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Homogeneity and Inhomogeneity of Sasakian Geometries

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Für Marie-Christin

Abstract

We study homogeneous and inhomogeneous manifolds with various Sasakian geometries. First we provide a new and more illustrative proof of the classification of homogeneous 3-Sasaki manifolds, which was originally obtained by Boyer, Galicki and Mann [BGM]. In doing so we construct an explicit one-to-one correspondence between simply connected homogeneous 3-Sasaki manifolds and simple complex Lie algebras via the theory of root systems. These results also yield an alternative derivation of the classification of homogeneous positive quaternionic Kähler manifolds due to Alekseevskii [Alek].
Subsequently we apply similar techniques to degenerate 3-(α, δ)-Sasaki manifolds to deduce several results which limit the number of homogeneous spaces with this geometry. We prove that this category contains no non-trivial compact examples as well as exactly one family of nilpotent Lie groups, namely the quaternionic Heisenberg groups.
By way of contrast we present a method to construct degenerate 3-(α, δ)-Sasaki manifolds as certain T^3-bundles over hyperkähler manifolds with integral Kähler classes, which is similar to the famous Boothby-Wang bundle [BW]. The manifolds obtained this way are necessarily inhomogeneous and we develop a way to quantify "how far away from homogeneous" they are. To this end we utilize that Sasakian geometries naturally come with the so-called characteristic foliation and elaborate a generalization of the famous Bochner technique for foliations [Boch]. Later it turned out that this Bochner technique for foliations also follows from results in the article [HR], but the applications to Sasakian geometries are new.

Zusammenfassung

Wir untersuchen homogene sowie inhomogene Mannigfaltigkeiten mit verschiedenen Sasakischen Geometrien. Zunächst geben wir einen neuen und anschaulicheren Beweis für die Klassifikation der homogenen 3-Sasaki-Mannigfaltigkeiten, welche ursprünglich von Boyer, Galicki und Mann [BGM] bewiesen wurde. Dabei konstruieren wir eine explizite Bijektion zwischen einfach zusammenhängenden homogenen 3-Sasaki-Mannigfaltigkeiten und einfachen komplexen Lie-Algebren mithilfe der Theorie der Wurzelsysteme. Diese Ergebnisse liefern zudem eine alternative Herleitung für die Klassifikation der homogenen positiven quaternionischen Kähler-Mannigfaltigkeiten von Alekseevskii [Alek].
Anschließend wenden wir ähnliche Techniken auf entartete 3-(α, δ)-Sasaki-Mannigfaltigkeiten an, um die Anzahl der homogenen Räume mit dieser Geometrie einzugrenzen. Wir beweisen, dass diese Kategorie keine nicht-trivialen kompakten Beispiele enthält sowie genau eine Familie von nilpotenten Lie-Gruppen, nämlich die quaternionischen Heisenberg-Gruppen.
Als Kontrast hierzu präsentieren wir eine Methode zur Konstruktion von entarteten 3-(α, δ)-Sasaki-Mannigfaltigkeiten als gewisse T^3-Bündel über Hyperkähler-Mannigfaltigkeiten mit integralen Kähler-Klassen, ähnlich dem berühmten Boothby-Wang-Bündel [BW]. Die auf diese Weise erhaltenen Mannigfaltigkeiten sind notwendigerweise inhomogen und wir entwickeln ein Verfahren, um zu quantifizieren „wie weit entfernt von Homogenität" sie sind. Zu diesem Zweck nutzen wir, dass Sasakische Geometrien stets mit der sogenannten charakteristischen Blätterung versehen sind, und entwickeln eine Verallgemeinerung der berühmten Bochner-Technik für Blätterungen [Boch]. Im Nachhinein stellte sich heraus, dass diese Bochner-Technik für Blätterungen auch aus bereits bekannten Resultaten in dem Artikel [HR] folgt, aber die Anwendungen auf Sasakische Geometrien sind neu.

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Introduction

> *"Symmetry, as wide or as narrow as you may define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty, and perfection."*
>
> Hermann Weyl [Weyl]

As illustrated by this quote, taking advantage of symmetry in order to understand patterns and structure is a pervasive concept throughout human history. In mathematics and specifically geometry this is exemplified particularly clearly by Felix Klein's seminal *Erlangen Program* from 1872 [Klei]. He proposed the investigation of geometric objects via their *symmetry groups* and established the framework of *homogeneous spaces* as those entities which admit the most symmetries.

One area where this approach has often proved successful is that of *special Riemannian geometries*. These are modeled by Riemannian manifolds which are endowed with certain *structure tensors* like vector fields, differential forms or endomorphisms. Such manifolds are called *homogeneous* if the group of all isometries which preserve the structure tensors acts transitively.

For example *quaternionic Kähler (qK) manifolds* can be characterized as Riemannian manifolds which locally admit three compatible almost complex structures that obey the multiplication rules of the quaternions. Under the additional assumption of positive scalar curvature these so-called *homogeneous positive qK manifolds* have been classified through the work of Wolf [Wolf] and Alekseevskii [Alek] in the 1960s.

Their results have also led to a corresponding classification for the odd-dimensional geometry of *3-Sasaki manifolds* by Boyer, Galicki and Mann in 1994 [BGM]. The first milestone of this thesis is to revisit this classification and provide a new proof which is more illustrative and self-contained. As Wolf and Alekseevskii had already pointed out for qK geometry, homogeneous 3-Sasaki manifolds also give rise to a *maximal root* of a certain *root system*. This connection ultimately manifests in a one-to-one correspondence between simply connected homogeneous 3-Sasaki manifolds and simple complex Lie algebras.

Subsequently we apply similar techniques to the newer geometry of *3-(α,δ)-Sasaki manifolds* which were invented in 2020 by Agricola and Dileo [AD]. These form a common generalization which accommodates both 3-Sasaki manifolds and certain other interesting examples like the *quaternionic Heisenberg groups*. This larger class of spaces still retains many favorable properties like *hypernormality* or the existence of a connection with totally skew-symmetric torsion which is well-adapted to the geometry [AD].

In this thesis we focus on the less studied case of so-called *degenerate* 3-(α,δ)-Sasaki manifolds where the three *almost contact metric structures* are more independent of each other in the sense that their *Reeb vector fields* commute. We prove that no non-trivial compact homogeneous degenerate 3-(α,δ)-Sasaki manifolds exist as well as that the quaternionic Heisenberg groups are the only nilpotent Lie groups with an invariant degenerate 3-(α,δ)-Sasaki structure.

As a counterpoint we explain how to use certain *hyperkähler manifolds* to construct degenerate 3-(α,δ)-Sasaki manifolds which are necessarily inhomogeneous. This approach is based on and similar to the famous *Boothby-Wang bundle* [BW]. Since interesting candidates for the base space of this bundle are known to exist [Cor], this opens up the possibility to construct many new examples of degenerate 3-(α,δ)-Sasaki manifolds.

Finally we devise a way to quantify "how far away from homogeneous" these and certain other manifolds are. Our approach uses the fact that Sasakian geometries are naturally endowed with the so-called *characteristic foliation*. We then develop and apply a generalization of the famous *Bochner technique* [Boch] for foliations with non-negative transverse Ricci curvature.

Structure

This thesis is divided into four chapters. In Chapter 1 we lay out all of the necessary preliminaries about the odd-dimensional geometries of Sasaki, 3-Sasaki and 3-(α,δ)-Sasaki manifolds. The remaining three chapters could theoretically be read mostly independently of each other.

Chapter 2 discusses the classification of homogeneous 3-Sasaki manifolds. After sketching the history of the problem (Section 2.1) we furnish several aspects of our new proof in Sections 2.2 to 2.4. The consequences of our approach for the isotropy groups, the non-simply connected case as well as the classification of homogenous positive qK manifolds are explored in Sections 2.5 to 2.7.

In Chapter 3 we investigate homogeneous and inhomogeneous degenerate 3-(α, δ)-Sasaki manifolds. Sections 3.1 and 3.2 limit the number of homogeneous and compact homogeneous spaces with this geometry and Section 3.3 does the same for nilpotent Lie groups. These "negative results" are contrasted by a contruction of degenerate 3-(α, δ)-Sasaki manifolds over hyperkähler manifolds in Section 3.4.

Chapter 4 is concerned with a Bochner technique for foliations and its relation to inhomogeneity of Sasakian geometries. After explaining prerequisites about Riemannian foliations and basic Hodge theory (Sections 4.1 and 4.2) we complete the proof of our generalization in Section 4.3. Finally we apply this method to degenerate 3-(α, δ)-Sasaki manifolds as well as certain Sasaki-η-Einstein spaces (Section 4.4).

The material in Chapters 2 and 3 is adapted from joint work with Oliver Goertsches and Leander Stecker in the publications [GRS1] and [GRS2], respectively. In particular the constructive result from Section 3.4 first appeared in [Ste] and then in [GRS2] and is only rendered here for the coherence of the arguments. The content of Chapter 4 is based on the author's preprint [Ros]. After its announcement on the arXiv it was pointed out to the author by Georges Habib that the main Theorem 4.31 also follows from work of his with Ken Richardson [HR, Proposition 6.7 & Theorem 6.16].

Chapter 1

Preliminaries about Odd-Dimensional Geometries

In this chapter we briefly outline the fundamentals of several interesting odd-dimensional geometries. We shall start with almost contact metric and Sasaki manifolds, which exist in every odd dimension $2n+1$ and have served as inspiration for numerous generalizations and modifications. Among the latter are 3-Sasaki and 3-(α, δ)-Sasaki manifolds, which only exist in dimensions $4n+3$ and will be introduced afterwards.

1.1 Sasaki Manifolds

Contact geometry was studied since the late 19th century by various authors following motivation from physics, especially classical mechanics [BG, Section 6.1]. The prominent special case of Sasaki manifolds was introduced in 1962 by Sasaki and Hatakeyama [SH] as an odd-dimensional analogue of Kähler manifolds. A lot more historical context as well as most of the important developments and results on the topic can be found in the comprehensive monograph *Sasakian Geometry* by Boyer and Galicki [BG].

Definition 1.1. Let $(M^{2n+1}, g, \xi, \eta, \varphi)$ be an odd-dimensional Riemannian manifold endowed with a unit length vector field ξ, its g-dual one-form η and an almost Hermitian structure φ on $\ker \eta$. Then M is an *almost contact metric manifold* if

$$\varphi\xi = 0\,, \qquad \varphi^2 = -\mathrm{id} + \xi \otimes \eta\,, \qquad g \circ (\varphi \times \varphi) = g - \eta \otimes \eta\,.$$

The structure tensors ξ and η are called *Reeb vector field* and *contact form*, respectively. The *fundamental two-form* is given by $\Phi(X, Y) := g(X, \varphi Y)$ and M is a *Sasaki manifold* if $[\varphi, \varphi] + d\eta \otimes \xi = 0$ as well as $d\eta = 2\Phi$.

Sasaki manifolds may always be oriented using the volume form $(d\eta)^n \wedge \eta$. A plethora of examples of such manifolds will be provided in subsequent chapters.

The relationship of Sasaki manifolds to Kähler geometry is twofold: On the one hand there is always a Kähler manifold "above" every Sasaki manifold, namely the Riemannian cone in dimension $2n+2$ [BG, Definition 6.5.15]. On the other hand there is often also a Kähler space "below" a Sasaki manifold in dimension $2n$: The Reeb vector field ξ spans an integrable distribution which induces the so-called *characteristic foliation* $\mathcal{F}$. Under certain regularity assumptions the space $M/\mathcal{F}$ of leaves of this foliation admits the structure of a Kähler orbifold or even manifold [BG, Theorem 7.1.3]. A partial converse to the latter construction is the famous *Boothby-Wang bundle* [BW]:

Theorem 1.2. *Let N be a Kähler manifold with integral Kähler class. Then a certain principal S^1-bundle over N admits a Sasaki structure.*

As mentioned in the introduction we are particularly interested in Sasaki manifold with a striking presence or absence of symmetries:

Definition 1.3. An *automorphism* of a Sasaki manifold M is an isometry $\phi : M \to M$ which satisfies one of the equivalent conditions $\phi_* \xi = \xi$, $\phi^* \eta = \eta$ or $\phi_* \circ \varphi = \varphi \circ \phi_*$. The collection of all such automorphisms constitutes a Lie group which we denote by $\mathrm{Aut}(M)$ and M is called *homogeneous* if $\mathrm{Aut}(M)$ acts transitively on M. The Lie algebra $\mathfrak{aut}(M)$ of $\mathrm{Aut}(M)$ is comprised of all complete Killing vector fields X which satisfy $\mathcal{L}_X \xi = 0$, $\mathcal{L}_X \eta = 0$ and $\mathcal{L}_X \varphi = 0$.

1.2 3-Sasaki Manifolds

3-Sasaki manifolds were conceived independently by Udrişte in 1969 [Udri] and Kuo in 1970 [Kuo].

Definition 1.4. A Riemannian manifold (M^{4n+3}, g) endowed with three Sasaki structures $(\xi_i, \eta_i, \varphi_i)_{i=1,2,3}$ is a *3-Sasaki manifold* if $g(\xi_i, \xi_j) = \delta_{ij}$ and $[\xi_i, \xi_j] = 2\xi_k$, where (i, j, k) is a cyclic permutation of $(1, 2, 3)$.

3-Sasaki manifolds are always spin as well as Einstein with a positive Einstein constant [BGM, Theorem A & Corollary 2.7] and they satisfy stringent topological constraints [BG, Section 13.5].

Again there is a twofold connection to important even-dimensional geometries: The Riemannian cone over any 3-Sasaki manifold admits a hyperkähler structure [BG, Definition 13.1.8] and the Reeb vector fields induce the characteristic foliation whose space of leaves is often a positive qK orbifold or manifold [BG, Theorem 13.3.13]. Conversely the *Konishi bundle* over any positive qK manifold admits a 3-Sasaki structure [Koni].

Definition 1.5. An *automorphism* of a 3-Sasaki manifold is an isometry $\phi : M \to M$ which satisfies one of the equivalent conditions $\phi_* \xi_i = \xi_i$, $\phi^* \eta_i = \eta_i$ or $\phi_* \circ \varphi_i = \varphi_i \circ \phi_*$

for $i = 1, 2, 3$. The collection of all such automorphisms constitutes a Lie group which we denote by $\mathrm{Aut}(M)$ and M is called *homogeneous* if $\mathrm{Aut}(M)$ acts transitively. The Lie algebra $\mathfrak{aut}(M)$ of $\mathrm{Aut}(M)$ is comprised of all complete Killing vector fields X which satisfy $\mathcal{L}_X \xi_i = 0$, $\mathcal{L}_X \eta_i = 0$ and $\mathcal{L}_X \varphi_i = 0$ for $i = 1, 2, 3$.

1.3 3-(α, δ)-Sasaki Manifolds

3-(α, δ)-Sasaki manifolds were introduced very recently in 2020 by Agricola and Dileo [AD] as a common generalization to accommodate both 3-Sasaki manifolds as well as other interesting examples like the quaternionic Heisenberg groups. For more information we refer the interested reader to the introductory articles [AD] and [ADS] as well as the thesis [Ste].

Definition 1.6. A Riemannian manifold (M^{4n+3}, g) endowed with three almost contact metric structures $(\xi_i, \eta_i, \varphi_i)_{i=1,2,3}$ is an *almost 3-contact metric manifold* if their interrelation is governed by the equations

$$\varphi_i \, \xi_j = \xi_k \,, \qquad \eta_i \circ \varphi_j = \eta_k \,, \qquad \varphi_i \circ \varphi_j = \varphi_k + \xi_i \otimes \eta_j \,,$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$. The *fundamental two-forms* are given by $\Phi_i(X, Y) := g(X, \varphi_i Y)$ and M is a *3-(α, δ)-Sasaki manifold* if there exist $\alpha, \delta \in \mathbb{R}$, $\alpha \neq 0$ such that

$$d\eta_i = 2\alpha\Phi_i + 2(\alpha - \delta)\eta_j \wedge \eta_k$$

for any cyclic permutation (i, j, k) of $(1, 2, 3)$. A 3-(α, δ)-Sasaki manifold is called *positive* if $\alpha\delta > 0$, *negative* if $\alpha\delta < 0$ and *degenerate* if $\delta = 0$. In the degenerate case the above condition simplifies to

$$d\eta_i = 2\alpha\Phi_i^{\mathcal{H}} \,,$$

where $\mathcal{H} := \bigcap_i \ker \eta_i$ and $\Phi_i^{\mathcal{H}}(X, Y) := \Phi_i(X_{\mathcal{H}}, Y_{\mathcal{H}})$.

One can show that 3-Sasaki manifolds are embedded in this definition as the special case $\alpha = \delta = 1$. The more general class of 3-(α, δ)-Sasaki manifolds still retains several favorable properties like orientability (e.g. via the volume form $(d\eta_1)^{2n} \wedge \eta_1 \wedge \eta_2 \wedge \eta_3$), hypernormality or the existence of a connection with skew torsion adapted to the geometry [AD].

It follows from the definition that $[\xi_i, \xi_j] = 2\delta\xi_k$ for any cyclic permutation (i, j, k) of $(1, 2, 3)$ [AD], which in part explains why there are pronounced qualitative differences between degenerate and non-degenerate 3-(α, δ)-Sasaki manifolds. The Reeb vector fields induce the characteristic foliation and the corresponding space of leaves locally admits a qK structure whose scalar curvature is positive/negative/zero if the 3-(α, δ)-Sasaki manifold is positive/negative/degenerate [ADS, Section 2.2]. In the degenerate case we will later discuss a partial converse to the latter construction akin to the Boothby-Wang bundle.

Definition 1.7. An *automorphism* of a 3-(α, δ)-Sasaki manifold is an isometry ϕ which satisfies one of the equivalent conditions $\phi_* \xi_i = \xi_i$, $\phi^* \eta_i = \eta_i$ or $\phi_* \circ \varphi_i = \varphi_i \circ \phi_*$ for $i = 1, 2, 3$. The collection of all such automorphisms constitutes a Lie group which we denote by $\mathrm{Aut}(M)$ and M is called *homogeneous* if $\mathrm{Aut}(M)$ acts transitively. The Lie algebra $\mathfrak{aut}(M)$ of $\mathrm{Aut}(M)$ is comprised of all complete Killing vector fields X which satisfy $\mathcal{L}_X \xi_i = 0$, $\mathcal{L}_X \eta_i = 0$ and $\mathcal{L}_X \varphi_i = 0$ for $i = 1, 2, 3$.

Since homogeneous non-degenerate 3-(α, δ)-Sasaki manifolds have already been studied intensively [ADS, Section 3], we instead focus on the degenerate case, where notably the Reeb vector fields are infinitesimal automorphisms.

Chapter 2

Classification of Homogeneous 3-Sasaki Manifolds

Homogeneous 3-Sasaki manifolds have successfully been classified after a long and complicated process culminating in a publication by Boyer, Galicki and Mann in 1994 [BGM]. In this chapter we provide a new, more direct and self-contained proof of this classification by constructing an explicit one-to-one correspondence between simply connected homogeneous 3-Sasaki manifolds and simple complex Lie algebras via the theory of root systems:

Theorem 2.1. *There is a one-to-one correspondence between simply connected homogeneous 3-Sasaki manifolds and simple complex Lie algebras.*
Given a simple complex Lie algebra $\mathfrak{u}$, choose a maximal root α of $\mathfrak{u}$ and let $\mathfrak{v}$ denote the direct sum of the subspace $\ker\alpha$ and the root spaces of roots perpendicular to α. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the compact real forms of $\mathfrak{u}$ and $\mathfrak{v}$, respectively, and write $\mathfrak{k} \cong \mathfrak{sp}(1)$ for the compact real form of the $\mathfrak{sl}(2,\mathbb{C})$-subalgebra defined by α. Let B denote the Killing form of $\mathfrak{g}$, set $\mathfrak{g}_1 = (\mathfrak{h}\oplus\mathfrak{k})^{\perp_B}$ and consider the reductive complement $\mathfrak{m} = \mathfrak{h}^{\perp_B} = \mathfrak{k}\oplus\mathfrak{g}_1$. Let G be the simply connected Lie group with Lie algebra $\mathfrak{g}$ and let $H \subset G$ the connected subgroup with Lie algebra $\mathfrak{h}$. Define a G-invariant Riemannian metric g on $M = G/H$ by extending the inner product on $T_{eH}M \cong \mathfrak{m}$ given by

$$g|_{\mathfrak{k}\times\mathfrak{k}} = -\frac{1}{4(n+2)}B\,, \qquad g|_{\mathfrak{g}_1\times\mathfrak{g}_1} = -\frac{1}{8(n+2)}B\,, \qquad g|_{\mathfrak{k}\times\mathfrak{g}_1} = 0\,.$$

Consider a basis X_1, X_2, X_3 of $\mathfrak{k}$ satisfying the commutator relations $[X_i, X_j] = 2X_k$ and extend $X_i \in \mathfrak{m} \cong T_{eH}M$ to a G-invariant vector field ξ_i on M. Let η_i denote the metric dual of ξ_i and φ_i the G-invariant endomorphism field defined by extending

$$\varphi_i|_{\mathfrak{k}} = \frac{1}{2}\,\mathrm{ad}_{X_i}\,, \qquad \varphi_i|_{\mathfrak{g}_1} = \mathrm{ad}_{X_i}\,.$$

Then $(g, \xi_i, \eta_i, \varphi_i)_{i=1,2,3}$ is a G-invariant 3-Sasaki structure on M.
Conversely, given a simply connected homogeneous 3-Sasaki manifold M, represented

as the quotient $\widetilde{G}/\widetilde{H}$*, where* $\widetilde{G}$ *is a connected Lie group acting effectively on* M*, then* $\widetilde{G} = \mathrm{Aut}_0(M)$*, the connected component of the* 3*-Sasaki automorphism group of* M*, and* M *is the unique space associated with the complexification of the Lie algebra of* $\widetilde{G}$*.*

Using this characterization we rediscover the list of homogeneous 3-Sasaki manifolds as given by Boyer, Galicki and Mann:

Corollary 2.2. *Every homogeneous 3-Sasaki manifold* $M = G/H$ *(not necessarily simply connected) is isomorphic to one of the following spaces:*

$$\frac{Sp(n+1)}{Sp(n)} \cong S^{4n+3}, \quad \frac{Sp(n+1)}{Sp(n)\times\mathbb{Z}_2} \cong \mathbb{R}P^{4n+3}, \quad \frac{SU(m)}{S(U(m-2)\times U(1))},$$
$$\frac{SO(k)}{SO(k-4)\times Sp(1)}, \quad \frac{G_2}{Sp(1)}, \quad \frac{F_4}{Sp(3)}, \quad \frac{E_6}{SU(6)}, \quad \frac{E_7}{Spin(12)}, \quad \frac{E_8}{E_7}.$$

To avoid redundancy, we need to assume $n \geq 0$*,* $m \geq 3$ *and* $k \geq 7$*.*

As a consequence we also arrive at the complete list of homogeneous positive qK manifolds as discovered by Alekseevskii:

Corollary 2.3. *Every homogeneous positive qK manifold is isometric to one of the spaces*

$$\frac{Sp(n+1)}{Sp(n)\times Sp(1)}, \quad \frac{SU(m)}{S(U(m-2)\times U(2))}, \quad \frac{SO(k)}{SO(k-4)\times SO(4)},$$
$$\frac{G_2}{SO(4)}, \quad \frac{F_4}{Sp(3)Sp(1)}, \quad \frac{E_6}{SU(6)Sp(1)}, \quad \frac{E_7}{Spin(12)Sp(1)}, \quad \frac{E_8}{E_7Sp(1)},$$

where the Riemannian metric and quaternionic structure are also determined by Theorem 2.1 via the Konishi bundle (see Section 2.7 for details).

The discussion of Theorem 2.1 and its consequences will be divided into several sections: We begin by summarizing the tortuous history of the classification in Section 2.1. We then introduce a certain $\mathbb{Z}$-grading of semisimple complex Lie algebras based on their root systems and use it we construct homogeneous 3-Sasaki manifolds from simple Lie algebras. The centerpiece of the chapter is the converse argument in Section 2.3. We complete the proof of Theorem 2.1 by showing that no proper subgroup of the identity component $\mathrm{Aut}_0(M)$ of the automorphism group can act transitively in Section 2.4. In Section 2.5 we compute the isotropy groups described in Corollary 2.2 explicitly for the classical spaces and Lie theoretically via Borel-de Siebenthal theory in the exceptional cases. In Section 2.6 we show that the only non-simply connected homogeneous 3-Sasaki manifolds are the real projective spaces $\mathbb{R}P^{4n+3}$, which are the $\mathbb{Z}_2$-quotient of the previously described space $S^{4n+3} = Sp(n+1)/Sp(n)$. Finally, since our arguments are independent of the classification of homogeneous positive qK manifolds, they allow for an alternative proof of the latter (Section 2.7).

2.1 History of the Classification

The earliest result concerning our topic was the classification for the related notion of (compact simply connected) homogeneous complex contact manifolds (so-called $\mathcal{C}$-spaces) by BOOTHBY in 1961 [Boot], who showed that these are in one-to-one correspondence with simple complex Lie algebras. The much more famous next step was the work of WOLF and ALEKSEEVSKII on qK manifolds in the 1960s. WOLF showed in 1961 that there is a one-to-one correspondence between $\mathcal{C}$-spaces and compact simply connected *symmetric* positive qK manifolds [Wolf, Theorem 6.1].

BOOTHBY and WOLF already emphasized the importance of the maximal root in the root system of a simple Lie algebra, which also plays a key role in our construction: WOLF demonstrated that the compact simply connected symmetric positive qK manifolds are precisely of the form $G/N_G(K)$, where G is a compact simple Lie group and $N_G(K)$ denotes the normalizer of the subgroup K corresponding to the compact real form of the subalgebra generated by the root spaces of a maximal root and its negative. These manifolds became known as WOLF spaces. As we will show in this chapter, the simply connected homogeneous 3-Sasaki manifolds are of the form $G/(C_G(K))_0$, where $(C_G(K))_0$ is the identity component of the centralizer $C_G(K)$ of K in G. In 1968 ALEKSEEVSKII fully classified compact *homogeneous* positive qK manifolds by demonstrating that they are necessarily of the form $G/N_G(K)$ [Alek, Theorem 1].

By 1994 BOYER, GALICKI and MANN transferred these results to the 3-Sasaki realm [BGM]. They combined the classification of homogeneous positive qK manifolds with the Konishi bundle to obtain the following diffeomorphism type classification:

Theorem 2.4 ([BGM, Theorem C]). *Every homogeneous 3-Sasaki manifold $M = G/H$ (not necessarily simply connected) is precisely one of the following:*

$$\frac{Sp(n+1)}{Sp(n)} \cong S^{4n+3}, \quad \frac{Sp(n+1)}{Sp(n)\times\mathbb{Z}_2} \cong \mathbb{R}P^{4n+3}, \quad \frac{SU(m)}{S(U(m-2)\times U(1))},$$
$$\frac{SO(k)}{SO(k-4)\times Sp(1)}, \quad \frac{G_2}{Sp(1)}, \quad \frac{F_4}{Sp(3)}, \quad \frac{E_6}{SU(6)}, \quad \frac{E_7}{Spin(12)}, \quad \frac{E_8}{E_7}.$$

To avoid redundancy, we need to assume $n \geq 0$, $m \geq 3$ and $k \geq 7$.

They also provided a more precise description of the 3-Sasaki structures in the four classical cases via a technique called 3-Sasaki reduction [BGM]. In 1996 BIELAWSKI [Biel] described the Riemannian structure on these spaces uniformly. Both for his result and for several later discussions, we need to recall the following construction: As was first described systematically by KOBAYASHI and NOMIZU [KN], the study of G-invariant geometric objects on a reductive homogeneous space $M = G/H = G/G_p$ can be greatly simplified by instead considering $\mathrm{Ad}(H)$-invariant algebraic objects on a fixed reductive complement $\mathfrak{m}$ of $\mathfrak{h}$ in $\mathfrak{g}$. More precisely the map $\psi : \mathfrak{m} \to T_pM$, $X \mapsto \overline{X}_p$ (where $\overline{X}_p$ denotes the fundamental vector field of the left G-action at p) is an isomorphism that allows us to translate between $\mathrm{Ad}(H)$-invariant tensors on $\mathfrak{m}$ and the restriction of G-invariant tensor fields to T_pM. While actually working on a more algebro-geometric problem (singularities of nilpotent varieties) and employing very different methods (e.g. Nahm's differential equation), Bielawski obtained the following

Theorem 2.5 ([Biel, Theorem 4]). *Given a homogeneous 3-Sasaki manifold $M = G/H$ with reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, there is a natural decomposition $\mathfrak{m} = \mathfrak{sp}(1) \oplus \mathfrak{m}'$ such that the metric on M corresponds to an inner product on $\mathfrak{m}$ of the form*

$$(X, Y) \mapsto -c\, B(X_{\mathfrak{sp}(1)}, Y_{\mathfrak{sp}(1)}) - \frac{c}{2}\, B(X_{\mathfrak{m}'}, Y_{\mathfrak{m}'})\,,$$

where B denotes the Killing form of $\mathfrak{g}$ and $c > 0$ is some constant.

In 2020 the work of Draper, Ortega and Palomo gave a new hands-on description of homogeneous 3-Sasaki manifolds [DOP]. Their study was based on the following

Definition 2.6 ([DOP, Definition 4.1]). A *3-Sasaki datum* is a pair $(\mathfrak{g}, \mathfrak{h})$ of real Lie algebras such that

1. $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a $\mathbb{Z}_2$-graded compact simple Lie algebra whose even part is a sum of two commuting subalgebras,

$$\mathfrak{g}_0 = \mathfrak{sp}(1) \oplus \mathfrak{h}\,;$$

2. there exists an $\mathfrak{h}^{\mathbb{C}}$-module W such that the complexified $\mathfrak{g}_0^{\mathbb{C}}$-module $\mathfrak{g}_1^{\mathbb{C}}$ is isomorphic to the tensor product of the natural $\mathfrak{sp}(1)^{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$-module $\mathbb{C}^2$ and W:

$$\mathfrak{g}_1^{\mathbb{C}} \cong \mathbb{C}^2 \otimes W\,.$$

Their main result is the following

Theorem 2.7 ([DOP, Theorem 4.2]). *Let $M = G/H$ be a homogeneous space such that H is connected and the Lie algebras $(\mathfrak{g}, \mathfrak{h})$ constitute a 3-Sasaki datum. Consider the reductive complement $\mathfrak{m} := \mathfrak{sp}(1) \oplus \mathfrak{g}_1$ and let $X_1, X_2, X_3 \in \mathfrak{m}$ denote the standard basis of $\mathfrak{sp}(1)$ and ξ_1, ξ_2, ξ_3 the corresponding G-invariant vector fields on M. If g and φ_i are the Riemannian metric and endomorphism fields described in Theorem 2.1 and $\eta_i = g(\xi_i, \cdot)$, then the tuple $(M, g, \xi_i, \eta_i, \varphi_i)_{i=1,2,3}$ constitutes a homogeneous 3-Sasaki structure.*

Furthermore they conducted a case-by-case study to show that every compact simple Lie algebra admits a 3-Sasaki datum, thus providing a detailed analysis of one homogeneous 3-Sasaki structure (it is at this point not clear if there could be more than one such structure on a given space) on each of the diffeomorphism types discovered by Boyer, Galicki and Mann.

We finish this section by giving an overview of the structure of our proof of Theorem 2.1: In Section 2.2 we first describe a way to construct a simply connected homogeneous 3-Sasaki manifold from a simple complex Lie algebra $\mathfrak{u}$ and a maximal root α of $\mathfrak{u}$. More precisely, we first utilize the theory of root systems to generate a complexified version $(\mathfrak{u}, \mathfrak{v})$ of a 3-Sasaki datum. We then pass to the compact real forms $(\mathfrak{g}, \mathfrak{h})$ to obtain a "real" 3-Sasaki datum in the sense of Definition 2.6 and apply Theorem 2.7.
For the converse argument in Sections 2.3 we start with a simply connected homogeneous 3-Sasaki manifold $M = G/H$, where G is a compact simply connected Lie group acting almost effectively and transitively on M via 3-Sasaki automorphisms. We prove that the Lie algebra $\mathfrak{g}$ and its complexification $\mathfrak{u} = \mathfrak{g}^{\mathbb{C}}$ are simple and that the 3-Sasaki structure gives rise to a maximal root α of $\mathfrak{u}$. We can therefore apply the previous construction and then show that this yields the same 3-Sasaki structure that we started with.
Section 2.4 completes the proof of Theorem 2.1 by showing that no subgroup of $\mathrm{Aut}_0(M)$ can act transitively. In particular, this proves that any two homogeneous 3-Sasaki manifolds $M = G/H$, $M' = G'/H'$ associated with two different simple complex Lie algebras $\mathfrak{g}^{\mathbb{C}} \neq (\mathfrak{g}')^{\mathbb{C}}$ are not isomorphic.

2.2 Constructing 3-Sasaki Manifolds from Lie Algebras

For the announced construction we first need certain basic facts about root systems. Let $\mathfrak{u}$ be a (finite-dimensional) semisimple complex Lie algebra. Its Killing form is non-degenerate and thus gives rise to an isomorphism $\mathfrak{u} \to \mathfrak{u}^*$ and a non-degenerate, symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{u}^*$. We fix a Cartan subalgebra $\mathfrak{c} \subset \mathfrak{u}$ and denote the corresponding root system and root spaces by $\Phi \subset \mathfrak{c}^*$ and $\mathfrak{u}_\alpha \subset \mathfrak{u}$ for $\alpha \in \mathfrak{c}^*$, respectively.

Each root $\alpha \in \Phi$ has an associated coroot $H_\alpha \in \mathfrak{c}$ defined as the unique element of $[\mathfrak{u}_\alpha, \mathfrak{u}_{-\alpha}]$ satisfying $\alpha(H_\alpha) = 2$. Furthermore $\mathfrak{s}_\alpha := \mathfrak{u}_\alpha \oplus \mathfrak{u}_{-\alpha} \oplus [\mathfrak{u}_\alpha, \mathfrak{u}_{-\alpha}]$ is a subalgebra of $\mathfrak{u}$ which is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. This isomorphism can be made explicit by choosing an $\mathfrak{sl}_2$-*triple*, i.e. vectors $X_\alpha \in \mathfrak{u}_\alpha, Y_\alpha \in \mathfrak{u}_{-\alpha}$ satisfying the commutation relations

$$[H_\alpha, X_\alpha] = 2X_\alpha\,, \quad [H_\alpha, Y_\alpha] = -2Y_\alpha\,, \quad [X_\alpha, Y_\alpha] = H_\alpha\,. \tag{2.1}$$

Moreover it can be shown that for any root $\alpha \in \Phi$ and any linear form $\beta \in \mathfrak{c}^*$:

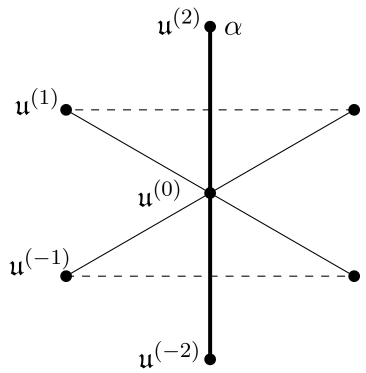
$$c_{\alpha\beta} := \beta(H_\alpha) = \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\,.$$

In particular $c_{\alpha\beta} = 0$ if and only if α and β are perpendicular to each other (with respect to $\langle \cdot, \cdot \rangle$). In case β is also a root $c_{\alpha\beta}$ is an integer, which we call the *Cartan number of*

β *with respect to* α. Fixing a root $\alpha \in \Phi$ we can therefore decompose

$$\mathfrak{u} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{u}^{(k)}\,, \quad \text{where} \quad \mathfrak{u}^{(k)} := \bigoplus_{\substack{\beta \in \mathfrak{c}^*\,, \\ c_{\alpha\beta} = k}} \mathfrak{u}_\beta\,.$$

Since $c_{\alpha\beta}$ is linear in β, this decomposition is in fact a $\mathbb{Z}$-grading, i.e. $[\mathfrak{u}^{(k)}, \mathfrak{u}^{(\ell)}] \subset \mathfrak{u}^{(k+\ell)}$. We also note that $\mathfrak{u}^{(k)}$ is precisely the k-eigenspace of $\mathrm{ad}(H_\alpha)$. One can visualize this grading using parallel copies of hyperplanes perpendicular to α, e.g. for the root system A_2:



The structure of this grading is related to the notion of maximality of the root α: Assuming we have chosen a set $\Delta \subset \Phi$ of simple roots, we may introduce a partial order $\leq$ on Φ by stipulating that $\alpha \leq \beta$ if and only if $\beta - \alpha$ is a linear combination of roots in Δ with non-negative coefficients. A root $\alpha \in \Phi$ is called *maximal* if there is a choice of simple roots such that there is no strictly larger root than α with respect to the induced partial order. The following lemma was adapted from [Wolf, Theorem 4.2]:

Lemma 2.8. *For any root $\alpha \in \Phi$, the following statements are equivalent:*

i) α is maximal.

ii) $|c_{\alpha\beta}| \leq 2$ for all roots $\beta \in \Phi$ and $c_{\alpha\beta} = \pm 2$ if and only if $\beta = \pm\alpha$.

Proof. *i)* $\implies$ *ii)*: It is well-known that for $\beta \in \Phi \setminus \{\pm\alpha\}$ the Cartan number is given by $c_{\alpha\beta} = p - q$, where $p, q \in \mathbb{N}_0$ are the greatest non-negative integers such that $\beta + r\alpha \in \Phi$ for every $r \in \{-p, \dots, q\}$ [Knap, Proposition 2.29]. Suppose there was some $\beta \in \Phi \setminus \{\alpha\}$ such that $c_{\alpha\beta} \geq 2$. Then $p \geq 2$, so that $\beta - \alpha, \beta - 2\alpha \in \Phi$ and their negatives $\alpha - \beta, 2\alpha - \beta \in \Phi$ are roots. In fact $\alpha - \beta$ has to be a *non-negative* linear combination of simple roots (for some choice of simple roots with respect to which α is maximal), since otherwise $\beta > \alpha$. But then $2\alpha - \beta \geq \alpha$ and maximality of α would imply $2\alpha - \beta = \alpha$, i.e. $\beta = \alpha$. For $\beta \in \Phi \setminus \{-\alpha\}$ such that $c_{\alpha\beta} \leq -2$ we apply this argument to $-\beta$.
ii) $\implies$ *i)*: We may choose a set of simple roots Δ in such a way that $c_{\alpha\beta} \geq 0$ for all $\beta \in \Delta$. This can be achieved by first choosing positive roots using a slight perturbation

of the hyperplane perpendicular to α. Let $\beta \in \Phi$ such that $\beta \geq \alpha$, i.e. $\beta - \alpha = \sum_{i=1}^{n} \lambda_i \alpha_i$, where $\lambda_i \geq 0$ and $\alpha_i \in \Delta$. Then,

$$c_{\alpha\beta} = c_{\alpha\alpha} + c_{\alpha(\beta-\alpha)} = 2 + \sum_{i=1}^{n} \lambda_i \underbrace{c_{\alpha\alpha_i}}_{\geq 0} \geq 2 \,.$$

Hypothesis *ii*) then implies $\beta = \alpha$, so that α is maximal. □

We remark that in an *irreducible* root system Φ the maximal root is unique up to the action of the Weyl group: This follows because in an irreducible root system the maximal root is uniquely determined after choosing simple roots and any two choices of simple roots can be mapped to each other by the Weyl group. Our goal is now to establish the following construction:

Theorem 2.9. *Let $\mathfrak{u}$ be a simple complex Lie algebra, α a maximal root in its root system, $\mathfrak{g}$ the compact real form of $\mathfrak{u}$ and $\mathfrak{k} \cong \mathfrak{sp}(1)$ the compact real form of the subalgebra $\mathfrak{s}_\alpha = \mathfrak{u}_\alpha \oplus \mathfrak{u}_{-\alpha} \oplus [\mathfrak{u}_\alpha, \mathfrak{u}_{-\alpha}] \cong \mathfrak{sl}(2,\mathbb{C})$. Let G denote the simply connected Lie group with Lie algebra $\mathfrak{g}$, K the connected subgroup with Lie algebra $\mathfrak{k}$ and $H = (C_G(K))_0$ the identity component of the centralizer $C_G(K)$ of K in G. Then the simply connected homogeneous space $M = G/H$ admits a homogeneous 3-Sasaki structure whose tensors are given by Theorem 2.1. All possible choices of a maximal root lead to isomorphic 3-Sasaki manifolds.*

Definition 2.10. A *complex 3-Sasaki datum* is a pair $(\mathfrak{u}, \mathfrak{v})$ of $\mathbb{C}$-Lie algebras such that

1. $\mathfrak{u} = \mathfrak{u}_{\text{even}} \oplus \mathfrak{u}_{\text{odd}}$ is a $\mathbb{Z}_2$-graded simple Lie algebra whose even part is a sum of two commuting subalgebras,
$$\mathfrak{u}_{\text{even}} = \mathfrak{v} \oplus \mathfrak{sl}(2,\mathbb{C})\,;$$
2. there exists a $\mathfrak{v}$-module W such that $\mathfrak{u}_{\text{odd}} \cong \mathbb{C}^2 \otimes W$ as $\mathfrak{u}_{\text{even}}$-modules.

Remark 2.11. We formulated the above definition in the given way because it allows us to branch off into two cases: Our primary interest here is be to consider the *compact* real forms $(\mathfrak{g}, \mathfrak{h})$ of $(\mathfrak{u}, \mathfrak{v})$ which then form a 3-Sasaki datum in the sense of Definition 2.6. On the other hand, one may also look at the real form $(\mathfrak{g}^*, \mathfrak{h})$ of $(\mathfrak{u}, \mathfrak{v})$ given by $\mathfrak{g}^* = \mathfrak{h} \oplus \mathfrak{s}_\alpha \oplus i\mathfrak{g}_1$ to obtain a generalized 3-Sasaki datum in the sense of [ADS]. These give rise to homogeneous negative 3-(α,δ)-Sasaki manifolds by a construction similar to Theorem 2.7, compare [ADS, Theorem 3.1.1].

Proposition 2.12. Let $\mathfrak{u}$ be a simple complex Lie algebra and $\alpha \in \Phi$ a maximal root in its root system. Set $\Phi_0 := \{\beta \in \Phi \,|\, c_{\alpha\beta} = 0\}$ as well as

$$\mathfrak{v} := \ker \alpha \oplus \bigoplus_{\beta \in \Phi_0} \mathfrak{u}_\beta \,.$$

Then $(\mathfrak{u}, \mathfrak{v})$ is a complex 3-Sasaki datum.

Proof. Using the above $\mathbb{Z}$-grading we let

$$\mathfrak{u}_{\text{even}} := \mathfrak{u}^{(-2)} \oplus \mathfrak{u}^{(0)} \oplus \mathfrak{u}^{(2)}, \quad \mathfrak{u}_{\text{odd}} := \mathfrak{u}^{(-1)} \oplus \mathfrak{u}^{(1)}.$$

Since $|c_{\alpha\beta}| \leq 2$ for all $\beta \in \Phi$ by Lemma 2.8, we have $\mathfrak{u} = \mathfrak{u}_{\text{even}} \oplus \mathfrak{u}_{\text{odd}}$. Because $\mathfrak{u}_{\text{even}}$ and $\mathfrak{u}_{\text{odd}}$ are comprised of the $\mathfrak{u}^{(k)}$ with even and odd k respectively, this decomposition is in fact a $\mathbb{Z}_2$-grading. We claim that

$$\mathfrak{u}_{\text{even}} = \mathfrak{s}_\alpha \oplus \mathfrak{v}$$

as a direct sum of Lie algebras, where $\mathfrak{s}_\alpha = \mathfrak{u}_\alpha \oplus \mathfrak{u}_{-\alpha} \oplus [\mathfrak{u}_\alpha, \mathfrak{u}_{-\alpha}]$. Since $c_{\alpha\beta} = \pm 2$ if and only if $\beta = \pm\alpha$, we have the following vector space decompositions:

$$\mathfrak{u}_{\text{even}} = \mathfrak{u}_\alpha \oplus \mathfrak{u}_{-\alpha} \oplus \mathfrak{c} \oplus \bigoplus_{\beta \in \Phi_0} \mathfrak{u}_\beta, \quad \mathfrak{c} = [\mathfrak{u}_\alpha, \mathfrak{u}_{-\alpha}] \oplus \ker \alpha.$$

In order to show that $\mathfrak{v}$ is indeed a sub*algebra* of $\mathfrak{u}$ note that $[\mathfrak{u}_\beta, \mathfrak{u}_\gamma] \subset \mathfrak{u}_{\beta+\gamma}$ for any $\beta, \gamma \in \Phi_0$. Now if $\beta + \gamma$ is a root, then $\beta + \gamma \in \Phi_0$, so $\mathfrak{u}_{\beta+\gamma} \subset \mathfrak{v}$. If $\beta + \gamma$ is not a root and not zero, then $\mathfrak{u}_{\beta+\gamma} = 0 \subset \mathfrak{v}$. If $\beta + \gamma = 0$, then $[\mathfrak{u}_\beta, \mathfrak{u}_{-\beta}] = \langle H_\beta \rangle \subset \ker \alpha$ because $\beta \in \Phi_0$. To check that $\mathfrak{s}_\alpha$ and $\mathfrak{v}$ commute we recall that $\mathfrak{v}$ is a subset of $\mathfrak{u}^{(0)} = \ker \operatorname{ad}_{H_\alpha}$. For $\beta \in \Phi_0$ we have $[\mathfrak{u}_{\pm\alpha}, \mathfrak{u}_\beta] \subset \mathfrak{u}_{\pm\alpha+\beta} \subset \mathfrak{u}^{(\pm 2)} = \mathfrak{u}_{\pm\alpha}$, so $\mathfrak{u}_{\pm\alpha}$ and $\mathfrak{u}_\beta$ commute.
We now verify the second condition from Definition 2.10 for the $\mathfrak{v}$-module $W := \mathfrak{u}^{(1)}$. We choose an $\mathfrak{sl}_2$-triple $(X_\alpha, Y_\alpha, H_\alpha)$ and identify it (in order) with the three standard matrices

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C}).$$

This fixes isomorphisms $\mathfrak{s}_\alpha \cong \mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{u}_{\text{even}} \cong \mathfrak{v} \oplus \mathfrak{sl}(2, \mathbb{C})$. We consider the following linear map:

$$\begin{aligned} \Psi : \mathfrak{u}_{\text{odd}} &= \mathfrak{u}^{(-1)} \oplus \mathfrak{u}^{(1)} \to \mathbb{C}^2 \otimes W, \\ X &= X^{(-1)} + X^{(1)} \mapsto (1,0) \otimes X^{(1)} + (0,1) \otimes [X_\alpha, X^{(-1)}]. \end{aligned}$$

If $\beta \in \Phi$ such that $c_{\alpha\beta} = -1$, then $\beta + \alpha$ must be a root and $[\mathfrak{u}_\alpha, \mathfrak{u}_\beta] = \mathfrak{u}_{\alpha+\beta}$. This shows that $\operatorname{ad}_{X_\alpha} : \mathfrak{u}_\beta \to \mathfrak{u}_{\beta+\alpha}$ and by extension Ψ are linear isomorphisms. It remains to be shown that Ψ preserves the $\mathfrak{u}_{\text{even}}$-module structure, where $\mathfrak{u}_{\text{even}} \cong \mathfrak{v} \oplus \mathfrak{sl}(2, \mathbb{C})$ acts on $\mathbb{C}^2 \otimes W$ via the above fixed isomorphism. We remind the reader of the commutator relations in (2.1).
If $Z \in \mathfrak{v} \subset \mathfrak{u}^{(0)}$, then ad_Z preserves the decomposition $\mathfrak{u}_{\text{odd}} = \mathfrak{u}^{(-1)} \oplus \mathfrak{u}^{(1)}$. Since $\mathfrak{v}$ and $\mathfrak{s}_\alpha$ are commuting subalgebras of $\mathfrak{u}$, so are their respective adjoint subrepresentations,

$$\begin{aligned} \Psi([Z, X]) &= (1,0) \otimes [Z, X^{(1)}] + (0,1) \otimes [X_\alpha, [Z, X^{(-1)}]] \\ &= (1,0) \otimes [Z, X^{(1)}] + (0,1) \otimes [Z, [X_\alpha, X^{(-1)}]] = Z \cdot \Psi(X). \end{aligned}$$

Here "·" denotes the adjoint representation of $\mathfrak{v}$ on W, whereas in the following equations it will signify the standard representation of $\mathfrak{sl}(2,\mathbb{C})$ on $\mathbb{C}^2$. Finally we check the representation of the basis $(X_\alpha, Y_\alpha, H_\alpha)$ of $\mathfrak{s}_\alpha$:

$$\Psi([X_\alpha, X]) = \Psi([X_\alpha, X^{(-1)}]) = (1,0) \otimes [X_\alpha, X^{(-1)}] = X_\alpha \cdot \Psi(X)\,.$$

By virtue of the Jacobi identity:

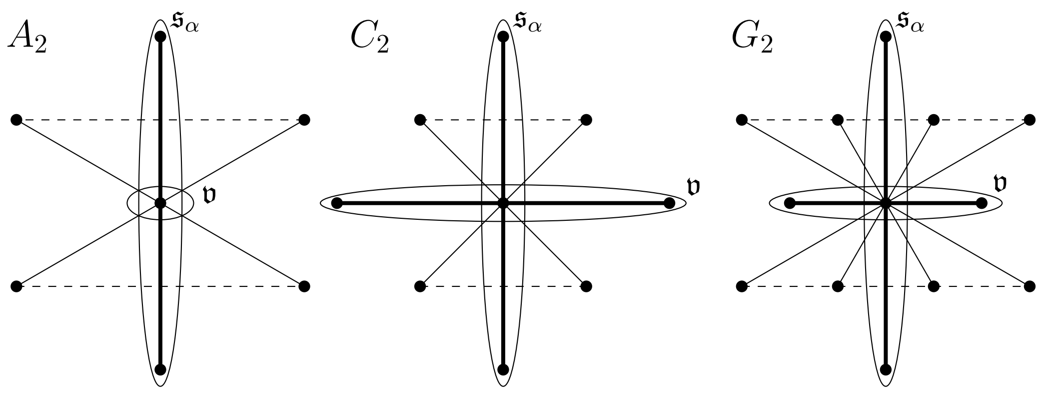
$$\begin{aligned}\Psi([Y_\alpha, X]) &= \Psi([Y_\alpha, X^{(1)}]) = (0,1) \otimes [X_\alpha, [Y_\alpha, X^{(1)}]] \\ &= (0,1) \otimes \big([\underbrace{[X^{(1)}, X_\alpha]}_{=0}, Y_\alpha] + [\underbrace{[X_\alpha, Y_\alpha]}_{=H_\alpha}, X^{(1)}]\big) \\ &= (0,1) \otimes X^{(1)} = Y_\alpha \cdot \Psi(X)\,.\end{aligned}$$

Ultimately:

$$\Psi([H_\alpha, X]) = \Psi(X^{(1)} - X^{(-1)}) = (1,0) \otimes X^{(1)} + (0,-1) \otimes [X_\alpha, X^{(-1)}] = H_\alpha \cdot \Psi(X)\,. \ \square$$

Proof of Theorem 2.9. Starting from a simple complex Lie algebra $\mathfrak{u}$ and a maximal root α Proposition 2.12 yields a complex 3-Sasaki datum $(\mathfrak{u}, \mathfrak{v})$. As mentioned in Remark 2.11 the compact real forms $(\mathfrak{g}, \mathfrak{h})$ constitute a "real" 3-Sasaki datum in the sense of Definition 2.6 and Theorem 2.7 endows $M = G/H$ with a homogeneous 3-Sasaki structure. Since $\mathfrak{v} = C_{\mathfrak{u}}(\mathfrak{s}_\alpha)$ and thus $\mathfrak{h} = C_{\mathfrak{g}}(\mathfrak{k})$, it follows that $H = (C_G(K))_0$. $\square$

Example 2.13. Let us illustrate the construction using the special case $\operatorname{rk}\mathfrak{u} = 2$: Here the only simple Lie algebras are $\mathfrak{sl}(3,\mathbb{C})$, $\mathfrak{sp}(4,\mathbb{C})$ and $\mathfrak{g}_2$ corresponding (in order) to the root systems A_2, C_2 and G_2. The following diagrams depict the subalgebras $\mathfrak{v}$ and $\mathfrak{s}_\alpha$ from the proposition in these three cases:



The corresponding homogeneous 3-Sasaki manifolds are (in order) the Aloff-Wallach space $W^{1,1} = SU(3)/S^1$, the 7-sphere $S^7 = Sp(2)/Sp(1)$ and the exceptional space $G_2/Sp(1)$.

We finish this section by showing that the maximal root is in fact an auxiliary choice:

Lemma 2.14. *All possible choices of a maximal root in Proposition 2.12 lead to isomorphic 3-Sasaki manifolds.*

Proof. Let $\mathfrak{u}$ be a simple complex Lie algebra, $\mathfrak{g}$ its compact real form, G the corresponding simply connected Lie group and $T \subset G$ a maximal torus. Let $\alpha, \widetilde{\alpha}$ denote two maximal roots in the root system Φ of $\mathfrak{u}$ with respect to the Cartan subalgebra given by the complexification of the Lie algebra of T. As mentioned above the maximal root of Φ is unique up to the action of the Weyl group $W(G) = N_G(T)/T$, so there is a representative $w \in N_G(T)$ such that $\mathrm{Ad}_w^{\mathbb{C}}(H_\alpha) = H_{\widetilde{\alpha}}$.
Because the Weyl group acts orthogonally on the root system $\mathrm{Ad}_w^{\mathbb{C}} : \mathfrak{u} \to \mathfrak{u}$ maps the $\mathbb{Z}$-grading $\mathfrak{u}^{(k)}$ with respect to α to the grading $\widetilde{\mathfrak{u}}^{(k)}$ with respect to $\widetilde{\alpha}$. This implies that $\mathrm{Ad}_w \mathfrak{h} = \widetilde{\mathfrak{h}}$, where $\mathfrak{h}, \widetilde{\mathfrak{h}} \subset \mathfrak{g}$ are the compact real forms of the subalgebras $\mathfrak{v}, \widetilde{\mathfrak{v}} \subset \mathfrak{u}$ considered in Proposition 2.12. Consequently $wHw^{-1} = \widetilde{H}$ for the corresponding connected subgroups $H, \widetilde{H} \subset G$ and we have a well-defined diffeomorphism $G/H \to G/\widetilde{H}$, $gH \mapsto wgw^{-1}\widetilde{H}$. One easily checks from the definitions in Theorem 2.1 that this map transforms one 3-Sasaki structure into the other. □

2.3 Deconstructing 3-Sasaki Manifolds

This section is the centerpiece of the chapter, where we explain a crucial step in the proof of Theorem 2.1:

Theorem 2.15. *Every simply connected homogeneous 3-Sasaki manifold arises from the construction described in Section 2.2.*

From now on let $(M^{4n+3}, g, \xi_i, \eta_i, \varphi_i)_{i=1,2,3}$ denote a simply connected homogeneous 3-Sasaki manifold and let G be a compact simply connected Lie group acting almost effectively (i.e. the kernel of the action is discrete and hence finite) and transitively on M by 3-Sasaki automorphisms. We show that the Lie algebra $\mathfrak{g}$ of G and its complexification $\mathfrak{u} = \mathfrak{g}^{\mathbb{C}}$ are simple and describe how the 3-Sasaki structure gives rise to a maximal root α of $\mathfrak{u}$ with respect to a suitably chosen Cartan subalgebra. We can then apply the construction from Section 2.2 and prove that this yields the same 3-Sasaki structure that we started with.

The prototypical example to have in mind is where G is the universal cover of $\mathrm{Aut}_0(M)$, the identity component of the 3-Sasaki automorphism group of M. Since M is complete and positively Einstein, it follows that M is compact, so by the Myers-Steenrod theorem the isometry group $\mathrm{Iso}(M)$ of M is a compact Lie group. The closed subgroup $\mathrm{Aut}(M) \subset \mathrm{Iso}(M)$ of 3-Sasaki automorphisms of M is thus also compact. Since M is connected, the identity component $\mathrm{Aut}_0(M)$ still acts transitively. The universal cover of $\mathrm{Aut}_0(M)$ acts almost effectively, transitively and by 3-Sasaki automorphisms. It will follow from the results that we are about to prove that the universal cover of $\mathrm{Aut}_0(M)$ is also compact. Later on we will show that in fact the effectively acting quotient of any group G satisfying the above assumptions is automatically the full identity component $\mathrm{Aut}_0(M)$ of the automorphism group.

Since G is compact, its Lie algebra $\mathfrak{g}$ is reductive, i.e. decomposes as a direct sum of a semisimple subalgebra and its center $Z(\mathfrak{g})$. We first show that $\mathfrak{g}$ itself is semisimple.

Lemma 2.16. *For $X, Y \in \mathfrak{g}$ the fundamental vector fields satisfy the equation*

$$d\eta_i(\overline{X}, \overline{Y}) = \eta_i([\overline{X}, \overline{Y}])\,.$$

Notably evaluating the left-hand side at a point $p \in M$ depends on $\overline{X}, \overline{Y}$ only through their values at p, while the right-hand side a priori depends on the values in a neighborhood of p.

Proof. The standard formula for the exterior derivative reads

$$d\eta_i(\overline{X}, \overline{Y}) = \overline{X}\big(\eta_i(\overline{Y})\big) - \overline{Y}\big(\eta_i(\overline{X})\big) - \eta_i([\overline{X}, \overline{Y}])\,.$$

The Leibniz rule for the Lie derivative implies

$$\overline{X}\big(\eta_i(\overline{Y})\big) = \mathcal{L}_{\overline{X}}\big(\eta_i(\overline{Y})\big) = \big(\mathcal{L}_{\overline{X}}\eta_i\big)(\overline{Y}) + \eta_i\big(\mathcal{L}_{\overline{X}}\overline{Y}\big) = \eta_i([\overline{X}, \overline{Y}])\,,$$

where $\mathcal{L}_{\overline{X}}\eta_i = 0$ because G acts by 3-Sasaki automorphisms. Applying the same reasoning to the second term yields $\overline{Y}\big(\eta_i(\overline{X})\big) = -\eta_i([\overline{X}, \overline{Y}])$. □

Proposition 2.17. The Lie algebra $\mathfrak{g}$ has trivial center and is therefore semisimple.

Proof. Let $X \in \mathfrak{g}$ such that $X \neq 0$. Since G acts almost effectively, there is a point $p \in M$ such that $\overline{X}_p \neq 0$ and thus an index $i \in \{1, 2, 3\}$ such that $\overline{X}_p$ is not proportional to $(\xi_i)_p$. We show that there exists some $Y \in \mathfrak{g}$ satisfying $\eta_i(\overline{[X, Y]}_p) \neq 0$, which implies $[X, Y] \neq 0$: Because G acts transitively we may choose some $Y \in \mathfrak{g}$ such that $\overline{Y}_p = \varphi_i \overline{X}_p$. From the previous lemma we have

$$\eta_i(\overline{[X, Y]}_p) = -d\eta_i(\overline{X}_p, \overline{Y}_p) = -d\eta_i(\overline{X}_p, \varphi_i \overline{X}_p)\,.$$

One of the Sasaki equations in Definition 1.1 reads $\varphi_i^2 \overline{X}_p = -\overline{X}_p + P_i \overline{X}_p$, where P_i denotes the orthogonal projection to the line through $(\xi_i)_p$. Hence:

$$\eta_i(\overline{[X, Y]}_p) = 2g_p\big(\overline{X}_p, \overline{X}_p - P_i \overline{X}_p\big) = 2\big\|\overline{X}_p - P_i \overline{X}_p\big\|^2 \neq 0\,. \qquad \square$$

Remark 2.18. The compactness assumption fails for homogeneous negative 3-(α, δ)-Sasaki manifolds. Thus unlike with the construction in the previous section, a classification cannot be achieved by the method described here. Indeed in [ADS] homogeneous negative 3-(α,δ)-Sasaki manifolds with a transitive action by a non-semisimple Lie group are constructed.

Since $\mathfrak{g}$ is now both semisimple and the Lie algebra of a compact Lie group, its Killing form B is negative definite. We fix a point $p \in M$ and let $H := G_p$ denote its isotropy group. We write $\theta : G \to M$, $g \mapsto g \cdot p$ for the orbit map, which has surjective differential $d\theta_e : T_eG \cong \mathfrak{g} \to T_pM$, $X \mapsto \overline{X}_p$. Let $\alpha_i := \theta^*\eta_i$ denote the pullback of the contact form along the orbit map, which we may view - depending on the context - as either a linear form on $\mathfrak{g}$ or as a left-invariant differential one-form on G. In their seminal 1958 article [BW] Boothby and Wang exhibited the following results:

Lemma 2.19 ([BW, Lemmata 2, 3, 4])**.** *The one-form α_i is $\mathrm{Ad}(H)$-invariant, satisfies $\alpha_i(\mathfrak{h}) = 0$ and $d\alpha_i$ has rank $4n+2$. Furthermore the Lie algebra of the closed subgroup $\{g \in G \mid \mathrm{Ad}_g^*\alpha_i = \alpha_i\}$ is given by $\ker d\alpha_i$, contains $\mathfrak{h}$ and has dimension $\dim \mathfrak{h} + 1$.*

We now let $\widetilde{X_i} \in \mathfrak{g}$ denote the Killing dual of α_i, i.e. $B(\widetilde{X_i}, \cdot) = \alpha_i$ and consider $X_i := \widetilde{X_i}/B(\widetilde{X_i}, \widetilde{X_i})$. Ad-invariance of B implies that $\{g \in G \mid \mathrm{Ad}_g X_i = X_i\}$ and $\{g \in G \mid \mathrm{Ad}_g^*\alpha_i = \alpha_i\}$ coincide, so

$$C_{\mathfrak{g}}(X_i) = \ker d\alpha_i = \mathfrak{h} \oplus \langle X_i \rangle\,.$$

Proposition 2.20. The fundamental vector fields $\overline{X_i}$ coincide with the Reeb vector fields ξ_i at the point p and obey the same commutator relations $[X_i, X_j] = 2X_k$, where (i,j,k) is a cyclic permutation of $(1,2,3)$.

Proof. Clearly $X_i \in C_{\mathfrak{g}}(X_i) = \ker d\alpha_i$, so that $(\overline{X_i})_p \in \ker(d\eta_i)_p$. Furthermore we have $1 = \alpha_i(X_i) = (\eta_i)_p(\overline{X_i})_p$. Thus $(\overline{X_i})_p$ statisfies the uniquely defining equations of the Reeb vector $(\xi_i)_p$. Phrased differently X_i (viewed as a left-invariant vector field on G) and ξ_i are θ-related. Consequently the Lie brackets $[X_i, X_j]$ and $[\xi_i, \xi_j] = 2\xi_k$ are also θ-related and in particular $\overline{[X_i, X_j]}_p = 2(\xi_k)_p = 2(\overline{X_k})_p$. Hence $[X_i, X_j]$ and $2X_k$ could only differ by an element of $\mathfrak{h}$. But $B(X_k, \mathfrak{h}) = \alpha_k(\mathfrak{h}) = 0$ and $B([X_i, X_j], \mathfrak{h}) = B(X_i, [X_j, \mathfrak{h}]) = 0$, so that also $B([X_i, X_j] - 2X_k, \mathfrak{h}) = 0$. □

Let $\mathfrak{s}$ be a maximal Abelian subalgebra of $\mathfrak{h}$. Since $C_{\mathfrak{g}}(X_1) = \mathfrak{h} \oplus \langle X_1 \rangle$, it follows that $\mathfrak{t} := \mathfrak{s} \oplus \langle X_1 \rangle$ is a maximal Abelian subalgebra of $\mathfrak{g}$. In particular we obtain that $\mathrm{rk}\, G = \mathrm{rk}\, H + 1$. The Riemannian metric g corresponds to an $\mathrm{Ad}(H)$-invariant and thus also $\mathrm{ad}(\mathfrak{h})$-invariant inner product on a reductive complement of our choice. The following lemma states that this inner product is even $\mathrm{ad}(\mathfrak{t})$-invariant:

Lemma 2.21. *For all $Y, Z \in \mathfrak{g}$ we have*

$$g_p(\overline{[X_i, Y]}_p, \overline{Z}_p) + g_p(\overline{Y}_p, \overline{[X_i, Z]}_p) = 0\,.$$

Proof. Since $\overline{X_i}$ is a Killing vector field (G acts isometrically) that coincides with ξ_i at p, we obtain

$$g_p(\overline{[X_i,Y]}_p, \overline{Z}_p) + g_p(\overline{Y}_p, \overline{[X_i,Z]}_p) = -g_p([\overline{X_i}, \overline{Y}]_p, \overline{Z}_p) - g_p(\overline{Y}_p, [\overline{X_i}, \overline{Z}]_p)$$
$$= -(\overline{X_i})_p\big(g(\overline{Y}, \overline{Z})\big) = -(\xi_i)_p\big(g(\overline{Y}, \overline{Z})\big)\,.$$

Because the Levi-Civita connection ∇ is metric and torsion free and all G-fundamental fields commute with ξ_i (G acts by 3-Sasaki automorphisms) we have

$$(\xi_i)_p\big(g(\overline{Y}, \overline{Z})\big) = g_p(\nabla_{(\xi_i)_p}\overline{Y}, \overline{Z}_p) + g_p(\overline{Y}_p, \nabla_{(\xi_i)_p}\overline{Z}) = g_p(\nabla_{\overline{Y}_p}\xi_i, \overline{Z}_p) + g_p(\overline{Y}_p, \nabla_{\overline{Z}_p}\xi_i)\,.$$

Finally $\nabla\xi_i = -\varphi_i$ and $g(\cdot, \varphi_i \cdot)$ is skew-symmetric. ☐

We now move on to the complex picture and let $\mathfrak{u} := \mathfrak{g}^{\mathbb{C}}$, $\mathfrak{v} := \mathfrak{h}^{\mathbb{C}}$, $\mathfrak{c} := \mathfrak{t}^{\mathbb{C}}$ and $\alpha := 2i\alpha_1^{\mathbb{C}}|_{\mathfrak{c}}$. Let us consider the vectors $H_\alpha, X_\alpha, Y_\alpha \in \mathfrak{u}$ defined by

$$H_\alpha := \frac{1}{i}X_1\,, \quad X_\alpha := \frac{1}{2i}(X_2 - iX_3)\,, \quad Y_\alpha := \frac{1}{2i}(X_2 + iX_3)\,,$$

which satisfy the commutation relations

$$[H_\alpha, X_\alpha] = 2X_\alpha\,, \quad [H_\alpha, Y_\alpha] = -2Y_\alpha\,, \quad [X_\alpha, Y_\alpha] = H_\alpha\,.$$

Proposition 2.22. The linear form α is a root of $\mathfrak{u}$ with respect to $\mathfrak{c}$, whose root space is given by $\mathfrak{u}_\alpha = \langle X_\alpha \rangle$. Furthermore $\mathfrak{u}_{-\alpha} = \langle Y_\alpha \rangle$ and H_α is the coroot of α.

Proof. First $[H_\alpha, X_\alpha] = 2X_\alpha = \alpha(H_\alpha)X_\alpha$. Since X_2, X_3 commute with $\mathfrak{h}$, the vector X_α commutes with $\mathfrak{v}$ and in particular with $\mathfrak{s}^{\mathbb{C}}$. Likewise α_1 vanishes on $\mathfrak{h}$, so that α vanishes on $\mathfrak{v}$ and in particular on $\mathfrak{s}^{\mathbb{C}}$. ☐

Let $\Phi \subset \mathfrak{c}^*$ denote the root system of $\mathfrak{u}$ with respect to $\mathfrak{c}$. We consider the $\mathbb{Z}$-grading of $\mathfrak{u}$ introduced in the previous section, viz.

$$\mathfrak{u}^{(k)} := \bigoplus_{\substack{\beta \in \mathfrak{c}^*, \\ c_{\alpha\beta} = k}} \mathfrak{u}_\beta\,.$$

Lemma 2.23. *The* 0- *and* ± 2-*components of the grading are given by* $\mathfrak{u}^{(0)} = \mathfrak{v} \oplus \langle H_\alpha \rangle$ *and* $\mathfrak{u}^{(\pm 2)} = \mathfrak{u}_{\pm\alpha}$*, respectively.*

Proof. $\mathfrak{u}^{(0)} = \ker \operatorname{ad}_{H_\alpha} = C_{\mathfrak{u}}(H_\alpha) = \mathfrak{v} \oplus \langle H_\alpha \rangle$. Suppose there was a root $\beta \neq \alpha$ such that $c_{\alpha\beta} = 2$. Then $\langle \beta, \alpha \rangle > 0$ and $\beta - \alpha$ was a root satisfying $c_{\alpha(\beta-\alpha)} = 0$. We would need to have $[\mathfrak{u}_\alpha, \mathfrak{u}_{\beta-\alpha}] = \mathfrak{u}_\beta$, but $\mathfrak{u}_{\beta-\alpha} \subset \mathfrak{u}^{(0)} = \mathfrak{v} \oplus \langle H_\alpha \rangle$ and $[\mathfrak{u}_\alpha, \mathfrak{v}] = 0$, $[\mathfrak{u}_\alpha, H_\alpha] = \mathfrak{u}_\alpha$. ☐

Proposition 2.24. The Lie algebras $\mathfrak{g}$ and $\mathfrak{u}$ are simple.

Proof. The semisimple Lie algebra $\mathfrak{g}$ decomposes as a direct sum $\mathfrak{g} = \mathfrak{g}_{(1)} \oplus \ldots \oplus \mathfrak{g}_{(m)}$ of simple ideals. Since the Killing form of $\mathfrak{g}$ is negative definite, the same applies to the ideals $\mathfrak{g}_{(i)}$, which thus cannot be the realification of a complex Lie algebra. Therefore their complexifications $\mathfrak{u}_{(i)} := \mathfrak{g}_{(i)}^{\mathbb{C}}$ are also simple and yield a similar decomposition $\mathfrak{u} = \mathfrak{u}_{(1)} \oplus \ldots \oplus \mathfrak{u}_{(m)}$ into simple ideals [Knap, Theorem 6.94]. Accordingly the root system is a disjoint union $\Phi = \Phi_1 \sqcup \ldots \sqcup \Phi_m$. We claim that $\mathfrak{g} = \mathfrak{g}_{(i)}$ (and hence $\mathfrak{u} = \mathfrak{u}_{(i)}$), where i is the unique index such that $\alpha \in \Phi_i$.
For $j \neq i$ the ideal $\mathfrak{g}_{(j)}$ commutes with $\mathfrak{g}_{(i)} \supset (\mathfrak{u}_\alpha \oplus \mathfrak{u}_{-\alpha}) \cap \mathfrak{g} \ni X_2, X_3$, so $\mathfrak{g}_{(j)} \subset \mathfrak{h} = \mathfrak{g}_p$. Since $\mathfrak{g}_{(j)}$ is an ideal and G is connected, it follows that $\mathfrak{g}_{(j)} = \mathrm{Ad}_g(\mathfrak{g}_{(j)}) \subset \mathrm{Ad}_g(\mathfrak{g}_p) = \mathfrak{g}_{g \cdot p}$ for all $g \in G$. Because the G-action is almost effective we must have $\mathfrak{g}_{(j)} = 0$. □

It is well-known that for any root system Φ and any roots $\alpha, \beta \in \Phi$ the Cartan numbers are bounded by $|c_{\alpha\beta}| \leq 3$. Furthermore the only *irreducible* case where $|c_{\alpha\beta}| = 3$ occurs is when $\mathfrak{g} = \mathfrak{g}_2$, α is one of the short roots and β is the long root that forms an angle of 150 (210) degrees with α. We relegate the proof that this case cannot actually occur in our situation to the end of this section.

In all the remaining cases we have therefore shown that α is a maximal root (cf. Lemma 2.8), so we may carry out the construction from Section 2.2. We now prove that the 3-Sasaki structure obtained this way indeed coincides with the original one we started with. We simplify the analysis by studying the reductive complement $\mathfrak{m} := \mathfrak{h}^{\perp_B}$.

Lemma 2.25. *The reductive complement* $\mathfrak{m}$ *decomposes* B*-orthogonally as*

$$\mathfrak{m} = \langle X_1, X_2, X_3 \rangle \oplus \bigoplus_{\substack{\beta \in \Phi, \\ c_{\alpha\beta} = 1}} (\mathfrak{u}_\beta \oplus \mathfrak{u}_{-\beta}) \cap \mathfrak{g} =: \mathfrak{k} \oplus \mathfrak{g}_1 \,.$$

Proof. $\supset$: Clearly X_1, X_2, X_3 are B-orthogonal to $\mathfrak{h}$. For $\beta, \gamma \in \mathfrak{c}^*$ with $\beta + \gamma \neq 0$ the subspaces $\mathfrak{u}_\beta$ and $\mathfrak{u}_\gamma$ are $B^{\mathbb{C}}$-orthogonal. This implies that for all $\beta \in \Phi$ with $c_{\alpha\beta} = 1$ the subspaces $\mathfrak{u}_{\pm\beta}$ are also $B^{\mathbb{C}}$-orthogonal to $\mathfrak{h}$.
$\subset$: By virtue of Lemma 2.23 both sides of the equation have dimension $4n + 3$. □

We can compare the structure tensors of the two 3-Sasaki structures in question via the isomorphism $\psi : \mathfrak{m} \to T_pM$, $X \mapsto \overline{X}_p$. Proposition 2.20 has already shown that the vectors X_i correspond to the Reeb vector fields ξ_i. Looking back at Theorem 2.1 we observe that equality of the contact forms is equivalent to the following

Lemma 2.26. $\alpha_i = -B(X_i, \cdot)/4(n+2)$.

Proof. By definition $\alpha_i = B(X_i, \cdot)/B(X_i, X_i)$. We have

$$\begin{aligned} B(X_1, X_1) &= B^{\mathbb{C}}(iH_\alpha, iH_\alpha) = -B^{\mathbb{C}}(H_\alpha, H_\alpha) = -\operatorname{tr} \mathrm{ad}^2_{H_\alpha} \\ &= -4 \cdot (\dim \mathfrak{u}^{(2)} + \dim \mathfrak{u}^{(-2)}) - 1 \cdot (\dim \mathfrak{u}^{(1)} + \dim \mathfrak{u}^{(-1)}) = -4(n+2)\,. \end{aligned}$$

We also have $B(X_2, X_2) = B(X_3, X_3) = -4(n+2)$, since we could have used the same arguments for a maximal torus of e.g. the form $\mathfrak{s} \oplus \langle X_2 \rangle$. $\square$

Because the contact forms coincide, so do their differentials, which are the fundamental two-forms. Since the Riemannian metrics are determined by the fundamental two-forms together with the almost complex structures it suffices to show that the latter coincide. Let $L_i : \mathfrak{m} \to \mathfrak{m}$ denote the $\mathrm{Ad}(H)$-invariant endomorphism of $\mathfrak{m}$ corresponding to the G-invariant endomorphism field φ_i, i.e. $L_i = \psi^{-1} \circ (\varphi_i)_p \circ \psi$. Recalling Theorem 2.1 the claim reduces to showing that

$$L_i|_{\mathfrak{k}} = \frac{1}{2}\mathrm{ad}_{X_i}\,, \qquad L_i|_{\mathfrak{g}_1} = \mathrm{ad}_{X_i}\,.$$

The first equation is clear from Proposition 2.20.

Proposition 2.27. The almost complex structures of the two 3-Sasaki structures in question coincide.

Proof. We first claim that L_1 is not only $\mathrm{ad}(\mathfrak{h})$- but even $\mathrm{ad}(\mathfrak{t})$-invariant, i.e. that the endomorphisms L_1 and ad_{X_1} commute on $\mathfrak{g}_1$. For all $Y, Z \in \mathfrak{g}_1$ we have

$$\begin{aligned} 2g_p(\overline{Y}_p, \overline{L_1 Z_p}) &= d\eta_1(\overline{Y}_p, \overline{Z}_p) \\ &= d\eta_1(\overline{[X_1, Y]_p}, \overline{[X_1, Z]_p}) \\ &= 2g_p(\overline{[X_1, Y]_p}, \overline{L_1[X_1, Z]_p}) \\ &= -2g_p(\overline{Y}_p, \overline{[X_1, L_1[X_1, Z]]_p})\,. \end{aligned}$$

In the second equation we used that ad_{X_1} corresponds to an almost complex structure on $\mathfrak{g}_1$ which is compatible with the common fundamental two-form $d\eta_1$. The last equation follows from Lemma 2.21. This shows that $L_1 = -\mathrm{ad}_{X_1} \circ L_1 \circ \mathrm{ad}_{X_1}$ on $\mathfrak{g}_1$ and consequently $\mathrm{ad}_{X_1} \circ L_1 = -\mathrm{ad}^2_{X_1} \circ L_1 \circ \mathrm{ad}_{X_1} = L_1 \circ \mathrm{ad}_{X_1}$.
Let β be a root such that $c_{\alpha\beta} = 1$. Since ad_{H_α} leaves $\mathfrak{u}_\beta$ invariant, so does $\mathrm{ad}^{\mathbb{C}}_{X_1}$. Because L_1 is $\mathrm{ad}(\mathfrak{t})$-invariant it follows that $L_1^{\mathbb{C}}$ is $\mathrm{ad}(\mathfrak{c})$-invariant and thus also leaves $\mathfrak{u}_\beta$ invariant. Now $\mathrm{ad}^{\mathbb{C}}_{X_1}$ and $L_1^{\mathbb{C}}$ are $\mathbb{C}$-linear maps on the one-dimensional subspace $\mathfrak{u}_\beta$ which square to $-\mathrm{id}$, so they must be given by multiplication with $\pm i$. Since both endomorphisms commute with complex conjugation, they act on $\mathfrak{u}_{-\beta} = \overline{\mathfrak{u}_\beta}$ by multiplication with $\mp i$. Therefore L_1 and ad_{X_1} coincide on $(\mathfrak{u}_\beta \oplus \mathfrak{u}_{-\beta}) \cap \mathfrak{g}$ up to sign. We finish the proof that $L_1 = \mathrm{ad}_{X_1}$ on $\mathfrak{g}_1$ by observing that for $Y \in \mathfrak{g}_1, Y \neq 0$ Lemma 2.16 implies

$$\begin{aligned} 2g_p(\overline{[X_1, Y]_p}, \overline{L_1 Y_p}) &= d\eta_1(\overline{[X_1, Y]_p}, \overline{Y}_p) \\ &= \eta_1(\overline{[[X_1, Y], Y]_p}) = -\alpha_1([[X_1, Y], Y]) \\ &= \frac{B(X_1, [[X_1, Y], Y])}{4(n+2)} = -\frac{B([X_1, Y], [X_1, Y])}{4(n+2)} > 0\,. \end{aligned}$$

Again we can repeat the arguments for the maximal tori $\mathfrak{s} \oplus \langle X_i \rangle$, $i = 2, 3$. Even though the root spaces look differently then, the subalgebra $\mathfrak{g}_1$ is still the same because it can be defined independently of the maximal torus as the B-orthogonal complement of $\mathfrak{k}$ in $\mathfrak{m}$ by virtue of Lemma 2.25. This proves that the almost complex structures in question also coincide for $i = 2, 3$. □

Remark 2.28. In later sections instead of working with the simply connected, almost effectively acting Lie group G with Lie algebra $\mathfrak{g}$, we may sometimes turn to a non-simply connected (possibly effectively acting) group $\widetilde{G}$ with Lie algebra $\mathfrak{g}$. For $\mathfrak{g} = \mathfrak{so}(k)$ using $\widetilde{G} = SO(k)$ instead of $G = Spin(k)$ allows us to describe the corresponding coset space more explicitly via matrices. If we consider a description $\widetilde{G}/\widetilde{H}$, then the isotropy group of the G-action on $\widetilde{G}/\widetilde{H}$ is given by the connected subgroup $H \subset G$ whose Lie algebra coincides with that of $\widetilde{H}$. This follows from the fact that M is simply connected via the long exact sequence of homotopy groups. Hence $\widetilde{G}/\widetilde{H}$ and G/H are governed by the same Lie algebraic data and are therefore isomorphic homogeneous 3-Sasaki manifolds.

We finish this section by closing the gap that we previously left:

Proposition 2.29. Even in the case of a homogeneous 3-Sasaki manifold with automorphism algebra $\mathfrak{g}_2$ the root described in Section 2.3 is maximal.

For the sake of contradiction let us assume that α was one of the short roots of $\mathfrak{g}_2$. Again we consider the reductive complement $\mathfrak{m} := \mathfrak{h}^{\perp_B}$ as well as the maps $\psi : \mathfrak{m} \to T_pM$, $X \mapsto \overline{X}_p$ and $L_i := \psi^{-1} \circ (\varphi_i)_p \circ \psi : \mathfrak{m} \to \mathfrak{m}$. Using the same arguments as in the proof of Lemma 2.25 we obtain the B-orthogonal decomposition

$$\mathfrak{m} = \langle X_1, X_2, X_3 \rangle \oplus \bigoplus_{\substack{\beta \in \Phi, \\ c_{\alpha\beta} \in \{1,3\}}} (\mathfrak{u}_\beta \oplus \mathfrak{u}_{-\beta}) \cap \mathfrak{g} .$$

Under the isomorphism $\psi : \mathfrak{m} \to T_pM$ this induces a decomposition of the tangent space:

$$T_pM = \langle \xi_1, \xi_2, \xi_3 \rangle \oplus \bigoplus_{\substack{\beta \in \Phi, \\ c_{\alpha\beta} \in \{1,3\}}} V_\beta ,$$

where $V_\beta := \psi((\mathfrak{u}_\beta \oplus \mathfrak{u}_{-\beta}) \cap \mathfrak{g})$.

Lemma 2.30. *The above decomposition of T_pM is g_p-orthogonal.*

Proof. If $Y \in (\mathfrak{u}_\beta \oplus \mathfrak{u}_{-\beta}) \cap \mathfrak{g}$, then

$$g_p((\xi_i)_p, \overline{Y}_p) = \eta_i(\overline{Y}_p) = \alpha_i(Y) = B(X_i, Y)/B(X_i, X_i) = 0 ,$$

Hence each V_β is g_p-orthogonal to $\langle \xi_1, \xi_2, \xi_3 \rangle$. If β_1, β_2 are roots such that $\beta_1 \neq -\beta_2$, then there exists some $X \in \mathfrak{c}$ such that $\beta_1(X) \neq -\beta_2(X)$. We extend ψ and g_p complex (bi-)linearly, let $Y \in \mathfrak{u}_{\beta_1}$, $Z \in \mathfrak{u}_{\beta_2}$ and complexify Lemma 2.21 to obtain

$$\beta_1(X) g_p(\psi Y, \psi Z) = g_p(\psi[X,Y], \psi Z) = -g_p(\psi Y, \psi[X,Z]) = -\beta_2(X) g_p(\psi Y, \psi Z) .$$

Since $\beta_1(X) \neq -\beta_2(X)$, it follows that $\psi\mathfrak{u}_{\beta_1}$ and $\psi\mathfrak{u}_{\beta_2}$ are g_p-orthogonal. This implies that for $\beta \neq \pm\gamma$ the subspaces V_β and V_γ are g_p-orthogonal. □

Lemma 2.31. *For all $Y, Z \in \mathfrak{g}$ we have*

$$g_p(\overline{Y}_p, \overline{L_i Z}_p) = 0 \iff B(X_i, [Y,Z]) = 0 .$$

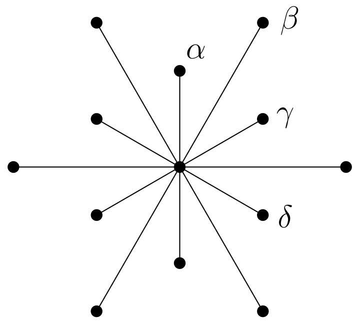
Proof. By virtue of Lemma 2.16:

$$2g_p(\overline{Y}_p, \overline{L_i Z}_p) = d\eta_i(\overline{Y}_p, \overline{Z}_p) = \eta_i([\overline{Y}, \overline{Z}]_p) = -\alpha_i([Y,Z]) = -\frac{B(X_i,[Y,Z])}{B(X_i,X_i)} .$$ □

Lemma 2.32. *For any root $\beta \in \Phi$ we have $\varphi_2 V_\beta \subset V_{\beta+\alpha} \oplus V_{\beta-\alpha}$.*

Proof. Let $\gamma \in \Phi$ such that $\gamma \neq \sigma\alpha + \tau\beta$ for all $\sigma, \tau \in \{\pm 1\}$. Then, $\sigma\beta + \tau\gamma \notin \{\pm\alpha\}$ for all $\sigma, \tau \in \{\pm 1\}$. Consequently the subspace $[\mathfrak{u}_\beta \oplus \mathfrak{u}_{-\beta}, \mathfrak{u}_\gamma \oplus \mathfrak{u}_{-\gamma}]$ is $B^{\mathbb{C}}$-orthogonal to $\mathfrak{u}_\alpha \oplus \mathfrak{u}_{-\alpha} \ni X_\alpha, Y_\alpha$ and thus also to $X_2 = i(X_\alpha + Y_\alpha)$ (see the equations above Proposition 2.22). The previous lemma now implies that $\varphi_2 V_\beta$ is g_p-orthogonal to V_γ. The claim then follows from Lemma 2.30. □

Proof of Proposition 2.29. Let us label some of the roots of $\mathfrak{g}_2$ according to the following diagram:



Lemma 2.32 implies that $\varphi_2 V_\beta \subset V_\gamma$. Since φ_2 is injective on the horizontal space, we in fact have $\varphi_2 V_\beta = V_\gamma$. Another application of Lemma 2.32 yields $\varphi_2 V_\gamma \subset V_\beta \oplus V_\delta$. If we can show that there exists some $Y \in V_\gamma$ such that $\varphi_2 Y$ has a non-trivial V_δ-component, then we arrive at a contradiction to the fact that $\varphi_2^2 = -\mathrm{id}$ on V_β.
Let Z^* denote the complex conjugate of a vector $Z \in \mathfrak{u}$. We can choose $X_\gamma \in \mathfrak{u}_\gamma$, $X_\delta \in \mathfrak{u}_\delta$ in such a way that

$$[X_\gamma, X_\delta^*] = X_\alpha = \frac{1}{2i}(X_2 - iX_3) .$$

We note that

$$X_\alpha^* = -\frac{1}{2i}(X_2 + iX_3) = -Y_\alpha\,, \quad i(X_\alpha - X_\alpha^*) = X_2\,,$$

and

$$B([X_\gamma + X_\gamma^*, i(X_\delta^* - X_\delta)], X_2) = B(i(X_\alpha - X_\alpha^*), X_2) = B(X_2, X_2) \neq 0\,.$$

Lemma 2.31 finally implies that $\varphi_2 Y$ has a V_δ-component for $Y := (\overline{X_\gamma + X_\gamma^*})_p$. $\square$

2.4 Why no Subgroup Acts Transitively

We have shown that any simply connected homogeneous 3-Sasaki manifold M is obtained from a complex 3-Sasaki datum as explained in Section 2.2. Hence M can be written as G/H, where G is a simply connected compact simple Lie group. Since G is simple, a finite quotient $\widetilde{G}$ of G acts effectively and thus forms a subgroup $\widetilde{G} \subset \mathrm{Aut}_0(M)$. We can now write $M = \widetilde{G}/\widetilde{H}$ and will show below that $\mathfrak{g} = \mathrm{Lie}(\widetilde{G}) = \mathfrak{aut}(M)$. This concludes the proof of Theorem 2.1 as M determines its 3-Sasaki datum.

Proposition 2.33. If a subgroup $\widetilde{G} \subset \mathrm{Aut}_0(M)$ acts transitively on a simply connected homogeneous 3-Sasaki manifold M, then $\widetilde{G} = \mathrm{Aut}_0(M)$.

Proof. We show that the Lie algebra $\mathfrak{g}$ of $\widetilde{G}$ is tied to purely geometric data of M. Recall the setup from Theorem 2.1: We have a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{m} = \mathfrak{h}^{\perp_B}$ and $\mathfrak{m} = \mathfrak{k} \oplus \mathfrak{g}_1$, where $\mathfrak{k}$ corresponds to the Reeb vector fields ξ_i and $\mathfrak{g}_1 = (\mathfrak{h} \oplus \mathfrak{k})^{\perp_B}$. Note that we have the commutator relations (cf. Proposition 2.12)

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h},\ [\mathfrak{h}, \mathfrak{k}] = 0,\ [\mathfrak{h}, \mathfrak{g}_1] \subset \mathfrak{g}_1,\ [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k},\ [\mathfrak{k}, \mathfrak{g}_1] \subset \mathfrak{g}_1, [\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{g}_0 = \mathfrak{h} \oplus \mathfrak{k}.$$

Consider the subspace $\mathfrak{m} + [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{g}$. Using the commutator relations we find that this is an ideal in $\mathfrak{g}$ and thus (Proposition 2.24) already $\mathfrak{g}$ itself. Hence the knowledge of $\mathfrak{m}$ embedded in the Lie algebra of Killing vector fields $\mathfrak{iso}(M)$ on M via $X \mapsto \overline{X}$ alone determines $\mathfrak{g} \subset \mathfrak{iso}(M)$.

We now characterize $\mathfrak{m}$ as the subset of Killing fields whose covariant derivatives obey a certain behavior at $o = e\widetilde{H}$. By analogy recall that in a symmetric space the analogue of $\mathfrak{m}$ can be characterized as the Killing fields whose covariant derivatives vanish at o. Let ∇ be the Levi-Civita connection on M and $\alpha : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ the associated Nomizu operator defined by

$$\overline{\alpha(X, Y)}_o = \nabla_{\overline{X}_o} \overline{Y} - \overline{[X, Y]}_o.$$

It satisfies

$$\alpha(X,Y)=\begin{cases}0\,, & X\in\mathfrak{k}\text{ and }Y\in\mathfrak{g}_1\,,\\ \frac{1}{2}[X,Y]_{\mathfrak{m}}\,, & X,Y\in\mathfrak{k}\text{ or }X,Y\in\mathfrak{g}_1\,,\\ [X,Y]_{\mathfrak{m}}\,, & X\in\mathfrak{g}_1\text{ and }Y\in\mathfrak{k},\end{cases}$$

see [DOP, Theorem 4.2]. By definition of the Nomizu operator we have

$$\nabla_{\overline{X}_o}\overline{Y}=\overline{\alpha(X,Y)}_o+\overline{[X,Y]}_o\,.$$

This means that

$$\nabla_{\overline{X}_o}\overline{Y}=\begin{cases}\frac{3}{2}\overline{[X,Y]}_o=-3\sum\eta_i(\overline{Y}_o)\varphi_i\overline{X}_o\,, & X\in\mathfrak{k}\,,\\ 2\overline{[X,Y]}_o=-2\sum\eta_i(\overline{Y}_o)\varphi_i\overline{X}_o\,, & X\in\mathfrak{g}_1\,,\end{cases}$$

for $Y\in\mathfrak{k}$ and

$$\nabla_{\overline{X}_o}\overline{Y}=\begin{cases}\overline{[X,Y]}_o=\sum_{i=1}^3\eta_i(\overline{X}_o)\varphi_i(\overline{Y}_o)\,, & X\in\mathfrak{k}\,,\\ \frac{3}{2}\overline{([X,Y]_{\mathfrak{k}})}_o=-\frac{3}{2}\sum_{i=1}^3 d\eta_i(\overline{X}_o,\overline{Y}_o)\xi_i\,, & X\in\mathfrak{g}_1\,,\end{cases}$$

for $Y\in\mathfrak{g}_1$, where we used Lemma 2.16 in the last equation. Hence the fundamental vector field of $Y\in\mathfrak{m}=\mathfrak{k}\oplus\mathfrak{g}_1$ satisfies

$$\begin{aligned}\nabla_v\overline{Y}=&-3\sum_{i,j=1}^3\eta_i(\overline{Y}_o)\eta_j(v)\varphi_i\xi_j-2\sum_{i=1}^3\eta_i(\overline{Y}_o)\varphi_i(v_{\mathcal{H}})\\&+\sum_{i=1}^3\eta_i(v)\varphi_i(\overline{Y}_o)_{\mathcal{H}}-\frac{3}{2}\sum_{i=1}^3 d\eta_i(v,\overline{Y}_o)\xi_i\end{aligned}\tag{2.2}$$

for all $v\in T_oM$, where $v_{\mathcal{H}}$ denotes the projection of v to $\mathcal{H}=\bigcap\ker\eta_i$. Note that $(\nabla\overline{Y})_o$ depends only on the value $\overline{Y}_o\in T_oM$. We now consider the maps

$$\mathfrak{m}\to\{Y\text{ Killing field on }M\mid Y\text{ satisfies (2.2) for all }v\in T_oM\}\to T_oM\,,$$

where the first map is $Y\mapsto\overline{Y}$ and the second is evaluation at o. The first map is injective beacuse the $\widetilde{G}$-action is effective. The evaluation map is also injective, as for Killing fields Y_1,Y_2 in the middle space with $(Y_1)_o=(Y_2)_o$ by Property (2.2) also $(\nabla Y_1)_o=(\nabla Y_2)_o$, which implies $Y_1=Y_2$. Because $\mathfrak{m}\cong T_oM$ both maps are isomorphisms. Hence:

$$\mathfrak{m}=\{Y\text{ Killing field on }M\mid Y\text{ satisfies (2.2) for all }v\in T_oM\}\subset\mathfrak{iso}(M).$$

Therefore we have shown that every connected Lie group $\widetilde{G}$ with Lie algebra $\mathfrak{g}$ acting effectively and transitively on M has the same Lie algebra, namely $\mathfrak{g}=\mathfrak{aut}(M)$. The corresponding connected subgroup of $\mathrm{Aut}(M)$ is then $\widetilde{G}=\mathrm{Aut}_0(M)$. □

2.5 Determining the Isotropy

After proving Theorem 2.1 we now derive the precise list given in Corollary 2.2. By virtue of Theorems 2.9 and 2.15 any simply connected homogeneous 3-Sasaki manifold can be written in the form G/H, where G is a simply connected simple Lie group and $H = (C_G(K))_0$, where $K \subset G$ is the connected subgroup with Lie algebra $\mathfrak{k} \cong \mathfrak{sp}(1)$ determined by a maximal root. In this section we determine the isotropy group H, thereby proving Corollary 2.2 in the simply connected case. The classical spaces are dealt with in the following

Proposition 2.34. For $G = Sp(n+1), SU(m)$ and $\widetilde{G} = SO(k)$ the isotropy groups are given by $H = Sp(n), S(U(m-2) \times U(1))$ and $\widetilde{H} = SO(k-4) \times Sp(1)$, respectively.

Proof. We use the explicit description of the root systems of the compact groups provided in [Tapp, Chapter 11].
$G = Sp(n+1)$: We may choose the maximal root α such that $\mathfrak{k} = \{\mathrm{diag}(0_n, \mathfrak{sp}(1))\}$ (by letting $\alpha = \gamma_{n+1}$ in Tapp's notation). Accordingly:

$$\begin{aligned} K &= \{\mathrm{diag}(I_n, Sp(1))\}\,, \\ C_G(K) &= \{\mathrm{diag}(Sp(n), \pm 1)\}\,, \\ H &= (C_G(K))_0 = \{\mathrm{diag}(Sp(n), 1)\}\,. \end{aligned}$$

$G = SU(m)$: We may choose α such that $\mathfrak{k} = \{\mathrm{diag}(0_{m-2}, \mathfrak{su}(2))\}$ (by letting $\alpha = \alpha_{m-1,m}$ in Tapp's notation). Accordingly:

$$\begin{aligned} K &= \{\mathrm{diag}(I_{m-2}, SU(2))\}\,, \\ C_G(K) = H &= \{\mathrm{diag}(SU(m-2), zI_2) \mid z \in U(1)\} \cap SU(m)\,. \end{aligned}$$

$\widetilde{G} = SO(k)$: We recall that there are two embeddings $Sp(1)^+$, $Sp(1)^- \subset SO(4)$, depending on whether $Sp(1)$ is viewed as acting on $\mathbb{H} \cong \mathbb{R}^4$ by multiplication from the left or right, respectively. We may choose α such that $\mathfrak{k} = \{\mathrm{diag}(0_{k-4}, \mathfrak{sp}^-(1))\}$ (by letting $\alpha = \alpha_{[k/2]-1,[k/2]}$ in Tapp's notation). Accordingly:

$$\begin{aligned} \widetilde{K} &= \{\mathrm{diag}(I_{k-4}, Sp^-(1))\}\,, \\ C_{\widetilde{G}}(\widetilde{K}) = \widetilde{H} &= \{\mathrm{diag}(SO(k-4), Sp^+(1))\}\,. \end{aligned}$$

□

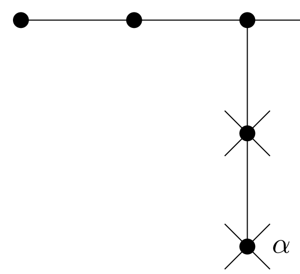
We now present a different method based on Borel-de Siebenthal theory which allows us to understand the isotopy algebra $\mathfrak{h}$ in the exceptional cases:
Using the same notation as before we let $\mathfrak{s} \subset \mathfrak{h}$ be a maximal Abelian subalgebra and consider the maximal Abelian subalgebra $\mathfrak{t} := \mathfrak{s} \oplus \langle X_1 \rangle$ of $\mathfrak{g}$. Let α denote the maximal

root that vanishes on $\mathfrak{s}^{\mathbb{C}}$. We fix a set of positive roots of $\mathfrak{g}$ using a slight perturbation of the hyperplane perpendicular to the maximal root α. By intersecting this hyperplane with $\mathfrak{s}$ we also obtain a notion of positive root for $\mathfrak{h}$. By definition any root of $\mathfrak{h}$ becomes a root of $\mathfrak{g}$ by extending it by zero on X_1, since $\mathfrak{h}$ commutes with X_1.

Proposition 2.35. The simple $\mathfrak{h}$-roots are precisely those simple $\mathfrak{g}$-roots which are perpendicular to α.

Proof. By our notions of positivity any $\mathfrak{h}$-simple root is also $\mathfrak{g}$-simple: If an $\mathfrak{h}$-root is the sum of two positive $\mathfrak{g}$-roots, then both of them have to lie in the hyperplane perpendicular to α. Conversely recall that by Proposition 2.12 the roots of $\mathfrak{h}$ are exactly those roots β perpendicular to the maximal root α. □

We can thus determine the isotropy type of H by deleting the nodes in the Dynkin diagram of G corresponding to simple roots that are not perpendicular to α. For each simple G these were determined by Borel and de Siebenthal in [BdS]: In the table on p. 219 they draw the Dynkin diagrams for every simple $\mathfrak{g}$, extended by the lowest root (which they denote by P). In order to find the isomorphism type of H one therefore only needs to erase this lowest root as well as all roots connected to it. As an example consider the Dynkin diagram of E_6:



Deleting α as well as the unique simple root connected to it results in the Dynkin diagram of $SU(6)$ and the homogeneous 3-Sasaki manifold corresponding to E_6 is $E_6/SU(6)$.

Remark 2.36. If one removes only the nodes in the Dynkin diagram of G that are connected to α but not α itself, then one obtains the Dynkin diagram of the normalizer $N_G(K)$ which then yields the Wolf space $G/N_G(K)$. Note that by the list in [BdS] in all cases except $G = SU(n)$ the maximal root α is connected to only one other node which means that in these cases the groups H and $N_G(K)$ are semisimple, whereas in case $G = SU(n)$ the groups H and $N_G(K)$ have one-dimensional center. Furthermore in the cases except $SU(n)$ the normalizer $N_G(K)$ is a maximal subgroup of maximal rank. The types of such groups are exactly those that were classified by Borel and de Siebenthal in [BdS]: Given a compact simple Lie group G one adds the lowest root to the Dynkin diagram and removes one other simple root from it.

Going through the list in [BdS] one obtains the Lie algebras of the isotropy groups of the homogeneous spaces occurring in Corollary 2.2. As we determined the isotropy groups in the classical cases above, in order to finish the proof of this corollary in the simply connected case, we only need to argue that in the exceptional cases the isotropy groups are simply connected. Ishitoya and Toda showed in [IT, Corollary 2.2] that in the cases $G = G_2, F_4, E_6, E_7, E_8$, we have $\pi_2(G/N_G(K)) = \mathbb{Z}_2$, which is, since G is simply connected, equivalent to $\pi_1(N_G(K)) = \mathbb{Z}_2$. (See also [BdS, Remarque II, p. 220] for how to compute the fundamental group of a maximal subgroup of G of maximal rank.) Moreover by [IT, Theorem 2.1] the normalizer $N_G(K)$ is of the form $N_G(K) = (H \times Sp(1))/\mathbb{Z}_2$ which finally implies that H is simply connected.

2.6 Non-Simply Connected 3-Sasaki Manifolds

Having treated the simply connected case of Corollary 2.2 our goal is now to prove the following

Theorem 2.37. *The only homogeneous 3-Sasaki manifolds which are not simply connected are the real projective spaces* $\mathbb{R}P^{4n+3}$.

Let $M = G/\overline{H}$ be a homogeneous 3-Sasaki manifold (not necessarily simply connected), where G is a simply connected compact Lie group and $\overline{H}$ is possibly disconnected. The universal cover of $G/\overline{H}$ is given by G/H, where H denotes the identity component of $\overline{H}$, and the homogeneous 3-Sasaki structure lifts to the simply connected space G/H. As shown in Section 2.3 the automorphism group G has to be simple. The vectors $X_i \in \mathfrak{g}$ from Section 2.3 span a subalgebra $\mathfrak{k} := \langle X_1, X_2, X_3 \rangle \cong \mathfrak{sp}(1)$ and we let $K \subset G$ denote the corresponding connected subgroup.
Since H is the identity component of $\overline{H}$, we have $\overline{H} \subset N_G(H)$. Furthermore the 3-Sasaki structure descends from G/H to $G/\overline{H}$, so $\overline{H} \subset C_G(K)$. Conversely any subgroup $\overline{H} \subset N_G(H) \cap C_G(K)$ containing H allows us to define a 3-Sasaki structure on $G/\overline{H}$. In summary the non-simply connected quotients of a given simply connected homogeneous 3-Sasaki manifold G/H are classified by the subgroups of the group

$$\big(N_G(H) \cap C_G(K)\big)/H\,.$$

Hence it suffices to show that this quotient is $\mathbb{Z}_2$ for $G = Sp(n+1)$ and trivial otherwise.

Lemma 2.38. *The numerator* $N_G(H) \cap C_G(K)$ *is the subgroup generated by* $H \cup Z(K)$.

Proof. Clearly $H \cup Z(K) \subset N_G(H) \cap C_G(K)$. The vector $X_1 \in \mathfrak{g}$ is the infinitesimal generator of a circle subgroup $S_1 \subset K$. Since $C_{\mathfrak{g}}(X_1) = \mathfrak{h} \oplus \langle X_1 \rangle$ and the centralizer of any torus (not necessarily maximal) in a compact connected Lie group is always connected, it follows that $C_G(S_1)$ is the subgroup generated by $H \cup S_1$. Consequently any $g \in C_G(K) \subset C_G(S_1)$ can be represented as $g = hg_1$ for some $h \in H$, $g_1 \in S_1$. Since $H \subset C_G(K)$, we have $g_1 = h^{-1}g \in C_G(K) \cap K = Z(K)$. □

Proposition 2.39. The quotient $(N_G(H) \cap C_G(K))/H$ is $\mathbb{Z}_2$ for $G = Sp(n+1)$ and trivial otherwise.

Proof. By the previous lemma, it suffices to check if $Z(K)$ is contained in H.
$G = Sp(n{+}1)$: We have already seen in Section 2.5 that the center $Z(K) = \{\mathrm{diag}(I_n, \pm 1)\}$ is **not** contained in $H = \{\mathrm{diag}(Sp(n), 1)\}$.
$G = SU(m)$: We have also shown that in this case $Z(K) = \{\mathrm{diag}(I_{m-2}, \pm I_2)\}$ is contained in $H = S(U(m-2) \times U(1))$.
$G = Spin(k)$: We have seen that for $\widetilde{G} = SO(k)$ the center $Z(K) = \{\mathrm{diag}(I_{k-4}, \pm I_4)\}$ is contained in $\widetilde{H} = SO(k-4) \times Sp(1)^+$. We now transfer this statement to $G = Spin(k)$: Denote the universal covering map by $\pi : Spin(k) \to SO(k)$. First we observe that the connected subgroup of $Spin(r)$ that maps onto a block-diagonally embedded $SO(r-1)$ is $Spin(r-1)$ for $r \geq 4$: This is because $S^{r-1} = SO(r)/SO(r-1)$ is 2-connected for $r \geq 4$ and hence equal to $Spin(r)/Spin(r-1)$ by the long exact sequence in homotopy. Thus for $k \geq 7$ the subgroups $SO(k-4)$ and $SO(4)$ lift to $Spin(k-4)$ and $Spin(4)$, respectively. As π is a $2:1$-covering the group covering $SO(k-4) \times SO(4)$ is $Spin(k-4) \times_{\mathbb{Z}_2} Spin(4)$, where the $\mathbb{Z}_2$-quotient means that the nontrivial elements in the kernels of the respective projections are identified. Thus $Spin(k-4) \times_{\mathbb{Z}_2} Spin(4) \cong Spin(k-4) \times_{\mathbb{Z}_2} (Sp(1)^+ \times Sp(1)^-)$. This implies that the center of $Sp(1)^-$ is contained in $H = Spin(k-4) \times Sp(1)^+$.
$G = G_2, F_4, E_6, E_7, E_8$: Ishitoya and Toda showed that the subgroup U of the corresponding symmetric base space G/U has to be of the form $U = (H \times Sp(1))/\mathbb{Z}_2$ and that the center $Z(Sp(1))$ is contained in H [IT, Theorem 2.1]. □

2.7 Positive QK Manifolds

We end this chapter by showing that the classification of homogeneous positive qK manifolds in Corollary 2.3, which had originally been the stepping stone for the classification of homogeneous 3-Sasaki manifolds, can in turn be derived from our results.
Let B be a positive qK manifold. We recall that qK manifolds may be characterized by a subbundle $Q \subset \mathrm{End}\,TB$ of the endomorphism bundle which admits a local frame satisfying the multiplication rules of the quaternions. In her 1975 article [Koni] Konishi showed that the $SO(3)$-principal fibre bundle $P \to B$ of oriented orthonormal frames of Q admits a 3-Sasaki structure. This construction is known as the *Konishi bundle* over B and constitutes the inverse of the fibration over the space of leaves of the characteristic foliation. Another natural and interesting bundle over B is the unit sphere bundle $Z := S(Q)$ in Q known as the *twistor fibration*. Its total space Z is both a complex contact manifold and a Fano variety [BG, Chapters 12, 13].
A *qK automorphism of* B is an isometry $\phi : B \to B$ such that conjugation with its

differential $d\phi$ leaves the bundle Q invariant. We call B a *homogeneous qK manifold* if there is a Lie group G acting transitively on B by qK automorphisms. We will first prove the following

Proposition 2.40. The Konishi bundle over a simply connected homogeneous positive qK manifold is a homogeneous 3-Sasaki manifold.

Let B be a simply connected homogeneous positive qK manifold, so that we may write $B = G/U$, where G is simply connected and U is connected. Because G acts on B by qK automorphisms, the G-action lifts to the Konishi bundle P. In particular the isotropy group U acts on the fiber F of P over the identity coset $eU \in B$.
Choose and fix a frame $p \in F$. This allows us to identify F with $SO(3)$ via the orbit bijection $\theta_p : SO(3) \to F$, $g \mapsto p \cdot g$. The $SO(3)$-left action on itself by left multiplication now induces a left action on F (which depends on the choice of p), viz. $g \cdot_p q := \theta_p(g\theta_p^{-1}(q))$. Since the U-action commutes with the $SO(3)$-right action on F, there exists a homomorphism $\rho : U \to V$ onto a subgroup $V \subset SO(3)$ (again all depending on p) such that $u \cdot q = \rho(u) \cdot_p q$ for all $q \in F$, namely $\rho(u) := \theta_p^{-1}(u \cdot p)$. Clearly $d := \dim V \in \{0, 1, 3\}$.

Lemma 2.41. $d = 3$.

Proof. Let us first assume that $d = 0$. Then the connected group U would act trivally on F. Hence we would obtain a well-defined global section $B \to P$, $gU \mapsto g \cdot p$, meaning the principal fiber bundle P was trivial. But the first Pontryagin class $p_1(P)$ of P is (up to a factor) given by the class of the fundamental four-form $\Omega \in \Omega^4(B)$ of B and is therefore non-trivial [Bess, Proposition 14.92].
If we suppose that $d = 1$, then V is a connected one-dimensional subgroup of $SO(3)$ and is thus comprised of rotations around a fixed axis $L \subset \mathbb{R}^3$. Choose a point $x \in L \cap S^2$. We view the frame $p \in F$ as a linear isometry $\mathbb{R}^3 \to Q$ and consider the mapping $B \to Z$, $gU \mapsto (g \cdot p)(x)$. This map is well-defined because

$$(u \cdot p)(x) = \big(p \cdot \theta_p^{-1}(u \cdot p)\big)(x) = \big(p \cdot \rho(u)\big)(x) = p\big(\rho(u)(x)\big) = p(x) \quad \forall\, u \in U\,.$$

We would thus obtain a global section of the twistor fibration which is impossible on compact positive qK manifolds [AMP, Theorem 3.8]. □

Proof of Proposition 2.40. From the previous lemma we know that U acts transitively on F and consequently G acts transitively on P. This action preserves the 3-Sasaki structure, since the Reeb vector fields are (by construction of the Konishi bundle) the infinitesimal generators of the $SO(3)$-action which commutes with G. □

Proof of Corollary 2.3. By Proposition 2.40 the Konishi bundle P over a simply connected homogeneous positive qK manifold B is a homogeneous 3-Sasaki manifold, i.e. one of the manifolds listed in Corollary 2.2. By dividing P by the action of the group $K \subset G$

from the previous sections we obtain the list in Corollary 2.3. The statement about the Riemannian metric and quaternionic structure follows from the fact that the Konishi bundle is a Riemannian fibration.

Let us now assume that $\overline{B} = G/\overline{U}$ was a non-simply connected homogeneous positive qK manifold, where $\overline{U}$ is disconnected. Then $\overline{B}$ is finitely covered by $B = G/U$, where U denotes the identity component of $\overline{U}$. The qK structure lifts from $\overline{B}$ to B and the Konishi bundles $P, \overline{P}$ over $B, \overline{B}$ form a diagram

$$\begin{array}{ccc} P & \dashrightarrow & \overline{P} \\ \downarrow & & \downarrow \\ B & \longrightarrow & \overline{B} \end{array}$$

This implies the existence of the dashed equivariant map $P \to \overline{P}$, so that $\overline{P}$ is a non-simply connected homogeneous 3-Sasaki manifold. By Theorem 2.37 the manifold $\overline{P}$ can only be $\mathbb{R}P^{4n+3}$ which leads to the same quotient $\overline{B} = Sp(n+1)/(Sp(n) \times Sp(1))$ as $P = S^{4n+3}$. □

Chapter 3

Homogeneous and Inhomogeneous 3-(α, δ)-Sasaki Manifolds

In this chapter we reapply some of the techniques from the previous one to a different geometry, namely degenerate 3-(α, δ)-Sasaki manifolds. This allows us to prove several results which limit the number of homogeneous manifolds with this geometry under certain assumptions. More precisely we show that no non-trivial compact examples exist as well as that there is exactly one family of nilpotent Lie groups with an invariant degenerate 3-(α, δ)-Sasaki structure, namely the quaternionic Heisenberg groups.
By way of contrast we demonstrate how to use hyperkähler manifolds to construct degenerate 3-(α, δ)-Sasaki manifolds which are then necessarily inhomogeneous. This method is akin to the famous Boothby-Wang bundle and may be seen as a counterpart to obtaining 3-Sasaki manifolds via the Konishi bundle over positive qK spaces.

Agricola, Dileo and Stecker have shown that all 3-(α, δ)-Sasaki manifolds locally submerge onto a qK space whose curvature depends on the sign of $\alpha\delta$ [ADS, Theorem 2.2.1]. This result motivated them to construct plentiful examples of homogeneous non-degenerate 3-(α, δ)-Sasaki manifolds over quaternionic Kähler spaces of non-vanishing curvature [ADS, Section 3]. To some extent this property also explains why it could be expected that there are rather few homogeneous degenerate 3-(α, δ)-Sasaki manifolds. This is because ideally such spaces would submerge onto homogeneous hyperkähler manifolds which are known to be completely flat, i.e. products of tori and Euclidean spaces [AK]. However, compared to the non-degenerate case it is less clear if the corresponding hyperkähler space will globally be a manifold. We identify two such scenarios where the base space is indeed a manifold and show that there are even more restrictions beyond those imposed by [AK].
About the structure of this chapter: Unlike in the non-degenerate case the Reeb vector fields are now elements of the automorphism algebra of a homogeneous degenerate 3-(α, δ)-Sasaki manifold because they commute with one another. In Section 3.1 we show that they comprise the center of the automorphism algebra and conclude that they

generate a *closed* subgroup of the automorphism group. Combining this with classical results due to ALEKSEEVSKII [Alek] as well as CONNER and RAYMOND [CR] we prove that the only connected *compact* homogeneous degenerate 3-(α,δ)-Sasaki manifold is the trivial example of the three-torus T^3 (Section 3.2). In Section 3.3 we study simply connected nilpotent Lie groups with a left-invariant degenerate 3-(α,δ)-Sasaki structure. Again using traditional results by WILSON [Wil] as well as MILNOR [Mil] we show that there is exactly one family of such spaces, the quaternionic Heisenberg groups. This parallels a result about nilpotent Sasaki Lie groups by ANDRADA, FINO and VEZZONI [AFV, Theorem 3.9]. Finally we provide the promised construction of degenerate 3-(α,δ)-Sasaki manifolds over hyperkähler manifolds in Section 3.4.

3.1 Homogeneous Spaces

Let $(M^{4n+3},g,\xi_i,\eta_i,\varphi_i)_{i=1,2,3}$ be a connected homogeneous degenerate 3-(α,δ)-Sasaki manifold, i.e. the identity component $G:=\mathrm{Aut}_0(M)$ of its group $\mathrm{Aut}(M)$ of 3-(α,δ)-Sasaki automorphisms acts transitively. The Lie algebra $\mathfrak{g}$ is (anti-)isomorphic to the space $\mathfrak{aut}(M)$ of all complete Killing vector fields on M which commute with the Reeb vector fields ξ_1,ξ_2,ξ_3 via the map $\mathfrak{g}\to\mathfrak{aut}(M)$, $X\mapsto\overline{X}$, where $\overline{X}$ denotes the fundamental vector field of the left G-action.

Since the automorphism group acts transitively, the Reeb vector fields are complete. Unlike in the 3-Sasaki and more generally the non-degenerate 3-(α,δ)-Sasaki case the Reeb vector fields commute with each other, so we have $\xi_1,\xi_2,\xi_3\in\mathfrak{aut}(M)$. In fact the corresponding elements $X_1,X_2,X_3\in\mathfrak{g}$ (i.e. $\overline{X_i}=\xi_i$) clearly even lie in the center $Z(\mathfrak{g})$ of $\mathfrak{g}$. We show that $\mathfrak{g}$ contains no other central elements using techniques very similar to those from Section 2.3: The same proof as for Lemma 2.16 yields

Lemma 3.1.

$$d\eta_i(\overline{X},\overline{Y})=\eta_i([\overline{X},\overline{Y}])\,,\qquad i=1,2,3,\quad X,Y\in\mathfrak{g}\,.$$

Lemma 3.2. *A vector field of the form* $\sum_{i=1}^3 f_i\,\xi_i$*, where* $f_1,f_2,f_3\in C^\infty(M)$*, lies in* $\mathfrak{aut}(M)$ *if and only if* f_1,f_2,f_3 *are constant functions.*

Proof. If $V=\sum_{i=1}^3 f_i\,\xi_i\in\mathfrak{aut}(M)$, then for $j=1,2,3$ and $W\in\Gamma(TM)$:

$$0=(\mathcal{L}_V\eta_j)(W)=\sum_{i=1}^3\Big(f_i\underbrace{(\mathcal{L}_{\xi_i}\eta_j)}_{=0}(W)+W(f_i)\underbrace{\eta_j(\xi_i)}_{=\delta_{ij}}\Big)=W(f_j)\,.$$

Since M is connected, it follows that f_1,f_2,f_3 have to be constant. ☐

Proposition 3.3. $Z(\mathfrak{g}) = \langle X_1, X_2, X_3 \rangle =: \mathfrak{k}$.

Proof. Let $X \in Z(\mathfrak{g})$ and choose some $p \in M$ and $Y \in \mathfrak{g}$ such that $\overline{Y}_p = \varphi_1 \overline{X}_p$ (which is possible, since G acts transitively). Then by virtue of Lemma 3.1:

$$0 = \eta_1(\overline{[X,Y]}_p) = -d\eta_1(\overline{X}_p, \varphi_1 \overline{X}_p) = 2\alpha \left\| (\overline{X}_p)_{\mathcal{H}} \right\|^2 ,$$

where $\mathcal{H} := \bigcap_i \ker \eta_i$ and $v_{\mathcal{H}}$ denotes the horizontal component of a tangent vector $v \in TM$. Since $\alpha \neq 0$ and $p \in M$ was arbitrary, the vector field $\overline{X}$ has to be of the form described in Lemma 3.2. Consequently $\overline{X} = \sum_{i=1}^3 a_i \xi_i$ for some $a_1, a_2, a_3 \in \mathbb{R}$ and $X = \sum_{i=1}^3 a_i X_i \in \mathfrak{k}$. ☐

Corollary 3.4. *The connected subgroup $K \subset G$ with Lie algebra $\mathfrak{k}$ is closed.*

Proof. Since G is connected, the Lie algebra of $Z(G)$ is given by $Z(\mathfrak{g})$. Thus Proposition 3.3 implies that $K = Z_0(G)$, the identity component of the center. Since $Z(G)$ is closed in G and $Z_0(G)$ is closed in $Z(G)$, it follows that K is closed in G. ☐

3.2 Compact Homogeneous Spaces

We start with the following observation:

Lemma 3.5. *The only connected compact three-dimensional degenerate 3-(α, δ)-Sasaki manifold is T^3.*

Proof. If $(M^3, g, \xi_i, \eta_i, \varphi_i)_{i=1,2,3}$ is a connected compact three-dimensional degenerate 3-(α, δ)-Sasaki manifold, then the Reeb vector fields generate an $\mathbb{R}^3$-action on M. The isotropy groups of this action are discrete (since the Reeb vector fields vanish nowhere), so any orbit is three-dimensional and thus by connectedness equal to all of M. Hence $M = \mathbb{R}^3/H$ is a connected compact Abelian Lie group (H is a normal subgroup, since $\mathbb{R}^3$ is Abelian), i.e. a torus. The structure tensors $(g, \xi_i, \eta_i, \varphi_i)_{i=1,2,3}$ are completely determined by the definition of an almost 3-contact metric structure. ☐

Let us now assume that $(M^{4n+3}, g, \xi_i, \eta_i, \varphi_i)_{i=1,2,3}$ is a connected compact homogeneous degenerate 3-(α, δ)-Sasaki manifold. Then its isometry group $\mathrm{Iso}(M)$ as well as the closed subgroups $\mathrm{Aut}(M)$ and $\mathrm{Aut}_0(M) =: G$ are also compact, as is K by Corollary 3.4. Thus the space M/K of K-orbits is at least an orbifold. However, since M is G-homogeneous, the K-action on M only has one isotropy type. Hence the orbit space M/K is a compact smooth manifold, in fact:

Proposition 3.6. M/K is a torus.

Proof. Agricola, Dileo and Stecker have shown that the space of leaves of the characteristic foliation of a degenerate 3-(α, δ)-Sasaki manifold (the foliation generated by the Reeb vector fields) locally admits a hyperkähler structure [ADS, Theorem 2.2.1]. Concretely the hyperkähler forms are induced by $d\eta_i$ which are basic by (1.6). Since X_1, X_2, X_3 are the infinitesimal generators of both K and the characteristic foliation, the space of leaves is given by the orbit space M/K. By construction the hyperkähler structure on M/K is also G-homogeneous. A result by Alekseevskii then concludes that M/K has to be a torus [Alek, Theorem 1 b)]. □

Theorem 3.7. *The only connected compact homogeneous degenerate 3-(α, δ)-Sasaki manifold is T^3.*

Proof. Let L be the kernel of the G-action on the hyperkähler quotient space M/K. Then G/L is a connected compact Lie group which acts effectively on a closed aspherical manifold (i.e. a manifold whose universal cover is contractible). It follows from the work of Conner and Raymond that G/L is also a torus [CR, Theorem 5.6].
Since L is a normal subgroup of G, its Lie algebra $\mathfrak{l}$ is an ideal in $\mathfrak{g}$. Because G is compact we may choose an $\mathrm{Ad}(G)$-invariant (and thus also $\mathrm{ad}(\mathfrak{g})$-invariant) inner product on $\mathfrak{g}$ and consider the complementary ideal $\mathfrak{m} := \mathfrak{l}^\perp$. Again by compactness of G we have $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{k}$ and $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{k}^\perp$ (independently of the choice of the inner product). Since $K \subset L$, it follows that $\mathfrak{m}$ is an ideal inside the semisimple algebra $[\mathfrak{g}, \mathfrak{g}]$ and is therefore itself semisimple. But on the other hand $\mathfrak{m}$ is isomorphic to the Lie algebra $\mathfrak{g}/\mathfrak{l}$ of G/L which is Abelian. Consequently $\mathfrak{m} = \{0\}$.
It follows that G/L has to be the trivial torus, meaning G acts trivially on M/K. Since this action is also transitive, we obtain that M/K is a singleton and M is three-dimensional. Lemma 3.5 finally implies that M is isomorphic to T^3. □

3.3 Nilpotent Lie Groups

In this section we investigate simply connected nilpotent Lie groups with a left-invariant degenerate 3-(α, δ)-Sasaki structure. We recall the specific example of the quaternionic Heisenberg groups, which were first described as naturally reductive spaces in [AFS] and later identified as a homogeneous degenerate 3-(α, δ)-Sasaki manifolds in [AD]:

Example 3.8. The *quaternionic Heisenberg group* of dimension $4n + 3$ is the simply connected Lie group H determined by the following Lie algebra $\mathfrak{h}$: As a set $\mathfrak{h} = Z \oplus \mathbb{H}^n$, where Z is the span of the imaginary quaternions i, j, k which we denote by ξ_1, ξ_2, ξ_3. We define an inner product g on $\mathfrak{h}$ by requiring that ξ_1, ξ_2, ξ_3 are orthonormal, $g|_{\mathbb{H}^n \times \mathbb{H}^n}$ is the standard Euclidean inner product on $\mathbb{H}^n$ and $g|_{Z \times \mathbb{H}^n} = 0$. Let η_i be the g-dual one-form of ξ_i and φ_i the endomorphism of $\mathfrak{h}$ such that $\varphi_i \xi_j = \xi_k$ for any cyclic permutation (i, j, k) of $(1, 2, 3)$ and $\varphi_i|_{\mathbb{H}^n}$ is quaternionic multiplication from the left by ξ_i. Finally the Lie bracket on $\mathfrak{h}$ is determined by the conditions $Z(\mathfrak{h}) = Z$, $[\mathbb{H}^n, \mathbb{H}^n] \subset Z$ (implying that $\mathfrak{h}$ is two-step nilpotent) and

$$g([X,Y],\xi_i) = g(\varphi_i X, Y)\,, \quad X, Y \in \mathbb{H}^n\,. \tag{3.1}$$

As shown in [AFS, Subsection 2.2] and [AD, Example 2.3.2] the corresponding left-invariant tensor fields $(g, \xi_i, \eta_i, \varphi_i)_{i=1,2,3}$ constitute a homogeneous degenerate 3-(α,δ)-Sasaki structure on H with $\alpha = 1/2$. As always one can modify this structure (and in particular the value of α) by applying an $\mathcal{H}$-homothetic deformation [AD, Definition 2.3.1]. In fact the presentation of the quaternionic Heisenberg group in [AFS] and [AD] already incorporates an $\mathcal{H}$-homothetic deformation with parameters $a = 1$, $b = \lambda^2 - 1$ and $c = \lambda$, where $\lambda > 0$.

Remark 3.9. In fact the quaternionic Heisenberg groups are the most elementary Lie groups with a degenerate 3-(α,δ)-Sasaki structure that one could imagine: The almost 3-contact metric structure is derived from the standard one on the quaternions $\mathbb{H}^n$, the Lie bracket is defined to be zero whenever possible and Equation (3.1) is just a restatement of the defining Condition (1.6). Maybe somewhat surprisingly we will show below that these are in fact the only simply connected nilpotent Lie group with a left-invariant degenerate 3-(α,δ)-Sasaki structure.

Now let N^{4n+3} denote a simply connected nilpotent Lie group with a left-invariant degenerate 3-(α,δ)-Sasaki structure $(g, \xi_i, \eta_i, \varphi_i)_{i=1,2,3}$. If $\mathcal{H} := \bigcap_i \ker \eta_i$ and $\mathcal{V} := \langle \xi_1, \xi_2, \xi_3 \rangle$, then the Lie algebra $\mathrm{Lie}(N) = \mathfrak{n} = \mathcal{H} \oplus \mathcal{V}$ decomposes both as vector spaces and as left-invariant distributions inside the tangent bundle.

Lemma 3.10. $Z(\mathfrak{n}) \subset \mathcal{V}$.

Proof. Since η_1 is left-invariant, we have $d\eta_1(X,Y) = -\eta_1([X,Y])$ for all $X, Y \in \mathfrak{n}$. Thus if $X \in Z(\mathfrak{n})$, then Condition (1.6) implies that for $Y := \varphi_1 X$:

$$0 = \eta_1([X,Y]) = -d\eta_1(X,Y) = 2\alpha \|X_{\mathcal{H}}\|^2\,.$$

□

Proposition 3.11. $Z(\mathfrak{n}) = \mathcal{V}$ and $\mathfrak{n}$ is two-step nilpotent.

Proof. Let V denote the connected subgroup of N with Lie algebra $\mathcal{V}$. Since N is simply connected and nilpotent, the (Lie group) exponential map of N is a global diffeomorphism [Knap, Theorem 1.127]. Hence V is a closed subgroup and N/V is (globally) a smooth manifold. Now N/V is also the space of leaves of the characteristic foliation of N, which admits a hyperkähler structure [ADS, Corollary 2.2.1] that is homogeneous as argued in the proof of Proposition 3.6.

Let W denote the kernel of the N-action on N/V. Then W is known as the *normal core of* V and it is the largest subgroup of V which is normal in N. The connected nilpotent quotient group N/W acts effectively and transitively by isometries on N/V, so N/V is isometric to N/W by virtue of a result due to WILSON [Wil, Theorem 2]. Consequently $\dim V = \dim W$, so $\mathcal{V} = \mathrm{Lie}(W)$ is an ideal in $\mathfrak{n}$. But $\eta_j([X, \xi_i]) = d\eta_j(\xi_i, X) = 0$ for all $X \in \mathfrak{n}$, so $\mathcal{V} = Z(\mathfrak{n})$. Moreover N/V is nilpotent and Ricci-flat, so a result by MILNOR implies that its Lie algebra $\mathfrak{n}/\mathcal{V} = \mathfrak{n}/Z(\mathfrak{n})$ is Abelian [Mil, Theorem 2.4]. Finally $[\mathfrak{n}, \mathfrak{n}] \subset Z(\mathfrak{n})$, so $\mathfrak{n}$ is two-step nilpotent. □

Theorem 3.12. *The only simply connected nilpotent Lie groups with a left-invariant degenerate 3-(α, δ)-Sasaki structure are the quaternionic Heisenberg groups endowed with the structure described in Example 3.8.*

Proof. Since N is simply connected, it suffices to construct a Lie algebra isomorphism $\psi : \mathfrak{n} \to \mathfrak{h}$ which preserves the structure tensors. First let ψ map the Reeb vector fields of $\mathfrak{n}$ to those of $\mathfrak{h}$. Both $\mathcal{H} \subset \mathfrak{n}$ and $\mathbb{H}^n \subset \mathfrak{h}$ are left quaternionic vector spaces with compatible inner products, so they admit orthonormal bases of the form $(e_1, \varphi_1 e_1, \varphi_2 e_1, \varphi_3 e_1, \ldots, e_n, \varphi_1 e_n, \varphi_2 e_n, \varphi_3 e_n)$. Let ψ map such a basis of $\mathcal{H}$ to a corresponding basis of $\mathbb{H}^n$.
From these definitions it is clear that ψ is a *linear* isomorphism which respects the structure tensors, so it only remains to check that ψ preserves the Lie bracket. To this end first note that by the previous proposition: $\psi(Z(\mathfrak{n})) = \psi(\mathcal{V}) = Z = Z(\mathfrak{h})$. Likewise $[\psi(\mathcal{H}), \psi(\mathcal{H})] = [\mathbb{H}^n, \mathbb{H}^n] \subset Z$ and $\psi[\mathcal{H}, \mathcal{H}] \subset \psi(\mathcal{V}) = Z$. Finally for all $X, Y \in \mathcal{H}$ we have (possibly after a suitable $\mathcal{H}$-homothetic deformation to match the values of α):

$$\begin{aligned} \eta_i^{\mathfrak{h}}([\psi X, \psi Y]) &= d\eta_i^{\mathfrak{h}}(\psi Y, \psi X) = 2\alpha g^{\mathfrak{h}}(\varphi_i^{\mathfrak{h}} \psi X, \psi Y) = 2\alpha g^{\mathfrak{h}}(\psi \varphi_i^{\mathfrak{n}} X, \psi Y) \\ &= 2\alpha g^{\mathfrak{n}}(\varphi_i^{\mathfrak{n}} X, Y) = d\eta_i^{\mathfrak{n}}(Y, X) = \eta_i^{\mathfrak{n}}([X, Y]) = \eta_i^{\mathfrak{h}}(\psi[X, Y]) . \end{aligned}$$

□

Remark 3.13. This theorem is very analogous to a result by ANDRADA, FINO and VEZZONI: They showed that the only simply connected nilpotent Lie groups with an invariant *Sasaki* structure are the odd-dimensional Heisenberg groups [AFV, Theorem 3.9]. We would like to warn the reader about the somewhat counterintuitive fact that, at least in the Sasaki realm, there exist (non-nilpotent) Lie groups with a left-invariant structure where the Reeb vector field does not lie in the center of the Lie algebra.

3.4 A Constructive Result

Whereas plentiful non-degenerate 3-(α, δ)-Sasaki structures are known on Konishi bundles over qK orbifolds of non-vanishing scalar curvature [ADS, Section 3], the only degenerate 3-(α, δ)-Sasaki manifolds studied so far are the quaternionic Heisenberg groups and derived spaces (cf. [AFS, Subsection 2.2], [AD, Example 2.3.2]). Now we present a method to construct many more interesting examples:

Let (N, g_N, I_1, I_2, I_3) be a hyperkähler manifold. Suppose that the fundamental two-forms represent integer classes $[\omega_i] \in H^2(N, \mathbb{Z})$. Then we obtain a Boothby-Wang bundle $S^1 \to P_i \xrightarrow{\pi_i} N$ for each Kähler structure [BW, Theorem 3]. In particular these bundles have Chern class $[\omega_i]$ and the total spaces P_i are equipped with Sasaki structures $(g_i, \tilde{\varphi}_i, \eta_i, \xi_i)$ such that ξ_i generates the fiber. Now consider the product bundle

$$T^3 \longrightarrow P_1 \times P_2 \times P_3 \xrightarrow{\Pi=(\pi_1,\pi_2,\pi_3)} N^3.$$

Let $M := \Pi^{-1}(\Delta N)$ denote the restriction of the product bundle $P_1 \times P_2 \times P_3$ to the diagonal $\Delta(N) = \{(x, x, x) \in N^3\}$ and consider M as a fiber bundle $M \xrightarrow{\pi} N$, where π is the composition

$$P_1 \times P_2 \times P_3 \supset M \xrightarrow{\Pi} \Delta(N) \xrightarrow{\cong} N, \quad (p_1, p_2, p_3) \mapsto (x, x, x) \mapsto x.$$

If we denote the fiber of P_i over $x \in N$ by $(P_i)_x := \pi_i^{-1}(\{x\}) \cong S^1$, then the bundle $M \xrightarrow{\pi} N$ can be seen as the fiber bundle with fiber $(P_1)_x \times (P_2)_x \times (P_3)_x$ over any given point $x \in N$. For $p = (p_1, p_2, p_3) \in M$ we have

$$T_pM \cong T_xN \oplus \langle \xi_1, \xi_2, \xi_3 \rangle$$

as ξ_i generates the fiber of P_i. The construction as a fiber product bundle ensures that the Reeb vector fields ξ_i are linearly independent and $[\xi_i, \xi_j] = 0$ as it ought to be for degenerate 3-(α, δ)-Sasaki manifolds.
Extend η_i trivially, i.e. $\ker \eta_i := TN \oplus \langle \xi_j, \xi_k \rangle$ with $j, k \neq i$ and $\eta_i(\xi_i) := 1$. Define the metric $g := \pi^* g_N + \eta_1^2 + \eta_2^2 + \eta_3^2$, so that $\pi\colon (M, g) \to (N, g_N)$ becomes a Riemannian submersion. Finally set $\varphi_i := \tilde{\varphi}_i + \eta_j \otimes \xi_k - \eta_k \otimes \xi_j$ for any cyclic permutation (i, j, k) of $(1, 2, 3)$.

Theorem 3.14. *Let (N, g_N, I_1, I_2, I_3) be a hyperkähler manifold with integer Kähler classes. Then $(M, g, \xi_i, \eta_i, \varphi_i)_{i=1,2,3}$ constructed as above is a degenerate 3-(α, δ)-Sasaki manifold with $\alpha = 1$, $\delta = 0$.*

Remark 3.15. The T^3-bundle M also appears in Fowdar's paper [Fow] in his construction of qK metrics on $\mathbb{R} \times M$ albeit without the geometric structure, see also [Cor, Lemma 2.1].
In [For1] Foreman obtains a so-called complex contact manifold by constructing a T^2-bundle over N in similar fashion. His construction imposes a less restrictive assumption than hyperkähler on N. However, in this special case the complex T^2-bundle he considers can be obtained from our construction as the quotient by one of the Reeb vector

fields. This should be considered as the analogue of the twistor space over a positive qK manifold and the 3-Sasaki space above (cf. [For2]).

Proof. Observe that $(g, \eta_i, \xi_i, \varphi_i)$ extends the almost contact metric structure on $T_{p_i}P_i \cong T_xN \oplus \langle \xi_i \rangle$ to $T_pM \cong T_xN \oplus \langle \xi_i, \xi_j, \xi_k \rangle$. The three almost contact metric structures are compatible on the vertical subspace by definition of φ_i. On the horizontal distribution φ_i projects to I_i, so $\varphi_i \circ \varphi_j|_{\mathcal{H}} = \varphi_k|_{\mathcal{H}}$.
The last identity to check is the differential Condition (1.6). As $(\xi_i, \eta_i, \tilde{\varphi}_i)$ defines a Sasaki structure on P_i we have $2g_i(X, \tilde{\varphi}_i Y) = d\eta_i(X, Y)$ for vertical vectors $X, Y \in T_{p_i}P_i$. This remains true for $X, Y \in \mathcal{H} \subset T_pM$ as φ_i is just $\tilde{\varphi}_i$ on $\mathcal{H}$. If either vector of X, Y lies in $\mathcal{V}$, then the left-hand side of (1.6) has to vanish. We have

$$\iota_{\xi_j} d\eta_i = L_{\xi_j}\eta_i - d(\iota_{\xi_j}\eta_i) = 0$$

as η_i is defined on the factor P_i of the product P. In particular it is invariant under ξ_j for $j \neq i$. Finally $\iota_{\xi_i} d\eta_i = 0$ from the Sasaki condition on P_i. □

Remark 3.16. We call this construction the 3-*Boothby-Wang bundle* over N. It is the inverse of the canonical submersion introduced in [ADS, Theorem 2.2.1].
In [Cor] CORTÉS shows that non-flat compact hyperkähler manifolds with integral Kähler classes exist in arbitrary dimensions. This implies the existence of plentiful compact degenerate 3-(α, δ)-Sasaki manifolds which are necessarily inhomogeneous by virtue of Theorem 3.7.

In summary our findings paint the somewhat curious picture that there are no non-trivial compact homogeneous examples in the degenerate 3-(α, δ)-Sasaki category, despite there being both interesting non-compact homogeneous (quaternionic Heisenberg groups) and compact inhomogeneous cases (3-Boothby-Wang bundles over compact inhomogeneous hyperkähler manifolds). Qualitatively this geometry is in stark contrast to the classical 3-Sasakian one where homogeneous examples are automatically compact and there exist as many as compact simple Lie groups (cf. Chapter 2).

As a proposal for future research an even more systematic treatment of this geometry would be desirable: Are the quaternionic Heisenberg groups maybe even the only non-compact homogeneous examples? Are there more ways to construct new degenerate 3-(α, δ)-Sasaki manifolds, for example via reduction? Can one classify the compact inhomogeneous examples in some way?

Chapter 4

A Bochner Technique for Foliations and Inhomogeneity

In the previous chapter we encountered certain degenerate 3-(α, δ)-Sasaki manifolds which were necessarily inhomogeneous. Now we develop a method to quantify "how far away from homogeneous" these and a number of other spaces are. One might also consider the conclusions we will arrive at as "rigidity" or "non-deformation" results. The key to obtaining these is to use the characteristic foliation generated by the Reeb vector field(s) of a Sasaki or 3-(α, δ)-Sasaki manifold and generalize the famous Bochner technique to foliations. Since this approach is of independent interest, we first develop it in a more general context and later specialize to Sasakian geometries.

The Bochner technique is a highly acclaimed method of proof in classical differential geometry which is attributed to BOCHNER [Boch] and sometimes also YANO [YB]. It discusses how Ricci curvature affects the types of vector fields that a manifold admits. A modern introduction to the topic can be found in [Pete, Chapter 7], which also served as inspiration for the generalization in this chapter.

Let (M, g) be a connected closed oriented Riemannian manifold and let Ric denote the Ricci curvature tensor of (M, g). The Bochner technique culminates in the following two theorems [Pete, Theorems 36 & 48]:

Theorem 4.1. *If* Ric *is negative semi-definite everywhere, then every Killing vector field is parallel. If additionally* Ric *is negative definite at one point, then no non-trivial Killing vector fields exist.*

Theorem 4.2. *If* Ric *is positive semi-definite everywhere, then every harmonic vector field (i.e. the g-dual one-form is harmonic) is parallel. If additionally* Ric *is positive definite at one point, then no non-trivial harmonic vector fields exist.*

By combining these results with the famous Hodge theorem, which states that the space of harmonic one-forms is isomorphic to the first cohomology group, we obtain the following interesting consequences:

Corollary 4.3. *If* Ric *is positive semi-definite everywhere and positive definite at one point, then the first Betti number* $b_1(M) = 0$*.*

Corollary 4.4. *If* Ric *vanishes everywhere, then the dimension of the isometry group of* (M, g) *is equal to* $b_1(M)$*.*

Over the years Theorems 4.1 and 4.2 have been adapted to work with various additional structures on M, in particular Riemannian foliations. We go through all of the necessary preliminaries about Riemannian foliations thoroughly in Section 4.1. Kamber and Tondeur devised the following analogue of Theorem 4.1 [KT2, Theorem B]:

Theorem 4.5. *Let* M *be a connected closed oriented manifold endowed with a Riemannian foliation* $(\mathcal{F}, g)$ *and let* Ric^T *denote the transverse Ricci curvature of* $(\mathcal{F}, g)$*. If* Ric^T *is negative semi-definite everywhere, then every transverse Killing vector field is transverse parallel. If additionally* Ric^T *is negative definite at one point, then no non-trivial transverse Killing vector fields exist.*

In fact Theorem 4.5 is even a generalization of Theorem 4.1, where the latter corresponds to the special case that $\mathcal{F}$ is the trivial foliation of M by singletons. In the direction of Corollary 4.3 there is the following vanishing theorem for basic cohomology which was discovered independently by Min-Oo, Ruh and Tondeur [Tond, Theorem 8.16] as well as Hebda [Hebd, Theorem 1]:

Theorem 4.6. *If* $(M, \mathcal{F}, g)$ *are as in Theorem 4.5 and* Ric^T *is positive definite everywhere, then the first basic Betti number* $b_1(\mathcal{F}) = 0$*.*

One main goal of this chapter is to develop the following generalization of Theorem 4.2:

Theorem 4.7. *Let* M *be a connected closed oriented manifold endowed with a transversely oriented harmonic Riemannian foliation* $(\mathcal{F}, g)$*. If* Ric^T *is positive semi-definite everywhere, then every basic harmonic vector field is transverse parallel. If additionally* Ric^T *is positive definite at one point, then no non-trivial basic harmonic vector fields exist.*

By combining Theorems 4.5 and 4.7 with a basic Hodge theorem we obtain the following variations of Corollary 4.4 and Theorem 4.6:

Corollary 4.8. *If* $(M, \mathcal{F}, g)$ *are as in Theorem 4.7 and* Ric^T *is positive semi-definite everywhere, then* $b_1(\mathcal{F}) \leq \operatorname{codim} \mathcal{F}$*. If additionally* Ric^T *is positive definite at one point, then* $b_1(\mathcal{F}) = 0$*.*

Corollary 4.9. *If* $(M, \mathcal{F}, g)$ *are as in Theorem 4.7 and* Ric^T *vanishes everywhere, then the dimension of the vector space* $\mathfrak{iso}(\mathcal{F})$ *of transverse Killing vector fields of* $(\mathcal{F}, g)$ *is equal to* $b_1(\mathcal{F})$.

Finally we apply Corollary 4.9 to two classes of spaces which naturally naturally satisfy all of the required conditions, namely degenerate 3-(α, δ)-Sasaki and certain so-called Sasaki-η-Einstein manifolds:

Theorem 4.10. *Let* M *be a connected closed degenerate 3-*(α, δ)*-Sasaki manifold with characteristic foliation* $\mathcal{F}$*. Then the dimension of the automorphism group* $\mathrm{Aut}(M)$ *is at most* $b_1(\mathcal{F}) + 3$.
In particular if M *arises via Theorem 3.14 as the 3-Boothby-Wang bundle over a compact hyperkähler manifold* N *with integral Kähler classes, then* $\dim \mathrm{Aut}(M) \leq b_1(N) + 3$.

Theorem 4.11. *Let* M *be a connected closed Sasaki-*η*-Einstein manifold with transverse Calabi-Yau structure and characteristic foliation* $\mathcal{F}$*. Then the dimension of the automorphism group* $\mathrm{Aut}(M)$ *is at most* $b_1(\mathcal{F}) + 1$.
In particular if M *arises as the Boothby-Wang bundle over a compact Calabi-Yau manifold* N *with integral Kähler class, then* $\dim \mathrm{Aut}(M) \leq b_1(N) + 1$.

About the structure of this chapter: We start with a self-contained explanation of the required fundamentals about Riemannian foliations (Section 4.1) and basic Hodge theory (Section 4.2). In Section 4.3 we complete the proof of the main Theorem 4.7 as well as its consequences and in Section 4.4 we provide the promised applications.

4.1 Riemannian Foliations

Let M be a smooth manifold and $(\mathcal{F}, g_T)$ a *Riemannian foliation* on M. This means that $\mathcal{F}$ is a foliation on M defined by an integrable subbundle $E \subset TM$ and g_T is a *transverse metric*, i.e. a symmetric positive semi-definite $(0,2)$-tensor field such that $\ker g_T = E$ and $\mathcal{L}_X g_T = 0$ for all $X \in \Gamma_\ell(E)$, where Γ_ℓ denotes the set of all local sections of a fiber bundle. In order to avoid having to deal with quotient bundles we shall choose and fix a so-called *bundle-like metric* g on M, i.e. a Riemannian metric such that $g(X^\perp, Y^\perp) = g_T(X, Y)$ for all $X, Y \in TM$, where $Z^\perp$ denotes the g-orthogonal projection of $Z \in TM$ to $E^\perp$.

Definition 4.12. The Lie algebra of *foliated vector fields* and the vector space of *transverse vector fields* are given by $\mathfrak{fol}(\mathcal{F}) := N_{\Gamma(TM)}\big(\Gamma(E)\big)$, the normalizer of $\Gamma(E)$ inside $\Gamma(TM)$, as well as $\mathfrak{trans}(\mathcal{F}) := \mathfrak{fol}(\mathcal{F}) \cap \Gamma(E^\perp)$, respectively. We call a function $f : M \to \mathbb{R}$ *basic* if $X(f) = 0$ for all $X \in \Gamma_\ell(E)$.

Lemma 4.13. *a) If* $X \in \mathfrak{fol}(\mathcal{F})$*, then* $f := \frac{1}{2}\, g_T(X, X)$ *is basic.*

b) If f *is basic, then its gradient (with respect to* g*) satisfies* $\nabla f \in \mathfrak{trans}(\mathcal{F})$.

Proof. a) For all $Y \in \Gamma_\ell(E)$: $Y(f) = g_T([Y,X],X) = 0$.

b) First $0 = X(f) = g(\nabla f, X)$ for all $X \in \Gamma_\ell(E)$, so $\nabla f \in \Gamma(E^\perp)$. Furthermore for all $X \in \Gamma(E)$, $Y \in \Gamma_\ell(E^\perp)$:

$$g_T([\nabla f, X], Y) = g_T(\nabla f, [X,Y]) - X\big(g_T(\nabla f, Y)\big) = [X,Y](f) - X\big(Y(f)\big) = 0\,.$$

□

Definition 4.14. Let ∇ denote the Levi-Civita connection of g. The *transverse Levi-Civita* or *Bott connection* ∇^T is the connection in the vector bundle $E^\perp$ given by

$$\nabla^T_X Y := \begin{cases} (\nabla_X Y)^\perp & , \ X \in \Gamma_\ell(E^\perp)\,, \\ [X,Y]^\perp & , \ X \in \Gamma_\ell(E)\,. \end{cases}$$

The condition $\mathcal{L}_X g_T = 0$ ensures that $[X,Y]^\perp$ is indeed tensorial in $X \in \Gamma_\ell(E)$. Note that if $Y \in \mathfrak{trans}(\mathcal{F})$, then $\nabla^T_X Y = [X,Y]^\perp = 0$ for all $X \in \Gamma_\ell(E)$. The connection ∇^T is the unique metric and torsion-free connection in $E^\perp$ [Tond, Theorem 5.9], i.e. for all $X \in \Gamma_\ell(TM)$ and $Y, Z \in \Gamma_\ell(E^\perp)$:

$$\begin{aligned} X\big(g_T(Y,Z)\big) &= g_T(\nabla^T_X Y, Z) + g_T(Y, \nabla^T_X Z)\,, \\ \nabla^T_Y Z - \nabla^T_Z Y &= [Y,Z]^\perp\,. \end{aligned}$$

Furthermore ∇^T may be characterized via a Koszul formula [KT1, Proposition 1.7], i.e. for all $X, Z \in \Gamma_\ell(TM)$, $Y \in \Gamma(E^\perp)$:

$$\begin{aligned} 2g_T(\nabla^T_X Y, Z) = {} & X\big(g_T(Y,Z)\big) + Y\big(g_T(Z,X)\big) - Z\big(g_T(X,Y)\big) \\ & + g_T([X,Y],Z) + g_T([Z,X],Y) - g_T([Y,Z],X)\,. \end{aligned}$$

Definition 4.15. Let f be a basic function. The *transverse Hessian* $\operatorname{Hess}_T f$ is the symmetric $(0,2)$-tensor field given by

$$\operatorname{Hess}_T f(X,Y) := g_T(\nabla^T_X \nabla f, Y)\,, \quad X, Y \in \Gamma_\ell(TM)\,.$$

Clearly $\iota_X \operatorname{Hess}_T f = 0$ for all $X \in \Gamma_\ell(E)$. The *transverse Laplacian* $\Delta_T f$ is defined as

$$\Delta_T f := \operatorname{tr}_g \operatorname{Hess}_T f = \sum_i \operatorname{Hess}_T f(E_i, E_i)\,,$$

where E_i is a local g-orthonormal frame. The *transverse Riemann curvature tensor* R^T is given by

$$R^T(X,Y)Z := \nabla^T_X \nabla^T_Y Z - \nabla^T_Y \nabla^T_X Z - \nabla^T_{[X,Y]} Z\,, \quad X, Y \in \Gamma_\ell(TM),\ Z \in \Gamma_\ell(E^\perp)\,.$$

Again $\iota_X R^T = 0$ for all $X \in \Gamma_\ell(E)$ [Tond, Proposition 3.6]. As usual:

$$R^T(X,Y,Z,V) := g_T\big(R^T(X,Y)Z, V\big)\,, \quad V \in TM\,.$$

Finally the *transverse Ricci curvature* Ric^T is defined as

$$\mathrm{Ric}^T(X,X) := \mathrm{tr}\,\big(Y \mapsto R^T(Y,X)X\big) = \sum_i R^T(E_i,X,X,E_i)\,, \quad X \in \Gamma_\ell(E^\perp)\,.$$

Remark 4.16. It is well-known that a Riemannian foliation can be characterized equivalently via local Riemannian submersions $\phi : U \to N$ onto a Riemannian model space (N, g_N). The transverse Riemann curvature tensor R^T then reflects the Riemann curvature tensor R^N of the local model N, as made precise by the following equation [Tond, Equation 5.40]:

$$\phi_* R^T(X,Y)Z = R^N(\phi_*X, \phi_*Y)\phi_*Z\,, \quad X,Y,Z \in \Gamma_\ell(E^\perp)\,.$$

Likewise Ric^T mirrors the Ricci curvature tensor Ric^N of N, viz.

$$\mathrm{Ric}^T(X,X) = \mathrm{Ric}^N(\phi_*X, \phi_*X) \circ \phi\,, \quad X \in \Gamma_\ell(E^\perp)\,.$$

Hence if the Riemannian foliation $(\mathcal{F}, g)$ was simply given by one (global) Riemannian submersion $\phi : M \to N$ onto a Riemannian manifold N, then we could just apply the ordinary Bochner technique to N instead of the more complicated approach presented here. However, the advantage of a Bochner technique for foliations is that it also works if the model space (globally) is not as well-behaved as a smooth manifold, which is often a non-trivial condition.

From now on let $n := \dim M$, $p := \mathrm{rk}\, E$ and $q := n - p$.

Definition 4.17. We call $\mathcal{F}$ *transversely orientable* if $E^\perp$ is orientable. Suppose that M is orientable and $\mathcal{F}$ is tranversely orientable. Then we shall orient M and $E^\perp$ using their *Riemannian volume forms* μ and μ_T, respectively. This means we choose a local oriented orthonormal frame $E_1, \ldots, E_n$ of TM such that $E_{p+1}, \ldots, E_n$ is an oriented frame of $E^\perp$ and require $\mu(E_1, \ldots, E_n) = \mu_T(E_{p+1}, \ldots, E_n) = 1$.
If $X \in \mathfrak{fol}(\mathcal{F})$ is a foliate vector field, then $\iota_Y \mathcal{L}_X \mu_T = 0$ for all $Y \in \Gamma_\ell(E)$. Thus $\mathcal{L}_X \mu_T$ may be viewed as a section of the vector bundle $\Lambda^q(E^\perp)^*$ which has rank one. Hence we can define the *transverse divergence* $\mathrm{div}_T\, X$ as the unique function which satisfies

$$\mathcal{L}_X \mu_T = \mathrm{div}_T\, X \cdot \mu_T\,.$$

Lemma 4.18. *For any transverse vector field $X \in \mathfrak{trans}(\mathcal{F})$:*

$$\mathrm{div}_T\, X = \mathrm{tr}\, \nabla^T X\,.$$

In particular for any basic function f:

$$\operatorname{div}_T \nabla f = \Delta_T f\,.$$

Proof. Since X is transverse, we have $\nabla^T_Y X = 0$ for all $Y \in \Gamma_\ell(E)$. If $E_{p+1}, \ldots, E_n$ is an oriented local orthonormal frame of $E^\perp$, then:

$$\begin{aligned}(\mathcal{L}_X \mu_T)(E_{p+1}, \ldots, E_n) &= X\big(\mu_T(E_{p+1}, \ldots, E_n)\big) - \sum_i \mu_T(E_{p+1}, \ldots, [X, E_i], \ldots, E_n) \\ &= -\sum_i g_T([X, E_i], E_i)\, \mu_T(E_{p+1}, \ldots, E_n) = -\sum_i g_T([X, E_i], E_i)\,.\end{aligned}$$

On the other hand:

$$\operatorname{tr} \nabla^T X = \sum_i g_T(\nabla^T_{E_i} X, E_i) = \sum_i g_T(\nabla^T_X E_i, E_i) - g_T([X, E_i], E_i) = -\sum_i g_T([X, E_i], E_i)\,.$$

□

Definition 4.19. A foliation $\mathcal{F}$ is called *taut* if there exists a Riemannian metric g on M such that the leaves of $\mathcal{F}$ are minimal submanifolds of M with respect to g. If a Riemannian foliation is taut, then g may be chosen to be bundle-like [Tond, Proposition 7.6], in which case $(\mathcal{F}, g)$ is called *harmonic*.

One key tool for us is the following transverse divergence theorem [Tond, Theorem 4.35]:

Theorem 4.20. *Let M be a closed oriented manifold endowed with a transversely oriented harmonic Riemannian foliation $(\mathcal{F}, g)$. Then for any foliate vector field $X \in \mathfrak{fol}(M)$:*

$$\int_M \operatorname{div}_T X \cdot \mu = 0\,.$$

4.2 Basic Hodge Theory

Let M be a smooth manifold endowed with a foliation $\mathcal{F}$ of codimension q defined by an integrable subbundle $E \subset TM$.

Definition 4.21. A differential k-form $\omega \in \Omega^k(M)$ is called *basic* if $\iota_X \omega = 0$ as well as $\mathcal{L}_X \omega = \iota_X d\omega = 0$ for all $X \in \Gamma_\ell(E)$. Note that for $f \in \Omega^0(M)$ this coincides with the Definition 4.12 of a basic function.
If ω is basic, then so is $d\omega$, meaning the basic differential forms constitute a subcomplex $\Omega_B(\mathcal{F})$ of the de Rham complex $\Omega(M)$. Clearly $\Omega^k_B(\mathcal{F}) = 0$ for $k > q$. We denote the restriction of d to $\Omega_B(\mathcal{F})$ by d_B. The cohomology ring of the complex $(\Omega_B(\mathcal{F}), d_B)$ is called the *basic cohomology of* $\mathcal{F}$ and will be denoted by $H_B(\mathcal{F})$. The *basic Betti numbers of* $\mathcal{F}$ are defined as $b_k(\mathcal{F}) := \dim H^k_B(\mathcal{F})$.

The inclusion $\Omega^1_B(\mathcal{F}) \to \Omega^1(M)$ induces an injective map $H^1_B(\mathcal{F}) \to H^1(M)$ [Tond, Proposition 4.1]. Furthermore, if M is closed and $(\mathcal{F}, g_T)$ is a Riemannian foliation on M, then $b_k(\mathcal{F}) < \infty$ for $k = 0, \dots, q$ [Tond, Chapter 4].

From now on we assume that M is closed and oriented, $(\mathcal{F}, g_T)$ is a transversely oriented Riemannian foliation on M and g is a bundle-like metric compatible with g_T. As usual the metric g induces an inner product on $\Lambda^k T^*_x M$ for every $x \in M$. We let $\mu \in \Omega^n(M)$ denote the Riemannian volume form of (M, g) and consider the inner product $\langle \cdot, \cdot \rangle$ on $\Omega^k(M)$ given by

$$\langle \omega, \omega' \rangle := \int_M g(\omega, \omega') \cdot \mu \,, \quad \omega, \omega' \in \Omega^k(M) \,.$$

We write $\langle \cdot, \cdot \rangle_B$ for the restriction of $\langle \cdot, \cdot \rangle$ to the subspace $\Omega^k_B(\mathcal{F}) \subset \Omega^k(M)$.

Definition 4.22. The *basic codifferential* $\delta_B : \Omega^k_B(\mathcal{F}) \to \Omega^{k-1}_B(\mathcal{F})$ is the formal adjoint of $d_B : \Omega^{k-1}_B(\mathcal{F}) \to \Omega^k_B(\mathcal{F})$ with respect to $\langle \cdot, \cdot \rangle_B$, viz.

$$\langle d_B \omega, \eta \rangle_B = \langle \omega, \delta_B \eta \rangle_B \,, \quad \omega \in \Omega^{k-1}_B(\mathcal{F}), \, \eta \in \Omega^k_B(\mathcal{F}) \,.$$

The *basic Laplacian* is given by

$$\Delta_B := d_B \delta_B + \delta_B d_B : \Omega^k_B(\mathcal{F}) \to \Omega^k_B(\mathcal{F}) \,.$$

A basic form $\omega \in \Omega^k_B(\mathcal{F})$ is called *basic harmonic* if $\Delta_B \omega = 0$. The vector space of all basic harmonic k-forms will be denoted by $\mathcal{H}^k_B(\mathcal{F})$.

Beware that Δ_B is not the restriction of the ordinary Laplacian $\Delta = d\delta + \delta d$ to $\Omega^k_B(\mathcal{F})$ [Tond, Equation 7.28]. On basic functions Δ_B differs from the previously defined transverse Laplacian Δ_T by a sign, see Remark 4.26. By definition of δ_B every $\omega \in \Omega^k_B(\mathcal{F})$ satisfies

$$\langle \Delta_B \omega, \omega \rangle_B = \langle d_B \delta_B \omega, \omega \rangle_B + \langle \delta_B d_B \omega, \omega \rangle_B = \langle d_B \omega, d_B \omega \rangle_B + \langle \delta_B \omega, \delta_B \omega \rangle_B \,.$$

Therefore $\Delta_B \omega = 0$ if and only if both $d_B \omega = 0$ and $\delta_B \omega = 0$. In particular we have a natural map $\mathcal{H}^k_B(\mathcal{F}) \to H^k_B(\mathcal{F})$. In case the bundle-like metric is chosen appropriately there is the following basic Hodge theorem [Tond, Theorem 7.51]:

Theorem 4.23. *Let M be a closed oriented manifold endowed with a transversely oriented harmonic Riemannian foliation $(\mathcal{F}, g)$. Then the natural map $\mathcal{H}^k_B(\mathcal{F}) \to H^k_B(\mathcal{F})$ is an isomorphism.*

In preparation for the Bochner technique in the next section we now specialize to one-forms: Recall the usual one-to-one correspondence between vector fields $X \in \Gamma(TM)$ and their g-dual one-forms $\omega_X := \iota_X g \in \Omega^1(M)$. One easily checks that $X \in \mathfrak{trans}(\mathcal{F})$ if and only if $\omega_X \in \Omega^1_B(\mathcal{F})$.

Lemma 4.24. *We have $d_B\omega_X = 0$ if and only if $\nabla^T X$ is g_T-symmetric, i.e.*

$$g_T(\nabla^T_Y X, Z) = g_T(Y, \nabla^T_Z X)\,, \quad Y, Z \in \Gamma_\ell(TM)\,.$$

Proof. The Koszul formula from Definition 4.14 can be rewritten as

$$2g_T(\nabla^T_Y X, Z) = (d_B\omega_X)(Y, Z) + (\mathcal{L}_X g_T)(Y, Z)\,, \quad Y, Z \in \Gamma_\ell(TM)\,.$$

Since $d_B\omega_X$ is skew-symmetric and $\mathcal{L}_X g_T$ is symmetric, this yields the claim. □

Lemma 4.25. *If $(\mathcal{F}, g)$ is harmonic, then $\delta_B\omega_X = -\operatorname{div}_T X$.*

Proof. By definition of δ_B we need to show that for all basic functions $f \in \Omega^0_B(\mathcal{F})$:

$$\int_M g(d_B f, \omega_X) \cdot \mu = -\int_M f \cdot \operatorname{div}_T X \cdot \mu\,.$$

Using Lemma 4.18 we calculate:

$$\operatorname{div}_T(f \cdot X) = f \cdot \operatorname{div}_T X + g(\nabla f, X) = f \cdot \operatorname{div}_T X + g(d_B f, \omega_X)\,.$$

If we integrate over M, then the left-hand side vanishes by Theorem 4.20, since $f \cdot X$ is foliate and $(\mathcal{F}, g)$ is harmonic, and the claim follows. □

Remark 4.26. Lemmas 4.18 and 4.25 imply that for all basic functions f:

$$\Delta_B f = \delta_B d_B f = \delta_B \omega_{\nabla f} = -\operatorname{div}_T \nabla f = -\Delta_T f\,.$$

Corollary & Definition 4.27. If $(\mathcal{F}, g)$ is harmonic, then ω_X is basic harmonic if and only if $\nabla^T X$ is g_T-symmetric and $\operatorname{div}_T X = 0$. In this case we call X *basic harmonic.*

4.3 A Bochner Technique for Foliations

From now on let M be a connected closed oriented manifold endowed with a transversely oriented harmonic Riemannian foliation $(\mathcal{F}, g)$ of codimension q.

Definition 4.28. A transverse vector field $X \in \mathfrak{trans}(\mathcal{F})$ is *transverse parallel* if $\nabla^T X = 0$.

Beware that a transverse vector field $X \in \mathfrak{trans}(\mathcal{F})$ which is parallel in the usual sense that $\nabla X = 0$ is also transverse parallel, but the converse is not true. By virtue of Lemma 4.18 and Corollary & Definition 4.27 every transverse parallel vector field is basic harmonic.

Lemma 4.29. *Transverse parallel vector fields have constant length. Hence they are uniquely determined by their value at one point.*

Proof. If $X \in \mathfrak{trans}(\mathcal{F})$ is transverse parallel and $f := \frac{1}{2}g(X,X) = \frac{1}{2}g_T(X,X)$, then for all $Y \in \Gamma_\ell(TM)$:

$$Y(f) = g_T(\nabla^T_Y X, X) = 0.$$

Since M is connected, this implies that f is constant. □

For an endomorphism field $A \in \Gamma(\mathrm{End}(TM))$ we set

$$|A|^2 := \mathrm{tr}(A \circ A^*) = \sum_i g\big(A(E_i), A(E_i)\big)\,,$$

where $E_1, \ldots, E_n$ is a local orthonormal frame.

Proposition 4.30. Let $X \in \mathfrak{trans}(\mathcal{F})$ be a basic harmonic vector field and consider the basic function $f := \frac{1}{2}\, g_T(X,X)$. Then:

a) $\nabla f = \nabla^T_X X$.

b) $\mathrm{Hess}_T\, f(Y,Y) = g_T(\nabla^T_Y X, \nabla^T_Y X) + R^T(Y,X,X,Y) + g_T(\nabla^T_X \nabla^T_Y X, Y) - g_T(\nabla^T_{\nabla^T_X Y} X, Y)$ for all $Y \in \Gamma_\ell(E^\perp)$.

c) $\Delta_T f = |\nabla^T X|^2 + \mathrm{Ric}^T(X,X)$.

Proof. a) By virtue of Lemma 4.24:

$$g(\nabla f, Y) = Y(f) = g_T(\nabla^T_Y X, X) = g_T(\nabla^T_X X, Y) = g(\nabla^T_X X, Y)\,, \quad Y \in \Gamma_\ell(TM)\,.$$

b) Part a) and Lemma 4.24 imply that for all $Y \in \Gamma_\ell(E^\perp)$:

$$\begin{aligned}
\mathrm{Hess}_T\, f(Y,Y) &= g_T(\nabla^T_Y \nabla f, Y) = g_T(\nabla^T_Y \nabla^T_X X, Y)\\
&= R^T(Y,X,X,Y) + g_T(\nabla^T_X \nabla^T_Y X, Y) + g_T(\nabla^T_{[Y,X]} X, Y)\\
&= R^T(Y,X,X,Y) + g_T(\nabla^T_X \nabla^T_Y X, Y) + g_T(\nabla^T_{\nabla^T_Y X} X, Y) - g_T(\nabla^T_{\nabla^T_X Y} X, Y)\\
&= R^T(Y,X,X,Y) + g_T(\nabla^T_X \nabla^T_Y X, Y) + g_T(\nabla^T_Y X, \nabla^T_Y X) - g_T(\nabla^T_{\nabla^T_X Y} X, Y)\\
&= g_T(\nabla^T_Y X, \nabla^T_Y X) + R^T(Y,X,X,Y) + g_T(\nabla^T_X \nabla^T_Y X, Y) - g_T(\nabla^T_{\nabla^T_X Y} X, Y)\,.
\end{aligned}$$

From the second to the third line we implicitly used that $\nabla^T_{[Y,X]} X = \nabla^T_{[Y,X]^\perp} X$, since $X \in \mathfrak{trans}(\mathcal{F})$.

c) If we sum b) over any local orthonormal frame, then the first two terms on the right-hand side yield $|\nabla^T X|^2$ and $\mathrm{Ric}^T(X,X)$ as desired.
Fix a point $x \in M$. As shown in [KTT, Section 3] there exists a local orthonormal frame $E_1, \ldots, E_n$ in a neighborhood of x such that $E_1, \ldots, E_p \in \Gamma_\ell(E)$, $E_{p+1}, \ldots, E_n \in \Gamma_\ell(E^\perp)$ and $(\nabla^T E_i)_x = 0$ for $i = p+1, \ldots, n$. If we sum b)

at x over such a frame, then the last term on the right-hand side vanishes and the third term reduces to

$$\sum_i g_T(\nabla^T_X \nabla^T_{E_i} X, E_i) = \sum_i X\big(g_T(\nabla^T_{E_i} X, E_i)\big) = X(\operatorname{div}_T X) = 0\,.$$

□

We can now finally come to our main result:

Theorem 4.31. *Let M be a connected closed oriented manifold endowed with a transversely oriented harmonic Riemannian foliation $(\mathcal{F}, g)$. If Ric^T is positive semi-definite everywhere, then every basic harmonic vector field is transverse parallel. If additionally Ric^T is positive definite at one point, then no non-trivial basic harmonic vector fields exist.*

Proof. Let $X \in \mathfrak{trans}(\mathcal{F})$ be a basic harmonic vector field and $f := \frac{1}{2} g_T(X, X)$. By virtue of Lemma 4.18, Theorem 4.20 and Proposition 4.30:

$$0 = \int_M \Delta_T f \cdot \mu = \int_M \Big(|\nabla^T X|^2 + \mathrm{Ric}^T(X, X)\Big) \cdot \mu \geq \int_M |\nabla^T X|^2 \cdot \mu \geq 0\,.$$

Therefore $|\nabla^T X|^2$ vanishes everywhere, meaning X is transverse parallel. Furthermore also $\mathrm{Ric}^T(X, X)$ vanishes everywhere, so if additionally Ric^T is positive definite at one point, then X vanishes at that point. But then X vanishes everywhere by virtue of Lemma 4.29. □

Remark 4.32. Note that Theorem 4.31 is indeed a generalization of Theorem 4.2: If $\mathcal{F}$ is the trivial foliation of M by singletons (i.e. the corresponding integrable distribution $E = 0$), then transverse orientability of $\mathcal{F}$ coincides with ordinary orientability of M, the Riemannian foliation $(\mathcal{F}, g)$ is trivially harmonic and $\mathrm{Ric}^T = \mathrm{Ric}$. Furthermore basic harmonic and transverse parallel vector fields are nothing else than ordinary harmonic and parallel vector fields in this case.

Corollary 4.33. *If $(M, \mathcal{F}, g)$ are as in Theorem 4.31 and Ric^T is positive semi-definite everywhere, then $b_1(\mathcal{F}) \leq q = \operatorname{codim} \mathcal{F}$. If additionally Ric^T is positive definite at one point, then $b_1(\mathcal{F}) = 0$.*

Proof. Theorem 4.23 states that $b_1(\mathcal{F}) = \dim \mathcal{H}^1_B(\mathcal{F})$. Fix a point $x \in M$ and consider the linear map $\mathcal{H}^1_B(\mathcal{F}) \to T_x E^\perp$, $\omega_X \mapsto X_x$. By virtue of Lemma 4.29 and Theorem 4.31 this map is injective, meaning $b_1(\mathcal{F}) \leq \dim T_x E^\perp = q$. If additionally Ric^T is positive definite at one point, then Theorem 4.31 even yields $b_1(\mathcal{F}) = 0$. □

We conclude this section by deriving Corollary 4.9, for which we first need the following

Definition 4.34. A transverse vector field $X \in \mathfrak{trans}(\mathcal{F})$ is *transverse Killing* if $\mathcal{L}_X g_T = 0$. We denote the vector space of all transverse Killing fields of $(\mathcal{F}, g)$ by $\mathfrak{iso}(\mathcal{F})$.

Again a transverse vector field $X \in \mathfrak{trans}(\mathcal{F})$ which is Killing in the usual sense that $\mathcal{L}_X g = 0$ is also transverse Killing, but the converse is not true. The same argument as in the proof of Lemma 4.24 shows that $X \in \mathfrak{trans}(\mathcal{F})$ is transverse Killing if and only if $\nabla^T X$ is g_T-skew-symmetric. This also demonstrates that every transverse parallel vector field is transverse Killing. Combining Theorems 4.5 and 4.31 yields the following

Corollary 4.35. *If $(M, \mathcal{F}, g)$ are as in Theorem 4.31 and Ric^T vanishes everywhere, then $\dim \mathfrak{iso}(\mathcal{F}) = b_1(\mathcal{F})$.*

4.4 Applications to Inhomogeneity

We conclude this thesis by applying Corollary 4.35 to two classes of spaces which naturally satisfy all of the required conditions, namely degenerate 3-(α, δ)-Sasaki and certain Sasaki-η-Einstein manifolds.

Definition 4.36. A Sasaki manifold M with contact form η is called *η-Einstein* if its Ricci curvature tensor satisfies $\mathrm{Ric} = ag + b\eta \otimes \eta$ for some constants $a, b \in \mathbb{R}$.

In order to apply Corollary 4.35 we limit ourselves to those Sasaki manifolds where the transverse Kähler structure of the characteristic foliation is Ricci-flat, i.e. Calabi-Yau. In this case the Sasaki manifold is not Einstein in the ordinary sense that $\mathrm{Ric} = ag$ but instead η-Einstein with $b \neq 0$ [BG, Theorem 11.1.3]. Examples of Sasaki-η-Einstein manifolds can be constructed as Boothby-Wang bundles over Calabi-Yau manifolds with integral Kähler class.

As outlined in Chapter 1, Sasaki and 3-(α, δ)-Sasaki manifolds are orientable and their characteristic foliation is transversely orientable and harmonic ([BG, Proposition 6.3.5], [AD, Corollary 2.3.1]). Hence Corollary 4.35 takes on the following form:

Corollary 4.37. *Let M be a connected closed degenerate 3-(α, δ)-Sasaki manifold or Sasaki-η-Einstein manifold with transverse Calabi-Yau structure. Then the characteristic foliation $\mathcal{F}$ satisfies $\dim \mathfrak{iso}(\mathcal{F}) = b_1(\mathcal{F})$.*

Finally we want to make this result more intelligible by clarifying the relationship between transverse Killing fields and infinitesimal automorphisms: Let M be a connected closed degenerate 3-(α, δ)-Sasaki manifold or Sasaki-η-Einstein manifold with transverse Calabi-Yau structure, let E be the integrable distribution spanned by the Reeb vector field(s) and $\mathcal{F}$ the characteristic foliation.

Lemma 4.38. *The orthogonal projection of any infinitesimal automorphism to $E^\perp$ is a transverse Killing field.*

Proof. Let $X \in \mathfrak{aut}(M)$ be an infinitesimal automorphism and let $X^\top$, $X^\perp$ denote its orthogonal projections to E, $E^\perp$, respectively. Because X commutes with the Reeb vector field(s) it follows that $X \in \mathfrak{fol}(\mathcal{F})$. Since $X^\top$ is trivially foliate, we obtain that $X^\perp = X - X^\top \in \mathfrak{fol}(\mathcal{F})$. This implies that $(\mathcal{L}_{X^\perp} g_T)(Y,Z) = 0$ if Y or Z lies in $\Gamma_\ell(E)$. Furthermore $\mathcal{L}_X g = 0$, since X is Killing and $\mathcal{L}_{X^\top} g_T = 0$ because g_T is a transverse metric. Hence if both $Y, Z \in \Gamma_\ell(E^\perp)$:

$$\begin{aligned}(\mathcal{L}_{X^\perp} g_T)(Y,Z) &= (\mathcal{L}_X g_T)(Y,Z)\\ &= X\big(g_T(Y,Z)\big) - g_T([X,Y],Z) - g_T(Y,[X,Z])\\ &= X\big(g(Y,Z)\big) - g([X,Y],Z) - g(Y,[X,Z])\\ &= (\mathcal{L}_X g)(Y,Z) = 0\,.\end{aligned}$$

□

Therefore $\pi : \mathfrak{aut}(M) \to \mathfrak{iso}(\mathcal{F})$, $X \mapsto X^\perp$ is a well-defined linear map. The kernel of π is given by $\mathfrak{aut}(M) \cap \Gamma(E)$ which is comprised of all the linear combinations of the Reeb vector field(s) with constant coefficients (cf. Lemma 3.2). Hence the rank-nullity theorem and Corollary 4.37 yield:

$$\dim \mathfrak{aut}(M) = \dim \operatorname{im} \pi + \dim \ker \pi \leq \dim \mathfrak{iso}(\mathcal{F}) + \operatorname{rk} E = b_1(\mathcal{F}) + \operatorname{rk} E\,.$$

We have thus arrived at our final two theorems:

Theorem 4.39. *Let M be a connected closed degenerate 3-(α,δ)-Sasaki manifold with characteristic foliation $\mathcal{F}$. Then the dimension of the automorphism group* $\mathrm{Aut}(M)$ *is at most $b_1(\mathcal{F}) + 3$.*
In particular if M arises via Theorem 3.14 as the 3-Boothby-Wang bundle over a compact hyperkähler manifold N with integral Kähler classes, then $\dim \mathrm{Aut}(M) \leq b_1(N) + 3$.

Theorem 4.40. *Let M be a connected closed Sasaki-η-Einstein manifold with transverse Calabi-Yau structure and characteristic foliation $\mathcal{F}$. Then the dimension of the automorphism group* $\mathrm{Aut}(M)$ *is at most $b_1(\mathcal{F}) + 1$.*
In particular if M arises as the Boothby-Wang bundle over a compact Calabi-Yau manifold N with integral Kähler class, then $\dim \mathrm{Aut}(M) \leq b_1(N) + 1$.

Remark 4.41. Consequently the 3-Boothby-Wang bundle M over a *simply connected* compact hyperkähler manifolds with integral Kähler classes is not only inhomogeneous but "as far from homogeneous as possible" in the sense that $\dim \mathrm{Aut}(M) = 3$.

One might ask if there is even equality $\dim \mathrm{Aut}(M) = b_1(\mathcal{F}) + \operatorname{rk} E$ in the above theorems. This is equivalent to the question if $\pi : \mathfrak{aut}(M) \to \mathfrak{iso}(\mathcal{F})$ is surjective or if every transverse Killing field can be extended to an infinitesimal automorphism. In the context of Sasaki manifolds this problem was further rephrased in [BG, Theorem 8.1.8], where they obtain that a transverse Killing field $X \in \mathfrak{iso}(\mathcal{F})$ extends to an infinitesimal automorphism if and only if the basic cohomology class $[\iota_X d\eta] \in H^1_B(\mathcal{F})$ vanishes. The same holds for degenerate 3-(α, δ)-Sasaki manifolds if the classes $[\iota_X d\eta_i]$ vanish for $i = 1, 2, 3$. In the special case $b_1(\mathcal{F}) = 0$ this leads to alternative proofs of the above theorems.

Bibliography

[AD] I. Agricola and G. Dileo. "Generalizations of 3-Sasakian manifolds and skew torsion". In: *Adv. Geom.* 20.3 (2020), pp. 331–374.

[ADS] I. Agricola, G. Dileo, and L. Stecker. "Homogeneous non-degenerate 3-(α, δ)-Sasaki manifolds and submersions over quaternionic Kähler spaces". In: *Ann. Global Anal. Geom.* 60.1 (2021), pp. 111–141.

[AFS] I. Agricola, A. C. Ferreira, and R. Storm. "Quaternionic Heisenberg groups as naturally reductive homogeneous spaces". In: *Int. J. Geom. Methods Mod. Phys.* 12.8 (2015). Id/No 1560007, p. 10.

[AFV] A. Andrada, A. Fino, and L. Vezzoni. "A class of Sasakian 5-manifolds". In: *Transform. Groups* 14.3 (2009), pp. 493–512.

[AK] D. V. Alekseevskii and B. N. Kimel'fel'd. "Structure of homogeneous Riemann spaces with zero Ricci curvature". In: *Funct. Anal. Appl.* 9 (1975), pp. 97–102.

[Alek] D. V. Alekseevskii. "Compact quaternion spaces". In: *Funkcional. Anal. i Priložen* 2.2 (1968), pp. 11–20.

[AMP] D. V. Alekseevskii, S. Marchiafava, and M. Pontecorvo. "Compatible almost complex structures on quaternion Kähler manifolds". In: *Ann. Global Anal. Geom.* 16.5 (1998), pp. 419–444.

[BdS] A. Borel and J. De Siebenthal. "Les sous-groupes fermés de rang maximum des groupes de Lie clos". In: *Comment. Math. Helv.* 23 (1949), pp. 200–221.

[Bess] A. L. Besse. *Einstein manifolds.* Classics in Mathematics. Reprint of the 1987 edition. Springer-Verlag, Berlin, 2008, pp. xii+516.

[BG] C. P. Boyer and K. Galicki. *Sasakian geometry.* Oxford Mathematical Monographs. Oxford University Press, Oxford, 2008, pp. xii+613.

[BGM] C. P. Boyer, K. Galicki, and B. M. Mann. "The geometry and topology of 3-Sasakian manifolds". In: *J. Reine Angew. Math.* 455 (1994), pp. 183–220.

[Biel] R. Bielawski. "On the hyper-Kähler metrics associated to singularities of nilpotent varieties". In: *Ann. Global Anal. Geom.* 14.2 (1996), pp. 177–191.

[Boch] S. Bochner. "Vector fields and Ricci curvature". In: *Bull. Amer. Math. Soc.* 52 (1946), pp. 776–797.

[Boot] W. M. Boothby. “Homogeneous complex contact manifolds”. In: *Proc. Sympos. Pure Math., Vol. III*. American Mathematical Society, Providence, R.I., 1961, pp. 144–154.

[BW] W. M. Boothby and H. C. Wang. “On contact manifolds”. In: *Ann. Math. (2)* 68 (1958), pp. 721–734.

[Cor] V. Cortés. *A note on quaternionic Kähler manifolds with ends of finite volume.* 2022. arXiv: 2205.13806.

[CR] P. E. Conner and F. Raymond. “Actions of compact Lie groups on aspherical manifolds”. In: *Topology of Manifolds, Proc. Univ. Georgia* (1971), pp. 227–264.

[DOP] C. Draper, M. Ortega, and F. J. Palomo. “Affine connections on 3-Sasakian homogeneous manifolds”. In: *Math. Z.* 294.1-2 (2020), pp. 817–868.

[For1] B. Foreman. “Boothby-Wang fibrations on complex contact manifolds”. In: *Differ. Geom. Appl.* 13.2 (2000), pp. 179–196.

[For2] B. Foreman. “Complex contact manifolds and hyperkähler geometry”. In: *Kodai Math. J.* 23.1 (2000), pp. 12–26.

[Fow] U. Fowdar. *Einstein metrics on bundles over hyperKähler manifolds.* 2021. arXiv: 2105.04254.

[GRS1] O. Goertsches, L. Roschig, and L. Stecker. “Revisiting the Classification of Homogeneous 3-Sasakian and Quaternionic Kähler Manifolds”. In: *European Journal of Mathematics* 9, 11 (2023).

[GRS2] O. Goertsches, L. Roschig, and L. Stecker. “On Degenerate 3-(α, δ)-Sasakian Manifolds”. In: *Complex Manifolds* 9.1 (2022), pp. 337–344.

[Hebd] J. J. Hebda. “Curvature and focal points in Riemannian foliations”. In: *Indiana Univ. Math. J.* 35.2 (1986), pp. 321–331.

[HR] G. Habib and K. Richardson. “Modified differentials and basic cohomology for Riemannian foliations”. In: *J. Geom. Anal.* 23.3 (2013), pp. 1314–1342.

[IT] K. Ishitoya and H. Toda. “On the cohomology of irreducible symmetric spaces of exceptional type”. In: *J. Math. Kyoto Univ.* 17.2 (1977), pp. 225–243.

[Klei] F. Klein. *Vergleichende Betrachtungen über neuere geometrische Forschungen.* Programm zum Eintritt in die philosophische Facultät und den Senat der Universität zu Erlangen. Verlag A. Deichert. 1872.

[KN] S. Kobayashi and K. Nomizu. *Foundations of differential geometry. Vol. II.* Wiley Classics Library. Reprint of the 1969 original, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1996, pp. xvi+468.

[Knap] A. W. Knapp. *Lie groups beyond an introduction.* Vol. 140. Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 1996, pp. xvi+604.

[Koni] M. Konishi. "On manifolds with Sasakian 3-structure over quaternion Kaehler manifolds". In: *Kōdai Math. Semin. Rep.* 26 (1974/75), pp. 194–200.

[KT1] F. W. Kamber and P. Tondeur. "Harmonic foliations". In: Lecture Notes in Math. 949 (1982), pp. 87–121.

[KT2] F. W. Kamber and P. Tondeur. "Infinitesimal automorphisms and second variation of the energy for harmonic foliations". In: *Tohoku Math. J. (2)* 34.4 (1982), pp. 525–538.

[KTT] F. W. Kamber, P. Tondeur, and G. Tóth. "Transversal Jacobi fields for harmonic foliations". In: *Michigan Math. J.* 34.2 (1987), pp. 261–266.

[Kuo] Y.-y. Kuo. "On almost contact 3-structure". In: *Tohoku Math. J. (2)* 22 (1970), pp. 325–332.

[Mil] J. W. Milnor. "Curvatures of left invariant metrics on Lie groups". In: *Adv. Math.* 21 (1976), pp. 293–329.

[Pete] P. Petersen. *Riemannian geometry.* Second. Vol. 171. Graduate Texts in Mathematics. Springer, New York, 2006, pp. xvi+401.

[Ros] L. Roschig. *A Bochner Technique For Foliations With Non-Negative Transverse Ricci Curvature.* 2023. arXiv: 2301.05914.

[SH] S. Sasaki and Y. Hatakeyama. "On differentiable manifolds with contact metric structures". In: *J. Math. Soc. Japan* 14 (1962), pp. 249–271.

[Ste] L. Stecker. "On 3-(α, δ)-Sasaki Manifolds and Their Canonical Submersions". PhD thesis. Phillipps-Universität Marburg, Nov. 2021.

[Tapp] K. Tapp. *Matrix groups for undergraduates.* Second. Vol. 79. Student Mathematical Library. AMS, Providence, RI, 2016, pp. viii+239.

[Tond] P. Tondeur. *Geometry of foliations.* Vol. 90. Monographs in Mathematics. Birkhäuser Verlag, Basel, 1997, pp. viii+305.

[Udri] C. Udrişte and C. Udrişte. "Structures presque coquaternioniennes". In: *Bull. Math. Soc. Sci. Math. R. S. Roumanie (N.S.)* 13(61).4 (1969), 487–507 (1970).

[Weyl] H. Weyl. *Symmetry.* Princeton: Princeton University Press, 2017, p. 5.

[Wil] E. N. Wilson. "Isometry groups on homogeneous nilmanifolds". In: *Geom. Dedicata* 12 (1982), pp. 337–346.

[Wolf] J. A. Wolf. "Complex homogeneous contact manifolds and quaternionic symmetric spaces". In: *J. Math. Mech.* 14 (1965), pp. 1033–1047.

[YB] K. Yano and S. Bochner. *Curvature and Betti numbers.* Annals of Mathematics Studies, No. 32. Princeton University Press, Princeton, N. J., 1953, pp. ix+190.