On the representation theory of braided Hopf-algebras.

Dissertation

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1 Introduction.

1.1 Abstract.

Since introduced in the late nineties in [FK99], Fomin-Kirillov algebras have earned a great amount of interest in the field of abstract algebras research.

Their connections to algebra combinatorics have been considered in [BLM16], [MPP14], [Pos99] and [GR97]. While that to Hopf algebras and Nichols algebras in [Gra], [AM03], [FP00] and [MS00] among others. Moreover, the algebras have made many interesting appearances in the field of Quantum groups theory [PV16], noncommutative geometry as in [Maj17] and [Maj19]. Some very interesting generalization approaches were taken in [Baz06] and [Lau16], also in what is commonly known as Hecke-Hopf-algebras [BK19].

Motivated by recent discoveries introduced by I. Heckenberger, L. Vendramin [HV18] and K. Wolf [Wol], this thesis will be addressing the topic of representation theory of PBW-deformations of Fomin-Kirillov algebras.

In particular, we shall utilize Gabriel Theorem in the purpose of developing an algebraic presentation in terms of path algebras for the special case of $n = 4$.

Furthermore, we dedicate some of our attention to studying the special connections Fomin-Kirillov subalgebras have with Iwahori-Hecke algebra and examine the consequential properties this has on other related structures.

1.2 The frame work.

Note 1. Here and throughout this work, K denotes a field of character not equal to 2. Furthermore, we simply denote the space of $n \times n$ K-matrices by K^n.

let n be a positive integer, we denote the permutations group on n letters by $\mathbb{S}_n$. Further, we set $[n] \coloneqq \{1, \cdots, n\}$.

The story so far: A motivation.

In the context of studying Schubert calculus, Fomin and Kirillov introduced $\mathcal{E}_n$ a family of quadratic K-algebras, that contains a commutative subalgebra isomorphic to the cohomology ring of a flag manifold. Commonly known as Fomin-Kirillov algebras, the authors defined:

Definition 1.1 ([FK99] Definition 2.1)**.** Let $n \geq 3$ be a positive integer. $\mathcal{E}_n$ is the quadratic K-algebra generated by x_{ij} for distinct $i, j \in [n]$, subject the following relations:

$$x_{ij} + x_{ji} = 0 \mid i, j \in [n] \text{ distinct}, \tag{1a}$$
$$x_{ij}^2 = 0 \mid i, j \in [n] \text{ distinct}, \tag{1b}$$
$$x_{ij}x_{kl} - x_{kl}x_{ij} = 0 \mid i, j, k, l \in [n] \text{ distinct}, \tag{1c}$$
$$x_{ij}x_{jk} + x_{jk}x_{ki} + x_{ki}x_{ij} = 0 \mid i, j, k \in [n] \text{ distinct}. \tag{1d}$$

Remark 1. While on the surface, it is rather straightforward to present $\mathcal{E}_n$ in terms of generators and relations, some of the structure's elementary properties remain challenging to approach, most interesting of such is that of dimensionality, as it is well-known that:

$$dim_K \mathcal{E}_n = \begin{cases} 2.3! & \mid n = 3, \\ 4!^2 & \mid n = 4, \\ 5!^2 4!^2 & \mid n = 5. \end{cases}$$

And conjectured to be infinite otherwise.

Moreover, the nature of $\mathcal{E}_n$ as a braided Hopf algebra over the symmetric group $\mathbb{S}_n$ indicates a strong connection to Nichols algebras over braided vector spaces, as it was proved that $\mathcal{E}_n$ is a Nichols-algebra for $n \leq 4$ in [MS00], and $n = 5$ in [Gra]. A statement that is conjectured to be true for $n \geq 6$ as well in [MS00], [Maj19].

It is also noteworthy that the algebra $\mathcal{E}_n$ happens to share some distinctive properties with other types of algebras, most famous of which is that of preprojective type of A_{n-1}, which shares the same number of indecomposable modules with $\mathcal{E}_n$ for $n \leq 5$ and is known to be of infinite representation-type otherwise[1].

Remark 2. We highlight that from the point of view of graded algebras, $\mathcal{E}_n$ remains a highly interesting candidate of an algebraic structure that is both naturally and symmetrically graded, moreover, the unique action of the symmetric group $\mathbb{S}_n$ on $\mathcal{E}_n$ defined as:

$$\sigma(xy) = (\sigma x)(\sigma y) \mid x, y \in \mathcal{E}_n,$$

where for all $\sigma \in \mathbb{S}_n$, we have:

$$\sigma(x_{ij}) = x_{\sigma(i)\sigma(j)} \mid i \neq j,$$

validates- among other reasons- the study of $\mathcal{E}_n$ from the viewpoint of PBW deformations. We recall that a PBW deformation of a graded algebra A is a filtered algebra D such that the associated graded algebra of D is isomorphic to A.

Definition 1.2. Let $\alpha_1, \alpha_2 \in K$. The *deformed Fomin-Kirillov* algebra, denoted by $\mathcal{D}_n(\alpha_1, \alpha_2)$, is the quadratic K-algebra generated by x_{ij} for distinct $i, j \in [n]$ subject the following relations:

$$x_{ij} + x_{ji} = 0 \mid i, j \in [n] \text{ distinct}, \tag{2a}$$
$$x_{ij}^2 = \alpha_1 \mid i, j \in [n] \text{ distinct}, \tag{2b}$$
$$x_{ij}x_{kl} - x_{kl}x_{ij} = 0 \mid i, j, k, l \in [n] \text{ distinct}, \tag{2c}$$
$$x_{ij}x_{jk} + x_{jk}x_{ki} + x_{ki}x_{ij} = \alpha_2 \mid i, j, k \in [n] \text{ distinct}. \tag{2d}$$

[1]We recognize this as Majids' conjecture, which while not having a precise expression, nonetheless highlights that the numerology is not accidental, further details can be explored in [BFZ05], [Wil] as well as in [GLS].

Remark 3. Computer based calculation has established that the algebra $\mathcal{D}_4(\alpha_1, \alpha_2)$ is $4!^2$-dimensional for all α_1, α_2.

PBW-deformations and semisimplicity. In 2018, motivated by understanding Nichols and Fomin-Kirillov algebras by means of PBW-deformations, Heckenberger and Vendramin established a framework objected to the classification and the study of representation theory of non-semisimple deformations of Fomin-Kirillov algebras. In particular, the authors proved:

Theorem 1.1. *[HV18, Theorem 2.11] The algebra $\mathcal{D}_3(\alpha_1, \alpha_2)$ is semi-simple if and only if:*

$$(3\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2) \neq 0,$$

in this case $\mathcal{D}_3(\alpha_1, \alpha_2) \cong (K^2)^3$

Further, solved:

Proposition 1.2. *[HV18, Theorems 2.15, 2.16] The following hold:*

1. *The algebra $\mathcal{D}_3(\alpha, -\alpha)$ is isomorphic to the product of three copies of the preprojective algebra of type A_2.*
2. *The algebra $\mathcal{D}_3(\alpha, 3\alpha)$ is isomorphic to the path algebra of the double Kronecker quiver bounded by the relations of the coinvariant ring of $\mathbb{S}_3$.*

Later that year, Wolf in [Wol] continued the study by examining the case of $\mathcal{D}_4(\alpha_1, \alpha_2)$, where it was proved that the algebra $\mathcal{D}_4(\alpha_1, \alpha_2)$ is semisimple if:

$$\alpha_1(3\alpha_1 - \alpha_2)(\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2) \neq 0.$$

Further, he *conjectured* that[2]:

Conjecture 1.1. [Wol, Corollary 2.32] The algebra $\mathcal{D}_4(\alpha_1, \alpha_2)$ is semisimple if and only if:

$$(\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2) \neq 0.$$

Remark 4. It has been calculated that the radical of $\mathcal{D}_4(\alpha, -\alpha)$ is generated by the commutator, that is, $\sigma[x_{12}, x_{13}]$ for all $\sigma \in \mathbb{S}_4$, while that of $\mathcal{D}_4(\alpha, \alpha)$ is generated by:

$$\sigma(x_{12}x_{13} + x_{12}x_{14} + x_{12}x_{23} + x_{13}x_{23} + x_{14}x_{12} + \alpha_1) \mid \sigma \in \mathbb{S}_4,$$

where both ideals are of 552-dimensional, and their corresponding quotient algebras are of 24-dimension.

[2]We remark here that recent computer based calculations verified Wolf's conjecture to be true, and this work assumes so as well.

The story henceforth: Main results.

After recalling some of the important prerequisites for this thesis in the next chapter, the body of work is presented and organize in two related but separate parts.

The first part deals with representation theory of non-semisimple $\mathcal{D}_4(\alpha_1, \alpha_2)$ and that of generic n in some cases from the viewpoint of Gabriel theorem.

As seen, we have couple of cases to discuss, first of which is that of $\mathcal{D}_n(\alpha, -\alpha)$ which will be proven to be basic and connected, hence admits an ordinary quiver presentation:

Theorem 3.11. The algebra $\mathcal{D}_n(\alpha, -\alpha)$ admits an ordinary quiver presentation of the form (Q_0, Q_1, s, t) where:

1. The vertices set is $\mathbb{S}_n$.
2. Let $\sigma, \tau \in \mathbb{S}_n$, there exists an arrow from σ to τ if $\tau = \overline{g}\sigma$ where $\overline{g}$ denotes a non-simple transposition of $\mathbb{S}_n$.

This in particular, enable us to discuss the application of Gabriel Theorem in the special case of $n = 4$. Indeed we prove that:

Theorem 3.19. The algebra $\mathcal{D}_4(\alpha, -\alpha)$ admits a projective module decomposition:

$$\mathcal{D}_4(\alpha, -\alpha) = \bigoplus_{\sigma \in \mathbb{S}_4} e_\sigma \mathcal{D}_4(\alpha, -\alpha),$$

where each copy of the projective module $e_\sigma \mathcal{D}_4(\alpha, -\alpha)$ is isomorphic to $\mathcal{N}_4$ the nil-Coxeter algebra associated with $\mathbb{S}_4$.

Next, we will shift our focus to $\mathcal{D}_4(\alpha, \alpha)$, which is not as straightforward as the previous case.

Indeed we start by proving that it is not basic, which as expected, complicates the discussion of the topic from the quiver point of view. Nonetheless yields $\mathcal{D}_4^b(\alpha, \alpha)$ an associated basic algebra, which is by default Morita equivalent to $\mathcal{D}_4(\alpha, \alpha)$.

We shall show that this associated version is connected and admits an ordinary quiver presentation of the form:

Theorem 3.28. The algebra $\mathcal{D}_4^b(\alpha, \alpha)$ admits an ordinary quiver presentation of the form (Q_0, Q_1, s, t) where:

1. The vertices set is $\mathbb{S}_3 = \langle s_2, s_3 \rangle$.
2. Let $\sigma, \tau \in \mathbb{S}_3$, the number of arrows from σ to τ is n where:

$$n = \begin{cases} 1 & \mid \tau = s_3\sigma, \\ 2 & \mid \tau = s_2\sigma, \\ 0 & \mid \text{ otherwise.} \end{cases}$$

Which enables us to propose:

Proposition 3.33. The algebra $\mathcal{D}_4^b(\alpha,\alpha)$ admits a projective module decomposition:

$$\mathcal{D}_4^b(\alpha,\alpha) = \bigoplus_{\sigma\in\mathbb{S}_3} e_\sigma \mathcal{D}_4^b(\alpha,\alpha),$$

where each copy of the projective module $e_\sigma\mathcal{D}_4^b(\alpha,\alpha)$ is isomorphic to the quotient algebra $K\langle s_1,s_2,s_3\rangle/ker(\pi)$, where:

$$\begin{aligned} ker(\pi) = \{&s_1^2 + q_1(s_1s_3s_2)^2,\\ &s_2^2 + q_2(s_2s_3s_1s_2) + q_3(s_1s_3s_2)^2,\\ &s_3^2 + q_4(s_1s_3s_2)^2,\\ &s_1s_3 - s_3s_1 + q_5(s_1s_3s_2)^2,\\ &s_2s_1s_2 - s_1s_2s_3 - s_3s_2s_1,\\ &s_2s_3s_2 - s_1s_2s_1 + s_3s_2s_3.\}. \end{aligned}$$

For some K-polynomials q_i.

The second part is dedicated to the study of some interesting Dynkin graph based subalgebras of $\mathcal{D}_n(\alpha_1,\alpha_2)$.

Initially, we will consider $\Lambda_n(\alpha_1,\alpha_2)$ the $\mathcal{D}_n(\alpha_1,\alpha_2)$-subalgebra generated by x_{ij} for $(ij) \in A_n$ the Dynkin quiver of type A_n, we will prove that:

Theorem 4.2. Let $n \geq 3$ be a positive integer. There exists some K-parameter q so that the following hold:

$$\Lambda_n(\alpha_1,\alpha_2) \cong H_q(n),$$

where $H_n(q)$ denotes the generic Iwahori-Hecke algebra on one parameter.

This result not only highlights the Berenstein and Kazhdan construction of new Hopf algebras that contain Hecke algebras as (left) coideal subalgebras but further lays the foundation for the study of other interesting examples of $\mathcal{D}_n(\alpha_1,\alpha_2)$-subalgebras.

Indeed, we start with formalizing a family of algebras that is isomorphic $\Lambda_n(\alpha_1,\alpha_2)$ and parameter compatible as well. This enables us to prove that for $\Delta_4(\alpha_1,\alpha_2)$ the $\mathcal{D}_4(\alpha_1,\alpha_2)$-subalgebra generated by x_{ij} for $(ij) \in D_4$ the Dynkin quiver of type D_4:

Proposition 4.22. The algebra $\Delta_4(\alpha_1,\alpha_2)$ is semisimple if and only if:

$$(3\alpha_1 - \alpha_2)(\alpha_1^2 - \alpha_2^2) = 0.$$

1.3 A note for the reader.

This work is mostly suited for an audience fairly familiar with basic notions and techniques of algebras and their representations.

In particular, we assume that the reader has a firm understanding of basic concepts such as modules, semisimplicity, decomposability, homological algebras, etc. These concepts will not be repeated for organizational reasons and we highly recommend [IHJ20], [AY17] and [Pie82] for an excellent source of references.

We further highlight that the theory of symmetric groups and their representations will play a central role in our discussion, to which we recommend [AB10], [CSST10] and [KT07] as well.

The first part of this work has been written by utilizing basic elements of representation theory via quivers and Gabriel's Theorem. Due to their importance we shall recall them briefly in the Preliminary chapter. We refer interested readers to [ASA06].

Finally, as one might suspect, some parts of this work are combinatorial by nature and require computations in low-dimensionality. We make a deliberate effort to verify every combinatorial claim, but should repetitiveness occurs, we shall provide the reader with pointers and sketches, to which we thank the reader for their understanding.

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Finally, I dedicate this work to my parents: Ghassan and Rim. Your love, faith and encouragement has never failed me, and I am forever thankful for you.

2 Preliminary.

By a K-algebra, we mean an associative unital algebra over K. In this section of our discussion, A denotes a finite-dimensional K-algebra.

Note 2. Let ρ, ρ' be two A-representations. One may view the space $Ext_A^1(\rho, \rho')$ as equivalence classes $Z^1(\rho, \rho')/B^1(\rho, \rho')$, where: $Z^1(\rho, \rho')$ denotes the space of (1)-cocycles, that is:

$$Z^1(\rho, \rho') = \{f : A \to Hom_K(\rho, \rho') \mid f(xy) = \rho(x)f(y) + f(x)\rho'(y)\}$$

and $B^1(\rho, \rho')$ denotes the space of coboundaries.

Note 3. We say that a K-algebra A is connected if it is not isomorphic to a direct product of two non-trivial algebras.

We also denote A's Jacobson's radical, that is, the intersection of all maximal ideals of A by $radA$.

Note 4. Let G_n denotes the finite group generated by $\{s_1, \cdots, s_n\}$ subject to the following set of relations:

$$\begin{aligned} s_i^2 = 1 &\mid i \in [n-1], \\ s_i s_j s_i - s_j s_i s_j = 0 &\mid |i-j| = 1, \\ s_i s_j - s_j s_i = 0 &\mid |i-j| \geq 2. \end{aligned}$$

Theorem 2.1. *Let $n \geq 2$ be a positive integer. There exists a group isomorphism $G_n \cong \mathbb{S}_n$.*

Definition 2.1. Let $n \geq 2$ be a positive integer, by $\mathcal{N}_n$ we denote the nil-Coxeter algebra of type A_{n-1}, that is, the K-algebra generated by s_i for $i \in [n-1]$ subject the following set of relations:

$$\begin{aligned} s_i^2 = 0 &\mid i \in [n-1], \\ s_i s_j s_i - s_j s_i s_j = 0 &\mid |i-j| = 1, \\ s_i s_j - s_j s_i = 0 &\mid |i-j| \geq 2. \end{aligned}$$

Remark 5. The notion of nil-Coxeter algebras is wildly attributed to [FS94] and has been studied in [Yan15] and generalized in [Kha17].

Note 5. We say that a finite-dimensional K-algebra A is representation-finite if the number of the isomorphism classes of indecomposable A-module is finite, and A-modules considered here are finite-dimensional over K. Otherwise, A is said to be representation-infinite.

Theorem 2.2. *[Dro80] Representation-infinite algebras are either tame or wild over algebraically closed fields.*

Lemma 2.3. *[Yan15, Theorem 2. III] The algebra $\mathcal{N}_n$ is of representation type wild for $n \geq 4$.*

2.1 Elementary algebra.

Graded and filtered algebras.

Definition 2.2. Let A be a filtered algebra, that is, an algebra with a family of subspaces $\{F_{i+1} \subseteq F_i \mid i \geq 0\}$ such that:

1. $1 \in F_0$,
2. $F_i F_j \subseteq F_{i+j}$,
3. $\cup F_n = A$.

We define the associated graded algebra of A, denoted by grA by setting $(grA)_n = F_n/F_{n+1}$ and $grA = \bigoplus (grA)_n$.

Example 2.1. Let A be a finite-dimensional K-algebra. Then A is radically filtered as follows:

$$F_r = (radA)^r \subset \cdots \subset F_2 = (radA)^2 \subset F = radA \subset F_0 = A,$$

where r is the minimal positive integer such that $F_{r+1} = 0$[3]

Note 6. Here and throughout this work grA denotes the associated graded algebra of a K-algebra A with respect to the radical filtration.

Definition 2.3. Let M, N be filtered algebras, given $\phi : M \to N$ a filtered homomorphism, that is, $\phi(M_j) \subseteq \phi(M) \cap N_j$. If it happens that $\phi(M_j) = \phi(M) \cap N_j$ for each j applicable, then ϕ is called strict.

Example 2.2. Let N be a filtered algebra. If $\alpha : M \to N$ is an arbitrary homomorphism and M is given the induced filtration $M_j = \alpha^{-}(\alpha(M) \cap N_j)$ then α is a strict filtered homomorphism. Similarly, for α surjective and if N is given the induced filtration $N_j = \alpha(M_j)$, then α is strict as well.

Corollary 1. *[MRS01, Corollary 6.14]. Let $\phi : M \to N$ be a filtered homomorphism. Then $gr\phi$ is injective (surjective) if and only if ϕ is injective (surjective) and ϕ is strict.*

Idempotents and indecomposable decompositions.

Definition 2.4. An element $e \in A$ is said to be an idempotent if $e^2 - e = 0$.

Note 7. Let $e, e_1, e_2 \in A$ be idempotents, we say:

1. e is central if $xe - ex = 0$ for all $x \in A$,
2. e_1, e_2 are orthogonal if $e_1 e_2 = e_2 e_1 = 0$,
3. e is primitive if e cannot be written as a sum of nonzero orthogonal A-idempotents.

[3]such r exists for Jacobson radicals are nilpotent

Remark 6. Since the algebra A is finite-dimensional, as a module it admits a direct sum decomposition of the form:

$$A = \bigoplus_{i=1}^{n} P_i,$$

where for $i \in [n]$, P_i are indecomposable ideals of A.

One may see that $P_i = e_i A$ for $\{e_i \mid i \in [n]\}$ are primitive pairwise-orthogonal idempotents of A such $1 = \sum_{i=1}^{n} e_i$.

Conversely every set of primitive pairwise-orthogonal idempotents of A induces a decomposition of such form.

This decomposition is called an indecomposable decomposition of A and such a set is often called a complete set of primitive pairwise-orthogonal idempotents.

Definition 2.5. Let M be a generic A-module we define the top of M as:

$$topM := M/radM.$$

Remark 7. It is well-known that for e an A-idempotent, the $A/radA$-module $top(eA)$ is simple and $rad(eA) = e(radA) \subseteq eA$ is the unique maximal proper submodule of eA.

Corollary 2. *[ASA06, Corollary 5.17] Suppose that $A = \bigoplus_{i=1}^{n} P_i$ is a decomposition of A into indecomposable modules. The following hold:*

1. *Every simple A-module is isomorphic to one of the modules $S_i = top(e_i A)$ for some $i \in n$.*
2. *Every indecomposable projective A-module is isomorphic to one of the modules $P_i = e_i A$ for some $i \in n$. Moreover, $e_i A \cong e_j A$ if and only if $S_i \cong S_j$.*

Basic algebras.

Definition 2.6. A finite-dimensional K-algebra A is said to be basic if $e_i A \ncong e_j A$ for all distinct i, j and $\{e_i \mid i \in [n]\}$ a complete set of primitive orthogonal A-idempotents.

Proposition 2.4. *[ASA06, Proposition 6.2] The following hold:*

1. *A finite-dimensional K-algebra A is basic if and only if the algebra $A/radA$ is isomorphic to a finite product of copies of K.*
2. *Every simple module over a basic K-algebra is 1-dimensional.*

Definition 2.7. Let $\{e_i \mid i \in [n]\}$ be a complete set of primitive orthogonal A-idempotents. A basic algebra associated with A is defined as:

$$A^b = e_A A e_A \mid e_A = \sum_{i=1}^{a} e_{j_i},$$

where e_{j_i} are chosen in a way such that $e_{j_i} \ncong e_{j_t}$ for distinct i, t and each module $e_S A$ is isomorphic to one of the modules $e_{j_i} A$ for some $i \in [a]$.

Remark 8. One may verify that A^b -the basic algebra associated with A- does not depend on the choice of the set of A-idempotent.

Lemma 2.5. *[ASA06, Corollary 6.10] Let A^b be a basic K-algebra associated with A. The algebra A^b is basic and its modules category is K-linearly equivalent to that of A.*

Remark 9. The previous equivalence between modules categories is often known as Morita equivalence, which is given by:

$$modA \xrightarrow[\simeq]{\mathfrak{F}} modA^b \xrightarrow[\simeq]{\mathfrak{G}} modA$$

where $\mathfrak{F}$ (respectively, $\mathfrak{G}$) is a functor that yields an equivalence, that is:

1. full (respectively faithful), that is, for any two A-modules M_1, M_2, the map induced by $\mathfrak{F}$:

$$Hom_{modA}(M_1, M_2) \to Hom_{modA^b}(\mathfrak{F}(M_1), \mathfrak{F}(M_2)),$$

   is surjective (respectively injective).

2. dense, that is, given an $modA^b$-module M, then M is isomorphic to $\mathfrak{F}(M')$ for some A-module M'.

In particular, such equivalence preserves simplicity and exactness.

2.2 Quivers and path algebras.

Definition 2.8. A quiver is a quadruple $Q = (Q_0, Q_1, s, t)$ with Q_0 and Q_1 finite sets and two maps $s, t : Q_1 \to Q_0$. The elements of Q_0 and Q_1 are called vertices and arrows of Q respectively. We say an arrow α in Q_1 starts in $s(\alpha)$ and terminates in (or targets) $t(\alpha)$.

Note 8. A quiver Q is said to be finite if both Q_0, Q_1 are finite sets.

Example 2.3. For $n \geq 2$. The Dynkin quiver of type A_n is of the form:

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} n$$

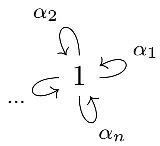
In particular, one sees that $s(\alpha_i) = i$ and $t(\alpha_i) = i + 1$ for all $i \in [n]$.

Note 9. Let Q be a quiver. A path of length $m \geq 1$ in Q is a tuple $(\alpha_1, \cdots, \alpha_m)$ of arrows of Q such that $s(\alpha_i) = t(\alpha_{i+1})$ for all $i \in [m-1]$, we write such path as $\alpha_1 \cdots \alpha_m$ if no misunderstanding occurs.

Additionally, for each vertex i of Q there exists a path e_i of trivial length such that $s(e_i) = t(e_i) = i$.

Note 10. A path of length $m \geq 1$ is called a cycle whenever its source and target coincide. A cycle of length 1 is called a loop. A quiver is called acyclic if it contains no cycles.

Example 2.4. For $n \geq 1$. The n-loop quiver is of the form:



Definition 2.9. Let Q be a quiver. The path algebra KQ of Q is the K-algebra whose underlying K-vector space has as its basis the set of all Q-paths of length $l \geq 0$ in Q such that the product of two basis vectors $\alpha_1 \cdots \alpha_m$ and $\beta_1 \cdots \beta_{m'}$ is trivial if $t(\alpha_m) \neq s(\beta_1)$ and equal to the composed path $\alpha_1 \cdots \alpha_m \beta_1 \cdots \beta_{m'}$ otherwise. The product of basis elements is then extended to arbitrary elements of KQ by distributivity.

Remark 10. By its definition. There is a direct sum decomposition:

$$KQ = \bigoplus_{i=1} KQ_i,$$

of all K-vector space KQ, where, for each $m \geq 0$, KQ_m is the subspace of KQ generated by the set Q_m of all paths of length m.

It is rather easy to see that:

$$(KQ_{m_1})(KQ_{m_2}) \subseteq KQ_{m_1+m_2} \mid m_1, m_2 \geq 0,$$

as the product in KQ of a path of length m_1 by a path of length m_2 is either 0 or a path of length $m_1 + m_2$.

This is often expressed by saying that the decomposition defines a grading on KQ or that KQ is a graded K-algebra.

Lemma 2.6. *[ASA06, Lemma 1.4] Let Q be a quiver and KQ be its path algebra. Then the following hold:*

1. *KQ is an associative algebra,*
2. *KQ has an identity element if and only if Q_0 is finite,*
3. *KQ is finite-dimensional if and only if Q is finite and acyclic.*

Corollary 3. *[ASA06, Corollary 1.5] Let Q be a finite-quiver. Then KQ has an identity element of the form:*

$$1 = \sum_{i \in Q_0} e_i,$$

furthermore, the set $\{e_i \mid i \in Q_0\}$ of all trivial length paths forms a complete set of primitive orthogonal idempotents of KQ.

Definition 2.10. Let Q be a finite and connected quiver. The two-sided ideal of the path algebra KQ generated as an ideal by the arrows of Q is called the arrow ideal of KQ and denoted by R_Q.

Definition 2.11. A two-sided ideal I of KQ is said to be admissible if there exists some $m \geq 2$ such that:

$$R_Q^m \subseteq I \subseteq R_Q^2.$$

Note 11. If I is an admissible ideal of KQ, the quotient algebra KQ/I is said to be a bound quiver algebra.

Definition 2.12. Let Q be a quiver. A relation in Q with coefficients in K is a K-linear combination of paths of length at least two having the same starting and terminating vertex.

Lemma 2.7. *[ASA06, Lemma 2.10] Let Q be a finite quiver, R_Q be the arrow ideal of KQ, and I an admissible ideal of KQ. Then:*

$$rad(KQ/I) = R_Q/I.$$

Corollary 4. *[ASA06, Corollary 2.11] For each $m \geq 1$, we have:*

$$(radKQ/I)^m = (R_Q/I)^m.$$

Gabriel's theorem and ordinary quivers.

Definition 2.13. Let A be a basic and connected finite-dimensional K-algebra and $\{e_i \mid i \in [n]\}$ a complete set of primitive orthogonal A-idempotents. The (ordinary) quiver of A, denoted by Q_A, is defined as follows:

1. The vertices of Q_A are numbers $\{1, \cdots, n\}$ which are in bijective correspondence with the A-idempotents e_i for $i \in [n]$.
2. Given two vertices $i, j \in Q_A$, the arrows $\alpha : i \to j$ are in bijective correspondence with the vectors in a basis of the K-vector space $e_i(radA/radA^2)e_j$.

Theorem 2.8. *[ASA06, Theorem 3.7] Let A be a basic and connected finite-dimensional K-algebra. There exists an admissible ideal I of KQ_A such that $A \cong KQ_A/I$.*

Note 12. The previous theorem is commonly known as Gabriel's Theorem.

Remark 11. In the purpose of proving Gabriel's theorem, one sets for each arrow $\alpha : i \to j$ some $x_\alpha \in radA$ chosen so that $\{x_\alpha + (radA)^2 \mid \alpha : i \to j\}$ forms a basis of $e_i(radA/(radA)^2)e_j$. To this, one considers two morphisms:

1. $\phi_0 : (Q_A)_0 \to A$ the map defined by:

$$\phi_0(i) = e_i \mid i \in (Q_A)_0.$$

2. $\phi_1 : (Q_A)_1 \to A$ the map defined by:

$$\phi_1(\alpha) = x_\alpha \mid \alpha \in (Q_A)_1.$$

The universal property of path algebras implies the existence of a unique K-algebra homomorphism $\phi : KQ_A \to A$ that extends two morphisms ϕ_0, ϕ_1, which we call Gabriel's theorem morphism.

The theorem is proven once shown that ϕ is surjective with an admissible kernel.

Lemma 2.9. *[ASA06, Lemma 2.12] Let $A = KQ/I$ be a bounded path algebra and let $i, j \in Q_0$. The following hold:*

1. *There exists an isomorphism of K-vector spaces:*

$$Ext^1_A(S_i, S_j) \cong e_i(radA/radA^2)e_j.$$

2. *The number of arrows in Q starting in i terminating in j is equal to the dimension of $Ext^1_A(S_i, S_j)$*

3 On the deformed Fomin-Kirillov-algebras.

3.1 The representation theory of $\mathcal{D}_n(\alpha, -\alpha)$

Note 13. For convenience, we denote $\Lambda := \mathcal{D}_n(\alpha, -\alpha)$, we further normalize the K-parameter α to 1_K.

Lemma 3.1. *Let $\sigma \in \mathbb{S}_n$. The algebra homomorphism:*

$$\begin{aligned} \rho_\sigma &: \Lambda \to K \\ x_{ij} &\mapsto \rho_\sigma(x_{ij}) = \begin{cases} +1 \mid \sigma(i) < \sigma(j), \\ -1 \mid \sigma(i) > \sigma(j). \end{cases} \end{aligned}$$

is a well defined 1-dimensional representation of Λ.

Proof. The proof follows by verifying that subjecting ρ_σ to the defining relations of Λ yields no contradictions in K.

Indeed as (2b) and (2c) hold directly, one only has to check that for distinct $i, j, k \in [n]$, then:

$$\begin{aligned} -1 &= \rho_\sigma(x_{ij}x_{jk} + x_{jk}x_{ki} + x_{ki}x_{ij}) \\ &= \rho_\sigma(x_{ij})\rho_\sigma(x_{jk}) + \rho_\sigma(x_{jk})\rho_\sigma(x_{ki}) + \rho_\sigma(x_{ki})\rho_\sigma(x_{ij}), \end{aligned}$$

which is easily verified as assuming the distinction of $i, j, k \in [n]$, implies that the inequality regarding $\sigma(i), \sigma(j), \sigma(k)$ has exactly one of six possibilities:

$$\begin{aligned} &\sigma(i) < \sigma(j) < \sigma(k), && \sigma(i) < \sigma(k) < \sigma(j), \\ &\sigma(j) < \sigma(i) < \sigma(k), && \sigma(j) < \sigma(k) < \sigma(i), \\ &\sigma(k) < \sigma(i) < \sigma(j), && \sigma(k) < \sigma(j) < \sigma(i). \end{aligned}$$

□

Lemma 3.2. *Let ρ a 1-dimensional Λ-representation. Then there exists $\sigma \in \mathbb{S}_n$ such that $\rho = \rho_\sigma$.*

Proof. Let ρ be a 1-dimensional Λ-representation, (2b) implies that $\rho(x_{ij}) = \pm 1$.

Furthermore, for distinct $i, j, k \in [n]$, then (2d) implies that one of the following possibilities occurs:

$$\begin{aligned} &+\rho(x_{ij}) = +\rho(x_{jk}) = +\rho(x_{ik}) = +1, && +\rho(x_{ij}) = -\rho(x_{jk}) = +\rho(x_{ik}) = +1, \\ &-\rho(x_{ij}) = +\rho(x_{jk}) = +\rho(x_{ik}) = +1, && +\rho(x_{ij}) = +\rho(x_{jk}) = -\rho(x_{ik}) = +1, \\ &+\rho(x_{ij}) = -\rho(x_{jk}) = -\rho(x_{ik}) = +1, && -\rho(x_{ij}) = -\rho(x_{jk}) = -\rho(x_{ik}) = +1. \end{aligned}$$

Which in and of itself asserts the claim. □

Remark 12. The previous lemma can be alternatively proven by setting: $l_i := |\mathbb{L}_i|$, $r_i := |\mathbb{R}_i|$, where:

$$\mathbb{L}_i := \{1 \leq j \leq i-1 \mid \rho(x_{ji}) = -1\} \subseteq \{0, \cdots, i-1\},$$
$$\mathbb{R}_i := \{i+1 \leq j \leq n \mid \rho(x_{ij}) = -1\} \subseteq \{0, \cdots, n-i\},$$

and defining the mapping σ on $[n]$ where $\sigma(i) = i + r_i - l_i$, the claim follows by showing that σ is indeed a permutation such that $\rho = \rho_\sigma$.

Lemma 3.3. *Let $\sigma, \tau \in \mathbb{S}_n$, then for all $x \in \Lambda$ we have:*

$$\rho_\sigma(\tau(x)) = \rho_{\sigma\tau}(x).$$

Proof. Since ρ_σ, ρ_τ and the group action of $\mathbb{S}_n$ are multiplicative, it is enough to verify the claim for a generator x_{ij} with $i, j \in [n]$ distinct, which hold directly since:

$$\rho_\sigma(\tau(x_{ij})) = \rho_\sigma(x_{\tau(i)\tau(j)}) = \rho_{\sigma\tau}(x_{ij}).$$

□

Theorem 3.4. *Given $\sigma, \tau \in \mathbb{S}_n$. If $\tau = \overline{g}\sigma$ where $\overline{g}$ denotes a non-simple transposition of $\mathbb{S}_n$, then $dim_K Ext^1_\Lambda(\rho_\sigma, \rho_\tau) = 1$, and 0 in any other case.*

Remark 13. In the purpose of proving Theorem 3.4, we start by utilizing Lemma 3.3, which implies that we may set $\sigma = e$ with no further restrictions.

Furthermore, we understand for $\tau \in \mathbb{S}_n$, that generic elements of the space of extensions $Ext^1_\Lambda(\rho_e, \rho_\tau)$ are of the form:

$$\begin{aligned} \rho &: \Lambda \to K^2 \\ x_{ij} &\mapsto \rho(x_{ij}) = \begin{bmatrix} \rho_e & f_{(e;\tau)} \\ 0 & \rho_\tau \end{bmatrix} (x_{ij}), \end{aligned}$$

such that for $f_{ij} = f_{(e;\tau)}(x_{ij})$, the following hold:

$$f_{ij}(1 + \rho_\tau(x_{ij})) = 0, \tag{3a}$$
$$f_{ij}(1 - \rho_\tau(x_{kl})) - f_{kl}(1 - \rho_\tau(x_{ij})) = 0, \tag{3b}$$
$$f_{ij}(\rho_\tau(x_{jk}) - 1) + f_{jk}(1 - \rho_\tau(x_{ik})) - f_{ik}(1 + \rho_\tau(x_{ij})) = 0, \tag{3c}$$
$$f_{ij}(1 - \rho_\tau(x_{ik})) + f_{jk}(\rho_\tau(x_{ij}) - 1) - f_{ik}(\rho_\tau(x_{jk}) + 1) = 0. \tag{3d}$$

where $i < j \in [n]$ in (3a) $i < j, k < l \in [n]$ distinct in (3b), and $i < j < k \in [n]$ in both (3c) and (3d).

Proposition 3.5. *If τ is a non-simple transposition, then:*

$$dim_k Ext^1_\Lambda(\rho_e, \rho_\tau) = 1.$$

Proof. Let $1 \leq s, t \leq n$, where $t > s + 1$, and assume that $\tau = (s, t)$ a non-simple transposition, then $\rho_\tau(x_{ij}) = -1$ if and only if:

$$(i = s \text{ and } j \leq t) \text{ or } (i \geq s \text{ and } j = t).$$

Therefore, for all $i < j$, (3a) implies that $f_{ij} = 0$ except those of the form:

$$f_{s(s+1)}, \cdots, f_{st} \text{ and } f_{(s+1)t}, \cdots, f_{(t-1)t}.$$

Further, given $1 \leq r \leq t - s - 1$, then $f_{s(s+r)} = f_{(s+r)t}$ implied by (3c) for $(i = s, j = s + r, k = t)$.

Similarly, for $2 \leq r \leq t - s - 1$, one obtains $f_{s(s+1)} = f_{s(s+r)}$ being implied by (3d) for $(i = s, j = s + 1, k = s + r)$.

Therefore, we are in the situation where:

$$f_{s(s+1)} = \cdots = f_{s(t-1)} = f_{(s+1)t} = \cdots = f_{(t-1)t}.$$

The proof then concludes by observing that up to a base change, one may assume that $f_{s(s+1)} = 0$. □

Remark 14. Given $\tau \in \mathbb{S}_n$ such that $\tau \neq \overline{g}$. Then τ has one of the following form:

1. $\tau = e$,
2. τ is a simple transposition,
3. τ is a cycle of length $p \geq 3$,
4. τ has at least two disjoint cycles of length $p, q \geq 2$.

Proposition 3.6. *If $\tau = e$. Then $dim_k Ext^1_\Lambda(\rho_e, \rho_\tau) = 0$.*

Proof. Assuming that $\tau = e$, this would imply that $\rho_\tau(x_{ij}) = 1$ for all distinct $1 \leq i < j \leq n$.

Now we have $f_{ij} = 0$ directly via (3a) which in and of itself holds the claim. □

Proposition 3.7. *If τ is a simple transposition. Then $dim_k Ext^1_\Lambda(\rho_e, \rho_\tau) = 0$.*

Proof. Assuming that $\tau = (s, s+1)$ for $s \in [n-1]$, this would imply that $\rho_\tau(x_{ij}) = 1$ for all $1 \leq i < j \leq n$ except for $(i, j) = (s, s+1)$.

Now we have $f_{ij} = 0$ for all $1 \leq i < j \leq n$ except for $(i, j) = (s, s+1)$ directly via (3a. Up to a base change, one may assume that $f_{s(s+1)} = 0$, which asserts the claim. □

Proposition 3.8. *If τ is a cycle of length $p \geq 3$. Then $dim_k Ext^1_\Lambda(\rho_e, \rho_\tau) = 0$.*

Proof. Assume that $\tau = (a_1, \cdots, a_p)$ is a cycle of length $p \geq 3$ where $1 \leq a_1, \cdots, a_p \leq n$, $a_i \neq a_j$ for all $i \neq j$ ordered such that $a_1 < a_j$ for all $2 \leq j \leq p$.

Since $a_1 < a_p$ and $\tau(a_p) = a_1 < a_2 = \tau(a_1)$, we see that $\rho_\tau(x_{a_1 a_p}) = -1$. We make a basis change so that $f_{a_1 a_p} = 0$.

For all $k < l$ such that $\{a_1, a_p, k, l\} \subset [n]$ distinct, we see by (3b) that $f_{kl} = 0$.

For $i < a_1$, we see that $\tau(i) = i < a_2 = \tau(a_1)$, which implies that $\rho_\tau(x_{ia_1}) = 1$, that is, $f_{ia_1} = 0$ implied by (3a).

Similarly, $\tau(i) = i < a_1 = \tau(a_p)$, which implies that $\rho_\tau(x_{ia_p}) = 1$, that is, $f_{ia_p} = 0$ implied by (3a).

For $a_p < i$, we see that $\tau(a_p) = a_1 < i = \tau(i)$, which implies that $\rho_\tau(x_{a_p i}) = +1$, that is, $f_{a_p i} = 0$ implied by (3a). Using this, we see that $f_{a_1 i} = 0$ being implied by (3d) for $(i = a_1, j = a_p, k = i)$.

Now if $a_p = a_1 + 1$, then all possible cases for i has been considered and the claim follows. If not, then for all $a_1 < i < a_p$ we have $\tau(i) > \tau(a_p)$ and hence $f_{a_1 i} = f_{ia_p}$ via (3c) for $(i = a_1, j = i, k = a_p)$.

We have two cases to consider here as well: either $\rho_\tau(x_{a_1 i}) = +1$, that is, $f_{a_1 i} = 0$ via (3a), effectively holding the claim, or $\rho_\tau(x_{a_1 i}) = -1$ and we have $\tau(a_1) > \tau(i) > \tau(a_p)$, in other words:

$$\tau(a_1) = a_2 > \tau(i) > a_1 = \tau(a_p),$$

and by differentiating possible orderings we have the following cases to discuss:

1. $a_1 < i < a_p < a_2$. At first, we have $a_p < a_2$ where $a_1 < \tau(a_2)$, therefore, $\rho_\tau(x_{a_p a_2}) = +1$ and $f_{a_p a_2} = 0$ via (3a).

   Now if we have $\tau(a_2) < a_2$, we see that $f_{a_1 a_2} = f_{ia_p}$ by (3b) to which applying (3d) for $(i = a_1, j = a_p, k = a_2)$ yields that $f_{a_1 a_2} = 0$, holding the claim.

   Otherwise, $\tau(a_2) > a_2$, in which case $\rho_\tau(x_{a_1 a_2}) = +1$, so that $f_{a_1 a_2} = 0$ via (3a), this implies that:

   $$f_{a_1 i}(\rho_\tau(x_{ia_2}) - 1) = 0,$$

   seen by (3c) for $(i = a_1, j = i, k = a_2)$.

   Now should $\tau(i) > \tau(a_2)$ be, then the claim falls, otherwise, $\rho_\tau(x_{ia_2}) = +1$ and $f_{ia_2} = 0$ via (3a).

   By repeating the previous argument finitely many times, one sees that there exists some $2 \le k \le p - 1$ such that $\tau(i) > \tau(a_k)$ since otherwise we get a contradiction to $\tau(a_p) = a_1 < a_j$ for all $2 \le j \le p$, in which case the claim hold by applying (3c) for $(i = a_1, j = i, k = a_k)$.

2. $a_1 < i < a_2 < a_p$. At first, we have $i < a_2$ to which one concludes that $f_{ia_2} = 0$ being implied by (3b).

   If $\tau(i) > a_3 = \tau(a_2)$, then (3c) for $(i = a_1, j = i, k = a_2)$ implies $f_{a_1 i} = 0$.

   Otherwise, $\tau(i) < a_3 = \tau(a_2)$ to which one sees that $f_{ia_2} = 0$ implied by (3b) which one uses in (3c) for $(i = i, j = a_2, k = a_p)$ to see $f_{a_2 a_p} = f_{ia_p}$, similarly, (3c) for $(i = a_1, j = i, k = a_2)$ yield $f_{a_1 i} = f_{a_1 a_2}$.

   Finally, (3d) for $(i = a_1, j = a_2, k = a_p)$ imply that:

   $$2f_{a_1 a_2} + f_{a_2 a_p}(\rho_\tau(x_{a_1 a_2}) - 1) = 0.$$

Now if $\rho_\tau(x_{a_1a_2}) = 1$ then the claim follows, otherwise, one argues as before on the existence of some $3 \leq k \leq p-1$ such that $a_1 < a_k$ and $\tau(a_1) < \tau(a_k)$, since otherwise we get a contradiction to $a_1 < a_j$ for all $2 \leq j \leq p$, in which case the claim hold by applying (3d) for $(i = a_1, j = a_k, k = a_p)$.

3. $a_1 < a_2 < i < a_p$. At first, we have $a_2 < i$ to which we hold that $f_{a_2i} = 0$ via (3b).

   If we have $\tau(a_2) = a_3 > \tau(i)$ to which one concludes that $f_{ia_p} = 0$ via (3d) for $(i = a_2, j = i, k = a_p)$.

   Otherwise, $\tau(a_2) = a_3 < \tau(i)$ to which one holds:

$$f_{a_1i}(1 + \rho_\tau(x_{a_1a_2})) = 0,$$

   seen by (3c) for $(i = a_1, j = a_2, k = i)$, and again, two cases to consider.

   Indeed should $\rho_\tau(x_{a_1a_2}) = +1$ be, the claim falls directly, otherwise, $a_2 > a_3$ to which one argues as before on the existence of some $3 \leq k \leq p-1$ such that $a_1 < a_k$ and $\tau(a_1) < \tau(a_k)$.

Therefore, with all possible cases considered for τ a cycle of length $p \geq 3$ we have $dim_k Ext^1_\Lambda(\rho_e, \rho_\tau) = 0$ as claimed. □

Proposition 3.9. *Let $p, q \geq 2$ and τ be given so that it has at least two disjoint cycles of length p, q. Then $dim_k Ext^1_\Lambda(\rho_e, \rho_\tau) = 0$.*

Proof. Assume that τ has disjoint cycles $(a_1, \cdots, a_p), (b_1, \cdots, b_q)$ for $p, q \geq 2$ ordered such that $a_1 < a_j$ for $2 \leq j \leq p$, $b_1 < b_j$ for $2 \leq j \leq q$ and $a_1 < b_1$.

We start with $a_1 < a_p$ and $\tau(a_p) = a_1 < a_2 = \tau(a_1)$ which implies that $\rho_\tau(x_{a_1a_p}) = -1$, similarly, $b_1 < b_q$ and $\tau(b_q) = b_1 < b_2 = \tau(b_1)$ which implies that $\rho_\tau(x_{b_1b_q}) = -1$.

For convenience of reference, we denote $a_1 = s, a_p = t, b_1 = s', b_q = t'$, and consider a change of basis such that $f_{st} = 0$.

Moreover, we see that $\rho_\tau(x_{st}) = -1$ which implies that $f_{kl} = 0$ for $k < l$, where $\{k, l, s', t'\} \subset [n]$ distinct obtained via (3b), in particular, $f_{s't'} = 0$.

Similarly, we see that $\rho_\tau(x_{s't'}) = -1$ which implies that $f_{kl} = 0$ for $k < l$, where $\{k, l, s', t'\} \subset [n]$ distinct obtained via (3b).

And we are left with the following cases of $f_{ss'}, f_{st'}, f_{min(s',t')max(s',t')}$ and $f_{min(t',t)max(t',t)}$.

For $f_{ss'}$. If $\rho_\tau(x_{ss'}) = +1$ then $f_{ss'} = 0$ via (3a), otherwise, $\rho_\tau(x_{ss'}) = -1$ to which one concludes that $f_{ss'} = 0$ via (3c) for $(i = s, j = s', k = t')$.

Finally, the remaining cases are processed by differentiating possible orderings of t, s', t':

1. If $s < s' < t' < t$, then:

$$f_{s't} = 0 \text{ via } (3c) \text{ for } (i = s, j = s', k = t),$$
$$f_{t't} = 0 \text{ via } (3d) \text{ for } (i = s', j = t', k = t),$$
$$f_{st'} = 0 \text{ via } (3d) \text{ for } (i = s, j = t', k = t).$$

2. If $s < s' < t < t'$, then we have $\tau(t) = \tau(a_p) = a_1 < b_1 = \tau(b_q) = \tau(t')$ which implies that $\rho_\tau(x_{tt'}) = +1$. This in particular yield:

$$\begin{aligned} f_{tt'} &= 0 \text{ via } (3a) \text{ for } (i = t, j = t'), \\ f_{s't} &= 0 \text{ via } (3c) \text{ for } (i = s, j = s', k = t), \\ f_{st'} &= 0 \text{ via } (3d) \text{ for } (i = s, j = t, k = t'). \end{aligned}$$

3. If $s < t < s' < t'$, then we have $\rho_\tau(x_{tt'}) = +1$ and:

$$\begin{aligned} f_{tt'} &= 0 \text{ via } (3a) \text{ for } (i = t, j = t'), \\ f_{ts'} &= 0 \text{ via } (3d) \text{ for } (i = s, j = t, k = s'), \\ f_{st'} &= 0 \text{ via } (3d) \text{ for } (i = s, j = t, k = t'). \end{aligned}$$

Therefore, given τ an $\mathbb{S}_n$-element such that it has at least two disjoint cycles of length $p, q \geq 2$, we have $dim_k Ext^1_\Lambda(\rho_e, \rho_\tau) = 0$ as claimed. □

Remark 15. We remark that the claim made in Theorem 3.4 has been proved per Propositions 3.5, 3.6, 3.7, 3.8 and 3.9.

Corollary 5. *The algebra* Λ *is connected.*

Proof. This is a natural consequence of Theorem 3.4. □

The special case of $\mathcal{D}_4(\alpha, -\alpha)$

Note 14. Here and throughout this section, we consider the special case of $\Lambda := \mathcal{D}_4(\alpha, -\alpha)$ where the K-parameter α remains normalized to 1_K.

Furthermore, we set $t_1 = (1, 3)$, $t_2 = (1, 4)$ and $t_3 = (2, 4)$ the non-simple transpositions of $\mathbb{S}_4$.

Proposition 3.10. *The algebra* Λ *is basic.*

Proof. This follows since the Jacobson radical of the finite-dimensional algebra Λ is generated by the commutator. In particular, the algebra Λ has a complete system of simple representations: $\{\rho_\sigma \mid \sigma \in \mathbb{S}_4\}$. □

Remark 16. The algebra Λ is connected as discussed earlier in Corollary 5.

Theorem 3.11. *The algebra* Λ *admits an ordinary quiver* Q_Λ *of the following form:*

1. *The vertices set are denoted by* $\mathbb{S}_4$*-permutations.*
2. *There exists an arrow from* σ *to* τ *if and only if* $\tau = t_i.\sigma$ *for* $i \in [3]$.

Proof. This is a natural consequence of Theorem 3.4 and Proposition 3.10. □

Note 15. Given $\sigma \in \mathbb{S}_4$. We call the arrow that starts in σ and terminate in $t_i\sigma$ by $\alpha(\sigma; t_i)$. Furthermore, we interpret the action of the symmetric group $\mathbb{S}_4$ on paths by changing the starting vertex, that is, given $r := \alpha(\sigma; t_i)\alpha(t_i\sigma; t_j)\cdots$, then, $\tau r = \alpha(\tau\sigma; t_i)\alpha(t_i\tau\sigma; t_j)\cdots$.

Remark 17. Due to its complexity, drawing the ordinary quiver might be tiresome, we nonetheless remark that by its definition there exist distinctive sections of said quiver worthy of highlighting.

Example 3.1. Since $t_i^2 = e$, we see that there exist 3 distinctive loops at any given any vertex $\sigma \in \mathbb{S}_4$, that is:

$$\begin{tikzcd}
 & \sigma \arrow[dl, shift left] \arrow[dl, leftarrow, shift right] \arrow[d, shift left] \arrow[d, leftarrow, shift right] \arrow[dr, shift left] \arrow[dr, leftarrow, shift right] & \\
t_1\sigma & t_2\sigma & t_3\sigma
\end{tikzcd}$$

Example 3.2. Since $t_1t_3 - t_3t_1 = 0$, we see that there exists a distinctive square at any given any vertex $\sigma \in \mathbb{S}_4$, that is:

$$\begin{tikzcd}
 & \sigma \arrow[dl, shift left] \arrow[dl, leftarrow, shift right] \arrow[dr, shift left] \arrow[dr, leftarrow, shift right] & \\
t_1\sigma \arrow[dr, shift left] \arrow[dr, leftarrow, shift right] & & t_3\sigma \arrow[dl, shift left] \arrow[dl, leftarrow, shift right] \\
 & t_1t_3\sigma &
\end{tikzcd}$$

Example 3.3. Since $t_it_{i+1}t_i - t_{i+1}t_it_{i+1} = 0$ for $i \in [2]$, we see that there exist two distinctive hexagons at any given any vertex $\sigma \in \mathbb{S}_4$, for $i = 1$ we draw:

$$\begin{tikzcd}
 & t_1\sigma \arrow[r, shift left] \arrow[r, leftarrow, shift right] & t_2t_1\sigma \arrow[dr, shift left] \arrow[dr, leftarrow, shift right] & \\
\sigma \arrow[ur, shift left] \arrow[ur, leftarrow, shift right] \arrow[dr, shift left] \arrow[dr, leftarrow, shift right] & & & t_1t_2t_1\sigma \\
 & t_2\sigma \arrow[r, shift left] \arrow[r, leftarrow, shift right] & t_1t_2\sigma \arrow[ur, shift left] \arrow[ur, leftarrow, shift right] &
\end{tikzcd}$$

Note 16. Denote by ϕ Gabriel's theorem morphism associated with Λ. We remind the reader that Gabriel's theorem states that $ker(\phi)$ is a two-sided admissible ideal of KQ_Λ with an associated quotient that is isomorphic to Λ.

Remark 18. We observe that the algebra KQ_Λ admits an indecomposable decomposition of the form:

$$KQ_\Lambda = \bigoplus_{\sigma \in \mathbb{S}_4} e_\sigma KQ_\Lambda.$$

where the action of $\mathbb{S}_4$ on KQ_Λ permutes the indecomposable projectives.

Definition 3.1. Let Γ denotes the indecomposable projective Λ-representation defined to be a quotient of $KQ_\Gamma := e_e KQ_\Lambda$ by the kernel of $\pi := \phi_{|_{e_e}}$.

Note 17. The previous remark implies in particular that from the viewpoint of representation theory, the study of Λ can be reduced to that of Γ.

Lemma 3.12. *The algebra* Γ *is 24-dimensional.*

Proof. This statement follows by observing that the algebra Λ is basic $4!^2$-dimensional, the set $\{e_\sigma \mid \sigma \in \mathbb{S}_4\}$ is complete of primitive orthogonal idempotents, and the action of $\mathbb{S}_4$ on KQ_Λ, to which one concludes that all indecomposable projective Λ-representation are of the same dimension. $\square$

Note 18. Given $\sigma, \tau = t_i\sigma$ for some $i \in [3]$, we proved that the space of extensions $Ext^1_\Lambda(\rho_\sigma, \rho_\tau)$ is 1-dimensional, viewed as equivalence class of (1)-cocycles up to the space of coboundaries we shall rename the single generator of such space by $\overline{f_{(\sigma;t_i)}}$ and omit the bar notation should no confusion occurs.

Example 3.4. Let $i \in [3]$. We remind the reader that the a generic element of the space $Ext^1_\Lambda(\rho_e, \rho_{t_i})$ takes the form:

$$\begin{aligned} \rho : \Lambda &\to K^2 \\ x_{st} &\mapsto \rho(x_{st}) = \begin{bmatrix} \rho_e & f_{(e;t_i)} \\ 0 & \rho_{t_i} \end{bmatrix} (x_{st}), \end{aligned}$$

where $f_{(e;t_i)}(x_{st}) = 0$ for all $(st) \neq t_i$. Moreover, one verifies that generic elements of the space $Ext^1_\Lambda(\rho_{t_i}, \rho_e)$ are given by:

$$\begin{aligned} \rho : \Lambda &\to K^2 \\ x_{st} &\mapsto \rho(x_{st}) = \begin{bmatrix} \rho_{t_i} & f_{(t_i;t_i)} \\ 0 & \rho_e \end{bmatrix} (x_{st}), \end{aligned}$$

where $f_{(t_i;t_i)}(x_{st}) = 0$ for all $(st) \neq t_i$, up to an isomorphism held by a proper change of basis.

Note 19. We remind the reader that grA denotes the associated graded algebra of a K-algebra A with respect to the radical filtration.

Remark 19. Path algebras has a natural grading by paths, which in particular enable us to discuss $gr\Gamma$ -which is 24-dimensional- first by utilizing the notion of extensions between simple modules up to higher powers of the radical.

Indeed, we shall compute the kernel of the graded morphism $gr\pi$ which should enable us to achieve two results:

1. $gr\Gamma \cong \mathcal{N}_4$, the nil-Coxeter algebra associated with $\mathbb{S}_4$.
2. $\Gamma \cong \mathcal{N}_4$.

Proposition 3.13. *The following hold:*

$$r_1 \coloneqq \alpha(e;t_1)\alpha(t_1;t_1) \in ker(gr\pi).$$

Proof. Assume for a contradiction that $r_1 \notin ker(gr\pi)$, that is:

$$\begin{aligned} gr\pi(\alpha(e;t_1)\alpha(t_1;t_1)) &= \pi(\alpha(e;t_1)\alpha(t_1;t_1)) + rad\Lambda^3, \\ &= f_{(e;t_1)}f_{(t_1;t_1)} + rad\Lambda^3 \neq 0, \end{aligned}$$

that is, $f_{(e;t_1)}f_{(t_1;t_1)} \notin rad\Lambda^3$.

The assumption implies that the space of paths of length 2 such that start and terminate in e is at least 1-dimensional, and at most 3-dimensional by Theorem 3.11, let:

$$X = \{\alpha(e;t_i)\alpha(t_i;t_i) \mid i \in [3]\},$$

be a generating set. We have 3 cases to discuss:

If $dimX = 1$. Consider the right KQ_Γ-module defined as a quotient of e_eKQ_Γ by the right-ideal generated by:

$$\{e_e(radKQ_\Gamma)^3, e_e(radKQ_\Gamma)^2e_\sigma, \mid e \neq \sigma \in \mathbb{S}_4\}.$$

Our assumption implies that such module exists and is 5-dimensional graded module generated by:

$$\{e_e, e_e\alpha(e;t_1), e_e\alpha(e;t_2), e_e\alpha(e;t_3), e_e\alpha(e;t_1)\alpha(t_1;t_1)\}.$$

that is, there exists a 5-dimensional graded Λ-representation:

$$\rho = \begin{bmatrix} \rho_e & f_{(e;t_1)} & f_{(e;t_2)} & f_{(e;t_3)} & g_{(e,e)} \\ 0 & \rho_{t_1} & 0 & 0 & f_{(t_1;t_1)} \\ 0 & 0 & \rho_{t_2} & 0 & f_{(t_2;t_2)} \\ 0 & 0 & 0 & \rho_{t_3} & f_{(t_3;t_3)} \\ 0 & 0 & 0 & 0 & \rho_e \end{bmatrix},$$

up to the third power of the radical, where $g_{(e,e)}$ is a set of K-parameters determined by the defining relations of Λ.

If $dimX = 2$, say with no loss of generality that:

$$\{\alpha(e;t_1)\alpha(t_1;t_1)KQ_\Gamma, \alpha(e;t_2)\alpha(t_2;t_2)KQ_\Gamma\},$$

is a basis of X.

Consider the KQ_Γ-module defined as a quotient of e_eKQ_Γ by the right-ideal generated by:

$$\{e_e(radKQ_\Gamma)^3, e_e(radKQ_\Gamma)^2e_\sigma, e_e\alpha(e;t_2), \mid e \neq \sigma \in \mathbb{S}_4\}.$$

Our assumption implies that such module exists and is 4-dimensional graded module generated by:

$$\{e_e, e_e\alpha(e;t_1), e_e\alpha(e;t_3), e_e\alpha(e;t_1)\alpha(t_1;t_1)\},$$

that is, there exists a 4-dimensional graded Λ-representation:

$$\rho = \begin{bmatrix} \rho_e & f_{(e;t_1)} & f_{(e;t_3)} & g_{(e,e)} \\ 0 & \rho_{t_1} & 0 & f_{(t_1;t_1)} \\ 0 & 0 & \rho_{t_3} & f_{(t_3;t_3)} \\ 0 & 0 & 0 & \rho_e \end{bmatrix},$$

up to the third power of the radical, where $g_{(e,e)}$ is a set of K-parameters determined by the defining relations of Λ.

Finally if $dimX = 3$. Consider the KQ_Γ-module defined as a quotient of e_eKQ_Γ by the right-ideal generated by:

$$\{e_e(radKQ_\Gamma)^3, e_e(radKQ_\Gamma)^2e_\sigma, \mid e \neq \sigma \in \mathbb{S}_4\},$$

along side:

$$\{e_e\alpha(e;t_i) \mid i = 2,3\}.$$

Consider the KQ_Γ-module defined as a quotient of e_eKQ_Γ by the right-ideal generated by:

$$\{e_e, e_e\alpha(e;t_1), e_e\alpha(e;t_1)\alpha(t_1;t_1)\},$$

that is, consider the 3-dimensional graded Λ-representation:

$$\rho = \begin{bmatrix} \rho_e & f_{(e;t_1)} & g_{(e,e)} \\ 0 & \rho_{t_1} & f_{(t_1;t_1)} \\ 0 & 0 & \rho_e \end{bmatrix},$$

up to the third power of the radical, where $g_{st(e,e)}$ is a set of K-parameters determined by the defining relations of Λ.

We observe here that the computations required for the defining relations of the algebra to hold are very similar in the previous three cases, therefore, we shall showcase them in the case of $dimX = 3$ and leave the others as an exercise.

Indeed one sees that the module described is of the form:

$$\rho(x_{12}) = \begin{bmatrix} +1 & 0 & g_{(e;e)}(x_{12}) \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{bmatrix}, \qquad \rho(x_{23}) = \begin{bmatrix} +1 & 0 & g_{(e;e)}(x_{23}) \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{bmatrix}$$

$$\rho(x_{13}) = \begin{bmatrix} +1 & f_{(e;t_1)}(x_{13}) & g_{(e;e)}(x_{13}) \\ 0 & -1 & f_{(t_1;t_1)}(x_{13}) \\ 0 & 0 & +1 \end{bmatrix},$$

to which one computes that:

$$g_{(e,e)}(x_{ij}) = \begin{cases} 0 & \mid (i,j) = (1,2), (2,3), \\ -\frac{(f_{(e;t_1)}f_{(t_1;t_1)})(x_{13})}{2} & \mid (i,j) = (1,3), \end{cases}$$

so that (2b) holds, while for (2d) to hold one must have $g_{(e,e)}(x_{13}) = 0$ up to the third power of the radical, that is:

$$(f_{(e;t_1)}f_{(t_1;t_1)})(x_{13})) + rad\Lambda^3 = 0,$$

that is, $f_{(e;t_1)}f_{(t_1;t_1)} \in rad\Lambda^3$ a contradiction that assures the claim. □

Corollary 6. *For $i \in [3]$ and $\sigma \in \mathbb{S}_4$. The following hold:*

1. $r_i \coloneqq \alpha(e;t_i)\alpha(t_i;t_i) \in ker(gr\pi)$.
2. $\sigma.r_i = \alpha(\sigma;t_i)\alpha(t_i\sigma,t_i) \in ker(gr\pi)$.

Proposition 3.14. *The following hold:*

$$r_4 \coloneqq \alpha(e;t_1)\alpha(t_1;t_3) - \alpha(e;t_3)\alpha(t_3;t_1) \in ker(gr\pi).$$

Proof. Assume for a contradiction that $r_4 \notin ker(gr\pi)$, that is:

$$\begin{aligned} gr\pi(\alpha(e;t_1)\alpha(t_1;t_3) - \alpha(e;t_3)\alpha(t_3;t_1)) &= (f_{(e;t_1)}f_{(t_1;t_3)} - f_{(e;t_3)}f_{(t_3;t_1)}) + rad\Lambda^3, \\ &\neq 0, \end{aligned}$$

that is, $f_{(e;t_1)}f_{(t_1;t_3)} - f_{(e;t_3)}f_{(t_3;t_1)} \notin rad\Lambda^3$.

In other words, we assume that the space of paths of length 2 that start in e and terminate in t_1t_3 is 2-dimensional.

Consider the right KQ_Γ-module defined as the quotient of KQ_Γ by the right-ideal generated by:

$$\{e_e(radKQ_\Gamma)^3, e_e(radKQ_\Gamma)^2e_\sigma, e_e\alpha(e;t_2) \mid \sigma \neq t_1t_3\}.$$

Our assumption implies that such module is a 5-dimensional graded module generated by:

$$\{e_e, e_e\alpha(e;t_1), e_e\alpha(e;t_1)\alpha(t_1;t_3), e_e\alpha(e;t_3), e_e\alpha(e;t_3)\alpha(t_3;t_1)\},$$

that is, there exists a 5-dimensional graded Λ-representation:

$$\rho = \begin{bmatrix} \rho_e & f_{(e;t_1)} & f_{(e;t_3)} & g_{(e,t_3t_1)} & g_{(e,t_1t_3)} \\ 0 & \rho_{t_1} & 0 & f_{(t_1;t_3)} & 0 \\ 0 & 0 & \rho_{t_3} & 0 & f_{(t_3;t_1)} \\ 0 & 0 & 0 & \rho_{t_3t_1} & 0 \\ 0 & 0 & 0 & 0 & \rho_{t_1t_3} \end{bmatrix},$$

up to the third power of the radical, where $g_{(e,t_1t_3)}$ and $g_{(e,t_3t_1)}$ are sets of K-parameters determined by the defining relations of Λ. In particular, one computes that up the third power of the radical:

$$\rho(x_{12}) = \begin{bmatrix} +1 & 0 & 0 & g_{(e;t_3t_1)}(x_{12}) & g_{(e;t_1t_3)}(x_{12}) \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & +1 \end{bmatrix},$$

along side:

$$\rho(x_{34}) = \begin{bmatrix} +1 & 0 & 0 & g_{(e;t_3t_1)}(x_{34}) & g_{(e;t_1t_3)}(x_{34}) \\ 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & +1 \end{bmatrix},$$

to which (2b) and (2c) hold if:

$$\begin{aligned} 0 &= g_{(e;t_3t_1)}(x_{12}) = g_{(e;t_1t_3)}(x_{12}), \\ &= g_{(e;t_3t_1)}(x_{34}) = g_{(e;t_1t_3)}(x_{34}), \end{aligned}$$

while that of:

$$\rho(x_{14}) = \begin{bmatrix} +1 & 0 & 0 & g_{(e;t_3t_1)}(x_{14}) & g_{(e;t_1t_3)}(x_{14}) \\ 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

$$\rho(x_{23}) = \begin{bmatrix} +1 & 0 & 0 & g_{(e;t_3t_1)}(x_{23}) & g_{(e;t_1t_3)}(x_{23}) \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

to which (2b) and (2c) hold if:

$$\begin{aligned} x &= g_{(e;t_3t_1)}(x_{14}) = g_{(e;t_3t_1)}(x_{23}), \\ y &= g_{(e;t_1t_3)}(x_{14}) = g_{(e;t_1t_3)}(x_{23}), \end{aligned}$$

for some field parameters x, y. Finally:

$$\rho(x_{13}) = \begin{bmatrix} +1 & f_{(e;t_1)}(x_{13}) & 0 & g_{(e;t_3t_1)}(x_{13}) & g_{(e;t_1t_3)}(x_{13}) \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & f_{(t_1;t_3)}(x_{13}) \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

$$\rho(x_{24}) = \begin{bmatrix} +1 & 0 & f_{(e;t_3)}(x_{24}) & g_{(e;t_3t_1)}(x_{24}) & g_{(e;t_1t_3)}(x_{24}) \\ 0 & +1 & 0 & f_{(t_3;t_1)}(x_{24}) & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

to which (2b), (2c) and (2d) for $(i = 1, j = 2, k = 3)$ hold if:

$$g_{(e,t_3t_1)}(x_{ij}) = \begin{cases} x & \mid (i,j) = (1,3), \\ x - \frac{f_{(e;t_1)}f_{(t_1;t_3)}(x_{24})}{2} & \mid (i,j) = (2,4). \end{cases}$$

$$g_{(e,t_1t_3)}(x_{ij}) = \begin{cases} y & \mid (i,j) = (1,3), \\ y + \frac{f_{(e;t_3)}f_{(t_3;t_1)}(x_{24})}{2} & \mid (i,j) = (2,4). \end{cases}$$

However, for (2d) in case of $(i = 1, j = 2, k = 4)$ to hold one must have:

$$\begin{aligned} 0 &= f_{(e;t_1)}f_{(t_1;t_3)} + rad\Lambda^3, \\ &= f_{(e;t_3)}f_{(t_3;t_1)} + rad\Lambda^3, \end{aligned}$$

as well, a contradiction which proves that the space of paths of length 2 that start in e and terminate in t_1t_3 is not 2-dimensional.

We see that it is 1-dimensional, with a basis of the form:

$$\{\alpha(e;t_1)\alpha(t_1;t_3) = \alpha(e;t_3)\alpha(t_3;t_1)\}.$$

Which is evident by the existence of the 4-dimensional graded Λ-representation:

$$\rho = \begin{bmatrix} \rho_e & f_{(e;t_1)} & f_{(e;t_3)} & g_{(e;t_1t_3)} \\ 0 & \rho_{t_1} & 0 & f_{(t_1;t_3)} \\ 0 & 0 & \rho_{t_3} & f_{(t_3;t_1)} \\ 0 & 0 & 0 & \rho_{t_1t_3} \end{bmatrix},$$

up to the third power of the radical, where $g_{(e;t_1t_3)}$ is a set of K-parameters that is determined by the defining relations of Λ. Indeed, one verifies that up to an isomorphism, the defining relations of the algebra hold with no contradictions once we set $g_{(e;t_1t_3)}(x_{ij}) = 0$ where:

$$g_{(e;t_1t_3)}(x_{13}) = \frac{f_{(e;t_1)}(x_{13})f_{(t_1;t_3)}(x_{24}) - f_{(e;t_3)}(x_{24})f_{(t_3;t_1)}(x_{13})}{2},$$

up to the third power of the radical. That is:

$$(f_{(e;t_1)}f_{(t_1;t_3)} - f_{(e;t_3)}f_{(t_3;t_1)}) + rad\Lambda^3 = 0,$$

that is, $gr\pi(\alpha(e;t_1)\alpha(t_1;t_3) - \alpha(e;t_3)\alpha(t_3;t_1)) = 0$, and $r_4 \in ker(gr\pi)$ as claimed. $\square$

Remark 20. Corollary 6 and Proposition 3.14 imply that the space of $KQ_\Gamma/gr\pi$-paths of length 2 that start in e is exactly 5-dimensional with a basis of the form:

$$\{\alpha(e;t_1)\alpha(t_1;t_2), \alpha(e;t_3)\alpha(t_3;t_2), \alpha(e;t_1)\alpha(t_1;t_3), \alpha(e;t_2)\alpha(t_2;t_1), \alpha(e;t_2)\alpha(t_2;t_3)\},$$

up to an isomorphism. In particular, we detail that the existence of the $KQ_\Gamma/gr\pi$-path $\alpha(e;t_1)\alpha(t_1;t_2)$ implies that of a 3-dimensional graded Λ-representation:

$$\rho = \begin{bmatrix} \rho_e & f_{(e;t_1)} & g_{(e;t_2t_1)} \\ 0 & \rho_{t_1} & f_{(t_1;t_2)} \\ 0 & 0 & \rho_{t_2t_1} \end{bmatrix},$$

up to the third power of the radical, where $g_{(e;t_2t_1)}$ is a set of K-parameters. We make a basis change such that:

$$\begin{aligned} &g_{(e;t_2t_1)} : \Lambda \to K, \\ &\qquad x_{ij} \mapsto g_{(e;t_2t_1)}(x_{ij}) = \begin{cases} 0 & \mid (i,j) \neq (1,4), \\ \frac{f_{(e;t_1)}(x_{13})f_{(t_1;t_2)}(x_{34})}{2} & \mid (i,j) = (1,4). \end{cases} \end{aligned} \tag{4}$$

While the existence of the $KQ_\Gamma/gr\pi$-path $\alpha(e;t_2)\alpha(t_2;t_1)$, implies that of a 3-dimensional graded Λ-representation:

$$\rho = \begin{bmatrix} \rho_e & f_{(e;t_2)} & g_{(e;t_1t_2)} \\ 0 & \rho_{t_2} & f_{(t_2;t_1)} \\ 0 & 0 & \rho_{t_1t_2} \end{bmatrix},$$

up to the third power of the radical, where $g_{(e;t_2t_1)}$ is a set of K-parameters. We make a basis change such that:

$$\begin{aligned} g_{(e;t_1t_2)} &: \Lambda \to K, \\ x_{ij} &\mapsto g_{(e;t_1t_2)}(x_{ij}) = \begin{cases} 0 & \mid (i,j) \neq (1,3), \\ \frac{f_{(e;t_2)}(x_{14}) f_{(t_2;t_1)}(x_{34})}{2} & \mid (i,j) = (1,3). \end{cases} \end{aligned} \tag{5}$$

Corollary 7. *Up to an isomorphism, the space of $KQ_\Gamma/gr\pi$-paths of length 2 that start in σ is exactly 5-dimensional with a basis of the form:*

$$\{\alpha(\sigma;t_1)\alpha(\sigma t_1;t_2), \alpha(\sigma;t_3)\alpha(\sigma t_3;t_2), \alpha(\sigma;t_1)\alpha(\sigma t_1;t_3), \alpha(\sigma;t_2)\alpha(\sigma t_2;t_1), \alpha(\sigma;t_2)\alpha(\sigma t_2;t_3)\}.$$

Remark 21. As it plays a role in our discussion, we shall detail that the existence of the $KQ_\Gamma/gr\pi$-path $\alpha(t_1;t_2)\alpha(t_1t_2;t_1)$ implies that of a 3-dimensional graded Λ-representation:

$$\rho = \begin{bmatrix} \rho_{t_1} & f_{(t_1;t_2)} & g_{(t_1;t_1t_2t_1)} \\ 0 & \rho_{t_2t_1} & f_{(t_2t_1;t_1)} \\ 0 & 0 & \rho_{t_1t_2t_1} \end{bmatrix},$$

up to the third power of the radical, where $g_{(t_1;t_1t_2t_1)}$ is a set of K-parameters. We make a basis change such that:

$$\begin{aligned} g_{(t_1;t_1t_2t_1)} &: \Lambda \to K, \\ x_{ij} &\mapsto g_{(t_1;t_1t_2t_1)}(x_{ij}) = \begin{cases} 0 & \mid (i,j) \neq (1,3), \\ \frac{f_{(t_1;t_2)}(x_{34}) f_{(t_2t_1;t_1)}(x_{14})}{-2} & \mid (i,j) = (1,3). \end{cases} \end{aligned} \tag{6}$$

While the existence of the $KQ_\Gamma/gr\pi$-path $\alpha(t_2;t_1)\alpha(t_2t_1;t_2)$ implies that of a 3-dimensional graded Λ-representation:

$$\rho = \begin{bmatrix} \rho_{t_2} & f_{(t_2;t_1)} & g_{(t_2;t_2t_1t_2)} \\ 0 & \rho_{t_1t_2} & f_{(t_1t_2;t_2)} \\ 0 & 0 & \rho_{t_2t_1t_2} \end{bmatrix},$$

up to the third power of the radical, where $g_{(t_2;t_1t_2t_1)}$ is a set of K-parameters. We make a basis change such that:

$$\begin{aligned} g_{(t_2;t_2t_1t_2)} &: \Lambda \to K, \\ x_{ij} &\mapsto g_{(t_2;t_2t_1t_2)}(x_{ij}) = \begin{cases} 0 & \mid (i,j) \neq (1,4), \\ \frac{f_{(t_2;t_1)}(x_{34}) f_{(t_1t_2;t_2)}(x_{13})}{-2} & \mid (i,j) = (1,4). \end{cases} \end{aligned} \tag{7}$$

Proposition 3.15. *The following hold:*

1. $r_5 \coloneqq \alpha(e;t_1)\alpha(t_1;t_2)\alpha(t_2t_1;t_1) - \alpha(e;t_2)\alpha(t_2;t_1)\alpha(t_1t_2;t_2) \in ker(gr\pi)$,
2. $r_6 \coloneqq \alpha(e;t_3)\alpha(t_3;t_2)\alpha(t_2t_3;t_3) - \alpha(e;t_2)\alpha(t_2;t_3)\alpha(t_3t_2;t_2) \in ker(gr\pi)$.

Proof. By means of the symmetric group action, the second claim is proven in a similar way to the first one, which we prove as follows.

Assume for a contradiction that $r_5 \notin ker(gr\pi)$, that is:

$$gr\pi(\alpha(e;t_1)\alpha(t_1;t_2)\alpha(t_2t_1;t_1) - \alpha(e;t_2)\alpha(t_2;t_1)\alpha(t_1t_2;t_2)) \neq 0,$$

that is:

$$(f_{(e;t_1)}f_{(t_1;t_2)}f_{(t_2t_1;t_1)} - f_{(e;t_2)}f_{(t_2;t_1)}f_{(t_1t_2;t_2)}) + rad\Lambda^4 \neq 0,$$

that is:

$$(f_{(e;t_1)}f_{(t_1;t_2)}f_{(t_2t_1;t_1)} - f_{(e;t_2)}f_{(t_2;t_1)}f_{(t_1t_2;t_2)}) \notin rad\Lambda^4.$$

In other words, we assume that the space of paths of length 3 that start in e and terminate in $t_1t_2t_1$ is 2-dimensional.

Consider the right KQ_Γ-module defined as the quotient of KQ_Γ by the right-ideal generated by:

$$\{e_e(radKQ_\Gamma)^4, e_e(radKQ_\Gamma)^3e_\sigma, e_e\alpha(e;t_3) \mid \sigma \neq t_1t_2t_1\}.$$

Our assumption implies that such module is a 7-dimensional graded module generated by:

$$\begin{aligned}\{e_e, &e_e\alpha(e;t_1), e_e\alpha(e;t_1)\alpha(t_1;t_2), e_e\alpha(e;t_1)\alpha(t_1;t_2)\alpha(t_2t_1;t_1)\\ &e_e\alpha(e;t_2), e_e\alpha(e;t_2)\alpha(t_2;t_1), e_e\alpha(e;t_2)\alpha(t_2;t_1)\alpha(t_1t_2;t_2)\}.\end{aligned}$$

That is, there exists a 7-dimensional graded Λ-representation:

$$\rho = \begin{bmatrix} \rho_e & f_{(e;t_1)} & f_{(e;t_2)} & g_{(e;t_2t_1)} & g_{(e;t_1t_2)} & g_{(e,t_1t_2t_1)} & g_{(e,t_2t_1t_2)} \\ 0 & \rho_{t_1} & 0 & f_{(t_1;t_2)} & 0 & g_{(t_1;t_1t_2t_1)} & 0 \\ 0 & 0 & \rho_{t_2} & 0 & f_{(t_2;t_1)} & 0 & g_{(t_2;t_2t_1t_2)} \\ 0 & 0 & 0 & \rho_{t_2t_1} & 0 & f_{(t_2t_1;t_1)} & 0 \\ 0 & 0 & 0 & 0 & \rho_{t_1t_2} & 0 & f_{(t_1t_2;t_2)} \\ 0 & 0 & 0 & 0 & 0 & \rho_{t_1t_2t_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho_{t_2t_1t_2} \end{bmatrix},$$

up to the fourth power of the radical, where $g_{(e;t_2t_1)}, g_{(e;t_1t_2)}, g_{(t_1;t_1t_2t_1)}$ and $g_{(t_2;t_2t_1t_2)}$ are the set of K-parameters that has been detailed previously in (4), (5), (6) and (7). While $g_{(e,t_1t_2t_1)}$ and $g_{(e,t_2t_1t_2)}$ are sets of K-parameters that are determined by the defining relations of Λ.

Indeed we see that up to the fourth power of the radical:

$$\rho(x_{12}) = \begin{bmatrix} +1 & 0 & 0 & 0 & 0 & g_{(e,t_1t_2t_1)}(x_{12}) & g_{(e,t_2t_1t_2)}(x_{12}) \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & +1 \end{bmatrix},$$

along side:

$$\rho(x_{23}) = \begin{bmatrix} +1 & 0 & 0 & 0 & 0 & g_{(e,t_1t_2t_1)}(x_{23}) & g_{(e,t_2t_1t_2)}(x_{23}) \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & +1 \end{bmatrix}$$

$$\rho(x_{24}) = \begin{bmatrix} +1 & 0 & 0 & 0 & 0 & g_{(e,t_1t_2t_1)}(x_{24}) & g_{(e,t_2t_1t_2)}(x_{24}) \\ 0 & +1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & +1 \end{bmatrix},$$

to which (2b) hold if for $(i,j) = (1,2), (2,3), (2,4)$:

$$0 = g_{(e,t_1t_2t_1)}(x_{ij}) = g_{(e,t_2t_1t_2)}(x_{ij}),$$

while that of:

$$\rho(x_{13}) = \begin{bmatrix} +1 & f_{(e;t_1)}(x_{13}) & 0 & 0 & g_{(e;t_1t_2)}(x_{13}) & g_{(e,t_1t_2t_1)}(x_{13}) & g_{(e,t_2t_1t_2)}(x_{13}) \\ 0 & -1 & 0 & 0 & 0 & g_{(t_1;t_1t_2t_1)}(x_{13}) & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & f_{(t_1t_2;t_2)}(x_{13}) \\ 0 & 0 & 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & +1 \end{bmatrix},$$

to which (2b) hold if:

$$\begin{aligned}
g_{(e,t_1t_2t_1)}(x_{13}) &= \frac{f_{(e;t_1)}(x_{13})f_{(t_1;t_2)}(x_{34})f_{(t_2t_1;t_1)}(x_{14})}{4}, \\
g_{(e,t_2t_1t_2)}(x_{13}) &= \frac{f_{(e;t_2)}(x_{14})f_{(t_2;t_1)}(x_{34})f_{(t_1t_2;t_2)}(x_{13})}{4}.
\end{aligned}$$

and:

$$\rho(x_{14}) = \begin{bmatrix}
+1 & 0 & f_{(e;t_2)}(x_{14}) & g_{(e;t_2t_1)}(x_{14}) & 0 & g_{(e,t_1t_2t_1)}(x_{14}) & g_{(e,t_2t_1t_2)}(x_{14}) \\
0 & +1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & g_{(t_2t_1;t_1)}(x_{14}) \\
0 & 0 & 0 & -1 & 0 & f_{(t_2t_1;t_1)}(x_{14}) & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & +1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & +1
\end{bmatrix},$$

to which (2b) hold if:

$$\begin{aligned}
g_{(e,t_1t_2t_1)}(x_{14}) &= \frac{f_{(e;t_1)}(x_{13})f_{(t_1;t_2)}(x_{34})f_{(t_2t_1;t_1)}(x_{14})}{+4}, \\
g_{(e,t_2t_1t_2)}(x_{14}) &= \frac{f_{(e;t_2)}(x_{14})f_{(t_2;t_1)}(x_{34})f_{(t_1t_2;t_2)}(x_{13})}{-4}.
\end{aligned}$$

However, should such condition hold, then (2d) for $(i = 1, j = 2, k = 3)$ would hold if:

$$\begin{aligned}
0 &= f_{(e;t_1)}(x_{13})f_{(t_1;t_2)}(x_{34})f_{(t_2t_1;t_1)}(x_{14}) + rad\Lambda^4, \\
0 &= f_{(e;t_2)}(x_{14})f_{(t_2;t_1)}(x_{34})f_{(t_1t_2;t_2)}(x_{13}) + rad\Lambda^4.
\end{aligned}$$

A contradiction which proves that the space of paths of length 3 that start in e and terminate in $t_1t_2t_1$ is not 2-dimensional.

We see that it is 1-dimensional, with a basis of the form:

$$\{\alpha(e;t_1)\alpha(t_1;t_2)\alpha(t_2t_1;t_1) = \alpha(e;t_2)\alpha(t_2;t_1)\alpha(t_1t_2;t_2)\}.$$

Which is evident by the existence of the 6-dimensional graded Λ-representation:

$$\rho = \begin{bmatrix}
\rho_e & f_{(e;t_1)} & f_{(e;t_2)} & g_{(e;t_2t_1)} & g_{(e;t_1t_2)} & g_{(e,t_1t_2t_1)} \\
0 & \rho_{t_1} & 0 & f_{(t_1;t_2)} & 0 & g_{(t_1;t_1t_2t_1)} \\
0 & 0 & \rho_{t_2} & 0 & f_{(t_2;t_1)} & g_{(t_2;t_1t_2t_1)} \\
0 & 0 & 0 & \rho_{t_2t_1} & 0 & f_{(t_2t_1;t_1)} \\
0 & 0 & 0 & 0 & \rho_{t_1t_2} & f_{(t_1t_2;t_2)} \\
0 & 0 & 0 & 0 & 0 & \rho_{t_1t_2t_1}
\end{bmatrix},$$

up to the fourth power of the radical, where $g_{(e;t_2t_1)}, g_{(e;t_1t_2)}, g_{(t_1;t_1t_2t_1)}$ and $g_{(t_2;t_2t_1t_2)}$ are the set of K-parameters that has been detailed previously in (4), (5), (6) and (7).

And $g_{(e,t_1t_2t_1)}$ is a set of K-parameter that is determined by the defining relations of the algebra.

Indeed one verifies that up to an isomorphism, the defining relations of the algebra hold with no contradiction once we set $g_{(e,t_1t_2t_1)}(x_{ij}) = 0$ for all (i, j), where $g_{(e,t_1t_2t_1)}(x_{13}) = -g_{(e,t_1t_2t_1)}(x_{14})$ equal to:

$$\frac{f_{(e;t_1)}(x_{13})f_{(t_1;t_2)}(x_{34})f_{(t_2t_1;t_1)}(x_{14}) - f_{(e;t_2)}(x_{14})f_{(t_2;t_1)}(x_{34})f_{(t_1t_2;t_2)}(x_{13})}{4},$$

up to the fourth power of the radical. That is:

$$(f_{(e;t_1)}f_{(t_1;t_2)}f_{(t_2t_1;t_1)} - f_{(e;t_2)}f_{(t_2;t_1)}f_{(t_1t_2;t_2)}) + rad\Lambda^4 = 0,$$

that is:

$$gr\pi(\alpha(e;t_1)\alpha(t_1;t_2)\alpha(t_2t_1;t_1) - \alpha(e;t_2)\alpha(t_2;t_1)\alpha(t_1t_2;t_2)) = 0,$$

and $r_5 \in ker(gr\pi)$. □

Corollary 8. *Up to a higher power of the radical, the following holds:*

$$\{\sigma.r_i \mid \sigma \in \mathbb{S}_4, i \in [6]\} \subseteq ker(gr\pi).$$

Remark 22. Corollaries 6, 8 and Propositions 3.14, 3.15 imply that up to a higher power of the radical, the space of $KQ_\Gamma/gr\pi$-paths of length 3 that start in e is exactly 6-dimensional with a basis of the form:

$$\begin{aligned}\{&\alpha(e;t_1)\alpha(t_1;t_2)\alpha(t_2t_1;t_1), \alpha(e;t_1)\alpha(t_1;t_2)\alpha(t_2t_1;t_3),\\ &\alpha(e;t_1)\alpha(t_1;t_3)\alpha(t_3t_1;t_2), \alpha(e;t_3)\alpha(t_3;t_2)\alpha(t_2t_3;t_3),\\ &\alpha(e;t_3)\alpha(t_3;t_2)\alpha(t_2t_3;t_1), \alpha(e;t_2)\alpha(t_2;t_1)\alpha(t_1t_2;t_3)\},\end{aligned}$$

up to an isomorphism.

Lemma 3.16. *Up to a higher power of the radical, the algebra $gr\Gamma$ is isomorphic to the nil-Coxeter algebra associated with $\mathbb{S}_4$. In particular, we have:*

$$\{\sigma.r_i \mid \sigma \in \mathbb{S}_4, i \in [6]\} = ker(gr\pi).$$

Proof. This follows by mapping graded paths:

$$\alpha(e;t_{i_1})\alpha(t_{i_1};t_{i_2})\cdots\alpha(t_{i_{j-1}}\cdots t_{i_1};t_{i_j}),$$

to Coxeter words $s_{i_1}\cdots s_{i_{j-1}}s_{i_j}$. In particular, such mapping is a surjective algebra homomorphism, the claim is then asserted by remarking that both of $gr\Gamma$ and $\mathcal{N}_4$ are 24-dimensional as seen in Lemma 3.12. □

Lemma 3.17. *The following hold:*

1. *Up to the fifth power of the radical, the space of $e_e KQ_\Gamma/ker(gr\pi)$-paths of length 4 that start in e is 5-dimensional. In particular,:*

$$X := \{\alpha(e;t_1)\alpha(t_1;t_2)\alpha(t_2t_1;t_1)\alpha(t_1t_2t_1;t_3), \alpha(e;t_1)\alpha(t_1;t_2)\alpha(t_2t_1;t_3)\alpha(t_3t_2t_1;t_2),$$
$$\alpha(e;t_1)\alpha(t_1;t_3)\alpha(t_3t_1;t_2)\alpha(t_2t_3t_1;t_1), \alpha(e;t_2)\alpha(t_2;t_1)\alpha(t_1t_2;t_3)\alpha(t_3t_1t_2;t_2),$$
$$\alpha(e;t_2)\alpha(t_2;t_3)\alpha(t_3t_2;t_2)\alpha(t_2t_3t_2;t_1)\},$$

is a 5-dimensional basis.

2. *Up to the sixth power of the radical, the space of $e_e KQ_\Gamma/ker(gr\pi)$-paths of length 5 such that starts in e is 3-dimensional. In particular:*

$$X := \{\alpha(e;t_1)\alpha(t_1;t_2)\alpha(t_2t_1;t_1)\alpha(t_1t_2t_1;t_3)\alpha(t_3t_1t_2t_1;t_2)$$
$$\alpha(e;t_1)\alpha(t_1;t_2)\alpha(t_2t_1;t_3)\alpha(t_3t_2t_1;t_2)\alpha(t_2t_3t_2t_1;t_1)$$
$$\alpha(e;t_2)\alpha(t_2;t_1)\alpha(t_1t_2;t_3)\alpha(t_3t_1t_2;t_2)\alpha(t_2t_3t_1t_2;t_1)\},$$

is a 3-dimensional basis.

3. *Up to the seventh power of the radical, the space of $e_e KQ_\Gamma/ker(gr\pi)$-paths of length 6 such that start in e is 1-dimensional generated by a single graded element:*

$$\alpha(e;t_1)\alpha(t_1;t_2)\alpha(t_2t_1;t_1)\alpha(t_1t_2t_1;t_3)\alpha(t_3t_1t_2t_1;t_2)\alpha(t_2t_3t_1t_2t_1;t_1).$$

4. *The space of $e_e KQ_\Gamma/ker(gr\pi)$-paths of length at least 7 such that start in e is null-dimensional.*

Proof. The claims made here are natural consequences of the isomorphism described in Proposition 3.16. Indeed, one computes that $\mathcal{N}_4$ has a basis of the following form:

$$\{1, s_1, s_2, s_3, s_1s_2, s_1s_3, s_2s_1, s_2s_3, s_3s_2,$$
$$s_1s_2s_1, s_1s_2s_3, s_1s_3s_2, s_2s_1s_3, s_2s_3s_2, s_3s_2s_1,$$
$$s_1s_2s_1s_3, s_1s_2s_3s_2, s_1s_3s_2s_1, s_2s_1s_3s_2, s_2s_3s_2s_1,$$
$$s_1s_2s_1s_3s_2, s_1s_2s_3s_2s_1, s_2s_1s_3s_2s_1, s_1s_2s_1s_3s_2s_1\},$$

ordered by degree, to which the claim falls directly. □

Remark 23. Alternatively, one may approach Lemma 3.17 without utilizing the theory of nil-Coxeter algebras, albeit, should such approach be taken, the computations become quite challenging in higher dimensions. In spirit of completion, we shall sketch an example on such approach as follows.

Proposition 3.18. *Up to the fifth power of the radical, the space of $e_e KQ_\Gamma/ker(gr\pi)$-paths of length 4 that start in e is 5-dimensional.*

Proof. The claim follows once shown that up to the fifth power of the radical:

$$\begin{aligned}X \coloneqq \{&\alpha(e;t_1)\alpha(t_1;t_2)\alpha(t_2t_1;t_1)\alpha(t_1t_2t_1;t_3),\alpha(e;t_1)\alpha(t_1;t_2)\alpha(t_2t_1;t_3)\alpha(t_3t_2t_1;t_2),\\ &\alpha(e;t_1)\alpha(t_1;t_3)\alpha(t_3t_1;t_2)\alpha(t_2t_3t_1;t_1),\alpha(e;t_2)\alpha(t_2;t_1)\alpha(t_1t_2;t_3)\alpha(t_3t_1t_2;t_2),\\ &\alpha(e;t_2)\alpha(t_2;t_3)\alpha(t_3t_2;t_2)\alpha(t_2t_3t_2;t_1)\},\end{aligned}$$

is a 5-dimensional basis of the described space. That is, to prove that each X-expression is not in the $ker(gr\pi)$. As the other expressions are verified in a rather similar fashion, we only show:

$$\alpha(e;t_1)\alpha(t_1;t_2)\alpha(t_2t_1;t_1)\alpha(t_1t_2t_1;t_3) \notin ker(gr\pi),$$

to avoid repetitiveness.

Indeed, one observes that such claim is equivalent to that of saying that the space of $e_eKQ_\Gamma/ker(gr\pi)$-paths of length 4 that start in e and terminate in $t_3t_1t_2t_1$ is exactly 1-dimensional, the following:

$$\begin{aligned}\{&\alpha(e;t_1)\alpha(t_1;t_2)\alpha(t_2t_1;t_1)\alpha(t_1t_2t_1;t_3),\alpha(e;t_2)\alpha(t_2;t_1)\alpha(t_1t_2;t_2)\alpha(t_2t_1t_2;t_3),\\ &\alpha(e;t_1)\alpha(t_1;t_2)\alpha(t_2t_1;t_3)\alpha(t_3t_2t_1;t_1)\},\end{aligned}$$

provide a generating set of the described set.

Now Proposition 3.14 implies the linear dependency of the first and third expressions, while Proposition 3.15 implies linear dependency of the first and second expressions.

Furthermore, as in Proposition 3.14 one sees that the space of paths of length 2 that starts in t_2t_1 and terminate in $t_3t_1t_2t_1$ is exactly 1-dimensional, this is evident by the existence of the graded representation:

$$\rho = \begin{bmatrix} \rho_{t_2t_1} & f_{(t_2t_1;t_1)} & g_{(t_2t_1,t_3t_1t_2t_1)} \\ 0 & \rho_{t_1t_2t_1} & f_{(t_1t_2t_1;t_3)} \\ 0 & 0 & \rho_{t_3t_1t_2t_1} \end{bmatrix},$$

where $g_{(t_2t_1,t_3t_1t_2t_1)}$ is a set of K-parameters that encloses the linearly dependency of $f_{(t_2t_1;t_1)}f_{(t_1t_2t_1;t_3)}$ on $f_{(t_1t_2;t_2)}f_{(t_2t_1t_2;t_3)}$ up to the third power of the radical.

Furthermore, one sees that the space of paths of length 3 that starts in t_1 and terminate in $t_3t_1t_2t_1$ is exactly 1-dimensional, this is evident by the existence of the graded representation:

$$\rho = \begin{bmatrix} \rho_{t_1} & f_{(t_1;t_2)} & g_{(t_1,t_1t_2t_1)} & g_{(t_1,t_3t_1t_2t_1)} \\ 0 & \rho_{t_2t_1} & f_{(t_2t_1;t_1)} & g_{(t_2t_1,t_3t_1t_2t_1)} \\ 0 & 0 & \rho_{t_1t_2t_1} & f_{(t_1t_2t_1;t_3)} \\ 0 & 0 & 0 & \rho_{t_3t_1t_2t_1} \end{bmatrix},$$

where $g_{(t_1,t_3t_1t_2t_1)}$ is a set of K-parameters determined by the algebras relations.

Finally, one considers a graded 5-dimensional Λ-representation of the form:

$$\rho = \begin{bmatrix} \rho_e & f_{(e;t_1)} & g_{(e,t_2t_1)} & g_{(e,t_1t_2t_1)} & g_{(e,t_3t_1t_2t_1)} \\ 0 & \rho_{t_1} & f_{(t_1;t_2)} & g_{(t_1,t_1t_2t_1)} & g_{(t_1,t_3t_1t_2t_1)} \\ 0 & 0 & \rho_{t_2t_1} & f_{(t_2t_1;t_1)} & g_{(t_2t_1,t_3t_1t_2t_1)} \\ 0 & 0 & 0 & \rho_{t_1t_2t_1} & f_{(t_1t_2t_1;t_3)} \\ 0 & 0 & 0 & 0 & \rho_{t_3t_1t_2t_1} \end{bmatrix},$$

up to the fifth power of the radical, which should exists with no contradictions, therefore confirming that the generating set is a basis and the claim falls, where the $g_{(e,t_3t_1t_2t_1)}$ is a set of K-parameters determined by the defining relations of Λ. □

Theorem 3.19. *There exits an algebra isomorphism $gr\Gamma \cong \Gamma$. In particular, the algebra Γ is isomorphic to the nil-Coxeter algebra associated with $\mathbb{S}_4$.*

Proof. This is indicated by observing that elements of the ideal $ker(gr\pi)$ are minimal and maximal in the precise length sense, that is, there exists no basis vectors of the space of $KQ_\Gamma/ker(gr\pi)$-paths such that share the same starting and terminating vertex and of different length. □

3.2 The representation theory of $\mathcal{D}_4(\alpha, \alpha)$

Note 20. Here and throughout this section, we consider the special case of $\Lambda := \mathcal{D}_4(\alpha, \alpha)$ where the K-parameter α remain normalized to 1_K.

Proposition 3.20. *The algebra Λ is not basic.*

Proof. We show this claim by showing that there exists no 1-dimensional Λ-representation, which is rather a stronger result.

Indeed, assume for a contradiction that ρ is a 1-dimensional Λ-representation say of the form:

$$\begin{aligned} \rho : \Lambda &\to K \\ x_{ij} &\mapsto \rho(x_{ij}) = y_{ij}. \end{aligned}$$

On one hand, (2b) implies that $y_{ij}^2 = 1$, that is, $y_{ij} = \pm 1$ for all distinct $i, j \in [4]$. On the other hand (2d) implies for $i < j < k \in [4]$, that:

$$y_{ij} y_{jk} - (y_{jk} y_{ik} + y_{ik} y_{ij}) = 1.$$

And we have two cases to discuss:

1. The case of $y_{ij} y_{jk} = +1$ implies that $y_{ij} - y_{jk} = 0$ and $y_{jk} y_{ik} + y_{ik} y_{ij} = 0$.
2. The case of $y_{ij} y_{jk} = -1$ implies that $y_{ij} + y_{jk} = 0$ and $y_{jk} y_{ik} + y_{ik} y_{ij} = -2$.

A clear contradiction on both cases proving that indeed, there exists no 1-dimensional Λ-modules and the algebra Λ is not basic. ☐

Remark 24. We recall that the algebra $\mathcal{D}_3(+1, +1)$ is a proper subalgebra of Λ which is semisimple with a complete system of simple 2-dimensional modules given as $\{\rho_\sigma \mid \sigma \in \mathbb{S}_3\}$ where:

$$\rho_\sigma(x_{ij}) = \begin{cases} \begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix} & \mid \sigma(i,j) = (3,1), \\ \begin{bmatrix} 0 & +1 \\ +1 & 0 \end{bmatrix} & \mid \sigma(i,j) \in \{(1,2),(2,3)\}. \end{cases}$$

This means in particular that there exists no Λ-modules of odd dimensions.

Proposition 3.21. *Given $\sigma \in \mathbb{S}_4$. The algebra homomorphism $\rho_\sigma : \Lambda \to K^2$ defined by mapping a generator x_{ij} for distinct $i, j \in [4]$ to:*

$$\rho_\sigma(x_{ij}) = \begin{cases} \begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix} & \mid \sigma(i,j) \in \{(1,3),(4,2)\}, \\ \begin{bmatrix} 0 & +1 \\ +1 & 0 \end{bmatrix} & \mid \sigma(i,j) \in \{(1,2),(2,3),(3,4),(4,1)\}, \end{cases}$$

is a well-defined 2-dimensional simple Λ-representation.

Proof. Should ρ be a well-defined 2-dimensional Λ-representation, the simplicity falls directly as Λ has no 1-dimensional representations per Proposition 3.20.

Therefore, one must verify that subjecting ρ_σ to the defining relations does not yield any contradictions for all $\sigma \in \mathbb{S}_4$.

As (2b), and (2c) holds directly, one remarks that (2d) holds by observing that for distinct $\sigma(i), \sigma(j), \sigma(k) \in [4]$, then one of the following situations occurs:

$$\begin{aligned} \rho_\sigma(x_{ij}x_{jk}) &= 1, & \rho_\sigma(x_{jk}x_{ki} + x_{ki}x_{ij}) &= 0. \\ \rho_\sigma(x_{jk}x_{ki}) &= 1, & \rho_\sigma(x_{ij}x_{jk} + x_{ki}x_{ij}) &= 0. \\ \rho_\sigma(x_{ki}x_{ij}) &= 1, & \rho_\sigma(x_{ij}x_{jk} + x_{jk}x_{ki}) &= 0. \end{aligned}$$

Which implies the claim. $\square$

Lemma 3.22. *Let $\tau, \sigma \in \mathbb{S}_4$. Then for all $x \in \Lambda$ we have:*

$$(\tau.\rho_\sigma)(x) = (\rho_\sigma(\tau^{-}.x)) = \rho_{\sigma.\tau^{-}}(x).$$

Proof. As ρ_τ, $\rho_{\tau\sigma}$ and the group action by $\mathbb{S}_4$ are multiplicative, the claim is asserted by remarking that:

$$(\tau.\rho_\sigma)(x_{ij}) = (\rho_\sigma(\tau^{-}.x_{ij})) = \rho_{\sigma\tau^{-}}(x_{ij}).$$

for all x_{ij} generating Λ. $\square$

Note 21. Denote by $\mathbb{V}$ the Klein four-subgroup of the symmetric group $\mathbb{S}_4$, that is, the normal subgroup generated by the permutations $\nu_1 = (1,3)(2,4)$ and $\nu_2 = (1,4)(2,3)$, further, we set $\nu_3 = \nu_1\nu_2$.

Furthermore, we remind the reader that $\mathbb{S}_4/\mathbb{V} \cong \mathbb{S}_3$, which for convenience sake we choose to be generated by s_2, s_3.

Proposition 3.23. *The following hold:*

1. *Let $\sigma \in \mathbb{S}_4$, $\rho_\sigma \cong \nu_i.\rho_\sigma$ via conjugation with $m_{\sigma\nu_i\sigma^{-}}$ for $i \in [3]$ where:*

$$m_{\nu_1} := \begin{bmatrix} 0 & +1 \\ +1 & 0 \end{bmatrix}, \qquad m_{\nu_2} := \begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad m_{\nu_3} := \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix}.$$

2. *Given distinct $\sigma, \tau \in \mathbb{S}_3 \cong \mathbb{S}_4/\mathbb{V}$, then $\rho_\sigma \ncong \rho_\tau$.*

Proof. 1. Let $\sigma \in \mathbb{S}_4$, consider an arbitrary $x \in \Lambda$, then:

$$\nu_i.\rho_\sigma(x) = \rho_{\sigma\nu_i^{-}}(x) = \rho_{\sigma\nu_i\sigma^{-}\sigma}(x) = \sigma^{-}.\rho_{\sigma\nu_i\sigma^{-}}(x).$$

Now $\sigma\nu_i\sigma^{-} \in \mathbb{V}$ since $\mathbb{V}$ is a normal subgroup of $\mathbb{S}_4$, in other words, $\sigma\nu_i\sigma^{-} = \nu_j$ for some $j \in [3]$.

The claim is then asserted since $\rho_{\nu_j} \cong \rho_e$ via m_{ν_j}, that is:

$$\rho_{\nu_j}(x) = m_{\nu_j}.\rho_e(x).m_{\nu_j}^{-} \mid j \in [3].$$

2. Given distinct $\sigma, \tau \in \mathbb{S}_3 \cong \mathbb{S}_4/\mathbb{V}$, say with no loss of generality that $\sigma = e$. The claim is then asserted by verifying that there exists no nonzero homomorphism in the space of $Hom_K(\rho_e, \rho_\tau)$, that is, there exists no $f \in K^4$ such that the following hold:

$$\rho_e(x_{ij})f - f\rho_\tau(x_{ij}) = 0 \mid \text{ distinct } i, j \in [4], \tag{8}$$

Since the different cases are solved in a rather similar fashion, we shall avoid repetitiveness and solve the claim for $\tau = s_3$ and leave the other as an exercise for interested readers.

Indeed, should $\tau = s_3$ be. Then (8) for $(i, j) = (1, 2), (3, 4)$ hold if:

$$\begin{bmatrix} 0 & +1 \\ +1 & 0 \end{bmatrix} f = f \begin{bmatrix} 0 & +1 \\ +1 & 0 \end{bmatrix} = f \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},$$

that is, $f = 0$ and the space $Hom_K(\rho_e, \rho_{s_3})$ is generically null-dimensional, that is, $\rho_e \ncong \rho_{s_3}$.

□

Proposition 3.24. *The following hold:*

1. *Simple Λ-modules are exactly 2-dimensional.*
2. *Given ρ a 2-dimensional simple Λ-representation. Then there exists $\sigma \in \mathbb{S}_4$ such that $\rho = \rho_\sigma$.*

Proof. The claims are natural consequences of Proposition 3.23 and the fact that the semisimple quotient of Λ, that is, the quotient of Λ by its radical, is 24-dimensional, to which one easily concludes that simple Λ-modules are exactly 2-dimensional. Furthermore $\{\rho_\sigma \mid \sigma \in \mathbb{S}_3\}$ is a complete system of simple Λ-representations. □

Corollary 9. *For $i \in [3]$. The following hold:*

$$Ext^1_\Lambda(\rho_\sigma, \rho_\tau) \cong Ext^1_\Lambda(\nu_i\rho_\sigma, \nu_i\rho_\tau) \mid \sigma, \tau \in \mathbb{S}_3.$$

Theorem 3.25. *Given $\sigma, \tau \in \mathbb{S}_3$. The following hold:*

$$dim_K Ext^1_\Lambda(\rho_\sigma, \rho_\tau) = \begin{cases} 1 & \mid \tau = s_3\sigma, \\ 2 & \mid \tau = s_2\sigma, \\ 0 & \mid \textit{ otherwise.} \end{cases}$$

Remark 25. In the purpose of proving Theorem 3.25, we start by utilizing Lemma 3.22, which implies that we may set $\sigma = e$ with no further restrictions.

Furthermore, we see that for $\tau \in \mathbb{S}_3$, generic elements of the space of extensions $Ext^1_\Lambda(\rho_e, \rho_\tau)$ are of the form:

$$\begin{aligned}
&\rho : \Lambda \to K^{4\times 4} \\
&x_{ij} \mapsto \rho(x_{ij}) = \begin{bmatrix} \rho_e & f_{(e;\tau)} \\ 0 & \rho_\tau \end{bmatrix} (x_{ij}) \mid f_{(e;\tau)}(x_{ij}) = f_{ij} = \begin{bmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{bmatrix},
\end{aligned}$$

such that the defining relations of the algebra hold.

Note 22. Often in our computations we make basis change of the last two canonical generating vectors of K^4, that is, we say:

$$e_3' := \lambda_1 e_1 + \lambda_2 e_2 \pm e_3, \qquad\qquad e_4' := \lambda_3 e_1 + \lambda_4 e_2 \pm e_4.$$

for some field parameters $\lambda_1, \cdots, \lambda_4$.

Example 3.5. One computes that with a basis change of K^4 by:

$$\lambda_1 = +b_{12}, \qquad\qquad \lambda_2 = +d_{12}, \qquad\qquad \lambda_3 = +b_{13},$$

a generic element of the space $Ext^1_\Lambda(\rho_e, \rho_e)$ takes the form:

$$\begin{aligned}
&\rho : \Lambda \to K^{4\times 4} \\
&x_{ij} \mapsto \rho(x_{ij}) = \begin{bmatrix} \rho_e & f_{(e;e)} \\ 0 & \rho_e \end{bmatrix} (x_{ij}) \mid f_{(e;e)}(x_{ij}) = f_{ij} = \begin{bmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{bmatrix},
\end{aligned}$$

where:

$$f_{12} = \begin{bmatrix} a_{12} & 0 \\ c_{12} & 0 \end{bmatrix}, \qquad\qquad f_{13} = \begin{bmatrix} a_{13} & 0 \\ c_{13} & d_{13} \end{bmatrix},$$

to which one computes that:

$$\begin{aligned}
f(x_{12}) = f(x_{34}) = 0, \\
f(x_{13}) = f(x_{24}) = 0, \\
f(x_{14}) = f(x_{23}) = 0.
\end{aligned}$$

Where the first line is implied by (2b) and (2c), the last two lines are implied by (2b), (2c) and (2d) for $(i = 1, j = 2, k = 3)$, $(i = 1, j = 3, k = 4)$ and $(i = 4, j = 3, k = 1)$. That is, generic elements of the space $Ext^1_\Lambda(\rho_e, \rho_e)$ are isomorphic to the trivial extension.

Proposition 3.26. *Given $\sigma \in \mathbb{S}_3$ a non-simple transposition. Then the space of extensions $Ext^1_\Lambda(\rho_e, \rho_\tau)$ is of null-dimension.*

Proof. As the case of $\tau = e$ has already been discussed in the previous example, the proof is done case by case and is fairly similar as before, we detail the remaining three cases for τ by taking:

$$\rho : \Lambda \to K^{4\times 4}$$
$$x_{ij} \mapsto \rho(x_{ij}) = \begin{bmatrix} \rho_e & f_{(e;\tau)} \\ 0 & \rho_\tau \end{bmatrix}(x_{ij}) \mid f_{(e;\tau)}(x_{ij}) = f_{ij} = \begin{bmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{bmatrix},$$

a generic element of the space $Ext^1_\Lambda(\rho_e, \rho_\tau)$.

For $\tau = s_3 s_2 s_3$. One computes that with a basis change of K^4 by:

$$\lambda_1 = +a_{24}, \qquad \lambda_2 = +c_{13}, \qquad \lambda_3 = -a_{12}, \qquad \lambda_4 = -c_{12},$$

implies that generically ρ satisfies:

$$f_{12} = \begin{bmatrix} 0 & b_{12} \\ 0 & d_{12} \end{bmatrix}, \qquad f_{13} = \begin{bmatrix} a_{13} & b_{13} \\ 0 & d_{13} \end{bmatrix}, \qquad f_{24} = \begin{bmatrix} 0 & b_{24} \\ c_{24} & d_{24} \end{bmatrix}.$$

to which one computes that:

$$f_{12} = f_{34} = 0,$$
$$f_{13} = f_{24} = f_{14} = f_{23} = 0.$$

where the first line is implied by (2b) and (2c), while the second one is implied by (2b) and (2d) for the cases of $(i = 1, j = 2, k = 3)$, $(i = 1, j = 2, k = 4)$, and $(i = 1, j = 3, k = 4)$.

Next, for $\tau = s_3 s_2$. One computes that with a basis change of K^4 by:

$$\lambda_1 = a_{14}, \qquad \lambda_2 = c_{23}, \qquad \lambda_3 = a_{13}, \qquad \lambda_4 = c_{13},$$

implies that generically ρ satisfies:

$$f_{13} = \begin{bmatrix} 0 & b_{13} \\ 0 & d_{13} \end{bmatrix}, \qquad f_{14} = \begin{bmatrix} 0 & b_{14} \\ c_{14} & d_{14} \end{bmatrix}, \qquad f_{23} = \begin{bmatrix} a_{23} & b_{23} \\ 0 & d_{23} \end{bmatrix},$$

to which one computes that:

$$f_{13} = f_{24} = 0,$$
$$f_{12} = f_{23} = f_{14} = 0.$$

Where the first line is implied by (2b), (2c), the second line is implied by (2b), and (2d) for the cases of $(i = 1, j = 2, k = 3)$, and $(i = 4, j = 2, k = 1)$, finally, $f_{34} = 0$ is held by (2b), and (2d) for the special cases of $(i = 1, j = 3, k = 4)$, and $(i = 4, j = 3, k = 2)$.

Finally, for $\tau = s_2 s_3$. One computes that with a basis change of K^4 by:

$$\lambda_1 = -b_{24}, \qquad \lambda_2 = -d_{24}, \qquad \lambda_3 = -a_{13}, \qquad \lambda_4 = -c_{13},$$

implies that generically ρ satisfies:

$$f_{13} = \begin{bmatrix} 0 & b_{13} \\ 0 & d_{13} \end{bmatrix}, \qquad\qquad f_{24} = \begin{bmatrix} a_{24} & 0 \\ c_{24} & 0 \end{bmatrix},$$

to which one computes that:

$$f_{13} = f_{24} = 0,$$
$$f_{12} = f_{34} = 0.$$

Where the first line is implied by (2b), the second line is implied by (2b), (2c), and (2d) for (i= 1, j= 2, k= 3), $f_{14} = 0$ is implied by (2b), and (2d) for the cases of $(i = 1, j = 2, k = 4)$, and $(i = 4, j = 3, k = 1)$, finally, $f_{23} = 0$ is held by (2b), and (2d) for the special cases of $(i = 2, j = 3, k = 4)$, and $(i = 3, j = 2, k = 1)$.

That is, and with all possible cases considered, for τ a non-simple transposition, a generic $\rho \in Ext^1_\Lambda(\rho_e, \rho_\tau)$ is isomorphic to zero and the space is null-dimensional as claimed. □

Proposition 3.27. *The following hold:*

$$dim_K Ext^1_\Lambda(\rho_e, \rho_{s_i}) = \begin{cases} 1 & | \; i = 3, \\ 2 & | \; i = 2. \end{cases}$$

Proof. Given $\rho \in Ext^1_\Lambda(\rho_e, \rho_{s_i})$ a generic element.

For $i = 3$. One computes that with a basis change of K^4 by:

$$\lambda_1 = +a_{14}, \qquad \lambda_2 = +d_{12}, \qquad \lambda_3 = -a_{13}, \qquad \lambda_4 = +d_{12},$$

implies that the defining relations of the algebra hold if:

$$f_{ij} = \begin{cases} 0 & | \; (i, j) \neq (3, 4), \\ c^2.diag(+1, +1) & | \; (i, j) = (3, 4). \end{cases}$$

for some K-parameter c^2, that is to say, the space $Ext^1_\Lambda(\rho_e, \rho_{s_3})$ is 1-dimensional.

Finally, for $i = 2$. One computes that with a basis change of K^4 by:

$$\lambda_1 = +b_{24}, \qquad \lambda_2 = +d_{24}, \qquad \lambda_3 = +a_{13}, \qquad \lambda_4 = +c_{13},$$

implies that the defining relations of the algebra hold if:

$$f_{ij} = \begin{cases} 0 & | \; (i, j) \neq (1, 4), (2, 3), \\ c^1 diag(-1, +1) & | \; (i, j) = (1, 4), \\ c^3 diag(+1, +1) & | \; (i, j) = (2, 3). \end{cases}$$

for some K-parameter $c^1, c3$, that is to say, the space $Ext^1_\Lambda(\rho_e, \rho_{s_2})$ is 2-dimensional. □

Remark 26. We remark that the claim made in Theorem 3.25 has been proved per Propositions 3.26, and 3.27.

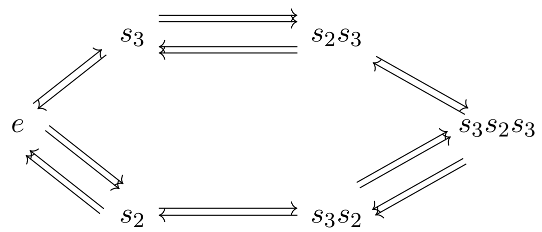
Note 23. Given $\sigma \in \mathbb{S}_3$. The space of extensions $Ext^1_\Lambda(\rho_\sigma, \rho_{s_3\sigma})$ is 1-dimensional, we say it is generated by $\{f^2_{(\sigma,s_3)}\}$, similarly, the space $Ext^1_\Lambda(\rho_\sigma, \rho_{s_2\sigma})$ is 2-dimensional, we say it is generated by $\{f^i_{(\sigma,s_2)} \mid i = 1, 3\}$.

Theorem 3.28. *The algebra $\Lambda^b(\alpha, \alpha)$ admits an ordinary quiver presentation of the form (Q_0, Q_1, s, t) where:*

1. *The vertices set is $\mathbb{S}_3$.*
2. *Given $\sigma, \tau \in \mathbb{S}_3$. The number of arrows from σ to τ is 1 if $\tau = s_3\sigma$, 2 if $\tau = s_2\sigma$ and 0 otherwise.*

Proof. The claim follows since Λ and Λ^b are Morita equivalent, which preserves simplicity and extensions. In particular, the vertices are indexed by $\mathbb{S}_3$ since $\{\rho_\sigma \mid \sigma \in \mathbb{S}_3\}$ is a complete set of simple Λ-modules per Proposition 3.24 up to the isomorphism described in Proposition 3.23. While the number of arrows follows from Theorem 3.25. $\square$

Remark 27. We draw the ordinary quiver of Λ^b as follows:



Note 24. Given $\sigma \in \mathbb{S}_3$. We call the two arrows starting in σ and terminating in $s_2\sigma$ by $\beta_i(\sigma; s_2)$ for $i = 1, 3$, furthermore, $\beta_2(\sigma; s_3)$ denotes the single arrow starting in σ terminating in $s_3\sigma$.

Moreover, we interpret the action of the symmetric group $\mathbb{S}_3$ on paths by changing the starting vertex.

Furthermore, we denote by ϕ Gabriel's theorem morphism associated with Λ^b.

Finally, we remind the reader that Gabriel's theorem states that $ker(\phi)$ is a two-sided admissible ideal of KQ_{Λ^b} with an associated quotient that is isomorphic to Λ^b, which is by default Morita equivalent to Λ, that is, both modules categories of Λ and Λ^b are equivalent.

Remark 28. As seen before, we observe that the algebra KQ_{Λ^b} admits an indecomposable decomposition of the form:

$$KQ_{\Lambda^b} = \bigoplus_{\sigma \in \mathbb{S}_3} KQ_{\Lambda^b},$$

which in particular, reduces the study of Λ^b to that of Γ^b, the indecomposable projective Λ^b-module understood to be a quotient of $KQ_{\Gamma^b} := e_e KQ_{\Lambda^b}$ by the kernel of $\pi := \phi|_{e_e}$.

Lemma 3.29. *The algebra* Γ^b *is 24-dimensional.*

Proof. We start by observing that the algebra Λ is $4!^2$-dimensional.

Now, the isomorphism given by the Klein four subgroup action -as described in Proposition 3.23- implies that the associated basic algebra Λ^b is $4!^2/4$-dimensional, to which one sees that the set $\{e_\sigma \mid \sigma \in \mathbb{S}_3\}$ is complete of primitive orthogonal idempotents in Λ^b, therefore, all indecomposable projective Λ^b-representation are of the same dimension.

In particular, this yields the claim and show that the graded algebra $gr\Gamma^b$ is 24-dimensional as well. $\square$

Note 25. Given $\sigma \in \mathbb{S}_3$. We proved that:

$$dim_K Ext^1_\Lambda(\rho_\sigma, \rho_{s_i\sigma}) = \begin{cases} 1 & \mid i = 3, \\ 2 & \mid i = 2. \end{cases}$$

Viewed as equivalence class of (1)-cocycles up to the space of coboundaries we shall rename the single generator of such space by $\{\overline{f^2_{(\sigma;s_3)}}, \overline{f^i_{(\sigma;s_2)}} \mid i = 1, 3.\}$ and omit the bar notation should no confusion occurs.

Remark 29. Since $\{f^2_{(\sigma;s_3)}, f^i_{(\sigma;s_2)}\}$ are 2-dimensional, distinguishing the parameters becomes necessary unlike it was in the first part of our discussion. That being said, we notice that such writing in practice leads to tiresome reading, therefore, we will omit writing the K-parameters multiplied with $f^2_{(\sigma;s_3)}, f^i_{(\sigma;s_2)}$ and once needed, we shall denote them simply by $c^2_{(\sigma;s_3)}, c^i_{(\sigma;s_2)}$, for $i = 1, 3$.

Proposition 3.30. *Let:*

$$\begin{aligned} r_i &:= \beta_i(e; s_2)\beta_i(s_2; s_2) \mid i = 1, 3, \\ r_2 &:= \beta_2(e; s_3)\beta_2(s_3; s_3), \\ r_4 &:= \beta_1(e; s_2)\beta_3(s_2; s_2) - \beta_3(e; s_2)\beta_1(s_2; s_2). \end{aligned}$$

Then $r_i \in ker(gr\pi)$ *for* $i \in [4]$. *In particular, the space of* $KQ_{\Gamma^b}/ker(gr\pi)$*-paths of length 2 that start in* e *is exactly 5-dimensional with a basis of the form:*

$$\begin{aligned} \{&\beta_1(e; s_2)\beta_3(s_2; s_2), \beta_1(e; s_2)\beta_2(s_2; s_3), \beta_3(e; s_2)\beta_2(s_2; s_3), \\ &\beta_2(e; s_3)\beta_1(s_3; s_2), \beta_2(e; s_3)\beta_3(s_3; s_2)\}. \end{aligned}$$

Proof. The first part of the claim is an alternative interpretation of claiming that the space of $KQ_{\Gamma^b}/ker(gr\pi)$-paths of length 2 such that start and terminate in e is exactly 1-dimensional with a basis of the form:

$$\{\beta_1(e; s_2)\beta_3(s_2; s_2) = \beta_3(e; s_2)\beta_1(s_2; s_2)\},$$

which we shall show by existence.

Indeed let:

$$\rho = \begin{bmatrix} \rho_e & f^2_{(e;s_3)} & f^i_{(e;s_2)} & g_{(e;e)} \\ 0 & \rho_{s_3} & 0 & f^2_{(s_3;s_3)} \\ 0 & 0 & \rho_{s_2} & f^i_{(s_2;s_2)} \\ 0 & 0 & 0 & \rho_e \end{bmatrix}, \tag{9}$$

be an 8-dimensional graded (up to the third power of the radical) Λ-map, where $g_{(e;e)}$ is a set of K^2-parameters, of the form:

$$g_{(e;e)}(x_{ij}) = \begin{bmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{bmatrix} \in K^2.$$

We see that:

1. Subjecting ρ to (2b) hold once:

$$diag(g_{(e;e)}(x_{13})) = diag(g_{(e;e)})(x_{24}) = 0,$$

along side:

$$\begin{aligned} a_{12} &= -d_{12}, & b_{12} &= -d_{12}, & a_{14} &= -d_{14}, \\ a_{23} &= -d_{23}, & a_{34} &= -d_{34}. \end{aligned}$$

as well as:

$$\begin{aligned} c_{14} &= -b_{14} + c^1_{(e;s_2)} c^1_{(s_2;s_2)}, \\ c_{23} &= -b_{23} - c^3_{(e;s_2)} c^3_{(s_2;s_2)}, \\ c_{34} &= -b_{34} - c^2_{(e;s_3)} c^2_{(s_3;s_3)}. \end{aligned}$$

2. In addition to (2b), subjecting ρ to (2c) hold once:

$$d_{12} = +d_{34}, \qquad c_{13} = -c_{24}, \qquad d_{14} = -d_{23}, \qquad b_{24} = -b_{13},$$

along side:

$$\begin{aligned} b_{23} &= -b_{14} - (1/2) \begin{bmatrix} -c^1_{(e;s_2)} c^1_{(s_2;s_2)} + c^1_{(e;s_2)} c^3_{(s_2;s_2)} \\ -c^3_{(e;s_2)} c^1_{(s_2;s_2)} + c^3_{(e;s_2)} c^3_{(s_2;s_2)} \end{bmatrix}, \\ b_{34} &= -c_{12} - (1/2) \left[c^2_{(e;s_3)} c^2_{(s_3;s_2)} \right]. \end{aligned}$$

3. In addition to (2b), and (2c) subjecting ρ to (2d) for $(i = 1, j = 2, k = 4)$ holds

once we set:

$$
\begin{aligned}
b_{14} &= c^1_{(e;s_2)}c^1_{(s_2;s_2)} + c_{12},\\
c_{24} &= 2c^1_{(e;s_2)}c^1_{(s_2;s_2)} - 2d_{23} + \frac{b_{13}}{2},\\
d_{34} &= c_{13} - b_{14} + d_{23}.
\end{aligned}
$$

and:

$$c^1_{(e;s_2)}c^1_{(s_2;s_2)} = 0,$$

Furthermore, (2d) for $(i = 4, j = 3, k = 1)$ holds if:

$$c^2_{(e;s_3)}c^2_{(s_3;s_3)} = 0,$$

finally, (2d) for $(i = 1, j = 2, k = 3)$ holds if both:

$$
\begin{aligned}
c^3_{(e;s_2)}c^3_{(s_2;s_2)} &= 0,\\
c^1_{(e;s_2)}c^3_{(e;s_2)} - c^3_{(e;s_2)}c^1_{(s_2;s_2)} &= 0.
\end{aligned}
$$

With all the other relations satisfied.

In other words, ρ as defined in (9) is actually a graded Λ-representation, if and only if for $i = 1, 3$, the following hold:

$$
\begin{aligned}
f^i_{(e;s_2)}f^i_{(s_2;s_2)} + (rad\Lambda)^3 &= 0,\\
f^2_{(e;s_3)}f^2_{(s_3;s_3)} + (rad\Lambda)^3 &= 0,\\
(f^1_{(e;s_2)}f^3_{(s_2;s_2)} - f^3_{(e;s_2)}f^1_{(s_2;s_2)}) + (rad\Lambda)^3 &= 0.
\end{aligned}
$$

That is, for $i = 1, 3$, the following paths are in $ker(gr\pi)$:

$$
\begin{aligned}
\beta_i(e;s_2)\beta_i(s_2;s_2),\\
\beta_2(e;s_3)\beta_2(s_3;s_3),\\
\beta_1(e;s_2)\beta_3(s_2;s_2) - \beta_3(e;s_2)\beta_1(s_2;s_2).
\end{aligned}
$$

Since Λ and Λ^b -which contains Γ^b as an indecomposable projective module- are morita equivalent.

Finally, we see that the space $KQ_{\Gamma^b}/ker(gr\pi)$-paths of length 2 that start in e and terminate in s_3s_2 is 2-dimensional with a basis of the form:

$$\{\beta_1(e;s_2)\beta_2(s_2;s_3), \beta_3(e;s_2)\beta_2(s_2;s_3)\},$$

which corresponds to the existence of a graded Λ-representation:

$$\rho = \begin{bmatrix} \rho_e & f^i_{(e;s_2)} & g^i_{(e;s_3s_2)} \\ 0 & \rho_{s_2} & f^2_{(s_2;s_3)} \\ 0 & 0 & \rho_{s_3s_2} \end{bmatrix},$$

where:

$$g^1(e;s_3s_2) = \begin{cases} 0 & \mid (i,j) \neq (1,2), \\ \frac{-c^1_{(e;s_2)}c^2_{(s_2;s_3)}}{2}.diag(+1,+1) & \mid (i,j) = (1,2). \end{cases} \tag{10}$$

$$g^3(e;s_3s_2) = \begin{cases} 0 & \mid (i,j) \neq (3,4), \\ \frac{+c^3_{(e;s_2)}c^2_{(s_2;s_3)}}{2}.diag(+1,-1) & \mid (i,j) = (3,4). \end{cases} \tag{11}$$

While those of length 2 that start in e and terminate in s_2s_3 is 2-dimensional with a basis of the form:

$$\{\beta_2(e;s_3)\beta_1(s_3;s_2), \beta_2(e;s_3)\beta_3(s_3;s_2)\},$$

which corresponds to the existence of a graded Λ-representation:

$$\rho = \begin{bmatrix} \rho_e & f^2_{(e;s_3)} & g^i_{(e;s_2s_3)} \\ 0 & \rho_{s_3} & f^i_{(s_3;s_2)} \\ 0 & 0 & \rho_{s_2s_3} \end{bmatrix},$$

where:

$$g^1(e;s_2s_3) = \begin{cases} 0 & \mid (i,j) \neq (1,4), \\ \frac{c^2_{(e;s_3)}c^1_{(s_3;s_2)}}{2}.diag(+1,+1) & \mid (i,j) = (1,4). \end{cases} \tag{12}$$

$$g^3(e;s_2s_3) = \begin{cases} 0 & \mid (i,j) \neq (2,3), \\ \frac{c^2_{(e;s_3)}c^3_{(s_3;s_2)}}{2}.diag(+1,-1) & \mid (i,j) = (2,3). \end{cases} \tag{13}$$

□

Corollary 10. *The following hold:*

$$\{\sigma r_i \mid \sigma \in \mathbb{S}_3\} \subseteq ker(gr\pi).$$

Example 3.6. Given $\sigma \in \mathbb{S}_3$, then there exists two distinct KQ_{Γ^b}-paths of the form: On one hand:

$$\sigma \Rrightarrow s_2\sigma \longrightarrow s_3s_2\sigma$$

to which one corresponds a graded Λ-representation:

$$\rho = \begin{bmatrix} \rho_\sigma & f^i_{(\sigma;s_2)} & g^i_{(\sigma;s_3s_2)} \\ 0 & \rho_{s_2\sigma} & f^2_{(s_2\sigma;s_3)} \\ 0 & 0 & \rho_{s_3s_2\sigma} \end{bmatrix},$$

up to the third power of the radical. In particular, we compute that:

$$g^1(s_3;s_3s_2) = \begin{cases} 0 & \mid (i,j) \neq (1,2), \\ \frac{-c^1_{(s_3;s_2)}c^2_{(s_2s_3;s_3)}}{2}.diag(+1,+1) & \mid (i,j) = (1,2). \end{cases} \tag{14}$$

$$g^3(s_3;s_3s_2) = \begin{cases} 0 & \mid (i,j) \neq (3,4), \\ \frac{+c^3_{(s_3;s_2)}c^2_{(s_2s_3;s_3)}}{2}.diag(-1,+1) & \mid (i,j) = (3,4). \end{cases} \tag{15}$$

On the other hand:

$$\sigma \longrightarrow s_3\sigma \Longrightarrow s_2s_3\sigma$$

to which one corresponds a graded Λ-representation:

$$\rho = \begin{bmatrix} \rho_\sigma & f^2_{(\sigma;s_3)} & g^i_{(\sigma;s_2s_3)} \\ 0 & \rho_{s_3\sigma} & f^i_{(s_3\sigma;s_2)} \\ 0 & 0 & \rho_{s_2s_3\sigma} \end{bmatrix},$$

up to the third power of the radical. In particular, we compute that:

$$g^1(s_2;s_2s_3) = \begin{cases} 0 & \mid (i,j) \neq (1,4), \\ \frac{c^2_{(s_2;s_3)}c^1_{(s_3s_2;s_2)}}{2}.diag(+1,+1) & \mid (i,j) = (1,4). \end{cases} \tag{16}$$

$$g^3(s_2;s_2s_3) = \begin{cases} 0 & \mid (i,j) \neq (2,3), \\ \frac{c^2_{(s_2;s_3)}c^3_{(s_3s_2;s_2)}}{2}.diag(-1,+1) & \mid (i,j) = (2,3). \end{cases} \tag{17}$$

Proposition 3.31. *Let:*

$$r_5 := \begin{bmatrix} -\beta_1(e;s_2)\beta_2(s_2;s_3)\beta_3(s_3s_2;s_2) \\ -\beta_3(e;s_2)\beta_2(s_2;s_3)\beta_1(s_3s_2;s_2) \\ +\beta_2(e;s_3)\beta_1(s_3;s_2)\beta_2(s_2s_3;s_3) \end{bmatrix}, \quad r_6 := \begin{bmatrix} +\beta_1(e;s_2)\beta_2(s_2;s_3)\beta_1(s_3s_2;s_2) \\ -\beta_3(e;s_2)\beta_2(s_2;s_3)\beta_3(s_3s_2;s_2) \\ +\beta_2(e;s_3)\beta_3(s_3;s_2)\beta_2(s_2s_3;s_3) \end{bmatrix}.$$

then $r_i \in ker(gr\pi)$ for $i = 5,6$.

Proof. The claim falls once we show that the space of $KQ_{\Gamma^b}/ker(gr\pi)$-paths of length 3 such that start in e and terminate in $s_3s_2s_3$ is exactly 2-dimensional with a basis of the form:

$$\{\beta_1(e;s_2)\beta_2(s_2;s_3)\beta_3(s_3s_2;s_2), \beta_1(e;s_2)\beta_2(s_2;s_3)\beta_1(s_3s_2;s_2)\}.$$

Which we shall show by existence.

Indeed let:

$$\rho = \begin{bmatrix} \rho_e & f^i_{(e;s_2)} & g^i(e;s_3s_2) & f^2_{(e;s_3)} & g^i(e;s_2s_3) & g(e;s_3s_2s_3) \\ 0 & \rho_{s_2} & f^2_{(s_2;s_3s_2)} & 0 & 0 & g^i(s_2;s_2s_3) \\ 0 & 0 & \rho_{s_3s_2} & 0 & 0 & f^i_{(s_3s_2;s_3)} \\ 0 & 0 & 0 & \rho_{s_3} & f^i_{(s_3;s_2)} & g^i(s_3;s_3s_2) \\ 0 & 0 & 0 & 0 & \rho_{s_2s_3} & f^i_{(s_2s_3;s_3)} \\ 0 & 0 & 0 & 0 & 0 & \rho_{(s_3s_2s_3)} \end{bmatrix} \tag{18}$$

be a graded (up to the fourth power of the radical), 12-dimensional Λ-map, where $g^i(e;s_3s_2)$, $g^i(e;s_2s_3)$, $g^i(s_2;s_2s_3)$ and $g^i(s_3;s_3s_2)$ are the sets of K^2-parameters determined in (10), (11), (12), (13), (14), (15), (16) and (17). And $g(e;s_3s_2s_3)$ is a set of K^2-parameters, of the form:

$$g(e;s_3s_2s_3)(x_{ij}) = \begin{bmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{bmatrix}.$$

We see that:

1. Subjecting ρ to (2b) hold once set:

   $$a_{13} = 0, \qquad d_{13} = 0, \qquad b_{24} = 0, \qquad c_{24} = 0,$$

   and:

   $$a_{12} = d_{12}, \qquad a_{34} = d_{34}, \qquad a_{14} = d_{14}, \qquad a_{23} = d_{23},$$

   along side:

   $$\begin{aligned} b_{12} &= c_{12} - \frac{c^1_{(e;s_2)} c^2_{(s_2;s_3)} c^1_{(s_3s_2;s_2)}}{2}, \\ c_{14} &= b_{14} + \frac{c^1_{(e;s_2)} c^2_{(s_2;s_3)} c^1_{(s_3s_2;s_2)}}{2}, \\ b_{23} &= c_{23} + \frac{(-c^3_{(e;s_2)} c^2_{(s_2;s_3)} c^3_{(s_3s_2;s_2)} + c^2_{(e;s_3)} c^3_{(s_3;s_2)} c^2_{(s_2s_3;s_3)})}{2}, \\ b_{34} &= c_{34} + \frac{(+c^3_{(e;s_2)} c^2_{(s_2;s_3)} c^3_{(s_3s_2;s_2)} - c^2_{(e;s_3)} c^3_{(s_3;s_2)} c^2_{(s_2s_3;s_3)})}{2}. \end{aligned}$$

2. In addition to (2b), subjecting ρ to (2c) hold once we set:

   $$a_{14} = -a_{23}, \qquad d_{12} = +d_{34},$$

along side:

$$c_{12} = +c_{34} + 1/4 \begin{bmatrix} +c^1_{(e;s_2)}c^2_{(s_2;s_3)}c^1_{(s_3s_2;s_2)} - c^1_{(e;s_2)}c^2_{(s_2;s_3)}c^3_{(s_3s_2;s_2)} \\ -c^3_{(e;s_2)}c^2_{(s_2;s_3)}c^1_{(s_3s_2;s_2)} + c^3_{(e;s_2)}c^2_{(s_2;s_3)}c^3_{(s_3s_2;s_3)} \\ +c^2_{(e;s_3)}c^1_{(s_3;s_2)}c^2_{(s_2s_3;s_3)} - c^2_{(e;s_3)}c^3_{(s_3;s_2)}c^2_{(s_2s_3;s_3)} \end{bmatrix},$$

$$c_{23} = -b_{14} + 1/4 \begin{bmatrix} -c^1_{(e;s_2)}c^2_{(s_2;s_3)}c^1_{(s_3s_2;s_2)} - c^1_{(e;s_2)}c^2_{(s_2;s_3)}c^3_{(s_3s_2;s_2)} \\ -c^3_{(e;s_2)}c^2_{(s_2;s_3)}c^1_{(s_3s_2;s_2)} + c^3_{(e;s_2)}c^2_{(s_2;s_3)}c^3_{(s_3s_2;s_2)} \\ +c^2_{(e;s_3)}c^1_{(s_3;s_2)}c^2_{(s_2s_3;s_3)} - c^2_{(e;s_3)}c^3_{(s_3;s_2)}c^2_{(s_2s_3;s_3)} \end{bmatrix}.$$

3. In addition to (2b), and (2c), subjecting ρ to (2d) for $(i = 1, j = 2, k = 3)$ hold once we have:

$$d_{34} = b_{13} - c_{13} - b_{14} - a_{23} - c_{34},$$

$$b_{14} = c_{13} + 2a_{23} + (1/2)\left[-c^3_{(e;s_2)}c^2_{(s_2;s_3)}c^3_{(s_3s_2;s_2)} + c^2_{(e;s_3)}c^3_{(s_3;s_2)}c^2_{(s_2s_3;s_3)}\right],$$

$$d_{24} = a_{24} + 2c_{34} + (1/4) \begin{bmatrix} +c^1_{(e;s_2)}c^2_{(s_2;s_3)}c^1_{(s_3s_2;s_2)} - c^1_{(e;s_2)}c^2_{(s_2;s_3)}c^3_{(s_3s_2;s_2)} \\ -c^3_{(e;s_2)}c^2_{(s_2;s_3)}c^1_{(s_3s_2;s_2)} + c^3_{(e;s_2)}c^2_{(s_2;s_3)}c^3_{(s_3s_2;s_2)} \\ +c^2_{(e;s_3)}c^1_{(s_3;s_2)}c^2_{(s_2s_3;s_3)} - c^2_{(e;s_3)}c^3_{(s_3;s_2)}c^2_{(s_2s_3;s_3)} \end{bmatrix}.$$

However, the (2d) for all the other cases hold should:

$$\left[+c^1_{(e;s_2)}c^2_{(s_2;s_3)}c^1_{(s_3s_2;s_2)} - c^3_{(e;s_2)}c^2_{(s_2;s_3)}c^3_{(s_3s_2;s_2)} + c^2_{(e;s_3)}c^3_{(s_3;s_2)}c^2_{(s_2s_3;s_3)}\right] = 0,$$

$$\left[+c^1_{(e;s_2)}c^2_{(s_2;s_3)}c^3_{(s_3s_2;s_2)} + c^3_{(e;s_2)}c^2_{(s_2;s_3)}c^1_{(s_3s_2;s_2)} - c^2_{(e;s_3)}c^1_{(s_3;s_2)}c^2_{(s_2s_3;s_3)}\right] = 0.$$

In other words, ρ as defined in (18) is actually a graded Λ-representation, if and only if:

$$\begin{bmatrix} +f^1_{(e;s_2)}f^2_{(s_2;s_3)}f^1_{(s_3s_2;s_2)} \\ -f^3_{(e;s_2)}f^2_{(s_2;s_3)}f^3_{(s_3s_2;s_2)} \\ +f^2_{(e;s_3)}f^3_{(s_3;s_2)}f^2_{(s_2s_3;s_3)} \end{bmatrix} + (rad\Lambda)^4 = 0, \quad \begin{bmatrix} -f^1_{(e;s_2)}f^2_{(s_2;s_3)}f^3_{(s_3s_2;s_2)} \\ -f^3_{(e;s_2)}f^2_{(s_2;s_3)}f^1_{(s_3s_2;s_2)} \\ +f^2_{(e;s_3)}f^1_{(s_3;s_2)}f^2_{(s_2s_3;s_3)} \end{bmatrix} + (rad\Lambda)^4 \quad = 0.$$

In other words, the following paths are in $ker(gr\pi)$:

$$\begin{bmatrix} -\beta_1(e;s_2)\beta_2(s_2;s_3)\beta_3(s_3s_2;s_2) \\ -\beta_3(e;s_2)\beta_2(s_2;s_3)\beta_1(s_3s_2;s_2) \\ +\beta_2(e;s_3)\beta_1(s_3;s_2)\beta_2(s_2s_3;s_3) \end{bmatrix}, \qquad \begin{bmatrix} +\beta_1(e;s_2)\beta_2(s_2;s_3)\beta_1(s_3s_2;s_2) \\ -\beta_3(e;s_2)\beta_2(s_2;s_3)\beta_3(s_3s_2;s_2) \\ +\beta_2(e;s_3)\beta_3(s_3;s_2)\beta_2(s_2s_3;s_3) \end{bmatrix}.$$

Since Λ and Λ^b -which contains Γ^b as an indecomposable projective module- are morita equivalent. $\square$

Corollary 11. *For $i = 5, 6$. The following hold:*

$$\{\sigma r_i \mid \sigma \in \mathbb{S}_3\} \subseteq ker(gr\pi).$$

Lemma 3.32. *Up to a higher power of the radical, the algebra $gr\Gamma^b$ is isomorphic to the free associative algebra $K\langle s_1, s_2, s_3\rangle$ bounded by the ideal J' generated by:*

$$\begin{aligned}
\iota_i &:= s_i^2, \mid i \in [3],\\
\iota_4 &:= s_1s_3 - s_3s_1,\\
\iota_5 &:= s_2s_1s_2 - s_1s_2s_3 - s_3s_2s_1,\\
\iota_6 &:= s_2s_3s_2 - s_1s_2s_1 + s_3s_2s_3.
\end{aligned}$$

Proof. We start with computing that $K\langle s_1, s_2, s_3\rangle/J'$ is 24-dimensional with an ordered basis:

$$\begin{aligned}
\{&1, s_1, s_3, s_2, s_1s_3, s_1s_2, s_3s_2, s_2s_1, s_2s_3,\\
&s_1s_3s_2, s_1s_2s_1, s_1s_2s_3, s_3s_2s_1, s_3s_2s_3, s_2s_1s_3,\\
&s_1s_3s_2s_1, s_1s_3s_2s_3, s_1s_2s_1s_3, s_3s_2s_1s_3, s_2s_1s_3s_2,\\
&s_1s_3s_2s_1s_3, s_1s_2s_1s_3s_2, s_3s_2s_1s_3s_2, (s_1s_3s_2)^2\}
\end{aligned}$$

to which mapping graded paths:

$$\beta_{i_1}(e; s_{j_1})\beta_{i_2}(s_{j_1}; s_{j_2}) \cdots \beta_{i_k}(s_{j_{k-1}} \cdots s_{j_1}; s_{j_k}),$$

to words $s_{i_1} \cdots s_{i_{k-1}} s_{i_k}$ gives a surjective algebra homomorphism, the claim is then asserted by remarking that both of $gr\Gamma^b$ and $K\langle s_1, s_2, s_3\rangle/J'$ are 24-dimensional as seen in Lemma 3.29. $\square$

Corollary 12. *The following hold:*

1. *Up to a fourth power of the radical, the space of $KQ_{\Gamma^b}/ker(gr\pi)$-paths of length 3 that start in e is exactly 6-dimensional with a basis of the form:*

$$\begin{aligned}
\{&\beta_1(e; s_2)\beta_3(s_2; s_2)\beta_2(e; s_3), \beta_1(e; s_2)\beta_2(s_2; s_3)\beta_1(s_3s_2; s_2),\\
&\beta_1(e; s_2)\beta_2(s_2; s_3)\beta_3(s_2s_3; s_2), \beta_3(e; s_2)\beta_2(s_2; s_3)\beta_1(s_3s_2; s_2),\\
&\beta_3(e; s_2)\beta_2(s_2; s_3)\beta_3(s_3s_2; s_2), \beta_2(e; s_3)\beta_1(s_3; s_2)\beta_3(s_2s_3; s_2)\}.
\end{aligned}$$

2. *Up to the fifth power of the radical, the space of $KQ_{\Gamma^b}/ker(gr\pi)$-paths of length 4 that start in e is exactly 5-dimensional with a basis of the form:*

$$\begin{aligned}
\{&\beta_1(e; s_2)\beta_3(s_2; s_2)\beta_2(e; s_3)\beta_1(s_3; s_2), \beta_1(e; s_2)\beta_3(s_2; s_2)\beta_2(e; s_3)\beta_3(s_3; s_2),\\
&\beta_1(e; s_2)\beta_2(s_2; s_3)\beta_1(s_3s_2; s_2)\beta_3(s_2s_3s_2; s_2),\\
&\beta_3(e; s_2)\beta_2(s_2; s_3)\beta_1(s_3s_2; s_2)\beta_3(s_2s_3s_2; s_2),\\
&\beta_2(e; s_3)\beta_1(s_3; s_2)\beta_3(s_2s_3; s_2)\beta_2(s_3; s_3).\}.
\end{aligned}$$

3. *Up to the sixth power of the radical, the space of $KQ_{\Gamma^b}/ker(gr\pi)$-paths of length 5 that start in e is exactly 3-dimensional with a basis of the form:*

$$\begin{aligned}\{&\beta_1(e;s_2)\beta_3(s_2;s_2)\beta_2(e;s_3)\beta_1(s_3;s_2)\beta_3(s_2s_3;s_2),\\ &\beta_1(e;s_2)\beta_2(s_2;s_3)\beta_1(s_3s_2;s_2)\beta_3(s_2s_3s_2;s_2)\beta_2(s_3s_2;s_3),\\ &\beta_3(e;s_2)\beta_2(s_2;s_3)\beta_1(s_3s_2;s_2)\beta_3(s_2s_3s_2;s_2)\beta_2(s_3s_2;s_3)\}.\end{aligned}$$

4. *Up to the seventh power of the radical, the space of $KQ_{\Gamma^b}/ker(gr\pi)$-paths of length 6 that start in e is exactly 1-dimensional with a basis of the form:*

$$\{\beta_1(e;s_2)\beta_3(s_2;s_2)\beta_2(e;s_3)\beta_1(s_3;s_2)\beta_3(s_2s_3;s_2)\beta_2(s_3;s_3)\}$$

5. *Up to a higher power of the radical, the space of $KQ_{\Gamma^b}/ker(gr\pi)$-paths of length 7 that start in e is null-dimensional.*

Note 26. We highlight two previously mentioned (basis elements) paths:

$$\begin{aligned}u_1 &:= \beta_2(e;s_3)\beta_1(s_3;s_2)\beta_3(s_2s_3;s_2)\beta_2(s_3;s_3)\\ u_2 &:= \beta_1(e;s_2)\beta_3(s_2;s_2)\beta_2(e;s_3)\beta_1(s_3;s_2)\beta_3(s_2s_3;s_2)\beta_2(s_3;s_3).\end{aligned}$$

which are unique in the sense that both paths start and terminate in e. For $i \in [2]$. Let v_i be the image of u_i in $K\langle s_1, s_2, s_3\rangle/J'$, that is, let:

$$v_1 := s_2s_3s_1s_2, \qquad v_2 := (s_1s_3s_2)^2.$$

Remark 30. Lemma 3.32 implies that there exits some K-polynomials q_i for $i \in [8]$ so that Γ^b is isomorphic to the free associative algebra $K\langle s_1, s_2, s_3\rangle$ modulo by the ideal J generated by:

$$\begin{aligned}\{&\iota_1 + q_1v_1 + q_2v_2, \iota_2 + q_3v_1 + q_4v_2,\\ &\iota_3 + q_5v_1 + q_6v_2, \iota_4 + q_7v_1 + q_8v_2,\\ &\iota_5, \iota_6.\}.\end{aligned}$$

Now, on one hand we compute:

$$\begin{aligned}0 &= (s_1\iota_4 - \iota_1s_3)\\ &= -s_1(q_7v_1 + q_8v_2) - (q_1v_1 + q_2v_2)\\ &= q_7(s_1s_2s_3s_1s_2) + q_1(s_3s_2s_3s_1s_2),\end{aligned}$$

also:

$$\begin{aligned}0 &= (s_1\iota_3 - \iota_4s_3)\\ &= -s_1(q_5v_1 + q_6v_2) - (q_7v_1 + q_8v_2)s_3\\ &= q_5(s_1s_2s_3s_1s_2) + q_7(s_3s_2s_3s_1s_2),\end{aligned}$$

that is, $q_1 = q_5 = q_7 = 0$. While on the other hand, for any nonzero K-polynomial q, one has:

$$(s_1 + q(s_3 s_2 s_3 s_1 s_2))^2 - (s_1^2 + 2qv_2) = 0$$
$$(s_2 + q(s_3 s_1 s_2))^2 - (s_2^2 + qv_1 + qv_2) = 0,$$

while there exists no basis element v such that:

$$(s_3 + qv)^2 - (s_3^2 + q'v_2) = 0.$$

Proposition 3.33. *The algebra Γ^b is isomorphic to the free associative algebra $K\langle s_1, s_2, s_3\rangle$ bounded by the ideal J generated by:*

$$\{\iota_1 + q_2 v_2, \iota_2 + q_3 v_1 + q_4 v_2,$$
$$\iota_3 + q_6 v_2, \iota_4 + q_8 v_2, \iota_5, \iota_6.\}.$$

for q_i some K-polynomials.

4 On the deformed Fomin-Kirillov-subalgebras.

Note 27. Let $\mathcal{Y}$ be a subalgebra of $\mathcal{D}_n(\alpha_1, \alpha_2)$ and let M be a $\mathcal{Y}$-module given generically. We denote by M^σ the module M under the action of σ an automorphism of $\mathcal{Y}$, that is:

$$M^\sigma(x) = M(\sigma x) \mid x \in \mathcal{Y}.$$

Definition 4.1. For any graph G with n-vertices, the deformed Fomin-Kirillov algebra $\mathcal{D}_G(\alpha_1, \alpha_2)$ is the subalgebra of $\mathcal{D}_n(\alpha_1, \alpha_2)$ generated by x_{ij} for all $(i,j) \in G$.

Example 4.1. Let K_n denote the complete graph with n vertices. Then $\mathcal{D}_n(\alpha_1, \alpha_2) = \mathcal{D}_{K_n}(\alpha_1, \alpha_2)$.

Note 28. Should no confusion occurs, we denote x_{ii+1} by simply x_i.

Remark 31. One may remark that the previous definition cover the special cases of "the non-deformed" Fomin-Kirillov algebra $\mathcal{E}_G$, the subalgebra of $\mathcal{E}_n = \mathcal{D}_n(0,0)$.

4.1 Relation with Iwahori-Hecke algebra

Note 29. By R we denote a commutative ring, we remind the reader that K denotes a field of character not equal to 2.

Definition 4.2. Given $n \geq 3$ a positive integer, Let $H_q(n)$ denotes the one-parameter Iwahori-Hecke-algebra, that is, the associative unital R-algebra generated by T_i for $i \in [n-1]$ subject to the following set of relations:

$$T_i^2 - (q-1)T_i + q = 0 \mid i \in [n-1], \tag{19a}$$

$$T_iT_j - T_jT_i = 0 \mid |i-j| \geq 2, \tag{19b}$$

$$T_iT_{i+1}T_i - T_{i+1}T_iT_{i+1} = 0 \mid |i-j| = 1. \tag{19c}$$

Lemma 4.1. *Given distinct $i, j, k \in [n]$. The following holds in $\mathcal{D}_n(\alpha_1, \alpha_2)$:*

$$x_{ij}x_{jk}x_{ij} - x_{jk}x_{ij}x_{jk} - \alpha_2(x_{ij} - x_{jk}) = 0.$$

Proof.

$$\begin{aligned} x_{ij}x_{jk}x_{ij} &= (\alpha_2 - x_{jk}x_{ki} - x_{ki}x_{ij})x_{ij}, \\ &= \alpha_2 x_{ij} - x_{jk}x_{ki}x_{ij} - \alpha_1 x_{ki}, \\ &= \alpha_2 x_{ij} - \alpha_1 x_{ki} - x_{jk}(\alpha_2 - x_{ij}x_{jk} - x_{jk}x_{ki}), \\ &= \alpha_2 x_{ij} - \alpha_2 x_{jk} + x_{jk}x_{ij}x_{jk}, \end{aligned}$$

implying that:

$$x_{ij}x_{jk}x_{ij} - x_{jk}x_{ij}x_{jk} = \alpha_2(x_{ij} - x_{jk}),$$

as claimed. $\square$

Corollary 13. *For $i \in [n-1]$. The following holds in $\mathcal{D}_n(\alpha_1, \alpha_2)$:*

$$x_i x_{i+1} x_i - x_{i+1} x_i x_{i+1} - \alpha_2(x_i - x_{i+1}) = 0.$$

Proof. This follows from Lemma 4.1 in the special case of $j = i+1$. □

Definition 4.3. Let $\Lambda_n(\alpha_1, \alpha_2)$ be $\mathcal{D}_G(\alpha_1, \alpha_2)$ for G the Dynkin-graph of type A_n, that is, the K-algebra generated by x_i for $i \in [n-1]$ subject the following set of relations:

$$x_i^2 - \alpha_1 = 0 \mid i \in [n-1], \tag{20a}$$
$$x_i x_j - x_j x_i = 0 \mid |j-i| > 1, \tag{20b}$$
$$x_i x_{i+1} x_i - x_{i+1} x_i x_{i+1} - \alpha_2(x_i - x_{i+1}) = 0 \mid i \in [n-2]. \tag{20c}$$

Theorem 4.2. *Given $n \geq 3$ positive integer. There exists some K-parameter q so that the following hold:*

$$\Lambda_n(\alpha_1, \alpha_2) \cong H_q(n).$$

Proof. Given K-parameters $a \neq 0, b$. We consider $x_i - ay_i - b = 0$ a K-linear transformation.

At first, this would imply that for $i \in [n-1]$, we have:

$$\begin{aligned} x_i^2 - \alpha_1 = 0 &\implies (ay_i + b)^2 - \alpha_1 = 0, \\ &\implies a^2 y_i^2 + b^2 + 2aby_i - \alpha_1 = 0, \\ &\implies y_i^2 + (2b/a)y_i + ((b^2 - \alpha_1)/a^2) = 0. \end{aligned}$$

Further, for $|j-i| \geq 2$, we have:

$$x_i x_j - x_j x_i = 0 \implies y_i y_j - y_j y_i = 0,$$

directly. Finally, for $|j-i| = 1$, we see that:

$$x_i x_{i+1} x_i - x_{i+1} x_i x_{i+1} - \alpha_2(x_i - x_{i+1}) = 0,$$

implies that:

$$\begin{aligned} 0 &= a^3(y_i y_j y_i - y_j y_i y_j) + ab^2(y_j - y_i) + a\alpha_2(y_j - y_i), \\ &= a^3(y_i y_j y_i - y_j y_i y_j) + a(\alpha_2 + b^2)(y_j - y_i). \end{aligned}$$

In other words. The algebra $\Lambda_n(\alpha_1, \alpha_2)$ is the K-algebra generated by y_i for $i \in [n-1]$ subject the following set of relations:

$$\begin{aligned} y_i^2 + (2b/a)y_i + ((b^2 - \alpha_1)/a^2) = 0 \mid i \in [n-1], \\ a^3(y_i y_{i+1} y_i - y_{i+1} y_i y_{i+1}) - a(b^2 + \alpha_2)(y_{i+1} - y_i) = 0 \mid i \in [n-2], \\ y_i y_j - y_j y_i = 0 \mid |j-i| \geq 2. \end{aligned}$$

to which, setting $b = \sqrt{-\alpha_2}$, $a = \sqrt{\alpha_1}+\sqrt{-\alpha_2}$ implies an isomorphism to a one parameter Iwahori-Hecke-algebra parametrized by:

$$q = \frac{\sqrt{\alpha_1} - \sqrt{-\alpha_2}}{\sqrt{\alpha_1} + \sqrt{-\alpha_2}}.$$

□

Remark 32. Theorem 4.2 suggests a deeper connection that Iwahori-Hecke-algebras share with $\mathcal{D}_4(\alpha_1, \alpha_2)$-subalgebras.

Indeed, a further analysis of such algebras inspired by compatibility of parameters has suggested the consideration of a family of algebras $H_n(\alpha_1, \alpha_2)$, which enable us to study the algebra $\Lambda_n(\alpha_1, \alpha_2)$ and other examples of subalgebras of $\mathcal{D}_4(\alpha_1, \alpha_2)$ in a more direct approach.

Definition 4.4. Given $n \geq 3$ a positive integer, let α_1, α_2 be R-parameters. Define $H_n(\alpha_1, \alpha_2)$ as the associative unital R-algebra generated by T_i for $i \in [n-1]$ subject to the following set of relations:

$$T_i^2 - \alpha_1 = 0 \mid i \in [n-1], \tag{21a}$$

$$T_i T_j - T_j T_i = 0 \mid 2 \leq |i-j|, \tag{21b}$$

$$T_i T_j T_i + T_j T_i T_j + \alpha_2(T_i + T_j) = 0 \mid 1 = |i-j|. \tag{21c}$$

Proposition 4.3. *The following hold in $H_3(\alpha_1, \alpha_2) = \langle T_1, T_2 \rangle$:*

1. *If $\alpha_1+\alpha_2 \neq 0$. Let M be a nonzero 1-dimensional $H_3(\alpha_1, \alpha_2)$-module, then $M(T_1+T_2) = 0$.*
2. *If $3\alpha_1 + \alpha_2 \neq 0$. Let M be a nonzero 2-dimensional $H_3(\alpha_1, \alpha_2)$-module. Then, either $M(T_1 + T_2) = 0$, or, $M(T_1T_2 + T_2T_1 + \alpha_2 - \alpha_1) = 0$.*

Proof. Let M be a nonzero $H_3(\alpha_1, \alpha_2) = \langle T_1, T_2 \rangle$-module:

1. Should $\alpha_1 + \alpha_2 \neq 0$ be. Assume that $dim_K M = 1$, then:

   $$M(T_1 + T_2) = 0,$$

   since $M(T_1T_2T_1 + T_2T_1T_2 + \alpha_2(T_1 + T_2)) = 0$, $M(T_i) \in K$ and $\alpha_1 + \alpha_2 \neq 0$.
2. Should $(3\alpha_1 + \alpha_2 \neq 0)$ be. Assume that $dim_K M = 2$, we start with remarking that the following hold in $H_3(\alpha_1, \alpha_2)$:

   $$(T_1 + T_2)^3 - (3\alpha_1 - \alpha_2)(T_1 + T_2) = 0.$$

   Now either M is an extension of 1-dimensional $H_3(\alpha_1, \alpha_2)$-module by another, in which case we see that $M(T_1 + T_2)^2 = 0$, and since $(3\alpha_1 + \alpha_2 \neq 0)$, we get $M(T_1 + T_2) = 0$, or it has no 1-dimensional $H_3(\alpha_1, \alpha_2)$-submodule in which case:

   $$M((T_1 + T_2)^2 - (3\alpha_1 - \alpha_2)) = 0,$$

or alternatively, $M(T_1T_2 + T_2T_1 - \alpha_1 + \alpha_2) = 0$, as claimed.

$\square$

Proposition 4.4. *Let $\rho : H_3(\alpha_1, \alpha_2) = \langle T_1, T_2 \rangle \to A$ be an algebra map. If $\alpha_1^2 - \alpha_2^2 \neq 0$ then the following hold:*

1. *If $\rho(T_1T_2T_1 + \alpha_2 T_2) = 0$ then $\alpha_1 = 0$.*
2. *If $\rho(T_1)$ is a constant, then so is $\rho(T_2)$.*

Proof. Assuming that $\alpha_1^2 - \alpha_2^2 \neq 0$:

1. Assume that $\rho(T_1T_2T_1 + \alpha_2T_2) = 0$, then multiplying with T_1, T_2 implies:

$$\begin{aligned} 0 &= \rho(\alpha_1 T_2T_1 + \alpha_2 T_1T_2), \\ &= \rho(\alpha_1 T_1T_2 + \alpha_2 T_2T_1), \end{aligned}$$

   that is:

$$(\alpha_1 - \alpha_2)\rho(T_2T_1 - T_1T_2) = 0$$

   that is, $\rho(T_2T_1) = \rho(T_1T_2)$, that is:

$$(\alpha_1 + \alpha_2)\rho(T_2) = 0$$

   in other words, $\rho(T_2) = 0$ and $\alpha_1 = 0$ as claimed.

2. Assume that $\rho(T_1)$ is a constant, then $T_1T_2T_1 + T_2T_1T_2 + \alpha_2(T_1 + T_2) = 0$ implies that:

$$0 = \alpha_1\rho(T_2) + \alpha_1\rho(T_1) + \alpha_2\rho(T_2 + T_1) = (\alpha_1 + \alpha_2)(\rho(T_1) + \rho(T_2)),$$

   asserting the claim.

$\square$

Lemma 4.5. *Given $n \geq 4$ a positive integer, let $\rho : H_n(\alpha_1, \alpha_2) \to A$ be an algebra map. If $\alpha_1^2 - \alpha_2^2 \neq 0$ and $\rho(T_1T_2 + T_2T_1) - (\alpha_1 - \alpha_2) = 0$ then $\rho(T_1) = \rho(T_3)$.*

Proof. For $i \in [3]$, set $t_i = \rho(T_i)$. Assume that $\alpha_1^2 - \alpha_2^2 \neq 0$ and $t_1t_2 + t_2t_1 + \alpha_2 - \alpha_1 = 0$. We start with:

$$\begin{aligned} 0 &= t_3(t_1t_2 + t_2t_1 + \alpha_2 - \alpha_1) - (t_1t_3 - t_3t_1)t_2, \\ &= t_3t_2t_1 + t_1t_3t_2 + (\alpha_2 - \alpha_1)t_3. \end{aligned}$$

Now on one hand, we compute that:

$$\begin{aligned}
0 =&t_3t_2(t_3t_1 - t_1t_3) - (t_3t_2t_3 + t_2t_3t_2 + \alpha_2t_3 + \alpha_2t_2)t_1,\\
=&-t_3t_2t_1t_3 - t_2t_3t_2t_1 - \alpha_2(t_3t_1 + t_2t_1),\\
=&t_1t_3t_2t_3 + (\alpha_2 - \alpha_1)t_3^2 + t_2t_1t_3t_2 + (\alpha_2 - \alpha_1)t_2t_3 - \alpha_2t_1t_3 - \alpha_2t_2t_1,\\
=&-t_1t_2t_3t_2 - \alpha_2t_1t_3 - \alpha_2t_1t_2 + \alpha_1(\alpha_2 - \alpha_1) + t_2t_1t_3t_2\\
&+ (\alpha_2 - \alpha_1)t_2t_3 - \alpha_2t_1t_3 - \alpha_2t_2t_1,\\
=&t_2t_1t_3t_2 + (\alpha_2 - \alpha_1)t_3t_2 - 2\alpha_2t_1t_3 - \alpha_2(\alpha_1 - \alpha_2)\\
&+ \alpha_1(\alpha_2 - \alpha_1) + t_2t_1t_3t_2 + (\alpha_2 - \alpha_1)t_2t_3,\\
=&2t_2t_1t_3t_2 + (\alpha_2 - \alpha_1)(t_2t_3 + t_3t_2) - 2\alpha_2t_1t_3 + (\alpha_2^2 - \alpha_1^2).
\end{aligned} \tag{22}$$

.

While on the other hand, we see that:

$$\begin{aligned}
0 =&t_2t_3t_2(t_3t_2t_1 + t_1t_3t_2 + (\alpha_2 - \alpha_1)t_3)\\
&- t_2(t_3t_2t_3 + t_2t_3t_2 + \alpha_2t_3 + \alpha_2t_2)t_2t_1,\\
=&t_2t_3t_2t_1t_3t_2 + (\alpha_2 - \alpha_1)t_2t_3t_2t_3\\
&- \alpha_1^2t_3t_1 - \alpha_2t_2t_3t_2t_1 - \alpha_1\alpha_2t_2t_1,\\
=&-t_2t_1t_3t_2t_3t_2 - (\alpha_2 - \alpha_1)t_2t_3^2t_2 - (\alpha_2 - \alpha_1)t_2^2t_3t_2\\
&- \alpha_2(\alpha_2 - \alpha_1)t_2(t_2 + t_3) - \alpha_1^2t_1t_3 + \alpha_2t_2t_1t_3t_2\\
&+ \alpha_2(\alpha_2 - \alpha_1)t_2t_3 - \alpha_1\alpha_2t_2t_1,\\
=&\alpha_1t_2t_1t_2t_3 + \alpha_2t_2t_1t_3t_2 + \alpha_1\alpha_22t_2t_1 + \alpha_1^2(\alpha_1 - \alpha_2)\\
&+ \alpha_1(\alpha_1 - \alpha_2)t_3t_2 + \alpha_1\alpha_2(\alpha_1 - \alpha_2)\\
&+ \alpha_2(\alpha_1 - \alpha_2)t_2t_3 - \alpha_1^2t_1t_3 + \alpha_2t_2t_1t_3t_2\\
&+ \alpha_2(\alpha_2 - \alpha_1)t_2t_3 - \alpha_1\alpha_2t_2t_1,\\
=&2\alpha_2t_2t_1t_3t_2 + \alpha_1(\alpha_1 - \alpha_2)t_3t_2\\
&+ \alpha_1(\alpha_1 - \alpha_2)t_2t_3 - 2\alpha_1^2t_1t_3 + \alpha_1(\alpha_1^2 - \alpha_2^2).
\end{aligned} \tag{23}$$

Taking the difference between (23) and (α_2(22)) implies that:

$$0 = (\alpha_1^2 - \alpha_2^2)(t_2t_3 + t_3t_2 - 2t_1t_3 + (\alpha_1 + \alpha_2)).$$

Now, we assume that $\alpha_1^2 - \alpha_2^2 \neq 0$, therefore, multiplying with t_1 from left/right and taking the sum yields:

$$\begin{aligned}
0 =& t_1(t_2t_3 + t_3t_2 - 2t_1t_3 + (\alpha_1 + \alpha_2)) \\
&+ (t_2t_3 + t_3t_2 - 2t_1t_3 + (\alpha_1 + \alpha_2))t_1, \\
=& t_1t_2t_3 + t_1t_3t_2 - 2\alpha_1t_3 + (\alpha_1 + \alpha_2)t_1 \\
&+ t_2t_1t_3 - t_1t_3t_2 + (\alpha_1 - \alpha_2)t_3 - 2\alpha_1t_3 + (\alpha_1 + \alpha_2)t_1, \\
=& - 2(\alpha_1 + \alpha_2)t_3 + 2(\alpha_1 + \alpha_2)t_1.
\end{aligned}$$

holding the claim. □

Definition 4.5. There exists a unique 1-dimensional $\Lambda_n(\alpha_1, \alpha_2)$-module ξ which we define as:

$$\begin{aligned}
\xi : \Lambda_n(\alpha_1, \alpha_2) \to K&, \\
x_i \mapsto \xi(x_i) = \sqrt{\alpha_1}&,
\end{aligned}$$

that is, a module generated by a vector ξ_1 with an action of $\Lambda_n(\alpha_1, \alpha_2)$ given as $\xi_1 x_i = \sqrt{\alpha_1}\xi_1$.

Remark 33. Assume that $\alpha_1(3\alpha_1 - \alpha_2) = 0$. Let:

$$\Lambda_4(\alpha_1, \alpha_2) = \mathcal{D}_{A_4}(\alpha_1, \alpha_2) \mid A_4 = 2 \to 1 \to 3 \to 4.$$

There exists a unique 2-dimensional indecomposable $\Lambda_4(\alpha_1, \alpha_2)$-module ϵ which has a proper submodule ξ. In particular, should such condition hold, we see that ξ is a proper submodule of ϵ.

Example 4.2. If $\alpha_1 = 0$. Then ϵ may be defined as:

$$\epsilon(x_{21}) = \epsilon(x_{13}) = \epsilon(x_{34}) = \begin{bmatrix} 0 & 0 \\ \sqrt{\alpha_2} & 0. \end{bmatrix}$$

In other words, ϵ has a basis of the form $\{\xi_1, \xi_2\}$ such:

$$\begin{aligned}
\xi_1(x_{21}) = \xi_1(x_{13}) = \xi_1(x_{34}) &= 0, \\
\xi_2(x_{21}) = \xi_2(x_{13}) = \xi_2(x_{34}) &= \sqrt{\alpha_2}\xi_1.
\end{aligned}$$

Example 4.3. If $3\alpha_1 - \alpha_2 = 0$. Then one may define:

$$\epsilon(x_{21}) = \epsilon(x_{34}) = \begin{bmatrix} +\sqrt{\alpha_1} & 0 \\ 0 & -\sqrt{\alpha_1} \end{bmatrix}, \qquad \epsilon(x_{13}) = \begin{bmatrix} +\sqrt{\alpha_1} & 0 \\ +1 & -\sqrt{\alpha_1} \end{bmatrix}.$$

In other words, ϵ has a basis of the form $\{\xi_1, \xi_2\}$ such:

$$\begin{aligned}
&\xi_1(x_{21}) = \xi_1(x_{13}) = \xi_1(x_{34}) = \sqrt{\alpha_1}\xi_1, \\
&\xi_2(x_{21}) = \xi_2(x_{34}) = -\sqrt{\alpha_1}\xi_2, \xi_2(x_{13}) = \xi_1 - \sqrt{\alpha_1}\xi_2.
\end{aligned}$$

Lemma 4.6. *Given a positive integer $n \geq 3$, there exists an algebraic embedding of $H_n(\alpha_1, \alpha_2)$ into $\mathcal{D}_n(\alpha_1, \alpha_2)$.*

Proof. The claim falls directly by considering that for each $k \in [n]$ and each injective map $\sigma : [k] \to [n]$ there exists an algebra map:

$$\begin{aligned} H_\sigma : H_n(\alpha_1, \alpha_2) &\to \mathcal{D}_n(\alpha_1, \alpha_2), \\ T_i &\mapsto (-1)^i x_{\sigma(i)\sigma(i+1)}. \end{aligned}$$

$\square$

Theorem 4.7. $\Lambda_n(\alpha_1, \alpha_2) \cong H_n(\alpha_1, \alpha_2)$ *for all* $n \geq 3$.

Remark 34. Naturally, one may observe that in the special case of $\alpha_1 = -\alpha_2$, we get an alternative point of view of what is commonly known in the literature as the 0-Hecke algebras. We recommend [Car86] for interested readers in such topic.

Example 4.4. As an easy example, we verified that the algebra $\Lambda_4(\alpha, -\alpha)$ admits an ordinary quiver Q_{Λ_4} of the form:

$$\begin{tikzcd}
+-- \arrow[dr, shift left, "\alpha_1"] & & & -++ \arrow[dl, shift left, "\beta_1'"] \\
+++ & -+- \arrow[ul, shift left, "\alpha_1'"] \arrow[dl, shift left, "\alpha_2"] \arrow[r, shift left, "\alpha"] & +-+ \arrow[l, shift left, "\alpha'"] \arrow[ur, shift left, "\beta_1"] \arrow[dr, shift left, "\beta_2"] & --- \\
--+ \arrow[ur, shift left, "\alpha_2'"] & & & ++- \arrow[ul, shift left, "\beta_2'"]
\end{tikzcd}$$

Furthermore, there exists an isomorphism $\Lambda_4(\alpha, -\alpha) \cong KQ_{\Lambda_4}/I$ for I the two-sided ideal generated by:

$$\alpha\alpha' - \alpha_i\alpha_i', \alpha\alpha' - \beta_i\beta_i', \alpha_i\alpha, \beta_i\alpha,$$

for $i \in [2]$. In particular, the algebra is 4!-dimensional. We corresponds the projective dimension as follows:

$$\begin{tikzcd}
3 \arrow[dr, no head] & & & 3 \\
1 & 5 \arrow[r, no head] & 5 \arrow[ur, no head] \arrow[dr, no head] & 1 \\
3 \arrow[ur, no head] & & & 3
\end{tikzcd}$$

Application in $\mathcal{D}_3(\alpha_1, \alpha_2)$.

Remark 35. Using the result held in Theorem 4.7. We shall demonstrate an alternative approach of the fact that the algebra $\mathcal{D}_3(\alpha_1, \alpha_2)$ is semisimple if and only if:

$$(3\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2) \neq 0.$$

Which was proven in [HV18] using direct construction and techniques of Clifford algebras.

Note 30. We remind the reader that the algebra $\mathcal{D}_3(\alpha_1, \alpha_2)$ is 12-dimensional.

Definition 4.6. Let S be the $\mathcal{D}_3(\alpha_1, \alpha_2)$-module induced by the $\Lambda_3(\alpha_1, \alpha_2)$-module ξ as defined in Definition 4.5, that is:

$$S \coloneqq \xi \otimes_{\Lambda_3(\alpha_1,\alpha_2)} \mathcal{D}_3(\alpha_1, \alpha_2).$$

Remark 36. One may easily see that S is 2-dimensional with a basis of the form: $\{1, x_{13}\}$. Furthermore, one may easily see that the action of x_{23} is identical to that of x_{12}.

Example 4.5. The action of $\mathcal{D}_3(\alpha_1, \alpha_2)$ is constructed as follows:

$$\begin{aligned}
\xi_1 x_{12} &= \sqrt{\alpha_1}\xi_1, \\
\xi_1 x_{31} x_{12} &= \xi_1(\alpha_2 - x_{12}x_{23} - x_{23}x_{31}), \\
&= (\alpha_2 - \alpha_1)\xi_1 - \sqrt{\alpha_1}\xi_1 x_{31}, \\
\xi_1 x_{31} &= \xi_1 x_{31}, \\
\xi_1 x_{31} x_{31} &= \alpha_1 \xi_1.
\end{aligned}$$

in other words, for $s_{ij} = S(x_{ij})$:

$$s_{12} = s_{23} = \begin{bmatrix} \sqrt{\alpha_1} & 0 \\ \alpha_2 - \alpha_1 & -\sqrt{\alpha_1} \end{bmatrix}, \qquad s_{31} = \begin{bmatrix} 0 & 1 \\ \alpha_1 & 0 \end{bmatrix}.$$

Note 31. We remind the reader that if two matrices have a common eigenvector then their commutator is singular.

Proposition 4.8. *If* $(3\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2) \neq 0$*. Then* S *is a simple* $\mathcal{D}_3(\alpha_1, \alpha_2)$*-module.*

Proof. We observe that S is simple if and only if there exists a common eigenvector of s_{12}, s_{31}. One computes that:

$$det(s_{12}s_{31} - s_{31}s_{12}) = (3\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2).$$

Therefore, should $det(s_{12}s_{31} - s_{31}s_{12}) \neq 0$ then S contains no 1-dimensional subrepresentation and therefore simple. $\square$

Proposition 4.9. *If* $(3\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2) \neq 0$*. Then* $S^{(1,2)}, S^{(2,3)}$ *are both simple* $\mathcal{D}_3(\alpha_1, \alpha_2)$*-modules pairwise non-isomorphic to* S*.*

Proof. We start the proof by highlighting that in $\mathcal{D}_3(\alpha_1, \alpha_2)$, the following hold:

$$s_{12}s_{23} + s_{23}s_{12} - 2\alpha_1 = 0.$$

Therefore, should a nonzero homomorphism $f \in Hom_K(S, S^{(1,2)})$ exists, then:

$$(s_{12}s_{23} + s_{23}s_{12} - 2\alpha_1)f = (s_{21}s_{13} + s_{13}s_{21} - 2\alpha_1) = 0,$$

which one easily computes to be not, in other words, $f = 0$ and $S^{(1,2)} \ncong S$. The other cases are solved in an identical fashion. ☐

Remark 37. The previous lemma can alternatively be proven by showing that there exists no nonzero $f \in K^2$ such that:

$$S(x_{ij})f - fS^{(1,2)}(x_{ij}) = 0,$$
$$S(x_{ij})f - fS^{(2,3)}(x_{ij}) = 0.$$

Corollary 14. *If* $(3\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2) \neq 0$*. Then the algebra* $\mathcal{D}_3(\alpha_1, \alpha_2)$ *is semisimple.*

Proposition 4.10. *If* $(3\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2) = 0$*. Then the algebra* $\mathcal{D}_3(\alpha_1, \alpha_2)$ *is not semisimple.*

Proof. We prove the claim by showing that if such condition holds then there exists a nonzero space of extensions between two simple $\mathcal{D}_3(\alpha_1, \alpha_2)$-modules. That is, by showing that $rad\mathcal{D}_3(\alpha_1, \alpha_2)$ is non-trivial.

1. If $\alpha_1 = -\alpha_2$. Let ρ_+, ρ_- be two fixed $\mathcal{D}_3(\alpha_1, \alpha_2)$-representations defined as: $\rho_+ = -\rho_- = +1$. Then one may easily verify that:

$$dim_K Ext^1_{\mathcal{D}_3(\alpha_1,\alpha_2)}(\rho_+, \rho_-) = 1.$$

2. If $3\alpha_1 = \alpha_2$. Let ρ_+, ρ_- be two $\mathcal{D}_3(\alpha_1, \alpha_2)$-representations defined as:

$$\rho_+(x_{ij}) = -\rho_-(x_{ij}) = \begin{cases} +1 & \mid (i,j) = (1,2), (2,3), \\ -1 & \mid (i,j) = (3,1). \end{cases}$$

   Then one may verify that:

$$dim_K Ext^1_{\mathcal{D}_3(\alpha_1,\alpha_2)}(\rho_+, \rho_-) = 2.$$

☐

4.2 $\Gamma_4(\alpha_1, \alpha_2)$. An intermediate algebra.

Lemma 4.11. *Given distinct $i, j, k \in [n]$. The following holds in $\mathcal{D}_n(\alpha_1, \alpha_2)$:*

$$x_{li}x_{ik}x_{ij} - x_{ij}x_{lj}x_{jk} + x_{jk}x_{ki}x_{kl} - x_{kl}x_{jl}x_{li} = 0.$$

Proof. This is a direct consequence of (2d), in particular, one sees:

$$\begin{aligned} 0 &= (\alpha_2 - x_{ik}x_{kl} - x_{kl}x_{li})x_{ij} - x_{ij}(\alpha_2 - x_{jk}x_{kl} - x_{kl}x_{lj}) \\ &\quad + (\alpha_2 - x_{ki}x_{ij} - x_{ij}x_{jk})x_{kl} - x_{kl}(\alpha_2 - x_{li}x_{ij} - x_{ij}x_{jl}), \end{aligned}$$

to which the claim falls directly. □

Corollary 15. *Given distinct $i, j, k, l \in [n]$. The following holds in $\mathcal{D}_n(\alpha_1, \alpha_2)$:*

$$x_{ij}x_{jk}x_{kl} + x_{jk}x_{kl}x_{li} + x_{kl}x_{li}x_{ij} + x_{li}x_{ij}x_{jk} = \alpha_2(x_{ij} + x_{jk} + x_{kl} + x_{li}).$$

Proof. One may easily sees that the claim made in Lemma 4.11, can be further developed via (2d) as:

$$\begin{aligned} +x_{li}x_{ik}x_{ij} &= -\alpha_2 x_{li} + x_{li}x_{ij}x_{jk} + x_{li}x_{jk}x_{ki}, \\ -x_{ij}x_{lj}x_{jk} &= -\alpha_2 x_{ij} + x_{ij}x_{jk}x_{kl} + x_{ij}x_{kl}x_{li}, \\ +x_{jk}x_{ki}x_{kl} &= -\alpha_2 x_{jk} + x_{jk}x_{kl}x_{li} + x_{jk}x_{ki}x_{ik}, \\ -x_{kl}x_{jl}x_{li} &= -\alpha_2 x_{kl} + x_{kl}x_{li}x_{ij} + x_{kl}x_{ij}x_{jl}. \end{aligned}$$

to which, (2c) implies the claim. □

Note 32. Let $\Gamma_4(\alpha_1, \alpha_2)$ denotes $\mathcal{D}_G(\alpha_1, \alpha_2)$ for the special case of:

$$G = I_4 = \begin{tikzcd} 4 & 3 \arrow[l] & 2 \arrow[dl] \\ & 1 \arrow[ul] \arrow[u] & \end{tikzcd}$$

Definition 4.7. The algebra $\Gamma_4(\alpha_1, \alpha_2)$ is the K-algebra generated by x_{ij} for $(i, j) \in I_4$, subject the following set of relations:

$$x_{ij} + x_{ji} = 0, \tag{24a}$$
$$x_{ij}^2 - \alpha_1 = 0, \tag{24b}$$
$$x_{12}x_{34} - x_{34}x_{12} = 0, \tag{24c}$$
$$x_{12}x_{13}x_{12} + x_{13}x_{12}x_{13} + \alpha_2(x_{12} + x_{13}) = 0, \tag{24d}$$
$$x_{12}x_{14}x_{12} + x_{14}x_{12}x_{14} + \alpha_2(x_{12} + x_{14}) = 0, \tag{24e}$$
$$x_{13}x_{34} + x_{34}x_{41} + x_{41}x_{13} - \alpha_2 = 0, \tag{24f}$$
$$x_{34}x_{13} + x_{41}x_{34} + x_{13}x_{41} - \alpha_2 = 0. \tag{24g}$$

for $(i, j) \in I_4$.

Remark 38. It was computed in [BLM16] that $\mathcal{E}_{I_4}$ is 96-dimensional. Our computations confirm that $\Gamma_4(\alpha_1, \alpha_2)$ shares the same dimension as well.

Note 33. Denote by:

$$\begin{aligned} u &\coloneqq x_{12}x_{13} + x_{13}x_{14} + x_{14}x_{12}, \\ v &\coloneqq x_{12}x_{14} + x_{14}x_{13} + x_{13}x_{12}. \end{aligned}$$

Remark 39. Our calculations using Gröbner basis techniques -in particular, strong normal forms- via [Gap] and [CK] confirms that:

1. $(u + \alpha_2)^2 = (v + \alpha_2)^2 = -4\alpha_2\alpha_2$,
2. $(u + \alpha_2)(v + \alpha_2) + (v + \alpha_2)(u + \alpha_2) = 2(3\alpha_1^2 + \alpha_2^2)$.

Proposition 4.12. *If $\alpha_1(\alpha_1^2 - \alpha_2^2) \neq 0$, then $\Gamma_4(\alpha_1, \alpha_2)$ has no i-dimensional nonzero modules for $i \in [2]$.*

Proof. Assume that $\alpha_1(\alpha_1^2 - \alpha_2^2) \neq 0$. We observe that there exists an embedding of $H_3(\alpha_1, \alpha_2)$ into $\Gamma_4(\alpha_1, \alpha_2)$ in the obvious way.

Let M be a 1-dimensional $\Gamma_4(\alpha_1, \alpha_2)$-module. Then $M|_{H_3(\alpha_1,\alpha_2)}$ is a 1-dimensional $H_3(\alpha_1, \alpha_2)$. By Proposition 4.3 we see that $M(x_{ij}) = 0$ for all $(i, j) \in I_4$, this in particular means that $\alpha_1 = \alpha_2 = 0$ by the defining relations of the algebra.

Let M be a 2-dimensional $\Gamma_4(\alpha_1, \alpha_2)$-module. Since $\alpha_1 \neq 0$, $[x_{12}, x_{34}] = 0$ implies that both of $M(x_{12}), M(x_{34})$ are simultaneously diagonalizable.

Should $M(x_{12})$ be a constant, then so is $M(x_{34})$ and consequently both of $M(x_{13})$ and $M(x_{14})$ as well implied by the defining relations of the algebra, in other words, M is a direct sum of two 1-dimensional $\Gamma_4(\alpha_1, \alpha_2)$-modules, a clear contradiction.

Therefore- and since $dim_K M = 2$- we see that $M(x_{34}) = \pm M(x_{12})$, say with no loss of generality that $M(x_{34}) = M(x_{12})$. This implies that:

$$\begin{aligned} 0 =& M(x_{12}x_{13}x_{12} + x_{13}x_{12}x_{13} + \alpha_2(x_{12} + x_{13})), \\ =& M(x_{12}x_{31}x_{12} + x_{31}x_{12}x_{31} + \alpha_2(x_{12} + x_{31})). \end{aligned}$$

that is:

$$M(x_{12}x_{13}x_{12} + \alpha_2 x_{13}) = 0,$$

which according to Proposition 4.4 implies that $\alpha_1 = 0$, a contradiction. □

Proposition 4.13. *If $\alpha_1 = 0$, then $\Gamma_4(\alpha_1, \alpha_2)$ has no i-dimensional nonzero modules for $i \in [2]$.*

Proof. Assume that $\alpha_1 = 0 \neq \alpha_2$. Clearly $(3\alpha_1 - \alpha_2)(\alpha_1^2 - \alpha_2^2) \neq 0$.

At first, we see that there exists no 1-dimensional $\Gamma_4(\alpha_1, \alpha_2)$-module, this is indicated by Proposition 4.3, similarly as seen in Proposition 4.12

Let M be a 2-dimensional $\Gamma_4(\alpha_1, \alpha_2)$-module. We have six possible cases to discuss:

1. If $M|_{\langle x_{12},x_{13}\rangle}$, $M|_{\langle x_{13},x_{14}\rangle}$, $M|_{\langle x_{14},x_{12}\rangle}$, $M|_{\langle x_{31},x_{34}\rangle}$, and $M|_{\langle x_{41},x_{43}\rangle}$ are extensions of 1-dimensional $H_3(\alpha_1,\alpha_2)$-modules by another. By Proposition 4.3 we see that $M(x_{ij}) = 0$ for all $(i,j) \in I_4$, a contradiction as discussed earlier.
2. If $M|_{\langle x_{12},x_{13}\rangle}$, $M|_{\langle x_{13},x_{14}\rangle}$, $M|_{\langle x_{14},x_{12}\rangle}$, $M|_{\langle x_{31},x_{34}\rangle}$ are extensions of 1-dimensional $H_3(\alpha_1,\alpha_2)$-modules by another and $M|_{\langle x_{41},x_{43}\rangle}$ is a simple $H_3(\alpha_1,\alpha_2)$-module. By Proposition 4.3 we see that:

$$\begin{aligned} M(x_{12}+x_{13}) = M(x_{13}+x_{14}) = M(x_{14}+x_{12}) = M(x_{31}+x_{34}) = 0,\\ M(x_{41}x_{43}+x_{43}x_{41}+\alpha_2-\alpha_1) = 0, \end{aligned}$$

   that is, $M(x_{41}) = M(x_{13}) = -M(x_{34})$ by the first line, to which the second line implies that, $\alpha_2 = 0$, a contradiction.
3. If $M|_{\langle x_{12},x_{13}\rangle}$, $M|_{\langle x_{13},x_{14}\rangle}$, $M|_{\langle x_{14},x_{12}\rangle}$ are extensions of 1-dimensional $H_3(\alpha_1,\alpha_2)$-modules by another and $M|_{\langle x_{31},x_{34}\rangle}$, $M|_{\langle x_{41},x_{43}\rangle}$ are simple $H_3(\alpha_1,\alpha_2)$-module. Proposition 4.3 implies that:

$$\begin{aligned} M(x_{12}) = M(x_{13}) = M(x_{14}) = 0,\\ M(x_{31}x_{34}+x_{34}x_{31}+\alpha_2-\alpha_1) = 0,\\ M(x_{41}x_{43}+x_{43}x_{41}+\alpha_2-\alpha_1) = 0, \end{aligned}$$

   that is, as before, $\alpha_2 = 0$, a contradiction.
4. If $M|_{\langle x_{12},x_{13}\rangle}$, $M|_{\langle x_{13},x_{14}\rangle}$ are extensions of 1-dimensional $H_3(\alpha_1,\alpha_2)$-modules by another and $M|_{\langle x_{14},x_{12}\rangle}$, $M|_{\langle x_{31},x_{34}\rangle}$, $M|_{\langle x_{41},x_{43}\rangle}$ are simple $H_3(\alpha_1,\alpha_2)$-module. Proposition 4.3 implies that:

$$\begin{aligned} M(x_{12}+x_{13}) = M(x_{13}+x_{14}) = 0,\\ M(x_{14}x_{12}+x_{12}x_{14}+\alpha_2-\alpha_1) = 0,\\ M(x_{31}x_{34}+x_{34}x_{31}+\alpha_2-\alpha_1) = 0,\\ M(x_{41}x_{43}+x_{43}x_{41}+\alpha_2-\alpha_1) = 0, \end{aligned}$$

   that is, $\alpha_2 = 0$, a contradiction implied by the first two lines.
5. If $M|_{\langle x_{12},x_{13}\rangle}$ is an extensions of 1-dimensional $H_3(\alpha_1,\alpha_2)$-modules by another and $M|_{\langle x_{13},x_{14}\rangle}$, $M|_{\langle x_{14},x_{12}\rangle}$, $M|_{\langle x_{31},x_{34}\rangle}$, $M|_{\langle x_{41},x_{43}\rangle}$ are simple $H_3(\alpha_1,\alpha_2)$-module. Proposition 4.3 implies that:

$$\begin{aligned} M(x_{12}+x_{13}) = 0,\\ M(x_{14}x_{13}+x_{13}x_{14}+\alpha_2-\alpha_1) = 0,\\ M(x_{14}x_{12}+x_{12}x_{14}+\alpha_2-\alpha_1) = 0,\\ M(x_{31}x_{34}+x_{34}x_{31}+\alpha_2-\alpha_1) = 0,\\ M(x_{41}x_{43}+x_{43}x_{41}+\alpha_2-\alpha_1) = 0, \end{aligned}$$

   that is, $\alpha_2 = 0$, a contradiction held by the first three lines.

6. If $M|_{\langle x_{12},x_{13}\rangle}$, $M|_{\langle x_{13},x_{14}\rangle}$, $M|_{\langle x_{14},x_{12}\rangle}$, $M|_{\langle x_{31},x_{34}\rangle}$, and $M|_{\langle x_{41},x_{43}\rangle}$ are simple $H_3(\alpha_1,\alpha_2)$-module. Proposition 4.3 implies that:

$$\begin{aligned}
M(x_{12}x_{13}+x_{13}x_{12}+\alpha_2-\alpha_1)&=0,\\
M(x_{14}x_{13}+x_{13}x_{14}+\alpha_2-\alpha_1)&=0,\\
M(x_{14}x_{12}+x_{12}x_{14}+\alpha_2-\alpha_1)&=0,\\
M(x_{31}x_{34}+x_{34}x_{31}+\alpha_2-\alpha_1)&=0,\\
M(x_{41}x_{43}+x_{43}x_{41}+\alpha_2-\alpha_1)&=,0
\end{aligned}$$

recall here that:

$$\begin{aligned}
(u+\alpha_2)^2=(v+\alpha_2)^2=-4\alpha_1\alpha_2 \mid u&=x_{12}x_{13}+x_{13}x_{14}+x_{14}x_{12},\\
\mid v&=x_{13}x_{12}+x_{14}x_{13}+x_{12}x_{14}.
\end{aligned}$$

In particular, we see that for $u' \coloneqq u+\alpha_2$ and $v' \coloneqq v+\alpha_2$, we have

$$\begin{aligned}
M(u'+v')&=M(x_{12}x_{13}+x_{13}x_{14}+x_{14}x_{12}+x_{13}x_{12}+x_{14}x_{13}+x_{12}x_{14}),\\
&=3\alpha_1-\alpha_2.
\end{aligned}$$

This allow us to see that $M(v')=M((3\alpha_1-\alpha_2)-u')$, to which we write:

$$\begin{aligned}
M(v')^2&=M((3\alpha_1-\alpha_2)-u')^2,\\
&=M(u')^2-2(3\alpha_1-\alpha_2)M(u')+(3\alpha_1-\alpha_2)^2,
\end{aligned}$$

that is to say $2M(u')=(3\alpha_1-\alpha_2)$, similarly, one sees that $2M(v')=(3\alpha_1-\alpha_2)$. But should this occur then $u'v'+v'u'=2(3\alpha_1^2+\alpha_2^2)$ would imply that:

$$\begin{aligned}
12\alpha_1^2+4\alpha_2^2&=2M(u'v'+v'u'),\\
&=2M(u')M(v')+2M(v')M(u'),\\
&=(3\alpha_1-\alpha_2)^2,\\
&=(9\alpha_1^2+\alpha_2^2-6\alpha_1\alpha_2),
\end{aligned}$$

that is, $3\alpha_1^2+3\alpha_2^2+6\alpha_1\alpha_2=0$, in other words:

$$0=\alpha_1^2+\alpha_2^2+2\alpha_1\alpha_2=(\alpha_1+\alpha_2)^2.$$

A contradiction. To which we conclude the claim.

□

Note 34. Let:

$$\Lambda_4(\alpha_1,\alpha_2)=\mathcal{D}_{A_4}(\alpha_1,\alpha_2) \mid A_4=2\to 1\to 3\to 4.$$

Definition 4.8. Let S be the 4-dimensional $\Gamma_4(\alpha_1, \alpha_2)$-module induced by the $\Lambda_4(\alpha_1, \alpha_2)$-module ξ, as defined in Definition 4.5, that is:

$$S := (\xi \otimes_{\Lambda_4(\alpha_1,\alpha_2)} \Gamma_4(\alpha_1, \alpha_2))$$

Proposition 4.14. *If* $(\alpha_1^2 - \alpha_2^2) \neq 0$*, then* S *is a simple* $\Gamma_4(\alpha_1, \alpha_2)$*-module.*

Proof. This is a direct consequence of Propositions 4.12 and 4.13. ☐

Remark 40. One may verify that S has a generating set of the following form:

$$\{\xi_1, \xi_1 x_{14}, \xi_1 x_{14} x_{12}, \xi_1 x_{14} x_{12} x_{13}\}.$$

Example 4.6. One may compute the action of x_{12} on the generating vector $\xi_1 x_{14} x_{12} x_{13}$ can be computed as:

$$(\alpha_1 - \alpha_2)^2(\xi_1.1) + (\xi_1 x_{14})(\sqrt{\alpha_1}\alpha_2) + (\xi_1 x_{14} x_{12})(-\alpha_2) + (\xi_1 x_{14} x_{12} x_{13})(\sqrt{\alpha_1}),$$

to which, one writes:

$$\begin{aligned} S :&\Gamma_4(\alpha_1, \alpha_2) \to K^4, \\ x_{12} \mapsto S(x_{12}) &= \begin{bmatrix} -\sqrt{\alpha_1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \alpha_1 & 0 & 0 \\ (\alpha_1 - \alpha_2)^2 & \sqrt{\alpha_1}\alpha_2 & -\alpha_2 & \sqrt{\alpha_1} \end{bmatrix}, \end{aligned}$$

similarly, we present:

$$\begin{aligned} S(x_{13}) &= \begin{bmatrix} \sqrt{\alpha_1} & 0 & 0 & 0 \\ \alpha_1 - \alpha_2 & -\sqrt{\alpha_1} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \alpha_1 & 0 \end{bmatrix}, \\ S(x_{14}) &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ \alpha_1 & 0 & 0 & 0 \\ \sqrt{\alpha_1}\alpha_2 & -\alpha_2 & \sqrt{\alpha_1} & 0 \\ -\alpha_2^2 & -\sqrt{\alpha_1}\alpha_2 & \alpha_1 - \alpha_2 & -\sqrt{\alpha_1} \end{bmatrix}, \\ S(x_{34}) &= \begin{bmatrix} \sqrt{\alpha_1} & 0 & 0 & 0 \\ \alpha_1 - \alpha_2 & -\sqrt{\alpha_1} & 0 & 0 \\ \sqrt{\alpha_1}(\alpha_2 - \alpha_1) & 0 & -\sqrt{\alpha_1} & 0 \\ \alpha_2(\alpha_2 - \alpha_1) & \sqrt{\alpha_1}(3\alpha_2 - \alpha_1) & \alpha_2 - \alpha_1 & \sqrt{\alpha_1} \end{bmatrix}. \end{aligned}$$

This in particular, enable us to establish useful expressions such as:

$$S(x_{13}x_{14} + x_{14}x_{13} + \alpha_2 - \alpha_1) = 0.$$

Remark 41. If $\alpha_1(3\alpha_1 - \alpha_2) = 0$, then ϵ -as defined in Examples 4.2 and 4.3 , is a $\Lambda_4(\alpha_1, \alpha_2)$-module, which is special in the sense that it has ξ as a proper submodule.

The algebra $\Gamma_4(\alpha_1, \alpha_2)$ is free on $\Lambda_4(\alpha_1, \alpha_2)$ which enable us to define an induced module by ϵ we call X, that is:

$$X \coloneqq (\epsilon \otimes_{\Lambda_4(\alpha_1,\alpha_2)} \Gamma_4(\alpha_1, \alpha_2)).$$

In particular, should such condition hold, then X is not a simple $\Gamma_4(\alpha_1, \alpha_2)$-module as S becomes a proper submodule of X.

Proposition 4.15. *If $\alpha_1(3\alpha_1 - \alpha_2) = 0$, then X is an indecomposable $\Gamma_4(\alpha_1, \alpha_2)$-module.*

Proof. Assume that $\alpha_1(3\alpha_1 - \alpha_2) = 0$.

If $\alpha_1 = 0$. We start with remarking that $X/S \cong S$ as X/S is a 4-dimensional quotient generated by $\epsilon/\xi \cong \xi$ in $\alpha_1 = 0$. Furthermore, we see that $S(x_{12}x_{34}) = 0$, while:

$$\xi_2(x_{12}x_{34}) = -\xi_2(x_{21}x_{34}) = -\alpha_2\xi_1 \neq 0,$$

which implies that X does not decompose into a direct sum of S and its complement should it exists. In other words, X is an indecomposable module.

If $3\alpha_1 - \alpha_2 = 0$. We remark that while $X/S \ncong S$, both S and its quotient X/S are annihilated at $x_{13}x_{14} + x_{14}x_{13} + 2\alpha_1$, however, we have:

$$\begin{aligned}
\xi_2(x_{13}x_{14} + x_{14}x_{13} + 2\alpha_1) &= (\xi_1 - \sqrt{\alpha_1}\xi_2)x_{14} - \xi_2(x_{41}x_{13}) + 2\alpha_1\xi_2, \\
&= \xi_1 x_{14} + \sqrt{\alpha_1}\xi_1, \\
&\neq 0.
\end{aligned}$$

effectively showing that X does not decompose into a direct sum of S and its complement should it exists. □

Corollary 16. *If $\alpha_1(3\alpha_1 - \alpha_2) = 0$, then $\Gamma_4(\alpha_1, \alpha_2)$ is not a semisimple algebra.*

Proposition 4.16. *If $\alpha_1(\alpha_1^2 - \alpha_2^2) \neq 0$, then $S^{(3,4)}$ is a simple $\Gamma_4(\alpha_1, \alpha_2)$-module non isomorphic to S.*

Proof. If $(\alpha_1^2 - \alpha_2^2) \neq 0$, the simplicity of $S^{(3,4)}$ is straightforward.

One may verify that should $\alpha_1 \neq 0$, then S is annihilated at $(x_{12}+\alpha_1)(x_{13}+\alpha_1)(x_{14}+x_{34})$, while that of:

$$S((x_{12} + \alpha_1)(x_{14} + \alpha_1)(x_{13} + x_{43})) \neq 0,$$

essentially showing that the morphism space $Hom_K(S, S^{(3,4)})$ is trivial. □

Remark 42. The previous lemma can be alternatively proven by showing that there exists no nonzero $f \in K^4$ such that:

$$S(x_{ij})f - fS^{(3,4)}(x_{ij}) = 0.$$

Proposition 4.17. *If $\alpha_1^2 - \alpha_2^2 \neq 0$, then there exists no algebra map:*

$$f : \Gamma_4^{(2,4)}(\alpha_1, \alpha_2) \to A,$$

such that $f(x_{13}x_{14} + x_{14}x_{13} + \alpha_2 - \alpha_1) = 0$ for nonzero A.

Proof. Assume that $\alpha_1^2 - \alpha_2^2 \neq 0$.

Assume for a contradiction there exists such f, let f_{ij} denotes $f(x_{ij})$.

We start with highlighting the algebraic embedding of $H_4(\alpha_1, \alpha_2)$ into $\mathcal{D}_n(\alpha_1, \alpha_2)$ which maps:

$$T_1 \mapsto x_{31}, \qquad T_2 \mapsto x_{41}, \qquad T_3 \mapsto x_{24}.$$

Lemma 4.5 would imply that $f_{31} = f_{24}$. Further, we have:

$$\begin{aligned} 0 =& + f_{14}f_{24}f_{14} + f_{24}f_{14}f_{24} + \alpha_2(f_{14} + f_{24}), \\ =& - f_{14}f_{13}f_{14} + f_{13}f_{14}f_{13} + \alpha_2(f_{14} - f_{13}). \end{aligned}$$

implying that, $0 = f_{14}f_{13}f_{14} + \alpha_2 f_{13}$, which we multiply from both right and left with f_{14} to get:

$$\begin{aligned} 0 =& \alpha_1 f_{13}f_{14} + \alpha_2 f_{14}f_{13}, \\ =& \alpha_2 f_{13}f_{14} + \alpha_1 f_{14}f_{13}. \end{aligned}$$

Now $\alpha_1^2 - \alpha_2^2 \neq 0$, implies that:

$$0 = \alpha_1(f_{13} - f_{14}) = \alpha_2(f_{13} - f_{14}),$$

that is, $f_{13} = f_{14} = f_{24} = 0$. The contradiction falls directly at this point by (2b) and (2d), which effectively hold if and only if $\alpha_1 = \alpha_2 = 0$. □

Definition 4.9. Let S' be the $\Gamma_4^{(2,4)}(\alpha_1, \alpha_2)$-module induced by S, as defined in Definition 4.8, that is:

$$S' := (S \otimes_{\Lambda_4(\alpha_1,\alpha_2)} \Gamma_4^{(2,4)}(\alpha_1, \alpha_2)).$$

Our aim at this point is proving the following results:

Proposition 4.18. *If $(3\alpha_1 - \alpha_2)(\alpha_1^2 - \alpha_2^2) \neq 0$, then S' is a simple $\Gamma_4^{(2,4)}(\alpha_1, \alpha_2)$-module.*

Which requires further insights into the star based graph of D_4.

The Dynkin type D_4

Lemma 4.19. *Given distinct $i, j, k, l \in [n]$. The following holds in $\mathcal{D}_n(\alpha_1, \alpha_2)$:*

$$\begin{bmatrix} +x_{ij}x_{kj}x_{lj}x_{ij} \\ +x_{kj}x_{lj}x_{ij}x_{kj} \\ +x_{lj}x_{ij}x_{kj}x_{lj} \end{bmatrix} + \alpha_2 \begin{bmatrix} +x_{ij}x_{kj} \\ +x_{kj}x_{lj} \\ +x_{lj}x_{ij} \end{bmatrix} + \alpha_2(\alpha_1 + \alpha_2) = 0.$$

Proof.

$$\begin{aligned}
x_{ij}x_{kj}x_{lj}x_{ij} =& x_{ij}(x_{kj}x_{lj})x_{ij}, \\
=& x_{ij}(-x_{kj}x_{jl})x_{ij}, \\
=& -x_{ij}(\alpha_2 - x_{jl}x_{lk} - x_{lk}x_{kj})x_{ij}, \\
=& x_{ij}(-\alpha_2 + x_{jl}x_{lk} + x_{lk}x_{kj})x_{ij}, \\
=& x_{ij}x_{jl}x_{lk}x_{ij} + x_{ij}x_{lk}x_{kj}x_{ij} - \alpha_1\alpha_2, \\
=& ((x_{ij}x_{jl}x_{ij})x_{lk}) + (x_{lk}(x_{ij}x_{kj}x_{ij})) - \alpha_1\alpha_2, \\
=& +(x_{jl}x_{ij}x_{jl}x_{lk} + \alpha_2(x_{ij}x_{lk} - x_{jl}x_{lk})) \\
& + (-x_{lk}x_{kj}x_{ij}x_{kj} - \alpha_2(x_{lk}x_{ij} + x_{lk}x_{kj})) - \alpha_1\alpha_2, \\
=& -\alpha_1\alpha_2 - \alpha_2(x_{jl}x_{lk} + x_{lk}x_{kj}) \\
& + (x_{jl}x_{ij}x_{jl}x_{lk}) - (x_{lk}x_{kj}x_{ij}x_{kj}), \\
=& -\alpha_1\alpha_2 - \alpha_2(\alpha_2 - x_{kj}x_{jl}) \\
& + (x_{jl}x_{ij}(x_{jl}x_{lk})) - ((x_{lk}x_{kj})x_{ij}x_{kj}), \\
=& -\alpha_1(\alpha_1 + \alpha_2) + \alpha_2 x_{kj}x_{jl} \\
& + (x_{jl}x_{ij}(\alpha_2 - x_{lk}x_{kj} - x_{kj}x_{jl})) \\
& - ((\alpha_2 - x_{kj}x_{jl} - x_{jl}x_{lk})x_{ij}x_{kj}), \\
=& -\alpha_1(\alpha_1 + \alpha_2) + \alpha_2 x_{kj}x_{jl} \\
& + (\alpha_2 x_{jl}x_{ij} - x_{jl}x_{ij}x_{kj}x_{jl}) \\
& - (\alpha_2 x_{ij}x_{kj} - x_{kj}x_{jl}x_{ij}x_{kj}).
\end{aligned}$$

□

Note 35. Let $\Delta_4(\alpha_1, \alpha_2)$ denotes $\mathcal{D}_G(\alpha_1, \alpha_2)$ for the special case of:

$$G = D_4 = \begin{tikzcd} 4 & 3 & 2 \\ & 1 \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}$$

in particular, $\Delta_4(\alpha_1, \alpha_2)$ is a proper subalgebra of $\Gamma_4(\alpha_1, \alpha_2)$.

Definition 4.10. The algebra $\Delta_4(\alpha_1, \alpha_2)$ is the K-algebra generated by x_{ij} for $(i, j) \in$

D_4 subject the following set of relations:

$$x_{ij} + x_{ji} = 0 \mid (i,j) \in D_4, \tag{25a}$$
$$x_{12}^2 - \alpha_1 = x_{13}^2 - \alpha_1 = x_{14}^2 - \alpha_1 = 0, \tag{25b}$$
$$x_{12}x_{13}x_{12} + x_{13}x_{12}x_{13} + \alpha_2(x_{12} + x_{13}) = 0, \tag{25c}$$
$$x_{12}x_{14}x_{12} + x_{14}x_{12}x_{14} + \alpha_2(x_{12} + x_{14}) = 0, \tag{25d}$$
$$x_{14}x_{13}x_{43} + x_{13}x_{14}x_{13} + \alpha_2(x_{14} + x_{13}) = 0, \tag{25e}$$
$$\begin{bmatrix} +x_{12}x_{13}x_{14}x_{12} \\ +x_{13}x_{14}x_{12}x_{13} \\ +x_{14}x_{12}x_{13}x_{14} \end{bmatrix} + \alpha_2(\begin{bmatrix} +x_{14}x_{12} \\ +x_{13}x_{14} \\ +x_{12}x_{13} \end{bmatrix} + \alpha_1 + \alpha_2) = 0, \tag{25f}$$
$$\begin{bmatrix} +x_{12}x_{14}x_{13}x_{12} \\ +x_{13}x_{12}x_{14}x_{13} \\ +x_{14}x_{13}x_{12}x_{14} \end{bmatrix} + \alpha_2(\begin{bmatrix} +x_{14}x_{13} \\ +x_{13}x_{12} \\ +x_{12}x_{14} \end{bmatrix} + \alpha_1 + \alpha_2) = 0. \tag{25g}$$

Remark 43. It was computed in [BLM16] that $\mathcal{E}_{D_4}$ is 48-dimensional. Our computations confirm that $\Delta_4(\alpha_1, \alpha_2)$ shares the same dimension as well.

Proposition 4.20. *If $(3\alpha_1 - \alpha_2)(\alpha_1^2 - \alpha_2^2) \neq 0$, then $\Delta_4(\alpha_1, \alpha_2)$ has no i-dimensional nonzero modules for $i \in [2]$.*

Proof. This is proven in an identical fashion as in Proposition 4.13. □

Note 36. The restriction of the $\Gamma_4(\alpha_1, \alpha_2)$-module S onto $\Delta_4(\alpha_1, \alpha_2)$ is a 4-dimensional $\Delta_4(\alpha_1, \alpha_2)$-module, we denote it by $S|_\Delta$ should no confusion occurs.

Corollary 17. *If $(3\alpha_1 - \alpha_2)(\alpha_1^2 - \alpha_2^2) \neq 0$, then $S|_\Delta$ is a simple $\Delta_4(\alpha_1, \alpha_2)$-module.*

Proposition 4.21. *If $S|_\Delta$ is simple, then $S|_\Delta^\sigma$ is a simple $\Delta_4(\alpha_1, \alpha_2)$-module pairwise non isomorphic to S for $\sigma = (2,4), (2,3)$.*

Proof. Assume that $(3\alpha_1 - \alpha_2)(\alpha_1^2 - \alpha_2^2) \neq 0$, the simplicity of $S|_\Delta^{(2,4)}$ falls immediately.
Furthermore, $S|_\Delta^{(2,4)} \ncong S|_\Delta$ is implied since:

$$S|_\Delta(x_{31}x_{41} + x_{41}x_{31}) + (\alpha_2 - \alpha_1) = 0,$$

while:

$$S|_\Delta(x_{31}x_{21} + x_{21}x_{31}) + (\alpha_2 - \alpha_1) \neq 0.$$

The other case is solved in a similar fashion. □

Remark 44. The previous lemma can be alternatively proven by showing that there exists no nonzero $f \in K^4$ such that:

$$S(x_{ij})f - fS^\sigma(x_{ij}) = 0, \mid \sigma = (2,4), (2,3).$$

Corollary 18. *If* $(3\alpha_1 - \alpha_2)(\alpha_1^2 - \alpha_2^2) \neq 0$, *then* $\Delta_4(\alpha_1, \alpha_2)$ *is semisimple.*

Proposition 4.22. $\Delta_4(\alpha_1, \alpha_2)$ *is semisimple only if:*

$$(3\alpha_1 - \alpha_2)(\alpha_1^2 - \alpha_2^2) \neq 0.$$

Proof. Assume that $(3\alpha_1 - \alpha_2)(\alpha_1^2 - \alpha_2^2) = 0$, since $\alpha_1 \neq 0$, we normalize that $\alpha_1 = 1$. We have three cases to discuss, we see:

1. If $\alpha_1 + \alpha_2 = 0$. In this case, we recall that the algebra $\mathcal{D}_4(\alpha, -\alpha)$-and consequently all of its subalgebras- is basic if such condition hold, therefore all simple $\Delta_4(\alpha, -\alpha)$-representation are given by the sign.

   In particular, for $i \in [2]$, there exists two non-isomorphic fixed simple representations ρ_i such that:

   $$\rho_1(x_{ij}) = -\rho_2(x_{ij}) = +1 \mid (i,j) \in D_4,$$

   where the space of extensions $Ext_K(\rho_1, \rho_2)$ is nonzero up to an isomorphism.

2. If $\alpha_1 - \alpha_2 = 0$. The algebra $\Delta_4(\alpha_1, \alpha_2)$ is not basic, we present a 2-dimensional module M of the form:

   $$M(x_{12}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad M(x_{13}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \rho(x_{14}) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},$$

   which is clearly simple as $\Delta_4(+1, +1)$ has no 1-dimensional modules.

   Furthermore, there exists a non-isomorphic simple representations M' such that:

   $$M'(x_{12}) = M(x_{13}), \qquad M'(x_{13}) = M(x_{12}), \qquad M'(x_{14}) = M(x_{14}),$$

   where the space of extensions $Ext_K(M, M')$ is 1-dimensional up to an isomorphism.

3. If $3\alpha_1 = \alpha_2 = +3$. The algebra $\Delta_4(\alpha_1, \alpha_2)$ is again not basic, we present a 2-dimensional module M of the form:

   $$M(x_{12}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad M(x_{13}) = \begin{bmatrix} -1 & 0 \\ -1 & +1 \end{bmatrix}, \qquad M(x_{14}) = \begin{bmatrix} -1 & +4 \\ 0 & +1 \end{bmatrix},$$

   which is clearly simple as $\Delta_4(+1, +3)$ has no 1-dimensional modules.

   Furthermore, there exists a non-isomorphic simple representations M' such that:

   $$M'(x_{12}) = M(x_{13}), \qquad M'(x_{13}) = M(x_{12}), \qquad M'(x_{14}) = M(x_{14}),$$

   where the space of extensions $Ext_K(M, M')$ is 2-dimensional up to an isomorphism.

□

Theorem 4.23. *The algebra $\Delta_4(\alpha_1, \alpha_2)$ is semisimple if and only if:*

$$(3\alpha_1 - \alpha_2)(\alpha_1^2 - \alpha_2^2) \neq 0.$$

Example 4.7. Interesting examples can be considered in the non semisimple algebra $\Delta_4(\alpha_1, \alpha_2)$, in particular, one may verify that $\Delta_4(\alpha, -\alpha)$ admits an ordinary quiver Q_{Δ_4} of the form:

$$\begin{tikzcd}
-++ \arrow[dr, shift left, "x_1"] & & & +-- \arrow[dl, shift left, "x_3"] \\
+-+ \arrow[r, shift right, "y"'] & --- \arrow[l, shift right, "x"'] \arrow[ul, shift left, "y_1"] \arrow[dl, shift left, "y_2"] \arrow[r, shift right, "\beta_1,\beta_2"'] & +++ \arrow[l, shift right, "\alpha_1,\alpha_2"'] \arrow[ur, shift left, "y_3"] \arrow[dr, shift left, "y_4"] \arrow[r, shift right, "y'"'] & -+- \arrow[l, shift right, "x'"'] \\
++- \arrow[ur, shift left, "x_2"] & & & --+ \arrow[ul, shift left, "x_4"]
\end{tikzcd}$$

Furthermore, there exists an isomorphism $\Delta_4(\alpha, -\alpha) \cong KQ_{\Delta_4}/I$, where I is the two-sided ideal generated by:

$$\begin{aligned}
\alpha_1\beta_1, \beta_1\alpha_1, x_i y_i, y_i x_i, \alpha_2 x_1, \alpha_1 x_2, \beta_2 y_3, \beta_1 y_4,\\
\alpha_1\beta_2 - \alpha_2\beta_1, \beta_1\alpha_2 - \beta_2\alpha_1, \alpha_1 x - \alpha_2 x, \beta_1 y' - \beta_2 y',\\
\alpha_2\beta_1 - \alpha_1\beta_2 - \alpha_2\beta_2, \beta_2\alpha_1 - \beta_1\alpha_2 - \beta_2\alpha_2,
\end{aligned}$$

for $\mid i \in [4]$.

Remark 45. We are now ready to prove Proposition 4.18. We remind the reader that we claimed that should:

$$(3\alpha_1 - \alpha_2)(\alpha_1^2 - \alpha_2^2) \neq 0,$$

be, then S' is a simple $\Gamma_4^{(2,4)}(\alpha_1, \alpha_2)$-module.

Indeed should such condition hold, then $\Delta_4(\alpha_1, \alpha_2)$ is a semisimple proper subalgebra of $\Gamma_4^{(2,4)}(\alpha_1, \alpha_2)$, and when viewed as a $\Delta_4(\alpha_1, \alpha_2)$-module, S' decomposes into a direct sum of two simple $\Delta_4(\alpha_1, \alpha_2)$-modules. The claim then is assured once we prove that $S|_\Delta$ does not extend to a $\Gamma_4^{(2,4)}(\alpha_1, \alpha_2)$-module which is proven to be true by Proposition 4.17.

This in particular, allow us to see that:

Corollary 19. *If $\alpha_1(3\alpha_1 - \alpha_2)(\alpha_1^2 - \alpha_2^2) \neq 0$, then the algebra $\Gamma_4(\alpha_1, \alpha_2)$ is semisimple.*

our final result.

5 Appendix

5.1 Conclusion remarks.

From our point of view, the appearance of the Nil-Coxeter algebra in Theorem **3.19** was surprising, to say the least, especially in contrast to the result of [HV18] where it was proven that the lower dimension case of the same parameters deformation was given by the pre-projective algebra of type A_2, two algebras with no apparent direct connection.

That being said, the opportunity of examining the theory from quiver theory point of view was very interesting, as seen, Gabriel theorem introduced an algebra morphism kernel which was minimal and maximal in the precise length sense, that is, there exists exactly one non-zero path connecting any two vertices in our quiver, a property that was not invariant going into the second deformation as seen in Proposition **3.33**.

Indeed, the inability to give a precise description of the non-basic non-semisimple deformation in terms of Gabriel theory highlights above all other things the irregularity one might face while dealing between what seem to be very similar structures.

From representation theory perspective, it is truly remarkable to have an isomorphism between the Dynkin based subalgebra and that of Iwahori-Hecke algebras, we highlight its practicality in advancing the discussion of the topic as doing representation theory in the classical scene might grow to be tiresome in higher dimensions, something we encountered previously in our work as well.

We foresee the techniques developed in the second part of our work, that is, the idea of inducing and restricting- to be highly valuable and productive in the field and highly recommend it as an alternative should high dimensionality calculations be too complicated.

Finally, we conclude our work by revising some important literature about $\mathcal{E}_n$, the "non-deformed" Fomin-Kirillov algebras and their connections to Nichols algebras.

5.2 Fomin-Kirillov and Nichols algebras.

Via braided vectors spaces.

Definition 5.1. A braided vector space is a pair (V, c), where V is a vector space and $c \in GL(V \otimes V)$ is a solution of the braid equation:

$$(c \otimes id)(id \otimes c)(c \otimes id) = (id \otimes c)(c \otimes id)(id \otimes c).$$

Example 5.1. Let V be a complex vector space with basis $\{x_i \mid i \in [n]\}$. Let $q_{ij} \in \mathbb{C}^\times$ where $i, j \in [n]$ distinct. A braided vector spaces of diagonal type (V, c) is given by:

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i.$$

Example 5.2. Let G be a finite group and $V = \mathbb{C}G$ the complex vector space with basis $\{g \mid g \in G\}$. Then (V, c) is a braided vector space where:

$$c(g \otimes h) = ghg^{-} \otimes g \mid g, h \in G.$$

Definition 5.2. The braid group $\mathbb{B}_n$ is given by generators $\{b_i \mid i \in [n]\}$ and relations:

$$\begin{aligned} b_i b_{i+1} b_i - b_{i+1} b_i b_{i+1} = 0 \mid i \in [n-1], \\ b_i b_j - b_j b_i = 0 \mid 1 < |i-j|. \end{aligned}$$

Remark 46. There exists a surjection $\mathbb{B}_n \to \mathbb{S}_n$ defined by mapping b_i to s_i.

Lemma 5.1. *There exists a set-theoretical section $\mathbb{S}_n \xrightarrow{\mu} \mathbb{B}_n$ mapping s_i to b_i, such that $\mu(xy) = \mu(x)\mu(y)$ if $l(xy) = l(x) + l(y)$.*

Remark 47. Given n a positive integer. Let (V, c) be a braided vector space. Consider:

$$c_i := id_{V^{\otimes(i-1)}} \otimes c \otimes id_{V^{\otimes(n-i-1)}} \in Aut(V^{\otimes n}) \mid i \in [n-1],$$

that is:

$$c_i(\otimes_{j=1}^{n} v_j) = (\otimes_{j=1}^{i-1} v_j) c(v_i \otimes v_{i+1})(\otimes_{j=i+2}^{n} v_j).$$

One may verify that:

$$\begin{aligned} \rho_n : \mathbb{B}_n \to Aut(V^{\otimes n}), \\ b_i \mapsto \rho_n(b_i) = c_i, \end{aligned}$$

is a $\mathbb{B}_n$-representation by showing that subjecting $\{c_i \mid i \in [n-1]\}$ to the defining relations of $\mathbb{B}_n$ holds no contradictions.

Definition 5.3. Let (V, c) be a braided vector space. Define $\mathfrak{B}(V, c)$ the Nichols algebra of (V, c) as:

$$\mathfrak{B}(V, c) := \bigoplus_n T^n(V)/Ker(\mathfrak{S}_n) = K \bigoplus V \bigoplus_{n \geq 2} V^{\otimes n}/Ker(\mathfrak{S}_n),$$

where $\mathfrak{S}_n$ is the quantum symmetrizer:

$$\mathfrak{S}_n := \sum_{\sigma \in \mathbb{S}_n} \rho_n \mu(\sigma).$$

Example 5.3. Let V be a complex vector space and let $V \otimes V \xrightarrow{flip} V \otimes V$ be the linear map defined by mapping $x \otimes y$ to $y \otimes x$.

1. The Nichols algebra of the braided vector space $(V, flip)$ is the Symmetric algebra $S(V)$.
2. The Nichols algebra of the braided vector space $(V, flip^{-})$ is the Exteriors algebra $\bigwedge(V)$.

Connection with Fomin-Kirillov algebras. Let V_n be the vector space with basis $\{v_{ij} \mid 1 \le i < j \le n\}$ and consider the map $c \in GL(V_n \otimes V_n)$ defined by:

$$c(v_\sigma \otimes v_\tau) = \xi(\sigma,\tau) v_{\sigma\tau\sigma^-} \otimes v_\sigma \mid \xi(\sigma,\tau) = \begin{cases} +1 & \mid \sigma(i) < \sigma(j), \\ -1 & \mid \sigma(i) > \sigma(j). \end{cases}$$

where σ and τ are transpositions and $\tau = (ij)$ with $i < j$.

Since (V_n, c) is a braided vector space, it is possible to consider the Nichols algebra $\mathfrak{B}(V_n)$.

Remark 48. One has a surjective homomorphism of algebras $\mathcal{E}_n \to \mathfrak{B}(V_n)$. It is well known that the surjection is an isomorphism for $n \le 5$. We refer readers interested in Nichols algebra to [IHJ20].

Via Quandles.

Definition 5.4. A set X together with a binary operation $\triangleright$ is said to be a quandle if the following hold:

$$x \triangleright x = x,$$
$$x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z).$$

for all $x, y, z \in X$ such that for all $x \in X$ the (auto)-mapping $y \mapsto x \triangleright y$ is bijective.

Definition 5.5. Given $(X, \triangleright)$ a quandle, the enveloping group of X -denoted by G_X- is the generic group generated by $\{g_x \mid x \in X\}$ subject the defining relation $g_x g_y = g_{x \triangleright y} g_x$ for all $x, y \in X$.

Note 37. Given $(X, \triangleright)$ a quandle, let V_X denote the K-vector space with a basis $\{v_x \mid x \in X\}$.

Definition 5.6. Given $(X, \triangleright)$ a quandle, a 2-cocycle on X is a map $q : X \times X \to K^\times$ such that:

$$q(y,z)q(x, y \triangleright z) = q(x,z)q(x \triangleright y, x \triangleright z) \mid x, y, z \in X.$$

Note 38. Given $(X, \triangleright)$ a quandle with q a 2-cocycle, the map $c \in Aut_K(V_X \otimes V_X)$, defined linearly by:

$$c(v_x \otimes v_y) = q(x,y) v_{x \triangleright y} \otimes v_x \mid x, y \in X.$$

is a braiding on V_X, that is, (V_X, c) is a braided vector space.

Note 39. Given $(X, \triangleright)$ a quandle with q a 2-cocycle, denote the Nichols algebra attached to (V_X, c) by $\mathfrak{B}_X$.

Remark 49. One can show that the mapping c as defined is a braiding if and only if q is 2-cocycle.

Example 5.4. Let G be a group and let $X \subseteq G$ be the conjugacy class of an element on G. Then X becomes a quandle via conjugation, that is:

$$x \rhd y = xyx^{-} \mid x, y \in X.$$

Example 5.5. Consider the quandle $X = \{0, 1, 2, 3\}$. The 2-cocycle is constant $q = -1$. It can be shown that $\mathfrak{B}_X$ is the K-algebra generated by $\{x_i \mid i \in X\}$, subject to the following set of relations:

$$\begin{aligned} x_0^2 = x_1^2 = x_2^2 = x_3^2 &= 0, \\ x_1x_3 + x_3x_2 + x_2x_{=}x_0x_3 + x_3x_1 + x_1x_0 &= 0, \\ x_2x_3 + x_3x_0 + x_0x_2 = x_1x_2 + x_2x_0 + x_0x_1 &= 0, \\ (x_2x_1 + x_1x_0 + x_0x_2)^3 &= 0. \end{aligned}$$

In particular, the algebra $\mathfrak{B}_X$ is 72-dimensional. Furthermore, the Hilbert polynomial of $\mathfrak{B}_X$ is given as: $(1+t)^2(1+t+t^2)(1+t^3)$.

Connection with Fomin-Kirillov algebras. As discussed earlier. The Fomin-Kirillov algebra $\mathcal{E}_n$ is isomorphic to the Nichols algebra given by the quandle:

$$X = \{(i, j) \mid 1 \leq i < j \leq n\} \subseteq \mathbb{S}_n \mid \sigma \rhd \tau = \sigma\tau\sigma^{-},$$

for all $\sigma, \tau \in X$, and the 2-cocycle q defined by:

$$q(\pi, (i, j)) = \begin{cases} +1 & \mid \pi(i) < \pi(j), \\ -1 & \mid \pi(i) > \pi(j). \end{cases} \mid 1 \leq i < j \leq n, \pi \in X.$$

Remark 50. While discussing the theory of Nichols algebras over Quandles limits our ability to realize the structure itself via the many tools braided Hopf-theory offers, it does give an access to a special class of algebras by discussing alternative 2-cocycles.

Example 5.6. Consider X the quandle of the six 2-cycles of $\mathbb{S}_4$, which of which form a conjugacy class. The 2-cocycle is constant $q = -1$. It can be shown that $\mathfrak{B}_X$ is the K-algebra generated by $\{x_{ij} \mid i < j \in [4]\}$, subject to the following set of relations:

$$\begin{aligned} x_{ij}^2 &= 0 \mid i, j \in [4], \\ [x_{ij}, x_{kl}] &= 0 \mid i, j, k, l \in [4] \text{ distinct}, \\ x_{12}x_{13} + x_{13}x_{23} + x_{23}x_{12} = x_{12}x_{14} + x_{14}x_{24} + x_{24}x_{12} &= 0, \\ x_{12}x_{23} + x_{23}x_{13} + x_{13}x_{12} = x_{12}x_{24} + x_{24}x_{14} + x_{14}x_{12} &= 0, \\ x_{13}x_{14} + x_{14}x_{34} + x_{34}x_{13} = x_{13}x_{34} + x_{34}x_{14} + x_{14}x_{13} &= 0, \\ x_{23}x_{24} + x_{24}x_{34} + x_{34}x_{23} = x_{23}x_{34} + x_{34}x_{24} + x_{24}x_{23} &= 0. \end{aligned}$$

Remark 51. The PBW-deformation of this example along that of X the quandle of the six 4-cycles of $\mathbb{S}_4$ with a the same fixed 2-cocycle has been studied in details in [Wol].

We refer readers interested in Nichols algebra via quandles to [Gra].

5.3 Miscellaneous

On the representation type of Fomin-Kirillov algebras.

Theorem 5.2. *[BLM16, Theorem 6.1] The only minimal relations in $\mathcal{E}_{A_n}$ are the quadratic and braid type relations. In other words, we have:*

$$\mathcal{E}_{A_n} \cong \mathcal{N}_n,$$

the nil-Coxeter algebra of type A_{n-1}.

Corollary 20. *The algebra $\mathcal{E}_n$ is of representation type wild for $n \geq 4$.*

5.4 German summary

Seit die Fomin-Kirillov-Algebren Ende der neunziger Jahre im [FK99] vorgestellt wurden, haben Sie ein grosses Interesse auf dem Gebiet der abstrakten Algebraforschung geweckt.

Ihre Verbindung zur Algebra-Kombinatorik wurden sowohl in [BLM16], [MPP14], [Pos99] und [GR97] berücksichtigt, als auch unter Anderem im [Gra], [AM03], [FP00] und [MS00] zur Hopf und Nichols algebras.

Darüber hinaus haben die Algebren viele interessante Erscheinungen auf dem Gebiet der Quantengruppentheorie [PV16], der nichtkommutative Geometrie [Maj17] und [Maj19] gezeigt. In [Baz06] und [Lau16] wurden einige interessante Verallgemeinerunsansätze dieser Algebren behandelt. In [BK19] wurden Sie auch auf die sogenannten Hecke-Hopf-Algebren angewendet.

Motiviert durch neuere Entdeckungen von I. Heckenberger, L. Vendramin [HV18] und K. Wolf [Wol], diese Doktorarbeit dem Thema der Darstellungstheorie von PBW-Deformationen von Fomin-Kirillov-Algebren behandelt.

Diese Arbeit in zwei Teile unterteilt. Im ersten Teil wurde die Darstellungstheorie von nicht-Halbeinfache $\mathcal{D}_4(\alpha_1, \alpha_2)$ und des generischen n in einigen Fällen aus der Sicht des Gabriel-Theorems entwickelt.

Es zeigte sich, dass zwei Fälle besprochen werden mussten. Der erste Fall war $\mathcal{D}_n(\alpha, -\alpha)$ der sich als basic und connected erwiesen hat, und daher eine graphische Darstellung wie in **Theorem 3.11** beschrieben hat.

Dies ermöglichte die Diskussion der, die Anwendung des Satzes von Gabriel im Spezialfall $n = 4$ zu diskutieren wie in **Theorem 3.19** gezeigt.

Als nächstes wurde $\mathcal{D}_4(\alpha, \alpha)$ betrachtet, das nicht-Basic war und eine zugehörige Basic Algebra hat, die Morita-äquivalent ist. Es zeigte sich, dass diese zugehörige Version connected ist und eine graphische Darstellung der in **Theorem 3.28** beschriebenen Form zulässt. Dadurch konnten seine Darstellung in **Proposition 3.33** vorgeschlagen werden.

Der zweite Teil war dem Studium einiger interessanter auf Dynkin-Graphen basierender Unteralgebren von $\mathcal{D}_n(\alpha_1, \alpha_2)$ gewidmet. Insbesondere haben wir $\Lambda_n(\alpha_1, \alpha_2)$, eine

Unteralgebra von $\mathcal{D}_n(\alpha_1, \alpha_2)$ basierend auf dem Dynkin-Graphen vom Typ A_n betrachtet. Wir haben bewiesen, dass diese bis auf einem Parameter isomorph zur generischen Iwahori-Hecke-Algebra, siehe **Theorem 4.2**.

Abschliessend wurde haben wir eine Familie von Algebren betrachtet, die isomorph zu $\Lambda_n(\alpha_1, \alpha_2)$ und auch parameterkompatibel ist. Dies ermöglichte uns, eine Äquivalenz der Halbeinfachheit von $\Delta_4(\alpha_1, \alpha_2)$ zu finden, einer weiteren Unteralgebra von $\mathcal{D}_n(\alpha_1, \alpha_2)$ basierend auf dem Dynkin-Graphen vom Typ D_n, wie in **Proposition 4.22** diskutiert.

5.5 Declaration

I hereby declare, that I have not made any doctoral attempts prior to this one. I assure, that I have written this thesis myself and without any external assistance and used no sources or aids other than those indicated. Moreover I assure, that I have not submitted the dissertation in its present or similar form to any other domestic or foreign university in connection with a doctoral application or for other examination purposes.

Marburg, January 2022.
Abdalla Alia.

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