On 3-(α, δ)-Sasaki Manifolds and Their Canonical Submersions

Dissertation

zur Erlangung des Doktorgrades
der Naturwissenschaften (Dr. rer. nat.)

am Fachbereich Mathematik und Informatik
der Philipps-Universität Marburg

vorgelegt von

Leander Stecker
geboren am 18.11.1991 in München

unter Betreuung von

Prof. Dr. habil. Ilka Agricola

On 3-(α, δ)-Sasaki Manifolds and Their Canonical Submersions

Dissertation

zur Erlangung des Doktorgrades
der Naturwissenschaften (Dr. rer. nat.)

am Fachbereich Mathematik und Informatik
der Philipps-Universität Marburg
Hochschulkennziffer: 1180

vorgelegt von

Leander Stecker
geboren am 18.11.1991 in München

Erstgutachter:	Prof. Dr. Ilka Agricola
Zweitgutachter:	Prof. Dr. Uwe Semmelmann
Eingereicht:	13.08.2021
Disputation:	23.11.2021

Abstract

We investigate 3-(α,δ)-Sasaki manifolds through their canonical connection. These manifolds are almost 3-contact metric manifolds satisfying the condition

$$d\eta_i = 2\alpha\Phi_i + 2(\alpha-\delta)\eta_j \wedge \eta_k,$$

thus generalizing 3-Sasaki manifolds. They admit a canonical metric connection ∇ with parallel skew torsion uniquely characterized by

$$\nabla_X \varphi_i = \beta(\eta_k(X)\varphi_j - \eta_j(X)\varphi_k), \qquad \beta \in \mathbb{R}. \tag{1}$$

This condition restricts the holonomy $\mathfrak{hol}(\nabla) \subset (\mathfrak{sp}(n)\oplus\mathfrak{sp}(1))\oplus\mathfrak{so}(3)$ and thereby induces a locally defined Riemannian submersion π along the Reeb orbits. This canonical submersion connects the geometries on the base N and total space by the key equation

$$\nabla^{g_N}_X Y = \pi_*(\nabla_{\overline{X}}\overline{Y}). \tag{2}$$

Combining (1) and (2), we obtain that for the projections $\check{\varphi}_i$ of φ_i

$$\nabla^{g_N}_X \check{\varphi}_i = 2\delta(\check{\eta}_k(X)\check{\varphi}_j - \check{\eta}_j(X)\check{\varphi}_k),$$

defining a quaternionic Kähler structure on the base N. We show that the scalar curvature of N is given by $\mathrm{scal}^{g_N} = 16n(n+2)\alpha\delta$, leading to vastly different behavior if $\alpha\delta = 0$, $\alpha\delta > 0$ or $\alpha\delta < 0$. These cases are called degenerate, positive and negative respectively.

Non-degenerate homogeneous 3-(α,δ)-Sasaki manifolds are a particularly well-behaved class to investigate, since the underlying homogeneous quaternionic Kähler manifolds of non-vanishing scalar curvature are classified. The first option are symmetric spaces of compact ($\mathrm{scal}^{g_N} > 0$) and non-compact ($\mathrm{scal}^{g_N} < 0$) type. We treat this case in a unified manner showing that pairs of positive and negative homogeneous 3-(α,δ)-Sasaki manifolds appear as quotients of simple Lie groups. The other option are Alekseevsky spaces which admit a transitive solvable group action. These are harder to deal with than those admitting a transitive simple group action, but a careful construction gives rise to additional negative homogeneous 3-(α,δ)-Sasaki manifolds.

In a third part we exploit identity (2) once more applying it to investigate the curvature. This yields a decomposition of the canonical curvature operator

$$\mathcal{R} = \alpha\beta\mathcal{R}_\perp + \mathcal{R}_{\mathrm{par}}$$

where $\mathcal{R}_\perp$ is controlled by the 3-(α,δ)-Sasaki structure and $\mathcal{R}_{\mathrm{par}}$ encodes the Riemannian curvature of the quaternionic Kähler base. If $\mathcal{R}^{g_N} \geq 0$ or $\mathcal{R}^{g_N} \leq 0$ and α,δ are suitable parameters, the decomposition allows us to derive the same properties for $\mathcal{R}$. By the formula $\mathcal{R}^g = \mathcal{R} + \frac{1}{4}\mathcal{G}_T + \frac{1}{4}\mathcal{S}_T$ we can also control positivity of $\mathcal{R}^g$. These results are tailor-made for the first class of homogeneous examples as their symmetric base automatically satisfies $\mathcal{R}^{g_N} \geq 0$ or $\mathcal{R}^{g_N} \leq 0$ depending on whether they are compact or not. We thus obtain semi-definite curvature operators on all of them and strongly positive curvature in some instances.

Zusammenfassung

Wir untersuchen 3-(α,δ)-Sasaki Mannigfaltigkeiten anhand ihrer kanonischen Zusammenhänge. Diese sind eine Klasse von metrischen fast 3 Kontaktmannigfaltigkeiten, welche

$$d\eta_i = 2\alpha\Phi_i + 2(\alpha-\delta)\eta_j \wedge \eta_k$$

erfüllen, und daher eine Verallgemeinerung von 3-Sasaki Mannigfaltigkeiten darstellen. Sie besitzen einen Zusammenhang ∇ mit paralleler, schief-symmetrischer Torsion, der eindeutig durch

$$\nabla_X \varphi_i = \beta(\eta_k(X)\varphi_j - \eta_j(X)\varphi_k), \qquad \beta \in \mathbb{R} \tag{1}$$

gegeben ist. Dies führt zu einer Reduktion der Holonomie $\mathfrak{hol}(\nabla) \subset (\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)) \oplus \mathfrak{so}(3)$ und induziert dadurch eine lokal definierte Riemannsche Submersion π entlang der Reeb Orbits. Die so gegebene kanonische Submersion verbindet die Geometrie der Basis N mit dem Totalraum durch die zentrale Beziehung

$$\nabla^{g_N}_X Y = \pi_*(\nabla_{\overline{X}}\overline{Y}). \tag{2}$$

Kombinieren wir (1) und (2), so erhalten wir für die Projektionen $\check{\varphi}_i$ von φ

$$\nabla^{g_N}_X \check{\varphi}_i = 2\delta(\check{\eta}_k(X)\check{\varphi}_j - \check{\eta}_j(X)\check{\varphi}_k)$$

und definieren dadurch eine quaternionisch Kähler Struktur auf der Basis N. Wir zeigen, dass die Skalarkrümmung von N durch $\mathrm{scal}^{g_N} = 16n(n+2)\alpha\delta$ gegeben ist, was zu wesentlich unterschiedlichem Verhalten in den Fällen $\alpha\delta = 0$, $\alpha\delta > 0$ und $\alpha\delta < 0$ führt. Wir bezeichnen diese als den degenerierten, positiven oder negativen Fall.

Nicht-degenerierte homogene 3-(α,δ)-Sasaki Mannigfaltigkeiten sind besonders gut zu untersuchen, da die ihnen zugrunde liegenden homogenen quaternionisch Kähler Räume von nicht verschwindender Skalarkrümmung klassifiziert sind. Die erste Option sind symmetrische Räume von kompaktem ($\mathrm{scal}^{g_N} > 0$) und nicht kompaktem ($\mathrm{scal}^{g_N} < 0$) Typ. Wir geben eine vereinheitlichte Beschreibung dieses Falles und zeigen dabei, wie Paare von positiven und negativen 3-(α,δ)-Sasaki Mannigfaltigkeiten als Quotienten einfacher Lie-Gruppen entstehen. Die andere Option sind Alekseevsky Räume, also homogene Räume mit transitiver Wirkung einer auflösbaren Gruppe. Obwohl diese schwieriger zu handhaben sind als solche mit transitiver Wirkung einer einfachen Gruppe, erhalten wir durch behutsame Konstruktion weitere Beispiele negativer homogener 3-(α,δ)-Sasaki Mannigfaltigkeiten.

Im dritten Teil nutzen wir nochmals (2), um die Krümmung zu untersuchen. Dies führt zu einer Zerlegung des kanonischen Krümmungsoperators

$$\mathcal{R} = \alpha\beta\mathcal{R}_\perp + \mathcal{R}_{\mathrm{par}},$$

wobei $\mathcal{R}_\perp$ durch die 3-(α,δ)-Sasaki Struktur bestimmt ist und $\mathcal{R}_{\mathrm{par}}$ die Riemannsche Krümmung der quaternionisch Kähler Basis widerspiegelt. Falls $\mathcal{R}^{g_N} \geq 0$ oder $\mathcal{R}^{g_N} \leq 0$ und entsprechende Parameter α, δ gegeben sind, erhalten wir aus der Zerlegung die jeweilige Eigenschaft auch für $\mathcal{R}$. Via $\mathcal{R}^g = \mathcal{R} + \frac{1}{4}\mathcal{G}_T + \frac{1}{4}\mathcal{S}_T$ haben wir auch Kontrolle über die Positivität von $\mathcal{R}^g$. Diese Ergebnisse sind maßgeschneidert für den ersten Typ von homogenen Beispielen, da ihre symmetrische Basis automatisch entweder $\mathcal{R}^{g_N} \geq 0$ oder $\mathcal{R}^{g_N} \leq 0$ erfüllt, je nachdem, ob sie kompakt ist oder nicht. Wir erhalten semi-definite Krümmungsoperatoren auf all diesen und in manchen Fällen sogar stark positive Krümmung.

Acknowledgements

First and foremost I would like to thank my supervisor Ilka Agricola, for inviting me to come to Marburg and introducing me the wonderful world of 3-(α, δ)-Sasaki manifolds, that has turned out to be so fruitful. On that note I want to thank the co-author of our almost two publications, Giulia Dileo, without whom we could not have made the progress we did. I'd like to also thank my second referee Uwe Semmelmann for a thorough read-through and many helpful comments.

I want to thank all the awesome people on A8. Whether active, former or loosely connected, all of you have made not only my work, but my time in Marburg an amazing one. Thank you Leo, without you I would not have gone to Marburg, and I am glad I did. Thank you Marius, my brother in math, travel mate and tex support. Thank you Stefan, I enjoyed sharing an office with you and the great blackboard chess games we had. Thank you Gibbo - it has only been a year, but man we talked a lot - math and other. Thank you Chief Oliver, for all the invaluable support and advice you have provided.

And finally I would like to thank my dear family. Thank you Flo for leading the way in math and outside, with all the advice you gave. Danke, liebe Eltern, dass ihr immer für mich da seid und mich unterstützt.

Contents

1 Introduction **1**
1.1 Motivation . . . 1
1.2 Outline . . . 3

2 Connections with Skew Torsion **5**
2.1 Decomposition of the space of Torsion Tensors . . . 5
2.2 Skew Torsion with Reducible Holonomy . . . 9

3 Almost Contact Metric Geometry **14**
3.1 Sasaki Manifolds . . . 14
3.2 Almost 3-Contact Metric Manifolds . . . 18
3.3 3-(α, δ)-Sasaki Manifolds . . . 22

4 The Canonical Submersion **29**
4.1 The Model Case of α-Sasaki manifolds . . . 29
4.2 The Canonical Submersion onto qK spaces . . . 31
4.3 The Global Picture . . . 35

5 Homogeneous 3-(α, δ)-Sasaki manifolds **39**
5.1 Homogeneous 3-(α, δ)-Sasaki manifolds over symmetric quaternionic Kähler spaces . . . 39
5.2 3-(α, δ)-Sasaki over Alekseevsky Spaces . . . 43
5.3 Examples . . . 48
5.4 Nomizu Maps . . . 54

6 Curvature **59**
6.1 Decomposition of the Canonical Curvature . . . 59
6.2 Special Eigenvalues of $\mathcal{R}$ and $\mathcal{R}^g$ . . . 61
6.3 Positivity of Curvature Operators . . . 66
6.4 The Homogeneous Case . . . 72
6.5 The Inhomogeneous Case . . . 74

Bibliography **77**

1 Introduction

1.1 Motivation

The investigation of geometry via transformation groups dates back to F. Klein's Erlangen Program [Kle72]. For geometries with tensorial data, the linear group G transforming adapted frames into one another was mainly investigated. This yields a reduction of the frame bundle to a principal G-bundle nowadays called a G-structure, where for Riemannian manifolds one has $G \subset \mathrm{SO}(n)$. However, soon people became aware that, unlike in the Euclidean case, G-structures are not necessarily preserved by parallel transport along curves. The thereby defined new group of transformations was coined the holonomy group of a manifold by E. Cartan in [Car26a].

Cartan's understanding of the holonomy group would prove to be crucial shortly thereafter, as he managed to show that for a symmetric space $M = G/H$ the Riemannian holonomy group is equal to H. This observation gave rise to his celebrated classification of symmetric spaces in [Car26b]. A second development promoting the importance of the holonomy group was its remarkable connection to curvature. In fact, W. Ambrose and I. M. Singer in [AS53] showed that the holonomy algebra is generated by elements

$$\mathcal{P}_\gamma^{-1} \circ R(X,Y) \circ \mathcal{P}_\gamma$$

for $X, Y \in TM$ and parallel transport $\mathcal{P}_\gamma$ along some piecewise smooth curve γ.

It was apparent that in order to understand Riemannian geometry, an investigation into possible Riemannian holonomy groups was mandated. Around the same time of the Ambrose-Singer Theorem, G. de Rham proved a splitting theorem [Rha52], that made a classification at all feasible.

Theorem (de Rham Splitting Theorem)**.** *Suppose that (M, g) is a simply connected manifold such that the Riemannian holonomy representation is reducible on the tangent space, i.e. $TM = T_1 \oplus T_2$. Then*

$$(M, g) \cong (M_1, g_1) \times (M_2, g_2).$$

This theorem restricted the question to irreducible holonomy representations and an answer was finally given in [Ber55] by M. Berger.

Theorem (Berger Holonomy Theorem)**.** *Let (M, g) be a non-symmetric, irreducible and simply connected Riemannian manifold. Then the holonomy group Hol^g is one of the subgroups of $\mathrm{SO}(n)$ in the table below.*

Name	$\mathrm{Hol}^{\mathrm{g}}$	$\dim M$
generic manifold	$\mathrm{SO}(n)$	n
Kähler	U(k)	$n = 2k$
Calabi-Yau	$\mathrm{SU}(k)$	$n = 2k$
quaternionic Kähler	$\mathrm{Sp}(k)\mathrm{Sp}(1)$	$n = 4k$
hyperkähler	$\mathrm{Sp}(k)$	$n = 4k$
parallel G_2	G_2	$n = 7$
parallel Spin(7)	Spin(7)	$n = 8$
	[Spin(9)]	$[n = 16]$

Table: Possible Riemannian holonomy groups.

Berger's list only gave possible Riemannian holonomy groups, but did not show whether these occurred as holonomy groups of compact non-symmetric manifolds. In fact, with the exception of holonomy Spin(9) manifolds that were shown to be necessarily symmetric by D. Alekseevsky and independently by R. Brown and A. Gray all entries of Berger's list exist. However, at the time of Berger's Theorem the only class known to exist were Kähler manifolds. The program finding these examples spanned the second half of the 20th century. Among others, Contributions were made by S.-T. Yau in 1978, for holonomy groups $\mathrm{SU}(n)$ and $\mathrm{Sp}(n)$, by D. Alekseevsky constructing non-symmetric homogeneous quaternionic Kähler spaces in 1975, and, finally, by D. Joyce who constructed compact manifolds with holonomy G_2 and Spin(7) in 1996.

The remarkable shortness of the list is both its strength and weakness alike. The non-generic classes show remarkable geometric properties. The most prominent are Kähler manifolds defined by a parallel almost complex structure, $\nabla^g J = 0$, and have been studied intensively by complex and algebraic geometry methods. With bigger codimension of $\mathrm{Hol}^{\mathrm{g}} \subset \mathrm{SO}(n)$ generally the conditions on these classes tend to be even more restrictive. This, in particular, applies to the quaternionic counterparts in the form of quaternionic Kähler and hyperkähler manifolds.

However, since these classes are so few and examples exceptionally rare, the holonomy is generic for almost all Riemannian manifolds - in odd dimension $n \neq 7$, the list even fails to account for any specialized geometries at all. This is unsatisfying at best. A particularly notable case can be found in Sasaki manifolds. Already at their introduction in [SH62] by S. Sasaki and Y. Hatakeyama they have been deemed odd dimensional analogues to Kähler manifolds. Yet neither they nor their quaternionic counterparts, 3-Sasaki and 3-(α, δ)-Sasaki manifolds, have special Riemannian holonomy.

Given this discrepancy, people have looked for generalizations of the classical Riemannian holonomy theory, for instance, the notion of weak holonomy due to A. Gray. The approach we follow is to look at the holonomy group of a connection other than the Levi-Civita connection. Folklore is that many exceptional geometries admit metric connections with torsion

better adapted to their geometry. This thesis can be read as an exhibitory case underlining that assessment. We investigate 3-(α, δ)-Sasaki manifolds. Introduced in [AD20], they are defined as almost 3-contact metric manifolds satisfying the condition

$$d\eta_i = 2\alpha\Phi_i + 2(\alpha - \delta)\eta_j \wedge \eta_k, \quad \text{where } \alpha, \delta \in \mathbb{R},\ \alpha \neq 0.$$

They generalize the special case of 3-Sasaki manifolds for values $\alpha = \delta = 1$. Naturally they are endowed with a metric connection ∇ of parallel skew torsion whose holonomy representation is reducible. Unlike in the de Rham theorem, we do not locally get a product. However, we will show that the reducible holonomy leads instead to a locally defined Riemannian submersion over a quaternionic Kähler manifold coupling the canonical connection to the Levi-Civita connection on the base by

$$\nabla^{g_N}_X Y = \pi_*(\nabla_{\overline{X}}\overline{Y}).$$

We show how this coupling and thereby the canonical connection lead to a deeper understanding of the geometry of 3-(α, δ)-Sasaki manifolds.

1.2 Outline

In Chapter 2 we recall the notion of connections with skew torsion and some classical examples. The curvature tensor for connections with skew torsion is investigated. We compare results on the reducibility of the holonomy representation of a connection with skew torsion in the spirit of de Rham and Berger. This leads to the investigation of connections with parallel skew torsion whose holonomy is reducible but non-decomposable, started in [CMS21]. We give a slight generalization, Corollary 2.2.7, of their submersion theorem which will be the key tool in the investigation of canonical submersions.

Chapter 3 is a compendium of almost contact metric and almost 3-contact metric manifolds, related definitions, classical results and well adapted connections. In Section 3.3 we focus purely on our main geometry, 3-(α, δ)-Sasaki manifolds, together with their canonical connections. We provide an overview of the state of the art and derive some immediate formulas for the canonical curvature needed later.

The key Chapter 4 applies the submersion result from Corollary 2.2.7 to 3-(α, δ)-Sasaki manifolds and their canonical connection. In Section 4.1 we discuss the technique in the simpler case of Sasaki manifolds, reproving that any Sasaki manifold locally admits a Riemannian submersion over a Kähler space. The main application is described in Section 4.2. We obtain a structure theorem for 3-(α, δ)-Sasaki manifolds showing that these admit a locally defined Riemannian submersion over a quaternionic Kähler space with scalar curvature a positive multiple of $\alpha\delta$. As the result is a priori local, we investigate its global behavior in Section 4.3. We show that for non-degenerate 3-(α, δ)-Sasaki manifolds the patches of the canonical submersion give rise to a globally defined submersion over an orbifold. For degenerate 3-(α, δ)-Sasaki manifolds Theorem 4.3.3 gives a model construction of such spaces submerging onto hyperkähler manifolds.

Non-degenerate homogeneous 3-(α, δ)-Sasaki are investigated in Chapter 5. We give explicit expressions of 3-(α, δ)-Sasaki manifolds over all known homogeneous quaternionic Kähler manifolds of non-vanishing scalar curvature. This yields a construction over Wolf spaces, deforming the description given in [DOP20], as well as their non-compact duals. Both are given in Theorem 5.1.2, showing the duality of both cases. Independently, over Alekseevsky spaces we give a construction in Theorem 5.2.3 using a description of the latter given by V. Cortés in [Cor00]. We provide detailed descriptions of the 7-dimensional Aloff-Wallach space and its negative counterpart which submerge onto the 4-dimensional Wolf space $\mathrm{SU}(3)/S(\mathrm{U}(2) \times \mathrm{U}(1))$, respectively its non-compact dual. We further display explicitly the homogeneous 3-(α, δ)-Sasaki space $\hat{\mathcal{T}}(1)$ in dimension 19 sitting above the lowest dimensional non-symmetric Alekseevsky space $\mathcal{T}(1)$. In Section 5.4 we compute the Nomizu map associated to the canonical connection, a necessary tool for further investigation of these spaces. In the symmetric base case we find the Nomizu map of the Levi-Civita connection as well.

Chapter 6 is then a discussion of the curvature operators of 3-(α, δ)-Sasaki manifolds. The discussion is led by a decomposition given in Proposition 6.1.2 introduced through the canonical submersion. Section 6.2 is devoted to the structure tensor part of this decomposition. We compute special eigenvalues and eigenvectors of the canonical and Riemannian curvature operators. We obtain, in particular, a characterization of the Einstein property in terms of eigenvalues in Theorem 6.2.6. Section 6.3 relates the Riemannian curvature of the quaternionic Kähler base to the canonical and Riemannian curvature operators of a 3-(α, δ)-Sasaki manifold. In particular, Theorem 6.3.3 shows that the canonical curvature is non-negative if the base is and either $\alpha\beta \geq 0$. If the base satisfies a certain positivity condition we prove in Theorem 6.3.9 that the 3-(α, δ)-Sasaki manifold above even has strongly positive curvature. Section 6.4 is an application to the homogeneous spaces described in Chapter 5. Finally, we give an outlook into possible application on non-homogeneous 3-(α, δ)-Sasaki manifolds.

Chapter 5 and the main result of 4.2 have been published in the joint work [ADS21a] with I. Agricola and G. Dileo. A publication [ADS21b] with the same authors is in preparation including most of Chapter 6.

2 Connections with Skew Torsion

2.1 Decomposition of the space of Torsion Tensors

Let (M^n, g) be a Riemannian manifold. If we investigate a non-symmetric geometric structure outside the scope of Berger's list of special holonomies the Riemannian holonomy will always be of general $\mathrm{SO}(n)$ type. For further investigation we, thus, look for more general connections better adapted to the geometry. Since the space of connections on TM is an affine space with associated vector space $(T^*M)^2 \otimes TM$ we may write any connection ∇ on TM as

$$\nabla_X Y = \nabla^g_X Y + A(X, Y),$$

relative to the Levi-Civita connection ∇^g. As we consider a Riemannian manifold any suitable connection ∇ should be metric, i.e. $\nabla g = 0$. Viewing A via the metric as a $(3, 0)$-tensor the condition $\nabla g = 0$ can be expressed as $A \in T^*M \otimes \Lambda^2 T^*M$.

Recall that the Levi-Civita connection is the unique metric connection that is torsion-free. Hence, the torsion

$$T^\nabla(X, Y, Z) := g(T^\nabla(X, Y), Z) = g(\nabla_X Y - \nabla_Y X - [X, Y], Z)$$

of ∇ is non-trivial whenever $\nabla \neq \nabla^g$. The space of torsion tensors $\mathcal{T}$ is then $\Lambda^2 T^*M \otimes T^*M$. The group $\mathrm{SO}(n)$ acts on both spaces. In fact, the spaces are isomorphic as $\mathrm{SO}(n)$-modules via

$$T^\nabla(X, Y, Z) = A(X, Y, Z) - A(Y, X, Z). \tag{2.1}$$

If $n \geq 3$ the space of torsion tensors decomposes under the action of $\mathrm{SO}(n)$ into 3 irreducible representations

$$\mathcal{T} = TM \oplus \Lambda^3 T^*M \oplus \mathcal{T}^1.$$

We will only be concerned with the second summand. For a more extensive survey including the other classes see [Agr06].

Definition 2.1.1. A metric connection ∇ on a Riemannian manifold (M, g) is said to have *skew symmetric torsion*, or *skew torsion* for short, if $T^\nabla \in \Lambda^3 T^*M$.

For a connection with skew torsion ∇ the isomorphism (2.1) shows that $A = \frac{1}{2} T^\nabla$ and hence

$$\nabla_X Y = \nabla^g_X Y + \frac{1}{2} T(X, Y, \cdot).$$

A consequence is that the geodesic curves γ of (M, g) are geodesic with respect to ∇ as well. Indeed,

$$\nabla_{\dot\gamma} \dot\gamma = \nabla^g_{\dot\gamma} \dot\gamma + \frac{1}{2} T(\dot\gamma, \dot\gamma, \cdot) = \nabla^g_{\dot\gamma} \dot\gamma$$

As we will see often an additional assumption on the connection is necessary for a meaningful investigation.

Definition 2.1.2. A metric connection ∇ with torsion T is said to have *parallel* torsion if $\nabla T = 0$.

We take a short detour for an important example of connections with skew torsion. Consider a homogeneous space $M = G/H$ with a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. The Nomizu Theorem characterizes G-invariant connections, see [KN96, Theorem X.2.1, Proposition X.2.3, p.191]

Theorem 2.1.3. *Let $M = G/H$ be a homogeneous space with reductive complement $\mathfrak{m}$. Then there is a bijective correspondence*

$$\{\nabla \quad G\text{-invariant connection on } M\} \qquad \overset{1:1}{\longleftrightarrow} \qquad \left\{\Lambda^{\nabla} \colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m} \quad \mathfrak{h}\text{-invariant}\right\}$$

given by $\Lambda^{\nabla}_X Y = \nabla_{X_0} Y - [X, Y]_0$. The torsion tensor of a connection ∇ is given on fundamental vector fields $X, Y \in \mathfrak{m}$ by

$$T_0^{\nabla}(X, Y) = \Lambda^{\nabla}_X Y - \Lambda^{\nabla}_Y X - [X, Y]_{\mathfrak{m}}, \tag{2.2}$$

where the subscript $\mathfrak{m}$ denotes projection to $\mathfrak{m} \subset \mathfrak{g}$. The connection is metric if and only if $\Lambda_X \in \mathfrak{so}(\mathfrak{m})$ for every $X \in \mathfrak{m}$.

Corollary 2.1.4. *Let $M = G/H$ be a homogeneous space with reductive complement $\mathfrak{m} \subset \mathfrak{g}$. Then there is a unique G-invariant connection given by $\Lambda \equiv 0$. It has torsion $T(X, Y) = -[X, Y]_{\mathfrak{m}}$.*

We will call this connection the Ambrose-Singer connection of the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

Example 2.1.5. A naturally reductive space is a homogeneous space that admits a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and a G-invariant metric induced by an $\operatorname{ad}\mathfrak{h}$-invariant scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{m}$ such that

$$\langle [X, Y]_{\mathfrak{m}}, Z \rangle = -\langle Y, [X, Z]_{\mathfrak{m}} \rangle \qquad \forall X, Y, Z \in \mathfrak{m}. \tag{2.3}$$

Equation (2.3) shows that the Ambrose-Singer connection has skew torsion exactly if the homogeneous space is naturally reductive. The torsion and curvature of this connection are always parallel. In fact, if there exists a metric connection with parallel skew torsion and curvature on a manifold M, it has to be locally isometric to a naturally reductive homogeneous space, see [Tri92].

Note that the Ambrose-Singer connection depends on the groups G, H and the reductive complement $\mathfrak{m}$. So a Riemannian homogeneous space $(M = G/H, g)$ might be naturally reductive, but not with respect to the data G, H and $\mathfrak{m}$.

We discuss another important class of geometries.

Definition 2.1.6. Let (M^n, g) be a Riemannian manifold with orthogonal frame bundle $L(M, g)$. Suppose $G \subset \mathrm{SO}(n)$ is a subgroup. A *G-structure* is a G-principal subbundle $P \subset L(M, g)$.

Most of the time G-structures will be defined in such a way that G is the stabilizer of some tensor on TM. For instance this is true for the classes appearing in Berger's holonomy list. Now the Levi-Civita connection preserves the G-structure exactly in those cases, paralleliz-ing the defining tensors. For other geometries we would like to have connections preserving their G-structure as well. Of course this comes with the cost of some non-trivial torsion tensor.

Definition 2.1.7. A metric connection ∇ on a Riemannian manifold (M, g) with G-structure is called the *characteristic connection* if it is the unique connection with skew torsion preserving the G-structure.

Example 2.1.8. Important examples of manifolds admitting characteristic connections are nearly Kähler manifolds and nearly parallel G_2-manifolds. Their connections first occurred in [Gra76] for nearly Kähler manifolds, and for nearly parallel G_2-manifolds in [FI02]. T. Friedrich and S. Ivanov also show in their work that these are indeed characteristic connections. Both examples have parallel torsion.

The most important examples for this thesis are almost contact metric manifolds and their connections. These will be discussed in Chapter 3.

Let us now discuss curvature for metric connections with torsion. For proofs of the formulas we refer to the survey [Agr06] and the references therein. We define the curvature tensor of a metric connection ∇ as the $(4, 0)$-tensor given by

$$R^{\nabla}(X, Y, Z, V) := g((\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z, V).$$

The curvature of the Levi-Civita connection will be called Riemannian curvature and denoted by a superscript g. With this reference point we compare the curvature R^{∇} to R^g. In order to simplify the formulas we restrict our discussion to connections with skew torsion. Then

$$\begin{aligned} R^g(X, Y, Z, V) = R^{\nabla}(X, Y, Z, V) &- \frac{1}{2}(\nabla_X T)(Y, Z, V) + \frac{1}{2}(\nabla_Y T)(X, Z, V) \\ &- \frac{1}{4} g(T(X, Y), T(Z, V)) - \frac{1}{4}\sigma_T(X, Y, Z, V). \end{aligned} \tag{2.4}$$

Here and in many instances forward we use the 4-form σ_T associated to a skew torsion tensor defined by

$$\begin{aligned} \sigma_T(X, Y, Z, V) &:= \overset{X,Y,Z}{\mathfrak{S}} g(T(X, Y), T(Z, V)) \\ &:= g(T(X, Y), T(Z, V)) + g(T(Y, Z), T(X, V)) + g(T(Z, X), T(Y, V)), \end{aligned}$$

where $\mathfrak{S}$ denotes the cyclic sum, i.e. the sum over all even permutations of (X, Y, Z). These curvature tensors inherit some symmetries of the Riemannian curvature. By definition they

are skew-symmetric in the first two entries and for a metric connection in the last two entries as well. If we assume ∇ is a connection with parallel skew torsion (2.4) also implies the pair symmetry of R^∇

$$R^\nabla(X,Y,Z,V) = R^\nabla(Z,V,X,Y). \tag{2.5}$$

One also obtains a modified first Bianchi identity

$$\overset{X,Y,Z}{\mathfrak{S}} R(X,Y,Z,V) = dT(X,Y,Z,V) - \sigma_T(X,Y,Z,V) + (\nabla_V T)(X,Y,Z),$$

which once again simplifies under the assumption of parallel torsion. In that case we find $dT = 2\sigma_T$ and thus the Bianchi identity becomes

$$\overset{X,Y,Z}{\mathfrak{S}} R(X,Y,Z,V) = \sigma_T(X,Y,Z,V). \tag{2.6}$$

Let $\langle\cdot,\cdot\rangle$ denote the metric on $\Lambda^2 M$ induced by g. We may then define the curvature operator $\mathcal{R}^\nabla\colon \Lambda^2 M \to \Lambda^2 M$ of the connection ∇ via

$$\langle \mathcal{R}^\nabla(X\wedge Y), Z\wedge V\rangle = -R^\nabla(X,Y,Z,V).$$

From now on we will assume that ∇ has parallel skew torsion such that $\mathcal{R}^\nabla \in \mathrm{Sym}(\Lambda^2 M)$ by (2.5). Let us motivate the additional $-$ sign in the definition of $\mathcal{R}$. Recall that the sectional curvature $\sec^\nabla$ of a plane spanned by $X, Y \in TM$ is given by

$$\sec^\nabla(X,Y) := \frac{R^\nabla(X,Y,Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2}.$$

With the given sign conventions we have that the sectional curvature $\sec^\nabla > 0$ if $\mathcal{R}^\nabla > 0$ is positive definite as a linear operator $\Lambda^2 M \to \Lambda^2 M$.

Definition 2.1.9. Let (M,g) be a Riemannian manifold and ∇ a connection with parallel skew torsion. We say that

a) (M,g) has a *positive ∇-curvature operator* if $\mathcal{R}^\nabla > 0$,

b) (M,g) has *strongly positive ∇-curvature* if $\mathcal{R}^\nabla + \omega > 0$ for some $\omega \in \Lambda^4 M$,

c) (M,g) has *positive ∇-curvature* if $\sec^\nabla(X,Y) > 0$ for all $X, Y \in TM$.

In accordance with usual notion we omit ∇ if we consider the Levi-Civita connection.

Observe that a 4-form $\omega \in \Lambda^4 M$ regarded as an operator $\Lambda^2 M \to \Lambda^2 M$ is always symmetric and $\omega(X,Y,Y,X) = 0$. Thus, part b) is a reasonable definition and

positive ∇-curvature operator $\Rightarrow$ strongly positive ∇-curvature $\Rightarrow$ positive ∇-curvature.

As for the Levi-Civita connection we may define holonomy for an arbitrary connection ∇. Along any smooth path $\gamma\colon [0,1] \to M$ we find a unique parallel vector field $X\colon [0,1] \to TM$ along γ with fixed initial value $X(0) = X_0$. This is as usual an application of the Picard-Lindelöf theorem. Evaluating X at the endpoint defines the parallel transport $P^\nabla_\gamma\colon T_{\gamma(0)}M \to T_{\gamma(1)}M$.

Definition 2.1.10. The *holonomy group* of a connection ∇ at a point $p \in M$ is given by

$$\mathrm{Hol}^{\nabla}(p) = \{P_\gamma^{\nabla} \mid \gamma \text{ a closed loop in } M \text{ with endpoint } p\} \subset \mathrm{Gl}(T_pM).$$

The reduced holonomy group $\mathrm{Hol}_0^{\nabla}(p)$ of ∇ is the connected component of the identity inside $\mathrm{Hol}^{\nabla}(p)$ and $\mathfrak{hol}^{\nabla}(p)$ its Lie algebra.

On a connected manifold the (reduced) holonomy groups of any two points are conjugate. Thus, we will just talk about the (reduced) holonomy group of a manifold M. Also we will usually drop the mention of ∇ when it is clear from context which connection is meant. In case ∇ is a metric connection, parallel transport preserves the metric and thus $\mathrm{Hol}_0^{\nabla} \subset \mathrm{SO}(n)$.

The notion of holonomy is closely connected to curvature. In fact, the classical Ambrose-Singer theorem [AS53] tells us that the Lie algebra $\mathfrak{hol}$ of the holonomy group is generated by the curvature.

Theorem 2.1.11 ([AS53])**.** *The holonomy algebra $\mathfrak{hol}$ of a metric connection ∇ is the subalgebra of $\mathfrak{so}(T_pM)$ generated by*

$$P_\gamma^{-1} \circ \mathcal{R}(P_\gamma X, P_\gamma Y) \circ P_\gamma$$

where γ runs through all piece-wise smooth curves from p to q and $\mathcal{R}(X,Y) \in \Lambda^2 M \cong \mathfrak{so}(T_qM)$ is the curvature operator evaluated at $X, Y \in T_qM$.

Since $\mathrm{Hol}_0 \subset \mathrm{SO}(n)$ for any metric connection ∇, the type of torsion tensor T is Hol_0-invariant. Under the additional assumption $\nabla T = 0$ the torsion tensor itself is invariant under the holonomy.

2.2 Skew Torsion with Reducible Holonomy

A key assumption in Berger's holonomy theorem was the irreducibility of Hol_0^g. This assumption was warranted as otherwise by de Rham splitting M is locally a Riemannian product manifold. For connections with torsion we do not have de Rham splitting. In this case we need a refined splitting condition on the tangent space.

Theorem 2.2.1 ([Sto17])**.** *Let (M,g) be a Riemannian manifold and ∇ a metric connection $\nabla = \nabla^g + A$, where $A \in V \otimes \mathfrak{so}(V)$. If the tangent space splits into ∇-parallel distributions $TM = V_1 \oplus V_2$ such that*

$$A = A_1 + A_2, \qquad A_i \in V_i \otimes \mathfrak{so}(V_i),$$

then (M,g) is locally a product manifold.

Conversely given two manifolds $(M_1, g_1), (M_2, g_2)$ both admitting metric connections ∇^1, ∇^2 with $\nabla^i = \nabla^{g_i} + A_i$, then the product

$$(M, g) = (M_1, g_1) \times (M_2, g_2)$$

with metric connection $\nabla = \nabla^g + A_1 + A_2$ satisfies the conditions of Theorem 2.2.1. In the case of skew torsion this warrants the following definition.

Definition 2.2.2. Let (M, g) be a connected Riemannian manifold and ∇ a metric connection with skew torsion T. Then the triple (M, g, ∇) is called

a) *decomposable*, if there exist ∇-parallel distributions $TM = V_1 \oplus V_2$ such that $T = T_1 + T_2$ with $T_i \in \Lambda^3 V_i$,

b) *reducible*, if the representation of $\mathfrak{hol}^\nabla$ on TM is reducible,

c) *irreducible*, if it is not reducible.

Clearly decomposable implies reducible, but if $T \neq 0$ they are not equal.

Corollary 2.2.3. *The triple (M, g, ∇) with skew torsion is locally isometric to a product*

$$(M, g, \nabla) = (M_1, g_1, \nabla^1) \times (M_2, g_2, \nabla^2)$$

if and only if it is decomposable.

On the other end, in [CS04, Theorem 5.14] R. Cleyton and A Swann have classified the irreducible case under the assumption of parallel torsion.

Theorem 2.2.4 ([CS04])**.** *Let (M, g) be a Riemannian manifold and ∇ a connection with non-zero parallel torsion such that the holonomy representation is irreducible. Then one of the following statements holds:*

a) (M, g) is locally isometric to a non-symmetric, isotropy irreducible homogeneous space,

b) (M, g) is locally isometric to one of the irreducible symmetric spaces $(G \times G)/G$ or $G^{\mathbb{C}}/G$ with $G = \mathrm{Hol}^\nabla(M, g)$,

c) (M, g) has weak holonomy $\mathrm{SU}(3)$ or G_2.

Remark 2.2.5. In case a) ∇ has skew torsion if and only if it is the Ambrose-Singer connection of a naturally reductive homogeneous space as discussed in Example 2.1.5. Case c) are the connections with skew torsion on nearly Kähler, respectively nearly parallel G_2-manifolds mentioned in Example 2.1.8. Case b) is also known to carry connections with skew torsion.

This leaves us with the reducible, non-decomposable case of parallel skew torsion. These have been investigated in [CMS21]. We recall their notation.

Suppose the tangent space TM decomposes under the action of the holonomy group Hol of ∇^T into a sum of irreducible representations $\mathfrak{v}_1, \ldots, \mathfrak{v}_r, \mathfrak{h}_1, \ldots, \mathfrak{h}_s$.

Definition 2.2.6. An irreducible submodule of the holonomy representation is called *vertical*, adequately denoted by $\mathfrak{v}_j$, if the subspace $\mathfrak{k}_{\mathfrak{v}_j} = \mathfrak{so}(\mathfrak{v}_j) \cap \mathfrak{hol}$ of $\mathfrak{hol}$ acting purely on $\mathfrak{v}_j$ is trivial.

Conversely, a subspace $\mathfrak{h}_a$ is called *horizontal* if the subspace $\mathfrak{k}_{\mathfrak{h}_a} = \mathfrak{so}(\mathfrak{h}_a) \cap \mathfrak{hol} \neq \{0\}$ of $\mathfrak{hol}$ acting purely on $\mathfrak{h}_a$ is non-trivial.

We need a slight generalization of the results obtained in [CMS21]. Suppose the tangent space decomposes into $TM = \mathfrak{v}_1 \oplus \cdots \oplus \mathfrak{v}_r \oplus \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_s$ as before. Let $TM = \mathcal{V}_\Gamma \oplus \mathcal{H}_\Gamma$ be a decomposition such that

$$\mathcal{H}_\Gamma := \bigoplus_{a=1}^{s} \mathfrak{h}_a \oplus \bigoplus_{j \in \Gamma_0 \setminus \Gamma} \mathfrak{v}_j, \qquad \mathcal{V}_\Gamma := \bigoplus_{j \in \Gamma} \mathfrak{v}_j, \tag{2.7}$$

for some subset $\Gamma \subset \Gamma_0 = \{1, \ldots, r\}$. Suppose further that for the decomposition (2.7)

$$0 = \mathrm{pr}_{\mathcal{H}_\Gamma \otimes \Lambda^2 \mathcal{V}_\Gamma} T \in \mathcal{H}_\Gamma \otimes \Lambda^2 \mathcal{V}_\Gamma \subset \Lambda^3(\mathcal{H}_\Gamma \oplus \mathcal{V}_\Gamma). \tag{2.8}$$

i.e. the projection of T onto the space $\mathcal{H}_\Gamma \otimes \Lambda^2 \mathcal{V}_\Gamma$ vanishes. This condition turns out to be sufficient to prove Lemmas 3.7-3.10 and Remark 3.11 from [CMS21]. In fact, they prove that the maximal decomposition $\mathcal{H}_{\Gamma_0} = \bigoplus_{a=1}^{s} \mathfrak{h}_a$, $\mathcal{V}_{\Gamma_0} = \bigoplus_{j=1}^{r} \mathfrak{v}_j$ always satisfies (2.8). We obtain

Corollary 2.2.7. *Suppose the decomposition $TM = \mathcal{V}_\Gamma \oplus \mathcal{H}_\Gamma$ from* (2.7) *fulfills condition* (2.8)*. Then*

a) the distribution $\mathcal{V}_\Gamma$ is the vertical distribution of a locally defined Riemannian submersion $(M, g) \xrightarrow{\pi} (N, g_N)$ with totally geodesic leaves,

b) there exists a 3-form $\sigma \in \Lambda^3 N$ satisfying $\pi^* \sigma = \mathrm{pr}_{\Lambda^3 \mathcal{H}_\Gamma} T$,

c) $\nabla^\sigma := \nabla^{g_N} + \frac{1}{2}\sigma$ defines a connection with parallel skew torsion σ on N. In particular, we have

$$\nabla^\sigma_X Y = \pi_*(\nabla^T_{\overline{X}} \overline{Y}), \tag{2.9}$$

for the horizontal lifts $\overline{X}, \overline{Y} \in TM$ of the vectors fields $X, Y \in TN$.

Remark 2.2.8. Let us clearify on the notion of locally defined Riemannian submersion. It means that for every point $p \in M$ there is an open neighborhood $U \subset M$, a local submanifold S through p, transverse to the foliation, and a Riemannian submersion $\pi_p \colon U \to S$ where S is equipped with the metric on $\mathcal{H}$. However, we consider the π_p as patches of a joint map $\pi \colon M \to N$ where N is the set of leaves. There arise some difficulties as N might not be a manifold unlike the individual transverse sections S.

Mostly we are interested in cases where the foliation tangent to $\mathcal{V}_\Gamma$ is given by the orbits of a compact Lie group G acting on M. Then the quotient M/G is necessarily an orbifold. For a proof of that fact and a thorough introduction into orbifolds we refer to [MM03]. An orbifold of the form M/G is locally homeomorphic to T/H where T is a local transverse section through some point $p \in M$ and H is the leaf holonomy group in p. That means H is the finite group given by the return maps in T along paths in the leaf L_p through p.

Proof. We note that by (2.8) and the invariance of $\mathcal{V}_\Gamma$ under ∇ for any vertical vector fields $V, W \in \mathcal{V}_\Gamma$ we have

$$\nabla^g_V W = \nabla^T_V W - \frac{1}{2}T(V,W) \in \mathcal{V}_\Gamma.$$

Therefore, the distribution $\mathcal{V}_\Gamma$ is integrable and a curve is geodesic on the integral submanifold tangent to $\mathcal{V}_\Gamma$ if and only if it is a geodesic in M. The integral submanifolds give rise to a foliation and hence to a submersion π from a small neighborhood $U \subset M$ to a local transverse section S. We show that the metric restricted to $\mathcal{H}_\Gamma \times \mathcal{H}_\Gamma$ is constant along vertical vector fields and therefore projectable. For $X, Y \in \mathcal{H}$ we have

$$\begin{aligned}(L_V g)(X,Y) &= V(g(X,Y)) - g([V,X],Y) - g(X,[V,Y]) = g(\nabla^g_X V, Y) + g(X, \nabla^g_Y X)\\ &= g(\nabla^T_X V, Y) + g(X, \nabla^T_Y V) = 0\end{aligned}$$

since $\mathcal{V}$ is preserved by ∇^T. This proves that π is a Riemannian submersion.

To prove the second assertion we denote $T^{\mathcal{H}} = \mathrm{pr}_{\Lambda^3\mathcal{H}_\Gamma} T$. If we show that $T^{\mathcal{H}}$ is constant along the fibers it projects to a well-defined 3-form σ. Let $V \in \mathcal{V}_\Gamma$. Then

$$T^{\mathcal{H}}(X,Y,T(V,Z)) = 0 \tag{2.10}$$

whenever either $X, Y, Z \in \mathcal{V}_\Gamma$. Indeed, by (2.8) we have that $T(V,Z) \in \mathcal{V}_\Gamma$ if $Z \in \mathcal{V}_\Gamma$ and, hence, $T(V,Z) \lrcorner T^{\mathcal{H}} = 0$. Now consider $X, Y, Z \in \mathcal{H}_\Gamma$. Since ∇^T preserves $\mathcal{V}_\Gamma$ and $\mathcal{H}_\Gamma$ the curvature $R^{\nabla^T}(X,Y,Z,V) = 0$ and the Bianchi identity for connections with parallel skew torsion, (2.6), implies

$$\begin{aligned}0 &= -\overset{X,Y,Z}{\mathfrak{S}} R^{\nabla^T}(X,Y,Z,V) = -\sigma_T(X,Y,Z,V) = -\overset{X,Y,Z}{\mathfrak{S}} g(T(X,Y),T(Z,V))\\ &= \overset{X,Y,Z}{\mathfrak{S}} T(X,Y,T(V,Z)) = \overset{X,Y,Z}{\mathfrak{S}} T^{\mathcal{H}}(X,Y,T(V,Z)),\end{aligned}$$

where the last step used again that $T(V,Z) \in \mathcal{H}_\Gamma$. Thus, (2.10) holds for any $X, Y, Z \in TM$.

As the subspaces $\mathcal{V}_\Gamma$, $\mathcal{H}_\Gamma$ are preserved by ∇^T so are the components of tensors on TM. In particular, $\nabla^T T = 0$ implies $\nabla^T T^{\mathcal{H}} = 0$. Use (2.10) and $\nabla^T_Z V \in \mathcal{V}_\Gamma$ to obtain

$$\begin{aligned}L_V T^{\mathcal{H}}(X,Y,Z) &= V(T^{\mathcal{H}}(X,Y,Z)) - \overset{X,Y,Z}{\mathfrak{S}} T^{\mathcal{H}}(X,Y,L_V Z)\\ &= (\nabla^T_V T^{\mathcal{H}})(X,Y,Z) + \overset{X,Y,Z}{\mathfrak{S}} T^{\mathcal{H}}(X,Y,\nabla^T_V Z)\\ &\qquad - \overset{X,Y,Z}{\mathfrak{S}} T^{\mathcal{H}}(X,Y,\nabla^T_V Z - \nabla^T_Z V - T(V,Z))\\ &= \overset{X,Y,Z}{\mathfrak{S}} T^{\mathcal{H}}(X,Y,T(V,Z)) = 0.\end{aligned}$$

Equation (2.9) follows directly from $\nabla^{g_N}_X Y = \pi_*(\nabla^g_{\overline{X}} \overline{Y})$ for Riemannian submersions, see for instance [Pet06, Prop. 13]. Finally we conclude that σ is ∇^σ-parallel:

$$\begin{aligned}(\nabla^\sigma_A \sigma)(X,Y,Z) &= A(\sigma(X,Y,Z)) - \overset{X,Y,Z}{\mathfrak{S}} \sigma(X,Y,\nabla^\sigma_A Z)\\ &= A(\pi \circ T^{\mathcal{H}}(\overline{X},\overline{Y},\overline{Z})) - \overset{X,Y,Z}{\mathfrak{S}} \sigma(X,Y,\pi_*(\nabla^T_{\overline{A}} \overline{Z}))\\ &= \overline{A}(T^{\mathcal{H}}(\overline{X},\overline{Y},\overline{Z})) - \overset{X,Y,Z}{\mathfrak{S}} T^{\mathcal{H}}(\overline{X},\overline{Y},\nabla^T_{\overline{A}} \overline{Z})\\ &= (\nabla^T_{\overline{A}} T^{\mathcal{H}})(\overline{X},\overline{Y},\overline{Z}) = 0\end{aligned}$$

□

Riemannian submersions have been studied intensively in [ONe66]. The author assigns to a submersion $\pi\colon M \to N$ with $\mathcal{V} = \ker d\pi$ and $\mathcal{H} = \mathcal{V}^\perp$ the tensors, now called O'Neill tensors,

$$\mathcal{A}_X Y = (\nabla^g_{X_\mathcal{H}} Y_\mathcal{H})_\mathcal{V} + (\nabla^g_{X_\mathcal{H}} Y_\mathcal{V})_\mathcal{H}, \qquad \mathcal{T}_X Y = (\nabla^g_{X_\mathcal{V}} Y_\mathcal{H})_\mathcal{V} + (\nabla^g_{X_\mathcal{V}} Y_\mathcal{V})_\mathcal{H}.$$

Here the subscripts denote projection on the respective subspaces. In fact $\mathcal{A}$ and $\mathcal{T}$ characterize the submersion uniquely.

Theorem 2.2.9 ([ONe66])**.** *Given two Riemannian submersions $\pi, \tilde{\pi}\colon M \to N$. If their respective O'Neill tensors are equal $\mathcal{A} = \tilde{\mathcal{A}}$, $\mathcal{T} = \tilde{\mathcal{T}}$ and $d_x\pi = d_x\tilde{\pi}$ for some point $x \in M$ then $\pi = \tilde{\pi}$.*

For the submersion of Corollary 2.2.7 the O'Neill-tensors $\mathcal{A}$ and $\mathcal{T}$ simplify a lot:

Lemma 2.2.10. *The O'Neill tensors $\mathcal{A}$ and $\mathcal{T}$ associated to the submersion defined by $TM = \mathcal{V}_\Gamma \oplus \mathcal{H}_\Gamma$ are given by*

$$g(\mathcal{A}_X Y, Z) = -\frac{1}{2}(T(X_{\mathcal{H}_\Gamma}, Y_{\mathcal{H}_\Gamma}, Z_{\mathcal{V}_\Gamma}) + T(X_{\mathcal{H}_\Gamma}, Y_{\mathcal{V}_\Gamma}, Z_{\mathcal{H}_\Gamma})), \qquad \mathcal{T} = 0.$$

Proof. Since $\mathcal{H}_\Gamma$ and $\mathcal{V}_\Gamma$ are ∇^T-holonomy invariant $(\nabla^T_X Y_{\mathcal{H}_\Gamma})_{\mathcal{V}_\Gamma} = (\nabla^T_X Y_{\mathcal{V}_\Gamma})_{\mathcal{H}_\Gamma} = 0$. Thus, $g(\nabla^g_X Y_{\mathcal{H}_\Gamma}, Z_{\mathcal{V}_\Gamma}) = -\frac{1}{2}T(X, Y_{\mathcal{H}_\Gamma}, Z_{\mathcal{V}_\Gamma})$ and $g(\nabla^g_X Y_{\mathcal{V}_\Gamma}, Z_{\mathcal{H}_\Gamma}) = -\frac{1}{2}T(X, Y_{\mathcal{V}_\Gamma}, Z_{\mathcal{H}_\Gamma})$. The first expression follows directly. The identity $\mathcal{T} = 0$ is then an immediate consequence of condition (2.8). □

The vanishing of $\mathcal{T}$ does not come as a surprise since it is equivalent to the fibers being totally geodesic.

3 Almost Contact Metric Geometry

This chapter is concerned with certain odd dimensional geometries. Namely the classes of almost contact metric manifolds in every odd dimension and of almost 3-contact metric manifold in dimensions $4n+3$. A complete account of almost contact metric and Sasaki geometry can be found in [BG08]. Apart from G_2-geometry, odd dimensions do not appear in Berger's list, implying that the Levi-Civita connection is not well adapted. Hence, we will consider suitable connections with torsion alongside these classes.

3.1 Sasaki Manifolds

We begin with almost contact metric geometry often considered as odd dimensional analogue to even dimensional almost Hermitian manifolds.

Definition 3.1.1. An *almost contact metric structure* on (M^{2n+1}, g) is a triple (ξ, η, φ) where $\xi \in TM$ is a unit length vector field, $\eta \in T^*M$ and $\varphi \in \mathrm{End}(TM)$ such that

$$\eta(\xi) = 1, \qquad \varphi(\xi) = 0, \qquad \eta \circ \varphi = 0, \qquad \varphi^2 = -\mathrm{id} + \xi \otimes \eta,$$
$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).$$

We call ξ the Reeb vector field of $(M, g, \xi, \eta, \varphi)$.

Remark 3.1.2. Usually one defines (strict) almost contact structures (ξ, η, φ) on M without reference to a metric g and omits the last compatibility condition. We will always have a metric present so we will not take this detour. For a discussion in absence of a metric see, for instance, [BG08, Chapter 6].

An almost contact metric manifold can be thought of as a manifold with a codimension 1 distribution $\mathcal{H} = \langle \xi \rangle^\perp$ called contact or horizontal distribution admitting an almost Hermitian structure. Accordingly the distribution $\mathcal{V} := \langle \xi \rangle$ is denoted the vertical distribution. In analogy to Hermitian manifolds we call

$$\Phi(X, Y) = g(X, \varphi Y) \tag{3.1}$$

the fundamental 2-form of the almost contact metric structure $(M, g, \xi, \eta, \varphi)$. We also define a Nijenhuis tensor by

$$N(X, Y) := [\varphi X, \varphi Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y] + \varphi^2[X, Y] + d\eta(X, Y)\xi. \tag{3.2}$$

We will use this ad hoc definition for the Nijenhuis tensor. It can be motivated as the obstruction to integrability on the cone, see Remark 3.1.10.

Definition 3.1.3. An almost contact metric manifold $(M, g, \xi, \eta, \varphi)$ is called

a) *normal if* $N \equiv 0$,

b) *quasi-Sasaki manifold* if it is normal and $d\Phi = 0$,

c) *α-contact metric manifold* if $d\eta = 2\alpha\Phi$ for a constant $\alpha \neq 0$,

d) *α-K-contact metric manifold* if ξ is a Killing field and $d\eta = 2\alpha\Phi$ for a constant $\alpha \neq 0$,

e) *α-Sasaki manifold* if it is normal and $d\eta = 2\alpha\Phi$ for a constant $\alpha \neq 0$.

For the value $\alpha = 1$ α-Sasaki/α-(K-)contact metric manifolds are simply called Sasaki/(K-)contact metric manifolds.

Remark 3.1.4. There are obvious inclusion relations between these classes. One can show that also α-Sasaki implies α-K-contact, in particular the Reeb vector field of a Sasaki structure is Killing, see [BG08, Prop. 6.5.14]. The inclusion tree is as follows.

$$\begin{tikzcd}
 & \alpha\text{-K-contact} \arrow[r, Rightarrow] & \alpha\text{-contact} \arrow[dd, bend left] \\
\alpha\text{-Sasaki} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \\
 & \text{quasi-Sasaki} \arrow[r, Rightarrow] & \text{normal} \arrow[uu, bend left] \arrow[ul, Rightarrow]
\end{tikzcd}$$

By Cartans formula

$$\xi \lrcorner\, d\eta = L_\xi \eta = \xi \lrcorner\, (L_\xi g). \tag{3.3}$$

In particular, when ξ is Killing we have $\xi \lrcorner\, d\eta = 0$. We consider two standard examples of Sasaki structures in detail.

Example 3.1.5. Let $\mathbb{R}^{2n+1}$ and fix coordinates $(x_1, y_1, \ldots, x_n, y_n, z)$. Set

$$\eta = dz - \sum_{i=1}^{n} (x_i dy_i - y_i dx_i)$$

and consider the metric $g = \eta^2 + \sum dx_i^2 + dy_i^2$. Then $\xi = \partial_z$, $Y_i = \partial_{y_i} + x_i \partial_z$ and $X_i = \partial_{x_i} - y_i \partial_z$ make up an orthonormal basis dual to η, dx_i, dy_i. We complete the almost contact metric structure with

$$\varphi = \sum_{i=1}^{n} X_i \wedge Y_i = \sum_{i=1}^{n} X_i \otimes dy_i - Y_i \otimes dx_i$$

Immediately we have $d\eta = 2\sum dx_i \wedge dy_i = 2\Phi$. Normality is proved on the basis ξ, X_i, Y_i independently. These have commutators

$$[X_i, X_j] = [X_i, Y_j] = [Y_i, Y_j] = 0,\ i \neq j, \qquad [X_i, \xi] = [Y_i, \xi] = 0, \qquad [X_i, Y_i] = 2\xi. \tag{3.4}$$

Then by value of (3.3) we have $\xi \lrcorner N = 0$. We also have $N(X_i, X_j) = N(Y_i, Y_j) = N(X_i, Y_j) = 0$ for $i \neq j$ as $d\eta = 2\sum dx_i \wedge dy_i$. Finally

$$N(X_i, Y_i) = -[X_i, Y_i] + d\eta(X_i, Y_i)\xi = -2\xi + 2\xi = 0.$$

We remark that by (3.4) the basis vectors generate the Heisenberg algebra

$$\mathfrak{h}^{n,\mathbb{C}} = \left\{ \begin{pmatrix} 0 & \overline{X}^t & Z \\ 0 & 0 & X \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{gl}(n+2, \mathbb{C}), \quad X \in \mathbb{C}^n,\ Z \in i\mathbb{R} \right\}.$$

Example 3.1.6. Equip the odd dimensional sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ with its standard metric. Consider its Hopf bundle

$$\begin{aligned} S^1 \to \qquad S^{2n+1} \quad &\to \mathbb{C}P^n \\ (z_0, \ldots, z_n) &\mapsto [z_0 : \ldots : z_n]. \end{aligned}$$

Denote by $\xi \in TS^{2n+1}$ the unit length vector field generating the action of S^1, η its metric dual and $\varphi\colon (Z_0, \ldots, Z_n) \mapsto \mathrm{pr}_{TS^{2n+1}}(-Z_0 i, \ldots, -Z_n i)$ defined by multiplication with $-i$ form the right. Here we have identified $T_{(z_1,\ldots,z_n)}S^{2n+1} = (z_1, \ldots, z_n)^\perp$. These clearly define an almost contact metric manifold. Set $z_j = x_j + iy_j$ then η takes the form $\eta = \sum x_i dy_i - y_i dx_i$ and hence

$$d\eta = 2\sum dx_i \wedge dy_i = 2\Phi.$$

Note that the last equality holds as both sides are considered only on TS^{2n+1}. Normality follows immediately from the equivalent condition we will give in Remark 3.1.10 since the cone is $C(S^{2n+1}) = \mathbb{R}^+ \times S^{2n+1} = \mathbb{C} \setminus \{0\}$.

One observes that an almost contact metric manifold has its structure group reduced to a subgroup of $\mathrm{U}(n) \times \mathrm{id} \subset \mathrm{SO}(2n+1)$. Conversely any reduction to $\mathrm{U}(n) \times \mathrm{id}$ gives rise to an almost contact metric manifold. With this reduction we can ask which almost contact metric manifolds admit characteristic connections as defined in Definition 2.1.7. A characteristic connection ∇ has to preserve the given $\mathrm{U}(n) \times \mathrm{id}$-structure. This is equivalent to

$$\nabla\xi = \nabla\eta = \nabla\varphi = 0.$$

The question was investigated and solved by T. Friedrich and S. Ivanov.

Theorem 3.1.7 ([FI02])**.** *An almost contact metric manifold $(M, g, \xi, \eta, \varphi)$ admits a characteristic connection ∇ if and only if*

- *ξ is a Killing field,*
- *the Nijenhuis-tensor N is skew-symmetric.*

In this case the torsion is given by

$$T = \eta \wedge d\eta + d^\varphi\Phi + N - \eta \wedge (\xi \lrcorner N),$$

where $d^\varphi\Phi(X, Y, Z) = -d\Phi(\varphi X, \varphi Y, \varphi Z)$.

Corollary 3.1.8. *The torsion tensor decomposes under $TM = \mathcal{V} \oplus \mathcal{H}$ as $T = T^{\mathcal{H}} + T^m$ where*

$$T^{\mathcal{H}} := \mathrm{pr}_{\Lambda^3\mathcal{H}} T = d^{\varphi}\Phi + (N - \eta \wedge (\xi \lrcorner N)), \qquad T^m := \mathrm{pr}_{\mathcal{V}\otimes\Lambda^2\mathcal{H}} T = \eta \wedge d\eta. \tag{3.5}$$

Proof. T is skew-symmetric and $\mathcal{V}$ 1-dimensional. Hence, only T^m and $T^{\mathcal{H}}$ can exist. We need to show that $d^{\varphi}\Phi + (N - \eta \wedge (\xi \lrcorner N)) \in \Lambda^3\mathcal{H}$. It is true for the first summand since $\mathcal{V} = \ker \varphi$. For the second summand we notice that $\eta \wedge (\xi \lrcorner N)$ is the projection of N to $\mathcal{V} \otimes \Lambda^2\mathcal{H}$:

$$N(\xi, X, Y) = \eta(\xi)(\xi \lrcorner N)(X, Y) = \eta \wedge (\xi \lrcorner N)(\xi, X, Y). \qquad \square$$

Observe that the class of α-Sasaki manifolds satisfies both conditions and the torsion tensor becomes $T = \eta \wedge d\eta$. In particular, $T^{\mathcal{H}} \equiv 0$. Since $d\eta = 2\alpha\Phi$ is automatically parallel. Thus we obtain

Corollary 3.1.9. *An α-Sasaki manifold $(M, g, \xi, \eta, \varphi)$ admits a characteristic connection ∇ with parallel torsion*

$$T = \eta \wedge d\eta.$$

We conclude this brief exposition of Sasaki and related almost contact metric manifolds by discussing their relation to Kähler geometry. In fact, almost contact manifolds are related to almost Hermitian manifolds in dimensions $2n + 2$ above and $2n$ below. In dimension $2n + 2$ one considers the Riemannian cone

$$C(M) = (\mathbb{R}_+ \times M, \mathrm{d}r^2 + r^2 g)$$

over $(M^{2n+1}, g, \xi, \eta, \varphi)$. This comes equipped with an almost Hermitian structure I induced from the almost contact metric structure on (M, g) via

$$IZ = -\xi, \quad IX = \varphi X + \eta(X)Z, \qquad X \in TM, \quad Z = r\frac{\partial}{\partial r}.$$

We will not be concerned with cones any further in this thesis yet they were key to the motivation of almost contact metric structures.

Remark 3.1.10. Several classes of almost contact manifolds are defined in such a way that the almost Hermitian manifold on the cone is of specific kind. In particular,

- the cone over a normal almost contact metric manifold is an integrable Hermitian space,
- the cone over a Sasaki manifold is a Kähler space.

In the dimension $2n$ below we consider the space of Reeb orbits $M/\mathcal{F}_\xi$, where $\mathcal{F}_\xi$ is the foliation induced by the flow of the Reeb vector field. This is a 1-dimensional foliation as the Reeb vector field vanishes nowhere. In general there is little hope that the leaf space is a manifold. However, under reasonable assumptions we obtain at least orbifolds.

We obtain the following result from [BG08, Theorem 7.1.4]. They assume an additional technical condition. A foliation $\mathcal{F}$ on a manifold M is called quasi-regular if there exists an $k \in \mathbb{N}$ such that for every $p \in M$ there is a neighborhood that intersects any given leaf in only finitely many plaques. If $k = 1$ the foliation is called regular. In other words, the orders of leaf holonomy groups are globally bounded by k.

Theorem 3.1.11 ([BG08, Theorem 7.1.4.])**.** *Let $(M, g, \xi, \eta, \varphi)$ be a quasi-regular K-contact manifold with compact leaves. Then*

a) the space of leaves $M/\mathcal{F}_\xi$ is a symplectic orbifold and the projection $\pi\colon M \to M/\mathcal{F}_\xi$ is a Riemannian orbifold submersion,

b) M is the total space of a principal S^1-orbibundle over $M/\mathcal{F}_\xi$ with connection 1-form η whose curvature is $d\eta = \pi^*\omega$ for a symplectic form ω on $M/\mathcal{F}_\xi$,

c) if the characteristic foliation $\mathcal{F}_\xi$ is regular then the S^1 action is free and M is the total space of a principal S^1-bundle over a symplectic manifold defining an integral class $[\omega] \in H^2(M/\mathcal{F}_\xi, \mathbb{Z})$,

d) $(M, g, \xi, \eta, \varphi)$ is Sasaki if and only if $(M/\mathcal{F}_\xi, \omega)$ is Kähler.

We will revisit the relation between Sasaki manifolds and their underlying Kähler spaces in the light of canonical submersions in Section 4.1.

We recall that there is a converse construction. We start with a suitable symplectic manifold and construct a K-contact S^1-bundle, named Boothby-Wang bundle after its inventors.

Theorem 3.1.12 ([BW58])**.** *Suppose (N, ω) is a symplectic manifold such that in cohomology $[\omega] \in H^2(N, \mathbb{Z})$ defines an integral class. Then there exists a principal S^1-bundle $\pi\colon M \to N$ with a K-contact structure (g, ξ, η, φ) such that $d\eta = \pi^*\omega$.*

Corollary 3.1.13. *If (N, J, g) is a Kähler manifold with integral fundamental form $[\omega] \in H^2(N, \mathbb{Z})$ then the corresponding Boothby-Wang bundle $(M, g, \xi, \eta, \varphi)$ is Sasaki.*

3.2 Almost 3-Contact Metric Manifolds

Similar as we viewed almost contact metric manifold as an analogue to Hermitian geometry, we now want to look at almost 3-contact metric manifolds in dimension $4n+3$ as an analogue to hyperhermitian geometry mimicking quaternions.

Definition 3.2.1. Let (M^{4n+3}, g) be a Riemannian manifold. An *almost* 3*-contact metric structure* is a set of 3 almost contact metric structures $(\xi_i, \eta_i, \varphi_i)_{i=1,2,3}$ on (M, g) that satisfy the compatibility relations

$$\varphi_k = \varphi_i\varphi_j - \xi_i \otimes \eta_j = -\varphi_j\varphi_i + \xi_j \otimes \eta_i, \qquad \xi_k = \varphi_i\xi_j = -\varphi_j\xi_i, \qquad \eta_k = \eta_i \circ \varphi_j = -\eta_j \circ \varphi_i$$

for any even permutation (ijk) of (123).

We remark immediately that the compatible conditions force the Reeb vector fields to be orthogonal. We denote the spaces

$$\mathcal{V} = \langle \xi_1, \xi_2, \xi_3 \rangle \qquad \text{and} \qquad \mathcal{H} = \mathcal{V}^\perp = \bigcap_{i=1}^{3} \ker \eta_i$$

the vertical and horizontal distribution of TM. The compatibility conditions creates via $\varphi_1, \varphi_2, \varphi_3$ the structure of a quaternion vector space on the horizontal space $\mathcal{H}$. In particular, this reduces the structure group of M to $\mathrm{id}_3 \times \mathrm{Sp}(n)$. We can define adapted frames subordinate to this reduction.

Definition 3.2.2. An *adapted basis* of an almost 3-contact metric manifold is an orthonormal frame $e_1, \ldots, e_{4n+3}$ such that

$$e_i = \xi_i, \qquad e_{4l+i} = \varphi_i e_{4l}$$

for all $i = 1, 2, 3$ and $l = 1, \ldots, n$.

An almost 3-contact metric structure comes equipped with the tensors N_i and Φ_i, defined in (3.2) and (3.1), for each individual almost contact metric structure $(g, \xi_i, \eta_i, \varphi_i)$. Canonically we may define classes of almost 3-contact metric manifolds, requiring the additional conditions as in Definition 3.1.3 for each of the three structures individually.

Definition 3.2.3. An almost 3-contact metric manifold $(M, g, \xi_i, \eta_i, \varphi_i)$ is called

a) *hypernormal* if $(M, g, \xi_i, \eta_i, \varphi_i)$ is normal for $i = 1, 2, 3$,

b) 3*-contact* if $(M, g, \xi_i, \eta_i, \varphi_i)$ is contact for $i = 1, 2, 3$,

c) 3*-α-Sasaki* if $(M, g, \xi_i, \eta_i, \varphi_i)$ is α-Sasaki for $i = 1, 2, 3$,

d) 3*-Sasaki* if $(M, g, \xi_i, \eta_i, \varphi_i)$ is Sasaki for $i = 1, 2, 3$.

Apriori it is unclear whether these definitions have additional consequences beyond those following immediately from the individual classes. A strong result in this direction is given by Kashiwada, [Kas01], rendering the class of 3-contact metric manifolds essentially obsolete.

Theorem 3.2.4. *A* 3*-contact metric manifold is hypernormal and, thus,* 3*-Sasaki.*

The idea of interplay between the almost contact metric structures suggest that they are part of greater structure. Indeed, given 3 almost contact metric structures, $(\xi_i, \eta_i, \varphi_i)$, $i = 1, 2, 3$, every

$$\varphi_a \in \{\varphi_a = a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3, \quad a = (a_1, a_2, a_3) \in S^2\}$$

is an isometry that squares to $-\mathrm{id}$ on a codimension 1 distribution and defines an almost contact metric structure $(\xi_a, \eta_a, \varphi_a)$ on (M, g) where

$$\xi_a = a_1\xi_1 + a_2\xi_2 + a_3\xi_3, \qquad \eta_a = a_1\eta_1 + a_2\eta_2 + a_3\eta_3.$$

Definition 3.2.5. Let $(M, g, \xi_i, \eta_i, \varphi_i)_{i=1,2,3}$ be an almost contact metric structure. We denote

$$\Sigma_M := \{(\xi_a, \eta_a, \varphi_a), \quad a \in S^2\}$$

the *associated sphere* of almost contact metric structures.

Clearly these structures preserve the splitting $TM = \mathcal{V} \oplus \mathcal{H}$. More properties carry over to the sphere of almost contact metric structures.

Proposition 3.2.6 ([CDY16],[BG08, Theorem 13.2.1])**.** *Let $(M, g, \xi_i, \eta_i, \varphi_i)_{i=1,2,3}$ be an hypernormal almost 3-contact metric/3-α-Sasaski/3-Sasaki manifold. Then every almost contact metric structure in Σ_M is normal/α-Sasaki/Sasaki.*

Remark 3.2.7. Let $a, b, c \in \mathbb{R}^3$ be any oriented orthonormal basis and $(\xi_i, \eta_i, \varphi_i)_{i=a,b,c}$ their corresponding structures in Σ_M, then we again obtain an almost 3-contact metric manifold, compare the discussion in [BG08, Section 13.1]. By the proposition above each of these almost 3-contact metric manifolds is again hypernormal/3-α-Sasaki/3-Sasaki if the original one is.

Let us now consider the standard example of a 3-Sasaki manifold.

Example 3.2.8. Consider the sphere $S^{4n+3} \subset \mathbb{H}^{n+1}$. For each of the three complex generators $i, j, k \in \mathbb{H}$ we obtain a Sasaki structure defined by right multiplication as described in Example 3.1.6. Indeed, they satisfy the compatibility conditions since i, j, k do and thus form a 3-Sasaki manifold.

The analogy to the case described in Example 3.1.6 goes further. The entire vertical distribution gives a foliation $\mathcal{F}$ with leaves diffeomorphic to S^3. In fact the projection to the leaf space is just the quaternionic Hopf fibration

$$S^3 \to S^{4n+3} \to \mathbb{H}P^n.$$

One might expect that in analogy to Sasaki manifolds the quotient space $M/\mathcal{F} = \mathbb{H}P^n$ is a hyperkähler manifold admitting three parallel almost complex structures. This is false, but it is quaternionic Kähler instead. Let us recall both definitions.

Definition 3.2.9. An *almost hyperhermitian manifold* $(M^{4n}, g, I_1, I_2, I_3)$ is a Riemannian manifold together with three almost complex structures $I_1, I_2, I_3 \in \mathrm{End}(TM)$ that are compatible with the metric g and satisfy the quaternion relation $I_i I_j = I_k$ for every even permutation (ijk) of (123).

A *hyperkähler manifold* is a hyperhermitian manifold $(M^{4n}, g, I_1, I_2, I_3)$ such that all almost complex structures are parallel $\nabla^g I_i = 0$.

A *quaternionic Kähler manifold* is a tupel $(M^{4n}, g, \mathcal{O})$ where $\mathcal{O} \subset \mathrm{End}(TM)$ is a subbundle locally spanned by an almost hyper Hermitian structure $\{I_1, I_2, I_3\}$ such that $\nabla^g I_i \in \mathcal{O}$. If $n = 1$ we additionally assume that (M, g) is Einstein and anti-self-dual.

We have encountered both hyperkähler and quaternionic Kähler manifold in Berger's holonomy theorem. Hyperkähler manifolds are exactly those with $\mathrm{Hol}^g \subset \mathrm{Sp}(n)$ and quaternionic Kähler manifolds those that satisfy $\mathrm{Hol}^g \subset \mathrm{Sp}(n)\mathrm{Sp}(1) := \mathrm{Sp}(n) \times \mathrm{Sp}(1)/(-\mathrm{id}, -\mathrm{id})$.

Remark 3.2.10. In the case $n = 1$ we have $\mathrm{Sp}(1)\mathrm{Sp}(1) = \mathrm{SO}(4)$ so the holonomy condition for quaternionic Kähler manifolds is empty. This motivates the now customary extra conditions placed in dimension 4.

As shown by C. Boyer, K. Galicki and B. Mann the submersion property of S^{4n+1} is no coincidence, but a general phenomenon.

Theorem 3.2.11 ([BGM94])**.** *Let $(M, g, \xi_i, \eta_i, \varphi_i)$ be a 3-Sasaki manifold. Then the vertical distribution is integrable and gives rise to a foliation $\mathcal{F}$ with totally geodesic leaves. If the vector fields ξ_i, $i = 1, 2, 3$, are complete, the leaf space $\mathcal{O} = M/\mathcal{F}$ is a quaternionic Kähler orbifold and the projection $\pi\colon M \to \mathcal{O}$ gives a G-principal orbifold bundle with either $G = \mathrm{SU}(2)$ or $G = \mathrm{SO}(3)$. Further, π is a Riemannian submersion and the scalar curvature of the $\mathcal{O}$ is* $\mathrm{scal}^g = 16n(n+2)$.

We now want to consider connections adapted to almost 3-contact metric manifolds. Recall that under the conditions of Theorem 3.1.7 the individual almost contact metric structures admit characteristic connections. However, even in the specific case of 3-Sasaki manifolds, we have connections ∇^i with torsion

$$T^{\varphi_i} = \eta_i \wedge d\eta_i \neq \eta_j \wedge d\eta_j = T^{\varphi_j}$$

if $i \neq j$. For instance, observe that $T^{\varphi_i}(\xi_j, X, Y) = 0$ for all $X, Y \in \mathcal{H}$. Hence, the assumption that all φ_i are parallel under a single connection is too rigid. Considering the sphere Σ_M of almost contact metric structures where individual structures are not preserved by a given connection reminds of the quaternionic Kähler case compared to hyperkähler manifolds. In analogy [AD20] investigated the less restrictive condition

$$\nabla_X \varphi_i = \beta(\eta_k(X)\varphi_j - \eta_j(X)\varphi_k) \tag{3.6}$$

for every even permutation (ijk) of (123). They obtain the following criterion.

Theorem 3.2.12 ([AD20])**.** *Let $(M, g, \xi_i, \eta_i, \varphi_i)_{i=1,2,3}$ be an almost 3-contact metric manifold. Then there exists a metric connection ∇ with skew torsion satisfying* (3.6) *for some function $\beta \in C^\infty(M)$ if and only if for all $i, j = 1, 2, 3$ and $X, Y, Z \in \mathcal{H}$ the following are satisfied*

a) N^{φ_i} is skew-symmetric on $\mathcal{H}$,

b) ξ_i is Killing,

c) $N^{\varphi_i}(X, Y, Z) + d^{\varphi_i}\Phi_i(X, Y, Z) = N^{\varphi_j}(X, Y, Z) + d^{\varphi_j}\Phi_j(X, Y, Z)$,

d) $A_{ij}(X, Y) = \sum \epsilon_{ijk}\beta\Phi_k(X, Y)$ for some fixed $\beta \in C^\infty(M)$,

where $A_{ij}(X, Y) := g((L_{\xi_j}\varphi_i)X, Y) + d\eta_j(X, \varphi_i Y) + d\eta_j(\varphi_i X, Y)$.

If this is satisfied ∇ is the unique such connection.

Definition 3.2.13. If the conditions in Theorem 3.2.12 are satisfied $(M, g, \xi_i, \eta_i, \varphi_i)_{i=1,2,3}$ is called *canonical almost 3-contact metric manifold* and ∇ is called the *canonical connection.*

3.3 3-(α, δ)-Sasaki Manifolds

The classes of almost 3-contact metric manifolds studied initially were defined by imposing additional conditions on the individual almost contact structures. We now want to include the possibility that the almost contact metric structures interact. This leads to the definition of 3-(α, δ)-Sasaki manifolds. These were initially investigated in [AD20].

Definition 3.3.1. An almost 3-contact metric manifold is called 3-(α, δ)-*Sasaki manifold* if

$$d\eta_i = 2\alpha\Phi_i + 2(\alpha - \delta)\eta_j \wedge \eta_k \tag{3.7}$$

for every even permutation (ijk) of (123) and constants $0 \neq \alpha \in \mathbb{R}$, $\delta \in \mathbb{R}$.

Notation. This definition is actually a slight abuse of notation. For a given 3-(α, δ)-Sasaki manifold α and δ are fixed numbers, but the class of 3-(α, δ)-Sasaki manifolds is not fixed to the specific value of α and δ. When $\alpha = \alpha_0, \delta = \delta_0$ are given we denote the manifold a 3-(α, δ)-Sasaki manifold with $\alpha = \alpha_0$ and $\delta = \delta_0$.

We note key properties for 3-(α, δ)-Sasaki manifolds.

Proposition 3.3.2 ([AD20])**.** *Let $(M, g, \xi_i, \eta_i, \varphi_i)$ be a 3-(α, δ)-Sasaki manifold. Then*

a) it is hypernormal,

b) the vertical distribution $\mathcal{V}$ is integrable with totally geodesic leaves,

c) the Reeb vector fields are Killing and for any even permutation (ijk) *of* (123)

$$L_{\xi_i}\varphi_i = 0, \qquad L_{\xi_i}\varphi_j = -L_{\xi_j}\varphi_i = 2\delta\varphi_k \tag{3.8}$$

$$[\xi_i, \xi_j] = -[\xi_j, \xi_i] = 2\delta\xi_k. \tag{3.9}$$

The first part can be considered as a generalization of Kashiwada's theorem to 3-(α,δ)-Sasaki manifolds. This means that whenever $\alpha = \delta$ a 3-(α,δ)-Sasaki manifold is 3-α-Sasaki and we can perceive 3-(α,δ)-Sasaki as generalizations thereof.

Proposition 3.3.3. *Let* $(M, g, \xi_i, \eta_i, \varphi_i)_{i=1,2,3}$ *be a* 3-(α,δ)*-Sasaki manifold. For any oriented orthonormal basis* $a, b, c \in \mathbb{R}^3$ *the corresponding almost contact metric structures* $(M, g, \xi_i, \eta_i, \varphi_i)_{i=a,b,c}$ *in* Σ_M *define again a* 3-(α,δ)*-Sasaki structure.*

Proof. By Remark 3.2.7 $(M, g, \xi_i, \eta_i, \varphi_i)_{i=a,b,c}$ is an almost 3-contact metric manifold. We need to verify the differential condition.

$$d\eta_a = d(a_1\eta_1 + a_2\eta_2 + a_3\eta_3) = 2\alpha\Phi_a + 2(\alpha-\delta)(a_1\eta_2\wedge\eta_3 + a_2\eta_3\wedge\eta_1 + a_3\eta_1\wedge\eta_2)$$

and using $a = b \times c$

$$\begin{aligned}\eta_b\wedge\eta_c &= (b_1\eta_1 + b_2\eta_2 + b_3\eta_3)\wedge(c_1\eta_1 + c_2\eta_2 + c_3\eta_3)\\ &= (b_1c_2 - b_2c_1)\eta_1\wedge\eta_2 + (b_3c_1 - b_1c_3)\eta_3\wedge\eta_1 + (b_2c_3 - b_3c_2)\eta_2\wedge\eta_3\\ &= a_3\eta_1\wedge\eta_2 + a_2\eta_3\wedge\eta_1 + a_1\eta_2\wedge\eta_3.\end{aligned}$$

□

Since $\mathcal{V}$ is integrable its commutators are again in $\mathcal{V}$. The distribution $\mathcal{H}$ is quite the opposite. Its non-integrability is strongly related to the fundamental forms.

Lemma 3.3.4. *For two horizontal vectors* $X, Y \in \mathcal{H}$ *we have*

$$[X,Y]_{\mathcal{V}} = -2\alpha\sum_{i=1}^{3}\Phi_i(X,Y)\xi_i.$$

Proof. Since the vertical distribution is spanned by the Reeb vector fields, we have

$$[X,Y]_{\mathcal{V}} = \sum_{i=1}^{3}\eta_i([X,Y])\xi_i = -\sum_{i=1}^{3}d\eta_i(X,Y)\xi_i = -2\alpha\sum_{i=1}^{3}\Phi_i(X,Y)\xi_i.$$

□

Let us take a closer look at the commutator relation (3.9). It suggests that the case $\delta = 0$ might be quite different. Indeed, in the case $\delta \neq 0$ the Reeb vector fields span a Lie algebra isomorphic to $\mathfrak{su}(2)$ while in the case $\delta = 0$ their span is isomorphic to the abelian Lie algebra $\mathbb{R}^3$. This motivates the first part of the following definition.

Definition 3.3.5. A 3-(α,δ)-Sasaki manifold is called

a) *degenerate* if $\delta = 0$ and *non-degenerate* if $\delta \neq 0$,

b) *positive* if $\alpha\delta > 0$ and *negative* if $\alpha\delta < 0$.

The motivation behind the second part of the definition comes from $\mathcal{H}$-homothetic deformations.

Proposition 3.3.6 ([AD20])**.** *Let $(M, g, \xi_i, \eta_i, \varphi_i)$ be a 3-(α, δ)-Sasaki manifold with $\alpha = \alpha_0$, $\delta = \delta_0$. Let $a, b, c \in \mathbb{R}$ with $a > 0$ and $a + b = c^2 > 0$ and consider the deformed tensors*

$$\xi_i' = \frac{1}{c}\xi_i, \qquad \eta_i' = c\eta_i, \qquad \varphi_i' = \varphi_i, \qquad g' = ag + b\sum_{i=1}^{3} \eta_i \otimes \eta_i. \tag{3.10}$$

Then $(M, g', \xi_i', \eta_i', \varphi_i')$ is again a 3-(α, δ)-Sasaki manifold with

$$\alpha = \frac{c}{a}\alpha_0, \qquad \delta = \frac{1}{c}\delta_0.$$

Definition 3.3.7. We will call (3.10) a *$\mathcal{H}$-homothetic deformation* and the 3-(α, δ)-Sasaki manifold $(M, g', \xi_i', \eta_i', \varphi_i')$ *$\mathcal{H}$-homothetic* to $(M, g, \xi_i, \eta_i, \varphi_i)$.

Remark 3.3.8. Let us remark two special cases of $\mathcal{H}$-homothetic deformations.

a) Let $b = 0$ and $c = \sqrt{a} > 0$. With respect to these parameters $\mathcal{H}$-homothetic deformation is merely a global scaling of the metric by $a = c^2$. To compensate the ξ_i are scaled by $\frac{1}{c}$. In terms of α and δ we have the new values $\alpha' = \alpha/c$ and $\delta' = \delta/c$.

b) If $b = 0$, $a = 1$ and $c = -1$ $\mathcal{H}$-homothetic deformation changes the sign of the Reeb vector fields ξ_i. If we consider the sphere of almost contact metric structures spanned by $(\xi_i, \eta_i, \varphi_i)_{i=1,2,3}$ then this deformation really only changes the orientation induced by the 3-(α, δ)-Sasaki structure. However, we have $\alpha' = -\alpha$, $\delta' = -\delta$. This reasons to often just consider $\alpha > 0$.

One observes that degenerate 3-(α, δ)-Sasaki manifolds are closed under $\mathcal{H}$-homothetic deformation and so is the sign of $\alpha\delta$ for non-degenerate 3-(α, δ)-Sasaki manifolds motivating the distinction between positive and negative ones. With careful choice of a, b, c one finds that this is the only restriction.

Proposition 3.3.9 ([AD20])**.** *A 3-(α, δ)-Sasaki manifold is*

a) positive, if and only if it is $\mathcal{H}$-homothetic to a 3-Sasaki manifold.

b) negative, if and only if it is $\mathcal{H}$-homothetic to a 3-(α, δ)-Sasaki manifold with $\alpha = 1$, $\delta = -1$.

c) degenerate, if and only if it is $\mathcal{H}$-homothetic to a 3-(α, δ)-Sasaki manifold with $\alpha = 1$, $\delta = 0$.

Positive 3-(α,δ)-Sasaki manifolds are therefore deformed 3-Sasaki manifolds. For examples of negative ones we refer to the structured investigation of homogeneous non-degenerate 3-(α,δ)-Sasaki manifolds in Chapter 5. For degenerate 3-(α,δ)-Sasaki manifolds the easiest example is the quaternionic Heisenberg group. This has been described as naturally reductive homogeneous almost 3-contact metric manifold in [AFS15] and framed as degenerate 3-(α,δ)-Sasaki manifold in [AD20].

Example 3.3.10. We consider the $4n+3$-dimensional quaternionic Heisenberg group

$$\mathrm{H}^{n,\mathbb{H}} = \left\{ \begin{pmatrix} 1 & \overline{x}^t & \frac{|x|^2}{2} + z \\ 0 & \mathrm{id}_n & x \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(n+2,\mathbb{H}), \quad x \in \mathbb{H}^n,\ z \in \mathrm{Im}\mathbb{H} \right\}.$$

One observes that there is an diffeomorphism $\mathrm{H}^{n,\mathbb{H}} \cong \mathbb{H}^n \times \mathrm{Im}\mathbb{H}$, in particular $\mathrm{H}^{n,\mathbb{H}}$ is the simply connected Lie group with Lie algebra

$$\mathfrak{h}^{n,\mathbb{H}} = \left\{ \begin{pmatrix} 0 & \overline{X}^t & Z \\ 0 & 0 & X \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{gl}(n+2,\mathbb{H}), \quad X \in \mathbb{H}^n,\ Z \in \mathrm{Im}\mathbb{H} \right\}.$$

Set $\xi_1,\xi_2,\xi_3 \in \mathfrak{h}^{n,\mathbb{H}}$ the components of Z and $e_{4r},\dots,e_{4r+3} \in \mathfrak{h}^{n,\mathbb{H}}$ the components of X_r, $r=1,\dots,n$. Then the only non-trivial commutators are

$$\begin{aligned} [e_{4r},e_{4r+1}] &= 2\xi_1, \quad [e_{4r+2},e_{4r+3}] = 2\xi_1, \quad [e_{4r},e_{4r+2}] = 2\xi_2, \\ [e_{4r+3},e_{4r+1}] &= 2\xi_2, \quad [e_{4r},e_{4r+3}] = 2\xi_3, \quad [e_{4r+1},e_{4r+2}] = 2\xi_3. \end{aligned}$$

Compare to the usual Heisenberg algebra in Example 3.1.5. The following should be regarded as a 3-fold conjunction of the Sasaki structure on $\mathfrak{h}^{2n,\mathbb{C}}$.

Consider $\xi_1,\xi_2,\xi_3,e_4,\dots,e_{4n+3}$ to be an orthonormal base of an inner product g on $\mathfrak{h}^{n,\mathbb{H}}$. Set η_i the metric dual to ξ_i and define endomorphisms φ_i by

$$\varphi_i = \xi_k \wedge \xi_j + \sum_{l=1}^{n} e_{4l+i} \wedge e_{4l} + e_{4l+k} \wedge e_{4l+j},$$

for every even permutation (ijk) of (123). Extend $(g,\xi_i,\eta_i,\varphi_i)$ left-invariantly to the entirety of $H^{n,\mathbb{H}}$. Then one can show that

$$d\eta_i = 2(\Phi_i + \eta_j \wedge \eta_k).$$

Hence, $(\mathrm{H}^{n,\mathbb{H}},g,\xi_i,\eta_i,\varphi_i)$ defines a homogeneous degenerate 3-(α,δ)-Sasaki manifold with $\alpha=1$, $\delta=0$.

In [AFS15] the authors show that $(H^{n,\mathbb{H}},g)$ is naturally reductive. Note that this is not trivial as the metric defined above is not biinvariant. Instead they consider the metric connection ∇ with skew torsion

$$T = \sum_{i=1}^{3} \eta_i \wedge d\eta_i - 8\eta_{123}.$$

They show that $\nabla T = \nabla R = 0$ and hence that ∇ is the Ambrose-Singer connection of the natural reductive structure on $H^{n,\mathbb{H}}$, compare Example 2.1.5. They further show that the holonomy is given by $\mathfrak{hol}^{\nabla} = \mathfrak{su}(2)$ acting irreducibly on $\mathcal{V} \cong \mathrm{Im}\mathbb{H}$ and through n copies of $\mathbb{H}$ on $\mathcal{H}$. A direct computation also shows

$$\nabla_X \varphi_i = 4(\eta_j(X)\xi_k - \eta_k(X)\xi_j)$$

implying that ∇ is a canonical connection as defined in Definition 3.2.13.

The natural reductive structure on a 3-(α, δ)-Sasaki manifold is an isolated instance, but the canonical connection is not. In [AD20] the authors show that every 3-(α, δ)-Sasaki manifold is canonical in the sense of Theorem 3.2.12. We compile the implications and further core properties they have shown for the canonical connection of a 3-(α, δ)-Sasaki manifold into the following theorem.

Theorem 3.3.11. *A 3-(α, δ)-Sasaki manifold admits a unique metric connection ∇ with skew torsion such that*

$$\nabla_X \varphi_i = \beta(\eta_k(X)\varphi_j - \eta_j(X)\varphi_k), \tag{3.11}$$

for every even permutation (ijk) of (123) and some function $\beta \in C^\infty(M)$.

The function is a constant given by $\beta = 2(\delta - 2\alpha)$. The connection ∇ preserves the splitting $TM = \mathcal{V} \oplus \mathcal{H}$ and its torsion is given by

$$T = 2\alpha \sum_{i=1}^{3} \eta_i \wedge \Phi_i - 2(\alpha - \delta)\eta_{123}. \tag{3.12}$$

In particular, the torsion is parallel $\nabla T = 0$.

For later use we note the formula for T as a $(2,1)$-tensor:

$$T(X,Y) = 2\alpha \sum_{i=1}^{3} \big(\eta_i(Y)\varphi_i X - \eta_i(X)\varphi_i(Y) + \Phi_i(X,Y)\xi_i\big) - 2(\alpha - \delta) \overset{i,j,k}{\mathfrak{S}} \eta_{ij}(X,Y)\xi_k \tag{3.13}$$

Corollary 3.3.12 ([AD20])**.** *The canonical torsion of a 3-(α, δ)-Sasaki manifold satisfies*

$$dT = 4\alpha^2 \sum_{i=1}^{3} \Phi_i \wedge \Phi_i + 8\alpha(\delta - \alpha) \overset{i,j,k}{\mathfrak{S}} \Phi_i \wedge \eta_{jk}. \tag{3.14}$$

In accordance with previous notion the connection ∇ defined by condition (3.11) is called the canonical connection of a 3-(α, δ)-Sasaki manifold.

Corollary 3.3.13. *The fundamental 4-form $\Psi := \sum \Phi_i \wedge \Phi_i$ is parallel. In particular, the holonomy algebra of ∇ satisfies*

$$\mathfrak{hol}(\nabla) \subset (\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)) \oplus \mathfrak{so}(3) \subset \mathfrak{so}(4n) \oplus \mathfrak{so}(3).$$

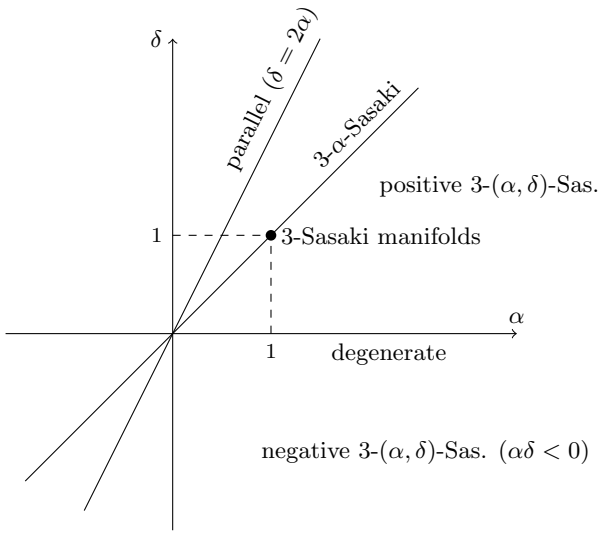
Using $(\nabla_X \varphi_i)\xi_i = -\varphi_i(\nabla_X \xi_i)$ and (3.11) reveals that the constant β controls all covariant derivatives of the structure tensors:

$$\begin{aligned} \nabla_X \varphi_i &= \beta(\eta_k(X)\varphi_j - \eta_j(X)\varphi_k) \\ \nabla_X \xi_i &= \beta(\eta_k(X)\xi_j - \eta_j(X)\xi_k) \\ \nabla_X \eta_i &= \beta(\eta_k(X)\eta_j - \eta_j(X)\eta_k) \end{aligned} \tag{3.15}$$

This warrants a seperate name if $\beta = 0$.

Definition 3.3.14. A 3-(α, δ)-Sasaki manifold with $\beta = 0$ is called *parallel* 3-(α, δ)-Sasaki.

For parallel 3-(α, δ)-Sasaki manifolds the holonomy even becomes a subspace $\mathfrak{hol}(\nabla) \subset \mathfrak{sp}(n)$. Since $\beta = 2(\delta - 2\alpha)$ they necessarily live in the realm of positive 3-(α, δ)-Sasaki spaces. The plane of 3-(α, δ)-Sasaki structures with the special cases is depicted in the following diagram. Note that the $\alpha = 0$ axis is not included. A possible extension has been discussed in [Dil21], but we will only consider $\alpha \neq 0$.



The curvature of the canonical connection is particularly well behaved on the defining tensors of a 3-(α, δ)-Sasaki manifold. We will make use of this to compute directly related curvature identities in the following two propositions. These, in turn, allowed us to prove the existence of the canonical submersion in [ADS21a].

Proposition 3.3.15. *Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be a 3-(α, δ)-Sasaki manifold. Let ∇ be the canonical connection and R the curvature tensor of ∇. Then, the following equations hold:*

$$\begin{aligned} R(X,Y)\varphi_i Z - \varphi_i R(X,Y)Z = {} & 2\alpha\beta\{\Phi_k(X,Y)\varphi_j Z - \Phi_j(X,Y)\varphi_k Z\} \\ & - 2\alpha\beta\{(\eta_i \wedge \eta_j)(X,Y)\varphi_j Z - (\eta_k \wedge \eta_i)(X,Y)\varphi_k Z\}, \end{aligned} \tag{3.16}$$

$$\begin{aligned} R(X,Y)\xi_i = {} & 2\alpha\beta\{\Phi_k(X,Y)\xi_j - \Phi_j(X,Y)\xi_k\} \\ & - 2\alpha\beta\{(\eta_i \wedge \eta_j)(X,Y)\xi_j - (\eta_k \wedge \eta_i)(X,Y)\xi_k\}, \end{aligned} \tag{3.17}$$

where $X, Y, Z \in TM$ and (ijk) is an even permutation of (123).

Proof. Applying the Ricci identity, (3.11) and (3.7), we have

$$\begin{aligned}
&R(X,Y)\varphi_i Z - \varphi_i R(X,Y)Z \\
&= (\nabla_X(\nabla_Y\varphi_i))Z - (\nabla_Y(\nabla_X\varphi_i))Z - (\nabla_{[X,Y]}\varphi_i)Z \\
&= \beta\{X(\eta_k(Y))\varphi_j Z + \eta_k(Y)(\nabla_X\varphi_j)Z - X(\eta_j(Y))\varphi_k Z - \eta_j(Y)(\nabla_X\varphi_k)Z\} \\
&\quad - \beta\{Y(\eta_k(X))\varphi_j Z + \eta_k(X)(\nabla_Y\varphi_j)Z - Y(\eta_j(X))\varphi_k Z - \eta_j(X)(\nabla_Y\varphi_k)Z\} \\
&\quad - \beta\{\eta_k([X,Y])\varphi_j Z - \eta_j([X,Y])\varphi_k Z\} \\
&= \beta\{d\eta_k(X,Y)\varphi_j Z - d\eta_j(X,Y)\varphi_k Z\} \\
&\quad + \beta^2\{\eta_k(Y)(\eta_i(X)\varphi_k Z - \eta_k(X)\varphi_i Z) - \eta_j(Y)(\eta_j(X)\varphi_i Z - \eta_i(X)\varphi_j Z)\} \\
&\quad - \beta^2\{\eta_k(X)(\eta_i(Y)\varphi_k Z - \eta_k(Y)\varphi_i Z) - \eta_j(X)(\eta_j(Y)\varphi_i Z - \eta_i(Y)\varphi_j Z)\} \\
&= 2\alpha\beta\{\Phi_k(X,Y)\varphi_j Z - \Phi_j(X,Y)\varphi_k Z\} \\
&\quad + \{2\beta(\alpha-\delta) + \beta^2\}\{(\eta_i\wedge\eta_j)(X,Y)\varphi_j Z - (\eta_k\wedge\eta_i)(X,Y)\varphi_k Z\},
\end{aligned}$$

which gives (3.16), since $\beta = 2(\delta - 2\alpha)$. □

Proposition 3.3.16. *The curvature tensor R of the canonical connection of a 3-(α,δ)-Sasaki manifold satisfies for any $X, Y, Z \in \mathcal{H}$ and $i, j, k, l = 1, 2, 3$ the identities*

$$R(X,\xi_i,Y,\xi_j) = R(X,Y,Z,\xi_i) = R(\xi_i,\xi_j,\xi_k,X) = 0, \tag{3.18}$$

$$R(\xi_i,\xi_j,\xi_k,\xi_l) = -4\alpha\beta(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \tag{3.19}$$

$$R(\xi_i,\xi_j,X,Y) = \pm 2\alpha\beta\Phi_k(X,Y), \tag{3.20}$$

$$R(X,Y,Z,\varphi_i Z) + R(X,Y,\varphi_j Z,\varphi_k Z) = \mp 2\alpha\beta\Phi_i(X,Y)\|Z\|^2, \tag{3.21}$$

where $\pm$ refers to an even, respectively odd, permutation (ijk) of (123).

Proof. Considering the symmetries of R we immediately obtain the first three expressions from equation (3.17). Using (3.16) for φ_j we obtain

$$\begin{aligned}
R(X,Y,\varphi_j Z,\varphi_k Z) &= g(\varphi_j R(X,Y)Z,\varphi_k Z) \\
&\quad + 2\alpha\beta(\Phi_i(X,Y)g(\varphi_k Z,\varphi_k Z) - \Phi_k(X,Y)g(\varphi_i Z,\varphi_k Z) \\
&= -R(X,Y,Z,\varphi_i Z) + 2\alpha\beta\Phi_i(X,Y)\|Z\|^2.
\end{aligned}$$

□

4 The Canonical Submersion

We come to the first core chapter of the thesis. It combines the preliminary chapters applying results on connections with skew torsion to 3-(α, δ)-Sasaki manifolds. We will obtain the structure Theorem 4.2.5 establishing a connection to quaternionic Kähler spaces. This theorem will in later chapters give us the means to investigate curvature properties of 3-(α, δ)-Sasaki manifolds.

4.1 The Model Case of α-Sasaki manifolds

We first want to obtain a canonical submersion in the case that $(M, g, \xi, \eta, \varphi)$ is a Sasaki manifold. This will serve as a model for the more involved 3-(α, δ)-Sasaki case later. For now assume $(M, g, \xi, \eta, \varphi)$ is an almost contact metric manifold satisfying the conditions in Theorem 3.1.7, hence admitting a characteristic connection ∇. Recall that for the characteristic connection

$$\nabla\xi = 0, \qquad \nabla\eta = 0.$$

This implies that $\mathcal{V} = \langle\xi\rangle$ and $\mathcal{H} = \ker\eta$ are invariant under $\mathfrak{hol}^{\nabla}$. Furthermore, the representation of $\mathfrak{hol}^{\nabla}$ on $\mathcal{V}$ is trivial. Hence $\mathcal{V}$ is vertical with regard to the notion in Corollary 2.2.7. We further assume that ∇ has parallel skew torsion. Since $\mathcal{V}$ is 1-dimensional $\Lambda^2\mathcal{V} = 0$ and $0 = \mathrm{pr}_{\mathcal{H}\otimes\Lambda^2\mathcal{V}}T$. Thus, we can apply Corollary 2.2.7 for the decomposition $TM = \mathcal{V} \oplus \mathcal{H}$. We obtain

Proposition 4.1.1. *Let $(M, g, \xi, \eta, \varphi)$ be an almost contact metric manifold with characteristic connection ∇ such that $\nabla T = 0$. Then*

a) there exists a locally defined Riemannian submersion $\pi\colon M \to N$ with fibers tangent to ξ,

b) N admits a Hermitian structure $J = \pi_* \circ \varphi \circ s_*$ for an arbitrary local section $s\colon N \to M$ of π and

c) N admits a connection ∇^σ with skew torsion σ such that $\pi_*\sigma = T^{\mathcal{H}} = d^\varphi\Phi + (N - \eta \wedge (\xi \lrcorner N))$ and for all $X, Y \in TN$

$$\nabla^\sigma_X Y = \pi_*(\nabla_{\overline{X}}\overline{Y}).$$

Proof. The only assertion still to prove is that J gives an almost complex structure compatible with g_N. The compatibility follows immediately as $\pi_*\colon \mathcal{H} \to TN$ and $\mathrm{pr}_{\mathcal{H}} \circ s_*\colon TN \to \mathcal{H}$ are isometric. We check that $J^2 = -\operatorname{id}$. Since s is a section of π we have $s_* \circ \pi_* = \operatorname{id}$ on the image of s and thus

$$J^2 = \pi_* \circ \varphi \circ s_* \circ \pi_* \circ \varphi \circ s_* = \pi_* \circ \varphi^2 \circ s_* = \pi_* \circ (-\operatorname{id} + \eta \otimes \xi) \circ s_* = -\operatorname{id}. \qquad \square$$

We now reprove the result from Theorem 3.1.11 that every Sasaki manifold with compact Reeb orbits fibers over a Kähler orbifold.

Theorem 4.1.2. *Let $(M, g, \xi, \eta, \varphi)$ be a Sasaki manifold. Then there exists a locally defined Riemannian submersion $\pi\colon M \to N$ such that N admits a Kähler structure given by $J = \pi_* \circ \varphi \circ s_*$ where $s\colon N \to M$ is a local section of the submersion π.*

Proof. We recall from Corollary 3.1.9 that a Sasaki structure admits a characteristic connection with parallel torsion and we may use Proposition 4.1.1. Hence we obtain a locally defined $\pi\colon M \to N$ with (N, g_N, J) a Hermitian manifold. Further $T^{\mathcal{H}} = 0$ so

$$\nabla^{g_N}_X Y = \pi_*(\nabla_{\overline{X}} \overline{Y}).$$

It remains to see that J is parallel. We first compute the covariant derivative of a basic vector field Y with respect to ξ. We have

$$\nabla_\xi Y = \nabla_Y \xi + L_\xi Y + T(\xi, Y) = \eta \wedge d\eta(\xi, Y, \cdot) = -2\varphi Y.$$

Observe that the horizontal lift of a vector field is necessarily basic and $\overline{\pi_*(\varphi(s_*Y))}_{\mathcal{H}} = (\varphi(s_*Y))_{\mathcal{H}}$ when the right side is defined, that is on the image $s(N)$. Set $\hat{X} := \overline{X} - s_*X$ the vertical part of $-s_*X$. Note that both φ and ∇ preserve the splitting $TM = \mathcal{H} \oplus \mathcal{V}$. Then

$$\begin{aligned}
(\nabla^{g_N}_X J)Y &= \nabla^{g_N}_X (JY) - J(\nabla^{g_N}_X Y) = \pi_*(\nabla_{\overline{X}} \overline{JY}) - \pi_*(\varphi(s_*(\pi_*(\nabla_{\overline{X}} \overline{Y})))) \\
&= \pi_*(\nabla_{\overline{X}}(\overline{\pi_*(\varphi(s_*Y))})) - \pi_*(\varphi(\nabla_{\overline{X}} \overline{Y})) \\
&= \pi_*(\nabla_{\hat{X}}(\overline{\pi_*(\varphi(s_*Y))})) + \pi_*(\nabla_{s_*X}(\varphi(s_*Y))) - \pi_*(\varphi(\nabla_{\hat{X}} \overline{Y})) - \pi_*(\varphi(\nabla_{s_*X} \overline{Y})) \\
&= -2\eta(\hat{X})\pi_*(\varphi(\overline{\pi_*(\varphi(s_*Y))}) - \varphi^2 \overline{Y}) + \pi_*(\nabla_{s_*X}(\varphi(s_*Y)) - \varphi(\nabla_{s_*X}(s_*Y))) \\
&= -2\eta(\hat{X})\pi_*(\varphi^2(s_*Y) - \varphi^2(s_*Y)) + \pi_*((\nabla_{s_*X}\varphi)s_*Y) = 0. \qquad \square
\end{aligned}$$

Corollary 4.1.3. *If the Reeb orbits of M are compact, M is a principal S^1 orbifold-bundle over a Kähler orbifold.*

The statement essentially comes down to a theorem by Wadsley.

Theorem 4.1.4 ([Wad75])**.** *Let $\mathbb{R}$ act on M such that each orbit is a circle. Then there exists an action of S^1 on M with the same orbits as the $\mathbb{R}$-action if and only if there is a Riemannian metric such that the orbits are totally geodesic submanifolds.*

Proof of Corollary 4.1.3. Since the Reeb vector fields are geodesic and nowhere vanishing, by Wadsley's theorem they are the orbits of a locally free circle action. In this case the leaf space has the structure of an orbifold, see [MM03, Corollary 2.16]. Locally the fibers of the canonical submersion agree with the Reeb orbits. Thus, the leaf space agrees with the locally defined base U of the canonical submersion up to a quotient by a finite group. In fact, these local patches U are orbifold charts for N. □

4.2 The Canonical Submersion onto qK spaces

We now discuss the situation for 3-(α,δ)-Sasaki manifolds. By (3.11) the holonomy representation of the canonical connection ∇ of a 3-(α,δ)-Sasaki manifold splits into the horizontal and vertical subspaces $\mathcal{H}$ and $\mathcal{V}$. In the non-parallel case $\mathcal{V}$ is irreducible, in the parallel case it decomposes into 3 trivial 1-dimensional representations. In either case the curvature properties stated in Proposition 3.3.16 allow us to prove:

Lemma 4.2.1. *The vertical distribution $\mathcal{V}$ of a 3-(α,δ)-Sasaki manifold is vertical with respect to Definition 2.2.6.*

Proof. By the Ambrose-Singer Theorem 2.1.11 the holonomy algebra $\mathfrak{hol}$ of the holonomy group $\mathrm{Hol}(p)$ at a point p is given by

$$\mathfrak{hol} = \{P_\gamma^{-1} \circ \mathcal{R}(P_\gamma X, P_\gamma Y) \circ P_\gamma \mid \gamma \text{ some path from } p \text{ to } q,\ X,Y \in T_pM\} \subset \mathfrak{so}(T_pM)$$

where P_γ denotes parallel transport along γ and $\mathcal{R}(X,Y) \in \Lambda^2 M \cong \mathfrak{so}(T_qM)$ the curvature operator. The horizontal and vertical distribution are invariant under parallel transport with respect to the canonical connection. Thus, we may assume γ to be trivial when investigating the holonomy action on these distributions. By (3.18) we know that the holonomy is only non-trivial if $X,Y \in \mathcal{V}$ or $X,Y \in \mathcal{H}$. In the first case (3.19) and (3.20) show that every element of $\mathfrak{hol}$ acting non-trivially on $\mathcal{V}$ must also act non-trivially on $\mathcal{H}$. The action of an element $\mathcal{R}(X,Y)$, $X,Y \in \mathcal{H}$, on $\mathcal{V}$ is again given by (3.20). Any such element of $\mathfrak{hol}$ acts non-trivially on $\mathcal{V}$ if $\beta \neq 0$ and $\Phi_i(X,Y) \neq 0$ for some $i = 1,2,3$. In this case $\mathcal{R}(X,Y)$ is also a non-trivial operator on $\mathcal{H}$ by (3.21). □

Proposition 4.2.2. *The decomposition $TM = \mathcal{H} \oplus \mathcal{V}$ of a 3-(α,δ)-Sasaki manifold M satisfies the conditions in Corollary 2.2.7. In particular, there exists a locally defined Riemannian submersion $\pi\colon M \to N$ such that*

$$\nabla^{g_N}_X Y = \pi_*(\nabla_{\overline{X}} \overline{Y}). \tag{4.1}$$

Definition 4.2.3. We will call $\pi\colon M \to N$ the *canonical submersion* of a 3-(α,δ)-Sasaki manifold.

Proof of Proposition 4.2.2. By (3.11) and Lemma 4.2.1 the decomposition $TM = \mathcal{V} \oplus \mathcal{H}$ is of type (2.7) with respect to the canonical connection ∇. By (3.12) the projection of the torsion onto $\mathcal{H} \otimes \Lambda^2\mathcal{V}$ vanishes, satisfying (2.8). Therefore the conditions of Corollary 2.2.7 are satisfied. Moreover, (3.12) shows that the projection of T onto $\Lambda^3\mathcal{H}$ vanishes so the connection ∇^σ in (2.9) for the canonical submersion is the Levi-Civita connection ∇^{g_N} on N. □

We observe that the canonical submersion is an almost contact metric 3-submersion in the sense of [Wat84]. Though we never make explicit use of this property. We give a preliminary lemma needed to prove that the base of the canonical submersion admits a qK structure.

Lemma 4.2.4. *For any vertical vector field $X \in \mathcal{V}$ and for any basic vector field $Y \in \mathcal{H}$ we have*

$$(\nabla_X Y)_{\mathcal{H}} = -2\alpha \sum_{i=1}^{3} \eta_i(X)\varphi_i Y.$$

Proof. We first use the identity $g(\nabla^g_X Y, Z) = -\frac{1}{2}g([Y,Z],X)$ for any vector fields $X \in \mathcal{V}, Y, Z \in \mathcal{H}$, with Y and Z projectable, of a Riemannian submersion [Pet06, Proposition 13]. Note that the horizontal and vertical distributions of the Riemannian submersion agree with the same notion in the 3-(α,δ)-Sasaki setting. Further, we make use of Lemma 3.3.4 to obtain

$$g(\nabla^g_{\xi_i} Y, Z) = -\frac{1}{2}g([Y,Z],\xi_i) = \frac{1}{2}d\eta(Y,Z) = \alpha\Phi_i(Y,Z).$$

Therefore

$$\begin{aligned} g(\nabla_X Y, Z) &= g(\nabla^g_X Y, Z) + \frac{1}{2}T(X,Y,Z) = \sum_{i=1}^{3} \eta_i(X)(\alpha\Phi_i(Y,Z) + \alpha\Phi_i(Y,Z)) \\ &= -2\alpha \sum_{i=1}^{3} \eta_i(X) g(\varphi_i Y, Z). \end{aligned}$$

□

Theorem 4.2.5. *The base N of the canonical submersion $\pi\colon M \to N$ of any 3-(α,δ)-Sasaki manifold M carries a quaternionic Kähler structure locally defined by*

$$\check{\varphi}_i = \pi_* \circ \varphi_i \circ s_*, \quad i = 1,2,3,$$

where $s\colon U \to M$ is any local section of π. The covariant derivatives of the almost complex structures $\check{\varphi}_i$ are given by

$$\nabla^{g_N}_X \check{\varphi}_i = 2\delta(\check{\eta}_k(X)\check{\varphi}_j - \check{\eta}_j(X)\check{\varphi}_k),$$

where $\check{\eta}_i(X) = \eta_i(s_*X)$ for $i = 1,2,3$.

Proof. Let s be a local section of the canonical submersion $\pi\colon M \to N$, hence $\pi_* \circ s_* = \mathrm{id}$ and $\mathrm{Im}(s_* \circ \pi_* - \mathrm{id}) \subset \mathcal{V}$ on the image $s(N) \subset M$. Define

$$\check{\varphi}_i = \pi_* \circ \varphi_i \circ s_*$$

for $i = 1, 2, 3$. The horizontal and vertical distributions, $\mathcal{H}$ and $\mathcal{V}$, are invariant under φ_i. Thus, $\pi_* \circ \varphi_i = \check{\varphi}_i \circ \pi_*$ on $s(N)$. This yields

$$\check{\varphi}_i \check{\varphi}_j = \check{\varphi}_i \circ (\pi_* \circ \varphi_j \circ s_*) = \pi_* \circ (\varphi_i \varphi_j) \circ s_*.$$

Now use that $(\varphi_i|_{\mathcal{H}})^2 = -\mathrm{id}|_{\mathcal{H}}$ and $(\varphi_i|_{\mathcal{H}})(\varphi_j|_{\mathcal{H}}) = \pm\varphi_k|_{\mathcal{H}}$ with sign $\pm$ depending on whether (ijk) is an even or odd permutation of (123). This shows $\check{\varphi}_i^2 = -\mathrm{id}$ and $\check{\varphi}_i \check{\varphi}_j = \pm\check{\varphi}_k$.

Finally, by means of (4.1) and (3.11), we show that the quaternionic structure is parallel. First

$$\begin{aligned}(\nabla^{g_N}_X \check{\varphi}_i)Y &= (\nabla^{g_N}_X (\check{\varphi}_i Y)) - (\check{\varphi}_i (\nabla^{g_N}_X Y)) = \pi_* \nabla_{\overline{X}} \overline{(\check{\varphi}_i Y)} - \check{\varphi}_i \left(\pi_*(\nabla_{\overline{X}} \overline{Y})\right) \\ &= \pi_* \nabla_{\overline{X}} \overline{(\pi_* (\varphi_i (s_* Y)))} - \pi_* \left(\varphi_i \left(s_* \left(\pi_* \left(\nabla_{\overline{X}} \overline{Y}\right)\right)\right)\right).\end{aligned}$$

By the properties of any Riemannian submersion we have that $\overline{(\pi_* (\varphi_i (s_* Y)))} = (\varphi_i (s_* Y))_{\mathcal{H}}$ wherever the right side is defined, that is on the image $s(N) \subset M$. Thus, we take the covariant derivatives in the direction of $s_* X$ resulting in a vertical correction term $\hat{X} = \overline{X} - s_* X \in \mathcal{V}$. Recall that ∇ and φ_i preserve the horizontal and the vertical distribution. Using Lemma 4.2.4, we obtain

$$\begin{aligned}\nabla_{\overline{X}} \overline{(\pi_* (\varphi_i (s_* Y)))} &= \nabla_{s_* X} (\varphi_i (s_* Y))_{\mathcal{H}} + \nabla_{\hat{X}} (\overline{(\pi_* (\varphi_i (s_* Y)))}) \\ &= (\nabla_{s_* X} (\varphi_i (s_* Y)))_{\mathcal{H}} - 2\alpha \sum_{l=1}^{3} \eta_l(\hat{X}) \varphi_l (\varphi_i (s_* Y))_{\mathcal{H}}.\end{aligned}$$

For the second summand, the horizontal projection is given by

$$\begin{aligned}\left(\varphi_i \left(s_* \left(\pi_* \left(\nabla_{\overline{X}} \overline{Y}\right)\right)\right)\right)_{\mathcal{H}} &= \varphi_i \left(s_* \left(\pi_* \left(\nabla_{\overline{X}} \overline{Y}\right)\right)\right)_{\mathcal{H}} \\ &= \varphi_i (\nabla_{s_* X} (s_* Y))_{\mathcal{H}} + \varphi_i \nabla_{\hat{X}} (\overline{Y}) \\ &= (\varphi_i (\nabla_{s_* X} (s_* Y)))_{\mathcal{H}} - 2\alpha \sum_{l=1}^{3} \eta_l(\hat{X}) \varphi_i (\varphi_l (s_* Y))_{\mathcal{H}}.\end{aligned}$$

Recombining both identities we obtain

$$\begin{aligned}(\nabla^{g_N}_X \check{\varphi}_i)Y &= \pi_* \big(\nabla_{s_* X} (\varphi_i (s_* Y)) - \varphi_i (\nabla_{s_* X} (s_* Y)) \\ &\qquad + 2\alpha \sum_{l=1}^{3} \eta_l(\hat{X}) (\varphi_i \varphi_l (s_* Y)_{\mathcal{H}} - \varphi_l \varphi_i (s_* Y)_{\mathcal{H}})\big) \\ &= \pi_* \big((\nabla_{s_* X} \varphi_i) s_* Y - 2\alpha \sum_{l=1}^{3} \eta_l (s_* X) (\varphi_i \varphi_l (s_* Y)_{\mathcal{H}} - \varphi_l \varphi_i (s_* Y)_{\mathcal{H}})\big) \\ &= (\beta + 4\alpha) \big((\eta_k (s_* X) \circ s) \pi_* (\varphi_j (s_* Y)) - (\eta_j (s_* X) \circ s) \pi_* (\varphi_k (s_* Y))\big) \\ &= 2\delta (\check{\eta}_k (X) \check{\varphi}_j - \check{\eta}_j (X) \check{\varphi}_k) Y.\end{aligned}$$

Here we used the defining identity (3.11) of the canonical connection for any even permutation (ijk) of (123). Therefore, the quaternionic structure is parallel and N is quaternionic Kähler. □

Remark 4.2.6. A priori the locally defined quaternionic structure may depend on the chosen section s. Indeed, the individual almost complex structures $\check{\varphi}_i$ vary with s. However, following the work of P. Piccinni and I. Vaisman [PV01], one can see that the quaternionic structure is preserved under the Bott connection $\mathring{D}\colon \mathcal{V}\times\mathcal{H} \to \mathcal{H}$ defined by $\mathring{D}_V X = [V, X]_{\mathcal{H}}$, since for any even permutation (ijk) of (123)

$$(\mathring{D}_{\xi_i}\varphi_j)X = [\xi_i, \varphi_j X]_{\mathcal{H}} - \varphi_j[\xi_i, X]_{\mathcal{H}} = ((L_{\xi_i}\varphi_j)X)_{\mathcal{H}} = 2\delta\varphi_k X$$

where we have used identity (3.8) in the last step. This implies that the quaternionic structure is projectable and, thus, independent of choices.

Moreover, we can identify a unique quaternionic Kähler structure to each leaf not only to a point in the image of some patch of π.

We observe one special case from both Theorem 4.2.5 and the remark.

Corollary 4.2.7. *A 3-(α, δ)-Sasaki manifold fibers locally over a hyperkähler manifold if it is degenerate. In this case the almost complex structures are independent of the choice of local section s.*

Remark 4.2.8. Apart from the degenerate case the induced quaternionic Kähler structure is hyperkähler if and only if $s_*X \in \mathcal{H}$ for all $X \in TN$. Such a section exists if and only if the horizontal distribution is tangent to $s(N)$ and, thus, integrable. By Lemma 3.3.4 this can not happen for any 3-(α, δ)-Sasaki manifold.

We should also mention the more subtle case of $n = 1$. Recall that one requires 4-dimensional quaternionic Kähler spaces to be Einstein and anti-self-dual. We show that this is true for the base of the canonical submersion of a 7-dimensional 3-(α, δ)-Sasaki manifold. This permits ot omit the restriction in Theorem 4.2.5.

Proposition 4.2.9. *If $n = 1$, the base (N, g_N) of the canonical submersion is quaternionic Kähler.*

Proof. We need to prove that (N, g_N) is Einstein and self-dual. For the latter we refer to Remark 6.3.2 when we investigate the curvature more closely. For now let us check that (N, g_N) is Einstein. In [AD20, Proposition 2.3.3] it is shown that the Ricci curvature of M is

$$\mathrm{Ric}_g = 2\alpha(2\delta(n+2) - 3\alpha)g + 2(\alpha - \delta)((2n+3)\alpha - \delta)g|_{\mathcal{V}}. \tag{4.2}$$

For submersions with O'Neill tensor $\mathcal{T} = 0$ and for some orthonormal base $e_1, \ldots, e_{4n}$ of $\mathcal{H}$ the Ricci tensor is

$$\mathrm{Ric}_g(X, Y) = \pi^*\mathrm{Ric}_{g_N}(X, Y) - 2\sum_{j=1}^{4n} g(\mathcal{A}_X e_j, \mathcal{A}_Y e_j), \tag{4.3}$$

see [Bes08, Proposition 9.36]. This implies that the space (N, g_N) is Einstein if and only if $\sum_{j=1}^{4n} g(\mathcal{A}_X e_j, \mathcal{A}_Y e_j)$ is a multiple of g_N. By Lemma 2.2.10 and (3.12) for $X, Y \in \mathcal{H}$

$$\mathcal{A}_X Y = -\frac{1}{2} T(X,Y)_{\mathcal{V}} = -\alpha \sum_{i=1}^{3} \Phi_i(X,Y)\xi_i. \tag{4.4}$$

Rewriting this formulas we obtain

$$\sum_{j=1}^{4n} g(\mathcal{A}_X e_j, \mathcal{A}_Y e_j) = \alpha^2 \sum_{j=1}^{4n} \sum_{i=1}^{3} \Phi_i(X, e_j)\Phi_i(Y, e_j) = 3\alpha^2 g(X,Y)$$

where we have used that $(\varphi_i e_j)_{j=1,\dots,4n}$ is again an orthonormal basis of $\mathcal{H}$. □

The formulas used above also give us an immediate interpretation of positive and negative 3-(α, δ)-Sasaki with respect to the base.

Theorem 4.2.10. *Let $\pi\colon M \to N$ be the canonical submersion of a 3-(α, δ)-Sasaki manifold. Then*

$$\mathrm{scal}_{g_N} = 16n(n+2)\alpha\delta.$$

Proof. Let $e_1, \dots, e_{4n}$ be a local adapted frame for $\mathcal{H}$. Then by (4.4)

$$\sum_{i,j=1}^{4n} g(A_{e_i} e_j, A_{e_i} e_j) = \alpha^2 \sum_{i,j=1}^{4n} \sum_{k=1}^{3} \Phi_k^2(e_i, e_j) = \alpha^2 \cdot 3 \cdot 4n = 12n\alpha^2.$$

Combining the identities (4.2) and (4.3) we have

$$\begin{aligned}\mathrm{scal}_{g_N} &= \sum_{i=1}^{4n} \mathrm{Ric}_{g_N}(\pi_* e_i, \pi_* e_i) = \sum_{i=1}^{4n} \mathrm{Ric}_g(e_i, e_i) + 2 \sum_{i,j=1}^{4n} g(A_{e_i} e_j, A_{e_i} e_j) \\ &= 4n \cdot 2\alpha(2\delta(n+2) - 3\alpha) + 24n\alpha^2 = 16n(n+2)\alpha\delta.\end{aligned}$$ □

Remark 4.2.11. In particular, we recover the scalar curvature result $\mathrm{scal}_{g_N} = 16n(n+2)$ known in the 3-Sasaki case, see Theorem 3.2.11.

4.3 The Global Picture

A priori the canonical submersion of a 3-(α, δ)-Sasaki manifold is a purely local statement. Yet, on a sufficiently small open set V it is given by the projection to the space of plaques tangent to $\mathcal{V}$. This suggests to investigate the space of leaves tangent to $\mathcal{V}$ globally.

Definition 4.3.1. The foliation $\mathcal{F}$ with leaves tangent to $\mathcal{V}$ will be called the *canonical foliation* of a 3-(α, δ)-Sasaki manifold.

In general the space of leaves $M/\mathcal{F}$ does not have a smooth structure. However, in the non-degenerate case we always have a submersion over a quaternion Kähler orbifold.

Theorem 4.3.2. *The local patches of the canonical submersion $\tilde{\pi}\colon M \supset U \to S$ of a non-degenerate, complete 3-(α,δ)-Sasaki manifold globally form a submersion $\pi\colon M \to N$ where N is a quaternion Kähler orbifold. The local transverse sections give rise to orbifold charts.*

Proof. We observe that by (3.15)

$$\nabla^g_{\xi_i}\xi_i = \nabla_{\xi_i}\xi_i = 0.$$

Thus, the flow lines of the ξ_i are geodesics and by completeness of M they extend for all time. In the non-degenerate case the Reeb vector fields generate a $\mathfrak{su}(2)$-subalgebra of $\mathfrak{X}(M)$. Since their flow is complete they generate an action by the compact group SU(2) on M.

As the ξ_i are everywhere linearly independent the action of SU(2) is locally free and therefore induces a foliation $\mathcal{F}$. The quotient $N = M/\mathcal{F}$ of such a foliation induced by a compact group is necessarily an orbifold [MM03, Corollary 2.16.]. It remains to check that the projection onto the quotient space agrees with the canonical submersion up to the quotient by a finite group. To this extent we note that the locally defined canonical submersion projects onto the leaves of the restricted foliation $\mathcal{F}|_V$ and the map to N is then just the quotient by the finite leaf holonomy group. □

For degenerate 3-(α,δ)-Sasaki manifolds the canonical submersion does not behave as nice globally. This boils down to the fact that the Reeb vector fields generate an abelian Lie algebra, so we can not expect the leaves to be compact. Therefore the leaf holonomy group might be infinite and hence the quotient is not an orbifold anymore. That being said let us consider the model case of a degenerate 3-(α,δ)-Sasaki manifold with compact fibers. That is a T^3-fiber bundle over a suitable hyperkähler manifold.

Let (N, g, I_1, I_2, I_3) be a hyperkähler manifold. Suppose that the fundamental two-forms are integer classes $[\omega_i] \in H^2(N,\mathbb{Z})$. Then from Corollary 3.1.13 we obtain a Boothby-Wang bundle $S^1 \to P_i \xrightarrow{\pi_i} N$ for each Kähler structure. In particular, these bundles have Chern class $[\omega_i]$ and the total spaces P_i are equipped with Sasaki structures $(g_i, \tilde{\varphi}_i, \eta_i, \xi_i)$, such that ξ_i generates the fiber.

Now consider the product bundle

$$T^3 \longrightarrow P_1 \times P_2 \times P_3 \xrightarrow{\Pi=(\pi_1,\pi_2,\pi_3)} N^3.$$

Set $M = \Pi^{-1}(\Delta N)$ the restriction of $P_1 \times P_2 \times P_3$ to the diagonal $\Delta(N) = \{(x,x,x) \in N^3\}$ and consider M as a fiber bundle $M \xrightarrow{\pi} N$, where π is the composition

$$P_1 \times P_2 \times P_3 \supset M \xrightarrow{\Pi} \Delta(N) \xrightarrow{\cong} N, \quad (p_1,p_2,p_3) \mapsto (x,x,x) \mapsto x.$$

If we denote the fiber of P_i over $x \in N$ by $(P_i)_x := \pi_i^{-1}(\{x\}) \cong S^1$, then the bundle $M \xrightarrow{\pi} N$ can be seen as the fiber bundle with fiber $(P_1)_x \times (P_2)_x \times (P_3)_x$ over any given point $x \in N$. Evidently for $p = (p_1, p_2, p_3) \in M$ we have

$$T_pM \cong T_xN \oplus \langle \xi_1, \xi_2, \xi_3 \rangle$$

as ξ_i generates the fiber of P_i. The construction as a product bundle ensures that the Reeb vector fields ξ_i are independent from one another, $[\xi_i, \xi_j] = 0$, as they ought to be for degenerate 3-(α, δ)-Sasaki manifolds. Compare (3.9).

Extend η_i trivially, i.e. $\ker \eta_i = TN \oplus \langle \xi_j, \xi_k \rangle$, with $j, k \neq i$, and $\eta_i(\xi_i) = 1$. Set the metric $g := \pi^* g_N + \eta_1^2 + \eta_2^2 + \eta_3^2$, then $\pi \colon (M, g) \to (N, g_N)$ becomes a Riemannian submersion and, in fact, the canonical submersion.

Theorem 4.3.3. *Let (N, g_N, I_1, I_2, I_3) be a hyperkähler manifold with integer fundamental forms. Construct (M, g, η_i, ξ_i) as above and set $\varphi_i = \tilde{\varphi}_i + \eta_j \otimes \xi_k - \eta_k \otimes \xi_j$ for any cyclic permutation (ijk) of* (123).
Then $(M, g, \xi_i, \eta_i, \varphi_i)$ defines a degenerate 3-(α, δ)-*Sasaki manifold with $\alpha = 1$, $\delta = 0$.*

Remark 4.3.4. In [For00a] the author obtains a so-called complex contact manifold by constructing a T^2-bundle over N in similar fashion. His construction imposes a less restrictive assumption than hyperkähler on N. However, in this special case the complex T^2-bundle he considers can be obtained from our construction as the quotient by one of the Reeb vector fields. This should be considered as the analogue of the twistor space over a positive quaternion Kähler manifold and the 3-Sasaskian space above, compare [For00b].

Proof. Observe that $(g, \eta_i, \xi_i, \varphi_i)$ extends the almost contact metric structure on $T_{p_i}P_i \cong T_xN \oplus \mathbb{R}\xi_i$ to $T_pM \cong T_xN \oplus \langle \xi_i, \xi_j, \xi_k \rangle$. The 3 a.c.m. structures are compatible on the vertical subspace by definition of φ_i. On the horizontal distribution φ_i projects to I_i, thus $\varphi_i \varphi_j|_{\mathcal{H}} = \varphi_k|_{\mathcal{H}}$.

The last identity to check is the differential condition $d\eta_i = 2\Phi_i + 2\eta_j \wedge \eta_k$. Since $\eta_j \wedge \eta_k = -\Phi_i|_{\mathcal{V}}$ we need to show

$$d\eta_i = 2\Phi_i|_{\mathcal{H}}. \tag{4.5}$$

As $(\xi_i, \eta_i, \tilde{\varphi}_i)$ defines a Sasaki structure on P_i we have for vertical vectors $X, Y \in T_{p_i}P_i$ that $g_i(X, \tilde{\varphi}_i Y) = d\tilde{\eta}_i(X, Y)$. This remains true for $X, Y \in \mathcal{H} \subset T_pM$ as φ_i is just $\tilde{\varphi}_i$ on $\mathcal{H}$. If either vector $X, Y \in \mathcal{V}$, the left hand side should vanish as does the right hand side of (4.5). We have

$$\xi_j \lrcorner\, d\eta_i = L_{\xi_j}\eta_i - d(\xi_j \lrcorner\, \eta_i) = 0$$

as η_i is defined on the factor P_i of the product P. In particular, it is invariant under ξ_j for $j \neq i$. Finally, $\xi_i \lrcorner\, d\eta_i = 0$ from the Sasaki condition on P_i. □

As shown in Corollary 4.2.7 locally any degenerate 3-(α, δ)-Sasaki manifold submerges onto a hyperkähler space. The above construction describes the converse starting from a hyperkähler manifold. By construction the Reeb vector fields span the fiber of the bundle $T^3 \to M \xrightarrow{\pi} N$. Therefore the canonical submersion agrees globally with π mapping onto a hyperkähler manifold, namely the initial manifold N. Observe that starting with a compact hyperkähler manifold in this way one again obtains a compact degenerate 3-(α, δ)-Sasaki manifold.

On the non-compact side consider the quaternionic Heisenberg group $H^{n,\mathbb{H}} \cong \mathbb{H}^n \times \mathrm{Im}\mathbb{H}$ from example Example 3.3.10. Here the canonical submersion is just the projection to $\mathbb{H}^n$. Accordingly the fibers are the non-compact $\mathbb{R}^3 = \mathrm{Im}\mathbb{H}$. One can take the quotient by $\mathbb{Z}^3$ in the space $\mathbb{R}^3 = \mathrm{Im}\mathbb{H}$ generated by the Reeb vector fields. This quotient can than be seen as the 3-Bothby-Wang bundle from Theorem 4.3.3 over the quaternion vector space $\mathbb{H}^n$ with its canonical hyperkähler structure.

In preparation for the next chapter we consider automorphisms of 3-(α, δ)-Sasaki manifold.

Definition 4.3.5. A 3-(α, δ)-*Sasaki automorphism* ϕ is an isometry $\phi\colon (M, g) \to (M, g)$ that preserves the tensor fields ξ_i, η_i, φ_i. A manifold is called a homogeneous 3-(α, δ)-Sasaki manifold if the automorphism group acts transitively.

Theorem 4.3.6. *If $(M, g, \xi_i, \eta_i, \varphi_i)_{i=1,2,3}$ is a non-degenerate homogeneous 3-(α, δ)-Sasaki manifold. Then the space of leaves N is a homogeneous quaternionic Kähler manifold.*

Proof. Any automorphism ϕ of the 3-(α, δ)-Sasaki structure preserves the vertical space and thus maps leaves to leaves. Now let ϕ map L_1 to L_2. It then also maps a local transverse section around $x \in L_1$ to a local transverse section of $\phi(x) \in L_2$. For every closed loop in $\pi_1(L_1, x)$ we get a closed loop in $\pi_1(L_2, \phi(x))$. This induces an homomorphism of the leaf holonomy groups and an isomorphism considering the inverse ϕ^{-1}. Since the automorphism group acts transitive all leaf holonomy groups are isomorphic and thus necessarily trivial. Therefore the quotient is a manifold $N = M/\mathcal{F}$ with an induced transitive action, hence homogeneous.

We check that the quaternionic structure is preserved under the induced action. Let $\phi\colon M \to M$ be an automorphism and $\check{\phi}\colon N \to N$ its induced map. By Remark 4.2.6 the quaternionic structure is independent of the lift into the fiber. So for leaves $L_1, L_2 = \check{\phi} L_1 \in N$ define the quaternionic structure by the lift to points $p_1 \in L_1$, $p_2 = \phi(p_1) \in L_2$. Then ϕ preserves the almost complex structures φ_i and thus $\check{\phi}$ the quaternionic structures. $\square$

5 Homogeneous 3-(α, δ)-Sasaki manifolds

Theorem 4.3.6 and Theorem 4.2.10 show that the base N of a non-degenerate homogeneous 3-(α, δ)-Sasaki manifold is a homogeneous quaternionic Kähler space of non-vanishing scalar curvature. There are two families of such spaces known: Compact qK symmetric spaces, named Wolf spaces, their non-compact duals and Alekseevsky spaces. The latter are homogeneous qK spaces admitting a solvable transitive group action. D. Alekseevsky conjectured that all homogeneous qK spaces with negative scalar curvature are Alekseevsky spaces [Ale75]. Only recently a proof of this conjecture has been given by C. Böhm and R. Lafuente in [BL21]. Note that therefore the non-compact qK symmetric spaces are included in the class of Alekseevsky. We will give independent constructions of homogeneous 3-(α, δ)-Sasaki manifolds over symmetric base spaces and such fibering over Alekseevsky spaces.

5.1 Homogeneous 3-(α, δ)-Sasaki manifolds over symmetric quaternionic Kähler spaces

Let G/G_0 be a real symmetric space, i.e. $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ on the level of Lie algebras. Suppose there exists a connected subgroup $H \subset G_0$ such that $\mathfrak{g}_0$ splits into a direct sum of Lie algebras $\mathfrak{g}_0 = \mathfrak{h} \oplus \mathfrak{sp}(1)$. Finally, assume that $\mathfrak{g}_1^{\mathbb{C}} = \mathbb{C}^2 \otimes_{\mathbb{C}} W$, for some $\mathfrak{h}^{\mathbb{C}}$-module W of $\dim_{\mathbb{C}} W = 2n$, and the adjoint action of $\mathfrak{g}_0^{\mathbb{C}}$ is given by

$$\mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{sp}(1)^{\mathbb{C}} \ni (A, B) \cdot ((z_1, z_2) \otimes w) = B(z_1, z_2) \otimes w + (z_1, z_2) \otimes Aw,$$

where $\mathfrak{sp}(1)^{\mathbb{C}} = \mathfrak{su}(2)^{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$ acts by multiplication on $\mathbb{C}^2$. We will call (G, G_0, H) *generalized 3-Sasaki data.*

Remark 5.1.1. a) For compact G this is called 3-Sasaki data in [DOP20, Definition 12, p. 12].

b) Consider the homogeneous space $M = G/H$. The assumptions above imply that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with $\mathfrak{m} = \mathfrak{sp}(1) \oplus \mathfrak{g}_1$ is a reductive decomposition. We rename the spaces $\mathcal{V} = \mathfrak{sp}(1)$ and $\mathcal{H} = \mathfrak{g}_1$ to express their role as vertical and horizontal subspaces of a 3-(α, δ)-Sasaki manifold via $T_0 M \cong \mathfrak{m}$. For clarity we restate the bracket relations between all these spaces. We have $\mathfrak{g} = \mathfrak{h} \oplus \mathcal{V} \oplus \mathcal{H}$, where $\mathfrak{h}$ and $\mathcal{V}$ are commuting subalgebras. Thus they form the joint subalgebra $\mathfrak{h} \oplus \mathcal{V} = \mathfrak{g}_0 \subset \mathfrak{g}$. The full set of commutator relations is

$$\begin{array}{ccc} [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} & [\mathcal{V}, \mathcal{V}] \subset \mathcal{V} & [\mathfrak{h}, \mathcal{V}] = 0 \\ [\mathfrak{h}, \mathcal{H}] \subset \mathcal{H} & [\mathcal{V}, \mathcal{H}] \subset \mathcal{H} & [\mathcal{H}, \mathcal{H}] \subset \mathcal{V} \oplus \mathfrak{h}. \end{array}$$

In particular, both $\mathcal{V}$ and $\mathcal{H}$ are $\mathfrak{h}$-invariant.

c) Since G/G_0 is a symmetric space there exists a dual symmetric space G^*/G_0 for every generalized 3-Sasaki data (G, G_0, H). The Lie algebras can then be identified as

$$\mathfrak{g}^* = \mathfrak{h} \oplus \mathcal{V} \oplus i\mathcal{H} \subset \mathfrak{g}^{\mathbb{C}}. \tag{5.1}$$

It is then clear that (G^*, G_0, H) is generalized 3-Sasaki data as well. This yields pairs of compact and non-compact generalized 3-Sasaki data. For clarity we will use G for the compact one and G^* for the non-compact one.

d) By [DOP20] any 3-Sasaki data gives rise to a homogeneous 3-Sasaki manifold. They were completely determined in [BGM94] by the fact that they are fiber bundles over the quaternionic Kähler base space G/G_0. The non-compact G^* are thus given as the isometry group of the non-compact quaternionic Kähler symmetric spaces [Bes08, p. 409]. Alltogether, we obtain Table 5.1.1.

$\boldsymbol{G}$	$\boldsymbol{G^*}$	$\boldsymbol{H}$	$\boldsymbol{G_0}$	**dim**	
$\mathrm{Sp}(n+1)$	$\mathrm{Sp}(n,1)$	$\mathrm{Sp}(n)$	$\mathrm{Sp}(n)\mathrm{Sp}(1)$	$4n+3$	$n \geq 0$
$\mathrm{SU}(n+2)$	$\mathrm{SU}(n,2)$	$S(\mathrm{U}(n) \times \mathrm{U}(1))$	$S(\mathrm{U}(n) \times \mathrm{U}(2))$	$4n+3$	$n \geq 1$
$\mathrm{SO}(n+4)$	$\mathrm{SO}(n,4)$	$\mathrm{SO}(n) \times \mathrm{Sp}(1)$	$\mathrm{SO}(n)\mathrm{SO}(4)$	$4n+3$	$n \geq 3$
G_2	G_2^2	$\mathrm{Sp}(1)$	$\mathrm{SO}(4)$	11	
F_4	F_4^{-20}	$\mathrm{Sp}(3)$	$\mathrm{Sp}(3)\mathrm{Sp}(1)$	31	
E_6	E_6^2	$\mathrm{SU}(6)$	$\mathrm{SU}(6)\mathrm{Sp}(1)$	43	
E_7	E_7^{-5}	$\mathrm{Spin}(12)$	$\mathrm{Spin}(12)\mathrm{Sp}(1)$	67	
E_8	E_8^{-24}	E_7	$\mathrm{E}_7\mathrm{Sp}(1)$	115	

Table 5.1.1: Complete table of generalized 3-Sasaki data

Theorem 5.1.2. *Consider some generalized 3-Sasaki data (G, G_0, H) and $0 \neq \alpha, \delta \in \mathbb{R}$. Additionally suppose $\alpha\delta > 0$ if G is compact and $\alpha\delta < 0$ if G is non-compact.*

Let $\kappa(X, Y) = \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$ denote the Killing form on $\mathfrak{g}$. Then define the inner product g on the tangent space $T_pM = T_p(G/H) \cong \mathfrak{m}$ by

$$g|_{\mathcal{V}} = \frac{-\kappa}{4\delta^2(n+2)}, \qquad g|_{\mathcal{H}} = \frac{-\kappa}{8\alpha\delta(n+2)}, \qquad \mathcal{V} \perp \mathcal{H}.$$

Let $\xi_i = \delta\sigma_i \in \mathcal{V} = \mathfrak{sp}(1)$, where the σ_i are the elements of $\mathfrak{sp}(1) = \mathfrak{su}(2)$ given by

$$\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

Define endomorphisms $\varphi_i \in \mathrm{End}_{\mathfrak{h}}(\mathfrak{m})$ *for* $i = 1,2,3$ *by*

$$\varphi_i|_{\mathcal{V}} = \frac{1}{2\delta}\mathrm{ad}\,\xi_i, \qquad \varphi_i|_{\mathcal{H}} = \frac{1}{\delta}\mathrm{ad}\,\xi_i.$$

Together with $\eta_i = g(\xi_i,\cdot)$ *the collection* $(G/H,\varphi_i,\xi_i,\eta_i,g)$ *defines a homogeneous* 3-(α,δ)-*Sasaki structure.*

Before we proceed with the proof, we collect some observations.

Remark 5.1.3. a) In case $\alpha\delta > 0$ the given 3-(α,δ)-Sasaki structure is obtained via a $\mathcal{H}$-homothetic deformation with parameters $a = \frac{1}{\alpha\delta}$, $b = \frac{1}{\delta^2} - \frac{1}{\alpha\delta}$, $c = \frac{1}{\delta}$ from the 3-Sasaki structure given in [DOP20].

b) Consider a homogeneous 3-(α,δ)-Sasaki manifold $(G/H,\varphi_i,\xi_i,\eta_i,g)$ with $\alpha\delta > 0$ such that the isotropy group H is connected, i.e. $G/H \neq \mathbb{RP}^{4n+3}$. Then a $\mathcal{H}$-homothetic deformation with $a = \alpha\delta$ and $c = \delta$ induces a homogeneous 3-Sasaki manifold with $G/H \neq \mathbb{RP}^{4n+3}$ and thus is given by the model in [DOP20]. By definition of $\mathcal{H}$-homothetic deformations the above inverse deformation will restore the original objects. Thus, $(G/H,\varphi_i,\xi_i,\eta_i,g)$ is given by the construction in the theorem.

c) Usually the real representation $\mathfrak{g}_1$ of $\mathfrak{h}$ will be irreducible and will only become reducible when complexified, thus we cannot describe the action of $\mathcal{V} = \mathfrak{sp}(1)$ on $\mathcal{H}$ easily. From the complexified action we still find that the relations $\mathrm{ad}\,\xi_i^2 = -\delta^2\mathrm{id}$ and $\mathrm{ad}\,\xi_i \circ \mathrm{ad}\,\xi_j = \pm\delta\mathrm{ad}\,\xi_k$ when (ijk) is an even, resp. odd permutation of (123) hold on $\mathcal{H}$.

d) The Riemannian metric on $\mathcal{H}$ is a fixed multiple of the Killing form on $\mathfrak{g}$ and thus the projection onto the symmetric orbit space

$$G/H \to G/G_0$$

is a Riemannian submersion. Indeed, this is the canonical submersion obtained in Theorem 4.2.5.

e) The real projective space $\mathbb{R}P^{4n+3} = \frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n)\times\mathbb{Z}_2}$ and its non compact dual $\frac{\mathrm{Sp}(n,1)}{\mathrm{Sp}(n)\times\mathbb{Z}_2}$ also admit 3-(α,δ)-Sasaki structures. They are obtained as the quotient of $S^{4n+3} = \frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n)}$, resp. $\frac{\mathrm{Sp}(n,1)}{\mathrm{Sp}(n)}$ by the action of $\mathbb{Z}_2$ inside the fiber. Since the action is discrete these spaces cannot be discerned in the Lie algebra picture. Note that all relevant tensors are invariant under the $\mathbb{Z}_2$ action and thus local results obtained for $S^{4n+3} = \frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n)}$, resp. $\frac{\mathrm{Sp}(n,1)}{\mathrm{Sp}(n)}$, remain true on $\mathbb{R}P^{4n+3}$ and its non compact dual.

f) Since the metric is a multiple of the Killing form and the Killing form is ad-invariant $[X,\cdot\,]$ will be metric if it preserves $\mathcal{H}$ and $\mathcal{V}$. This is precisely the case if $X \in \mathcal{V}$. For $X \in \mathcal{H}$, we compute with $Y \in \mathcal{V}, Z \in \mathcal{H}$

$$g([X,Y],Z) = \frac{-1}{8\alpha\delta(n+2)}\kappa([X,Y],Z) = \frac{1}{8\alpha\delta(n+2)}\kappa(Y,[X,Z]) = -\frac{\delta}{2\alpha}g(Y,[X,Z]).$$

Thus $[X,\cdot\,] \in \mathfrak{so}(\mathfrak{m})$ if and only if $\delta = 2\alpha$, i.e. we are in the parallel case. This is exactly the condition that our homogeneous space is naturally reductive. This can only occur if $\alpha\delta > 0$, i.e. we are in the positive case.

g) Elements of the Lie algebra $\mathfrak{g}$ are often identified with the respective fundamental vector field induced by the left action of G. The vector fields ξ_i defined on $\mathfrak{m}$ are not fundamental vector fields of the G-action, but left-invariant vector fields. For instance, observe that they can never generate automorphisms, since they rotate the other two Reeb vector fields. In fact, we can identify them as a subgroup of the centralizer of H in G acting by right multiplication.

Proof (of Theorem 5.1.2). If G is compact, $\kappa < 0$. If G is of non-compact type, we have $\kappa|_{\mathcal{V}} < 0$ while $\kappa|_{\mathcal{H}} > 0$ by (5.1) . Thus, in both cases the given metric g is indeed positive definite.

Remark 5.1.3 shows $\mathrm{tr}(\mathrm{ad}^2\xi_i|_{\mathcal{H}}) = \mathrm{tr}(-\delta^2\mathrm{id}|_{\mathcal{H}}) = -4n\delta^2$. On $\mathcal{V}$ we have

$$[\xi_i,[\xi_i,\xi_j]] = \pm 2\delta[\xi_i,\xi_k] = -4\delta^2\xi_j,$$

whenever (ijk) is an even, respectively odd, permutation of (123). Thus, $\mathrm{tr}(\mathrm{ad}^2\xi_i|_{\mathcal{V}}) = -8\delta^2$ and therefore

$$g(\xi_i,\xi_i) = \frac{-\mathrm{tr}(\mathrm{ad}^2\xi_i)}{4\delta^2(n+2)} = \frac{-\mathrm{tr}(\mathrm{ad}^2\xi_i|_{\mathcal{H}}) - \mathrm{tr}(\mathrm{ad}^2\xi_i|_{\mathcal{V}})}{4\delta^2(n+2)} = \frac{8\delta^2 + 4\delta^2 n}{4\delta^2(n+2)} = 1.$$

On the contrary we have $\mathrm{tr}(\mathrm{ad}\,\xi_i \circ \mathrm{ad}\,\xi_j|_{\mathcal{H}}) = \mathrm{tr}(\pm\mathrm{ad}\,\xi_k|_{\mathcal{H}}) = 0$ as its trace on the complexification vanishes. And similar $[\xi_i,[\xi_j,\xi_k]] = 0$ if (ijk) is any permutation of (123) or $[\xi_i,[\xi_j,\xi_k]] = 4\delta^2\xi_j$ if $i = k \neq j$. In any case $\mathrm{tr}(\mathrm{ad}\,\xi_i \circ \mathrm{ad}\,\xi_j) = 0$ and, hence, $g(\xi_i,\xi_j) = 0$ if $i \neq j$.

Next we check that the endomorphisms φ_i are metric almost complex structures on the complement to ξ_i. Note that they vanish on their corresponding ξ_i. Furthermore,

$$\varphi_i^2(\xi_j) = \frac{1}{4\delta^2}[\xi_i,[\xi_i,\xi_j]] = -\xi_j,$$
$$\varphi_i^2|_{\mathcal{H}} = \frac{1}{\delta^2}\mathrm{ad}^2\xi_i|_{\mathcal{H}} = \frac{-\delta^2}{\delta^2}\mathrm{id} = -\mathrm{id}.$$

Since $\mathcal{H}$ and $\mathcal{V}$ are invariant under φ_i we check orthogonality on each component individually. On $\mathcal{H}$ use the associativity of κ to find

$$\kappa(\varphi_i X,\varphi_i Y) = -\kappa(X,\frac{1}{\delta^2}\mathrm{ad}^2\xi_i Y) = \kappa(X,Y)$$

and thus $g(\varphi_i X,\varphi_i Y) = \frac{-\kappa(\varphi_i X,\varphi_i Y)}{8\alpha\delta(n+2)} = \frac{-\kappa(X,Y)}{8\alpha\delta^2(n+2)} = g(X,Y)$. On $\mathcal{V}$ we have

$$g(\varphi_i\xi_j,\varphi_i\xi_{j'}) = g(\frac{1}{2\delta}\mathrm{ad}\,\xi_i(\xi_j),\frac{1}{2\delta}\mathrm{ad}\,\xi_i(\xi_{j'})) = g(\pm\xi_k,\pm\xi_{k'}) = g(\xi_j,\xi_{j'})$$

if (ijk), $(ij'k')$ are according permutations of (123) and the left side vanishes whenever j or j' equals i.

Next we check the compatibility conditions of the 3 almost contact metric structures. Suppose (ijk) is an even permutation of (123) then $\varphi_i\xi_j = \xi_k$ and together with the invariance of

$\mathcal{H}$ under φ_i we conclude $\eta_i \circ \varphi_j = \eta_k$. Further, $\varphi_i\varphi_j|_{\mathcal{H}} = \frac{1}{\delta^2}\operatorname{ad}\xi_i \circ \operatorname{ad}\xi_j|_{\mathcal{H}} = \frac{1}{\delta}\operatorname{ad}\xi_k|_{\mathcal{H}} = \varphi_k|_{\mathcal{H}}$ and on $\mathcal{V}$ we have

$$\begin{aligned}
\varphi_i\varphi_j\xi_i &= \frac{1}{4\delta^2}[\xi_i,[\xi_j,\xi_i]] = \xi_j = \varphi_k\xi_i = \varphi_k\xi_i + \eta_j(\xi_i)\xi_i,\\
\varphi_i\varphi_j\xi_j &= 0 = \xi_i - \xi_i = \varphi_k\xi_j - \eta_j(\xi_j)\xi_i,\\
\varphi_i\varphi_j\xi_k &= \frac{1}{4\delta^2}[\xi_i,[\xi_j,\xi_k]] = \frac{1}{2\delta}[\xi_i,\xi_i] = 0 = \varphi_k\xi_k + \eta_j(\xi_k)\xi_i.
\end{aligned}$$

We have thus shown that the given structure is a homogeneous almost 3-contact metric structure. It remains to show the 3-(α,δ)-Sasaki condition $\mathrm{d}\eta_i = 2\alpha\Phi_i + 2(\alpha-\delta)\eta_j\wedge\eta_k$, for any even permutation (ijk) of (123). We show this case by case. Note that the last summand vanishes whenever either entry is in $\mathcal{H}$. Let $X \in \mathcal{H}$. Then, since $\operatorname{ad}\xi_j X \in \mathcal{H}$,

$$\begin{aligned}
\mathrm{d}\eta_i(\xi_j,X) &= \xi_j(\eta_i(X)) - X(\eta_i(\xi_j)) - \eta_i(\operatorname{ad}\xi_j X) = -\eta_i(\operatorname{ad}\xi_j X) = 0,\\
2\alpha\Phi_i(\xi_j,X) &= 2\alpha g(\xi_j,\varphi_i X) = \frac{2\alpha}{\delta}g(\xi_j,\operatorname{ad}\xi_i X) = 0.
\end{aligned}$$

For $X,Y \in \mathcal{H}$ we use associativity of κ

$$\begin{aligned}
\mathrm{d}\eta_i(X,Y) &= X(\eta_i(Y)) - Y(\eta_i(X)) - \eta_i([X,Y]) = -g(\xi_i,[X,Y])\\
&= \frac{1}{4\delta^2(n+2)}\kappa(\xi_i,[X,Y]) = \frac{-1}{4\delta^2(n+2)}\kappa(\operatorname{ad}\xi_i Y, X),\\
2\alpha\Phi_i(X,Y) &= 2\alpha g(X,\varphi_i Y) = \frac{2\alpha}{\delta}g(X,\operatorname{ad}\xi_i Y) = \frac{-2\alpha}{8\alpha\delta^2(n+2)}\kappa(X,\operatorname{ad}\xi_i Y)\\
&= \frac{-1}{4\delta^2(n+2)}\kappa(X,\operatorname{ad}\xi_i Y).
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\mathrm{d}\eta_i(\xi_j,\xi_k) &= \xi_j(\eta_i(\xi_k)) - \xi_k(\eta_i(\xi_j)) - \eta_i([\xi_j,\xi_k]) = -\eta_i(2\delta\xi_i) = -2\delta,\\
2\alpha\Phi_i(\xi_j,\xi_k) &= 2\alpha g(\xi_j,\varphi_i\xi_k) = -2\alpha g(\xi_j,\xi_j) = -2\alpha,\\
2(\alpha-\delta)\eta_{i+1}\wedge\eta_{i+2}(\xi_j,\xi_k) &= 2(\alpha-\delta) = 2\alpha - 2\delta.
\end{aligned} \tag{5.2}$$

□

5.2 3-(α,δ)-Sasaki over Alekseevsky Spaces

In order to construct the second family of homogeneous 3-(α,δ)-Sasaki manifolds we recall the unified construction of Alekseevsky spaces due to V. Cortés [Cor00]. Let $q \in \mathbb{N}$. Set $V = \mathbb{R}^{3,q}$ the real vector space with signature $(3,q)$. Let $C\ell^0(V)$ denote the even Clifford algebra over V. Depending on $q \mod 4$ there exist exactly one or two inequivalent irreducible $C\ell^0(V)$-modules. Accordingly, let $l \in \mathbb{N}$, if $q \not\equiv 3 \mod 4$, or $l^+, l^- \in \mathbb{N}$, if $q \equiv 3 \mod 4$. Then set

$$\mathfrak{g} = \mathfrak{so}(V) \oplus V \oplus \mathbb{R}D \oplus W,$$

where W is the sum of l equivalent irreducible $C\ell^0(V)$-modules (or the sum of l^+, l^- irreducible $C\ell^0(V)$-modules if there are two inequivalent ones) and D a derivation with eigenvalue decomposition $\mathfrak{so}(V) \oplus V \oplus W$ and respective eigenvalues $(0,1,1/2)$. The action of

$\mathfrak{so}(V)$ on V is given by the standard representation and $\mathfrak{so}(V)$ acts on W via the isomorphism $\mathfrak{so}(V) \cong \mathfrak{spin}(V) \subset C\ell^0(V)$, $e \wedge e' \mapsto -\frac{1}{2}ee'$ where e, e' are orthogonal. V commutes with itself and W. Finally the commutators $[W, W]$ are given by some non-degenerate $\mathfrak{so}(V)$-equivariant map $\Pi\colon \Lambda^2 W \to V$ where $\mathfrak{so}(V)$ acts on W as $\mathfrak{spin}(V)$.

Let $\mathfrak{g} = \mathfrak{so}(V) \oplus V \oplus \mathbb{R}D \oplus W$ with $W = \mathbb{R}^l \otimes W_0$ for an irreducible $C\ell^0(V)$-module W_0. Similar in the case that there exist two inequivalent irreducible $C\ell^0(V)$-modules W_+, W_-, we have $W = (\mathbb{R}^{l^+} \otimes W^+) \oplus (\mathbb{R}^{l^-} \otimes W^-)$.

Remark 5.2.1. Note that Π is unique up to rescaling along the irreducible summands of W [Cor00, Theorem 5]. This rescaling leads to an isomorphism of the Lie algebras $\mathfrak{g}(\Pi)$ and $\mathfrak{g}(\Pi')$ corresponding to two such maps Π and Π'. The isomorphism extends to an isomorphism of the 3-(α,δ)-Sasaki structures defined later on. We will prove this in Theorem 5.2.6. Thus, we will ignore the ambiguity in Π from here on.

Notation. On $V = \mathbb{R}^{3,q}$ fix an ONB $\hat{e}_1, \hat{e}_2, \hat{e}_3, e_1, \dots, e_q$ with signature $(+,+,+,-,\dots,-)$. Then with the identification $\mathfrak{so}(V) \cong \Lambda^2 V$ we also obtain a standard basis of the space $\mathfrak{so}(V)$ given by $\{\hat{e}_i \wedge \hat{e}_j, \hat{e}_i \wedge e_k, e_k \wedge e_l\}_{\substack{i,j=1,2,3\\ k,l=1,\dots,3}}$.

Denote $\sigma_i = 2\hat{e}_k \wedge \hat{e}_j$ for any even permutation (ijk) of (123). Using the identification $\mathrm{End}(V) = V \otimes V^*$ this implies $[\sigma_i, \hat{e}_j] = 2\hat{e}_k$ and $[\sigma_i, \sigma_j] = 2\sigma_k$ where again (ijk) is an even permutation of (123).

We further set $\mathcal{V} = \mathfrak{so}(3) \subset \mathfrak{so}(3,q)$, $\mathcal{H}_0$ the subspace generated by the elements D and $\hat{e}_i + \sigma_i$ and $\mathcal{H}_1$ the subspace generated by $e_1, \dots, e_q \in V$ and $e_i \wedge \hat{e}_j \in \mathfrak{so}(3,q)$.

The 4-dimensional spaces $\mathcal{H}_0$ and $\langle e_l, e_l \wedge \hat{e}_j\rangle \subset \mathcal{H}_1$ will form the quaternionic subspaces inside $\mathfrak{so}(V) \oplus V \oplus \mathbb{R}D \subset \mathfrak{g}$. Accordingly, we show that they have the only commutators with non-trivial $\mathcal{V}$-part.

Lemma 5.2.2. *The only non-trivial projections on $\mathcal{V}$ of commutators are*

$$\pi_{\mathcal{V}}([\sigma_i, \sigma_j]) = \pm 2\sigma_k, \quad \pi_{\mathcal{V}}([D, \hat{e}_i + \sigma_i]) = -\sigma_i, \quad \pi_{\mathcal{V}}([\hat{e}_i + \sigma_i, \hat{e}_j + \sigma_j]) = \mp 2\sigma_k,$$
$$\pi_{\mathcal{V}}([e_l, \hat{e}_i \wedge e_l]) = -\sigma_i, \quad \pi_{\mathcal{V}}([\hat{e}_i \wedge e_l, \hat{e}_j \wedge e_l]) = \pm\frac{1}{2}\sigma_k, \quad \pi_{\mathcal{V}}([w_1, w_2]) = \pi_{\mathcal{V}}(\Pi(w_1, w_2))$$

for all permutations (ijk) of (123) with $\pm$ indicating the sign of the permutation, $l = 1, \dots, q$ and $w_1, w_2 \in W$.

Proof. The full list of commutators of basis vectors is

$$\begin{aligned}
[\sigma_i, \sigma_j] &= \pm 2\sigma_k, & [\sigma_i, D] &= 0, & [\hat{e}_i + \sigma_i, \hat{e}_i \wedge e_l] &= e_l,\\
[\sigma_i, e_l] &= 0, & [\sigma_i, \hat{e}_i \wedge e_l] &= 0, & [\hat{e}_i \wedge e_l, W] &= W,\\
[D, e_l] &= e_l, & [D, \hat{e}_i \wedge e_l] &= 0, & [\hat{e}_i \wedge e_l, \hat{e}_j \wedge e_m] &= 0,\\
[D, W] &= W, & [\sigma_i, \hat{e}_i + \sigma_i] &= 0, & [\hat{e}_i + \sigma_i, W] &= W,\\
[\sigma_i, W] &= W, & [\hat{e}_i + \sigma_i, e_l] &= 0, & [\hat{e}_i \wedge e_l, \hat{e}_i \wedge e_m] &= -e_l \wedge e_m,\\
[e_l, e_m] &= 0, & [e_l, W] &= W, & [e_l, \hat{e}_i \wedge e_m] &= 0,
\end{aligned}$$

$$[\sigma_i, \hat{e}_j + \sigma_j] = \pm 2(\hat{e}_k + \sigma_k), \qquad [\sigma_i, \hat{e}_j \wedge e_l] = \pm 2\hat{e}_k \wedge e_l,$$
$$[D, \hat{e}_i + \sigma_i] = \hat{e}_i = (\hat{e}_i + \sigma_i) - \sigma_i, \qquad [\hat{e}_i + \sigma_i, \hat{e}_j \wedge e_l] = \pm 2\hat{e}_k \wedge e_l,$$
$$[\hat{e}_i \wedge e_l, \hat{e}_j \wedge e_l] = -\hat{e}_i \wedge \hat{e}_j = \pm\frac{1}{2}\sigma_k, \qquad [e_l, \hat{e}_i \wedge e_l] = \hat{e}_i = (\hat{e}_i + \sigma_i) - \sigma_i$$

and finally

$$[\hat{e}_i + \sigma_i, \hat{e}_j + \sigma_j] \;=\; [\sigma_i, \hat{e}_j] - [\sigma_j, \hat{e}_i] + [\sigma_i, \sigma_j] \;=\; \pm 4\hat{e}_k \pm 2\sigma_k = \pm 4(\hat{e}_k + \sigma_k) \mp 2\sigma_k,$$

where (ijk) is a permutation of (123) with $\pm$ indicating the sign of the permutation and $l, m = 1, \ldots, q$ with $l \neq m$. For the commutator $[W, W]$ we have $[w_1, w_2] = \Pi(w_1, w_2) \in V \subset \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{V}$. □

By [Cor00, Proposition 3] the adjoint action of $\mathfrak{g}$ on $\mathfrak{r} = \mathbb{R}D \oplus V \oplus W \subset \mathfrak{g}$ is faithful. Thus, $\mathfrak{g}$ is a subalgebra $\mathfrak{g} \subset \mathrm{der}(\mathfrak{r})$. Set G the subgroup $G \subset \mathrm{Aut}(\mathfrak{r})$ with Lie Algebra $\mathfrak{g}$. Let $\mathfrak{h} = \mathfrak{so}(q) \subset \mathfrak{so}(V) \subset \mathfrak{g}$ and $H \subset G$ the corresponding connected subgroup. Then both G and H are closed subgroups of $\mathrm{Aut}(\mathfrak{r})$. This follows from [Cor00, Corollary 3] and the fact that H is closed in $\mathrm{Spin}_0(V) \subset G$. In particular, G/H is a homogeneous space. We now define the desired negative 3-(α,δ)-Sasaki structure on $M = G/H$.

Theorem 5.2.3. *Let $\alpha, \delta \in \mathbb{R}$ with $\alpha\delta < 0$. Let G, H with Lie algebras $\mathfrak{g}, \mathfrak{h}$ as above. Then $\mathfrak{m} = \mathcal{V} \oplus \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus W$ is a reductive complement to $\mathfrak{h}$ in $\mathfrak{g}$. Set*

$$\xi_1 = \delta\sigma_1, \qquad \xi_2 = \delta\sigma_2, \qquad \xi_3 = \delta\sigma_3.$$

Define the almost complex structures $\varphi_i \colon \mathfrak{m} \to \mathfrak{m}$ on $\mathcal{V}$, $\mathcal{H}_0$, $\mathcal{H}_1$ and W individually. For any permutation (ijk) of (123) with signature $\pm$ we set

$$\varphi_i(\sigma_j) = \pm\sigma_k, \qquad \varphi_i(\sigma_i) = 0, \tag{$\mathcal{V}$}$$
$$\varphi_i(2D) = \hat{e}_i + \sigma_i, \qquad \varphi_i(\hat{e}_i + \sigma_i) = -2D, \qquad \varphi_i(\hat{e}_j + \sigma_j) = \pm(\hat{e}_k + \sigma_k), \tag{$\mathcal{H}_0$}$$
$$\varphi_i(e_l) = 2\hat{e}_i \wedge e_l, \qquad \varphi_i(2\hat{e}_i \wedge e_l) = -e_l, \qquad \varphi_i(\hat{e}_j \wedge e_l) = \pm\hat{e}_k \wedge e_l, \tag{$\mathcal{H}_1$}$$
$$\varphi_i|_W = \rho(\sigma_i), \tag{W}$$

where ρ is the Clifford-multiplication on W.

Define a scalar product $g_{[e]}$ by declaring the following vectors to be an orthonormal basis of $\mathcal{V} \oplus \mathcal{H}_0 \oplus \mathcal{H}_1$:

$$\delta\sigma_i, \sqrt{-4\alpha\delta}D, \sqrt{-\alpha\delta}(\hat{e}_i + \sigma_i), \sqrt{-4\alpha\delta}\,\hat{e}_i \wedge e_l, \sqrt{-\alpha\delta}\,e_l.$$

On W we set the scalar product

$$g_{[e]}|_{W\times W}(s,t) = (-2\alpha\delta)^{-1}b(s,t) := (-2\alpha\delta)^{-1}\langle \hat{e}_i, \Pi(\rho(\hat{e}_j\hat{e}_k)s, t)\rangle,$$

where $\langle , \rangle$ is the scalar product on V and (ijk) is any even permutation of (123). We set W orthogonal to $\mathcal{V} \oplus \mathcal{H}_0 \oplus \mathcal{H}_1$. Set $\eta_i = g(\xi_i, \cdot\,)$ the dual to ξ_i.

Then $(G/H, g, \xi_i, \eta_i, \varphi_i)$ defines a homogeneous 3-(α,δ)-Sasaki manifold.

Proof. We first note that the defined scalar product is positive definite and $\mathrm{Spin}(q)$-invariant. This is clear on $\mathcal{V}\oplus\mathcal{H}_0\oplus\mathcal{H}_1$ and it is shown for b in [Cor00, Theorem 1 and Proposition 9]. Thus, the scalar product extends to an invariant Riemannian metric on G/H. The invariance under H of the ξ_i is obvious. For an invariant 3-a.c.m. structure, it remains to check that the φ_i are invariant as well. $\mathrm{Spin}(q)$ acts trivial on $\mathcal{V}\oplus\mathcal{H}_0$ and on $\mathcal{H}_1$ by its adjoint action on $e_l\in\mathbb{R}^q\subset V$. On W it acts by Clifford multiplication with orthogonal vectors $\mathbb{R}^q$ twice, thus commuting with the Clifford multiplication defining the almost complex structures on W.

The endomorphisms φ_i are compatible with the metric by definition on $\mathcal{V}\oplus\mathcal{H}_0\oplus\mathcal{H}_1$ and by $\mathrm{Spin}(q)\cdot\mathrm{Spin}(3)$-invariance of b on W. Next we check the compatibility conditions of the 3 almost contact structures. Again on $\mathcal{V}\oplus\mathcal{H}_0\oplus\mathcal{H}_1$ this is a direct consequence of the definition and on W we have

$$\rho(\sigma_i)\rho(\sigma_j)w=\hat{e}_k\cdot\hat{e}_j\cdot\hat{e}_i\cdot\hat{e}_k\cdot w=(-1)^2\hat{e}_j\cdot\hat{e}_k\cdot\hat{e}_k\cdot\hat{e}_i\cdot w=-\hat{e}_j\cdot\hat{e}_i\cdot w=\rho(\sigma_k)w.$$

Finally we need to check the defining condition $\mathrm{d}\eta_i=2\alpha\Phi_i+2(\alpha-\delta)\eta_j\wedge\eta_k$. By bilinearity it suffices to check it for any pair of two basis vectors individually. On $\mathcal{V}\times\mathcal{V}$ this is exactly the same computation as in the 3-(α,δ)-Sasaki structure over symmetric bases (compare (5.2)). Apart from $\mathcal{V}\times\mathcal{V}$ the equation reduces to $\mathrm{d}\eta_i=2\alpha\Phi_i$. Note that the left hand side reduces to checking the commutators. From Lemma 5.2.2 and the definition of the φ_i we see that both sides vanish for all mixed terms regarding the decomposition $\mathcal{V}\oplus\mathcal{H}_0\oplus\mathcal{H}_1\oplus W$ of the tangent space. Similarly on $\mathcal{H}_1$ if the index l of $\hat{e}_i\wedge e_l$, respectively e_l, is not the same both sides vanish. On $\mathcal{H}_0\times\mathcal{H}_0$ we compute

$$\begin{aligned}\mathrm{d}\eta_i(D,\hat{e}_i+\sigma_i)&=-\eta_i([D,\hat{e}_i+\sigma_i])=-\eta_i(-\sigma_i)=\frac{1}{\delta}g(\delta\sigma_i,\delta\sigma_i)=\frac{1}{\delta},\\ 2\alpha\Phi_i(D,\hat{e}_i+\sigma_i)&=2\alpha g(D,\varphi_i(\hat{e}_i+\sigma_i))=\frac{2\alpha}{-2\alpha\delta}g(\sqrt{-4\alpha\delta}\,D,-\sqrt{-\alpha\delta}\,2D)=\frac{1}{\delta}.\end{aligned}$$

In similar fashion for the remaining pairs in $\mathcal{H}_0\times\mathcal{H}_0$ and on $\mathcal{H}_1\times\mathcal{H}_1$ we have

$$\begin{aligned}\frac{2}{\delta}&=-\eta_k(2\sigma_k)=\mathrm{d}\eta_k(\hat{e}_i+\sigma_i,\hat{e}_j+\sigma_j)=2\alpha\Phi_k(\hat{e}_i+\sigma_i,\hat{e}_j+\sigma_j)=\frac{2}{\delta},\\ \frac{1}{\delta}&=-\eta_i(-\sigma_i)=\mathrm{d}\eta_i(e_l,\hat{e}_i\wedge e_l)=2\alpha\Phi_i(e_l,\hat{e}_i\wedge e_l)=\frac{1}{\delta},\\ \frac{1}{2\delta}&=-\eta_k(-\frac{1}{2}\sigma_k)=\mathrm{d}\eta_k(\hat{e}_i\wedge e_l,\hat{e}_j\wedge e_l)=2\alpha\Phi_k(\hat{e}_i\wedge e_l,\hat{e}_j\wedge e_l)=\frac{1}{2\delta}\end{aligned}$$

for any even permutation (ijk) of (123). Finally, we look at $W\times W$. Let $w_1,w_2\in W$ and suppose $\Pi(w_1,w_2)=\sum_{r=1}^q a_re_r+\sum_{s=1}^3\hat{a}_s\hat{e}_s$. Then

$$\begin{aligned}\mathrm{d}\eta_i([w_1,w_2])&=-\eta_i(\Pi(w_1,w_2))=-\eta_i\left(\sum_{r=1}^q a_re_r+\sum_{s=1}^3\hat{a}_s\hat{e}_s\right)\\ &=-\eta_i\left(\sum_{s=1}^3\hat{a}_s((\hat{e}_s+\sigma_s)-\sigma_s)\right)=\frac{\hat{a}_i}{\delta}\end{aligned}$$

and

$$\begin{aligned}2\alpha\Phi_i(w_1,w_2) = 2\alpha g(w_1,\varphi_i w_2) &= \frac{2\alpha}{-2\alpha\delta}\left\langle \hat{e}_i, \Pi\left(w_1, \hat{e}_j\hat{e}_k\hat{e}_j\hat{e}_k w_2\right)\right\rangle \\ &= \frac{(-1)^3 2\alpha}{-2\alpha\delta}\langle \hat{e}_i, \Pi(w_1,w_2)\rangle = \frac{1}{\delta}\left\langle \hat{e}_i, \sum_{r=1}^{q} a_r e_r + \sum_{s=1}^{3}\hat{a}_s\hat{e}_s\right\rangle = \frac{\hat{a}_i}{\delta}. \qquad \square\end{aligned}$$

Remark 5.2.4. We try to motivate the construction. In [Cor00] Cortés shows that $\mathfrak{so}(V)\oplus V\oplus\mathbb{R}D$ is isomorphic to a subalgebra of $\mathfrak{so}(4,q+1)\cong\Lambda^2(V\oplus\langle e^+,e^-\rangle)$, with e^+,e^- unit length vectors of corresponding signature, given by the inclusion

$$\mathfrak{so}(V)\mapsto\Lambda^2 V,\qquad V\mapsto V\wedge(e^+-e^-),\qquad D\mapsto e^+\wedge e^-.$$

Now φ_i is modeled on $\mathfrak{so}(V)\oplus V\oplus\mathbb{R}D$ after the adjoint action with $\hat{e}_j\wedge\hat{e}_k+\hat{e}_i\wedge e^+\in\mathfrak{so}(3)_+\subset\mathfrak{so}(3)_+\oplus\mathfrak{so}(3)_-=\mathfrak{so}(4)$ in the known $\mathrm{SO}(4,q+1)/\mathrm{SO}(q+1)\mathrm{SO}(3)$ setting. However, this does not exist as an inner derivative in $\mathfrak{g}$ unlike in the (semi-)simple case. Therefore we prescribe the behavior manually.

We owe the reader a proof that the defined 3-(α,δ)-Sasaki structure is indeed independent of the choice of $\Pi\colon\Lambda^2W\to V$.

Remark 5.2.5. Any two suitable maps Π, Π' can be related by a $\Gamma\in\mathrm{Gl}(l)$ where l is the number of irreducible spin-invariant submodules W_0 of W. By [Cor00, Theorem 4] for any such map Π one can find a basis v_i of $\mathbb{R}^l$ such that Π decomposes as a sum of maps $\Lambda^2(v_i\otimes W_0)\to V$. Then [Cor00, Theorem 5] shows that any such map $\Lambda^2W_0\to V$ is unique up to scaling. This can be combined into a linear map $\hat{\Gamma}$ that induces an isomorphism $\Gamma\colon\mathfrak{g}(\Pi)\to\mathfrak{g}(\Pi')$ as long as we assume Π,Π' to be non-degenerate. The result in the case $l\equiv 3 \mod 4$ is the same with $\Gamma=\Gamma^+\times\Gamma^-$. Whenever we need to clarify the Lie algebra structure we will denote $\mathfrak{g}(\Pi)$.

Theorem 5.2.6. *Suppose $\Pi,\Pi'\colon\Lambda^2W\to V$ are two non-degenerate $\mathfrak{spin}(V)$-equivariant maps. Let $\hat{\Gamma}\colon\mathbb{R}^l\to\mathbb{R}^l$ be an automorphism, such that $\Pi'=\Gamma^*\Pi$ for the induced map $\Gamma\colon\mathfrak{g}(\Pi)\to\mathfrak{g}(\Pi')$, $\Gamma\big|_{\mathfrak{g}(V)}=\mathrm{id}_{\mathfrak{g}(V)}$, $\Gamma\big|_W=\hat{\Gamma}\otimes\mathrm{id}_{W_0}$. Then $\Gamma\colon\mathfrak{g}(\Pi)\to\mathfrak{g}(\Pi')$ induces an isomorphism of the homogeneous 3-(α,δ)-Sasaki manifolds.*

Proof. First note that $\Gamma\colon\mathfrak{g}(\Pi)\to\mathfrak{g}(\Pi')$ is an isomorphism of Lie algebras. Suppose $X\in\mathfrak{g}(V)$ and $v_1\otimes w_1, v_2\otimes w_2\in W$. Then

$$\begin{aligned}\Gamma([X,v_1\otimes w_1]_{\mathfrak{g}(\Pi)}) &= \Gamma(v_1\otimes\rho(X)w_1) = \hat{\Gamma}v_1\otimes\rho(X)w_1 = [\Gamma(X),\Gamma(v_1\otimes w_1)]_{\mathfrak{g}(\Pi')}\\ \Gamma([v_1\otimes w_1,v_2\otimes w_2]_{\mathfrak{g}(\Pi)}) &= \Pi(v_1\otimes w_1,v_2\otimes w_2) = (\Gamma^{-1})^*\Pi'(v_1\otimes w_1,v_2\otimes w_2)\\ &= \Pi'(\Gamma(v_1\otimes w_1),\Gamma(v_2\otimes w_2)) = [\Gamma(v_1\otimes w_1),\Gamma(v_2\otimes w_2)]_{\mathfrak{g}(\Pi')}.\end{aligned}$$

The map Γ fixes the isotropy so that the homogeneous spaces are diffeomorphic. Next we show that this linear isomorphism induces an isometry. It suffices to show that on W.

Denote the Riemannian metrics defined by Π, respectively Π' by g_Π, respectively $g_{\Pi'}$. Then we have

$$\begin{aligned} g_\Pi(v_1\otimes w_1, v_2\otimes w_2) &= \langle \hat{e}_1, \Pi(v_1\otimes w_1, \hat{e}_2\hat{e}_3(v_2\otimes w_2))\rangle \\ &= \langle (\hat{e}_1, (\Gamma^{-1})^*\Pi'(v_1\otimes w_1, v_2\otimes \hat{e}_2\hat{e}_3 w_2)\rangle \\ &= \langle (\hat{e}_1, \Pi'(\hat{\Gamma}v_1\otimes w_1, \hat{\Gamma}v_2\otimes \hat{e}_2\hat{e}_3 w_2)\rangle \\ &= (\Gamma^{-1})^* g_{\Pi'}(v_1\otimes w_1, v_2\otimes w_2). \end{aligned}$$

Finally we need to compare the 3 almost complex structures on W. But the φ act on W_0, while Γ acts on $\mathbb{R}^l$, so they commute. □

5.3 Examples

We begin with an example of the construction over a symmetric Wolf space.

Example 5.3.1. Our first example is the Aloff-Wallach space $W^{1,1} = \mathrm{SU}(3)/S^1 = G/H$. In this case the isotropy algebra $\mathfrak{h}$ inside $\mathfrak{g} = \mathfrak{su}(3)$ is the 1-dimensional space generated by

$$h = \begin{bmatrix} -i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 2i \end{bmatrix}.$$

We locate the space $\mathfrak{sp}(1) = \mathfrak{su}(2) \subset \mathfrak{su}(3)$ as the upper left 2-by-2 block. One checks that this is a splitting of $\mathfrak{su}(3)$ as necessary. Then for $\alpha, \delta > 0$ the Reeb vector fields are given by

$$\xi_1 = \delta \begin{bmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \xi_2 = \delta \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \xi_3 = \delta \begin{bmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

On the horizontal subspace we choose a basis vector

$$\tilde{e}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Then we normalize it $g(\tilde{e}_1, \tilde{e}_1) = \frac{-6\mathrm{tr}(\tilde{e}_1\cdot\tilde{e}_1)}{24\alpha\delta} = \frac{1}{2\alpha\delta}$, i.e. $e_1 = \sqrt{2\alpha\delta}\cdot\tilde{e}_1$, and generate an adapted basis:

$$e_2 = \sqrt{2\alpha\delta}\begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \qquad e_3 = \sqrt{2\alpha\delta}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \qquad e_4 = \sqrt{2\alpha\delta}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix}.$$

Example 5.3.2. Next consider the dual negative 3-(α,δ)-Sasaki space $\mathrm{SU}(2,1)/S^1$. We realize the Lie algebra

$$\mathfrak{su}(2,1) = \mathfrak{g}^* = \mathfrak{h} \oplus \mathfrak{sp}(1) \oplus i\mathcal{H} \subset \mathfrak{su}(3)^{\mathbb{C}}$$

as described in (5.1). Then as for the Aloff-Wallach space we identify the 1-dimensional isotropy $\mathfrak{h}$ generated by

$$h = \begin{bmatrix} -i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 2i \end{bmatrix}.$$

Analogously the Reeb vector fields are given by

$$\xi_1 = \delta \begin{bmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \xi_2 = \delta \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \xi_3 = \delta \begin{bmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

On the horizontal subspace we choose

$$\tilde{e}_1^* = i\tilde{e}_1 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} \subset i\mathcal{H}.$$

We have

$$g(\tilde{e}_1^*, \tilde{e}_1^*) = \frac{-i^2}{24\alpha\delta}\kappa(\tilde{e}_1, \tilde{e}_1) = -\frac{1}{2\alpha\delta}.$$

Thus we find an adapted base of $\mathrm{SU}(2,1)/S^1$ by $e_1^* = i\sqrt{-2\alpha\delta}\tilde{e}_1$ and

$$e_2^* = \sqrt{-2\alpha\delta} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \; e_3^* = \sqrt{-2\alpha\delta} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{bmatrix}, \; e_4^* = \sqrt{-2\alpha\delta} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

We now discuss the lowest dimensional example $\hat{\mathcal{T}}(1)$ of a negative homogeneous 3-(α,δ)-Sasaki manifold, fibering over the Alekseevsky space $\mathcal{T}(1)$, not obtained by the construction over symmetric spaces.

Remark 5.3.3. The first new example arising from the construction over Alekseevsky spaces appears only in dimension 19. Table 5.3.1 lists all homogeneous 3-(α,δ)-Sasaki manifolds obtained by Theorem 5.2.3 up to dimension 19 and, if existing, the isomorphic ones appearing in Table 5.1.1, i.e. obtained by Theorem 5.1.2 over non-compact symmetric spaces. The list gets more intricate with higher dimension, in particular, there appear two inequivalent even Clifford modules for $q = 3$ beginning in $\dim 27$ and for $q \geq 4$ we have $\dim W_q > 4$. Further, observe that the symmetric base cases $\mathrm{SU}(2,1)/\mathrm{U}(1)$, $\mathrm{G}_2^{(2)}/\mathrm{SO}(3)$ are not obtained by this construction.

We now give more concrete descriptions of the $C\ell^0(3,q)$-modules W_q for $q = 0,1,2$. Note that there are choices to be made though these lead to isomorphisms of the modules since all these modules are unique. Let $\mathbb{R}^{3,q} = \langle e_{\hat{1}}, e_{\hat{2}}, e_{\hat{3}}, e_1, \ldots, e_q \rangle$, where $e_{\hat{i}}$ have signature $+1$ while e_i have signature -1. Then we have $C\ell^0(3,0) = \mathbb{H}$, $C\ell^0(3,1) = \mathcal{M}_2(\mathbb{C})$, $C\ell^0(3,2) = \mathcal{M}_4(\mathbb{R})$ realized as follows. Table 5.3.2 lists the cases $q = 0$ and $q = 1$, while Table 5.3.3 is devoted to the case $q = 2$.

Dimension	parameters	$\mathfrak{g}$	$\mathfrak{h}$	alternative description
7	$q=0,\ l=0$	$\mathfrak{so}(3)\oplus\mathbb{R}^3\oplus\mathbb{R}D$	0	$\mathrm{Sp}(1,1)/\mathrm{Sp}(1)$
11	$q=0,\ l=1$	$\mathfrak{so}(3)\oplus\mathbb{R}^3\oplus\mathbb{R}D\oplus W_0$	0	$\mathrm{Sp}(2,1)/\mathrm{Sp}(2)$
	$q=1,\ l=0$	$\mathfrak{so}(3,1)\oplus\mathbb{R}^{3,1}\oplus\mathbb{R}D$	0	$\mathrm{SU}(2,2)/\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))$
15	$q=0,\ l=2$	$\mathfrak{so}(3)\oplus\mathbb{R}^3\oplus\mathbb{R}D\oplus 2W_0$	0	$\mathrm{Sp}(3,1)/\mathrm{Sp}(3)$
	$q=1,\ l=1$	$\mathfrak{so}(3,1)\oplus\mathbb{R}^{3,1}\oplus\mathbb{R}D\oplus W_1$	0	$\mathrm{SU}(3,2)/\mathrm{S}(\mathrm{U}(3)\times\mathrm{U}(1))$
	$q=2,\ l=0$	$\mathfrak{so}(3,2)\oplus\mathbb{R}^{3,2}\oplus\mathbb{R}D$	$\mathfrak{so}(2)$	$\mathrm{SO}_0(3,4)/\mathrm{SO}(3)\times\mathrm{SO}(3)$
19	$q=0,\ l=3$	$\mathfrak{so}(3)\oplus\mathbb{R}^3\oplus\mathbb{R}D\oplus 3W_0$	0	$\mathrm{Sp}(4,1)/\mathrm{Sp}(4)$
	$q=1,\ l=2$	$\mathfrak{so}(3,1)\oplus\mathbb{R}^{3,1}\oplus\mathbb{R}D\oplus 2W_1$	0	$\mathrm{SU}(4,2)/\mathrm{S}(\mathrm{U}(4)\times\mathrm{U}(1))$
	$q=2,\ l=1$	$\mathfrak{so}(3,2)\oplus\mathbb{R}^{3,2}\oplus\mathbb{R}D\oplus W_2$	$\mathfrak{so}(2)$	$\hat{\mathcal{T}}(1)$ non symmetric base
	$q=3,\ l=0$	$\mathfrak{so}(3,3)\oplus\mathbb{R}^{3,3}\oplus\mathbb{R}D$	$\mathfrak{so}(3)$	$\mathrm{SO}_0(4,4)/\mathrm{SO}(4)\times\mathrm{SO}(3)$

Table 5.3.1: 3-(α,δ)-Sasaki manifolds over Alekseevsky spaces of dim $\leq$ 19, see Remark 5.3.3.

$q=0$	$\deg 0:$	$[\,1\,]$	$+1$
	$\deg 2:$	$e_{\hat{1}\hat{2}}=[\,i\,],\quad e_{\hat{2}\hat{3}}=[\,j\,],\quad e_{\hat{3}\hat{1}}=[\,k\,]$	-1
$q=1$	$\deg 0:$	$\begin{bmatrix} 1 & \\ & 1\end{bmatrix}$	$+1$
	$\deg 2:$	$e_{\hat{1}\hat{2}}=\begin{bmatrix} i & \\ & -i\end{bmatrix},\quad e_{\hat{2}\hat{3}}=\begin{bmatrix} & i \\ i & \end{bmatrix},\quad e_{\hat{3}\hat{1}}=\begin{bmatrix} & -1 \\ 1 & \end{bmatrix}$	-1
		$e_{\hat{3}1}=\begin{bmatrix} 1 & \\ & -1\end{bmatrix},\quad e_{\hat{1}1}=\begin{bmatrix} & 1 \\ 1 & \end{bmatrix},\quad e_{\hat{2}1}=\begin{bmatrix} & i \\ -i & \end{bmatrix}$	$+1$
	$\deg 4:$	$e_{\hat{1}\hat{2}\hat{3}1}=\begin{bmatrix} i & \\ & i\end{bmatrix}$	-1

Table 5.3.2: Choice of $C\ell^0(3,q)$-representations for $q=0$ and $q=1$

The notation is as follow: We denote elements $e_{ij}=e_ie_j\in C\ell^0(V)$ and analogous for the action of elements in $C\ell^0(V)$ of higher degree. The last line denotes the square of elements in the respective row, which are invariant of choices unlike the matrices itself.

With this we can find the map $\Pi\colon\Lambda^2W_2\to\mathbb{R}^{3,2}$.

Theorem 5.3.4. *Set $W_2\cong\mathbb{R}^4=\langle E_1,E_2,E_3,E_4\rangle$ with the $\mathfrak{spin}(3,2)$-module structure above. Then the map $\Pi\colon\Lambda^2W_2\to\mathbb{R}^{3,2}$ given by*

$$\Pi(E_1\wedge E_2)=-\hat{e}_3-e_1,\quad \Pi(E_1\wedge E_3)=-\hat{e}_2,\quad \Pi(E_1\wedge E_4)=-\hat{e}_1+e_2,$$
$$\Pi(E_2\wedge E_3)=\hat{e}_1+e_2,\quad \Pi(E_4\wedge E_2)=\hat{e}_2,\quad \Pi(E_3\wedge E_4)=\hat{e}_3-e_1$$

is $\mathfrak{spin}(3,2)$-invariant and non-degenerate.

$\deg 0:$	$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$	$+1$
$\deg 2:$	$e_{\hat{1}\hat{2}} = \begin{bmatrix} & -1 & & \\ 1 & & & \\ & & & 1 \\ & & -1 & \end{bmatrix}$, $e_{\hat{2}\hat{3}} = \begin{bmatrix} & & 1 & -1 \\ & & -1 & \\ 1 & & & \\ & 1 & & \end{bmatrix}$, $e_{\hat{3}\hat{1}} = \begin{bmatrix} & -1 & & \\ & & & -1 \\ 1 & & & \\ & 1 & & \end{bmatrix}$, $e_{12} = \begin{bmatrix} & & 1 & \\ & & & -1 \\ -1 & & & \\ & 1 & & \end{bmatrix}$	-1
	$e_{\hat{3}1} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$, $e_{\hat{1}1} = \begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix}$, $e_{\hat{2}1} = \begin{bmatrix} & & 1 & -1 \\ & & & \\ -1 & & & \\ & 1 & & \end{bmatrix}$, $e_{\hat{3}2} = \begin{bmatrix} & & & 1 \\ & & & -1 \\ 1 & & & \\ & -1 & & \end{bmatrix}$	$+1$
	$e_{\hat{1}2} = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}$, $e_{\hat{2}2} = \begin{bmatrix} & -1 & & \\ -1 & & & \\ & & & -1 \\ & & -1 & \end{bmatrix}$	
$\deg 4:$	$e_{\hat{1}\hat{2}\hat{3}1} = \begin{bmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{bmatrix}$, $e_{\hat{1}\hat{2}\hat{3}2} = \begin{bmatrix} & & 1 & 1 \\ & & -1 & \\ -1 & & & \\ & & & \end{bmatrix}$	-1
	$e_{\hat{1}\hat{2}12} = \begin{bmatrix} & & 1 & 1 \\ & 1 & & \\ 1 & & & \\ & & & \end{bmatrix}$, $e_{\hat{2}\hat{3}12} = \begin{bmatrix} & -1 & & \\ -1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix}$, $e_{\hat{3}\hat{1}12} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}$	$+1$

Table 5.3.3: Choice of $C\ell^0(3,q)$-representations for $q=2$

Recall that the action of $\mathfrak{so}(3,q)$ on the $C\ell^0(3,q)$-module W, and thereby $W\wedge W$, is given by the isomorphism $\mathrm{ad}^{-1}\colon \mathfrak{so}(3,q)\to\mathfrak{spin}(3,q)=C\ell^0(3,q)$, $e_i\wedge e_j\mapsto -\frac{1}{2}e_{ij}$, where $i,j\in\{\hat{1},\hat{2},\hat{3},1,\dots,q\}$.

Proof. Non-degeneracy is clear. It suffices to check the invariance on a generating set of $C\ell^0(3,2)$. One such set is given by $e_{\hat{1}\hat{2}}, e_{\hat{2}\hat{3}}, e_{\hat{1}1}, e_{12}$. Each of these map certain subspaces of W_2 onto one another, hence their action on the exterior product of these subspaces vanishes. This yields the identities

$$\begin{aligned}
-2\hat{e}_1\wedge\hat{e}_2(\hat{e}_3\pm e_1) &= 0 = \Pi(e_{\hat{1}\hat{2}}(E_1\wedge E_2)) = \Pi(e_{\hat{1}\hat{2}}(E_3\wedge E_4)),\\
-2\hat{e}_2\wedge\hat{e}_3(\hat{e}_1\pm e_2) &= 0 = \Pi(e_{\hat{2}\hat{3}}(E_1\wedge E_4)) = \Pi(e_{\hat{2}\hat{3}}(E_2\wedge E_3)),\\
-2\hat{e}_1\wedge e_1(\hat{e}_2) &= 0 = \Pi(e_{\hat{1}1}(E_1\wedge E_3)) = \Pi(e_{\hat{1}1}(E_4\wedge E_2)),\\
-2e_1\wedge e_2(\hat{e}_2) &= 0 = \Pi(e_{12}(E_1\wedge E_3)) = \Pi(e_{12}(E_4\wedge E_2)).
\end{aligned}$$

The rest is just more computations. We start with $e_{\hat{1}\hat{2}}$:

$$\begin{aligned}
\Pi(e_{\hat{1}\hat{2}}(E_1\wedge E_3)) &= \Pi(E_2\wedge E_3)+\Pi(E_1\wedge -E_4) = \hat{e}_1+e_2+\hat{e}_1-e_2 = 2\hat{e}_1\\
&= -2\hat{e}_1\wedge\hat{e}_2(-\hat{e}_2) = -2\hat{e}_1\wedge\hat{e}_2(\Pi(E_1\wedge E_3)),\\
\Pi(e_{\hat{1}\hat{2}}(E_1\wedge E_4)) &= \Pi(E_2\wedge E_4)+\Pi(E_1\wedge E_3) = -\hat{e}_2-\hat{e}_2 = -2\hat{e}_2\\
&= -2\hat{e}_1\wedge\hat{e}_2(-\hat{e}_1+e_2) = -2\hat{e}_1\wedge\hat{e}_2(\Pi(E_1\wedge E_4)),\\
\Pi(e_{\hat{1}\hat{2}}(E_2\wedge E_3)) &= \Pi(-E_1\wedge E_3)+\Pi(E_2\wedge -E_4) = \hat{e}_2+\hat{e}_2 = 2\hat{e}_2\\
&= -2\hat{e}_1\wedge\hat{e}_2(\hat{e}_1+e_2) = -2\hat{e}_1\wedge\hat{e}_2(\Pi(E_2\wedge E_3)),\\
\Pi(e_{\hat{1}\hat{2}}(E_4\wedge E_2)) &= \Pi(E_3\wedge E_2)+\Pi(E_4\wedge -E_1) = -\hat{e}_1-e_2-\hat{e}_1+e_2 = -2\hat{e}_1\\
&= -2\hat{e}_1\wedge\hat{e}_2(\hat{e}_2) = -2\hat{e}_1\wedge\hat{e}_2(\Pi(E_4\wedge E_2)).
\end{aligned}$$

For $e_{\hat{2}\hat{3}}$:

$$\begin{aligned}
\Pi(e_{\hat{2}\hat{3}}(E_1\wedge E_2)) &= \Pi(E_4\wedge E_2)+\Pi(E_1\wedge -E_3) = \hat{e}_2+\hat{e}_2 = 2\hat{e}_2\\
&= -2\hat{e}_2\wedge\hat{e}_3(-\hat{e}_3-e_1) = -2\hat{e}_2\wedge\hat{e}_3(\Pi(E_1\wedge E_2)),\\
\Pi(e_{\hat{2}\hat{3}}(E_1\wedge E_3)) &= \Pi(E_4\wedge E_3)+\Pi(E_1\wedge E_2) = -\hat{e}_3+e_1-\hat{e}_3-e_1 = -2\hat{e}_3\\
&= -2\hat{e}_2\wedge\hat{e}_3(-\hat{e}_2) = -2\hat{e}_2\wedge\hat{e}_3(\Pi(E_1\wedge E_3)),\\
\Pi(e_{\hat{2}\hat{3}}(E_4\wedge E_2)) &= \Pi(-E_1\wedge E_2)+\Pi(E_4\wedge -E_3) = \hat{e}_3+e_1+\hat{e}_3-e_1 = 2\hat{e}_3\\
&= -2\hat{e}_2\wedge\hat{e}_3(\hat{e}_2) = -2\hat{e}_2\wedge\hat{e}_3(\Pi(E_4\wedge E_2)),\\
\Pi(e_{\hat{2}\hat{3}}(E_2\wedge E_4)) &= \Pi(E_3\wedge E_2)+\Pi(E_3\wedge -E_1) = -\hat{e}_2-\hat{e}_2 = -2\hat{e}_2\\
&= -2\hat{e}_2\wedge\hat{e}_3(\hat{e}_3-e_1) = -2\hat{e}_2\wedge\hat{e}_3(\Pi(E_3\wedge E_4)).
\end{aligned}$$

For $e_{\hat{1}1}$:

$$\begin{aligned}
\Pi(e_{\hat{1}1}(E_1\wedge E_2)) &= \Pi(E_3\wedge E_2)+\Pi(E_1\wedge E_4) = -\hat{e}_1-e_2-\hat{e}_1+e_2 = -2\hat{e}_1\\
&= -2\hat{e}_1\wedge e_1(-\hat{e}_3-e_1) = -2\hat{e}_1\wedge e_1(\Pi(E_1\wedge E_2)),\\
\Pi(e_{\hat{1}1}(E_1\wedge E_4)) &= \Pi(E_3\wedge E_4)+\Pi(E_1\wedge E_2) = \hat{e}_3-e_1-\hat{e}_3-e_1 = -2e_1\\
&= -2\hat{e}_1\wedge e_1(-\hat{e}_1+e_2) = -2\hat{e}_1\wedge e_1(\Pi(E_1\wedge E_4)),\\
\Pi(e_{\hat{1}1}(E_2\wedge E_3)) &= \Pi(E_4\wedge E_3)+\Pi(E_2\wedge E_1) = -\hat{e}_3+e_1+\hat{e}_3+e_1 = 2e_1\\
&= -2\hat{e}_1\wedge e_1(\hat{e}_1+e_2) = -2\hat{e}_1\wedge e_1(\Pi(E_2\wedge E_3)),\\
\Pi(e_{\hat{1}1}(E_3\wedge E_4)) &= \Pi(E_1\wedge E_4)+\Pi(E_3\wedge E_2) = -\hat{e}_1+e_2-\hat{e}_1-e_2 = -2\hat{e}_1\\
&= -2\hat{e}_1\wedge e_1(\hat{e}_3-e_1) = -2\hat{e}_1\wedge e_1(\Pi(E_3\wedge E_4)).
\end{aligned}$$

And finally for e_{12}:

$$\begin{aligned}
\Pi(e_{12}(E_1\wedge E_2)) &= \Pi(-E_3\wedge E_2)+\Pi(E_1\wedge E_4) = \hat{e}_1+e_2-\hat{e}_1+e_2 = 2e_2\\
&= -2e_1\wedge e_2(-\hat{e}_3-e_1) = -2e_1\wedge e_2(\Pi(E_1\wedge E_2)),\\
\Pi(e_{12}(E_1\wedge E_4)) &= \Pi(-E_3\wedge E_4)+\Pi(E_1\wedge -E_2) = -\hat{e}_3+e_1+\hat{e}_3+e_1 = 2e_1\\
&= -2e_1\wedge e_2(-\hat{e}_1+e_2) = -2e_1\wedge e_2(\Pi(E_1\wedge E_4)),\\
\Pi(e_{12}(E_2\wedge E_3)) &= \Pi(E_4\wedge E_3)+\Pi(E_2\wedge E_1) = -\hat{e}_3+e_1+\hat{e}_3+e_1 = 2e_1\\
&= -2e_1\wedge e_2(\hat{e}_1+e_2) = -2e_1\wedge e_2(\Pi(E_2\wedge E_3)),\\
\Pi(e_{12}(E_3\wedge E_4)) &= \Pi(E_1\wedge E_4)+\Pi(E_3\wedge -E_2) = -\hat{e}_1+e_2+\hat{e}_1+e_2 = 2e_2\\
&= -2e_1\wedge e_2(\hat{e}_3-e_1) = -2e_1\wedge e_2(\Pi(E_3\wedge E_4)).
\end{aligned}$$

□

To display the algebra $\mathfrak{g}$ with $q=2$, $l=1$ corresponding to $\hat{\mathcal{T}}(1)=G/H$ we use the inclusion $\mathfrak{g}\subset\mathfrak{der}(\mathfrak{r})\subset\mathfrak{gl}(\mathbb{R}^{3,2}\oplus\mathbb{R}D\oplus W_2)$. We give these elements as matrices with respect to the basis $\hat{e}_1,\hat{e}_2,\hat{e}_3,e_1,e_2,D,E_1,E_2,E_3,E_4$ of $\mathfrak{r}$.

Recall $\mathfrak{g} = \mathfrak{so}(3,2) \oplus \mathbb{R}^{3,2} \oplus \mathbb{R}D \oplus W_2$. We begin with $\mathfrak{so}(3,2)$:

$$2\hat{e}_1 \wedge \hat{e}_2 = \begin{bmatrix} 0 & 2 & & & & 0 & & & & \\ -2 & 0 & & & & 0 & & & & \\ & & 0 & & & 0 & & & & \\ & & & 0 & & 0 & & & & \\ & & & & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & & & 0 & 1 \\ & & & & & 0 & & & -1 & 0 \\ & & & & & 0 & 0 & 1 & & \\ & & & & & 0 & -1 & 0 & & \end{bmatrix}, \quad 2\hat{e}_3 \wedge \hat{e}_1 = \begin{bmatrix} 0 & & -2 & & & 0 & & & & \\ & 0 & & & & 0 & & & & \\ 2 & & 0 & & & 0 & & & & \\ & & & 0 & & 0 & & & & \\ & & & & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & & & 1 & 0 \\ & & & & & 0 & & & 0 & 1 \\ & & & & & 0 & -1 & 0 & & \\ & & & & & 0 & 0 & -1 & & \end{bmatrix},$$

$$2\hat{e}_2 \wedge \hat{e}_3 = \begin{bmatrix} 0 & & & & & 0 & & & & \\ & 0 & 2 & & & 0 & & & & \\ & -2 & 0 & & & 0 & & & & \\ & & & 0 & & 0 & & & & \\ & & & & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & & & 0 & 1 \\ & & & & & 0 & & & -1 & 0 \\ & & & & & 0 & 0 & 1 & & \\ & & & & & 0 & -1 & 0 & & \end{bmatrix},$$

$$2\hat{e}_1 \wedge e_1 = \begin{bmatrix} 0 & & & -2 & & 0 & & & & \\ & 0 & & & & 0 & & & & \\ & & 0 & & & 0 & & & & \\ -2 & & & 0 & & 0 & & & & \\ & & & & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & & & -1 & 0 \\ & & & & & 0 & & & 0 & -1 \\ & & & & & 0 & -1 & 0 & & \\ & & & & & 0 & 0 & -1 & & \end{bmatrix}, \qquad 2\hat{e}_1 \wedge e_2 = \begin{bmatrix} 0 & & & & -2 & 0 & & & & \\ & 0 & & & & 0 & & & & \\ & & 0 & & & 0 & & & & \\ & & & 0 & & 0 & & & & \\ -2 & & & & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & 1 & 0 & & \\ & & & & & 0 & 0 & -1 & & \\ & & & & & 0 & & & -1 & 0 \\ & & & & & 0 & & & 0 & 1 \end{bmatrix},$$

$$2\hat{e}_2 \wedge e_1 = \begin{bmatrix} 0 & & & & & 0 & & & & \\ & 0 & & -2 & & 0 & & & & \\ & & 0 & & & 0 & & & & \\ & -2 & & 0 & & 0 & & & & \\ & & & & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & & & 0 & 1 \\ & & & & & 0 & & & -1 & 0 \\ & & & & & 0 & 0 & -1 & & \\ & & & & & 0 & 1 & 0 & & \end{bmatrix}, \qquad 2\hat{e}_2 \wedge e_2 = \begin{bmatrix} 0 & & & & & 0 & & & & \\ & 0 & & & -2 & 0 & & & & \\ & & 0 & & & 0 & & & & \\ & & & 0 & & 0 & & & & \\ & -2 & & & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 1 & & \\ & & & & & 0 & 1 & 0 & & \\ & & & & & 0 & & & 0 & 1 \\ & & & & & 0 & & & 1 & 0 \end{bmatrix},$$

$$2\hat{e}_3 \wedge e_1 = \begin{bmatrix} 0 & & & & & 0 & & & & \\ & 0 & & & & 0 & & & & \\ & & 0 & -2 & & 0 & & & & \\ & & -2 & 0 & & 0 & & & & \\ & & & & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & -1 & 0 & & \\ & & & & & 0 & 0 & -1 & & \\ & & & & & 0 & & & 1 & 0 \\ & & & & & 0 & & & 0 & 1 \end{bmatrix}, \qquad 2\hat{e}_3 \wedge e_2 = \begin{bmatrix} 0 & & & & & 0 & & & & \\ & 0 & & & & 0 & & & & \\ & & 0 & & -2 & 0 & & & & \\ & & & 0 & & 0 & & & & \\ & & -2 & & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & & & -1 & 0 \\ & & & & & 0 & & & 0 & 1 \\ & & & & & 0 & -1 & 0 & & \\ & & & & & 0 & 0 & 1 & & \end{bmatrix},$$

$$2e_1 \wedge e_2 = \begin{bmatrix} 0 & & & & & 0 & & & & \\ & 0 & & & & 0 & & & & \\ & & 0 & & & 0 & & & & \\ & & & 0 & -2 & 0 & & & & \\ & & & 2 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & & & -1 & 0 \\ & & & & & 0 & & & 0 & 1 \\ & & & & & 0 & 1 & 0 & & \\ & & & & & 0 & 0 & -1 & & \end{bmatrix}.$$

Recall that $2e_1 \wedge e_2$ generates the isotropy algebra $\mathfrak{h}$.

Next we describe the element $2D$:

$$2D = \begin{bmatrix} 2 & & & & & 0 & & & & \\ & 2 & & & & 0 & & & & \\ & & 2 & & & 0 & & & & \\ & & & 2 & & 0 & & & & \\ & & & & 2 & 0 & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & 1 & & & \\ & & & & & 0 & & 1 & & \\ & & & & & 0 & & & 1 & \\ & & & & & 0 & & & & 1 \end{bmatrix}.$$

The generators of $V = \mathbb{R}^{3,2}$ are:

$$\hat{e}_1 = \begin{bmatrix} & & & & & -1 & & & & \\ & & & & & 0 & & & & \\ & & & & & 0 & & & & \\ & & & & & 0 & & & & \\ & & & & & 0 & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & & & & \\ & & & & & 0 & & & & \\ & & & & & 0 & & & & \\ & & & & & 0 & & & & \end{bmatrix}, \quad \hat{e}_2 = \begin{bmatrix} & & & & & 0 & & & & \\ & & & & & -1 & & & & \\ & & & & & 0 & & & & \\ & & & & & 0 & & & & \\ & & & & & 0 & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & & & & \\ & & & & & 0 & & & & \\ & & & & & 0 & & & & \\ & & & & & 0 & & & & \end{bmatrix}, \quad \hat{e}_3 = \begin{bmatrix} & & & & & 0 & & & & \\ & & & & & 0 & & & & \\ & & & & & -1 & & & & \\ & & & & & 0 & & & & \\ & & & & & 0 & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & & & & \\ & & & & & 0 & & & & \\ & & & & & 0 & & & & \\ & & & & & 0 & & & & \end{bmatrix},$$

$$e_1 = \begin{bmatrix} & & & & & 0 & & & & \\ & & & & & 0 & & & & \\ & & & & & 0 & & & & \\ & & & & & -1 & & & & \\ & & & & & 0 & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & & & & \\ & & & & & 0 & & & & \\ & & & & & 0 & & & & \\ & & & & & 0 & & & & \end{bmatrix}, \quad e_2 = \begin{bmatrix} & & & & & 0 & & & & \\ & & & & & 0 & & & & \\ & & & & & 0 & & & & \\ & & & & & 0 & & & & \\ & & & & & -1 & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & & & & \\ & & & & & 0 & & & & \\ & & & & & 0 & & & & \\ & & & & & 0 & & & & \end{bmatrix}.$$

Finally we have the 4 basis elements of W_2:

$$2E_1 = \begin{bmatrix} & & & & & 0 & & & & -2 \\ & & & & & 0 & & & -2 & \\ & & & & & 0 & & -2 & & \\ & & & & & 0 & 2 & & & \\ & & & & & 0 & & & & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & -1 & & & & \\ & & & & & 0 & & & & \\ & & & & & 0 & & & & \\ & & & & & 0 & & & & \end{bmatrix}, \quad 2E_2 = \begin{bmatrix} & & & & & 0 & & & 2 & \\ & & & & & 0 & & & & -2 \\ & & & & & 0 & 2 & & & \\ & & & & & 0 & 2 & & & \\ & & & & & 0 & & & 2 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & & & & \\ & & & & & -1 & & & & \\ & & & & & 0 & & & & \end{bmatrix}$$

$$2E_3 = \begin{bmatrix} & & & & & 0 & & -2 & & \\ & & & & & 0 & 2 & & & \\ & & & & & 0 & & & & 2 \\ & & & & & 0 & & & & -2 \\ & & & & & 0 & & -2 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & & & & \\ & & & & & 0 & & & & \\ & & & & & -1 & & & & \\ & & & & & 0 & & & & \end{bmatrix}, \quad 2E_4 = \begin{bmatrix} & & & & & 0 & 2 & & & \\ & & & & & 0 & & 2 & & \\ & & & & & 0 & & & -2 & \\ & & & & & 0 & & & 2 & \\ & & & & & 0 & -2 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & & & & \\ & & & & & 0 & & & & \\ & & & & & 0 & & & & \\ & & & & & -1 & & & & \end{bmatrix}$$

5.4 Nomizu Maps

Recall that by the Nomizu Theorem 2.1.3 invariant connections on reductive homogeneous spaces $M = G/H$ are in bijective correspondence with isotropy equivariant maps $\Lambda\colon \mathfrak{m}\times\mathfrak{m} \to \mathfrak{m}$. This section is meant to apply the Nomizu Theorem to both the canonical and the Levi-Civita connection.

For the following theorem we need a similar statement to Lemma 3.3.4 for two fundamental vector fields. Note that even in the case when $X, Y \in \mathfrak{g}$ are horizontal in the origin they fail to be horizontal in other points. Yet we have

Lemma 5.4.1. *Let G act by automorphisms on a 3-(α,δ)-Sasaki manifold M from the left, $X, Y \in \mathfrak{g}$ with fundamental vector fields $\hat{X}, \hat{Y} \in TM$. Then*

$$\widehat{[X,Y]}_{\mathcal{V}} = -\sum_{i=1}^{3} \big(2\alpha\Phi_i(\hat{X},\hat{Y}) + 2(\alpha-\delta)\eta_j \wedge \eta_k(\hat{X},\hat{Y})\big)\xi_i.$$

Proof. Since G is a group of automorphisms we have $L_{\hat{X}}\eta_i = 0$ so

$$\begin{aligned} d\eta_i(\hat{X},\hat{Y}) &= \hat{X}(\eta_i(\hat{Y})) - \hat{Y}(\eta_i(\hat{X})) - \eta_i([\hat{X},\hat{Y}]) \\ &= \eta_i([\hat{X},\hat{Y}]) - \eta_i([\hat{Y},\hat{X}]) - \eta_i([\hat{X},\hat{Y}]) = \eta_i([\hat{X},\hat{Y}]) = -\eta_i(\widehat{[X,Y]}). \end{aligned}$$

Using $\widehat{[X,Y]}_{\mathcal{V}} = \sum_{i=1}^3 \eta_i(\widehat{[X,Y]})\xi_i$ and (3.7) yields the result. □

Theorem 5.4.2. *The Nomizu map for the canonical connection $\Lambda^\nabla\colon \mathfrak{m}\times\mathfrak{m}\to\mathfrak{m}$ of a homogeneous 3-(α,δ)-Sasaki manifold with $\mathfrak{m} = \mathcal{V}\oplus\mathcal{H}$ is given by*

$$\Lambda^\nabla_X Y = \begin{cases} \Lambda^{g_N}_X Y & X,Y\in\mathcal{H} \\ \frac{\beta}{2\delta}[X,Y] & X,Y\in\mathcal{V} \\ [X,Y] - 2\alpha\sum_{i=1}^3 \eta_i(X)\varphi_i Y & X\in\mathcal{V}, Y\in\mathcal{H} \\ 0 & X\in\mathcal{H}, Y\in\mathcal{V}, \end{cases}$$

where $\Lambda^{g_N}\colon \mathcal{H}\times\mathcal{H}\to\mathcal{H}$ is the Nomizu map of the Levi-Civita connection on the homogeneous base of the canonical submersion.

Proof. We first prove that the torsion of Λ^∇ given by (2.2) agrees with the canonical torsion

$$T^\nabla = 2\alpha\sum_{i=1}^3 \eta_i\wedge\Phi_i^{\mathcal{H}} + 2(\delta-4\alpha)\eta_{123}.$$

We begin with the case $X,Y\in\mathcal{H}$. Evaluating Lemma 5.4.1 in the origin $T_0M\cong\mathfrak{m}$ we obtain $[X,Y]_{\mathcal{V}} = -2\alpha\sum_{i=1}^3\Phi_i(X,Y)\xi_i$. Thus,

$$\begin{aligned} \Lambda^\nabla_X Y - \Lambda^\nabla_Y X - [X,Y] &= \Lambda^{g_N}_X Y - \Lambda^{g_N}_Y X - [X,Y]_{\mathcal{H}} - [X,Y]_{\mathcal{V}} \\ &= T^{g_N}(X,Y) + 2\alpha\sum_{i=1}^3\Phi_i(X,Y)\xi_i = 2\alpha\sum_{i=1}^3\Phi_i(X,Y)\xi_i = T^\nabla(X,Y) \end{aligned}$$

as the torsion $T^{g_N} = 0$ of the Levi-Civita connection on the base vanishes. Suppose now that $X = \xi_i$ and $Y\in\mathcal{H}$. Then

$$\Lambda^\nabla_X Y - \Lambda^\nabla_Y X - [X,Y] = [X,Y] - 2\alpha\varphi_i Y - [X,Y] = -2\alpha\varphi_i Y = T^\nabla(X,Y).$$

Finally if both $X = \xi_i, Y = \xi_j\in\mathcal{V}$ and (ijk) is an even permutation of (123), we have

$$\begin{aligned} \Lambda^\nabla_X Y - \Lambda^\nabla_Y X - [X,Y] &= \frac{\beta}{2\delta}[X,Y] - \frac{\beta}{2\delta}[Y,X] - [X,Y] = (2-\frac{4\alpha}{\delta}-1)[\xi_i,\xi_j] \\ &= (1-\frac{4\alpha}{\delta})2\delta\xi_k = T^\nabla(X,Y). \end{aligned}$$

We further need to verify that $\Lambda^\nabla_X\in\mathfrak{so}(\mathfrak{m})$ for all $X\in\mathfrak{m}$, that is $g(\Lambda^\nabla_X Y,Z)+g(Y,\Lambda^\nabla_X Z) = 0$ for all $Y,Z\in\mathfrak{m}$. Suppose $X\in\mathcal{H}$ and $Y,Z\in\mathcal{H}$. Then $\Lambda^\nabla_X = \Lambda^{g_N}_X$ and $g|_{\mathcal{H}} = g_N$ thus Λ^∇_X is metric as $\Lambda^{g_N}_X$ is. If $Y\in\mathcal{V}$ and $Z\in\mathcal{H}$ we find that $\Lambda^\nabla_X Y = 0$ by definition while $\Lambda^\nabla_X Z\in\mathcal{H}$ is orthogonal to Y. Analogously, if $Y,Z\in\mathcal{V}$ the Nomizu map Λ^∇_X acts trivially on both

sides. Now suppose $X \in \mathcal{V}$. Then $X_0 = \sum a_i(\xi_i)_0$. Observe that the vector field $\tilde{X} = \sum a_i\xi_i$ is left-invariant and Killing by [AD20, Corollary 2.3.1]. Thus

$$g(L_XY,Z) + g(Y,L_XZ) = X(g(Y,Z)) = \tilde{X}(g(Y,Z)) = g(L_{\tilde{X}}Y,Z) + g(Y,L_{\tilde{X}}Z) = 0$$

and $L_X \in \mathfrak{so}(\mathfrak{m})$. In particular, if $Y,Z \in \mathcal{V}$ then $\Lambda^{\nabla}_X Y = \frac{\beta}{2\delta}L_XY$ is metric. If $Y \in \mathcal{H}$ we have

$$g(\Lambda^{\nabla}_X Y,Z) = g(L_XY - 2\alpha\sum \eta_i(X)\varphi_iY,Z) = -g(Y,L_XZ - 2\alpha\sum \eta_i(X)\varphi_iZ).$$

If $Z \in \mathcal{H}$ the right hand side is just $-g(Y,\Lambda^{\nabla}_X Z)$ by definition while for $Z \in \mathcal{V}$

$$L_XZ - 2\alpha\sum \eta_i(X)\varphi_iZ \in \mathcal{V}$$

is perpendicular to Y. Hence, $g(\Lambda^{\nabla}_X Y,Z) = 0 = -g(Y,\Lambda^{\nabla}_X Z)$. □

Remark 5.4.3. For symmetric spaces the Levi-Civita connection corresponds to the trivial Nomizu map $\Lambda^{g_N} = 0$. The Nomizu map of the Alekseevsky base is given in [Cor00, Lemma 5, p. 35].

In the case of a positive homogeneous 3-(α,δ)-Sasaki manifold or its non-compact sibling the Nomizu map Λ^{∇} simplifies drastically.

Proposition 5.4.4. *The canonical connection ∇ of a homogeneous 3-(α,δ)-Sasaki manifold over a Wolf space or its non-compact dual corresponds to the map*

$$\Lambda^{\nabla}_X = \begin{cases} 0, & X \in \mathcal{H} \\ \frac{\beta}{2\delta}\operatorname{ad} X, & X \in \mathcal{V}. \end{cases}$$

Proof. In the case of a Riemannian symmetric space the Levi–Civita connection agrees with the Ambrose–Singer connection. Thus, $\Lambda^{g_N} \equiv 0$. Now let $X \in \mathcal{V}$ and $Y \in \mathcal{H}$. Then

$$2\alpha\sum_{i=1}^{3}\eta_i(X)\varphi_i|_{\mathcal{H}} = \frac{2\alpha}{\delta}\sum_{i=1}^{3}\eta_i(X)\operatorname{ad}\xi_i = \frac{2\alpha}{\delta}\operatorname{ad} X.$$

It follows $\Lambda^{\nabla}_X Y = (1-\frac{2\alpha}{\delta})[X,Y] = \frac{\beta}{2\delta}[X,Y]$. □

Remark 5.4.5. As noted in Remark 5.1.3 the homogeneous 3-(α,δ)-Sasaki space is naturally reductive if and only if $\beta = 0$. Then the Ambrose-Singer connection is metric. In this case Proposition 5.4.4 shows that the canonical and Ambrose-Singer connections agree.

Proposition 5.4.6. *The Levi–Civita connection corresponds to the map*

$$\Lambda^{g}_X Y = \begin{cases} \frac{1}{2}[X,Y]_{\mathfrak{m}} & X,Y \in \mathcal{V} \text{ or } X,Y \in \mathcal{H} \\ (1-\frac{\alpha}{\delta})[X,Y] & X \in \mathcal{V}, Y \in \mathcal{H} \\ \frac{\alpha}{\delta}[X,Y] & X \in \mathcal{H}, Y \in \mathcal{V}. \end{cases}$$

Proof. Note that the correspondence is $\Lambda^{\nabla}_X Y = \nabla_X Y - [X,Y]$, for $X,Y \in \mathfrak{m}$, and the canonical connection is given by $\nabla = \nabla^g + \frac{1}{2}T$ where the canonical torsion is given by (3.12), or equivalently (3.13). Thus we have $\Lambda^g_X Y = \nabla^g_X Y - [X,Y] = \nabla_X Y - [X,Y] - \frac{1}{2}T(X,Y) = \Lambda^{\nabla}_X Y - \frac{1}{2}T(X,Y)$. Again we look at each case individually. Let $X,Y \in \mathcal{H}$. Then

$$\begin{aligned}
\frac{1}{2}T(X,Y) &= \alpha \sum_{i=1}^{3} \Phi_i(X,Y)\xi_i = \alpha \sum_{i=1}^{3} g(X,\varphi_i Y)\xi_i \\
&= \frac{\alpha}{\delta}\sum_{i=1}^{3} g(X,[\xi_i,Y])\xi_i = \frac{\alpha}{8\alpha\delta^2(n+2)}\sum_{i=1}^{3}\kappa(X,[Y,\xi_i])\xi_i \\
&= \frac{1}{8\delta^2(n+2)}\sum_{i=1}^{3}\kappa([X,Y],\xi_i)\xi_i \\
&= -\frac{1}{2}\sum_{i=1}^{3} g([X,Y]_{\mathfrak{m}},\xi_i)\xi_i = -\frac{1}{2}[X,Y]_{\mathfrak{m}},
\end{aligned}$$

where we have used that $\kappa(\mathfrak{h},\mathcal{V}) = 0$ and $[X,Y]_{\mathfrak{m}} \in \mathcal{V} = \operatorname{span}\{\xi_1,\xi_2,\xi_3\}$. Thus

$$\Lambda^g_X Y = \Lambda^{\nabla}_X Y - \frac{1}{2}T(X,Y) = \frac{1}{2}[X,Y]_{\mathfrak{m}}$$

For vertical vectors $X = \xi_i, Y = \xi_j$, (ijk) an even permutation of (123), we find

$$\begin{aligned}
\Lambda^g_{\xi_i}\xi_j &= \Lambda^{\nabla}_{\xi_i}\xi_j - (\delta - 4\alpha)\eta_{123}(\xi_i,\xi_j,\cdot) = \frac{\beta}{2\delta}[\xi_i,\xi_j] - (\delta-4\alpha)\xi_k \\
&= \beta\xi_k + (\delta - 4\alpha)\xi_k = \delta\xi_k = [\xi_i,\xi_j]
\end{aligned}$$

and by linearity for arbitrary $X,Y \in \mathcal{V}$. Let $X \in \mathcal{V}, Y \in \mathcal{H}$ then

$$\begin{aligned}
T(X,Y) &= 2\alpha\sum_{i=1}^{3}\eta_i \wedge \Phi^{\mathcal{H}}_i(X,Y,\cdot) = 2\alpha\sum_{i=1}^{3}\eta_i(X)\Phi^{\mathcal{H}}_i(Y,\cdot) \\
&= -2\alpha\sum_{i=1}^{3}\eta_i(X)\varphi_i Y = -\frac{2\alpha}{\delta}\left[\sum_{i=1}^{3}\eta_i(X)\xi_i, Y\right] = -\frac{2\alpha}{\delta}[X,Y]
\end{aligned}$$

and thus

$$\Lambda^g_X Y = \Lambda^{\nabla}_X Y - \frac{1}{2}T(X,Y) = (1 - \frac{2\alpha}{\delta})[X,Y] + \frac{\alpha}{\delta}[X,Y] = (1-\frac{\alpha}{\delta})[X,Y].$$

For the final expression $X \in \mathcal{H}, Y \in \mathcal{V}$ use the above identity for T with $X \leftrightarrow Y$. Then

$$\Lambda^g_X Y = \Lambda^{\nabla}_X Y - \frac{1}{2}T(X,Y) = \frac{1}{2}T(Y,X) = \frac{\alpha}{\delta}[X,Y]. \qquad \square$$

The Nomizu operators in Proposition 5.4.4 allow to immediately compute the curvature operators of Example 5.3.1 and Example 5.3.2 via

$$R(X,Y,Z,V) = g(\Lambda^{\nabla}_X\Lambda^{\nabla}_Y Z - \Lambda^{\nabla}_Y\Lambda^{\nabla}_X Z - \Lambda^{\nabla}_{[X,Y]_{\mathfrak{m}}}Z - \operatorname{ad}([X,Y]_{\mathfrak{h}})Z, V),$$

compare [KN96, p.191]. The discussion in the next chapter is more structural, but these examples should provide a comparison for the identities developed there.

Example 5.4.7. We give the canonical curvature operator of Example 5.3.1 and Example 5.3.2 with respect to the basis defined therein. The choice of basis vectors in a dual manner yields a matrix representation of $\mathcal{R}$ that is identical for both examples, yet in one case $\alpha\delta > 0$ in the other $\alpha\delta < 0$.

$$\mathcal{R}^{\nabla} = \begin{bmatrix} \alpha\beta+6\alpha\delta & \alpha\beta-6\alpha\delta & 0 & 0 & 0 & 0 & 0 & 2\alpha\beta & 0 \\ \alpha\beta-6\alpha\delta & \alpha\beta+6\alpha\delta & 0 & 0 & 0 & 0 & 0 & 2\alpha\beta & 0 \\ 0 & 0 & \alpha\beta & \alpha\beta & 0 & 0 & 0 & 0 & 2\alpha\beta \\ 0 & 0 & \alpha\beta & \alpha\beta & 0 & 0 & 0 & 0 & 2\alpha\beta \\ 0 & 0 & 0 & 0 & \alpha\beta & \alpha\beta & 2\alpha\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha\beta & \alpha\beta & 2\alpha\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\alpha\beta & 2\alpha\beta & 4\alpha\beta & 0 & 0 \\ 2\alpha\beta & 2\alpha\beta & 0 & 0 & 0 & 0 & 0 & 4\alpha\beta & 0 \\ 0 & 0 & 2\alpha\beta & 2\alpha\beta & 0 & 0 & 0 & 0 & 4\alpha\beta \end{bmatrix}$$

Here we omitted the trivial part $\mathcal{H} \otimes \mathcal{V}$ and considered the induced basis on $\Lambda^2\mathcal{H} \oplus \Lambda^2\mathcal{V} \subset \Lambda^2 M$ with the following order:

$$e_1 \wedge e_2,\ e_3 \wedge e_4,\ e_1 \wedge e_3,\ e_4 \wedge e_2,\ e_1 \wedge e_4,\ e_2 \wedge e_3,\ \xi_1 \wedge \xi_2,\ \xi_2 \wedge \xi_3,\ \xi_3 \wedge \xi_1.$$

6 Curvature

We take a closer look at the canonical curvature operator $\mathcal{R} = \mathcal{R}^{\nabla}$, i.e. the curvature operator of the canonical connection, of any 3-(α, δ)-Sasaki manifold. Through the canonical submersion we want to compare it with the Riemannian curvature of the quaternion Kähler base space. Meanwhile, on the total space we compare it to the Riemannian curvature operator via (2.4).

6.1 Decomposition of the Canonical Curvature

Recall that the canonical submersion $\pi\colon M \to N$ tells us that the Levi-Civita connection on N essentially is a projection of the canonical connection on M. Using this relation we can understand the missing purely horizontal part of $\mathcal{R}$, compare (3.21) in Proposition 3.3.16.

Theorem 6.1.1. *The canonical curvature on $\Lambda^2\mathcal{H} \otimes \Lambda^2\mathcal{H}$ is given by*

$$R(\overline{X}, \overline{Y}, \overline{Z}, \overline{V}) = R^{g_N}(X, Y, Z, V) + 4\alpha^2 \sum_{i=1}^{3} \Phi_i(\overline{X}, \overline{Y})\Phi_i(\overline{Z}, \overline{V}),$$

where $X, Y, Z, V \in TN$ with horizontal lifts $\overline{X}, \overline{Y}, \overline{Z}, \overline{V} \in \mathcal{H}$.

Proof. We note that by $\nabla^{g_N}_X Y = \pi_*(\nabla_{\overline{X}}\overline{Y})$ the vector field $\nabla_{\overline{X}}\overline{Y} \in \mathcal{H}$ is π-related to $\nabla^{g_N}_X Y$ and thus $\nabla_{\overline{X}}\overline{Y} = \overline{\pi_*(\nabla_{\overline{X}}\overline{Y})}$. We obtain

$$\begin{aligned}
g_N(\nabla^{g_N}_X \nabla^{g_N}_Y Z, V) &= g_N(\nabla^{g_N}_X \pi_*(\nabla_{\overline{Y}}\overline{Z}), V) = g(\nabla_{\overline{X}}\overline{\pi_*(\nabla_{\overline{Y}}\overline{Z})}, \overline{V}) = g(\nabla_{\overline{X}}\nabla_{\overline{Y}}\overline{Z}, \overline{V}),\\
g_N(\nabla^{g_N}_{[X,Y]} Z, V) &= g_N(\pi_*\nabla_{\overline{[X,Y]}}\overline{Z}, V) = g(\nabla_{[\overline{X},\overline{Y}]}\overline{Z} - \nabla_{[\overline{X},\overline{Y}]_{\mathcal{V}}}\overline{Z}, \overline{V})\\
&= g(\nabla_{[\overline{X},\overline{Y}]}\overline{Z}, \overline{V}) + 4\alpha^2 \sum_{i=1}^{3} \Phi_i(\overline{X}, \overline{Y})\Phi_i(\overline{Z}, \overline{V})
\end{aligned}$$

where we have used Lemma 3.3.4 and Lemma 4.2.4. Plugging these identities into the curvature we find

$$\begin{aligned}
R^{g_N}(X, Y, Z, V) &= g_N(\nabla^{g_N}_X \nabla^{g_N}_Y Z - \nabla^{g_N}_Y \nabla^{g_N}_X Z - \nabla^{g_N}_{[X,Y]} Z, V)\\
&= g(\nabla_{\overline{X}}\nabla_{\overline{Y}}\overline{Z} - \nabla_{\overline{Y}}\nabla_{\overline{X}}\overline{Z} - \nabla_{[\overline{X},\overline{Y}]}\overline{Z}, \overline{V}) - 4\alpha^2 \sum_{i=1}^{3} \Phi_i(\overline{X}, \overline{Y})\Phi_i(\overline{Z}, \overline{V})\\
&= R(\overline{X}, \overline{Y}, \overline{Z}, \overline{V}) - 4\alpha^2 \sum_{i=1}^{3} \Phi_i(\overline{X}, \overline{Y})\Phi_i(\overline{Z}, \overline{V}).
\end{aligned}$$

□

We now want to look at the canonical curvature as a curvature operator and consider its eigenvalues and definiteness. Recall that the canonical curvature operator $\mathcal{R}\colon \Lambda^2 M \to \Lambda^2 M$ defines a symmetric operator. Rewriting (3.18) as operator identities we obtain

$$\langle \mathcal{R}(X \wedge \xi_i), Y \wedge \xi_j \rangle = \langle \mathcal{R}(X \wedge Y), Z \wedge \xi_i \rangle = \langle \mathcal{R}(\xi_i \wedge \xi_j), \xi_k \wedge X \rangle = 0$$

showing that the canonical curvature operator vanishes on $\mathcal{V} \wedge \mathcal{H}$. Thus, it can be considered as a symmetric operator $\mathcal{R}\colon \Lambda^2\mathcal{V} \oplus \Lambda^2\mathcal{H} \to \Lambda^2\mathcal{V} \oplus \Lambda^2\mathcal{H}$. It does not restrict to the individual summands but we can accomplish a more nuanced decomposition.

Proposition 6.1.2. *The curvature operator $\mathcal{R}$ can be decomposed as*

$$\mathcal{R} = \alpha\beta\mathcal{R}_\perp + \mathcal{R}_{\text{par}}$$

where $\mathcal{R}_\perp$ is defined by

$$\mathcal{R}_\perp \coloneqq \overset{i,j,k}{\mathfrak{S}} (\Phi_i - \xi_{jk}) \otimes (\Phi_i - \xi_{jk}) \tag{6.1}$$

and $\mathcal{R}_{\text{par}}$ is trivial outside of the horizontal part, i.e. $\mathcal{R}_{\text{par}}|_{(\Lambda^2\mathcal{H})^\perp} = 0$.

Proof. (3.19) and (3.20) in terms of the curvature operator read

$$\langle \mathcal{R}(\xi_i \wedge \xi_j), \xi_k \wedge \xi_l \rangle = 4\alpha\beta \sum_{\mu=1}^{3} \Phi_\mu(\xi_i, \xi_j)\Phi_\mu(\xi_k, \xi_l), \tag{6.2}$$

$$\langle \mathcal{R}(\xi_i \wedge \xi_j), X \wedge Y \rangle = 2\alpha\beta \sum_{\mu=1}^{3} \Phi_\mu(\xi_i, \xi_j)\Phi_\mu(X, Y). \tag{6.3}$$

We observe that the right hand sides of (6.2) and (6.3) are of the form $C\sum_{i=1}^{3} \Phi_i \otimes \Phi_i$, where the coefficient C is either 4 or 2. Using the symmetry of $\mathcal{R}$ and setting the coefficient 1 in the purely horizontal part we obtain

$$\begin{aligned}
&\sum_{i=1}^{3} \big(4(\Phi_i|_{\Lambda^2\mathcal{V}} \otimes \Phi_i|_{\Lambda^2\mathcal{V}}) + 2(\Phi_i|_{\Lambda^2\mathcal{H}} \otimes \Phi_i|_{\Lambda^2\mathcal{V}}) \\
&\qquad\qquad + 2(\Phi_i|_{\Lambda^2\mathcal{V}} \otimes \Phi_i|_{\Lambda^2\mathcal{H}}) + (\Phi_i|_{\Lambda^2\mathcal{H}} \otimes \Phi_i|_{\Lambda^2\mathcal{H}})\big) \\
&= \overset{i,j,k}{\mathfrak{S}} (\Phi_i - \xi_{jk}) \otimes (\Phi_i - \xi_{jk}) =: \mathcal{R}_\perp.
\end{aligned}$$

By construction $\mathcal{R}_{\text{par}} \coloneqq \mathcal{R} - \alpha\beta\mathcal{R}_\perp$ is trivial on $(\Lambda^2\mathcal{H})^\perp$. □

By definition we immediately get

Lemma 6.1.3. *$\mathcal{R}_\perp$ has the single eigenvalue $2(n+2)$ with eigenspace generated by $\Phi_i - \xi_{jk}$ for $i = 1, 2, 3$.*

Proof. Immediately by definition $\Phi_i - \xi_{jk}$ are the only non-vanishing eigenvectors with eigenvalue $|\Phi_i - \xi_{jk}|^2$. From the identities

$$\langle \Phi_i, \Phi_j \rangle = (2n+1)\delta_{ij}, \qquad \langle \Phi_i, \xi_{jk} \rangle = \Phi_i(\xi_j, \xi_k) = -\epsilon_{ijk}, \tag{6.4}$$

we obtain the eigenvalue $2(n+2)$. □

Lemma 6.1.4. *The kernel of $\mathcal{R}_{\text{par}}$ contains the space generated by $Z \wedge \varphi_i Z + \varphi_j Z \wedge \varphi_k Z$, where $Z \in \mathcal{H}$.*

Proof. For $X, Y, Z \in \mathcal{H}$ compare (3.21) with (6.1) to obtain

$$\begin{aligned}\langle \mathcal{R}(X \wedge Y), Z \wedge \varphi_i Z + \varphi_j Z \wedge \varphi_k Z \rangle &= 2\alpha\beta\Phi_i(X,Y)\|Z\|^2 \\ &= \langle \alpha\beta\mathcal{R}_\perp(X \wedge Y), Z \wedge \varphi_i Z + \varphi_j Z \wedge \varphi_k Z \rangle\end{aligned}$$

for any even permutation (ijk) of (123). Thus, $Z \wedge \varphi_i Z + \varphi_j Z \wedge \varphi_k Z \in \ker \mathcal{R}_{\text{par}}$. □

6.2 Special Eigenvalues of $\mathcal{R}$ and $\mathcal{R}^g$

In this section we will determine eigenvalues of the canonical and Riemannian curvature operators, that only depend on α and δ. In particular, this will allow us to characterize the Einstein condition in terms of such eigenvalues.

Theorem 6.2.1. *The curvature operator $\mathcal{R}$ of the canonical connection of any 3-(α, δ)-Sasaki manifold $(M, \varphi_i, \xi_i, \eta_i, g)$ admits the following six orthogonal eigenforms:*

- $\Phi_i - \xi_{jk} = \Phi_i^{\mathcal{H}} - 2\xi_{jk}$, *$i = 1, 2, 3$, eigenform with eigenvalue $2\alpha\beta(n+2)$,*
- $\Phi_i + (n+1)\xi_{jk} = \Phi_i^{\mathcal{H}} + n\xi_{jk}$, *$i = 1, 2, 3$, eigenform with vanishing eigenvalue.*

Proof. We observe that $\Phi_i^{\mathcal{H}} = -\frac{1}{4}\sum_{r=4}^{4n+3} e_r \wedge \varphi_i e_r + \varphi_j e_r \wedge \varphi_k e_r$ for an adapted basis e_r. Lemma 6.1.4 then shows $\Phi_i \mathcal{H} \in \ker \mathcal{R}_{\text{par}}$. Also $\xi_{jk} \in \Lambda^2 \mathcal{V} \subset \ker \mathcal{R}_{\text{par}}$ so we only have to check that $\Phi_i - \xi_{jk}$ and $\Phi_i + (n+1)\xi_{jk}$ are eigenvectors of $\mathcal{R}_\perp$ with the respective eigenvalues. Lemma 6.1.3 provides just that under the observation $\langle \Phi_i + (n+1)\xi_{jk}, \Phi_i - \xi_{jk} \rangle = 0$. □

For later use we observe that the same argument yields

$$\mathcal{R}(\Phi_i) = \mathcal{R}_\perp(\Phi_i) = 2\alpha\beta(n+1)(\Phi_i - \xi_{jk}), \tag{6.5}$$

$$\mathcal{R}(\xi_{jk}) = \mathcal{R}_\perp(\xi_{jk}) = -2\alpha\beta(\Phi_i - \xi_{jk}). \tag{6.6}$$

Recall that by (2.4) applied to any connection with parallel torsion

$$R^g(X,Y,Z,V) = R^\nabla(X,Y,Z,V) - \frac{1}{4}g(T(X,Y),T(Z,V)) - \frac{1}{4}\sigma_T(X,Y,Z,V). \tag{6.7}$$

In particular, the Riemannian and canonical curvature operators, $\mathcal{R}^g$ and $\mathcal{R}$, are related by the operators $\mathcal{S}_T$ and $\mathcal{G}_T$ defined by

$$\langle \mathcal{S}_T(X \wedge Y), Z \wedge V\rangle := \sigma_T(X,Y,Z,V) = \frac{1}{2} dT(X,Y,Z,V), \tag{6.8}$$

$$\langle \mathcal{G}_T(X \wedge Y), Z \wedge V\rangle := G_T(X,Y,Z,V) = g(T(X,Y),T(Z,V)). \tag{6.9}$$

They act on the forms Φ_i and ξ_{jk} as follows.

Proposition 6.2.2. *Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be a 3-(α,δ)-Sasaki manifold. The torsion T of the canonical connection satisfies the following:*

$$\mathcal{S}_T(\xi_{jk}) = 2\alpha\beta(\Phi_i + \xi_{jk}), \tag{6.10}$$

$$\mathcal{S}_T(\Phi_i) = \left\{4\alpha^2(2n+1) - 2\alpha\beta\right\}\Phi_i + \left\{4\alpha^2(2n+1) + 2\alpha\beta(2n-1)\right\}\xi_{jk}. \tag{6.11}$$

Proof. First we show that for vector fields X, Y and every even permutation (ijk) of (123)

$$dT(X,Y,\xi_i,\xi_j) = 4\alpha\beta\{\Phi_k(X,Y) + (\eta_i \wedge \eta_j)(X,Y)\}, \tag{6.12}$$

which is equivalent to (6.10), taking into account (6.8). Indeed, we compute

$$\begin{aligned}
&\sum_{l=1}^{3}(\Phi_l \wedge \Phi_l)(X,Y,\xi_i,\xi_j)\\
&= 2\{\Phi_k(X,Y)\Phi_k(\xi_i,\xi_j) + \Phi_k(X,\xi_i)\Phi_k(\xi_j,Y) + \Phi_k(X,\xi_j)\Phi_k(Y,\xi_i)\}\\
&= 2\{-\Phi_k(X,Y) + \eta_j(X)\eta_i(Y) - \eta_i(X)\eta_j(Y)\}\\
&= -2\{\Phi_k(X,Y) + (\eta_i \wedge \eta_j)(X,Y)\}.
\end{aligned}$$

We can also compute

$$\begin{aligned}
&\overset{l,m,n}{\mathfrak{S}}(\Phi_l \wedge \eta_m \wedge \eta_n)(X,Y,\xi_i,\xi_j)\\
&= \overset{l,m,n}{\mathfrak{S}}\{\Phi_l(X,Y)\eta_{mn}(\xi_i,\xi_j) + \Phi_l(X,\xi_i)\eta_{mn}(\xi_j,Y) + \Phi_l(X,\xi_j)\eta_{mn}(Y,\xi_i)\\
&\qquad + \Phi_l(\xi_i,\xi_j)\eta_{mn}(X,Y) + \Phi_l(\xi_j,Y)\eta_{mn}(X,\xi_i) + \Phi_l(Y,\xi_i)\eta_{mn}(X,\xi_j)\}\\
&= \Phi_k(X,Y) - \Phi_k(X,\xi_i)\eta_i(Y) - \Phi_k(X,\xi_j)\eta_j(Y)\\
&\qquad - \eta_{ij}(X,Y) - \Phi_k(\xi_j,Y)\eta_j(X) + \Phi_k(Y,\xi_i)\eta_i(X)\\
&= \Phi_k(X,Y) - \eta_j(X)\eta_i(Y) + \eta_i(X)\eta_j(Y) - \eta_{ij}(X,Y) - \eta_i(Y)\eta_j(X) + \eta_j(Y)\eta_i(X)\\
&= \Phi_k(X,Y) + \eta_{ij}(X,Y).
\end{aligned}$$

Therefore, using (3.14) we get (6.12).

In order to prove (6.11), first we show that for every $X, Y \in \mathcal{H}$

$$\sum_{r=1}^{4n+3} dT(X,Y,e_r,\varphi_i e_r) = \{-16\alpha^2(2n+1) + 8\alpha\beta\}\Phi_i(X,Y), \tag{6.13}$$

where e_r, $r = 1, \dots, 4n+3$, is a local orthonormal frame. Indeed, we can compute

$$\begin{aligned}
&\sum_{r=1}^{4n+3}\sum_{l=1}^{3}(\Phi_l \wedge \Phi_l)(X, Y, e_r, \varphi_i e_r)\\
&= 2\sum_r\sum_l\{\Phi_l(X,Y)\Phi_l(e_r,\varphi_i e_r) + \Phi_l(X,e_r)\Phi_l(\varphi_i e_r, Y) + \Phi_l(X,\varphi_i e_r)\Phi_l(Y,e_r)\}\\
&= 2\sum_r \Phi_i(X,Y)\Phi_i(e_r,\varphi_i e_r) + 2\sum_l g(\varphi_l X, \varphi_i\varphi_l Y) - 2\sum_l g(\varphi_i\varphi_l X, \varphi_l Y)\\
&= -2(4n+2)\Phi_i(X,Y) - 2g(\varphi_i X, Y) - 4g(X,\varphi_i Y) + 2g(X,\varphi_i Y) + 4g(\varphi_i X, Y)\\
&= -8(n+1)\Phi_i(X,Y).
\end{aligned}$$

We also compute

$$\begin{aligned}
&\sum_{r=1}^{4n+3} \overset{l,m,n}{\mathfrak{S}} (\Phi_l \wedge \eta_m \wedge \eta_n)(X, Y, e_r, \varphi_i e_r)\\
&= \sum_r \overset{l,m,n}{\mathfrak{S}} \Phi_l(X,Y)(\eta_m \wedge \eta_n)(e_r, \varphi_i e_r)\\
&= \Phi_i(X,Y)\{(\eta_j \wedge \eta_k)(\xi_j, \xi_k) + (\eta_j \wedge \eta_k)(\xi_k, -\xi_j)\}\\
&= 2\Phi_i(X,Y).
\end{aligned}$$

Applying (3.14),we can deduce that

$$\sum_{r=1}^{4n+3} dT(X, Y, e_r, \varphi_i e_r) = -32\alpha^2(n+1)\Phi_i(X,Y) - 16\alpha(\alpha-\delta)\Phi_i(X,Y),$$

which gives (6.13), being $\beta = 2(\delta - 2\alpha)$. Now, from (6.8) and (6.13), we have

$$\begin{aligned}
\langle \mathcal{S}_T(\Phi_i), X \wedge Y\rangle &= -\frac{1}{2}\sum_{r=1}^{4n+3}\langle \mathcal{S}_T(e_r \wedge \varphi_i e_r), X \wedge Y\rangle = -\frac{1}{4}\sum_{r=1}^{4n+3} dT(X, Y, e_r, \varphi_i e_r)\\
&= \{4\alpha^2(2n+1) - 2\alpha\beta\}\langle \Phi_i, X \wedge Y\rangle.
\end{aligned}$$

From (3.14), one can see that for every $X, Y, Z \in \mathcal{H}$ and $r, s, t = 1, 2, 3$

$$dT(X, Y, Z, \xi_r) = 0, \qquad dT(X, \xi_r, \xi_s, \xi_t) = 0,$$

which imply that

$$\langle \mathcal{S}_T(\Phi_i), X \wedge \xi_s\rangle = -\frac{1}{2}\sum_{r=1}^{4n+3}\langle \mathcal{S}_T(e_r \wedge \varphi_i e_r), X \wedge \xi_s\rangle = -\frac{1}{4}\sum_{r=1}^{4n+3} dT(e_r, \varphi_i e_r, X, \xi_s) = 0.$$

Finally, using (6.10) and (6.4),

$$\begin{aligned}
\langle \mathcal{S}_T(\Phi_i), \xi_j \wedge \xi_k\rangle &= \langle \mathcal{S}_T(\xi_j \wedge \xi_k), \Phi_i\rangle = 2\alpha\beta\langle \Phi_i + \xi_{jk}, \Phi_i\rangle\\
&= 2\alpha\beta(2n + 1 + \Phi_i(\xi_j, \xi_k)) = 4\alpha\beta n
\end{aligned}$$

and analogously, $\langle \mathcal{S}_T(\Phi_i), \xi_i \wedge \xi_j\rangle = \langle \mathcal{S}_T(\Phi_i), \xi_i \wedge \xi_k\rangle = 0$, thus completing the proof of (6.11). □

Proposition 6.2.3. *Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be a 3-(α, δ)-Sasaki manifold. The torsion T of the canonical connection satisfies the following:*

$$\mathcal{G}_T(\xi_{jk}) = (\beta - 4\alpha)\left\{2\alpha\Phi_i + (\beta - 2\alpha)\xi_{jk}\right\}, \tag{6.14}$$

$$\mathcal{G}_T(\Phi_i) = \left\{8\alpha^2(n+1) - 2\alpha\beta\right\}\Phi_i + \left\{-8\alpha^2(n+1) - \beta^2 + 2\alpha\beta(2n+3)\right\}\xi_{jk}. \tag{6.15}$$

Proof. By direct computation using (3.12) or (3.13), one easily gets that for all vector fields X, Y and for every even permutation (ijk) of (123)

$$T(X, Y, \xi_i) = 2\alpha\Phi_i(X, Y) + 2(\delta - 3\alpha)(\eta_j \wedge \eta_k)(X, Y),$$
$$T(\xi_j, \xi_k) = 2(\delta - 4\alpha)\xi_i.$$

Hence,

$$\begin{aligned}\langle \mathcal{G}_T(\xi_{jk}), X \wedge Y\rangle &= g(T(\xi_j, \xi_k), T(X, Y)) = 2(\delta - 4\alpha)T(X, Y, \xi_i)\\ &= 2(\delta - 4\alpha)\{2\alpha\Phi_i(X, Y) + 2(\delta - 3\alpha)(\eta_j \wedge \eta_k)(X, Y)\},\end{aligned}$$

which gives (6.14), since $\beta = 2(\delta - 2\alpha)$.

In order to prove (6.15), notice that for every $X, Y, Z, V \in \mathcal{H}$, applying (3.13), we have

$$G_T(X, Y, Z, V) = 4\alpha^2 \sum_{l=1}^{3} \Phi_l(X, Y)\Phi_l(Z, V).$$

Then, choosing a local orthonormal frame of type e_r, $r = 1, \ldots, 4n$, ξ_i, ξ_j, ξ_k, and using $2\Phi_i = -\sum e_r \wedge \varphi_i e_r$, for every $X, Y \in \mathcal{H}$ we have

$$\begin{aligned}\langle \mathcal{G}_T(\Phi_i), X \wedge Y\rangle &= -\frac{1}{2}\sum_{r=1}^{4n} G_T(e_r, \varphi_i e_r, X, Y) - G_T(\xi_j, \xi_k, X, Y)\\ &= -2\alpha^2 \sum_{r=1}^{4n}\sum_{l=1}^{3} \Phi_l(e_r, \varphi_i e_r)\Phi_l(X, Y) - 2\alpha(\beta - 4\alpha)\Phi_i(X, Y)\\ &= 8\alpha^2 n\Phi_i(X, Y) - 2\alpha(\beta - 4\alpha)\Phi_i(X, Y)\\ &= \{8\alpha^2(n+1) - 2\alpha\beta\}\langle \Phi_i, X \wedge Y\rangle.\end{aligned}$$

Now, from (3.13), we also have that for every $X, Y, Z \in \mathcal{H}$ and $r, s, t = 1, 2, 3$,

$$G_T(X, Y, Z, \xi_r) = 0, \qquad G_T(\xi_r, \xi_s, \xi_t, X) = 0,$$

which give

$$\langle \mathcal{G}_T(\Phi_i), X \wedge \xi_s\rangle = -\frac{1}{2}\sum_{r=1}^{4n} G_T(e_r, \varphi_i e_r, X, \xi_s) - G_T(\xi_j, \xi_k, X, \xi_s) = 0.$$

Finally, using (6.14) and (6.4),

$$\langle \mathcal{G}_T(\Phi_i), \xi_{jk}\rangle = \langle \mathcal{G}_T(\xi_{jk}), \Phi_i\rangle = (\beta - 4\alpha)\{2\alpha(2n+1) - \beta + 2\alpha\},$$

which is coherent with (6.15). Analogously,

$$\langle \mathcal{G}_T(\Phi_i), \xi_{ij}\rangle = \langle \mathcal{G}_T(\Phi_i), \xi_{ki}\rangle = 0,$$

thus completing the proof of (6.15). □

Proposition 6.2.4. *Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be a 3-(α, δ)-Sasaki manifold. Then, the Riemannian curvature of M satisfies*

$$\mathcal{R}^g(\xi_{jk}) = -\left(2\alpha^2 + \alpha\beta\right)\Phi_i + \left(2\alpha^2 + \alpha\beta + \frac{1}{4}\beta^2\right)\xi_{jk}, \tag{6.16}$$

$$\mathcal{R}^g(\Phi_i) = \left\{\alpha^2(4n+3) + \alpha\beta(2n+1)\right\}\Phi_i - \left(\alpha + \frac{1}{2}\beta\right)^2\xi_{jk}. \tag{6.17}$$

Proof. Since $\mathcal{R}^g$ and $\mathcal{R}$ are related by (6.7), the result follows from direct computations using equations (6.6), (6.14), (6.10), and equations (6.5), (6.15), (6.11). □

Remark 6.2.5. Being $\beta = 2(\delta - 2\alpha)$, equations (6.16) and (6.17) can be rephrased as

$$\mathcal{R}^g(\xi_{jk}) = 2\alpha(\alpha - \delta)\Phi_i + \{\alpha^2 + (\alpha - \delta)^2\}\xi_{jk}, \tag{6.18}$$

$$\mathcal{R}^g(\Phi_i) = \{\alpha(\delta - \alpha)(4n+1) + \alpha\delta\}\Phi_i - (\alpha - \delta)^2\xi_{jk}. \tag{6.19}$$

The Riemannian Ricci tensor of a 3-(α, δ)-Sasaki manifold has been computed in [AD20, Proposition 2.3.3]:

$$Ric^g(X, Y) = 2\alpha\{2\delta(n+2) - 3\alpha\}g(X, Y) + 2(\alpha - \delta)\{(2n+3)\alpha - \delta\}\sum_{i=1}^{3}\eta_i(X)\eta_i(Y)$$

implying that the manifold is Einstein if and only if $\delta = \alpha$ or $\delta = \alpha(2n+3)$. In the following we provide one further characterization.

Theorem 6.2.6. *Let $(M, \varphi_i, \xi_i, \eta_i, g)$ be a 3-(α, δ)-Sasaki manifold. Then, the following conditions are equivalent:*

a) each 2-form $\Phi_i - \xi_{jk}$ is an eigenform of $\mathcal{R}^g$;

b) each 2-form $\Phi_i + (n+1)\xi_{jk}$ is an eigenform of $\mathcal{R}^g$;

c) either $\delta = \alpha$ or $\delta = (2n+3)\alpha$;

d) (M, g) is Einstein.

If $\delta = \alpha$, that is in the 3-α-Sasaki case, the six orthogonal eigenforms $\Phi_i - \xi_{jk}$, $\Phi_i + (n+1)\xi_{jk}$ admit the same eigenvalue $\lambda = \alpha^2$. If $\delta = \alpha(2n+3)$, then each $\Phi_i - \xi_{jk}$ has eigenvalue $\lambda_1 = \alpha^2(8n^2 + 16n + 9)$, while each $\Phi_i + (n+1)\xi_{jk}$ has eigenvalue $\lambda_2 = \alpha^2(2n+1)^2$.

Proof. The equivalence of c) and d) is known. From (6.18) and (6.19), we have that $\mathcal{R}^g(\Phi_i - \xi_{jk}) = a\Phi_i + b\xi_{jk}$ with

$$a = \alpha(\delta - \alpha)(4n+1) + \alpha\delta - 2\alpha(\alpha - \delta), \qquad b = -\alpha^2 - 2(\alpha - \delta)^2,$$

which give

$$a + b = 2(\delta - \alpha)\{\alpha(2n+3) - \delta\}.$$

Then $\Phi_i - \xi_{jk}$ is an eigenform of $\mathcal{R}^g$ if and only if $a + b = 0$, that is $\delta = \alpha$ or $\delta = (2n+3)\alpha$. In particular, if $\delta = \alpha$, the corresponding eigenvalue is $\lambda = a = \alpha^2$. If $\delta = \alpha(2n+3)$, the eigenvalue is $\lambda_1 = a = \alpha^2(8n^2 + 16n + 9)$.

Analogously, we can compute $\mathcal{R}^g(\Phi_i + (n+1)\xi_{jk}) = a'\Phi_i + b'\xi_{jk}$ with

$$a' = \alpha(\delta-\alpha)(4n+1)+\alpha\delta+2\alpha(\alpha-\delta)(n+1), \quad b' = -(\alpha-\delta)^2+(n+1)\alpha^2+(\alpha-\delta)^2(n+1),$$

from which

$$(n+1)a' - b' = n(\delta - \alpha)\{\alpha(2n+3) - \delta\},$$

thus proving the equivalence of b) and c). If $\delta = \alpha$, then the eigenform $\Phi_i + (n+1)\xi_{jk}$ has eigenvalue $\lambda = a' = \alpha^2$. If $\delta = \alpha(2n+3)$, the corresponding eigenvalue is $\lambda_2 = a' = \alpha^2(2n+1)^2$. □

6.3 Positivity of Curvature Operators

Recall from Proposition 6.1.2 the decomposition

$$\mathcal{R} = \alpha\beta\mathcal{R}_\perp + \mathcal{R}_{\text{par}},$$

where $\mathcal{R}_{\text{par}}$ is trivial on $(\Lambda^2\mathcal{H})^\perp$. Via horizontal lift we consider the Riemannian curvature operator $\mathcal{R}^{g_N}$ on the base N of the canonical submersion as a curvature operator $\Lambda^2\mathcal{H} \to \Lambda^2\mathcal{H}$. Via Theorem 6.1.1 we can then compare $\mathcal{R}_{\text{par}}$ with the Riemannian curvature operator to obtain

$$\mathcal{R}_{\text{par}} = \mathcal{R}^{g_N} - 2\alpha\delta\sum_{\mu=1}^{3}\Phi_\mu^{\mathcal{H}} \otimes \Phi_\mu^{\mathcal{H}}. \tag{6.20}$$

Analogous to $\mathcal{R}_\perp$ we set $\mathcal{R}_\perp^{\mathcal{H}} := \sum_{i=1}^3 \Phi_i^{\mathcal{H}} \otimes \Phi_i^{\mathcal{H}}$. Then

Lemma 6.3.1. *$\mathcal{R}_\perp^{\mathcal{H}}$ has the only non-zero eigenvalue $2n$ with eigenspace spanned by $\Phi_i^{\mathcal{H}}$, $i = 1, 2, 3$.*

Lemma 6.1.3, Lemma 6.1.4 and Lemma 6.3.1 show that $\mathcal{R}_{\text{par}}$ and $\mathcal{R}_\perp$ share eigenspaces and analogously $\mathcal{R}_{\text{par}}$ and $\mathcal{R}_\perp^{\mathcal{H}}$. Additionally, the eigenspaces to non-zero eigenvalues of $\mathcal{R}_{\text{par}}$ are inside the kernel of $\mathcal{R}_\perp$ and vice versa. The same holds for $\mathcal{R}_{\text{par}}$ and $\mathcal{R}_\perp^{\mathcal{H}}$. Thus the space $\text{Spec}(\mathcal{R})$ of eigenvalues of $\mathcal{R}$ is the union

$$\text{Spec}(\mathcal{R}) = \text{Spec}(\mathcal{R}_{\text{par}}) \cup \{0, 2\alpha\beta(n+2)\} \tag{6.21}$$

of the eigenvalues of $\mathcal{R}_{\text{par}}$ and the eigenvalues $\{0, 2\alpha\beta(n+2)\}$ of $\alpha\beta\mathcal{R}_\perp$. Similarly the eigenvalues of $\mathcal{R}^{g_N}$ are

$$\text{Spec}(\mathcal{R}^{g_N}) \cup \{0\} = \text{Spec}(\mathcal{R}_{\text{par}}) \cup \{0, 4\alpha\delta n\}. \tag{6.22}$$

Remark that $\text{Spec}(\mathcal{R})$ certainly contains 0 as $\Phi_i^{\mathcal{H}} + n\xi_{jk} \in \ker\mathcal{R}$, by Theorem 6.2.1, while $\text{Spec}(\mathcal{R}^{g_N})$ may or may not contain 0.

Remark 6.3.2. The discussion on the curvature operator $\mathcal{R}^{g_N}$ actually proved that the $\Phi_i^{\mathcal{H}}$ are eigenvectors of $\mathcal{R}^{g_N}$ with eigenvalue $4\alpha\delta n$. This shows that in the case $n = 1$ the space (N, g_N) is anti-self-dual. We, thus, finally completed the proof of Proposition 4.2.9 that 7-dimensional 3-(α, δ)-Sasaki manifolds have a quaternionic Kähler base.

Theorem 6.3.3. *Let M be a 3-(α, δ)-Sasaki manifold with canonical submersion $\pi\colon M \to N$.*

a) If N has a non-negative Riemannian curvature operator $\mathcal{R}^{g_N} \geq 0$ then $\mathcal{R}$ is non-negative if and only if $\alpha\beta \geq 0$.

b) If N has a non-positive Riemannian curvature operator $\mathcal{R}^{g_N} \leq 0$ then $\mathcal{R}$ is non-positive.

Proof. By (6.22) if $\mathcal{R}^{g_N}$ is either non-negative or non-positive then so is $\mathcal{R}_{\text{par}}$ and the sign of $\alpha\delta$. Using (6.21) we obtain part a) directly. For part b) note that if $\alpha\delta \leq 0$ then $\alpha\beta = 2\alpha\delta - 4\alpha^2 < 0$. □

In the following we investigate the eigenspaces and eigenvectors more closely. We need some notation. For $i = 1, 2, 3$ denote the 2-dimensional spaces $N_i := \operatorname{span}\{\Phi_i^{\mathcal{H}}, \xi_{jk}\}$. Then decompose the space of 2-forms into orthogonal subbundles

$$\Lambda^2 M = \Lambda_1^2 \oplus \Lambda_2^2 \oplus \Lambda_3^2,$$

where the Λ_i^2 are given by

$$\Lambda_1^2 = \bigoplus_{i=1}^{3} N_i, \quad \Lambda_2^2 = \Lambda^2\mathcal{H} \cap \{\Phi_1^{\mathcal{H}}, \Phi_2^{\mathcal{H}}, \Phi_3^{\mathcal{H}}\}^{\perp}, \quad \Lambda_3^2 = \mathcal{V} \wedge \mathcal{H}.$$

For a linear map $A\colon \Lambda^2 M \to \Lambda^2 M$ we denote $A_1 := A|_{\Lambda_1^2}$ and correspondingly for the other spaces.

Let us motivate this decomposition. The obvious $\Lambda^2(\mathcal{V} \oplus \mathcal{H}) = \Lambda^2\mathcal{V} \oplus \mathcal{V} \wedge \mathcal{H} \oplus \Lambda^2\mathcal{H}$ motivates $\Lambda_3^2 = \mathcal{V} \wedge \mathcal{H}$, in particular since $\mathcal{R}|_{\Lambda_3^2} = 0$. However, $\mathcal{R}$ does not restrict to $\Lambda^2\mathcal{V}$ and $\Lambda^2\mathcal{H}$, but to Λ_1^2 and Λ_2^2. In fact, the characterization $\mathcal{R} = \alpha\beta\mathcal{R}_{\perp} + \mathcal{R}_{\text{par}}$ is with respect to Λ_1^2 and Λ_2^2 as noted in the proof of Theorem 6.2.1. The space Λ_1^2 can be seen as controlled by the 3-(α, δ)-Sasaki structure, while Λ_2^2 reflects the geometry of the base N. This is emphasized by the fact that the common eigenforms discussed in Theorem 6.2.1 all lie in Λ_1^2.

We can now discuss strong non-negativity. Recall that by (6.7) the curvature operators $\mathcal{R}$ and $\mathcal{R}^g$ are related by

$$\mathcal{R}^g = \mathcal{R} + \frac{1}{4}\mathcal{G}_T + \frac{1}{4}\mathcal{S}_T.$$

In particular, (M, g) is strongly non-negative with 4-form $-\frac{1}{4}\sigma_T$ if and only if

$$\mathcal{R} + \frac{1}{4}\mathcal{G}_T \geq 0.$$

Observe that $\mathcal{G}_T$ is non-negative by definition so we directly have strong non-negativity, if $\mathcal{R}$ is non-negative. We can relax the condition on the base to strong non-negativity, but we need an additional assumption on the 4-form.

Definition 6.3.4. We call a 4-form $\omega \in \Lambda^4 N$ on a quaternionic Kähler space N *quaternionic non-negative* if ω, considered as an operator $\Lambda^2 N \to \Lambda^2 N$, restricts to a map from the quaternionic bundle $\mathcal{Q} \subset \Lambda^2 N$ to itself and this restriction is a non-negative operator.

Given a 3-(α, δ)-Sasaki manifold with canonical submersion $\pi\colon M \to N$ the quaternionic non-negativity of $\omega \in \Lambda^4 N$ implies that $\pi^*\omega$ admits a block diagonal structure $\pi^*\omega = (\pi^*\omega)_1 \oplus (\pi^*\omega)_2$ with $(\pi^*\omega)_1 \geq 0$. Note that $(\pi^*\omega)_3$ vanishes trivially as $\pi_*\mathcal{V} = 0$.

Theorem 6.3.5. *Let M be a 3-(α, δ)-Sasaki manifold with $\alpha\beta \geq 0$ such that the base N of the canonical submersion is strongly non-negative with respect to a quaternionic non-negative 4-form ω. Then M is strongly non-negative with 4-form $\pi^*\omega - \frac{1}{4}\sigma_T$.*

Proof. We follow the same line of thought as in Theorem 6.3.3. Decompose $\mathcal{R} + \pi^*\omega = \mathcal{R}_{\text{par}} + (\pi^*\omega)_2 + \mathcal{R}_\perp + (\pi^*\omega)_1$ and the condition on ω gives

$$\begin{aligned}\operatorname{Spec}(\mathcal{R} + \pi^*\omega) \cup \{0\} &= \operatorname{Spec}(\mathcal{R}_{\text{par}} + (\pi^*\omega)_2) \cup \operatorname{Spec}(\mathcal{R}_\perp + (\pi^*\omega)_1) \\ &\subset \operatorname{Spec}(\mathcal{R}^{g_N} + \omega) \cup \operatorname{Spec}(\mathcal{R}_\perp + (\pi^*\omega)_1) \subset [0, \infty[.\end{aligned}$$

Now $\mathcal{R}^g + \pi^*\omega - \frac{1}{4}\mathcal{S}_T = \mathcal{R} + \pi^*\omega + \frac{1}{4}\mathcal{G}_T \geq 0$ proving the theorem. □

Strong positivity requires a more deliberate investigation of how $\mathcal{G}_T$ acts on the spaces Λ^2_i. From equation (3.13) it follows that the torsion T of the canonical connection satisfies

$$T(X,Y) \in \mathcal{H} \;\Leftrightarrow\; X \wedge Y \in \mathcal{V} \wedge \mathcal{H} \quad \text{and} \quad T(X,Y) \in \mathcal{V} \;\Leftrightarrow\; X \wedge Y \in \Lambda^2\mathcal{V} \oplus \Lambda^2\mathcal{H}.$$

Thus, $\mathcal{G}_T$ preserves $\Lambda^2_3 = \mathcal{V} \wedge \mathcal{H}$ and, by Proposition 6.2.3, Λ^2_1 as well. Therefore $\mathcal{G}_T$ splits into a direct sum of operators $\mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \mathcal{G}_3$ on $\Lambda^2_1 \oplus \Lambda^2_2 \oplus \Lambda^2_3$.

Consider some adapted basis e_r, $r = 1, \ldots, 4n+3$ of M. We may define the quaternionic spaces $\mathcal{H}_l = \operatorname{span}\{e_{4l}, e_{4l+1}, e_{4l+2}, e_{4l+3}\}$, $l = 1, \ldots, n$, and accordingly we have

$$\Lambda^2_2 = \left(\bigoplus_{l=1}^{n} \Lambda^2\mathcal{H}_l\right) \cap \langle \Phi_i^{\mathcal{H}} \rangle^\perp \oplus \bigoplus_{k<l} \mathcal{H}_k \wedge \mathcal{H}_l, \qquad \Lambda^2_3 = \bigoplus_{l=1}^{n} \mathcal{V} \wedge \mathcal{H}_l.$$

Note that these descriptions depend on the choice of adapted basis unlike the spaces Λ^2_i themselves.

Lemma 6.3.6. *The linear operator $\mathcal{G}_2$ vanishes. The operator $\mathcal{G}_3$ has the unique non-vanishing eigenvalue $12\alpha^2$ with eigenspace generated by $e_l \wedge \xi_1 + \varphi_3 e_l \wedge \xi_2 - \varphi_2 e_l \wedge \xi_3$, $l = 4, \ldots, 4n+3$.*

Proof. The space $\left(\bigoplus \Lambda^2\mathcal{H}_l\right) \cap \langle \Phi_i^{\mathcal{H}} \rangle^\perp \subset \Lambda^2_2$ is spanned by $e_l \wedge \varphi_i e_l - e_r \wedge \varphi_i e_r$, $r, l = 4, \ldots, 4n+3$, $i = 1, 2, 3$. $\mathcal{G}_2$ vanishes on these. Indeed, by expression (3.13)

$$T(e_l \wedge \varphi_i e_l) = 2\alpha\Phi_i(e_l, \varphi_i e_l)\xi_i = -2\alpha\xi_i = T(e_r \wedge \varphi_i e_r) \tag{6.23}$$

for any $l = 4, \ldots, 4n+3$. $\mathcal{G}_2$ vanishes on $\mathcal{H}_k \wedge \mathcal{H}_l$ as well since $\Phi_i(X,Y) = 0$ whenever X and Y are in different quaternionic subspaces.

We observe that $\mathcal{G}_3$ is the sum of n identical copies $\hat{\mathcal{G}}_3$ for each space $\mathcal{V}\wedge\mathcal{H}_i$, since $T(X,Y)\in\mathcal{H}_i$ if $X\in\mathcal{H}_i$ and $Y\in\mathcal{V}$. Again using (3.13), we find

$$T(e_l\wedge\xi_i)=2\alpha\varphi_i e_l=T(\varphi_k e_l\wedge\xi_j)=-T(\varphi_j e_l\wedge\xi_k). \tag{6.24}$$

In particular,

$$\mathcal{G}_T(e_r\wedge\xi_i)=\mathcal{G}_T(\varphi_k e_r\wedge\xi_j)=\mathcal{G}_T(-\varphi_j e_r\wedge\xi_k)=4\alpha^2(e_r\wedge\xi_1+\varphi_3 e_r\wedge\xi_2-\varphi_2 e_r\wedge\xi_3)$$

for the adapted basis $e_{4l},\dots,e_{4l+3}$ of $\mathcal{H}_l$. Hence, the vectors $e_r\wedge\xi_1+\varphi_3 e_r\wedge\xi_2-\varphi_2 e_r\wedge\xi_3$ are four linearly independent eigenvectors with eigenvalue $12\alpha^2$. In fact, these are all non-zero eigenvectors since (6.24) shows that

$$4\le \operatorname{rk}(\hat{\mathcal{G}}_3)=\dim T(\mathcal{V}\wedge\mathcal{H}_l)\le\frac{\dim\mathcal{V}\wedge\mathcal{H}_l}{3}=\frac{12}{3}=4. \qquad \square$$

This implies that analogous to $\mathcal{R}_\perp$ the sum $\alpha\beta\mathcal{R}_\perp+\frac{1}{4}\mathcal{G}_T$ is orthogonal to $\mathcal{R}_{\text{par}}$. In particular, $\mathcal{R}+\frac{1}{4}\mathcal{G}_T+\pi^*\omega$ can never be non-negative unless $\mathcal{R}_{\text{par}}+(\pi^*\omega)_2$ is. Unlike for $\mathcal{R}_\perp$, the eigenspace decomposition of $\alpha\beta\mathcal{R}_\perp+\frac{1}{4}\mathcal{G}_T$ inside the spaces N_i varies with α and δ. Still we have

Lemma 6.3.7. *The operator $\alpha\beta\mathcal{R}_\perp+\frac{1}{4}\mathcal{G}_T$ is positive on N_i, $i=1,2,3$, if and only if $\alpha\beta>0$.*

Proof. From (3.12) and (6.23) we compute that

$$\begin{aligned}\mathcal{G}_T(e_l\wedge\varphi_i e_l)&=-4\alpha^2\Phi_i^{\mathcal{H}}-4\alpha(\delta-4\alpha)\xi_{jk}\\ \mathcal{G}_T(\xi_{jk})&=4\alpha(\delta-4\alpha)\Phi_i^{\mathcal{H}}+4(\delta-4\alpha)^2\xi_{jk}\end{aligned}$$

Thus on N_i, with respect to the basis $\Phi_i^{\mathcal{H}}$ and ξ_{jk}, the sum takes the matrix form

$$\alpha\beta\mathcal{R}_\perp+\frac{1}{4}\mathcal{G}_T=\begin{pmatrix}2n\alpha(2\delta-3\alpha) & -\alpha(3\delta-4\alpha)\\ -2n\alpha(3\delta-4\alpha) & \delta^2\end{pmatrix}.$$

Note that $|\Phi_i^{\mathcal{H}}|^2=2n$, so $\alpha\beta\mathcal{R}_\perp+\frac{1}{4}\mathcal{G}_T$ is self-adjoint. Hence, the restriction to the 2-dimensional space N_i is positive if and only if both the determinant and the trace are. We have

$$\operatorname{tr}_{N_i}\left(\alpha\beta\mathcal{R}_\perp+\frac{1}{4}\mathcal{G}_T\right)=2n\alpha(2\delta-3\alpha)+\delta^2.$$

Setting $c=\frac{\delta}{\alpha}\in\mathbb{R}$ one obtains the quadratic polynomial

$$\operatorname{tr}_{N_i}\left(\alpha\beta\mathcal{R}_\perp+\frac{1}{4}\mathcal{G}_T\right)=\alpha^2(c^2+4nc-6n).$$

Hence, positivity holds if $c>\sqrt{4n^2+6n}-2n$. For $\alpha\beta=2\alpha(\delta-2\alpha)=2\alpha^2(c-2)>0$ also $c>2$ and the condition above holds as

$$\sqrt{4n^2+6n}-2n<2.$$

The second condition reads

$$\det\left(\left(\alpha\beta\mathcal{R}_\perp + \frac{1}{4}\mathcal{G}_T\right)\Big|_{N_i}\right) = 2n\alpha(2\delta^3 - 3\alpha\delta^2 - \alpha(3\delta - 4\alpha)^2)$$
$$= 4n\alpha^4(c^3 - 6c^2 + 12c - 8) = 4n\alpha^4(c-2)^3 > 0.$$

This is exactly the case if $\alpha\beta = 2\alpha^2(c-2) > 0$. □

We recollect the results so far. Under $\mathcal{R}_\perp$, $\mathcal{R}_{\text{par}}$ and $\mathcal{G}_T$ the spaces Λ^2_1, Λ^2_2 and Λ^2_3 are invariant. Under the assumption that $\mathcal{R}^{g_N} + \omega > 0$ and $\alpha\beta > 0$ the results are summarized in the following table.

space	dim	$\alpha\beta\mathcal{R}_\perp$	$\frac{1}{4}\mathcal{G}_T$	$\mathcal{R}_{\text{par}} + (\pi^*\omega)_2$	$(\pi^*\omega)_1$
Λ^2_1	6	sum > 0		0	≥ 0?
Λ^2_2	$6n - 3 + \binom{n}{2} \cdot 16$	0	0	> 0	0
Λ^2_3	$n \cdot 12$	0	≥ 0	0	0

It now remains to show that a suitable 4-form provides strict positivity on the kernel of $\mathcal{G}_3$.

Lemma 6.3.8. *The operator $\mathcal{S}_T$ corresponding to σ_T is negative definite on the kernel of $\mathcal{G}_3$ if and only if $\alpha\beta > 0$.*

Proof. As in the proof of Lemma 6.3.6 we may split $\mathcal{G}_3$ into n copies of $\hat{\mathcal{G}}_3$ on each quaternionic subspace. Let $e_{4l}, \dots, e_{4l+3} \in \mathcal{H}_l$ be an adapted basis of one such subspace. Then, from the same proof, we find that $T(e_r \wedge \xi_i) = 2\alpha\varphi_i e_l = -T(\varphi_j e_r \wedge \xi_k)$. Thus

$$\ker \hat{\mathcal{G}}_3 = \ker T \cap (\mathcal{H}_l \wedge \mathcal{V}) = \operatorname{span}\{e_r \wedge \xi_i + \varphi_j e_r \wedge \xi_k \mid i = 1,2,3;\ r = 4l, \dots, 4l+3\}.$$

By definition of $\mathcal{S}_T$ we have

$$g(\mathcal{S}_T(e_r \wedge \xi_i), e_s \wedge \xi_a) = g(T(e_r, \xi_i), T(e_s, \xi_a)) + g(T(e_s, e_r), T(\xi_i, \xi_a)) - g(T(e_s, \xi_i), T(e_r, \xi_a)).$$

With $T = 2\alpha \sum_{i=1}^n \eta_i \wedge \Phi_i^{\mathcal{H}} + 2(\delta - 4\alpha)\eta_{123}$ we compute each term individually

$$g(T(e_r, \xi_i), T(e_s, \xi_a)) = \begin{cases} 4\alpha^2, & e_s = -\varphi_a\varphi_i e_r \\ 0 \end{cases},$$

$$g(T(e_s, e_r), T(\xi_i, \xi_a)) = \begin{cases} 2\alpha(\beta - 4\alpha), & e_s = \varphi_i\varphi_a e_r \text{ and } a \neq i \\ 0 \end{cases},$$

$$g(T(e_s, \xi_i), T(e_r, \xi_a)) = \begin{cases} 4\alpha^2, & e_s = -\varphi_i\varphi_a e_r \\ 0 \end{cases}.$$

We thus obtain the full expression

$$\begin{aligned}\mathcal{S}_T(e_r \wedge \xi_i) &= 4\alpha^2 \sum_{a=1}^{3}(-\varphi_a\varphi_i e_r + \varphi_i\varphi_a e_r) \wedge \xi_a + 2\alpha(\beta - 4\alpha)\sum_{a\neq i}\varphi_i\varphi_a e_r \wedge \xi_a \\ &= 2\alpha\beta \sum_{a\neq i}\varphi_i\varphi_a e_r \wedge \xi_a \\ &= 2\alpha\beta(\varphi_k e_r \wedge \xi_j - \varphi_j e_r \wedge \xi_k).\end{aligned}$$

Finally compute $\mathcal{S}_T$ on $\ker \hat{\mathcal{G}}_3$ to obtain the result

$$\begin{aligned}\mathcal{S}_T(e_r \wedge \xi_i + \varphi_j e_r \wedge \xi_k) &= 2\alpha\beta(\varphi_k e_r \wedge \xi_j - \varphi_j e_r \wedge \xi_k + \varphi_j\varphi_j e_r \wedge \xi_i - \varphi_i\varphi_j e_r \wedge \xi_j) \\ &= -2\alpha\beta(e_r \wedge \xi_i + \varphi_j e_r \wedge \xi_k).\end{aligned}$$

□

We can finally prove the main theorem of this section providing strong positivity on M given a sufficient positivity condition on N.

Theorem 6.3.9. *Let M be a 3-(α, δ)-Sasaki manifold with $\alpha\beta > 0$. Suppose further that the base N of the canonical submersion has strongly positive curvature with respect to either*

a) a quaternionic non-negative 4-form $\omega \in \Lambda^4 TN$ or

b) an arbitrarily small 4-form $\omega \in \Lambda^4 TN$.

Then M has strongly positive curvature with 4-form $\pi^*\omega - (\frac{1}{4} + \varepsilon)\sigma_T$, $\varepsilon > 0$ sufficiently small.

Remark 6.3.10. We consider a space strongly positive with respect to an arbitrarily small 4-form ω if positivity is obtained for all 4-forms $\varepsilon\omega$, $0 < \varepsilon \leq 1$. For sake of notation we will denote by ω the 4-form for a sufficiently small value of ε. Note that in this case $\mathcal{R}^{g_N} \geq 0$ since it is a closed condition.

Proof. In case a) the operator $\mathcal{R}^g + \pi^*\omega - \frac{1}{4}\sigma_T$ is non-negative by Theorem 6.3.3. In case b) we have that $\mathcal{R}^g + \frac{1}{4}\sigma_T \geq 0$ by Theorem 6.3.3 and Remark 6.3.10. Then Lemma 6.3.7 proves positivity on Λ^2_1. By assumption $\mathcal{R}^{g_N} + \omega > 0$ on $\Lambda^2 N$. Thus,

$$(\mathcal{R}_{\text{par}} + \pi^*\omega)_2 = \pi^*((\mathcal{R}^{g_N} + \omega)_{\mathcal{Q}^\perp}) > 0.$$

In case a) $(\pi^*\omega)_1 \geq 0$, in case b) $(\pi^*\omega)_1$ is sufficiently small such that in both cases $(\mathcal{R}^g + \pi^*\omega - \frac{1}{4}\sigma_T)_1 > 0$. Finally the addition of $-\varepsilon\sigma_T$ by Lemma 6.3.8 guarantees positivity on Λ^2_3. If we choose $\varepsilon > 0$ sufficiently small it will not disturb positivity on $\Lambda^2_1 \oplus \Lambda^2_2$. □

6.4 The Homogeneous Case

We like to apply the positivity discussion to homogeneous 3-(α, δ)-Sasaki manifolds, more precisely to those that fiber over Wolf spaces and their non-compact duals as constructed in Theorem 5.1.2.

Theorem 6.4.1. *Let $M = G/H$ be a homogeneous 3-(α, δ)-Sasaki manifold.*

a) If M is a positive 3-(α, δ)-Sasaki manifold then the canonical curvature operator $\mathcal{R}$ is non-negative if and only if $\alpha\beta \geq 0$. In this case M is strongly non-negative.

b) If M is a negative 3-(α, δ)-Sasaki manifold over a symmetric base then the canonical curvature operator $\mathcal{R}$ is non-positive.

Proof. In the positive case M has to fiber over a symmetric base. In this case the base is a compact symmetric space, hence the curvature operator $\mathcal{R}^{g_N}$ is non-negative. In part b) the base is a non-compact symmetric space by assumption, hence $\mathcal{R}^{g_N} \leq 0$. Therefore in both cases it fulfills the requirement of Theorem 6.3.3. In the positive case also Theorem 6.3.5 applies with $\omega = 0$. □

We will next focus on strong positivity. This is much more restrictive than strong non-negativity. In particular, strong positivity implies strict positive sectional curvature and homogeneous manifolds with strictly positive sectional curvature have been classified, see [Wal72], [Ber61]. Out of these only the 7-dimensional Aloff-Wallach-space $W^{1,1}$, the spheres S^{4n+3} and real projective spaces $\mathbb{R}P^{4n+3}$ admit homogeneous 3-(α, δ)-Sasaki structures. We will thus prove

Theorem 6.4.2. *The 3-(α, δ)-Sasaki spaces*

a) $W^{1,1} = \mathrm{SU}(3)/S^1$ with 4-form $-(\frac{1}{4} + \varepsilon)\sigma_T$ for small $\varepsilon > 0$,

b) S^{4n+3}, $\mathbb{R}P^{4n+3}$, $n \geq 1$, with 4-form $\frac{\alpha\delta}{4}\pi^*\Omega_N - (\frac{1}{4} + \varepsilon)\sigma_T$ for small $\varepsilon > 0$

with Ω_N the fundamental 4-form of the qK base, are strongly positive if and only if $\alpha\beta > 0$.

Remark 6.4.3. The strong positivity of all three spaces $W^{1,1}$ and S^{4n+3}, $\mathbb{R}P^{4n+3}$, can actually be proven by the Strong Wallach Theorem in [BM18]. We compare to our case:

a) Observe that all positive homogeneous 3-(α, δ)-Sasaki manifolds are given by a homogeneous fibration
$$\mathrm{SO}(3) = G_0/H \to G/H \to G/G_0.$$
In the case of S^{4n+3} the fiber is $\mathrm{Sp}(1)$ instead.

b) In their strong Wallach theorem [BM18] the autors consider the metrics $g_t = tQ|_{\mathcal{V}} + Q|_{\mathcal{H}}$ for $0 < t < 1$, where Q is a negative multiple of the Killing form. If we set $Q = \frac{-\kappa}{8\alpha\delta(n+2)}$ as in the 3-(α, δ)-Sasaki setting then $t = \frac{2\alpha}{\delta}$ and, thus, the condition $0 < t < 1$ is equivalent to $\beta > 0$.

c) We have $\dim G_0/H = 3$ and $G_0/H = \mathrm{SO}(3) = \mathbb{R}P^3$, S^3 in the case of S^{4n+3}, with a scaled standard metric. In particular the fiber is of positive sectional curvature.

d) They require a strong fatness property for the homogeneous fibration. Adapted to our notation the bundle is strongly fat if there is a 4-form τ such that $F+\tau\colon \mathcal{H}\otimes\mathcal{V} \to \mathcal{H}\otimes\mathcal{V}$ is positive definite, where F is given by

$$\begin{aligned} g(F(X\wedge\xi_i), Y\wedge\xi_j) &= g([X,\xi_i],[Z,\xi_j]) = \delta^2 g(\varphi_i X, \varphi_j Y) \\ &= \frac{\delta^2}{4\alpha^2} g(T(X\wedge\xi_i), T(Y\wedge\xi_j)) = \frac{\delta^2}{4\alpha^2} g(\mathcal{G}_T(X\wedge\xi_i), Y\wedge\xi_j). \end{aligned}$$

Thus by the previous lemma $\tau = -\varepsilon\sigma_T$ achieves strong fatness for sufficiently small ε.

e) The final condition is for the base to be one of $S^{4n}, \mathbb{R}P^{4n}, \mathbb{C}P^{2n}, \mathbb{H}P^n$. The only homogeneous 3-(α,δ)-Sasaki manifolds where this holds are S^{4n+3}, $\mathbb{R}P^n$, which fiber over $\mathbb{H}P^n$, and $W^{1,1}$, which fibers over $\mathbb{C}P^2$.

Note that a)-d) are valid for all positive homogeneous examples not only for the spheres, real projective spaces and $W^{1,1}$.

Proof of Theorem 6.4.2. Since our discussion is pointwise we will identify tensors on N with those on $\mathcal{H}$.

a) On any 7-dimensional homogeneous 3-(α,δ)-Sasaki manifold we have that [AD20, Theorem 4.4.1]

$$\sigma_T|_{\Lambda^4\mathcal{H}} = \frac{1}{2}\mathrm{d}T|_{\Lambda^4\mathcal{H}} = 2\alpha^2 \sum_{i=1}^{3} \Phi_i^{\mathcal{H}} \wedge \Phi_i^{\mathcal{H}} = 12\alpha^2 \mathrm{dVol}_N.$$

Observe that in this case also $\Lambda_2^2 = \Lambda_-^2$ is the (-1)-eigenspace of the volume form on N. In fact, for $\alpha\delta > 0$, (6.20) shows that $(\mathcal{R}^{g_N})_1$ is positive, so the addition of $\omega = -\varepsilon\sigma_T|_{\Lambda^4\mathcal{H}}$ provides positivity. Hence, we may apply Theorem 6.3.9. The stated 4-form is obtained by the observation that $\pi^*\omega$ can be included in the addition $-\varepsilon\sigma_T$ from Theorem 6.3.9.

b) The 4-form $\omega = \frac{\alpha\delta}{4}\Omega_N$ appears as possible 4-form in the strong positivity of $\mathbb{H}P^n$. In [BM18] they prove that the 4-form $\mathfrak{b}(\mu)$ suffices where μ[1] is given as the symmetric product of twice the A-tensor of the submersion $(S^{4n+3}, g_0) \to (\mathbb{H}P^n, g_B)$, $(\mathbb{R}P^{4n+3}, g_0) \to (\mathbb{H}P^n, g_B)$ respectively and $\mathfrak{b}$ is the Bianchi map

$$\mathfrak{b}\colon \mathrm{Sym}(\Lambda^2 M) \to \Lambda^4(M), \qquad \mathfrak{b}(T)(X,Y,Z,V) := \overset{X,Y,Z}{\mathfrak{S}} T(X,Y,Z,V).$$

Here the metric g_0 denotes up to global scaling by $\frac{1}{8\alpha\delta(n+2)}$ the standard round metric. Adapted to our notation

$$\mu(X\wedge Y, Z\wedge V) = g_0(A_X Y, A_Z V) = \frac{1}{4} g_0([X,Y]_{\mathfrak{m}}, [Z,V]_{\mathfrak{m}}) = \frac{\alpha\delta}{2} \sum_{i=1}^{3} \Phi_i^{\mathcal{H}}(X,Y)\Phi_i^{\mathcal{H}}(Z,V).$$

[1] Due to conflicting notation, we renamed this μ from α in [BM18].

and thus

$$\mathfrak{b}(\mu) = \frac{\alpha\delta}{2}\sum_{i=1}^{3}\mathfrak{b}(\Phi_i^{\mathcal{H}} \otimes \Phi_i^{\mathcal{H}}) = \frac{\alpha\delta}{4}\sum_{i=1}^{3}\Phi_i^{\mathcal{H}} \wedge \Phi_i^{\mathcal{H}} = \frac{\alpha\delta}{4}\pi^*\Omega_N.$$

Therefore we have $\mathcal{R}^{g_N} + \frac{\alpha\delta}{4}\Omega_N > 0$ on $\Lambda^2 N$. It remains to check that ω is quaternionic non-negative. We compute for a basis $e_1, \dots, e_{4n}$ of TN

$$\begin{aligned}
\omega(\Phi_i^{\mathcal{H}}) &= \frac{\alpha\delta}{4}\sum_{s=1}^{3}\Phi_i^{\mathcal{H}} \lrcorner (\Phi_s^{\mathcal{H}} \wedge \Phi_s^{\mathcal{H}}) \\
&= \frac{\alpha\delta}{2}\left(2n\Phi_i^{\mathcal{H}} - \frac{1}{2}\sum_{l=1}^{4n}\sum_{s=1}^{3}\left((e_l \lrcorner \Phi_s^{\mathcal{H}}) \wedge (\varphi_i e_l \lrcorner \Phi_s^{\mathcal{H}})\right)\right) \\
&= \frac{\alpha\delta}{2}\left(2n\Phi_i^{\mathcal{H}} + \frac{1}{2}\sum_{l=1}^{4n}\sum_{s=1}^{3}\varphi_s e_l \wedge \varphi_s\varphi_i e_l\right) \\
&= \frac{\alpha\delta}{2}\left(2n\Phi_i^{\mathcal{H}} + \frac{1}{2}\sum_{l=1}^{4n}(e_l \wedge \varphi_i e_l - \varphi_j e_l \wedge \varphi_k e_l + \varphi_k e_l \wedge \varphi_j e_l\right) \\
&= \frac{\alpha\delta}{2}\left(2n\Phi_i^{\mathcal{H}} + \frac{1}{2}\sum_{l=1}^{4n}(e_l \wedge \varphi_i e_l - 2\varphi_j e_l \wedge \varphi_i\varphi_j e_l\right) \\
&= \frac{\alpha\delta}{2}(2n+1)\Phi_i^{\mathcal{H}}
\end{aligned}$$

□

6.5 The Inhomogeneous Case

The theory we established in Section 6.3 goes beyond the homogeneous spaces. These can only be seen as the most tractable examples. For inhomogeneous 3-(α, δ)-Sasaki spaces the base of the canonical submersion will typically be inhomogeneous as well. In [LS94] the authors conjecture that all complete quaternionic Kähler manifolds of positive scalar curvature are Wolf spaces and therefore homogeneous. Hence, we are forced to consider actual orbifolds admitting a quaternionic Kähler structure with strongly positive curvature and either of the assumptions in Theorem 6.3.9. No such result has been obtained so far, yet there are good candidates. Accordingly, this section is meant as an exposition of these and outlook into further research.

Recall that strongly positive curvature implies positive sectional curvature. Quaternionic Kähler manifolds of such type were investigated by M. Berger in [Ber66]. As observed in [Dea04], Berger's argument is purely local and, thus, extends to orbifolds.

Theorem 6.5.1 ([Ber66],[Dea04])**.** *Let $n \geq 2$ and (M, g, Ω) a $4n$-dimensional quaternionic Kähler orbifold of positive sectional curvature. Then locally (M, g, Ω) is isometric to the quaternionic projective space $\mathbb{H}P^n$ with its usual metric and quaternionic structure up to a finite quotient.*

Let us first consider the trivial possible example.

Example 6.5.2. Consider the lens space $\mathbb{Z}_k \backslash S^{4n+3}$ where $\mathbb{Z}_k$ acts as a subgroup of S^1 generated by diagonal left multiplication of $e^{i\phi} = \cos(\phi) + i\sin(\phi)$ on $S^{4n+3} \subset \mathbb{H}^{n+1}$. Equip S^{4n+3} with the 3-(α, δ)-Sasaki structure defined by right multiplication as in Example 3.2.8. Then the action of $\mathbb{Z}_k$ preserves the structure and, thus, the lens spaces are 3-(α, δ)-Sasaki manifolds themselves. If $k = 2$ this is exactly $\mathbb{R}P^{4n+3}$. For k odd the quotients are $N_k = \mathbb{Z}_k \backslash \mathbb{H}P^n$. Our theory applies since N_k is locally isometric to $\mathbb{H}P^n$. However, the result is equally trivial as the lens spaces are locally just S^{4n+3}, so strong positivity for some 3-(α, δ)-Sasaki metrics is not surprising.

With that we now restrict ourselves to the case of positive 7-dimensional 3-(α, δ)-Sasaki manifolds over 4-dimensional quaternionic Kähler spaces. Recall that these are $\mathcal{H}$-homothetic to 7-dimensional 3-Sasaki manifolds. Accordingly we will use the machinery of 3-Sasaki geometry. We recall 3-Sasaki reduction invented in [BGM94] to construct new 3-Sasaki manifolds.

Definition 6.5.3. Let $(M, g, \xi_i, \eta_i, \varphi_i)_{i=1,2,3}$ be a 3-Sasaki manifold and G a compact connected Lie group such that $G \curvearrowright M$ by 3-Sasaki automorphisms. The map

$$\begin{aligned} \mu \colon M &\to \mathfrak{g}^* \otimes \mathbb{R}^3, \\ p &\mapsto (\mu_1(p), \mu_2(p), \mu_3(p)) \end{aligned}$$

such that for all $X \in \mathfrak{g}$ with fundamental vector field $\hat{X}$

$$\langle \mu_i(p), X \rangle = \frac{1}{2}\eta_i(\hat{X})$$

is called the 3-*Sasaki moment map* of the action $G \curvearrowright M$.

Theorem 6.5.4 ([BGM94])**.** *Let $(M, g, \xi_i, \eta_i, \varphi_i)_{i=1,2,3}$ be a 3-Sasaki manifold, G a compact connected Lie group such that $G \curvearrowright M$ by 3-Sasaki automorphisms. Assume that 0 is a regular value of μ and G acts freely on $\mu^{-1}(0)$.*

Then there is a unique 3-Sasaki structure $(\check{M}, \check{g}, \check{\xi}_i, \check{\eta}_i, \check{\varphi}_i)_{i=1,2,3}$ on the quotient manifold $\check{M} = \mu^{-1}(0)/G$ such that $\iota^* g = \pi^* \check{g}$ and $\check{\xi}_i = \pi_*(\xi_i|_{\mu^{-1}(0)})$.

Now let $G = S^1$ and $M = S^{11} \subset \mathbb{H}^3$ with the usual 3-Sasaki structure defined by right multiplication. Consider the action of $\rho_{(p_1,p_2,p_3)} \colon S^1 \curvearrowright S^{11}$ with positive weights $p_1, p_2, p_3 \in \mathbb{N}$ given by

$$\rho_{(p_1,p_2,p_3)}(z)(q_1, q_2, q_3) = (z^{p_1} q_1, z^{p_2} q_2, z^{p_3} q_3).$$

Clearly this action preserves the 3-Sasaki structure. We have

Corollary 6.5.5 ([BG08, p.500])**.** *The action of S^1 with weights (p_1, p_2, p_3) is free on $\mu^{-1}_{(p_1,p_2,p_3)}(0)$ if the p_i are pairwise coprime. Hence the space*

$$S(p_1, p_2, p_3) \coloneqq \mu^{-1}_{(p_1,p_2,p_3)}(0)/S^1_{(p_1,p_2,p_3)}$$

admits a 3-Sasaki structure.

Let us now explain why their underlying quaternionic Kähler orbifolds $\mathcal{O}(p_1, p_2, p_3)$ are good candidates to admit strongly positive curvature. In [BGM94] the authors prove that the spaces $S(p_1, p_2, p_3)$ are diffeomorphic to biquotients $SU(3)//S^1$, where S^1 acts from both sides as

$$\begin{aligned} S^1 \times \mathrm{SU}(3) &\to \mathrm{SU}(3) \\ z, \quad A \quad &\mapsto \mathrm{diag}(z^{p_1}, z^{p_2}, z^{p_3}) \cdot A \cdot \mathrm{diag}(1, 1, \overline{z}^{p_1+p_2+p_3}). \end{aligned}$$

J. Eschenburg investigated these biquotients in [Esc82] showing that they admit metrics of positive sectional curvature coming from SU(3).

Our considerations focus on a different construction of positive sectional curvature metrics on these space due to [Dea04].

Theorem 6.5.6 ([Dea04])**.** *Let $(M, g, \xi_i, \eta_i, \varphi_i)$ be a 3-Sasaki manifold and assume that the leaf space $N = M/\mathcal{F}$ has positive sectional curvature. Then (M, g_t) with $g_t = g|_{\mathcal{H}} + tg_{\mathcal{V}}$ for $t < 1/2$ have positive sectional curvature.*

We remark that the metric g_t is exactly the metric obtained by $\mathcal{H}$-homothetic deformation with parameters $a = 1$, $b < -1/2$ and $0 < c < 1/\sqrt{2}$, compare (3.10). Considering that for 3-Sasaki manifolds $\alpha = \delta = 1$ this corresponds to those metrics for 3-(α, δ)-Sasaki manifolds with $\beta > 0$. This makes Theorem 6.5.6 a positive sectional curvature version of Theorem 6.3.9.

O. Dearricott then proved positive sectional curvature of $\mathcal{O}(p_1, p_2, p_3)$ under an additional condition on the p_i.

Theorem 6.5.7 ([Dea04])**.** *If $\sqrt{2}\min\{p_1, p_2, p_3\} > \max\{p_1, p_2, p_3\}$ then $\mathcal{O}(p_1, p_2, p_3)$ has positive sectional curvature.*

We expect to show even strongly positive curvature of $\mathcal{O}(p_1, p_2, p_3)$ under possibly further conditions on the weights. We mention one final case strengthening this expectation. If $p_1 = p_2 = p_3 = 1$ the 3-Sasaki manifold $S(p_1, p_2, p_3) = W^{1,1} = \mathrm{SU}(3)/S^1$ is the homogeneous Aloff-Wallach space encountered before and we have implicitly used in the proof of Theorem 6.4.2 that $\mathcal{O}(p_1, p_2, p_3) = \mathbb{C}P^2$ is strongly positive with a sufficiently small multiple of the volume form.

Bibliography

[Agr06] Ilka Agricola. "The Srní lectures on non-integrable geometries with torsion". *Arch. Math. (Brno)* 42.suppl. (2006), pp. 5–84.

[AD20] Ilka Agricola and Giulia Dileo. "Generalizations of 3-Sasakian manifolds and skew torsion". *Adv. Geom.* 20.3 (2020), pp. 331–374.

[ADS21a] Ilka Agricola, Giulia Dileo, and Leander Stecker. "Homogeneous non-degenerate $3-(\alpha,\delta)$-Sasaki manifolds and submersions over quaternionic Kähler spaces". *Ann. Global Anal. Geom.* 60.1 (2021), pp. 111–141.

[ADS21b] Ilka Agricola, Giulia Dileo, and Leander Stecker. "Curvature properties of 3-(α,δ)-Sasaki manifolds". to appear. 2021.

[AFS15] Ilka Agricola, Ana Cristina Ferreira, and Reinier Storm. "Quaternionic Heisenberg groups as naturally reductive homogeneous spaces". *Int. J. Geom. Methods Mod. Phys.* 12.8 (2015), p. 10.

[Ale75] Dmitrii V. Alekseevskii. "Classification of quaternionic spaces with a transitive solvable group of motions". *Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya* 39.2 (1975), pp. 315–362.

[AS53] Warren Ambrose and Isadore M. Singer. "A theorem on holonomy". *Trans. Am. Math. Soc.* 75 (1953), pp. 428–443.

[Ber55] Marcel Berger. "Sur les groupes d'holonomie homogènes de variétés à connexion affine et des variétés riemanniennes". *Bull. Soc. Math. Fr.* 83 (1955), pp. 279–330.

[Ber61] Marcel Berger. "Les variétés riemanniennes homogènes normales simplement connexes à courbure strictement positive". *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze* 15.3 (1961), pp. 179–246.

[Ber66] Marcel Berger. "Trois remarques sur les variétés riemanniennes à courbure positive". *C. R. Acad. Sci., Paris, Sér. A* 263 (1966), pp. 76–78.

[Bes08] Arthur L. Besse. *Einstein manifolds.* Classics in Mathematics. Reprint of the 1987 edition. Springer-Verlag, Berlin, 2008, pp. xii+516.

[BM18] Renato G. Bettiol and Ricardo A. E. Mendes. "Strongly positive curvature". *Ann. Global Anal. Geom.* 53.3 (2018), pp. 287–309.

[BL21] Christoph Böhm and Ramiro A. Lafuente. "Non-compact Einstein manifolds with symmetry" (2021). arXiv: `2107.04210 [math.DG]`.

[BW58] William M. Boothby and Hsien C. Wang. "On contact manifolds". *Ann. Math. (2)* 68 (1958), pp. 721–734.

[BG08] Charles P. Boyer and Krzysztof Galicki. *Sasakian geometry.* Oxford Mathematical Monographs. Oxford University Press, Oxford, 2008, pp. xii+613.

[BGM94] Charles P. Boyer, Krzysztof Galicki, and Benjamin M. Mann. "The geometry and topology of 3-Sasakian manifolds". *J. Reine Angew. Math.* 455 (1994), pp. 183–220.

[CDY16] Beniamino Cappelletti-Montano, Antonio De Nicola, and Ivan Yudin. "Cosymplectic p-spheres". *J. Geom. Phys.* 100 (2016), pp. 68–79.

[Car26a] É. Cartan. "Les groupes d'holonomie des espaces généralisés." *Acta Math.* 48 (1926), pp. 1–42.

[Car26b] E. Cartan. "Sur une classe remarquable d'espaces de Riemann. I, II." *Bull. Soc. Math. Fr.* 54 (1926), pp. 214–264.

[CMS21] Richard Cleyton, Andrei Moroianu, and Uwe Semmelmann. "Metric connections with parallel skew-symmetric torsion". *Adv. Math.* 378 (2021), pp. 107519, 50.

[CS04] Richard Cleyton and Andrew Swann. "Einstein metrics via intrinsic or parallel torsion". *Math. Z.* 247.3 (2004), pp. 513–528.

[Cor00] Vicente Cortés. "A new construction of homogeneous quaternionic manifolds and related geometric structures". *Mem. Amer. Math. Soc.* 147.700 (2000), pp. viii+63.

[Dea04] Owen Dearricott. "Positive sectional curvature on 3-Sasakian manifolds". *Ann. Global Anal. Geom.* 25.1 (2004), pp. 59–72.

[Dil21] Giulia Dileo. "Remarkable Classes of Almost 3-Contact Metric Manifolds". *Axioms* 10.1 (2021).

[DOP20] Cristina Draper, Miguel Ortega, and Francisco J. Palomo. "Affine connections on 3-Sasakian homogeneous manifolds". *Math. Z.* 294.1-2 (2020), pp. 817–868.

[Esc82] Jost-Hinrich Eschenburg. "New examples of manifolds with strictly positive curvature". *Invent. Math.* 66 (1982), pp. 469–480.

[For00a] Brendan Foreman. "Boothby-Wang fibrations on complex contact manifolds". *Differ. Geom. Appl.* 13.2 (2000), pp. 179–196.

[For00b] Brendan Foreman. "Complex contact manifolds and hyperkähler geometry". *Kodai Math. J.* 23.1 (2000), pp. 12–26.

[FI02] Thomas Friedrich and Stefan Ivanov. "Parallel spinors and connections with skew-symmetric torsion in string theory". *Asian J. Math.* 6.2 (2002), pp. 303–335.

[Gra76] Alfred Gray. "The structure of nearly Kähler manifolds". *Math. Ann.* 223 (1976), pp. 233–248.

[Kas01] Toyoko Kashiwada. "On a contact 3-structure". *Math. Z.* 238.4 (2001), pp. 829–832.

[Kle72] Felix Klein. *Vergleichende Betrachtungen über neuere geometrische Forschungen.* Programm zum Eintritt in die philosophische Facultät und den Senat der Universität zu Erlangen. Erlangen. A. Deichert (1872). 1872.

[KN96] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry. Vol. I.* Wiley Classics Library. Reprint of the 1963 original, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1996, pp. xii+329.

[LS94] Claude LeBrun and Simon Salamon. "Strong rigidity of positive quaternion-Kähler manifolds". *Invent. Math.* 118.1 (1994), pp. 109–132.

[MM03] I. Moerdijk and J. Mrčun. *Introduction to foliations and Lie groupoids.* Vol. 91. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2003, pp. x+173.

[ONe66] Barrett O'Neill. "The fundamental equations of a submersion". *Mich. Math. J.* 13 (1966), pp. 459–469.

[Pet06] Peter Petersen. *Riemannian geometry.* Second. Vol. 171. Graduate Texts in Mathematics. Springer, New York, 2006, pp. xvi+401.

[PV01] Paolo Piccinni and Izu Vaisman. "Foliations with transversal quaternionic structures". *Ann. Mat. Pura Appl. (4)* 180.3 (2001), pp. 303–330.

[Rha52] Georges de Rham. "Sur la réductibilité d'un espace de Riemann". *Comment. Math. Helv.* 26 (1952), pp. 328–344.

[SH62] Shigeo Sasaki and Yoji Hatakeyama. "On differentiable manifolds with contact metric structures". *J. Math. Soc. Japan* 14 (1962), pp. 249–271.

[Sto17] Reinier Storm. "The classificatio of naturally reductive homogeneous spaces in dimension 7 and 8". PhD thesis. Marburg: Philipps-Universität, Feb. 2017.

[Tri92] Franco Tricerri. "Locally homogeneous Riemannian manifolds". *Rend. Semin. Mat., Torino* 50.4 (1992), pp. 411–426.

[Wad75] A. W. Wadsley. "Geodesic foliations by circles". *J. Differ. Geom.* 10 (1975), pp. 541–549.

[Wal72] Nolan R. Wallach. "Compact homogeneous Riemannian manifolds with strictly positive curvature". *Ann. of Math. (2)* 96 (1972), pp. 277–295.

[Wat84] Bill Watson. "Almost contact metric 3-submersions". *Internat. J. Math. Math. Sci.* 7.4 (1984), pp. 667–688.