the smooth quotient $\mathscr{M}_0$ is



and $\mathscr{M}_0$, which is a smooth symplectic manifold by Proposition Proposition 2.1.28, is symplectomorphic to $\mathbb{C}\mathrm{P}^2$ while the partial desingularization $\widetilde{\mathscr{M}_0}$ has momentum image



and is therefore symplectomorphic to $\mathbb{C}\mathrm{P}^2$ blown-up in the three corner points. Now the bundle $\widetilde{\mathrm{J}}^{-1}(0) \to \widetilde{\mathscr{M}_0}$ cannot be trivial since it is equal to the Hopf bundle, when restricted to the exceptional loci of the blow-up

$$\mathrm{Bl}_3^{\mathbb{C}}(\mathbb{C}\mathrm{P}^2) \longrightarrow \mathbb{C}\mathrm{P}^2$$

and the curvature class of $\widetilde{\mathscr{M}_0}$ does not vanish.

Finally, we want to see how to relate the cohomology of the cokernel $C(\mathscr{M}_0)$ which turned out to fulfill

$$H^*(C(\mathscr{M}_0)) \cong \bigoplus_{F \in \mathcal{F}_0} \operatorname{coker}\left(H^*(F) \to H^*(\widetilde{F})\right)$$

to relative de Rham theory defined in [6, p. 78]. Paraphrasing the construction, we make the following

Definition 6.1.19. Consider orbifolds M and N and let $f\colon M \to N$ be a smooth map. Define the complex

$$\begin{aligned} \Omega^q(f) &:= \Omega^q(N) \oplus \Omega^{q-1}(M) \\ d(\omega, \theta) &:= (d\omega, f^*\omega - d\theta). \end{aligned}$$

This complex sits inside the short exact sequence

$$0 \longrightarrow \Omega^{*-1}(M) \xrightarrow{\alpha} \Omega^*(f) \xrightarrow{\beta} \Omega^*(N) \longrightarrow 0,$$

where $\alpha(\theta) := (0, \theta)$ and $\beta(\omega, \theta) = \omega$. The cohomology $H^*(\Omega^*(f)) =: H^*(\Omega(f\colon M \to N)) =: H^*(f)$ is called the relative de Rham cohomology of $f\colon M \to N$ and the above short exact sequence induces a long exact sequence in the following way

$$\ldots \longrightarrow H^q(f) \xrightarrow{\beta^*} H^q(N) \xrightarrow{f^*} H^q(M) \xrightarrow{\alpha^*} H^{q+1}(f) \longrightarrow \ldots.$$

Now, since $\beta_0^{\mathbb{C},F}\colon \frac{E_F^{\mathbb{C}} \cap \widetilde{C}_{\mathbb{C}}}{G} \to F$ is a fibre bundle and a surjective orbifold submersion the pull-back $(\beta_0^{\mathbb{C},F})^*\colon \Omega(F) \to \Omega(\frac{E_F^{\mathbb{C}} \cap \widetilde{C}_{\mathbb{C}}}{G})$ is injective and we obtain a short exact sequence of complexes

$$0 \longrightarrow \Omega^*(F) \longrightarrow \Omega^*\left(\frac{E_F^{\mathbb{C}} \cap \widetilde{C}_{\mathbb{C}}}{G}\right) \longrightarrow \operatorname{coker}\left((\beta_0^{\mathbb{C},F})^*\right) \longrightarrow 0.$$

Consider the map

$$\begin{aligned} \psi = \left(\psi^k\colon \Omega^k\left(\beta_0^{\mathbb{C},F}\colon \frac{E_F^{\mathbb{C}} \cap \widetilde{C}_{\mathbb{C}}}{G} \to F\right) \longrightarrow \operatorname{coker}^{k-1}\left((\beta_0^{\mathbb{C},F})^*\right)\right) \\ (\omega, \theta) \longmapsto [\theta] = \theta + \Omega^*(F). \end{aligned}$$

The map ψ anticommutes with the differential because

$$\psi^k(d(\omega,\theta)) = \psi^k(d\omega, (\beta_0^{\mathbb{C},F})^*\omega - d\theta) = \left((\beta_0^{\mathbb{C},F})^*\omega - d\theta\right) + \Omega^*(F) = [-d\theta] = -d(\psi^{k-1}(\omega,\theta)),$$

and therefore induces a map $\psi\colon H^*(\beta_0^{\mathbb{C},F}) \to H^{*-1}\left(\operatorname{coker}\left((\beta_0^{\mathbb{C},F})^*\right)\right)$. Now, $H^*(\beta_0^{\mathbb{C},F})$ and $H^*(\operatorname{coker}\left((\beta_0^{\mathbb{C},F})^*\right))$ sit inside long exact sequences, which ψ relates by the diagram

$$\begin{tikzcd}
\dots \arrow[r] & H^{q-1}(F) \arrow[r, "(\beta_0^{\mathbb{C},F})^*"] \arrow[d, "\mathrm{id}"] & H^{q-1}\left(\frac{E_F^{\mathbb{C}}\cap\widetilde{C}_{\mathbb{C}}}{G}\right) \arrow[r, "\alpha^*"] \arrow[d, "\mathrm{id}"] & H^q(\beta_0^{\mathbb{C},F}) \arrow[r, "\beta^*"] \arrow[d, "\psi^q"] & H^q(F) \arrow[r, "(\beta_0^{\mathbb{C},F})^*"] \arrow[d, "\mathrm{id}"] & H^q\left(\frac{E_F^{\mathbb{C}}\cap\widetilde{C}_{\mathbb{C}}}{G}\right) \arrow[r] \arrow[d, "\mathrm{id}"] & \dots \\
\dots \arrow[r] & H^{q-1}(F) \arrow[r, "(\beta_0^{\mathbb{C},F})^*"] & H^{q-1}\left(\frac{E_F^{\mathbb{C}}\cap\widetilde{C}_{\mathbb{C}}}{G}\right) \arrow[r, "\mathrm{pr}"] & H^{q-1}\left(\operatorname{coker}((\beta_0^{\mathbb{C},F})^*)\right) \arrow[r, "\delta"] & H^q(F) \arrow[r, "(\beta_0^{\mathbb{C},F})^*"] & H^q\left(\frac{E_F^{\mathbb{C}}\cap\widetilde{C}_{\mathbb{C}}}{G}\right) \arrow[r] & \dots .
\end{tikzcd}$$

We need to check, that this diagram is commutative. Of course, the interesting squares are the ones containing ψ. For $[\theta] \in H^{q-1}\left(\frac{E_F^{\mathbb{C}}\cap\widetilde{C}_{\mathbb{C}}}{G}\right)$ we have

$$\psi^q(\alpha^*)([\theta]) = \psi^q([0,\theta]) = [\theta] = \mathrm{pr}([\theta])$$

and for $[\omega,\theta] \in H^q(\beta_0^{\mathbb{C},F})$,

$$\beta^*([\omega,\theta]) = [\omega] = \delta([\theta])$$

by the construction of the connecting homomorphism δ, see [6, p. 17]. With the five lemma we conclude that

$$\psi\colon H^*(\beta_0^{\mathbb{C},F}) \longrightarrow H^{*-1}\left(\operatorname{coker}((\beta_0^{\mathbb{C},F})^*)\right)$$

is an isomorphism. As a corollary we obtain

Corollary 6.1.20. *The cohomology of the cokernel $H^*(C(\mathscr{M}_0))$ is iomorphic to the direct sum of the relative cohomologies associated to the exceptional fibre bundles $\beta_0^{\mathbb{C},F} : \widetilde{F} \to F$ for $F \in \mathcal{F}_0$ where the degree is shifted by one, i.e.*

$$H^*(C(\mathscr{M}_0)) \cong \bigoplus_{F\in\mathcal{F}_0} H^{*+1}(\beta_0^{\mathbb{C},F}).$$

□

6.2 The Kirwan map

Let (M,σ) be a compact connected symplectic manifold carrying a Hamiltonian group action of a compact Lie group G with equivariant momentum map $\mathrm{J}\colon M \to \mathfrak{g}^*$ and consider the associated symplectic quotient $\mathscr{M}_0 := \mathrm{J}^{-1}(0)/G$. One of the main tools in the study of its cohomology is the *Kirwan map*, which in the case when 0 is a regular value of the momentum map is defined as the composition

$$\kappa : H_G^*(M) \xrightarrow{\iota^*} H_G^*(\mathrm{J}^{-1}(0)) \xrightarrow{\mathrm{Car}} H^*_{\mathrm{bas}\,G}(\mathrm{J}^{-1}(0)) \xrightarrow{(\pi^*)^{-1}} H^*(\mathscr{M}_0),$$

where $\iota : \mathrm{J}^{-1}(0) \to M$ and $\pi : \mathrm{J}^{-1}(0) \to \mathscr{M}_0$ are the natural injection and projection, respectively, and $\mathrm{Car}\colon H_G^*(\mathrm{J}^{-1}(0)) \to H^*_{\mathrm{bas}\,G}(\mathrm{J}^{-1}(0))$ is the *Cartan isomorphism* of equivariant and basic cohomology. For $G = S^1$ it is explicitely given by [14]

$$\mathrm{Car}\left(\sum_I \omega_I \cdot x^I\right) := \sum_I \omega_I \wedge \Omega^I - \alpha \wedge \sum_I i_{\overline{X}}\omega_I \wedge \Omega^I,$$

for $\alpha \in \Omega^1(M)$ a connection form, Ω its curvature and X a generator of $\mathfrak{g}$ with $\alpha(\overline{X}) = 1$. As a major application of our theory of resolution differential forms developed in the previous section, we are able to extend the above definition of the Kirwan map to the case where 0 is not necessarily a regular value of J: We define a linear map

$$\mathcal{K}\colon H^*_G(M) \to H^*(\widetilde{\Omega}^*(\mathscr{M}_0), d) \tag{6.2.1}$$

from the equivariant cohomology $H^*_G(M)$ to the resolution cohomology $H^*(\widetilde{\Omega}^*(\mathscr{M}_0), d)$ which we call the *resolution Kirwan map*. To do so, let us first look at our complex of resolution differential forms in the light of $\mathfrak{g}$-differential graded algebras, see Section 2.2.2 and [15, Section 4] and [23] for a systematic exposition.
In our setting, we introduce the $\mathfrak{g}$-differential graded algebras

$$\begin{aligned} \Omega^*(\mathrm{J}^{-1}(0)) &:= \left\{\omega \in \Omega(\mathrm{J}^{-1}(0)^\top) \;\middle|\; \exists \eta \in \Omega(M)\colon \iota_\top^* \eta = \omega\right\}, \\ \widetilde{\Omega}^*(\mathrm{J}^{-1}(0)) &:= \left\{\omega \in \Omega(\mathrm{J}^{-1}(0)^\top) \;\middle|\; \exists \eta \in \Omega(\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M))\colon (\iota^{\mathbb{C}}_\top)^* \eta = (\beta^{\mathbb{C}}_\top)^* \omega\right\}. \end{aligned} \tag{6.2.2}$$

These complexes of differential and resolution forms on $\mathscr{M}_0$ are isomorphic to the basic subcomplexes of these $\mathfrak{g}$-differential graded algebras, respectively, via the pullback associated to the quotient map $\pi_\top\colon J^{-1}(0)^\top \to \mathscr{M}_0^\top$ because G is connected. The associated equivariant cohomology groups are

$$H^*_G(\mathrm{J}^{-1}(0)) := H^*(C_G(\Omega^*(\mathrm{J}^{-1}(0))), d_G) \qquad \text{and} \qquad H^*(C_G(\widetilde{\Omega}^*(\mathrm{J}^{-1}(0))), d_G),$$

respectively, where d_G is the equivariant differential. The crucial difference between $\Omega^*(\mathrm{J}^{-1}(0))$ and $\widetilde{\Omega}^*(\mathrm{J}^{-1}(0))$ and driving force behind our investigations is that after fixing a connection form as in Corollary 5.2.6, $\Omega^*(\mathrm{J}^{-1}(0))$ is not invariant under multiplication with this connection form, while $\widetilde{\Omega}^*(\mathrm{J}^{-1}(0))$ is. This makes $\widetilde{\Omega}^*(\mathrm{J}^{-1}(0))$ a W^*-module, and even a $\mathfrak{g}$-differential graded algebra of type (C) in the sense of [23, Definition 3.4.1] and [23, Definition 2.3.4]. This is a powerful property, because for such a W^*-module A, the map

$$A_{\mathrm{bas}\,\mathfrak{g}} \longrightarrow C_G(A), \qquad \omega \longmapsto 1 \otimes \omega, \tag{6.2.3}$$

induces an isomorphism in cohomology with homotopy inverse given by the *Cartan map*

$$\mathrm{Car}\colon C_G(A) \longrightarrow A_{\mathrm{bas}\,\mathfrak{g}},$$

see [23, Sections 4 and 5]. In fact, there it is proved, that Car is a quasi-isomorphism. But clearly, $\mathrm{Car}(1 \otimes \omega) = \omega$ for $\omega \in A_{\mathrm{bas}\,\mathfrak{g}}$, so (6.2.3) induces the inverse to the Cartan map in cohomology. We are now ready to define the resolution Kirwan map.

Definition 6.2.1. The *resolution Kirwan map* $\mathcal{K}$ is defined as the composition of maps

$$\begin{tikzcd} H^*_G(M) \arrow[r, "\iota_\top^*"] \arrow[rrrd, "\mathcal{K}"'] & H^*_G\left(\mathrm{J}^{-1}(0)\right) \arrow[r, "\mathrm{inc}"] & H^*\left(C_G\big(\widetilde{\Omega}^*\left(\mathrm{J}^{-1}(0)\right)\big), d_G\right) \arrow[r, "\mathrm{Car}"] & H^*\left(\widetilde{\Omega}^*\left(\mathrm{J}^{-1}(0)\right)_{\mathrm{bas}\,\mathfrak{g}}, d\right) \arrow[d, "(\pi_\top^*)^{-1}"] \\ & & & H^*(\widetilde{\Omega}^*(\mathscr{M}_0), d), \end{tikzcd}$$

where inc denotes the map induced by the inclusion $C_G\big(\Omega^*\left(\mathrm{J}^{-1}(0)\right)\big) \hookrightarrow C_G\big(\widetilde{\Omega}^*\left(\mathrm{J}^{-1}(0)\right)\big)$.

Remark 6.2.2. Another angle from where to see the resolution Kirwan map is given by Proposition 6.1.6, Corollary 5.3.2 and Lemma 6.2.8. In fact, the resolution Kirwan map $\mathcal{K}$ is equal to the composition of $(\beta^{\mathbb{C}})^*\colon H_G^*(M) \to H_G^*(\mathrm{Bl}_{\mathrm{G}}^{\mathbb{C}}(M))$ with the regular Kirwan map of the blow-up $\kappa\colon H_G^*(\mathrm{Bl}_{\mathrm{G}}^{\mathbb{C}}(M)) \to H^*(\widetilde{\mathscr{M}_0})$, i.e.

$$\mathcal{K}\colon H_G^*(M) \xrightarrow{(\beta^{\mathbb{C}})^*} H_G^*(\mathrm{Bl}_{\mathrm{G}}^{\mathbb{C}}(M)) \xrightarrow{\kappa} H^*(\widetilde{\mathscr{M}_0}).$$

An immediate question is whether the resolution Kirwan map is non-trivial; more precisely, how large its image is. To study this question, recall from Section 6.1 that there is a natural map

$$H^*(\mathscr{M}_0;\mathbb{R}) \longrightarrow H^*(\widetilde{\Omega}^*(\mathscr{M}_0), d)$$

from the singular cohomology $H^*(\mathscr{M}_0;\mathbb{R})$ with real coefficients to the resolution cohomology, the target space of $\mathcal{K}$. While this natural map is not necessarily injective (see Remark 6.1.15), we have seen in Theorem 6.1.14 that it is at least injective in even degrees under appropriate assumptions. This shows that the image of $H^*(\mathscr{M}_0;\mathbb{R})$ in $H^*(\widetilde{\Omega}^*(\mathscr{M}_0), d)$ is an interesting space, even if in general it contains less information than the full singular cohomology $H^*(\mathscr{M}_0;\mathbb{R})$. Therefore, the following main result of this section can be seen as a weak form of "resolution Kirwan surjectivity":

Theorem 6.2.3. *The image of the resolution Kirwan map $\mathcal{K}\colon H_G^*(M) \to H^*(\widetilde{\Omega}^*(\mathscr{M}_0), d)$ contains the image of the natural map $H^*(\mathscr{M}_0;\mathbb{R}) \to H^*(\widetilde{\Omega}^*(\mathscr{M}_0), d)$.*

The proof is given in the following subsection. Let us collect some consequences of Theorem 6.2.3:

Remark 6.2.4 (Resolution Kirwan surjectivity onto subspaces of the singular cohomology). Suppose that a subspace $\mathcal{H} \subset H^*(\mathscr{M}_0;\mathbb{R})$ is *injectively* mapped into $H^*(\widetilde{\Omega}^*(\mathscr{M}_0), d)$ by the natural map $H^*(\mathscr{M}_0;\mathbb{R}) \to H^*(\widetilde{\Omega}^*(\mathscr{M}_0), d)$, and denote by $\mathbb{H} \subset H^*(\widetilde{\Omega}^*(\mathscr{M}_0), d)$ the image of $\mathcal{H}$. Then Theorem 6.2.3 allows us to build from the resolution Kirwan map $\mathcal{K}$ a *surjective* map

$$H_G^*(M) \ni \mathcal{K}^{-1}(\mathbb{H}) \longrightarrow \mathcal{H} \tag{6.2.4}$$

by identifying $\mathcal{H}$ with $\mathbb{H}$ and composing $\mathcal{K}$ with a linear projection $\mathrm{pr}_{\mathbb{H}} : H^*(\widetilde{\Omega}^*(\mathscr{M}_0), d) \to \mathbb{H}$. While in general there is no canonical choice of such a projection, it can still be useful to choose one.

Example 6.2.5 (Resolution Kirwan surjectivity in even degrees)**.** Assume that for all connected components $F \subset M^{S^1} \cap \mathrm{J}^{-1}(0)$ we have $H^{2k+1}(F) = 0$ for all $k \in \mathbb{N}$. Then Theorem 6.1.14 shows that the cohomology in even degrees $H^{\mathrm{ev}}(\mathscr{M}_0;\mathbb{R}) \subset H^*(\mathscr{M}_0;\mathbb{R})$ is injected into $H^*(\widetilde{\Omega}^*(\mathscr{M}_0), d)$ by the natural map. Identify it with its image $\mathbb{H} \subset H^*(\widetilde{\Omega}^*(\mathscr{M}_0), d)$ and choose a linear projection $\mathrm{pr}_{\mathbb{H}} : H^*(\widetilde{\Omega}^*(\mathscr{M}_0), d) \to \mathbb{H}$. Then, since $\mathcal{K}$ is degree-preserving, its composition with $\mathrm{pr}_{\mathbb{H}}$ restricts to a surjective map

$$H_G^{\mathrm{ev}}(M) \longrightarrow H^{\mathrm{ev}}(\mathscr{M}_0;\mathbb{R}). \tag{6.2.5}$$

In the case that the odd cohomology of $\mathscr{M}_0$ vanishes, we obtain a surjective linear map

$$\kappa\colon H_G^*(M) \longrightarrow H^*(\mathscr{M}_0;\mathbb{R}). \tag{6.2.6}$$

Remark 6.2.6. Note that in general the resolution Kirwan map $\mathcal{K}\colon H_G^*(M) \to H^*(\widetilde{\mathscr{M}_0})$ itself cannot be surjective as it can be seen as the composition of the map induced by the blow-down $\beta^{\mathbb{C}}$ composed with the regular Kirwan map of the blown-up Hamiltonian action

$$\mathcal{K}\colon H_G^*(M) \xrightarrow{(\beta^{\mathbb{C}})^*} H_G^*(\mathrm{Bl}_{\mathrm{G}}^{\mathbb{C}}(M)) \xrightarrow{\kappa} H^*(\widetilde{\mathscr{M}_0}).$$

Now, by [44, Proposition 2.4] the cohomology of the blow-up $\mathrm{Bl}_{\mathrm{G}}^{\mathbb{C}}(M)$ is generated by cohomology classes pulled back from M and the symplectic class of each exceptional bundle. These exceptional symplectic classes are not in the image of $(\beta^{\mathbb{C}})^*$. Thus, in the situation of Corollary 6.1.12 the only exceptional contributions of $\mathcal{K}$ are the curvature class terms introduced by the Cartan map and the symplectic class of $\widetilde{\mathscr{M}_0}$ is not in the image of $\mathcal{K}$, whereas the symplectic form $\omega_0^\top \in \Omega^2(\mathscr{M}_0)$ defines a cohomology class $[\omega_0^\top] \in H^2(\mathscr{M}_0;\mathbb{R}) \subset H^2(\widetilde{\mathscr{M}_0})$ which is contained in the image of $\mathcal{K}$ as the equivariant symplectic class of M is mapped to it.

Remark 6.2.7. In the case that the odd cohomology of all $F \in \mathcal{F}_0$ vanishes and the odd cohomology of $\mathscr{M}_0$ vanishes, we have a linear surjection

$$\kappa\colon H_G^*(M) \longrightarrow H^*(\mathscr{M}_0;\mathbb{R})$$

by (6.2.6). It would be interesting to know if κ is in fact a ring homomorphism, which is not clear from our construction since the isomorphism

$$H^*(\widetilde{\mathscr{M}_0}) \cong H^*\left(\mathscr{M}_0\right) \oplus \bigoplus_{F\in\mathcal{F}_0} \frac{\mathbb{R}[\widetilde{\sigma}_0|_{\widetilde{F}}, \widetilde{\Omega}|_{\widetilde{F}}]_{\geq 1}}{I_F}.$$

from Corollary 6.1.12 is linear but not necessarily multiplicative. This is a common drawback of a resolution Kirwan map as becomes clear from the approaches mentioned in Remark 7.2.11.

Proof of Theorem 6.2.3

We start with some preparations. We want to prove Theorem 6.2.3 by evoking the equivariant de Rham isomorphism on the one hand and surjectivity of

$$\iota^*\colon H_G^*(M;\mathbb{R}) \longrightarrow H_G^*(\mathrm{J}^{-1}(0);\mathbb{R}),$$

on the other hand, which was proved by Kirwan using Morse-Bott-Kirwan theory [27, Theorem 8.1]. Note that for any homomorphism Ψ between the cohomologies of two cochain complexes of $\mathbb{R}$-vector spaces there exists a cochain map φ inducing Ψ, which follows from the fact that every short exact sequence of $\mathbb{R}$-vector spaces splits (c.f. [62, Exercise 1.1.3 and beginning of Section 1.4]) as in Appendix A.1. Thus, the equivariant de Rham isomorphism

$$\Psi_{dR}^G\colon H_G^*(M) \longrightarrow H_G^*(M;\mathbb{R})$$

is induced by a cochain map

$$\Psi_{dR}^G\colon C_G(M) \longrightarrow S_G(M;\mathbb{R}),$$

where $S_G(M;\mathbb{R}) := S(M_G;\mathbb{R})$ denotes the complex of G-equivariant singular cochains of M with real coefficients which are singular cochains of the homotopy quotient M_G from Definition 2.2.3.

For every $\omega \in C_G(\Omega^*(\mathrm{J}^{-1}(0)))$ there is some $\eta \in C_G(M)$ such that $\omega = \iota_\top^*\eta$ and we now set

$$\varphi(\omega) := \iota^*(\Psi_{dR}^G\eta) \in S_G(\mathrm{J}^{-1}(0);\mathbb{R}).$$

This map is a well-defined cochain map since the following diagram

$$\begin{tikzcd} C_G(M) \arrow[r, "\Psi_{dR}^G"] \arrow[d, "\iota_\top^*"] & S_G(M;\mathbb{R}) \arrow[d, "\iota^*"] \\ C_G(\Omega(\mathrm{J}^{-1}(0))) \arrow[r, "\varphi"] \arrow[d] & S_G(\mathrm{J}^{-1}(0);\mathbb{R}) \arrow[d, "\mathrm{res}"] \\ C_G(\mathrm{J}^{-1}(0)^\top) \arrow[r, "\Psi_{dR}^G"] & S_G(\mathrm{J}^{-1}(0)^\top;\mathbb{R}), \end{tikzcd}$$

where res denotes restriction, commutes and $\mathrm{J}^{-1}(0)^\top$ is dense in $\mathrm{J}^{-1}(0)$. Thus, we obtain a well-defined induced map

$$\varphi\colon H^*_G(\mathrm{J}^{-1}(0)) \longrightarrow H^*_G(\mathrm{J}^{-1}(0);\mathbb{R}).$$

By the commutativity of

$$\begin{tikzcd}
H^*_G(M) \arrow[r, "\Psi^G_{dR}"] \arrow[d, "\iota^*_\top"] & H^*_G(M;\mathbb{R}) \arrow[d, two heads, "\iota^*"] \\
H^*_G(\mathrm{J}^{-1}(0)) \arrow[r, "\varphi"] & H^*_G(\mathrm{J}^{-1}(0);\mathbb{R})
\end{tikzcd}$$

one sees that φ is surjective. Next, we want to relate the cohomology of the equivariant resolution forms to the cohomology of the strict transform $\widetilde{C}_{\mathbb{C}}$. Analogous to Proposition 6.1.6, we have

Lemma 6.2.8. *The map Φ defined as the composition*

$$\Phi\colon H^*_G(\widetilde{C}_{\mathbb{C}};\mathbb{R}) \overset{(\Psi^G_{dR})^{-1}}{\longrightarrow} H^*_G(\widetilde{C}_{\mathbb{C}}) \overset{(\widetilde{\iota}_\top)^*}{\longrightarrow} H^*_G((\beta^{\mathbb{C}})^{-1}(\mathrm{J}^{-1}(0)^\top)) \overset{((\beta^{\mathbb{C}}_\top)^*)^{-1}}{\longrightarrow} H^*_G(C_G(\widetilde{\Omega}(\mathrm{J}^{-1}(0))), d_G)$$

is an isomorphism.

Proof. The inverse is given by

$$\omega \longmapsto (\beta^{\mathbb{C}}_\top)^*\omega = (\iota^{\mathbb{C}}_\top)^*\widetilde{\eta} \longmapsto (\iota^{\mathbb{C}})^*\widetilde{\eta} \longmapsto \Psi^G_{dR}((\iota^{\mathbb{C}})^*\widetilde{\eta}),$$

which is well-defined because $(\beta^{\mathbb{C}})^{-1}(\mathrm{J}^{-1}(0)^\top)$ is dense in $\widetilde{C}_{\mathbb{C}}$ and the strict transform $\widetilde{C}_{\mathbb{C}}$ is a G-invariant submanifold of $\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M)$. □

Lemma 6.2.9. *The pull-back $\pi^*_\top\colon \widetilde{\Omega}^*(\mathscr{M}_0) \to \widetilde{\Omega}^*\left(\mathrm{J}^{-1}(0)\right)_{\mathrm{bas}\,\mathfrak{g}}$ is an isomorphism of real algebras.*

Proof. Since the G-action is free on $\mathrm{J}^{-1}(0)^\top$, the pull-back

$$\pi^*_\top\colon \Omega(\mathscr{M}_0^\top) \longrightarrow \Omega^*(\mathrm{J}^{-1}(0)^\top)_{\mathrm{bas}\,\mathfrak{g}}$$

is an isomorphism by [18, Proposition 2.5]. Now, let $\omega \in \widetilde{\Omega}^*(\mathscr{M}_0)$. Then there is $\widetilde{\eta} \in \Omega^*(\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M))$ such that $(\beta^{\mathbb{C}}_\top)^*\pi^*_\top\omega = (\iota^{\mathbb{C}}_\top)^*\widetilde{\eta}$. Thus, $\pi^*_\top\omega$ is an element of $\widetilde{\Omega}^*\left(\mathrm{J}^{-1}(0)\right)_{\mathrm{bas}\,\mathfrak{g}}$. Analogously $(\pi^*_\top)^{-1}\theta \in \widetilde{\Omega}^*(\mathscr{M}_0)$ for any $\theta \in \Omega^*(\mathrm{J}^{-1}(0)))_{\mathrm{bas}\,\mathfrak{g}}$. □

Now we are ready to prove Theorem 6.2.3:

Proof of Theorem 6.2.3. Consider the commutative diagram

$$\begin{tikzcd}
H^*_G(M;\mathbb{R}) \arrow[r, two heads, "\iota^*"] & H^*_G(\mathrm{J}^{-1}(0);\mathbb{R}) \arrow[rr, "(\beta^{\mathbb{C}})^*"] & & H^*_G(\widetilde{C}_{\mathbb{C}};\mathbb{R}) \arrow[d, "\Phi"] & \\
H^*_G(M) \arrow[u, "\Psi^G_{dR}"] \arrow[r, "\iota^*_\top"] \arrow[drrrr, "\mathcal{K}"'] & H^*_G(\mathrm{J}^{-1}(0)) \arrow[u, "\varphi"'] \arrow[rr, "\mathrm{inc}"] & & H^*\left(C_G\left(\widetilde{\Omega}^*\left(\mathrm{J}^{-1}(0)\right)\right), d_G\right) \arrow[r, "\mathrm{Car}"] & H^*\left(\widetilde{\Omega}^*\left(\mathrm{J}^{-1}(0)\right)_{\mathrm{bas}\,\mathfrak{g}}, d\right) \arrow[d, "(\pi^*_\top)^{-1}"] \\
& & & & H^*(\widetilde{\Omega}^*(\mathscr{M}_0), d)
\end{tikzcd}$$

where the maps ι, $\iota'_{\mathbb{C}}$, and $\beta^{\mathbb{C}}$ form the commutative diagram

$$\begin{tikzcd} \mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M) \arrow[d, "\beta^{\mathbb{C}}"'] & \widetilde{C}_{\mathbb{C}} \arrow[l, "\iota^{\mathbb{C}}"'] \arrow[d, "\beta^{\mathbb{C}}"] \\ M & \mathrm{J}^{-1}(0). \arrow[l, "\iota"] \end{tikzcd}$$

Denote by $\mathbb{H} \subset H^*(\widetilde{\Omega}^*(\mathscr{M}_0), d)$ the image of $H^*(\mathscr{M}_0; \mathbb{R})$ and let $[\varrho] \in \mathbb{H}$ with $\pi_\top^* \varrho =: \omega = \iota_\top^* \eta$ for some $\eta \in \Omega^*(M)$. Then $\mathrm{Car}([1 \otimes \omega]) = [\omega]$. Since $\iota_\top^*(1 \otimes \eta) = 1 \otimes \omega$, we can regard $[1 \otimes \omega]$ as an element in $H^*_G(\mathrm{J}^{-1}(0))$ and $\varphi([1 \otimes \omega]) \in H^*_G(\mathrm{J}^{-1}(0); \mathbb{R})$. By Kirwan's surjectivity theorem [27, Theorem 8.1] there is $[\widetilde{\eta}'] \in H^*_G(M; \mathbb{R})$ with

$$\iota^*[\widetilde{\eta}'] = \varphi([1 \otimes \omega]).$$

By commutativity and setting $[\eta'] := (\Psi^G_{dR})^{-1}[\widetilde{\eta}']$, we have

$$\varphi([1 \otimes \omega]) = \iota^* \Psi^G_{dR}((\Psi^G_{dR})^{-1}[\widetilde{\eta}']) = \varphi([\iota_\top^* \eta']).$$

But this implies that in $H^*\big(C_G\big(\widetilde{\Omega}^*\left(\mathrm{J}^{-1}(0)\right)\big), d_G\big)$ we have

$$\begin{aligned} [1 \otimes \omega] &= \mathrm{inc}([1 \otimes \omega]) = \Phi((\beta^{\mathbb{C}})^*(\varphi([1 \otimes \omega]))) \\ &= \Phi((\beta^{\mathbb{C}})^*(\varphi([\iota_\top^* \eta']))) = \mathrm{inc}(\iota_\top^*[\eta']). \end{aligned}$$

In total, we conclude

$$\mathcal{K}([\eta']) = [\varrho].$$

□

6.3 Examples - Abelian polygon spaces

Let us finally see one interesting family of examples, where our theory applies, namely the family of Abelian polygon spaces, which have been of interest for symplectic and algebraic geometers as well as combinatorialists for a long time, see [26] and the abundant references therein.
To begin with, consider a product of spheres

$$\prod_{i=1}^n S^2_{\alpha_i} \subset \left(\mathbb{R}^3\right)^n$$

endowed with the product symplectic form, which it inherits from the symplectic forms $\sigma_i \in \Omega^2(S^2_{\alpha_i})$ and let S^1 act on this product diagonally where the circle acts on each 2-sphere as rotation around the z-axis. This action is Hamiltonian and a momentum map is given by

$$\begin{aligned} \mathrm{J} \colon \prod_{i=1}^n S^2_{\alpha_i} &\longrightarrow \mathbb{R} \\ (x_k, y_k, z_k)_k &\longmapsto \sum_{k=1}^n z_k. \end{aligned}$$

When all radii are equal to 1, the regularity of $0 \in \mathbb{R}$ depends on the parity of n and 0 is a regular value of the momentum map if and only if n is odd. Let us now deal with the stratified symplectic quotient

$$\mathscr{M}_0 := \frac{\mathrm{J}^{-1}(0)}{S^1}.$$

The singular stratum of this space consists of isolated points which are induced by the fixed point set

$$\mathrm{J}^{-1}(0) \cap \left(\prod_{i=1}^{n} S^2_{\alpha_i}\right)^{S^1} = \left\{((0,0,\pm\alpha_k))_k \in \mathrm{J}^{-1}(0)\right\}.$$

Now, our results, and in particular Example 6.2.5, apply and show that there is a surjective map

$$\mathcal{K}\colon H^{\mathrm{ev}}_{S^1}\left(\prod_{i=1}^{n} S^2_{\alpha_i}\right) \longrightarrow H^{\mathrm{ev}}(\mathscr{M}_0;\mathbb{R}).$$

Note that since the whole fixed point set $\left(\prod_{i=1}^{n} S^2_{\alpha_i}\right)^{S^1}$ consists of isolated points, our results will also hold for reduction parameters $\alpha \in \mathbb{R}$ which are different from 0 since we can shift the momentum map by a constant. This leads us to

Definition 6.3.1. In general the polygon space $\mathrm{Pol}(\alpha_1,\dots,\alpha_n)$, $(\alpha_1,\dots,\alpha_n) \in \mathbb{R}^n_+$, is defined as the family of piecewise linear paths in $\mathbb{R}^3$, whose ith step (which is of lengh α_i) can proceed in any direction subject to the polygon ending where it begins, considered up to rotation and translation. In other words

$$\mathrm{Pol}(\alpha_1,\dots,\alpha_n) := \left\{(x_1,\dots,x_n) \in \prod_{i=1}^{n} S^2_{\alpha_i} \,\middle|\, \sum_{i=1}^{n} x_i = 0\right\} \Big/ \mathrm{SO}(3)$$

These spaces have a natural symplectic structure as they arise as symplectic reductions of $\prod_{i=1}^{n} S^2_{\alpha_i}$. This is because

$$\begin{aligned} \prod_{i=1}^{n} S^2_{\alpha_i} &\longrightarrow \mathbb{R}^3 \\ (x_1,\dots,x_n) &\longmapsto \sum_{i=1}^{n} x_i \end{aligned}$$

is a momentum map of the diagonal SO(3) action and

$$\mathrm{Pol}(\alpha_1,\dots,\alpha_n) = \left(\prod_{i=1}^{n} S^2_{\alpha_i}\right) \Big/\!\!\Big/_{0} \mathrm{SO}(3) = \left(\prod_{i=1}^{n-1} S^2_{\alpha_i}\right) \Big/\!\!\Big/_{\alpha_n} \mathrm{SO}(3),$$

where the subscript indicates the reduction parameter. Thus, symplectic geometry is a powerful tool to study polygon spaces. These have been of interest naturally to combinatorialists but also to algebraic geometers since they play a role in studying the moduli space of n-times punctured Riemann spheres as well as the moduli space of n unordered weighted points in $\mathbb{CP}^1$, see [2]. In differential geometers, polygon spaces spark interest as they are connected to the moduli space of flat connections on Riemann-surfaces, see [2] again.
One important class of spaces associated to polygon spaces are the so called Abelian polygon spaces which are used to determine the cohomology ring of smooth polygon spaces in [26].

Definition 6.3.2. The *Abelian polygon space* $\mathrm{APol}(\alpha_1,\dots,\alpha_n)$ is defined as

$$\mathrm{APol}(\alpha_1,\dots,\alpha_n) := \left(\prod_{i=1}^{n-1} S^2_{\alpha_i}\right) \Big/\!\!\Big/_{\alpha_n} S^1.$$

Note that while the homology of singular polygon spaces has been studied in [33], where it was for example shown that the homlogy groups of singular polygon spaces generally do not fulfil Poincaré-duality, [33, Remark 1.7], the case of singular Abelian polygon spaces seems uncovered in the literature. But computing the local homology of a singular point p with the help of the local normal form theorem analogously to [25, p. 231] reveals

$$\begin{aligned} H_*(\mathrm{APol}(\alpha_1,\dots,\alpha_n),\mathrm{APol}(\alpha_1,\dots,\alpha_n)\setminus\{p\};\mathbb{R}) &= H_*(C(S^{2k-1}\times_{S^1}S^{2l-1}),\mathbb{R}\times\frac{S^{2k-1}\times S^{2l-1}}{S^1};\mathbb{R}) \\ &\cong \tilde{H}_*(\mathbb{R}\times\frac{S^{2k-1}\times S^{2l-1}}{S^1};\mathbb{R}) \ncong \tilde{H}_*(S^m;\mathbb{R}) \end{aligned}$$

and these Abelian polygon spaces are in general no real homology manifolds and thus no orbifolds by [55, p. 362]. In contrast to the regular situation, the polygon spaces $\mathrm{Pol}(\alpha_1,\dots,\alpha_n)$ need not be even cohomology spaces, i.e. $H^{2k-1}(\mathrm{Pol}(\alpha_1,\dots,\alpha_n))\neq 0$ for some k, as was pointed out in [33, Remark 1.7], and the cohomology ring $H^*(\mathrm{Pol}(\alpha_1,\dots,\alpha_n))$ remains unknown. Nevertheless, our results applied to the Abelian polygon spaces yield

Theorem 6.3.3. *There is a linear surjection*

$$H^{\mathrm{ev}}_{S^1}\left(\prod_{i=1}^{n-1}S^2_{\alpha_i}\right)\longrightarrow H^{\mathrm{ev}}(\mathrm{APol}(\alpha_1,\dots,\alpha_n);\mathbb{R}).$$

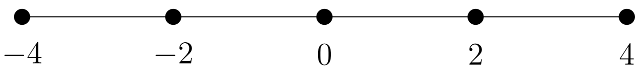
Proof. Apply Theorem 6.2.3 as in Example 6.2.5. □

Let us end this section by exploring one example, related to the ones above, in which $H^{\mathrm{odd}}(\mathscr{M}_0)\neq 0$.

Example 6.3.4. Consider as M the product of four 2-spheres of radius 1 equipped with the standard volume forms and with the standard diagonal rotational action of S^1 with speed 1. The standard momentum map is given as

$$\begin{aligned} \mathrm{J}\colon S^2\times S^2\times S^2\times S^2 &\longrightarrow \mathbb{R} \\ (x_i,y_i,z_i)_{1\leq i\leq 4} &\longmapsto z_1+z_2+z_3+z_4. \end{aligned}$$

The image of the moment map is the closed interval $[-4,4]$ and the critical values inside this interval are $\{-4,-2,0,2,4\}$:



If we intersect the level sets of these critical values with the fixed point set of the action we obtain

$$\begin{aligned} \mathrm{J}^{-1}(4)\cap M^{S^1} &= \{(N,N,N,N)\} \\ \mathrm{J}^{-1}(2)\cap M^{S^1} &= \{(S,N,N,N),(N,S,N,N),(N,N,S,N),(N,N,N,S)\} \\ \mathrm{J}^{-1}(0)\cap M^{S^1} &= \{x\in\{N,S\}^4 \mid \text{exactly two entries of } x \text{ are } N\} \\ \mathrm{J}^{-1}(-2)\cap M^{S^1} &= \{(N,S,S,S),(S,N,S,S),(S,S,N,S),(S,S,S,N)\} \\ \mathrm{J}^{-1}(-4)\cap M^{S^1} &= \{(S,S,S,S)\} \end{aligned}$$

and

$$\begin{aligned}
\#\left(\mathrm{J}^{-1}(4)\cap M^{S^1}\right) &= 1\\
\#\left(\mathrm{J}^{-1}(2)\cap M^{S^1}\right) &= 4\\
\#\left(\mathrm{J}^{-1}(0)\cap M^{S^1}\right) &= 6\\
\#\left(\mathrm{J}^{-1}(-2)\cap M^{S^1}\right) &= 4\\
\#\left(\mathrm{J}^{-1}(-4)\cap M^{S^1}\right) &= 1,
\end{aligned}$$

where $N = (0,0,1) \in S^2$ and $S = (0,0,-1) \in S^2$. Now we may look at the symplectic quotients $\mathscr{M}_\varepsilon$ where $\varepsilon \in [-4,4]$ and study them by invoking the local normal form theorem, the Duistermaat-Heckman theorem and [21, Section 2.3]. From the local normal from theorem one knows that

$$\mathscr{M}_\varepsilon \cong \mathbb{C}\mathrm{P}^3$$

for $\varepsilon > -4$ but close to -4, where the isomorphism is meant as diffeomorphism. By the Duistermaat-Heckman theorem it follows that $\mathscr{M}_\varepsilon \cong \mathbb{C}\mathrm{P}^3$ for all $\varepsilon \in (-4,-2)$. When we cross the critical value -2, as explained in [21, p. 35 f.], we are facing four fixed points where the positive weight space in the isotropy representation in each of these fixed points is complex one-dimensional. Therefore $\mathscr{M}_\varepsilon$ is diffeomorphic to the blow-up of $\mathbb{C}\mathrm{P}^3$ in four points when $\varepsilon \in (-2,0)$. In particular, the dimension of the second cohomology group of this space is $\dim H^2(\mathscr{M}_\varepsilon) = 5$. Now, when passing from $\varepsilon \in (-2,0)$ to 0 we have to collapse six isolated $\mathbb{C}\mathrm{P}^1$'s inside $\mathscr{M}_\varepsilon$. When decomposing $\mathscr{M}_0 = U \cup V$ and $\mathscr{M}_\varepsilon = U' \cup V'$, the Mayer-Vietoris sequences connected by the maps induced by collapsing look like

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & H^2(\mathscr{M}_0) & \longrightarrow & H^2(U)\oplus H^2(V) & \longrightarrow & H^2(U\cap V) & \longrightarrow & H^3(\mathscr{M}_0) & \longrightarrow & \dots\\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & H^2(\mathscr{M}_\varepsilon) & \longrightarrow & H^2(U')\oplus H^2(V') & \xrightarrow{A} & H^2(U'\cap V') & \longrightarrow & H^3(\mathscr{M}_\varepsilon) & \longrightarrow & \dots,
\end{array}$$

where U' is the union of disjoint tubular neighbourhoods of the six $\mathbb{C}\mathrm{P}^1$'s, V' is the slighty enlarged complement of U' inside $\mathscr{M}_\varepsilon$ and U and V are the corresponding images under the collapsing map. In this case $H^*(V) \cong H^*(V')$, $H^*(U\cap V) \cong H^*(U'\cap V')$, $H^*(U') = \bigoplus H^*(\mathbb{C}\mathrm{P}^1)$ and by the long exact cohomology sequence of the pair $(U', \bigcup \mathbb{C}\mathrm{P}^1)$, see [25, p. 199 f.], we have $H^k(U) = 0$ for $k > 1$. Now, $H^3(\mathscr{M}_\varepsilon) = 0$ and the map

$$A\colon H^2(U') \oplus H^2(V') \longrightarrow H^2(U' \cap V')$$

is surjective. Moreover, as $\dim H^2(U') = 6$, while $\dim H^2(\mathscr{M}_\varepsilon) = 5$, the kernel $\ker A$ can be at most 5-dimensional as it is the image $\mathrm{Im}\,(H^2(\mathscr{M}_\varepsilon) \to H^2(U') \oplus H^2(V'))$ by exactness. Therefore A cannot be the zero map. We find an element $a \in H^2(U')$ which is not in the image of the map $H^2(\mathscr{M}_\varepsilon) \to H^2(U')$ and therefore $(a',0) \in H^2(U') \oplus H^2(V')$ maps to some non-zero $v' \in H^2(U' \cap V')$. Then there is a unique non-zero element $v \in H^2(U \cap V)$, which maps to v'.
Claim: The image of v under the map $H^2(U\cap V) \to H^3(\mathscr{M}_0)$ is not zero.
Suppose this was not the case and v maps to zero. By exactness, there is an element $(0,w) \in H^2(U)\oplus H^2(V)$ (remember that $H^2(U) = 0$) which maps to v. Denote the image of $(0,w)$ under the map $0 \oplus H^2(V) \to H^2(U') \oplus H^2(V')$ by $(0,w')$. Since the above diagram is commutative we have that

$$(a', -w') = (a',0) - (0,w') \longmapsto v' - v' = 0.$$

Thus, there exists an element $b \in H^2(\mathscr{M}_\varepsilon)$ which maps to $(a', -w')$. But this is a contradiction as a' is not in the image of the map $H^2(\mathscr{M}_\varepsilon) \to H^2(U')$.

Conclusion: Looking back, our ansatz to extend Sjamaar's complex of differential forms on $\mathscr{M}_0$ using blow-ups was not too exciting in the case of real blow-ups, while it was fruitful when we performed symplectic blow-ups. In this case, the complex of resolution differential forms on $\mathscr{M}_0$ and the long exact sequence (6.1.11) allowed us to define the resolution Kirwan map and study its surjectivity properties. Even though the resolution Kirwan map is not as well-behaved as its regular counterpart, as Example 6.3.4 showed, it is still a very interesting object which deserves to be studied more deeply in the future.

Chapter 7

Comparison

Summary: We will now compare our results to previous work in the field. On the one hand we compare our resolution process to known resolutions of singular symplectic quotients while on the other hand we also want to look for similarities or differences to other defintions of singular Kirwan maps.

7.1 Desingularizations of symplectic quotients

Recall that we desingularized the action of G on M or the zero level set $\mathrm{J}^{-1}(0)$ by successively blowing up the minimal orbit type using real or complex blow-ups, which also resulted in the desingularitzations $\widehat{\mathscr{M}_0}$ and $\widetilde{\mathscr{M}_0}$ of the symplectic quotient.

7.1.1 Shift desingularization

By Sard's theorem, the set of regular values is dense in the momentum polytope. Now, we can shift our reduction parameter 0 a bit to a regular value and take the reduced space there. The comparison to this desingularization was already fully worked out by [46, Section 4.3]. At the heart of this comparison lies the seminal paper [20] by Guillemin and Sternberg.

7.1.2 Lerman-Tolman's small resolution

Instead of shifting the reduction parameter a bit, Lerman-Tolman [40] cleverly perturb the momentum map in order to construct a small resolution of the symplectic quotient. They start with the following

Definition 7.1.1. A simple stratified space is a topological Hausdorff space X with the following properties:

- The space X is a disjoint (set-theoretic) union of even-dimensional orbifolds, called strata.
- There exists an open dense oriented stratum $X^\top$, called the top stratum.
- The complement of $X^\top$ in X is a disjoint union of connected orbifolds, $X \setminus X^\top = \coprod Y_i$, called the singular strata.
- For each singular stratum Y there is a neighbourhood T of Y in X and a retraction map $\pi\colon T \to Y$ which is a C^0 fibre bundle with typical fibre $C(L)$ for some orbifold L, which depends on Y, where $C(L) := L \times [0,1)/\sim$ is the open cone over L. Note that Y embeds into T as the vertex section.

- There exists a diffeomorphism from the complement $T \setminus Y$ to $Q \times (0,1)$, where $Q \to Y$ is a C^∞ fibre bundle of orbifolds with typical fibre L, such that the following diagram commutes:

$$\begin{array}{ccc} T \setminus Y & \longrightarrow & Q \times (0,1) \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow \\ Y & \longrightarrow & Y. \end{array}$$

In Lerman-Tolman's work it is then shown in [40, Proposition 3.2] that a symplectic quotient $\mathscr{M}_0$ by a Hamiltonian circle action is in general a simple stratified space. The authors then proceed to study this space by constructing a so-called *small* resolution $f\colon \mathscr{M}_0' \to \mathscr{M}_0$. What we mean by a resolution is clarified by the following

Definition 7.1.2. Let $X = X^\top \cup \bigcup Y_i$ be a simple stratified space. A resolution $h\colon X' \to X$ is a continuous surjection from a smooth orbifold X', such that $h^{-1}(X^\top)$ is dense in X' and $h\colon h^{-1}(X^\top) \to X^\top$ is a diffeomorphism. A resolution $h\colon X' \to X$ is called small if and only if for all $r > 0$

$$\operatorname{codim}\left\{x \in X \,\middle|\, \dim\left(h^{-1}(x)\right) \geq r\right\} > 2r.$$

Indeed, Lerman-Tolman construct a small resolution of $\mathscr{M}_0$ by perturbing the momentum map $\mathrm{J}\colon M \to \mathbb{R}$. They construct a new map $\mathrm{J}'\colon M \to \mathbb{R}$ which is an S^1-invariant Morse-Bott function such that 0 is a regular value of J' and

$$f\colon \mathscr{M}_0' := (\mathrm{J}')^{-1}(0)/S^1 \longrightarrow \mathrm{J}^{-1}(0)/S^1 = \mathscr{M}_0$$

is a small resolution.

Remark 7.1.3. It is worthwile to think of $\beta_0^{\mathbb{C}}\colon \widetilde{\mathscr{M}_0} \to \mathscr{M}_0$ as a desingularization and compare it to other desingularizations of the singular quotient $\mathscr{M}_0$. Lerman-Tolman's desingularization [40] is rather different from ours, since they construct a small resolution of $\mathscr{M}_0$ by perturbing the momentum map. Let us consider $\beta_0^{\mathbb{C}}\colon \widetilde{\mathscr{M}_0} \to \mathscr{M}_0$ and some $x \in F$ for some fixed point stratum $F \subset \mathscr{M}_0$. As submanifold of M the fixed point component F has codimension $\operatorname{codim}_M(F) = 2k$. Thus, on the one hand

$$\operatorname{codim}_{\mathscr{M}_0}(F) = 2k - 2,$$

while on the other hand

$$\dim((\beta_0^{\mathbb{C}})^{-1}(x)) = \dim\left(\mathbb{C}\mathrm{P}^{k-1} /\!\!/ S^1\right) = 2k - 4$$

and even though $\beta_0^{\mathbb{C}}$ is a resolution of $\mathscr{M}_0$, it is in general not small, since our exceptional fibres are "too big". A similar dimension count shows that the shift-desingularization obtained by reducing M at a regular value near 0 is in general not small as well, see [21, Section 2.3].

7.2 Singular Kirwan maps and intersection cohomology

Let us begin this section by recalling the notion the complex of intersection differential forms of a simple stratified space.

Definition 7.2.1. Let $\pi\colon E \to B$ be a smooth submersion of orbifolds. The Cartan filtration $\mathbb{F}_k\Omega^*(E)$ of the complex of differential forms $\Omega^*(E)$ on E is given by

$$\mathbb{F}_k\Omega^*(E) := \left\{\omega \in \Omega^*(E) \middle| i_{\xi_0} \circ \ldots \circ i_{\xi_k}(\omega_e) = 0 = i_{\xi_0} \circ \ldots \circ i_{\xi_k}(d\omega_e)\right\},$$

for all $e \in E$ and all $\xi_0, \ldots, \xi_k \in \ker(d\pi_e)$.

Remark 7.2.2. The 0-th step of the Cartan filtration, $\mathbb{F}_0\Omega^*(E)$, consists of basic differential forms.

Before we head to the definition of intersection differential forms and finally intersection cohomology, we first have to define what a perversity is.

Definition 7.2.3. Let $X = X^\top \sqcup \coprod Y_i$ be a simple stratified space. A perversity $\overline{p}\colon \{Y_i\} \to \mathbb{N}$ is a function that assigns a non-negative integer to each singular straum Y_i. The middle perversity $\overline{m}$ is defined by

$$\overline{m}(Y_i) := \frac{1}{2}\big(\dim(X^\top) - \dim(Y_i)\big) - 1 =: \frac{1}{2}\operatorname{codim}(Y_i) - 1.$$

Definition 7.2.4. Consider a simple stratified space $\big(X = X^\top \sqcup \coprod Y_i, \{\pi_i\colon T_i \to Y_i\}\big)$ and a perversity $\overline{p}\colon \{Y_i\} \to \mathbb{N}$. The complex of intersection differential forms $I\Omega^*_{\overline{p}}(X)$ is a subcomplex of the complex of differential forms on the top stratum $\Omega^*(X^\top)$ defined as follows: $\omega \in I\Omega^*_{\overline{p}}(X)$ if and only if $\omega|_{U_i} \in \mathbb{F}_{\overline{p}(Y_i)}\Omega^*(U_i \cap X^\top)$ on some neighbourhood $U_i \subset T_i$ of every singular stratum Y_i, where the Cartan filtration $\mathbb{F}_{\overline{p}(Y_i)}\Omega^*(U_i \cap X^\top)$ is relative to the submersion $\pi_i|_{U_i}\colon U_i \cap X^\top = U_i \setminus T_i \to Y_i$ and the coboundary map is the exterior differential d. The intersection cohomology $IH^*_{\overline{p}}(X)$ of a simple stratified space X with perversity $\overline{p}$ is the cohomology of the complex $\big(I\Omega^*_{\overline{p}}(X), d\big)$.

Remark 7.2.5. Intersection differential forms $\omega \in I\Omega^q_{\overline{p}}(X)$ of degree $q > \dim(Y_i) + \overline{p}(Y_i)$ vanish in a neighbourhood of Y_i. In particular, all forms $\omega \in I\Omega^{\dim(X^\top)}_{\overline{m}}(X)$ are compactly supported and we may define an integration map $\int_{X^\top} : I\Omega^{\dim(X^\top)}_{\overline{m}}(X) \to \mathbb{R}$. Thus, the symplectic volume form of the top stratum does not define an intersection differential form! Moreover, if $\dim(X^\top) - 1 > \dim(Y_i) + \overline{p}(Y_i)$ as is the case for example when $\overline{p}$ is the middle perversity, the integration map gives a well-defined map $\int_{X^\top} : IH^{\dim(X^\top)}_{\overline{m}}(X) \to \mathbb{R}$, which after extension by zero in non-top degrees gives $\int_{X^\top} : IH^*_{\overline{m}}(X) \to \mathbb{R}$ and leads us to the definition of the intersection pairing defined as

$$\begin{aligned} IH^p_{\overline{m}}(X) \times IH^q_{\overline{m}}(X) &\longrightarrow \mathbb{R} \\ ([\alpha], [\beta]) &\longmapsto \int_{X^\top} \alpha \wedge \beta. \end{aligned}$$

The non-degeneracy of this pairing was a major motivation to study intersection cohomology when dealing with stratified spaces.

In their study of the intersection cohomology of $\mathscr{M}_0$, Lerman-Tolman consider another complex of differential forms on $\mathscr{M}_0^\top$, whose definition we will now recall:

Definition 7.2.6. Let $h\colon X' \to X$ be a resolution of a simple stratified space X. Let $X^\top$ be the top stratum of X, let $(X')^\top$ be its preimage $h^{-1}(X^\top)$ and let $\iota\colon (X')^\top \to X'$ denote the inclusion. By construction there are maps of complexes $h^*\colon I\Omega^*_{\overline{m}}(X) \to \Omega^*((X')^\top)$ and $\iota^*\colon \Omega^*(X') \to \Omega^*((X')^\top)$. We define the complex of resolution intersection forms to be

$$A^*_{\overline{m}}(X) := h^*\left(I\Omega^*_{\overline{m}}(X)\right) \cap \iota^*\left(\Omega^*(X')\right).$$

Remark 7.2.7. Corollary 6.1.7 reveals that the complexes $\widetilde{\Omega}(\mathscr{M}_0)$ (and $\widehat{\Omega}(\mathscr{M}_0)$) naturally extend the resolution intersection forms studied in [40]. These are intersection differential forms on $\mathscr{M}_0^\top$ whose pullback to a resolution of singularities of $\mathscr{M}_0$ extends to the whole resolution space, see [40, Definition 5.2]. Our definition drops this intersection condition using the partial desingularization as a resolution.

Remark 7.2.8. Small resolutions are so useful when studying stratified spaces because their cohomology is equal to the intersection cohomology of the stratified space with respect to the middle perversity, see [19, Section 6.2]. Lerman-Tolman actually reprove this statement for symplectic circle quotients using the complex $A^*_{\overline{m}}(X)$, see [40, Proposition 5.6].

By studying the perturbed momentum map $\mathrm{J}'\colon M \to \mathbb{R}$ using Morse-Bott-Kirwan theory, Lerman-Tolman find a map

$$\kappa'\colon H^*_{S^1}(M) \longrightarrow H^*(\mathscr{M}_0') \cong IH^*(\mathscr{M}_0)$$

and prove, see [40, Theorem 1 and Theorem 1′],

Theorem 7.2.9. *The map*

$$\kappa'\colon H^*_{S^1}(M) \longrightarrow H^*(\mathscr{M}_0') \cong IH^*(\mathscr{M}_0)$$

is surjective. Moreover, there is a ring structure on $IH^*(\mathscr{M}_0)$, such that κ' is a ring homomorphism.

Remark 7.2.10. Notice, that even though there is a ring structure on $IH^*(\mathscr{M}_0)$, namely the one induced by the isomorphism $H^*(\mathscr{M}_0') \cong IH^*(\mathscr{M}_0)$, this is no canonical structure by [19, Remark p. 121] and [35, Remark 7.6].

Remark 7.2.11. Such non-canonicity phenomenona occur in most approaches to a singular Kirwan map, see [36, Theorem 1 and Theorem 6], [63, Corollary 3.5] and [29, p. 234], and are in congruence to the maps (6.2.4), (6.2.5) being non-canonical, c.f. Remark 6.1.16. In the GIT case of [29], it turns out, see [29, p. 234], that there is a natural choice of projection $H^*(\widetilde{\mathscr{M}_0}) \to IH^*(\mathscr{M}_0)$ due to the Hard Lefschetz theorem.

The intersection cohomology of singular symplectic quotients has subsequently been studied in [29], [35] and [36]. In general their approach is quite close to ours and relies on the following (roughly summarized) construction: Consider a Hamiltonian S^1-action on M with symplectic quotient $\mathscr{M}_0$ and the partial desingularization $\widetilde{\mathscr{M}_0}$ obtained by reducing the blow-up $\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M)$ at the regular value 0. Since $\widetilde{\mathscr{M}_0}$ is an orbifold, there is an isomorphism $H^*(\widetilde{\mathscr{M}_0}) \cong IH^*(\widetilde{\mathscr{M}_0})$ and one considers the map

$$H^*_G(M) \longrightarrow H^*_G(\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M)) \longrightarrow H^*(\widetilde{\mathscr{M}_0}) \cong IH^*(\widetilde{\mathscr{M}_0}).$$

The crux consists in finding $IH^*(\mathscr{M}_0)$ as a summand inside $H^*(\widetilde{\mathscr{M}_0})$ and a projection

$$H^*(\widetilde{\mathscr{M}_0}) \longrightarrow IH^*(\mathscr{M}_0)$$

in order to define a singular Kirwan map as the composition

$$\kappa\colon H^*_G(M) \longrightarrow H^*_G(\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M)) \longrightarrow H^*(\widetilde{\mathscr{M}_0}) \longrightarrow IH^*(\mathscr{M}_0).$$

The existence of such a projection and the surjecitvity of the resulting Kirwan map is still not known, see [36, p. 1], but in some situations both problems have been solved, sometimes at the cost of non-canonicity:

- In [36] Kiem-Woolf showed the existence of a non-canonical map

$$H^*_G(\mathrm{J}^{-1}(0); \mathbb{R}) \longrightarrow IH^*(\mathscr{M}_0),$$

for a general Hamiltonian G-action on M with momentum map $\mathrm{J}\colon M \to \mathfrak{g}^*$, where the surjectivity remains unknown. Composing this with the map $\iota^*\colon H^*_G(M;\mathbb{R}) \to H^*_G(\mathrm{J}^{-1}(0);\mathbb{R})$ one obtains a non-canonical singular Kirwan

$$\kappa\colon H^*_G(M;\mathbb{R}) \longrightarrow IH^*(\mathscr{M}_0).$$

By [30, Theorem A.1] and references therein, this singular Kirwan map κ is surjective in case of an almost-balanced action. This is a technical condition, see [34, Definition 5.1], but for the special case where S^1 acts linearly on $\mathbb{C}\mathrm{P}^n$ it is equivalent to the number of positive weights being equal to the number of negative weights as explained in [34, p. 177].

- In [29], [63] and [39] symplectic quotients of nonsingular connected complex projective varieties equipped with the pull-back of the Fubini-Study form along their embedding into complex projective space and an action of a complex reductive group G are studied as quotients in the sense of geometric invariant theory. In this situation there is a surjective map
$$H^*_G(M) \longrightarrow IH^*(\mathscr{M}_0)$$
which is canonical due to the Hard Lefschetz theorem. In fact, in [29] Jeffrey-Kiem-Kirwan-Woolf studied actions of connected complex reductive groups on smooth connected complex projective varieties which are linear when lifted to an ample line bundle over the variety. Let $G \curvearrowright X$ be such an action and consider a maximal compact subgroup $K \subset G$. After endowing X with the symplectic form it inherits from an embedding into complex projective space, the K action on X turns out to be Hamiltonian with momentum map
$$\mathrm{J}\colon X \longrightarrow \mathfrak{k}^*$$
and the symplectic quotient $X/\!\!/K$ is homeomorphic to the geometric invariant-theoretic quotient of the G action which we denote by $(X/G)_{\mathrm{GIT}}$ by the Kempf-Ness theorem, see [49, Theorem 8.3]. They then use the successive blow-up of X along isotropy components to get $\widetilde{X}$, the induced partial desingularization $\widetilde{X/\!\!/K}$ and the map
$$H^*_K(X) \longrightarrow H^*_K(\widetilde{X}) \longrightarrow H^*(\widetilde{X/\!\!/K}),$$
where the first arrow is induced by the blow-down map and the second map is the regular Kirwan map. Since $\widetilde{X/\!\!/K}$ is an orbifold, its usual cohomology coincides with its intersection cohomology, i.e.
$$H^*(\widetilde{X/\!\!/K}) = IH^*(\widetilde{X/\!\!/K}).$$
By using algebro-geometric arguments involving the decomposition theorem of Beilinson-Bernstein-Deligne and the Hard Lefschetz theorem, see [29, p. 234], they conclude that $IH^*(X/\!\!/K)$ is canonically a direct summand of $IH^*(\widetilde{X/\!\!/K})$. Thus there is a canonical projection
$$H^*(\widetilde{X/\!\!/K}) \longrightarrow IH^*(X/\!\!/K)$$
which is then used to define the Kirwan map as
$$\kappa\colon H^*_K(X) \longrightarrow H^*_K(\widetilde{X}) \longrightarrow H^*(\widetilde{X/\!\!/K}) \longrightarrow IH^*(X/\!\!/K).$$
This map κ is surjective and is their instrument to study the intersection cohomology $IH^*((X/G)_{\mathrm{GIT}})$. Now, to span the arc between our theory and theirs, consider a Hamiltonian S^1-action on a compact connected symplectic manifold (M,σ). We then proceeded

to blow-up all the occuring fixed point components to obtain $\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M)$ and the partial desingularization $\widetilde{\mathscr{M}_0}$ of the symplectic quotient $\mathscr{M}_0$ of interest. We looked at the map

$$H^*_{S^1}(M) \longrightarrow H^*_{S^1}(\mathrm{Bl}^{\mathbb{C}}_{\mathrm{G}}(M)) \longrightarrow H^*(\widetilde{\mathscr{M}_0})$$

and related subspaces of $H^*(\widetilde{\mathscr{M}_0})$ to the real cohomology $H^*(\mathscr{M}_0;\mathbb{R})$ of the symplectic quotient. Thus, our approach is somewhat similar to the one of [29] but instead of looking for $IH^*(\mathscr{M}_0)$ we search for $H^*(\mathscr{M}_0;\mathbb{R})$ in the more general context of Hamiltonian circle actions on compact symplectic manifolds.

One is often interested in examining the intersection pairing

$$IH^*(\mathscr{M}_0) \times IH^*(\mathscr{M}_0) \longrightarrow \mathbb{R}$$

from Remark 7.2.5 and most of the above cited papers are concerned with this in one way or another, see [40, Theorem 1], [35, Theorem 1.3] or [29, Section 7]. In the surjective setting this pairing couples well with the Poincaré duality of intersection cohomology to give a description of the kernel of the Kirwan map, namely

$$\kappa([\omega]) = 0 \Leftrightarrow \langle \kappa([\omega]), \kappa([\eta])\rangle = 0 \,\forall [\eta] \in H^*_G(M).$$

Due to the absence of Poincaré duality for the real cohomology of $\mathscr{M}_0$, this determination of the kernel of the Kirwan map is not open to our resolution Kirwan map. Nevertheless we can make the following

Remark 7.2.12. In [35], Kiem and Woolf identified the intersection cohomology $IH^*(\mathscr{M}_0)$ with respect to the middle perversity with a subspace of $H^*_G(\mathrm{J}^{-1}(0);\mathbb{R})$. Namely, they found an isomorphism

$$IH^*(\mathscr{M}_0) \cong \left\{\eta \in H^*_G(\mathrm{J}^{-1}(0);\mathbb{R}) \;\middle|\; \eta|_F \in H^*(F;\mathbb{R}) \otimes \mathbb{R}[x]_{\leq 2d-2} \,\forall F \subset \mathrm{J}^{-1}(0)\right\},$$

where $H^*(F;\mathbb{R}) \otimes \mathbb{R}[x]_{\leq 2d_i-2} \subset H^*(F;\mathbb{R}) \otimes \mathbb{R}[x] \cong H^*_G(F;\mathbb{R})$ and $d := \min\{\ell^+_F, \ell^-_F\}$, which identifies the intersection pairing in $IH^*(\mathscr{M}_0)$ with the cup product in $H^*_G(\mathrm{J}^{-1}(0);\mathbb{R})$.
Remark 6.1.13 provides a description of singular cohomology inside resolution cohomology similiar in spirit as we characterize singular classes as those classes in resolution cohomology whose restriction to the exceptional fibres vanish in cohomology.

Another approach to the topology of singular symplectic quotients could be given by considering L^2-cohomology after endowing $\mathscr{M}_0$ with a suitable Riemannian structure. This is defined as the cohomology of the complex L^2 whose definition we will now recall:

Definition 7.2.13. The complex of L^2-differential forms on a simple stratified space X, where X is endowed with a suitable Riemannian structure, is defined as

$$L^2(X) := \left\{\omega \in \Omega^*(X^\top) \mid \forall x \in X \exists U_x \subset X\colon x \in U_x, \omega|_{U_x^\top} \in L^2(U_x^\top) \text{ and } d\omega|_{U_x^\top} \in L^2(U_x^\top)\right\},$$

where the neighbourhoods U_x should be open and $U_x^\top := U_x \cap X^\top$. This complex is closed under the usual exterior derivative of differential forms and its cohomology is called the L^2-cohomology of X.

The fact connecting this theory to the aforementioned approaches is that in certain situations L^2-cohomology coincides with intersection cohomology, see [19, Example p. 105]. In this context one might also hope to build a bridge between resolution differential forms and L^2-differential forms on $\mathscr{M}_0^\top$. Unfortunately we could not find a canonical map from resolution differential forms to L^2-differential forms on $\mathscr{M}_0$ or vice-versa for some suitable Riemannian structure on $\mathscr{M}_0$.

Chapter 8

Outlook

We want to close this thesis by pointing out some directions in which future work might be headed.

8.1 Riemann-Roch formulas of singular symplectic quotients

In the context of geometric quantization, Delarue-Ioos-Ramacher proved in [10] that the S^1-invariant Riemann-Roch number of a compact connected prequantized Hamiltonian S^1-manifold (M, σ) fulfills

$$\mathrm{RR}^{S^1}(M) = \int\limits_{\mathscr{M}_0^\top} e^{\sigma_0} \mathcal{K}(\mathrm{Td}_{\mathfrak{g}}(M)) + \mathcal{R},$$

where $\mathrm{Td}_{\mathfrak{g}}(M) \in \Omega_G^\omega(M)$ is the equivariant Todd class of M (which is an equivariant form whose Lie algebra parts are infinite power series on $\mathfrak{g}$ as explained in [22, Appendix I.2]), $\mathcal{K}\colon \Omega_G^\omega(M) \longrightarrow \Omega^*(\mathscr{M}_0^\top)$ is our resolution Kirwan map on the level of differential forms and $\mathcal{R}$ is an explicit remainder term, which would lead us to far off now. For the detailed definitions consider [10]. Important for us is that in case where all the weights of the isotropy representations of the fixed point components in $\mathcal{F}_0$ have the same absolute value, as in Proposition 5.2.9 and Proposition 5.3.5, the integrand $e^{\sigma_0}\mathcal{K}(\mathrm{Td}_{\mathfrak{g}}(M))$ is not only a differential form on the top stratum of $\mathscr{M}_0$, but is indeed a resolution differential form and can be interpreted as a term in resolution cohomology $H^*(\widetilde{\Omega}(\mathscr{M}_0), d)$! It will be very interesting to elaborate this connection of our theory of resolution cohomology and the associated resolution Kirwan map and the results of [10].

8.2 Extending the circle case

We worked things out for $G = S^1$. It will be worthwhile to consider more general Hamiltonian actions of tori or even non-Abelian compact Lie groups. The partial desingularization has already been worked out by Meinrenken-Sjamaar [46], Sjamaar's de Rham theory [57] is available and our ansatz carries over in the straight forward way but one would have to study the exceptional divisors carefully and see if anything can be said about their topology. This is more involved than in our case as the blow-up procedure gets more complicated when more isotropy types occur.

8.3 Examples

A good starting point for dwelling deeper into our theory would be examples. Consider certain classes of Hamiltonian actions, e.g. polygon spaces or toric symplectic manifolds, and try to understand singular symplectic quotients and their topology in these realms. Then, relate it to resolution cohomology and the resolution Kirwan map.

Appendix A

Appendix

A.1 Induced maps in cohomology

Proposition A.1.1. *Let* $(C^\bullet, d)$ *and* $(D^\bullet, d')$ *be cochain complexes over* $\mathbb{R}$ *and*

$$\psi\colon H^*(C^\bullet, d) \longrightarrow H^*(D^\bullet, d')$$

a homomorphism. Then there exists a homomorphism of complexes

$$\varphi\colon (C^\bullet, d) \longrightarrow (D^\bullet, d')$$

such that $\psi = \overline{\varphi}$, *i.e.* ψ *is induced by* φ.

Proof. Since the cochain complex

$$\dots \longrightarrow C^{k-1} \xrightarrow{d^{k-1}} C^k \xrightarrow{d^k} C^{k+1} \longrightarrow \dots$$

is made of real vector spaces we may decompose

$$C^k \cong \mathrm{Im}\,(d^k) \oplus \ker(d^k).$$

Now

$$H^k(C^\bullet, d) = \ker(d^k)/\mathrm{Im}\,(d^{k-1})$$

and we further decompose

$$C^k \cong \mathrm{Im}\,(d^k) \oplus H^k(C^\bullet, d) \oplus \mathrm{Im}\,(d^{k-1})$$

as in [62, Exercise 1.1.3 and beginning of Section 1.4]. Under this identification the differential reads as

$$\begin{aligned} d^k\colon \mathrm{Im}\,(d^k) \oplus H^k(C^\bullet, d) \oplus \mathrm{Im}\,(d^{k-1}) &\longrightarrow \mathrm{Im}\,(d^{k+1}) \oplus H^{k+1}(C^\bullet, d) \oplus \mathrm{Im}\,(d^k) \\ (u, v, w) &\longmapsto (0, 0, u)). \end{aligned}$$

If we decompose the cochain complex $(D^\bullet, d')$ in the same manner, we can define

$$\begin{aligned} \varphi^k\colon \mathrm{Im}\,(d^k) \oplus H^k(C^\bullet, d) \oplus \mathrm{Im}\,(d^{k-1}) &\longrightarrow \mathrm{Im}\,((d')^k) \oplus H^k(D^\bullet, d') \oplus \mathrm{Im}\,((d')^{k-1}) \\ (u, v, w) &\longmapsto (0, \psi(v), 0), \end{aligned}$$

which is a cochain map such that $\overline{\varphi} = \psi$, as was explained in [53]. $\square$

Appendix B

Glossary

Occuring spaces

- Symplectic compact connected ambient manifold M and submanifold N.
- Acting compact connected Lie group $G = S^1$.
- Lie algebra of the acting group $\mathfrak{g} \cong \mathbb{R}$.
- The fixed point set of the action M^G.
- The finite set of the fixed point components $\mathcal{F}$.
- The finite set of fixed point components contained in the zero level set $\mathcal{F}_0$.
- A component of the fixed point set F with $\operatorname{codim}(F) = 2k$.
- The real blow-up of M along all fixed point components $\mathrm{Bl}_{\mathrm{G}}^{\mathbb{R}}(M)$ with exceptional divisor $E_{\mathbb{R}} := \beta_{\mathbb{R}}^{-1}(M^G)$.
- The complex blow-up of M along all fixed point components $\mathrm{Bl}_{\mathrm{G}}^{\mathbb{C}}(M)$ with exceptional divisor $E_{\mathbb{C}} := \beta_{\mathbb{C}}^{-1}(M^G)$.
- The zero level set of the momentum map $\mathrm{J}^{-1}(0)$ with its smooth part $\mathrm{J}^{-1}(0)^{\top}$.
- The symplectic quotient $\mathscr{M}_0$ with smooth locus $\mathscr{M}_0^{\top}$.
- The strict transform of the zero level set $\widehat{C}_{\mathbb{R}} = \overline{\beta_{\mathbb{R}}^{-1}(\mathrm{J}^{-1}(0)^{\top})}$ associated to the real blow-up.
- The strict transform of the zero level set $\widetilde{C}_{\mathbb{C}} = \overline{\beta_{\mathbb{C}}^{-1}(\mathrm{J}^{-1}(0)^{\top})}$ associated to the complex blow-up.
- The exceptional loci of the blow-up $E_F^{\mathbb{R}} := \beta_{\mathbb{R}}^{-1}(F)$.
- The exceptional loci of the blow-up $E_F^{\mathbb{C}} := \beta_{\mathbb{C}}^{-1}(F)$ with fibers $\mathbb{C}\mathrm{P}_{\lambda,F}^{k-1}$ and their standard open subsets V_i. The subscript encodes the Hamiltonian action of S^1 on these projective spaces.
- The real partial desingularization $\widehat{\mathscr{M}_0} := \widehat{C}_{\mathbb{R}}/G$ with top part $\widehat{\mathscr{M}_0}^{\top} = \beta_{\mathbb{R}}^{-1}(\mathrm{J}^{-1}(0)^{\top})/G$.
- The complex partial desingularization $\widetilde{\mathscr{M}_0} := \widetilde{C}_{\mathbb{C}}/G$ with top part $\widetilde{\mathscr{M}_0}^{\top} = \beta_{\mathbb{C}}^{-1}(\mathrm{J}^{-1}(0)^{\top})/G$.
- The exceptional loci of the real partial desingularization $\widehat{F} = (\beta_0^{\mathbb{R}})^{-1}(F) = (E_F^{\mathbb{R}} \cap \widehat{C}_{\mathbb{R}})/G$.

- The exceptional loci of the partial desingularization $\widetilde{F} = (\beta_0^{\mathbb{C}})^{-1}(F) = (E_F^{\mathbb{C}} \cap \widetilde{C}_{\mathbb{C}})/G$.
- Symplectic cut at ε is $\overline{M}_{\mathrm{J}\geq\varepsilon}$.

Occuring maps

- Momentum map $\mathrm{J}\colon M \to \mathfrak{g}^*$.
- Real blow-down $\beta^{\mathbb{R}}\colon \mathrm{Bl}_{\mathrm{G}}^{\mathbb{R}}(M) \to M$.
- Complex blow-down $\beta^{\mathbb{C}}\colon \mathrm{Bl}_{\mathrm{G}}^{\mathbb{C}}(M) \to M$.
- Real partial desingularization $\beta_0^{\mathbb{R}}\colon \widehat{\mathscr{M}_0} \to \mathscr{M}_0$ with smooth part $(\beta_0^{\mathbb{R}})^{\top}\colon \widehat{\mathscr{M}_0}_{\top} \to \mathscr{M}_0^{\top}$.
- Complex partial desingularization $\beta_0^{\mathbb{C}}\colon \widetilde{\mathscr{M}_0} \to \mathscr{M}_0$ with smooth part $(\beta_0^{\mathbb{C}})_{\top}\colon \widetilde{\mathscr{M}_0}^{\top} \to \mathscr{M}_0^{\top}$.
- Quotient map $\pi\colon \mathrm{J}^{-1}(0) \to \mathscr{M}_0$ with smooth part $\pi_{\top}\colon \mathrm{J}^{-1}(0)^{\top} \to \mathscr{M}_0^{\top}$
- Exceptional bundles $\beta_F^{\mathbb{R}}\colon E_F^{\mathbb{R}} \to F$ of $\mathrm{Bl}_{\mathrm{G}}^{\mathbb{R}}(M)$.
- Exceptional bundles $\beta_F^{\mathbb{C}}\colon E_F^{\mathbb{C}} \to F$ of $\mathrm{Bl}_{\mathrm{G}}^{\mathbb{C}}(M)$.
- Exceptional bundles $\beta_0^{\mathbb{R},F}\colon \widehat{F} \to F$ of $\widehat{\mathscr{M}_0}$.
- Exceptional bundles $\beta_0^{\mathbb{C},F}\colon \widetilde{F} \to F$ of $\widetilde{\mathscr{M}_0}$.
- The universal bundle $L \to \mathbb{C}\mathrm{P}^{k-1}$.
- The normal bundle $\nu\colon Q \to N$.
- The (normal) frame bundle $P \to N$.
- $\pi'_{\top} = \pi_{\top} \circ \beta_{\top}^{\mathbb{C}}\colon (\beta^{\mathbb{C}})^{-1}(\mathrm{J}^{-1}(0)^{\top}) \to \mathscr{M}_0^{\top}$.

Occuring coordinates

- Real coordinates (w_i), (x_i), (y_i), (p_i).
- Complex coordinates (z_i).
- Real homogeneous coordinates (t_i).
- Complex homogeneous coordinates (l_i).
- Standard coordinates of real projective space $(v_j) := \left(\frac{t_j}{t_i}\right)$ in $V_i \subset \mathbb{R}\mathrm{P}^{k-1}$.
- Standard coordinates of complex projective space $(u_j) := \left(\frac{l_j}{l_i}\right)$ in $V_i \subset \mathbb{C}\mathrm{P}^{k-1}$.

Occuring Forms

- Symplectic forms: σ on M, $\tilde{\sigma}$ on $\mathrm{Bl}_{\mathrm{G}}^{\mathbb{C}}(M)$, σ_0 on $\mathscr{M}_0$ and $\widetilde{\sigma_0}$ on $\widetilde{\mathscr{M}_0}$.
- Fubini-Study form σ_{FS}.
- Connection form α.
- Curvature form $\Omega = d\alpha$.

Occuring $\mathfrak{g}$-differential graded algebras

- Differential forms on a manifold M are $\Omega(M)$.
- Differential forms on the symplectic quotient $\Omega(\mathscr{M}_0)$ in Sjamaar's sense.
- Real resolution forms $\widehat{\Omega}(\mathscr{M}_0)$.
- Resolution forms $\widetilde{\Omega}(\mathscr{M}_0)$.
- Differential forms on $\mathrm{J}^{-1}(0)$ are $\Omega(\mathrm{J}^{-1}(0))$ analogous to Sjamaar's definition.
- Resolution forms on $\mathrm{J}^{-1}(0)$ are $\widetilde{\Omega}(\mathrm{J}^{-1}(0))$ analogous to our definition.
- The basic subcomplex of a $\mathfrak{g}$-differential graded algebra (A, d) is $(A_{\mathrm{bas}\, G}, d)$.
- The Cartan complex associated with a $\mathfrak{g}$-differential graded algebra (A, d) is $(\mathrm{C}_G(A), d_G)$.

Miscellaneous

- The fundamental vector field associated with some $X \in \mathfrak{g}$ is $\overline{X}_p = \frac{d}{dt}|_{t=0} \exp(tX) \cdot p$.

Fundamental diagrams

$$
\begin{tikzcd}
E_F^{\mathbb{R}} \arrow[r] \arrow[d, "\beta_F^{\mathbb{R}}"] & \mathrm{Bl}_{\mathrm{G}}^{\mathbb{R}}(M) \arrow[d, "\beta^{\mathbb{R}}"] & \widehat{C}_{\mathbb{R}} \arrow[l, "\iota^{\mathbb{R}}"'] \arrow[r, "\pi^{\mathbb{R}}"] \arrow[d, "\beta^{\mathbb{R}}"] & \widehat{\mathscr{M}_0} \arrow[ddl, "\beta_0^{\mathbb{R}}"] & \widehat{F} \arrow[l] \arrow[ddl, "\beta_0^{\mathbb{R},F}"] \\
F \arrow[r] & M & \mathrm{J}^{-1}(0) \arrow[l, "\iota"'] \arrow[d, "\pi"] & & \\
 & & \mathscr{M}_0 & F \arrow[l] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
E_F^{\mathbb{C}} \arrow[r] \arrow[d, "\beta_F^{\mathbb{C}}"] & \mathrm{Bl}_{\mathrm{G}}^{\mathbb{C}}(M) \arrow[d, "\beta^{\mathbb{C}}"] & \widetilde{C}_{\mathbb{C}} \arrow[l, "\iota^{\mathbb{C}}"'] \arrow[r, "\pi^{\mathbb{C}}"] \arrow[d, "\beta^{\mathbb{C}}"] & \widetilde{\mathscr{M}_0} \arrow[ddl, "\beta_0^{\mathbb{C}}"] & \widetilde{F} \arrow[l] \arrow[ddl, "\beta_0^{\mathbb{C},F}"] \\
F \arrow[r] & M & \mathrm{J}^{-1}(0) \arrow[l, "\iota"'] \arrow[d, "\pi"] & & \\
 & & \mathscr{M}_0 & F \arrow[l] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \widetilde{C}_{\mathbb{C}} \arrow[r, "\pi^{\mathbb{C}}"] & \widetilde{\mathscr{M}_0} \\
\mathrm{Bl}_{\mathrm{G}}^{\mathbb{C}}(M) \arrow[d, "\beta^{\mathbb{C}}"] & (\beta^{\mathbb{C}})^{-1}\left(\mathrm{J}^{-1}(0)^{\top}\right) \arrow[l, "\iota_{\top}^{\mathbb{C}}"'] \arrow[u, "\widetilde{\iota}_{\top}"] \arrow[r, "\pi_{\top}^{\mathbb{C}}"] \arrow[d, "\beta_{\top}^{\mathbb{C}}"] & \widetilde{\mathscr{M}_0}^{\top} \arrow[u, "\iota_0^{\top}"'] \arrow[ddl, "(\beta_0^{\mathbb{C}})^{\top}"] \\
M & \mathrm{J}^{-1}(0)^{\top} \arrow[l, "\iota_{\top}"'] \arrow[d, "\pi_{\top}"] & \\
 & \mathscr{M}_0^{\top} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
H_G^*(M) \arrow[r, "\iota_\top^*"] \arrow[drrr, "\mathcal{K}"'] & H_G^*\left(\mathrm{J}^{-1}(0)\right) \arrow[r, "\mathrm{inc}"] & H^*\left(C_G\big(\widetilde{\Omega}^*\left(\mathrm{J}^{-1}(0)\right)\big), d_G\right) \arrow[r, "\mathrm{Car}"] & H^*\left(\widetilde{\Omega}^*\left(\mathrm{J}^{-1}(0)\right)_{\mathrm{bas}\,\mathfrak{g}}, d\right) \arrow[d, "\left(\pi_\top^*\right)^{-1}"] \\
 & & & H^*(\widetilde{\Omega}^*(\mathscr{M}_0), d)
\end{tikzcd}
$$

$$
\begin{tikzcd}
H_G^*(M;\mathbb{R}) \arrow[r, two heads, "\iota^*"] & H_G^*(\mathrm{J}^{-1}(0);\mathbb{R}) \arrow[r, "(\beta^{\mathbb{C}})^*"] & H_G^*(\widetilde{C}_{\mathbb{C}};\mathbb{R}) \arrow[d, "\Phi"] & \\
H_G^*(M) \arrow[u, "\Psi_{dR}^G"] \arrow[r, "\iota_\top^*"] \arrow[drrr, "\mathcal{K}"'] & H_G^*(\mathrm{J}^{-1}(0)) \arrow[u, "\varphi"] \arrow[r, "\mathrm{inc}"] & H^*\left(C_G\big(\widetilde{\Omega}^*\left(\mathrm{J}^{-1}(0)\right)\big), d_G\right) \arrow[r, "\mathrm{Car}"] & H^*\left(\widetilde{\Omega}^*\left(\mathrm{J}^{-1}(0)\right)_{\mathrm{bas}\,\mathfrak{g}}, d\right) \arrow[d, "\left(\pi_\top^*\right)^{-1}"] \\
 & & & H^*(\widetilde{\Omega}^*(\mathscr{M}_0), d)
\end{tikzcd}
$$

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