Some Representation stability results and generalizations

Dissertation

zur Erlangung des Doktorgrades der Naturwissenschaften (Dr. rer. nat)

vorgelegt dem
Fachbereich Mathematik und Informatik
der Philipps-Universität Marburg von

Artur Rapp
M. Sc. Mathematik
aus Tschimkent in Kasachstan

Marburg, 2020

Erstgutachter und Betreuer: Prof. Dr. Volkmar Welker
Zweitgutachter: Prof. Dr. John Shareshian
Eingereicht: 06.01.2020
Tag der Disputation: 08.06.2020
Erscheinungsort: Marburg
Erscheinungsjahr: 2020
Hochschulkennziffer: 1180

Contents

1. Introduction 2
2. Stability for arrangements defined by integer partitions 4
2.1. Main theorem and proof 4
2.2. Improved stability bounds for k-equal arrangements 7
2.3. Stability in the homology of k-equal partition lattices 11
3. Products of stabilizing representations 12
3.1. Introduction 12
3.2. Reduction to homogeneous symmetric functions 15
3.3. Proof of Proposition 3.15 20
4. Relative arrangements of linear subspaces 32
4.1. Introduction 32
4.2. Basics of diagrams of spaces 33
4.3. Goresky-MacPherson formula for relative arrangements 36
References 38
Erklärung 40
Deutsche Zusammenfassung 41
Lebenslauf 43

1. Introduction

Representation stability in the sense of Church and Farb (see [5] and [4]) is a property of sequences of symmetric group representations. For every number $n \in \mathbb{N}$ we write S_n for the symmetric group on $\{1, 2, ..., n\}$. The following basic facts about symmetric group representations can be found in [16]. The irreducible representations of S_n are indexed by integer partitions. An integer partition $\lambda \vdash n$ is a finite sequence $\lambda = (\lambda_1, ..., \lambda_l)$ of positive integers with $\lambda_1 \geq ... \geq \lambda_l$ and $|\lambda| := \sum_{i=1}^{l} \lambda_i = n$. We sometimes write $(1^{m_1}, ..., n^{m_n})$ for $\lambda = (\lambda_1, ..., \lambda_l)$ where every $m_i = m_i(\lambda)$ is the number of occurences of i in $(\lambda_1, ..., \lambda_l)$. We write S^λ for the irreducible representation corresponding to λ and s_λ for its Frobenius characteristic. We refer to [12] for background on symmetric functions. For every $n \in \mathbb{N}$ the functions s_λ with $|\lambda| = n$ are called Schur functions and form a $\mathbb{Z}$-basis of Λ_n, the group of symmetric functions whose monomials all have degree n. The function $h_n := s_{(n)}$ is called a complete homogeneous symmetric function and $e_n := s_{(1^n)}$ is called an elementary symmetric function. Now we introduce representation stability in the sense of Church and Farb. Let $\lambda = (\lambda_1, ..., \lambda_l) \vdash n$. Then $\lambda + (1) := (\lambda_1 + 1, \lambda_2, ..., \lambda_l) \vdash n + 1$. If an S_n-module V has a decomposition

$$V = \bigoplus_{\lambda \vdash n} a_\lambda S^\lambda$$

I thank Volkmar Welker for proposing the topic of this thesis and for the many helpful conversations and suggestions.

then we define

$$V + (1) = \bigoplus_{\lambda \vdash n} a_\lambda S^{\lambda + (1)}.$$

Similarly, if a symmetric function f has a decomposition

$$f = \sum_{\lambda \vdash n} a_\lambda s_\lambda$$

then we define

$$f + (1) = \sum_{\lambda \vdash n} a_\lambda s_{\lambda + (1)}.$$

Next, we look at sequences $\{V_n\}_{n \geq 0}$ of S_{n+n_0}-representations or sequences of their characteristics. Such a sequence stabilizes at $m \geq n_0$ if

$$V_n = V_{n-1} + (1) \text{ for all } n > m.$$

The sequence stabilizes sharply at $m \geq n_0$ if m is the smallest integer such that

$$V_n = V_{n-1} + (1) \text{ for all } n > m.$$

In Chapter 2, we consider arrangements of diagonal subspaces of $(\mathbb{R}^d)^n$ for natural numbers d and n. For a finite arrangement $\mathcal{A}$ of linear subspaces of $(\mathbb{R}^d)^n$, we define the union $U_{\mathcal{A}} = \cup_{V \in \mathcal{A}} V$ and the complement $\mathcal{M}_{\mathcal{A}} = (\mathbb{R}^d)^n \setminus U_{\mathcal{A}}$. The intersection lattice $L_{\mathcal{A}}$ is the set of intersections of arbitrarily many elements of $\mathcal{A}$ ordered by reverse inclusion. The least element $\hat{0}$ is $(\mathbb{R}^d)^n$, the empty intersection, and the greatest element $\hat{1}$ is the intersection of all elements of $\mathcal{A}$. For a subset T of $L_{\mathcal{A}}$ the join sublattice of $L_{\mathcal{A}}$ generated by T consists of all intersections of arbitrarily many elements of T also ordered by reverse inclusion. If $\mathcal{A}$ is the arrangement of diagonal subspaces given by all equations of the form $w_i = w_j$ for $1 \leq i < j \leq n$, $w = (w_1, ..., w_n) \in (\mathbb{R}^d)^n$, the intersection lattice $L_{\mathcal{A}}$ is isomorphic to the lattice Π_n of set partitions of $\{1, ..., n\}$. For a set partition π of $\{1, ..., n\}$ let W_π^d be the linear subspace of n-tuples $(w_1, ..., w_n)$ of points in $\mathbb{R}^d$ such that $w_i = w_j$ whenever i and j are in the same block of π. We also write π for the corresponding subspace W_π^d of $(\mathbb{R}^d)^n$. If $\pi \in \Pi_n$ is a set partition into the subsets $B_1, ..., B_l$ of $\{1, ..., n\}$ called blocks of π, we write $\pi = B_1|...|B_l$. In this notation, we have $\hat{0} = \{1\}|\{2\}|...|\{n\}$. The set partition $\pi = B_1|...|B_l$ is said to be finer than $\pi' = C_1|...|C_m$, if for every $1 \leq i \leq l$ there is a $1 \leq j \leq m$ such that $B_i \subseteq C_j$. We may reorder the sets $B_1, ..., B_l$ such that $\#B_1 \geq ... \geq \#B_l$. The integer partition $(\#B_1, ..., \#B_l)$ is then called the type of π. If Λ is a set of integer partitions of n, then Π_Λ is the join sublattice of Π_n generated by all set partitions of type λ for all $\lambda \in \Lambda$. For an integer partition λ we denote by $\mathcal{A}_\lambda^d$ the arrangement of all subspaces W_π^d such that π is of type λ. More generally, set $\mathcal{A}_\Lambda^d = \cup_{\lambda \in \Lambda} \mathcal{A}_\lambda^d$ for every finite set Λ of integer partitions of n. The complement $\mathcal{M}_\Lambda^d = (\mathbb{R}^d)^n \setminus \cup_{W \in \mathcal{A}_\Lambda^d} W$ is a real manifold. If $\Lambda = \{\lambda\}$, we write $\mathcal{M}_\lambda^d$ for $\mathcal{M}_\Lambda^d$. The action of the symmetric group S_n on n-tuples of points in $\mathbb{R}^d$ by permuting the coordinates induces an S_n-representation on the reduced singular cohomology $\tilde{H}^i(\mathcal{M}_\Lambda^d, \mathbb{C})$. Formulas for these S_n-representations were determined by Sundaram and Welker in [19]. We look into representation stability of these modules.

Our main purpose in Chapter 2 is to prove that sequences of these modules stabilize, and

to obtain stabilization bounds. This is the content of Theorem 2.1. The fact that this sequence stabilizes can also be deduced by results of Gadish ([8, Theorem A]) and Petersen ([13, Theorem 4.15]). Their theorems do not provide bounds. The case $\Lambda = \{(2, 1^{n-2})\}$ was proved by Church ([4, Theorem 1]) and for this case Hersh and Reiner provided the exact stabilization bounds ([11, Theorem 1.1]). The results of Chapter 2 are published in [14].
In Chapter 3, we define a generalized kind of stability for sequences of representations. Motivated by Lemma 2.2 where we show that the product of a stabilizing sequence with a constant sequence also stabilizes, we show in Chapter 3 that sequences obtained as products of stabilizing sequences fulfill certain recursive relations in a way that generalizes the definition of representation stability. We use methods from the theories of symmetric functions and polytopes. These results can be found in [15].
The aim of Chapter 4 is to look for further examples where representation stability occurs. We look at relative subspace arrangements, i.e. pairs of arrangements $(\mathcal{A}, \mathcal{B})$ such that $U_{\mathcal{B}} \subseteq U_{\mathcal{A}}$. For sequences $\{(\mathcal{A}_n, \mathcal{B}_n)\}_n$ of relative arrangements one can ask whether the sequence $\{\tilde{H}^i(U_{\mathcal{A}_n} \setminus U_{\mathcal{B}_n}, \mathbb{C})\}_n$ stabilizes. We take one step into this direction by deriving a Goresky-MacPherson like formula ([9],[19, Theorem 2.5(ii)],[2, Theorem 2.1]) for relative arrangements. Our formula corrects a formula from [21, Theorem 4.8] where too weak assumptions are formulated. For deriving the formula we use an approach from Ziegler and Živaljević ([23]). We study homotopy colimits. In particular, we give elementary proofs of G-equivariant versions of classical results from this theory. We see at an example that stability will not hold in general but we believe that it could hold for nice classes of relative arrangements.

2. Stability for arrangements defined by integer partitions

2.1. **Main theorem and proof.** For an integer partition λ we write $l(\lambda)$ for its length i.e. its number of parts. As in [11, Definition 2.5] let $\operatorname{rank}(\lambda) := |\lambda| - l(\lambda)$ be the rank of λ. Note that set partitions of type λ have $\operatorname{rank}(\lambda)$ as their poset rank in the partition lattice. Now we formulate the main theorem of this chapter.

Theorem 2.1. *Let Λ be a nonempty finite set of integer partitions of the number n_0 not containing (1^{n_0}). For every $n \geq n_0$ let $\Lambda^{(n)}$ be the set of all integer partitions of n obtained from integer partitions in Λ by adding $n - n_0$ parts of size 1. Let* $\operatorname{rank}(\Lambda) = \min\{\operatorname{rank}(\lambda) \mid \lambda \in \Lambda\}$*. For every i and $d \geq 2$ the sequence $\{\tilde{H}^i(\mathcal{M}^d_{\Lambda^{(n)}}, \mathbb{C})\}_n$ stabilizes at $4(i + 1 - \operatorname{rank}(\Lambda))/(d-1)$.*

The following lemma is a generalization of [11, Lemma 2.2]. For integer partitions ν, λ and μ with $\mu \subseteq \nu$, we write $LR^{\nu}_{\mu,\lambda}$ for the set of all Littlewood-Richardson tableaux of shape ν/μ and weight λ. A Littlewood-Richardson tableaux T of shape ν/μ and weight λ is a semistandard skew tableau of shape ν/μ whose boxes are labeled with λ_1 1's, λ_2 2's etc. and concatening the reversed rows of T from top to bottom yields a word w with the property: In every initial part of w the integer i occurs at least as often as $i + 1$ for every $i \geq 1$.

Lemma 2.2. *Let λ and α be integer partitions. For every $n \geq \alpha_1$ we consider the integer partition $(n, \alpha) = (n, \alpha_1, \alpha_2, ...)$. The sequence $\{s_{(n,\alpha)} s_\lambda\}_n$ stabilizes sharply at $\lambda_1 + \alpha_1$. In other words*

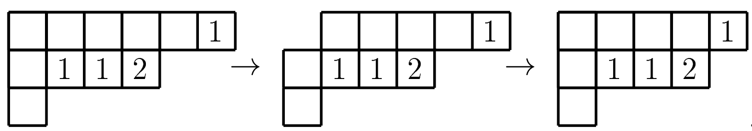
$$s_{(n,\alpha)} s_\lambda = s_{(n-1,\alpha)} s_\lambda + (1)$$

if and only if $n > \lambda_1 + \alpha_1$.

Proof. Suppose $n > \lambda_1 + \alpha_1$. Let ν be an integer partition of $n + |\lambda| + |\alpha|$ with $(n, \alpha) \subseteq \nu$. By the Littlewood-Richardson rule (see [12]) the multiplicity of s_ν in $s_{(n,\alpha)} s_\lambda$ is $\#LR^{\nu}_{(n,\alpha),\lambda}$. Let ν' be the integer partition of $n-1+|\lambda|+|\alpha|$ obtained from ν by replacing ν_1 by $\nu_1 - 1$. We define the map

$$\phi : LR^{\nu}_{(n,\alpha),\lambda} \to LR^{\nu'}_{(n-1,\alpha),\lambda}$$

by the following procedure: Remove the first empty box in the first row of the tableau and then move all other boxes of the first row one place to the left. The two steps are illustrated below with $n = 5,\ \alpha = (1, 1),\ \lambda = (3, 1)$ and $\nu = (6, 4, 1)$:



We want to show that the resulting tableau is indeed a Littlewoood-Richardson tableau so that ϕ is well defined. The only condition that has to be checked is that, in the first two rows, we have no two 1's lying in the same column. But this follows from the inequality $\nu_1 \geq n$, since $n > \lambda_1 + \alpha_1$ implies that n is larger than the number α_1 of empty boxes in the second row plus the number of 1's in the second row. Note that ϕ has an inverse map: Given a tableau in $LR^{\nu'}_{n-1,\lambda}$ we move the first row one place to the right and put an empty box in the gap. So ϕ is bijective and $\#LR^{\nu}_{(n,\alpha),\lambda} = \#LR^{\nu'}_{(n-1,\alpha),\lambda}$. This shows that $\{s_{(n,\alpha)} s_\lambda\}_n$ stabilizes at $\lambda_1 + \alpha_1$ or sooner. Now let $n = \lambda_1 + \alpha_1$ and $\nu = (n, n, \lambda_2 + \alpha_2, \lambda_3 + \alpha_3, ...)$. There is a Littlewood-Richardson tableau of shape $\nu/(n, \alpha)$ and weight λ: We look at the Ferrers diagram of ν and put λ_1 1's at the end of the second row, λ_2 2's at the end of the third row and so on. It follows that we have a Schur function s_ν with $\nu_1 = \nu_2$ and multiplicity greater than or equal to 1 in the decomposition of $s_{(n,\alpha)} s_\lambda$. This shows that $s_{(n,\alpha)} s_\lambda$ cannot equal $f + (1)$ for any symmetric function f, completing the proof of sharpness. ∎

Though the special case of Lemma 2.2 where $\alpha = ()$ ([11, Lemma 2.2]) suffices to prove our main results, Theorem 2.1 and Theorem 2.5, the general case might also be of interest as we show in Section 4.

Proof of Theorem 2.1. By [19, Theorem 2.5(ii)] and [2, Theorem 2.1] we have

$$\tilde{H}^i(\mathcal{M}^d_{\Lambda^{(n)}}, \mathbb{C}) = \bigoplus_{\pi \in (\Pi^{>\hat{0}}_{\Lambda^{(n)}})/S_n} \mathrm{Ind}^{S_n}_{(S_n)_\pi} (\tilde{H}_{\mathrm{codim}(\pi)-i-2}((\hat{0}, \pi), \mathbb{C}) \otimes \tilde{H}_{\mathrm{codim}(\pi)-1}(S^{dn-1} \cap \pi^\perp, \mathbb{C})).$$

$(\Pi^{>\hat{0}}_{\Lambda^{(n)}})/S_n$ is a set of representatives of the action of S_n on $\Pi_{\Lambda^{(n)}}$ excluding $\hat{0}$. $(S_n)_\pi$ is the stabilizer subgroup of π. $\tilde{H}_j((\hat{0}, \pi), \mathbb{C})$ is the reduced simplicial homology on the order complex $\Delta((\hat{0}, \pi))$ in degree $j \geq -1$. The number $\mathrm{codim}(\pi)$ is the codimension of π as a real subspace of $\mathbb{R}^{dn}$ and S^{dn-1} is the unit sphere in $\mathbb{R}^{dn}$. If π is of type $\mu = (1^{m_1(\mu)}, 2^{m_2(\mu)}, \ldots) \vdash n$, then its stabilizer $(S_n)_\pi$ is the product of wreath products $\prod_j S_{m_j(\mu)}[S_j]$ and $\mathrm{codim}(\pi) = d(n - l(\mu)) = d \cdot \mathrm{rank}(\mu)$. The length of a chain in Π_n from $\hat{0}$ to π is less than or equal to $\sum_{j=1}^{l(\mu)}(\mu_j - 1) = n - l(\mu) = \mathrm{rank}(\mu)$. Since the atoms in Π_Λ are of shape λ for $\lambda \in \Lambda$, the length of a chain in Π_Λ from $\hat{0}$ to π is less than or equal to $\mathrm{rank}(\mu) - \mathrm{rank}(\Lambda) + 1$ and contributes to homology in degree less than or equal to $\mathrm{rank}(\mu) - \mathrm{rank}(\Lambda) - 1$. It follows that if the homology $\tilde{H}_{\mathrm{codim}(\pi)-i-2}((\hat{0}, \pi), \mathbb{C})$ is not zero, then

$$-1 \leq d \cdot \mathrm{rank}(\mu) - i - 2 \leq \mathrm{rank}(\mu) - \mathrm{rank}(\Lambda) - 1$$

and then

$$(i+1)/d \leq \mathrm{rank}(\mu) \leq (i + 1 - \mathrm{rank}(\Lambda))/(d-1).$$

Let $\tilde{\mu}$ be the integer partition obtained from μ by removing the parts of size 1. The rank of μ and the rank of $\tilde{\mu}$ are the same. From [11, Proposition 2.8], we have $\mathrm{rank}(\tilde{\mu}) + 1 \leq |\tilde{\mu}| \leq 2 \cdot \mathrm{rank}(\tilde{\mu})$. This yields

$$1 + (i+1)/d \leq |\tilde{\mu}| \leq 2(i + 1 - \mathrm{rank}(\Lambda))/(d-1).$$

The subgroup $S_{m_1(\mu)}[S_1] \cong S_{m_1(\mu)}$ acts trivially on $\tilde{H}_{\mathrm{codim}(\pi)-i-2}((\hat{0}, \pi), \mathbb{C})$. The coordinates of vectors in the space $\pi^\perp$ which correspond to the singletons of π are zero. It follows that the above copy of $S_{m_1(\mu)}$ acts trivially on $\tilde{H}_{\mathrm{codim}(\pi)-1}(S^{dn-1} \cap \pi^\perp, \mathbb{C})$. Let $S^{(m_1(\mu))}$ be the trivial $S_{m_1(\mu)}$-module. We get the following isomorphism of $\prod_{j\geq 1} S_{m_j(\mu)}[S_j]$-modules:

$$\begin{aligned} &\tilde{H}_{\mathrm{codim}(\pi)-i-2}((\hat{0}, \pi), \mathbb{C}) \otimes \tilde{H}_{\mathrm{codim}(\pi)-1}(S^{dn-1} \cap \pi^\perp, \mathbb{C}) \\ \cong\ & S^{(m_1(\mu))} \otimes (\tilde{H}_{\mathrm{codim}(\pi)-i-2}((\hat{0}, \pi), \mathbb{C}) \otimes \tilde{H}_{\mathrm{codim}(\pi)-1}(S^{dn-1} \cap \pi^\perp, \mathbb{C})). \end{aligned}$$

We consider the interval $(\hat{0}, \pi)$ in $\Pi_{\Lambda^{(n)}}$. The atoms in $(\hat{0}, \pi)$ have at least $n - n_0$ singletons. If we delete $\min\{n - |\tilde{\mu}|, n - n_0\}$ many singletons from π, after renumbering we can view $(0, \pi)$ as an interval in $\Pi_{\Lambda^{(max\{|\tilde{\mu}|, n_0\})}}$. We may also ignore the coordinates of vectors in $\pi^\perp$ which correspond to the singletons of π. We have $\mathrm{codim}(\pi) = d \cdot \mathrm{rank}(\tilde{\mu})$. It follows that the $\prod_{j\geq 2} S_{m_j(\mu)}[S_j]$-module

$$\tilde{H}_{\mathrm{codim}(\pi)-i-2}((\hat{0}, \pi), \mathbb{C}) \otimes \tilde{H}_{\mathrm{codim}(\pi)-1}(S^{dn-1} \cap \pi^\perp, \mathbb{C})$$

does not depend on n and we write $V_{\tilde{\mu}}$ for it. Using the transitivity of induction on $\prod_{j\geq 1} S_{m_j(\mu)}[S_j] \leq S_{m_1(\mu)} \times S_{n-m_1(\mu)} \leq S_n$ we get:

$$\begin{aligned} &\mathrm{Ind}_{\prod_{j\geq 1} S_{m_j(\mu)}[S_j]}^{S_n}(S^{m_1(\mu)} \otimes V_{\tilde{\mu}}) \\ &= \mathrm{Ind}_{S_{m_1(\mu)} \times S_{n-m_1(\mu)}}^{S_n}(S^{m_1(\mu)} \otimes \mathrm{Ind}_{\prod_{j\geq 2} S_{m_j(\mu)}[S_j]}^{S_{n-m_1(\mu)}}(V_{\tilde{\mu}})) \\ &= \mathrm{Ind}_{S_{n-|\tilde{\mu}|} \times S_{|\tilde{\mu}|}}^{S_n}(S^{n-|\tilde{\mu}|} \otimes \mathrm{Ind}_{\prod_{j\geq 2} S_{m_j(\tilde{\mu})}[S_j]}^{S_{|\tilde{\mu}|}}(V_{\tilde{\mu}})). \end{aligned}$$

Let

$$f_{\tilde{\mu}} := \mathrm{ch}(\mathrm{Ind}_{\prod_{j\geq 2} S_{m_j(\tilde{\mu})}[S_j]}^{S_{|\tilde{\mu}|}}(V_{\tilde{\mu}})).$$

We have

$$\operatorname{ch}(\operatorname{Ind}^{S_n}_{S_{n-|\tilde{\mu}|}\times S_{|\tilde{\mu}|}}(S^{n-|\tilde{\mu}|}\otimes \operatorname{Ind}^{S_{|\tilde{\mu}|}}_{\prod_{j\geq 2} S_{m_j(\tilde{\mu})}[S_j]}(V_{\tilde{\mu}}))) \\ = h_{n-|\tilde{\mu}|}f_{\tilde{\mu}}$$

where $h_{n-|\tilde{\mu}|}=s_{(n-|\tilde{\mu}|)}$. It follows that the characteristic of $\tilde{H}^i(\mathcal{M}^d_{\Lambda^{(n)}},\mathbb{C})$ is

$$\sum_{\substack{\tilde{\mu} \text{ an integer partition with no parts of size } 1,\\ 1+(i+1)/d\leq|\tilde{\mu}|\leq 2(i+1-\operatorname{rank}(\Lambda))/(d-1)}} h_{n-|\tilde{\mu}|}f_{\tilde{\mu}}.$$

From Lemma 2.2, it follows that the sequence stabilizes at a number larger than $2|\tilde{\mu}|$ for every $\tilde{\mu}$ occurring in the sum. This is fulfilled at $4(i+1-\operatorname{rank}(\Lambda))/(d-1)$. ■

2.2. **Improved stability bounds for k-equal arrangements.** We consider the sequence $\{\tilde{H}^i(\mathcal{M}^d_{(k,1^{n-k})},\mathbb{C})\}_n$ for $k\geq 2$. Theorem 2.1 states that stabilization occurs at $4(i+2-k)/(d-1)$. First we have a closer look at the special case $k=2$. In this case, stabilization occurs at $4i/(d-1)$. We compare this to the known results in the literature which focus on the case $k=2$: By [4, Theorem 1] we have stabilization at $2i$ for $d\geq 3$ and stabilization at $4i$ for $d=2$. By [11, Theorem 1.1] we have the following for $i\geq 1$. The sequence is zero from the beginning, if $d-1$ does not divide i. Otherwise it stabilizes sharply at $3i/(d-1)$ for odd $d\geq 3$ and it stabilizes sharply at $3i/(d-1)+1$ for even $d\geq 2$.
Now we consider $\{\tilde{H}^i(\mathcal{M}^d_{(k,1^{n-k})},\mathbb{C})\}_n$ for general $k\geq 2$. The stability of this sequence was also considered by Gadish ([8, Example 6.11]) as an example of his general results. We want to determine smaller upper bounds than the ones given in Theorem 2.1 where stabilization occurs for $k\geq d+1$. Let $h_n=s_{(n)}$ be the complete homogeneous symmetric function, $e_n=s_{(1^n)}$ the elementary symmetric function and ω the involutive ring homomorphism of the ring of symmetric functions with $\omega(h_n)=e_n$. We write π_n for the characteristic of $\tilde{H}^{n-3}(\Delta(\Pi_n),\mathbb{C})$ and $l_n=\omega(\pi_n)$. For symmetric functions f and g we write $f[g]$ for the plethysm of these two functions.

Theorem 2.3. [19, Theorem 4.4(iii)] *Let $d\geq 2$, $k\geq 2, i\geq 0$ and $n\geq 1$. Let $U_k:=\sum_{j\geq k}s_{(j-k+1,1^{k-1})}$. For every $r,t\geq 1$ and $q\geq 0$ such that $i=(d-1)(n-r-q)+t(k-2)$ let $\psi_{n,q,r,t}$ be*

$$\begin{cases} \omega\left(\omega^k\left(e_r[\sum_{j\geq 1}l_j]\right)|_{\deg\ t}[U_k]\right)|_{\deg\ n-q}h_q & \text{if } d \text{ is even} \\ \left(\left(h_r[\sum_{j\geq 1}l_j]\right)|_{\deg\ t}[U_k]\right)|_{\deg\ n-q}h_q & \text{if } d \text{ is odd and } k \text{ is even}. \\ \left(\left((-1)^t h_r[\sum_{j\geq 1}(-1)^j\pi_j]\right)|_{\deg\ t}[U_k]\right)|_{\deg\ n-q}h_q & \text{if } d \text{ and } k \text{ are odd} \end{cases}$$

Then the characteristic of the S_n-representation on $\tilde{H}^i(\mathcal{M}^d_{(k,1^{n-k})},\mathbb{C})$ is

$$\sum_{r,t\geq 1,q\geq 0: i=(d-1)(n-r-q)+t(k-2)}\psi_{n,q,r,t}.$$

Lemma 2.4. *Let $d \geq 2$, $k \geq d+1, i \geq 0$ and $n \geq 1$. Let $r, t \geq 1$ and $q \geq 0$ be such that $i = (d-1)(n-r-q) + t(k-2)$. Let $U_k := \sum_{j \geq k} s_{(j-k+1,1^{k-1})}$ and $\psi_{n,q,r,t}$ be*

$$\begin{cases} \omega\left(\omega^k\left(e_r[\sum_{j\geq 1} l_j]\right)|_{\deg\ t}[U_k]\right)|_{\deg\ n-q} h_q & \text{if } d \text{ is even} \\ \left(\left(h_r[\sum_{j\geq 1} l_j]\right)|_{\deg\ t}[U_k]\right)|_{\deg\ n-q} h_q & \text{if } d \text{ is odd and } k \text{ is even .} \\ \left(\left((-1)^t h_r[\sum_{j\geq 1}(-1)^j \pi_j]\right)|_{\deg\ t}[U_k]\right)|_{\deg\ n-q} h_q & \text{if } d \text{ and } k \text{ are odd} \end{cases}$$

Then

(i) $\psi_{n,q,r,t} = \psi_{n-1,q-1,r,t} + (1)$ *if $q > n/2$ and $n \geq 2$.*
(ii) $\psi_{n,q,r,t} = \psi_{n-1,q-1,r,t} + (1)$ *if d is even, $q > tk$ and $n \geq 2$.*
(iii) $\psi_{n,q,r,t} = 0$ *if $r > t$ or $t > n/k$.*
(iv) $\psi_{n,q,r,t} = 0$ *if $q \leq n/2$ and $n > \frac{2i}{d-1}$.*
(v) $\psi_{n,q,r,t} = 0$ *if $k \geq d+2$, $q \leq tk$ and $n > \frac{ki}{k-d-1}$.*

Proof. (i) We have $\psi_{n,q,r,t} = f_{n-q}h_q$ for a symmetric function f_{n-q} of degree $n-q$ and $h_q = s_{(q)}$. From Lemma 2.2, we get

$$\psi_{n,q,r,t} = f_{n-q}h_q = f_{n-q}h_{q-1} + (1) = f_{(n-1)-(q-1)}h_{q-1} + (1) = \psi_{n-1,q-1,r,t} + (1)$$

if $q > n-q$ or equivalently $q > n/2$.
(ii) If d is even, then $\psi_{n,q,r,t} = \omega(f_t[U_k])|_{deg\ n-q}h_q$ for a symmetric function f_t of degree t. The partition of every Schur function in $U_k = \sum_{j\geq k} s_{(j-k+1,1^{k-1})}$ has length k. From [11, Proposition 4.3 (d)] it follows that for every s_λ with $\lambda \vdash n-q$ occurring in the Schur function decomposition of $\omega(f_t[U_k])|_{deg\ n-q}$ the first row of λ has length less than or equal to tk. If $q > tk$, it follows from Lemma 2.2 that

$$\begin{aligned} \psi_{n,q,r,t} &= \omega(f_t[U_k])|_{deg\ n-q}h_q = \omega(f_t[U_k])|_{deg\ n-q}h_{q-1} + (1) \\ &= \omega(f_t[U_k])|_{deg\ (n-1)-(q-1)}h_{q-1} + (1) = \psi_{n-1,q-1,r,t} + (1). \end{aligned}$$

(iii) If $r > t$ the terms $e_r[\sum_{j\geq 1} l_j]$, $h_r[\sum_{j\geq 1} l_j]$ and $(-1)^t h_r[\sum_{j\geq 1}(-1)^j\pi_j]$ only have terms of degree greater than t. Then the whole term $\psi_{n,q,r,t}$ is zero. U_k only has terms of degree greater than or equal to k. Then $f_t[U_k]$ for a symmetric function f_t of degree t has only terms of degree greater than or equal to tk. If $t > n/k$ then $tk > n \geq n-q$ and again $\psi_{n,q,r,t}$ is zero.
(iv) Suppose $\psi_{n,q,r,t} \neq 0$. We have to show that $q > n/2$ or $n \leq \frac{2i}{d-1}$. Suppose $q \leq n/2$. From $\psi_{n,q,r,t} \neq 0$ and (ii) we get $r \leq t$. From $q \leq n/2$ and $i = (d-1)(n-r-q) + t(k-2)$ we get

$$\frac{i}{1-d} + n/2 + \frac{t(k-2)}{d-1} \leq \frac{i}{1-d} + n - q + \frac{t(k-2)}{d-1} = r.$$

Using $r \leq t$ we get

$$\frac{i}{1-d} + n/2 + \frac{t(k-2)}{d-1} \leq t$$

and simplifying yields

$$n/2 \leq \frac{i}{d-1} + \frac{t(d+1-k)}{d-1}.$$

Using $k \geq d+1$ we get

$$n \leq \frac{2i}{d-1}.$$

(v) Let $k \geq d+2$. Suppose $\psi_{n,q,r,t} \neq 0$ and $q \leq tk$. We have to show that $n \leq \frac{ki}{k-d-1}$. From $q \leq tk$, $i = (d-1)(n-r-q) + t(k-2)$ and $r \leq t$ by (iii) we get

$$\frac{i}{1-d} + n - tk + \frac{t(k-2)}{d-1} \leq \frac{i}{1-d} + n - q + \frac{t(k-2)}{d-1} = r \leq t.$$

It follows that

$$\frac{i}{1-d} + n - tk + \frac{t(k-2)}{d-1} \leq t$$

and then

$$n \leq \frac{i}{d-1} + t(k + \frac{k-2}{1-d} + 1).$$

From (iii) we know $t \leq n/k$. It follows that

$$n \leq \frac{i}{d-1} + \frac{n}{k}(k + \frac{k-2}{1-d} + 1)$$

and then

$$n(\frac{k-2}{d-1} - 1) \leq \frac{ki}{d-1}.$$

Using $k \geq d+2$ we get

$$n \leq \frac{\frac{ki}{d-1}}{\frac{k-2}{d-1} - 1} = \frac{ki}{k-d-1}.$$

■

Theorem 2.5. *Let $d \geq 2$, $k \geq d+1$ and $i \geq 0$. The sequence $\{\tilde{H}^i(\mathcal{M}^d_{(k,1^{n-k})}, \mathbb{C})\}_n$ stabilizes at $\frac{2i}{d-1}$. If d is even and $k \geq d+2$, the sequence stabilizes at $\frac{ki}{k-d-1}$.*

Proof. From Theorem 2.3, we have that the characteristic of the S_n-representation on $\tilde{H}^i(\mathcal{M}^d_{(k,1^{n-k})}, \mathbb{C})$ is

$$\sum_{r,t \geq 1, q \geq 0 : i = (d-1)(n-r-q)+t(k-2)} \psi_{n,q,r,t}$$

where $\psi_{n,q,r,t}$ is as in the previous lemma. If $q > n/2$ then we get

$$\psi_{n,q,r,t} = \psi_{n-1,q-1,r,t} + (1)$$

from Lemma 2.4(i). From Lemma 2.4 (iv) we get $\psi_{n,q,r,t} = 0$ if $q \leq n/2$ and $n > \frac{2i}{d-1}$. Putting these facts together we get for $n > \frac{2i}{d-1}$:

$$\sum_{r,t\geq 1, q\geq 0: i=(d-1)(n-r-q)+t(k-2)} \psi_{n,q,r,t} = \sum_{r,t\geq 1, q\geq 1: i=(d-1)(n-r-q)+t(k-2)} \psi_{n,q,r,t} =$$

$$\sum_{r,t\geq 1, q\geq 1: i=(d-1)(n-r-q)+t(k-2)} \psi_{n-1,q-1,r,t} + (1) = \sum_{r,t\geq 1, q\geq 0: i=(d-1)(n-1-r-q)+t(k-2)} \psi_{n-1,q,r,t} + (1).$$

Now let d be even and $k \geq d+2$. If d is even and $q > tk$ we have

$$\psi_{n,q,r,t} = \psi_{n-1,q-1,r,t} + (1)$$

from Lemma 2.4(ii) and $\psi_{n,q,r,t} = 0$ if $q \leq tk$ and $n > \frac{ki}{k-d-1}$ from Lemma 2.4 (v). For $n > \frac{ki}{k-d-1}$ the same computation as above yields the stability property. ■

In Table 2.7, we give a list of sharp stability bounds for these representations.

Question 2.6. Is there an explicit formula for the sharp stability bound of $\tilde{H}^i(\mathcal{M}^d_{(k,1^{n-k})}, \mathbb{C})$ for general k, d, i?

Table 2.7 (Sharp stability bounds for $\tilde{H}^i(\mathcal{M}^2_{(k,1^{n-k})}, \mathbb{C})$). [1]
If k is fixed and i grows, the sequence of bounds appears to increase by 1 in most of the steps especially at the beginning and with large k. Later, there also appear steps with bound differences 2 or 3.

$k = 3:$

i	3	4	5	6	7	8	9	10	11	12	13	14
bound	6	7	8	11	13	14	16	18	20	21	23	25

$k = 4:$

i	5	6	7	8	9	10	11	12	13	14	15	16
bound	8	9	10	11	12	15	17	18	19	20	22	24

$k = 5:$

i	7	8	9	10	11	12	13	14	15	16	17	18	19	20
bound	10	11	12	13	14	15	16	19	21	22	23	24	25	26

[1] For the computations Maple 18.01 and the SF-package of J. R. Stembridge (www.math.lsa.umich.edu/ jrs) is used.

$k = 6:$

i	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
bound	12	13	14	15	16	17	18	19	20	23	25	26	27	28	29

$k = 7:$

i	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
bound	14	15	16	17	18	19	20	21	22	23	24	27	29	30	31	32	33

$k = 8:$

i	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
bound	16	17	18	19	20	21	22	23	24	25	26	27	28	31	33	34	35

$k = 9:$

i	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
bound	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	35	37

2.3. **Stability in the homology of k-equal partition lattices.** We showed in Lemma 2.2 that for integer partitions α and λ the sequence $\{s_{(n,\alpha)}s_\lambda\}_n$ stabilizes at $\alpha_1 + \lambda_1$. In this section we give an application of this fact in a situation where α is not the empty partition. For every $2 \leq k \leq n$ we consider the lattice $\Pi_{(k,1^{n-k})}$ of set partitions all of whose block sizes are 1 or greater than or equal to k ordered by reverse refinement. We have $\Pi_{(2,1^{n-2})} = \Pi_n$. Note that $\Pi_{k,1^{n-k}}$ is the intersection lattice of the subspace arrangement with complement $\mathcal{M}^d_{(k,1^{n-k})}$. We recall the following result on the homology of the order complex of $\Pi_{k,1^{n-k}}$:

Theorem 2.8. [18, Corollary 3.6] (i) *Let $3 \leq k \leq n$ and $1 \leq t \leq \lfloor n/k \rfloor$. The characteristic of $\tilde{H}_{n-3-t(k-2)}(\Pi_{(k,1^{n-k})}, \mathbb{C})$ tensored with the sign representation is given by the degree n term in*

$$\omega^k(l_t)\left[\sum_{j\geq k} s_{(j-k+1,1^{k-1})}\right].$$

(ii) *Let $2 = k \leq n$. The characteristic of $\tilde{H}_{n-3}(\Pi_n, \mathbb{C})$ tensored with the sign representation is given by the degree n term in*

$$\sum_{t=1}^{\lfloor n/2 \rfloor} l_t \left[\sum_{j\geq 2} s_{(j-1,1)}\right].$$

By [18, Lemma 3.2] the term $\omega^k(l_t)\left[\sum_{j\geq k} s_{(j-k+1,1^{k-1})}\right]|_{\deg\ n}$ decomposes into the sum

$$\sum_{\lambda} \phi_{k,t,n,\lambda}$$

where

$$\phi_{k,t,n,\lambda} = \omega^k(l_t)|_{\prod_{i\geq 1:\ m_i>0} S_{m_i}}\left[\bigotimes_{j\geq k} s_{(j-k+1,1^{k-1})}\right]$$

and the sum runs over all partitions $\lambda = (\lambda_1, ..., \lambda_t) = (n^{m_n}, ..., k^{m_k})$ of n with t parts and all parts greater than or equal to k. Now we apply Lemma 2.2 to $\{\phi_{k,t,n,\lambda}\}_{\lambda_1}$ where $\lambda_2, ..., \lambda_t, k$ and t are fixed and λ_1 and $n = n(\lambda_1) = \sum_{i\geq 1}\lambda_i$ grow.

Proposition 2.9. *The sequence $\{\phi_{k,t,n,\lambda}\}_{\lambda_1}$ stabilizes at $\lambda_1 = k + \sum_{i\geq 2}\lambda_i$.*

Proof. If $\lambda_1 > \lambda_2$ then $m_{\lambda_1} = 1$ and the restriction $\omega^k(l_t)|_{\prod_{i\geq 1:\ m_i>0} S_{m_i}}$ is the tensor product of the trivial S_1-module and a $\prod_{i\geq 1: i\neq\lambda_1,\ m_i>0} S_{m_i}$-module. We get $\phi_{k,t,n,\lambda} = s_{(\lambda_1-k+1,1^{k-1})}f$ for a symmetric function f of degree $\sum_{i\geq 2}\lambda_i$. It follows from Lemma 2.2 that $\{\phi_{k,t,n,\lambda}\}_{\lambda_1}$ stabilizes at $\lambda_1 = k + \sum_{i\geq 2}\lambda_i$. ■

3. Products of stabilizing representations

3.1. **Introduction.** In this chapter, we formulate all statements about S_n-representations in the world of symmetric functions. We refer to [12] for background on S_n-representations and symmetric functions. We extend the notation from Chapter 2: The componentwise sum of two partitions $\lambda = (\lambda_1, ..., \lambda_l)$ and $\mu = (\mu_1, ..., \mu_k)$ with $l \leq k$ is defined by $\lambda + \mu = (\lambda_1 + \mu_1, ..., \lambda_l + \mu_l, \mu_{l+1}, ..., \mu_k)$. For a fixed partition λ we denote by $s_\mu + \lambda$ the function $s_{\mu+\lambda}$ and extend this definition from the basis of Schur functions linearly to all symmetric functions. By $\Lambda_{\mathbb{N}}^{n_0,k}$ we denote the $\mathbb{Z}$-module of sequences $\{f_n\}_{n\in\mathbb{N}}$ with $f_n \in \Lambda_{nk+n_0}$ for all $n \in \mathbb{N}$. For every $n_0 \in \mathbb{N}$, every partition λ and every divisor m of $|\lambda|$ we define

$$\Delta_m^\lambda : \Lambda_{\mathbb{N}}^{n_0,|\lambda|/m} \to \Lambda_{\mathbb{N}}^{n_0,|\lambda|/m},\ \{f_n\}_{n\geq 0} \mapsto \{\Delta_m^\lambda f_{n+m}\}_{n\geq 0}$$

$$\text{where } \Delta_m^\lambda f_n = f_n - (f_{n-m} + \lambda) \text{ for all } n \geq m.$$

We write Δ^λ for Δ_1^λ.

Example 3.1. (i) Let $\{s_{(n,2)}\}_n \in \Lambda_{\mathbb{N}}^{2,1}$. Then

$$\Delta^{(1)} s_{(n,2)} = s_{(n,2)} - (s_{(n-1,2)} + (1)) = s_{(n,2)} - s_{(n,2)} = 0.$$

(ii) Let $\{s_{(3n,n)}\}_n \in \Lambda_{\mathbb{N}}^{0,4}$. Then

$$\Delta^{(4)} s_{(3n,n)} = s_{(3n,n)} - (s_{(3n-3,n-1)} + (4)) = s_{(3n,n)} - s_{(3n+1,n-1)}$$

and

$$\Delta^{(3,1)} s_{(3n,n)} = s_{(3n,n)} - (s_{(3n-3,n-1)} + (3,1)) = s_{(3n,n)} - s_{(3n,n)} = 0.$$

For a partition $\lambda = (\lambda_1, ..., \lambda_l)$ and $n \geq \lambda_1$ we write(n, λ) for $(n, \lambda_1, \lambda_2, ..., \lambda_l)$. We consider sequences of Schur functions of the form $\{s_{(n,\lambda)}\}_{n\geq\lambda_1} \in \Lambda_{\mathbb{N}}^{|\lambda|,1}$. Let $\alpha_1, ..., \alpha_k \in \mathbb{N}$ and $\lambda_1, ..., \lambda_k$ be number partitions. Then the sequence of products $\{s_{(n+\alpha_1,\lambda_1)} \cdots s_{(n+\alpha_k,\lambda_k)}\}_n$ is an element of $\Lambda_{\mathbb{N}}^{\alpha_1+|\lambda_1|+\cdots+\alpha_k+|\lambda_k|,k}$. We can apply difference operators $\Delta^{\mu}_{|\mu|/k}$ on it where $|\mu|$ is a multiple of k. Using these operators, representation stability in the sense of Church and Farb can be described in the following way: A sequence $\{f_n\}_n \in \Lambda_{\mathbb{N}}^{n_0,1}$ stabilizes at $N \in \mathbb{N}$ if

$$\Delta^{(1)}(f_n) = 0 \text{ for all } n > N.$$

This can be seen as a special case of a wider set of properties of symmetric function sequences. There are sequences who do not eventually become zero by applying $\Delta^{(1)}$ but by applying another of the difference operators defined above or a finite sequence of them. This is the case for sequences of componentwise products of two or three stabilizing sequences. Now, we are in position to formulate our main theorem.

Theorem 3.2. *Let $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq 0$ and $\lambda_1, \lambda_2, \lambda_3$ be number partitions. Then*

(a)

$$\Delta^{(2)}\Delta^{(1,1)}(s_{(n+\alpha_1,\lambda_1)}s_{(n+\alpha_2,\lambda_2)}) = 0 \textit{ for all } n > |\lambda_1| + |\lambda_2| + 1 - \alpha_2.$$

The set $\{\Delta^{(2)}, \Delta^{(1,1)}\}$ is minimal in the sense that the above sequence is not eventually zero if we remove one of the difference operators.

(b)

$$\Delta_2^{(3,3)}\Delta^{(3)}\Delta^{(2,1)}\Delta^{(1,1,1)}(s_{(n+\alpha_1,\lambda_1)}s_{(n+\alpha_2,\lambda_2)}s_{(n+\alpha_3,\lambda_3)}) = 0$$
$$\textit{for all } n > \max\{4, \alpha_1 - \alpha_2 + l(\lambda_1)\} + 2(|\lambda_1| + |\lambda_2| + |\lambda_3| + 1) - \alpha_3.$$

The set $\{\Delta_2^{(3,3)}, \Delta^{(3)}, \Delta^{(2,1)}, \Delta^{(1,1,1)}\}$ is minimal in the sense that the above sequence is not eventually zero if we remove one of the difference operators.

There is experimental evidence that an analogous statement about fourfold products holds. We formulate this in the following conjecture. We do not provide a complete proof of this statement but show how our methods indicate its validity until reaching a point where the number of cases to discuss is massive.

Conjecture 3.3. *Let $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 \geq 0$ and $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be number partitions. Then*

$$\Delta_3^{(4,4,4)}\Delta_2^{(3,3,2)}\Delta^{(4)}\Delta^{(3,1)}(\Delta^{(2,2)})^2(\Delta^{(2,1,1)})^2\Delta^{(1,1,1,1)}(s_{(n+\alpha_1,\lambda_1)}s_{(n+\alpha_2,\lambda_2)}s_{(n+\alpha_3,\lambda_3)}s_{(n+\alpha_4,\lambda_4)}) = 0$$
$$\textit{for sufficiently large } n.$$

The multiset $\{\Delta_3^{(4,4,4)}, \Delta_2^{(3,3,2)}, \Delta^{(4)}, \Delta^{(3,1)}, \Delta^{(2,2)}, \Delta^{(2,2)}, \Delta^{(2,1,1)}, \Delta^{(2,1,1)}, \Delta^{(1,1,1,1)}\}$ is minimal in the sense that the above sequence is not eventually zero if we remove one of the difference operators.

We show in Lemma 3.9 that every difference operator Δ_m^{λ} is linear so that we can expand the theorem to a wider set of symmetric function sequences. We get the following corollary.

Corollary 3.4. *Let $m_1, m_2, m_3, N_1, N_2, N_3 \in \mathbb{N}$ and $\{f_n^{(1)}\}_n, \{f_n^{(2)}\}_n, \{f_n^{(3)}\}_n$ be sequences such that $\{f_n^{(i)}\}_n \in \Lambda_{\mathbb{N}}^{m_i,1}$ stabilizes at N_i for all $i \in \{1, 2, 3\}$. Then*

(a)

$$\Delta^{(2)}\Delta^{(1,1)}(f_n^{(1)}f_n^{(2)}) = 0$$

for all $n > \max\{N_1, N_2, m_1 + m_2 + N_1 + N_2 + \max\{N_1, N_2\} - 2\}$.

The set $\{\Delta^{(2)}, \Delta^{(1,1)}\}$ *is minimal in the sense that the above sequence is not eventually zero if we remove one of the difference operators.*

(b)

$$\Delta_2^{(3,3)}\Delta^{(3)}\Delta^{(2,1)}\Delta^{(1,1,1)}(f_n^{(1)}f_n^{(2)}f_n^{(3)}) = 0$$

for all $n > \max\{N_1, N_2, 2(m_1 + m_2 + m_3 + \max\{m_1, m_2, m_3\}) + 3(N_1 + N_2 + N_3) - 7\}$.

The set $\{\Delta_2^{(3,3)}, \Delta^{(3)}, \Delta^{(2,1)}, \Delta^{(1,1,1)}\}$ *is minimal in the sense that the above sequence is not eventually zero if we remove one of the difference operators.*

In the same way, Conjecture 3.3 is equivalent to the following statement.

Conjecture 3.5. *Let* $m_1, ..., m_4, N_1, ..., N_4 \in \mathbb{N}$ *and* $\{f_n^{(1)}\}_n, \{f_n^{(2)}\}_n, \{f_n^{(3)}\}_n, \{f_n^{(4)}\}_n$ *be sequences such that* $\{f_n^{(i)}\}_n \in \Lambda_{\mathbb{N}}^{m_i,1}$ *stabilizes at* N_i *for all* $i \in \{1, 2, 3, 4\}$. *Then*

$$\Delta_3^{(4,4,4)}\Delta_2^{(3,3,2)}\Delta^{(4)}\Delta^{(3,1)}(\Delta^{(2,2)})^2(\Delta^{(2,1,1)})^2\Delta^{(1,1,1,1)}\left(f_n^{(1)}f_n^{(2)}f_n^{(3)}f_n^{(4)}\right) = 0$$

for sufficiently large n.

The multiset $\{\Delta_3^{(4,4,4)}, \Delta_2^{(3,3,2)}, \Delta^{(4)}, \Delta^{(3,1)}, \Delta^{(2,2)}, \Delta^{(2,2)}, \Delta^{(2,1,1)}, \Delta^{(2,1,1)}, \Delta^{(1,1,1,1)}\}$ *is minimal in the sense that the above sequence is not eventually zero if we remove one of the difference operators.*

We formulate a statement about products of arbitrarily many stabilizing sequences as a question.

Question 3.6. Let $k \geq 1$ and $\{f_n^{(1)}\}_n \in \Lambda_{\mathbb{N}}^{m_1,1}, ..., \{f_n^{(k)}\}_n \in \Lambda_{\mathbb{N}}^{m_k,1}$ be stabilizing sequences. Let $\{\lambda_1, ..., \lambda_r\}$ be the set of partitions of the numbers $k, 2k, ..., (k-1)k$. Is there a set of nonnegative integers $\{q_1, ..., q_{k-1}\}$ and a number N such that

$$(\Delta_{|\lambda_1|/k}^{\lambda_1})^{q_1} \cdots (\Delta_{|\lambda_r|/k}^{\lambda_r})^{q_r}\left(f_n^{(1)} \cdots f_n^{(k)}\right) = 0 \text{ for all } n > N$$

and $q_1 + ... + q_{k-1}$ is minimal with this property and how can we compute the numbers $q_1, ..., q_r$ and N?

By Lemma 3.9 (ii) this is equivalent to

Question 3.7. Let $k \geq 1$, $\alpha_1 \geq \cdots \geq \alpha_4 \geq 0$ and $\mu_1, \ldots, \mu_k$ be number partitions. We consider the sequence $\{s_{(n+\alpha_1,\mu_1)} \cdots s_{(n+\alpha_k,\mu_k)}\}_n \in \Lambda_{\mathbb{N}}^{\alpha_1+|\mu_1|+\cdots+\alpha_k+|\mu_k|,k}$. Let $\{\lambda_1, ..., \lambda_r\}$ be the set of partitions of the numbers $k, 2k, ..., (k-1)k$. Is there a set of nonnegative integers $\{q_1, ..., q_{k-1}\}$ and a number N such that

$$(\Delta_{|\lambda_1|/k}^{\lambda_1})^{q_1} \cdot (\Delta_{|\lambda_r|/k}^{\lambda_r})^{q_r}\left(s_{(n+\alpha_1,\mu_1)} \cdots s_{(n+\alpha_k,\mu_k)}\right) = 0 \text{ for all } n > N$$

and $q_1 + ... + q_{k-1}$ is minimal with this property and how can we compute the numbers $q_1, ..., q_r$ and N?

Corollary 3.4 can be applied to cohomology groups with coefficients in $\mathbb{C}$ of products of spaces $X_n \subseteq \mathbb{C}^n$ on which the symmetric group S_n acts and $\{\tilde{H}^i(X_n, \mathbb{C})\}_n$ is representation stable like the spaces considered in Chapter 2. If $X_n, Y_n, Z_n \subseteq \mathbb{C}^n$ are spaces with an action of S_n the group $S_n \times S_n$ acts on $X_n \times Y_n$ and the group $S_n \times S_n \times S_n$ acts on $X_n \times Y_n \times Z_n$. By the Künneth formula, we have the following equalities of $(S_n \times S_n)-$ and $(S_n \times S_n \times S_n)$-modules:

$$\tilde{H}^k(X_n \times Y_n, \mathbb{C}) = \bigoplus_{i+j=k} \tilde{H}^i(X_n, \mathbb{C}) \otimes \tilde{H}^j(Y_n, \mathbb{C}),$$
$$\tilde{H}^k(X_n \times Y_n \times Z_n, \mathbb{C}) = \bigoplus_{i+j+l=k} \tilde{H}^i(X_n, \mathbb{C}) \otimes \tilde{H}^j(Y_n, \mathbb{C}) \otimes \tilde{H}^l(Z_n, \mathbb{C}).$$

If all the sequences

$$\{\tilde{H}^i(X_n, \mathbb{C})\}_n, \{\tilde{H}^i(Y_n, \mathbb{C})\}_n, \{\tilde{H}^i(Z_n, \mathbb{C})\}_n$$

are representation stable for all i then inducing the $(S_n \times S_n)$-representations up to S_{2n} and the $(S_n \times S_n \times S_n)$-representations up to S_{3n} leads to the following corollary:

Corollary 3.8. *Let $\{X_n\}_n, \{Y_n\}_n, \{Z_n\}_n$ be sequences of topological spaces equipped with an S_n-action on X_n, Y_n and Z_n for every n such that the sequences $\{\tilde{H}^i(X_n, \mathbb{C})\}_n, \{\tilde{H}^i(Y_n, \mathbb{C})\}_n$ and $\{\tilde{H}^i(Z_n, \mathbb{C})\}_n$ are representation stable sequences of S_n-representation for every i then for every k the sequences $\{\tilde{H}^k(X_n \times Y_n, \mathbb{C})\}_n$ and $\{\tilde{H}^k(X_n \times Y_n \times Z_n, \mathbb{C})\}_n$ fulfill the following recurrence relations:*

$$\Delta^{(2)}\Delta^{(1,1)}\mathrm{Ind}_{S_n \times S_n}^{S_{2n}} \tilde{H}^k(X_n \times Y_n, \mathbb{C}) = 0,$$
$$\Delta_2^{(3,3)}\Delta^{(3)}\Delta^{(2,1)}\Delta^{(1,1,1)}\mathrm{Ind}_{S_n \times S_n \times S_n}^{S_{3n}} \tilde{H}^k(X_n \times Y_n \times Z_n, \mathbb{C}) = 0$$

for sufficiently large n.

The rest of the chapter is dedicated to the proof of Theorem 3.2 and Corollary 3.4.

3.2. **Reduction to homogeneous symmetric functions.** We show in the following lemma that the difference operators commute such that we are free to choose their order and that they are linear.

Lemma 3.9. *Let λ and μ be partitions, m a divisor of $|\lambda|$ and l a divisor of $|\mu|$ with $|\mu|/l = |\lambda|/m$. Let $n_0 \geq 0$ and $\{f_n\}_n \in \Lambda_{\mathbb{N}}^{n_0, |\lambda|/m}$. Then*

(i)
$$\Delta_m^\lambda \Delta_l^\mu (f_n) = \Delta_l^\mu \Delta_m^\lambda (f_n) = \text{ for all } n.$$

(ii) *The map $\Delta_m^\lambda : \Lambda_{\mathbb{N}}^{n_0, |\lambda|/m} \to \Lambda_{\mathbb{N}}^{n_0, |\lambda|/m}$ is linear.*

Proof. (i) We have

$$\Delta_m^\lambda \Delta_l^\mu f_n = \Delta_m^\lambda (f_n - (f_{n-l} + \mu)) = f_n - (f_{n-l} + \mu) - (f_{n-m} + \lambda) + (f_{n-l-m} + \mu + \lambda)$$
$$= \Delta_l^\mu (f_n - (f_{n-m} + \lambda)) = \Delta_l^\mu \Delta_m^\lambda f_n.$$

(ii) Δ_m^λ is the difference of the shift operator $\{f_n\}_n \mapsto \{f_{n+m}\}_n$ which is linear and the map $\{f_n\}_n \mapsto \{f_n + \lambda\}_n$ which is a linear extension. ■

It follows from Lemma 3.9 (i) that we can define iterated products $\prod_{\Delta\in D}\Delta$ over sets of difference operators $D=\{\Delta_{m_1}^{\lambda_1},\dots,\Delta_{m_r}^{\lambda_r}\}$. Now, we prove Corollary 3.4 using Theorem 3.2 and Lemma 3.9 (ii).

Proof. (a) The sequence $\{f_n^{(i)}\}_n \in \Delta_{\mathbb{N}}^{m_i,1}$ stabilizes at N_i for every $i\in\{1,2\}$. It follows that $f_n^{(i)}$ is a linear combination of Schur functions $s_{(n+\alpha_i,\lambda_i)}$ with $\alpha_i \geq 1-N_i$ and $\lambda_i \vdash m_i-\alpha_i$ for all $n\geq N_i$. Theorem 3.2 yields that

$$\Delta^{(2)}\Delta^{(1,1)}(s_{(n+\alpha_1,\lambda_1)}s_{(n+\alpha_2,\lambda_2)})=0 \text{ for all } n>|\lambda_1|+|\lambda_2|+1-\min\{\alpha_1,\alpha_2\}.$$

We have

$$|\lambda_1|+|\lambda_2|+1-\min\{\alpha_1,\alpha_2\}\leq m_1+m_2+N_1+N_2+\max\{N_1,N_2\}-2.$$

The claim follows from Lemma 3.9 (ii).
(b) The sequence $\{f_n^{(i)}\}_n \in \Delta_{\mathbb{N}}^{m_i,1}$ stabilizes at N_i for every $i\in\{1,2,3\}$. It follows that $f_n^{(i)}$ is a linear combination of Schur functions $s_{(n+\alpha_i,\lambda_i)}$ with $m_i\geq\alpha_i \geq 1-N_i$ and $\lambda_i \vdash m_i-\alpha_i$ for all $n\geq N_i$. Suppose $\alpha_1\geq\alpha_2\geq\alpha_3$. Theorem 3.2 yields that

$$\Delta_2^{(3,3)}\Delta^{(3)}\Delta^{(2,1)}\Delta^{(1,1,1)}(s_{(n+\alpha_1,\lambda_1)}s_{(n+\alpha_2,\lambda_2)}s_{(n+\alpha_3,\lambda_3)})=0$$

$$\text{for all } n>\max\{4,\alpha_1-\alpha_2+l(\lambda_1)\}+2(|\lambda_1|+|\lambda_2|+|\lambda_3|+1)-\alpha_3.$$

We have

$$\begin{aligned}&\max\{4,\alpha_1-\alpha_2+l(\lambda_1)\}+2(|\lambda_1|+|\lambda_2|+|\lambda_3|+1)-\alpha_3\\ \leq\ & m_1+N_2-1+m_1+N_1-1+2(m_1+m_2+m_3+N_1+N_2+N_3-2)+N_3-1\\ =\ & 4m_1+2m_2+2m_3+3(N_1+N_2+N_3)-7\end{aligned}$$

The claim follows from Lemma 3.9 (ii). ■

We want to show next that we can restrict to products of homogeneous symmetric functions $s_{(n)}$ if we additionally multiply with a constant sequence.

Lemma 3.10. *Let $k\in\mathbb{N}$ and $\lambda_1,\dots,\lambda_m$ be number partitions of multiples of k. If for all number partitions β and $\alpha_1,\dots,\alpha_{k-1}\in\mathbb{N}$ the sequence $\{s_{(n+\alpha_1)}\cdots s_{(n+\alpha_{k-1})}s_{(n)}s_\beta\}_n \in \Lambda_{\mathbb{N}}^{\alpha_1+\cdots+\alpha_{k-1}+|\beta|,k}$ fulfills*

$$\Delta_{|\lambda_1|/k}^{\lambda_1}\dots\Delta_{|\lambda_m|/k}^{\lambda_m}(s_{(n+\alpha_1)}\cdots s_{(n+\alpha_{k-1})}s_{(n)}s_\beta)=0$$

for all n greater than some number $n_0(\alpha_1,\dots,\alpha_{k-1},|\beta|)$ then for all number partitions $\mu_1,\dots,\mu_k$ and $\alpha_1,\dots,\alpha_{k-1}\in\mathbb{N}$ the sequence $\{s_{(n+\alpha_1,\mu_1)}\cdots s_{(n+\alpha_{k-1},\mu_{k-1})}s_{(n,\mu_k)}\}_n \in \Lambda_{\mathbb{N}}^{\alpha_1+\cdots+\alpha_{k-1}+|\mu_1|+\cdot+|\mu_k|,k}$ fulfills

$$\Delta_{|\lambda_1|/k}^{\lambda_1}\dots\Delta_{|\lambda_1|/k}^{\lambda_m}(s_{(n+\alpha_1,\mu_1)}\cdots s_{(n+\alpha_{k-1},\mu_{k-1})}s_{(n,\mu_k)})=0$$

for all
$n>\max\{n_0(\alpha_1+i_1,\dots,\alpha_k+i_k,|\mu_1|-i_1+\dots+|\mu_k|-i_k)\mid i_q\in\{0,\dots,l(\mu_q)\}$ *for all* $q\in\{1,\dots,k\}\}$.

Proof. Let $\alpha_k = 0$. The Jacobi-Trudi identity yields

$$\prod_{q=1}^{k} s_{(n+\alpha_q,\mu_q)}$$

$$= \prod_{q=1}^{k} \det \begin{pmatrix} s_{(n+\alpha_q)} & s_{(n+\alpha_q+1)} & \cdots & s_{(n+\alpha_q+l(\mu_q))} \\ s_{(\mu_{q,1}-1)} & s_{(\mu_{q,1})} & \cdots & s_{(\mu_{q,1}+l(\mu_q)-1)} \\ \cdots & \cdots & \cdots & \cdots \\ s_{(\mu_{q,l(\mu_q)}-l(\mu_q))} & \cdots & \cdots & s_{(\mu_{q,l(\mu_q)})} \end{pmatrix}$$

$$= \prod_{q=1}^{k} \left(\sum_{i=0}^{l(\mu_q)} s_{(n+\alpha_q+i)} (-1)^i \det(M_{\mu_q,i}) \right)$$

$$= \sum_{i_1=0}^{l(\mu_1)} \cdots \sum_{i_k=0}^{l(\mu_k)} \left(\prod_{q=1}^{k} s_{(n+\alpha_q+i_q)} \right) (-1)^{i_1+\ldots+i_k} \prod_{q=1}^{k} \det(M_{\mu_q,i_q})$$

where for $\gamma \in \{\mu_1, ..., \mu_k\}$ the matrix $M_{\gamma,i}$ is the matrix we get by deleting the ith column of

$$\begin{pmatrix} s_{(\gamma_1-1)} & s_{(\gamma_1)} & \cdots & s_{(\gamma_1+l(\gamma)-1)} \\ \cdots & \cdots & \cdots & \cdots \\ s_{(\gamma_{l(\gamma)}-l(\gamma))} & \cdots & \cdots & s_{(\gamma_{l(\gamma)})} \end{pmatrix}.$$

The degree of $\prod_{q=1}^{k} \det(M_{\mu_q,i_q})$ is $|\mu_1| - i_1 + ... + |\mu_k| - i_k$. It follows from the assumption that this sequence vanishes under $\Delta_{n_1}^{\lambda_1}...\Delta_{n_m}^{\lambda_m}$ for $n > \max\{n_0(\alpha_1 + i_1, ..., \alpha_k + i_k, |\mu_1| - i_1 + ... + |\mu_k| - i_k) \mid i_q \in \{0, ..., l(\mu_q)\}$ for all $q \in \{1, ..., k\}\}$. ∎

It follows that to prove Theorem 3.2 it is sufficient to prove the following

Proposition 3.11. *Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{N}$ and β be a number partition.*

(a) *The sequence $\{s_{(n+\alpha_1)}s_{(n)}s_\beta\}_n \in \Lambda_{\mathbb{N}}^{\alpha_1+|\beta|,2}$ fulfills*

$$\Delta^{(2)}\Delta^{(1,1)}(s_{(n+\alpha_1)}s_{(n)}s_\beta) = 0 \text{ for all } n > \beta_1 + 1.$$

(b) *The sequence $\{s_{(n+\alpha_1)}s_{(n+\alpha_2)}s_{(n)}s_\beta\}_n \in \Lambda_{\mathbb{N}}^{\alpha_1+\alpha_2+|\beta|,3}$ fulfills*

$$\Delta_2^{(3,3)}\Delta^{(3)}\Delta^{(2,1)}\Delta^{(1,1,1)}(s_{(n+\alpha_1)}s_{(n+\alpha_2)}s_{(n)}s_\beta) = 0$$

$$\text{for all } n > \max\{4, \alpha_1 - \alpha_2 + \beta_1 + 2\} + \beta_1.$$

(c) *The sequence $\{s_{(n+\alpha_1)}s_{(n+\alpha_2)}s_{(n+\alpha_3)}s_{(n)}s_\beta\}_n \in \Lambda_{\mathbb{N}}^{\alpha_1+\alpha_2+\alpha_3+|\beta|,4}$ fulfills*

$$\Delta_3^{(4,4,4)}\Delta_2^{(3,3,2)}\Delta^{(4)}\Delta^{(3,1)}(\Delta^{(2,2)})^2(\Delta^{(2,1,1)})^2\Delta^{(1,1,1,1)}(s_{(n+\alpha_1)}s_{(n+\alpha_2)}s_{(n+\alpha_3)}s_{(n)}s_\beta) = 0$$

$$\text{for sufficiently large } n.$$

For every symmetric function f and $l \in \mathbb{N}$ we write $f_{\leq l}$ for the part of the Schur function decomposition of f with partitions of length less than or equal to l and $f_{>l}$ for the part of the Schur function decomposition with partitions of length greater than l.

Lemma 3.12. *Let $k, n_0, l \in \mathbb{N}$ and $\{f_n\}_n \in \Lambda_{\mathbb{N}}^{n_0,k}$. Let $\lambda_1, ..., \lambda_m$ be number partitions of multiples of k. Suppose*

$$\Delta^{\lambda_1}_{|\lambda_1|/k} ... \Delta^{\lambda_m}_{|\lambda_m|/k}(f_n) = 0.$$

Then

$$\left(\prod_{i:l(\lambda_i)\leq l} \Delta^{\lambda_i}_{|\lambda_i|/k} \right) (f_n)_{\leq l} = 0.$$

Proof. We have

$$\Delta^{\lambda_1}_{|\lambda_1|/k} ... \Delta^{\lambda_m}_{|\lambda_m|/k} f_n = \Delta^{\lambda_1}_{|\lambda_1|/k} ... \Delta^{\lambda_m}_{|\lambda_m|/k}(f_n)_{\leq l} + \Delta^{\lambda_1}_{|\lambda_1|/k} ... \Delta^{\lambda_m}_{|\lambda_m|/k}(f_n)_{> l}$$

$$= \left(\prod_{i:l(\lambda_i)\leq l} \Delta^{\lambda_i}_{|\lambda_i|/k} \right) \left(\prod_{i:l(\lambda_i)> l} \Delta^{\lambda_i}_{|\lambda_i|/k} \right) (f_n)_{\leq l} + \Delta^{\lambda_1}_{|\lambda_1|/k} ... \Delta^{\lambda_m}_{|\lambda_m|/k}(f_n)_{> l}.$$

We can write

$$\left(\prod_{i:l(\lambda_i)> l} \Delta^{\lambda_i}_{|\lambda_i|/k} \right) (f_n)_{\leq l} = (f_n)_{\leq l} + g_n.$$

for a function g_n with only partitions of length greater than l in its Schur function decomposition. It follows

$$0 = \Delta^{\lambda_1}_{|\lambda_1|/k} ... \Delta^{\lambda_m}_{|\lambda_m|/k} f_n = \left(\prod_{i:l(\lambda_i)\leq l} \Delta^{\lambda_i}_{|\lambda_i|/k} \right) (f_n)_{\leq l} + \left(\prod_{i:l(\lambda_i)\leq l} \Delta^{\lambda_i}_{|\lambda_i|/k} \right) g_n$$

$$+ \Delta^{\lambda_1}_{|\lambda_1|/k} ... \Delta^{\lambda_m}_{|\lambda_m|/k}(f_n)_{> l}.$$

All partitions with length less than or equal to l appear in $\left(\prod_{i:l(\lambda_i)\leq l} \Delta^{\lambda_i}_{|\lambda_i|/k} \right) (f_n)_{\leq l}$ while all partitions with length greater than l appear in $\left(\prod_{i:l(\lambda_i)\leq l} \Delta^{\lambda_i}_{|\lambda_i|/k} \right) g_n + \Delta^{\lambda_1}_{|\lambda_1|/k} ... \Delta^{\lambda_m}_{|\lambda_m|/k}(f_n)_{> l}$ and it follows that each of these two parts must itself be zero. ■

Consider a semistandard skew tableau T of shape ν/β and weight $(n+\alpha_1, n+\alpha_2, ..., n+\alpha_k)$. We split T into two parts: The part of the first β_1 columns which we call the centre of T and denote it centre(T) and the rest which we call the arm of T and denote it arm(T). For example, let

$$T = \begin{array}{|c|c|c|c|c|c|} \hline \ & \ & 1 & 1 & 1 & 2 \\ \hline \ & 1 & 2 & 3 \\ \cline{1-4} 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array} .$$

Then

$$\text{centre}(T) = \begin{array}{|c|c|}\hline & \\ \hline & 1 \\ \hline 2 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \text{and } \text{arm}(T) = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2} \end{array}.$$

For fixed β and k, there are only finitely many tableaux appearing as centres of tableaux of shape ν/β for arbitrary ν and we denote this finite set by $c_k(\beta)$. We denote the set of semistandard Young tableaux of weight γ and arbitrary shape by $ST(\gamma)$ and the number of occurences of the Symbol i in the tableau $c \in c_k(\beta)$ by $c(i)$

Lemma 3.13. *Let $\alpha_1, ..., \alpha_k \in \mathbb{N}$ and β be a number partition. Then*

$$s_{(n+\alpha_1)} ... s_{(n+\alpha_k)} s_\beta = \sum_{c \in c_k(\beta)} (s_{(n+\alpha_1-c(1))} ... s_{(n+\alpha_k-c(k))})_{\leq \max\{i \mid \text{shape}(c)_i = \beta_1\}} + \text{shape}(c)$$

for all $n \in \mathbb{N}$.

Proof. We have

$$s_{(n+\alpha_1)} ... s_{(n+\alpha_k)} s_\beta = \sum_{T'} s_{\text{shape}(T')}$$

where the sum runs over all semistandard skew tableau T' of shape ν/β for any ν and weight $(n+\alpha_1, n+\alpha_2, ..., n+\alpha_k)$ We can rewrite this sum by splitting every such skew tableau into its centre and arm:

$$\sum_{T'} s_{\text{shape}(T')} = \sum_{c \in c_k(\beta)} \sum_T s_{\text{shape}(c)+\text{shape}(T)} = \sum_{c \in c_k(\beta)} \left(\left(\sum_T s_{\text{shape}(T)} \right) + \text{shape}(c) \right)$$

where the sum runs over all $T \in ST((n+\alpha_1 - c(1), ..., n+\alpha_k - c(k)))$ such that T has as most as many rows as c has rows of length β_1. Note that $\sum_T s_{\text{shape}(T)}$ is the part of $s_{(n+\alpha_1-c(1))} ... s_{(n+\alpha_k-c(k))}$ with Schur functions with partitions with at most as many rows as c has rows of length β_1. ■

The following lemma follows from the previous two lemmas.

Lemma 3.14. *Let $k \in \mathbb{N}$ and β be a number partition. Let $\lambda_1, ..., \lambda_m$ be number partitions of multiples of k. If for all numbers $\alpha_1, ..., \alpha_k \in \mathbb{N}$ there is a number $N(\alpha_1, ..., \alpha_k)$ such that the sequence $\{s_{(n+\alpha_1)} ... s_{(n+\alpha_k)}\} \in \Lambda_{\mathbb{N}}^{\alpha_1+...+\alpha_k, k}$ fulfills*

$$\Delta_{n_1}^{\lambda_1} \cdots \Delta_{n_m}^{\lambda_m} (s_{(n+\alpha_1)} \cdots s_{(n+\alpha_k)}) = 0 \textit{ for all } n > N(\alpha_1, ..., \alpha_k)$$

then the sequence $\{s_{(n+\alpha_1)} ... s_{(n+\alpha_k)} s_\beta\} \in \Lambda_{\mathbb{N}}^{\alpha_1+...+\alpha_k+|\beta|, k}$ fulfills

$$\Delta_{n_1}^{\lambda_1} \cdots \Delta_{n_m}^{\lambda_m} (s_{(n+\alpha_1)} \cdots s_{(n+\alpha_k)} s_\beta) = 0$$
$$\textit{for all } n > \max\{N(\alpha_1 - c(1), ..., \alpha_k - c(k)) \mid c \in c_k(\beta)\} + \beta_1.$$

This lemma shows that it is sufficient to prove the following proposition.

Proposition 3.15. *Let $\alpha_1 \geq \alpha_2 \geq 0$.*

(a) *The sequence* $\{s_{(n+\alpha_1)}s_{(n)}\}_n \in \Lambda_{\mathbb{N}}^{\alpha_1,2}$ *fulfills*

$$\Delta^{(2)}\Delta^{(1,1)}s_{(n+\alpha_1)}s_{(n)} = 0 \text{ for all } n > 1.$$

The set $\{\Delta^{(2)}, \Delta^{(1,1)}\}$ *is minimal in the sense that the above sequence is not eventually zero if we remove one of the difference operators.*

(b) *The sequence* $\{s_{(n+\alpha_1)}s_{(n+\alpha_2)}s_{(n)}\}_n \in \Lambda_{\mathbb{N}}^{\alpha_1+\alpha_2,3}$ *fulfills*

$$\Delta_2^{(3,3)}\Delta^{(3)}\Delta^{(2,1)}\Delta^{(1,1,1)}s_{(n+\alpha_1)}s_{(n+\alpha_2)}s_{(n)} = 0$$

$$\text{for all } n > \max\{4, \alpha_1 - \alpha_2 + 2\}.$$

The set $\{\Delta_2^{(3,3)}, \Delta^{(3)}, \Delta^{(2,1)}, \Delta^{(1,1,1)}\}$ *is minimal in the sense that the above sequence is not eventually zero if we remove one of the difference operators.*

We do not provide a complete proof of the following statement about fourfold products but show how our methods point to its validity until reaching a point where the number of cases to discuss is massive.

Conjecture 3.16. *Let* $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq 0$*. The sequence* $\{s_{(n+\alpha_1)}s_{(n+\alpha_2)}s_{(n+\alpha_3)}s_{(n)}\}_n \in \Lambda_{\mathbb{N}}^{\alpha_1+\alpha_2+\alpha_3,4}$ *fulfills*

$$\Delta_3^{(4,4,4)}\Delta_2^{(3,3,2)}\Delta^{(4)}\Delta^{(3,1)}(\Delta^{(2,2)})^2(\Delta^{(2,1,1)})^2\Delta^{(1,1,1,1)}s_{(n+\alpha_1)}s_{(n+\alpha_2)}s_{(n+\alpha_3)}s_{(n)} = 0$$

$$\text{for sufficiently large } n.$$

The multiset $\{\Delta_3^{(4,4,4)}, \Delta_2^{(3,3,2)}, \Delta^{(4)}, \Delta^{(3,1)}, \Delta^{(2,2)}, \Delta^{(2,2)}, \Delta^{(2,1,1)}, \Delta^{(2,1,1)}, \Delta^{(1,1,1,1)}\}$ *is minimal in the sense that the above sequence is not eventually zero if we remove one of the difference operators.*

3.3. **Proof of Proposition 3.15.** In the whole section n and $\alpha_1 \geq \alpha_2 \geq \alpha_3$ are natural numbers. In the next lemmas, we use partial matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2,(k-1)} & \\ \vdots & & & & \\ a_{(k-1),1} & a_{(k-1),2} & & & \\ a_{k,1} & & & & \end{pmatrix}.$$

For every $i \in \{1, ..., k-1\}$, we write a_i for the ith row of a and $|a_i|$ for the sum of the entries of the ith row. Let $k, n \in \mathbb{N}$ and $\alpha = (\alpha_1, ..., \alpha_k) \in \mathbb{N}_0^k$ with $\alpha_1 \geq ... \geq \alpha_k$. Let $P_{k,n,\alpha}$ be the set of all partial matrices $\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2,(k-1)} & \\ \vdots & & & & \\ a_{(k-1),1} & a_{(k-1),2} & & & \\ a_{k,1} & & & & \end{pmatrix}$ with real entries and

$$0 \leq a_{ij} \leq n + \alpha_{i+j-1} \text{ for all } 1 \leq i \leq k \text{ and } 1 \leq j \leq k - i + 1,$$

$$\sum_{j=1}^{m} a_{ij} \leq \sum_{j=1}^{m} a_{(i-1)j}, \text{ for all } 2 \leq i \leq k \text{ and } 1 \leq m \leq k-i+1,$$

$$\sum_{i=1}^{m} a_{i,(m+1-i)} = n + \alpha_m \text{ for all } 1 \leq m \leq k.$$

or written as vector inequalities and equalities:

$$\begin{pmatrix} -1 & 1 & & & & \\ -1 & 1 & -1 & 1 & & \\ \vdots & & & \ddots & \ddots & \\ -1 & 1 & \dots & \dots & -1 & 1 \end{pmatrix} \begin{pmatrix} a_{(i-1),1} \\ a_{i,1} \\ a_{(i-1),2} \\ a_{i,2} \\ \vdots \\ a_{(i-1),(k-i+1)} \\ a_{i,(k-i+1)} \end{pmatrix} \leq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \text{ for all } 2 \leq i \leq k,$$

$$\begin{pmatrix} 1 & & & & & & & & & \\ & 1 & 1 & & & & & & & \\ & & & 1 & 1 & 1 & & & & \\ & & & & & & \ddots & & & \\ & & & & & & 1 & 1 & \dots & 1 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{13} \\ a_{22} \\ a_{31} \\ \vdots \\ a_{1k} \\ \vdots \\ a_{(k-1),2} \\ a_{k,1} \end{pmatrix} = \begin{pmatrix} n+\alpha_1 \\ \vdots \\ n+\alpha_k \end{pmatrix}.$$

$P_{k,n,\alpha}$ is a convex polytope in $\mathbb{R}^{\binom{k+1}{2}}$. If M is a set of partial matrices we write $M^{\mathbb{Z}}$ for its subset of partial matrices with only integer valued entries.

Proposition 3.17. *Let $k, n \in \mathbb{N}$ and $\alpha = (\alpha_1, ..., \alpha_k) \in \mathbb{N}_0^k$ with $\alpha_1 \geq ... \geq \alpha_k$. The polytope $P_{k,n,\alpha}$ is $\binom{k}{2}$-dimensional.*

Proof. $P_{k,n,\alpha}$ is contained in the affine subspace defined by

$$\begin{pmatrix} 1 & & & & & & & & & & \\ & 1 & 1 & & & & & & & & \\ & & & 1 & 1 & 1 & & & & & \\ & & & & & & \ddots & & & & \\ & & & & & & & 1 & 1 & \dots & 1 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{13} \\ a_{22} \\ a_{31} \\ \vdots \\ a_{1k} \\ \vdots \\ a_{(k-1),2} \\ a_{k,1} \end{pmatrix} = \begin{pmatrix} n+\alpha_1 \\ \vdots \\ n+\alpha_k \end{pmatrix}$$

The above matrix has rank k. This implies that the affine subspace has dimension $\binom{k+1}{2} - k = \binom{k}{2}$. Otherwise, $P_{k,n,\alpha}$ contains the following $\binom{k}{2} + 1$ affine independent points. For every $i \in \{0, ..., k-1\}$ we construct $\min\{1, i\}$ points starting with

$$\begin{pmatrix} n+\alpha_1 & 0 & 0 & \dots & 0 & n+\alpha_{k-i+1} & \dots & n+\alpha_k \\ n+\alpha_2 & 0 & \dots & \dots & \dots & \dots & 0 & \\ \vdots & \vdots & & & & & & \\ n+\alpha_{k-i} & 0 & & & & & & \\ 0 & \vdots & & & & & & \\ \vdots & & & & & & & \\ 0 & & & & & & & \end{pmatrix}.$$

We get the next point by moving $n + \alpha_k$ diagonally left and down:

$$\begin{pmatrix} n+\alpha_1 & 0 & 0 & \dots & 0 & n+\alpha_{k-i+1} & \dots & n+\alpha_{k-1} & 0 \\ n+\alpha_2 & 0 & \dots & 0 & 0 & \dots & 0 & n+\alpha_k & \\ \vdots & \vdots & & & & & & & \\ n+\alpha_{k-i} & 0 & & & & & & & \\ 0 & \vdots & & & & & & & \\ \vdots & & & & & & & & \\ 0 & & & & & & & & \end{pmatrix}.$$

Now, we move $n+\alpha_{k-1}$ and $n+\alpha_k$ diagonally left and down:

$$\begin{pmatrix} n+\alpha_1 & 0 & 0 & \dots & 0 & n+\alpha_{k-i+1} & \dots & n+\alpha_{k-2} & 0 & 0 \\ n+\alpha_2 & 0 & \dots & 0 & 0 & \dots & 0 & n+\alpha_{k-1} & 0 \\ n+\alpha_3 & 0 & \dots & 0 & 0 & \dots & 0 & n+\alpha_k \\ \vdots & \vdots \\ n+\alpha_{k-i} & 0 \\ 0 & \vdots \\ \vdots \\ 0 \end{pmatrix}.$$

We go on moving the rightmost nonzero values diagonally left and down until getting the point

$$\begin{pmatrix} n+\alpha_1 & 0 & 0 & \dots & 0 & n+\alpha_{k-i+1} & \dots & 0 & 0 & 0 \\ n+\alpha_2 & 0 & \dots & 0 & & n+\alpha_{k-i+2} & 0 & \dots & 0 \\ n+\alpha_3 & 0 & \dots & 0 & 0 & n+\alpha_{k-i+3} & 0 & \dots \\ \vdots & & & & & \vdots \\ \vdots & 0 & 0 & 0 & 0 & n+\alpha_k \\ n+\alpha_{k-i} & 0 & \vdots \\ 0 & \vdots \\ \vdots \\ 0 \end{pmatrix}.$$

We constructed $1+1+2+\cdots(k-1) = \binom{k}{2}+1$ affine independent points lying in $P_{k,n,\alpha}$. ■

Lemma 3.18. *Let $k, n \in \mathbb{N}$ and $\alpha = (\alpha_1, ..., \alpha_k) \in \mathbb{N}_0^k$ with $\alpha_1 \geq ... \geq \alpha_k$. Then*

$$\prod_{i=1}^{k} s_{(n+\alpha_i)} = \sum_{A \in P^{\mathbb{Z}}_{k,n,\alpha}} s_{(|a_1|,...,|a_k|)}.$$

Proof. The product $\prod_{i=1}^{k} s_{(n+\alpha_i)}$ is the homogeneous symmetric function $h_{(n+\alpha_1,...,n+\alpha_k)}$. It follows from the transition matrix between the basis of Schur functions and the basis of homogeneous symmetric functions that $h_{(n+\alpha_1,...,n+\alpha_k)}$ is the sum $\sum_T s_{\text{shape}(T)}$ running over all semistandard Young tableaux T of weight $(n+\alpha_1, ..., n+\alpha_k)$. For every such tableau T let $a_{i,j}(T)$ be the number of $(i+j-1)$'s in the ith row. Then the map given by

$$T \mapsto \begin{pmatrix} a_{11}(T) & a_{12}(T) & a_{13}(T) & \dots & a_{1k}(T) \\ a_{21}(T) & a_{22}(T) & \dots & a_{2,(k-1)}(T) \\ \vdots \\ a_{(k-1),1}(T) & a_{(k-1),2}(T) \\ a_{k,1}(T) \end{pmatrix}$$

is a bijection between the semistandard Young tableaux T of weight $(n+\alpha_1,...,n+\alpha_k)$ and $P^{\mathbb{Z}}_{k,n,\alpha}$. ■

Lemma 3.19. *Let $k \geq 1$ and $\alpha = (\alpha_1,...,\alpha_k) \in \mathbb{N}_0^k$ with $\alpha_1 \geq ... \geq \alpha_k$. We consider the sequence $\{\prod_{i=1}^k s_{(n+\alpha_i)}\}_n \in \Lambda_{\mathbb{N}}^{|\alpha|,k}$. For all $n \geq 1$, we have*

$$\Delta^{(1^k)}\left(\prod_{i=1}^k s_{(n+\alpha_i)}\right)$$
$$= \sum_{A \in P^{\mathbb{Z}}_{k,n,\alpha}:\ a_{k,1}=0} s_{(|a_1|,...,|a_{k-1}|)}.$$

Proof. It follows from the previous lemma that

$$\Delta^{(1^k)}\left(\prod_{i=1}^k s_{(n+\alpha_i)}\right) = \sum_{A\in P^{\mathbb{Z}}_{k,n,\alpha}} s_{(|a_1|,...,|a_k|)} - \sum_{A\in P^{\mathbb{Z}}_{k,n-1,\alpha}} s_{(|a_1|+1,...,|a_k|+1)}.$$

There is an injection $P^{\mathbb{Z}}_{k,n-1,\alpha} \to P^{\mathbb{Z}}_{k,n,\alpha}$ given by

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2,(k-1)} & \\ \vdots & & & & \\ a_{(k-1),1} & a_{(k-1),2} & & & \\ a_{k,1} & & & & \end{pmatrix} \mapsto \begin{pmatrix} a_{11}+1 & a_{12} & a_{13} & \dots & a_{1k} \\ a_{21}+1 & a_{22} & \dots & a_{2,(k-1)} & \\ \vdots & & & & \\ a_{(k-1),1}+1 & a_{(k-1),2} & & & \\ a_{k,1}+1 & & & & \end{pmatrix}.$$

The matrices that are not hit by this map are those with a 0 in the first column. This property is equivalent to $a_{k,1} = 0$ because of $0 \leq a_{k,1} \leq a_{k-1,1} \leq ... \leq a_{11}$. ■

We look at the polytope $\Delta^{(1^k)}P_{k,n,\alpha} := \{A \in P_{k,n,\alpha} :\ a_{k,1} = 0\}$.

Proposition 3.20. *Let $k \geq 1$ and $\alpha = (\alpha_1,...,\alpha_k) \in \mathbb{N}_0^k$ with $\alpha_1 \geq ... \geq \alpha_k$. The set $\Delta^{(1^k)}P_{k,n,\alpha}$ is a facet of $P_{k,n,\alpha}$.*

Proof. $\Delta^{(1^k)}P_{k,n,\alpha}$ is a proper face of $P_{k,n,\alpha}$ because of $a_{k,1} = 0$ for all $A \in \Delta^{(1^k)}P_{k,n,\alpha}$ but not for every $A \in P_{k,n,\alpha}$ and $a_{k,1} \geq 0$ for all $A \in P_{k,n,\alpha}$. This face is $\left(\binom{k}{2} - 1\right)$-dimensional because the set of $\binom{k}{2} + 1$ many affine independent points of $P_{k,n,\alpha}$ given in the proof of Proposition 3.17 contains exactly one point that does not lie in $\Delta^{(1^k)}P_{k,n,\alpha}$. ■

Lemma 3.21. *Let $k \geq 1$ and $\alpha = (\alpha_1,...,\alpha_k) \in \mathbb{N}_0^k$ with $\alpha_1 \geq ... \geq \alpha_k$. We consider the sequence $\{\prod_{i=1}^k s_{(n+\alpha_i)}\}_n \in \Lambda_{\mathbb{N}}^{|\alpha|,k}$. For all $n \geq 2$, we have*

$$\Delta^{(2,1^{k-2})}\Delta^{(1^k)}\left(\prod_{i=1}^k s_{(n+\alpha_i)}\right)$$
$$= \sum_{A\in P^{\mathbb{Z}}_{k,n,\alpha}:\ a_{k,1}=0 \wedge (a_{1k}=0 \vee a_{k-1,1}=0)} s_{(|a_1|,...,|a_{k-1}|)}.$$

Proof. Lemma 3.19 yields

$$\Delta^{(2,1^{k-2})}\Delta^{(1^k)}\prod_{i=1}^{k}s_{(n+\alpha_i)}$$

$$=\sum_{A\in P^{\mathbb{Z}}_{k,n,\alpha}:\ a_{k,1}=0}s_{(|a_1|,\ldots,|a_{k-1}|)}-\sum_{A\in P^{\mathbb{Z}}_{k,n,\alpha}:\ a_{k,1}=0}s_{(|a_1|+2,|a_2|+1,\ldots,|a_{k-1}|+1)}.$$

There is an injective map $\{A\in P^{\mathbb{Z}}_{k,n-1,\alpha}:\ a_{k,1}=0\}\mapsto\{A\in P^{\mathbb{Z}}_{k,n,\alpha}:\ a_{k,1}=0\}$ given by

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1k}\\ a_{21} & a_{22} & \ldots & a_{2,(k-1)} & \\ \vdots & & & & \\ a_{(k-1),1} & a_{(k-1),2} & & & \\ 0 & & & & \end{pmatrix}\mapsto\begin{pmatrix} a_{11}+1 & a_{12} & a_{13} & \ldots & a_{1k}+1\\ a_{21}+1 & a_{22} & \ldots & a_{2,(k-1)} & \\ \vdots & & & & \\ a_{(k-1),1}+1 & a_{(k-1),2} & & & \\ 0 & & & & \end{pmatrix}.$$

The matrices that are not hit are those with $a_{i1}=0$ for a $1\le i\le k-1$ or $a_{1k}=0$. We can reduce the condition to $a_{1k}=0$ or $a_{k-1,1}=0$ because of $0\le a_{k-1,1}\le a_{k-2,1}\le\ldots\le a_{11}$. ■

Let $\Delta^{(2,1^{k-2})}\Delta^{(1^k)}P_{k,n,\alpha}=\{A\in\Delta^{(1^k)}P_{k,n,\alpha}:\ a_{1k}=0\vee a_{k-1,1}=0\}$.

Proposition 3.22. *Let $k\ge 1$ and $\alpha=(\alpha_1,\ldots,\alpha_k)\in\mathbb{N}_0^k$ with $\alpha_1\ge\ldots\ge\alpha_k$. The set $\Delta^{(2,1^{k-2})}\Delta^{(1^k)}P_{k,n,\alpha}$ is the union of the two facets $\{A\in\Delta^{(1^k)}P_{k,n,\alpha}:\ a_{1,k}=0\}$ and $\{A\in\Delta^{(1^k)}P_{k,n,\alpha}:\ a_{k-1,1}=0\}$ of $\Delta^{(1^k)}P_{k,n,\alpha}$.*

Proof. The two sets are proper subsets of $\Delta^{(1^k)}P_{k,n,\alpha}$ and faces because of $a_{1,k}\ge 0$ and $a_{k-1,1}\ge 0$ for all $A\in\Delta^{(1^k)}P_{k,n,\alpha}$. $\{A\in\Delta^{(1^k)}P_{k,n,\alpha}:\ a_{k-1,1}=0\}$ is a facet of $\Delta^{(1^k)}P_{k,n,\alpha}$ because the set of $\binom{k}{2}+1$ many affine independent points of $P_{k,n,\alpha}$ given in the proof of Proposition 3.17 contains exactly two points with $a_{k-1,1}>0$. For $\{A\in\Delta^{(1^k)}P_{k,n,\alpha}:\ a_{1,k}=0\}$ we slightly modify the list of affine independent points given in the proof of Proposition 3.17. For every $i\in\{2,\ldots,k-2\}$ we take the i points

$$\begin{pmatrix} n+\alpha_1 & 0 & 0 & \ldots & 0 & n+\alpha_{k-i+1} & \ldots & n+\alpha_{k-1} & 0\\ n+\alpha_2 & 0 & \ldots & 0 & 0 & \ldots & 0 & n+\alpha_k & \\ \vdots & \vdots & & & & & & & \\ n+\alpha_{k-i} & 0 & & & & & & & \\ 0 & \vdots & & & & & & & \\ \vdots & & & & & & & & \\ 0 & & & & & & & & \end{pmatrix},$$

$$\begin{pmatrix} n+\alpha_1 & 0 & 0 & \dots & 0 & n+\alpha_{k-i+1} & \dots & n+\alpha_{k-2} & 0 & 0 \\ n+\alpha_2 & 0 & \dots & 0 & 0 & \dots & 0 & n+\alpha_{k-1} & 0 & \\ n+\alpha_3 & 0 & \dots & 0 & 0 & \dots & 0 & n+\alpha_k & & \\ \vdots & \vdots & & & & & & & & \\ n+\alpha_{k-i} & 0 & & & & & & & & \\ 0 & \vdots & & & & & & & & \\ \vdots & & & & & & & & & \\ 0 & & & & & & & & & \end{pmatrix}, ...,$$

$$\begin{pmatrix} n+\alpha_1 & 0 & 0 & \dots & 0 & n+\alpha_{k-i+1} & \dots & 0 & 0 & 0 \\ n+\alpha_2 & 0 & \dots & 0 & & n+\alpha_{k-i+2} & 0 & \dots & 0 & \\ n+\alpha_3 & 0 & \dots & 0 & 0 & n+\alpha_{k-i+3} & 0 & \dots & & \\ \vdots & & & & & \vdots & & & & \\ \vdots & 0 & 0 & 0 & 0 & n+\alpha_k & & & & \\ n+\alpha_{k-i} & 0 & \vdots & & & & & & & \\ 0 & \vdots & & & & & & & & \\ \vdots & & & & & & & & & \\ 0 & & & & & & & & & \end{pmatrix}$$

and

$$\begin{pmatrix} n+\alpha_1 & 0 & \dots & 0 & 1 & n+\alpha_{k-i+1} & \dots & 0 & 0 & 0 \\ n+\alpha_2 & 0 & \dots & 0 & & n+\alpha_{k-i+2} & 0 & \dots & 0 & \\ n+\alpha_3 & 0 & \dots & 0 & 0 & n+\alpha_{k-i+3} & 0 & \dots & & \\ \vdots & & & & & \vdots & & & & \\ \vdots & 0 & 0 & 0 & 0 & n+\alpha_k & & & & \\ n+\alpha_{k-i}-1 & 0 & \vdots & & & & & & & \\ 0 & \vdots & & & & & & & & \\ \vdots & & & & & & & & & \\ 0 & & & & & & & & & \end{pmatrix}.$$

We additionally take the $k-1$ points

$$\begin{pmatrix} n+\alpha_1 & n+\alpha_2 & \dots & \dots & \dots & n+\alpha_{k-1} & 0 \\ 0 & 0 & \dots & 0 & 0 & n+\alpha_k & \\ 0 & \dots & \dots & 0 & 0 & & \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & & & \\ 0 & 0 & & & & & \\ 0 & & & & & & \end{pmatrix}, \dots, \begin{pmatrix} n+\alpha_1 & n+\alpha_2 & \dots & 0 & 0 & 0 & 0 \\ 0 & n+\alpha_3 & \dots & 0 & 0 & 0 & \\ 0 & \vdots & \dots & 0 & 0 & & \\ \vdots & & & & & & \\ 0 & n+\alpha_{k-2} & 0 & 0 & & & \\ 0 & n+\alpha_{k-1} & 0 & & & & \\ 0 & n+\alpha_k & & & & & \\ 0 & & & & & & \end{pmatrix}$$

and

$$\begin{pmatrix} n+\alpha_1 & n+\alpha_2-1 & \dots & 0 & 0 & 0 & 0 \\ 1 & n+\alpha_3 & \dots & 0 & 0 & 0 & \\ 0 & \vdots & \dots & 0 & 0 & & \\ \vdots & & & & & & \\ 0 & n+\alpha_{k-2} & 0 & 0 & & & \\ 0 & n+\alpha_{k-1} & 0 & & & & \\ 0 & n+\alpha_k & & & & & \\ 0 & & & & & & \end{pmatrix}.$$

These are $2+3+...+(k-2)+(k-1)=\binom{k}{2}-1$ many affine independent points. ■

Now we can prove Proposition 3.15.

Proof of Proposition 3.15. (*a*) It follows from Lemma 3.21 that for all $n \geq 2$:

$$\Delta^{(2)}\Delta^{(1,1)}s_{(n+\alpha_1)}s_{(n)}$$

$$=\sum_A s_{(|a_1|,|a_2|)}$$

running over all

$$A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & \end{pmatrix} \in P^{\mathbb{Z}}_{2,n,\alpha}$$

with

$$a_{21}=0,\ a_{11}=n+\alpha_1,\ a_{12}=n,$$
$$a_{11}=0 \text{ or } a_{12}=0.$$

But this cannot be for $n>0$. Therefore the sum is zero.
We show the minimality of the set $\{\Delta^{(2)},\Delta^{(1,1)}\}$ next. By Lemma 3.19, we have

$$\Delta^{(1,1)}s_{(n+\alpha_1)}s_{(n)}$$

$$=\sum_A s_{(|a_1|,|a_2|)}$$

running over all

$$A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & \end{pmatrix} \in P^{\mathbb{Z}}_{2,n,\alpha}$$

with

$$a_{21}=0,\ a_{11}=n+\alpha_1,\ a_{12}=n.$$

This is the matrix

$$A=\begin{pmatrix} n+\alpha_1 & n \\ 0 & \end{pmatrix}$$

and the above sum is not zero. The term $\Delta^{(2)}s_{(n+\alpha_1)}s_{(n)}$ does also not equal zero because the sum

$$s_{(n+\alpha_1)}s_{(n)}=\sum_{A\in P^{\mathbb{Z}}_{2,n,\alpha}} s_{(|a_1|,|a_2|)}$$

has the summand $s_{(n+\alpha_1,n)}$ that is not cancelled by applying $\Delta^{(2)}$ because $|a_1| \geq n-1+\alpha_1$ for all $A \in P^{\mathbb{Z}}_{2,n-1,\alpha}$.
(b) It follows from Lemma 3.21 that for all $n \geq 2$:

$$\Delta^{(2,1)}\Delta^{(1,1,1)}s_{(n+\alpha_1)}s_{(n+\alpha_2)}s_{(n)}$$

$$= \sum_{A} s_{(|a_1|,|a_2|)}$$

running over all

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & & \end{pmatrix} \in P^{\mathbb{Z}}_{3,n,\alpha}$$

with

$$a_{31} = 0,$$

$$a_{13} = 0 \text{ or } a_{21} = 0.$$

We want to apply the map $\Delta^{(3)}$ next. We treat the two sets $Q_{n,21} = \{A \in P^{\mathbb{Z}}_{3,n-1,\alpha} \mid a_{31} = 0 \wedge a_{21} = 0\}$ and $Q_{n,13} = \{A \in P^{\mathbb{Z}}_{3,n-1,\alpha} \mid a_{31} = 0 \wedge a_{13} = 0 \wedge a_{21} > 0\}$ separately. There is an injection $Q_{n-1,21} \to Q_{n,21}$ given by

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & \\ 0 & & \end{pmatrix} \mapsto \begin{pmatrix} a_{11}+1 & a_{12}+1 & a_{13}+1 \\ 0 & a_{22} & \\ 0 & & \end{pmatrix}.$$

The only matrix that is not hit by this map is $\begin{pmatrix} n+\alpha_1 & n+\alpha_2 & 0 \\ 0 & n & \\ 0 & & \end{pmatrix}$. It follows

$$\Delta^{(3)}\Delta^{(2,1)}\Delta^{(1,1,1)}s_{(n+\alpha_1)}s_{(n+\alpha_2)}s_{(n)}$$

$$= s_{(2n+|\alpha|,n)} + \sum_{A \in Q_{n,13}} s_{(|a_1|,|a_2|)} - \sum_{B \in Q_{n-1,13}} s_{(|b_1|,|b_2|)}$$

We have

$$Q_{n,13} = \left\{ \begin{pmatrix} n+\alpha_1 & n+\alpha_2-a_{21} & 0 \\ a_{21} & n & \\ 0 & & \end{pmatrix} \mid a_{21} \in \left\{1, ..., \left\lfloor \frac{n+|\alpha|}{2} \right\rfloor \right\} \right\}.$$

It follows that for every $B \in Q_{n-1,13}$ there is exactly one $A \in Q_{n,13}$ with $(|a_1|,|a_2|) = (|b_1|+3,|b_2|)$. It is the matrix A with $a_{21} = b_{21} - 1$. There is one matrix in $Q_{n-1,13}$ and one or two matrices in $Q_{n,13}$ not involved in this correspondence depending on the parity of $n+|\alpha|$. These matrices are

$$\begin{pmatrix} n-1+\alpha_1 & n-2+\alpha_2 & 0 \\ 1 & n-1 & \\ 0 & & \end{pmatrix} \in Q_{n-1,13},$$

$$\begin{pmatrix} n+\alpha_1 & (n+\alpha_2-\alpha_1)/2+1 & 0 \\ (n+|\alpha|)/2-1 & n & \\ 0 & & \end{pmatrix}, \begin{pmatrix} n+\alpha_1 & (n+\alpha_2-\alpha_1)/2 & 0 \\ (n+|\alpha|)/2 & n & \\ 0 & & \end{pmatrix} \in Q_{n,13}$$

$$\text{if } n+|\alpha| \text{ is even,}$$

$$\begin{pmatrix} n+\alpha_1 & (n+\alpha_2-\alpha_1+1)/2 & 0 \\ (n+|\alpha|-1)/2 & n & \\ 0 & & \end{pmatrix} \in Q_{n,13} \text{ if } n+|\alpha| \text{ is odd.}$$

Now in $\Delta^{(3)}\Delta^{(2,1)}\Delta^{(1,1,1)}(s_{(n+\alpha_1)}s_{(n+\alpha_2)}s_{(n)})$, the Schur function $s_{(2n+|\alpha|,n)}$ from before is subtracted and what is left is

$$s_{(3n+|\alpha|)/2+1,(3n+|\alpha|)/2-1)} + s_{((3n+|\alpha|)/2,(3n+|\alpha|)/2)}, \text{ if } n+|\alpha| \text{ is even}$$

$$s_{((3n+|\alpha|+1)/2,(3n+|\alpha|-1)/2)}, \text{ if } n+|\alpha| \text{ is odd.}$$

Applying $\Delta_2^{(3,3)}$ to this yields 0.

We show the minimality of the set $\{\Delta_2^{(3,3)}, \Delta^{(3)}, \Delta^{(2,1)}, \Delta^{(1^3)}\}$ next. We see above that $\Delta^{(3)}\Delta^{(2,1)}\Delta^{(1,1,1)}(s_{(n+\alpha_1)}s_{(n+\alpha_2)}s_{(n)})$ is not zero. The term $\Delta_2^{(3,3)}\Delta^{(3)}\Delta^{(2,1)}(s_{(n+\alpha_1)}s_{(n+\alpha_2)}s_{(n)})$ is not zero because the summand $s_{(n+\alpha_1,n+\alpha_2,n)}$ in the Schur function expansion of $s_{(n+\alpha_1)}s_{(n+\alpha_2)}s_{(n)}$ is not cancelled by $\Delta^{(3)}$ or $\Delta^{(2,1)}$ because $|a_1| \geq n-1+\alpha_2$ for all $A \in P^{\mathbb{Z}}_{3,n-1,\alpha}$ and it is not cancelled by $\Delta_2^{(3,3)}$ because $|a_1| \geq n-2+\alpha_2$ for all $A \in P^{\mathbb{Z}}_{3,n-2,\alpha}$. The term $\Delta_2^{(3,3)}\Delta^{(2,1)}\Delta^{(1^3)}(s_{(n+\alpha_1)}s_{(n+\alpha_2)}s_{(n)})$ is not zero because the summand $s_{(3n+|\alpha|)}$ in the Schur function expansion of $s_{(n+\alpha_1)}s_{(n+\alpha_2)}s_{(n)}$ is not cancelled. In order to show that the term $\Delta_2^{(3,3)}\Delta^{(3)}\Delta^{(1^3)}(s_{(n+\alpha_1)}s_{(n+\alpha_2)}s_{(n)})$ is not zero we show that the multiplicity of $s_{(2n+|\alpha|,n)}$ in its Schur function decomposition is not zero. We denote the multiplicity of a Schur function s_λ in a symmetric function f by $\mathrm{mult}(\lambda, f)$. Now, we have

$$\mathrm{mult}((2n+|\alpha|,n), \Delta_2^{(3,3)}\Delta^{(3)}\Delta^{(1^3)}(s_{(n+\alpha_1)}s_{(n+\alpha_2)}s_{(n)}))$$

$$= \mathrm{mult}((2n+|\alpha|,n), \Delta^{(1^3)}(s_{(n+\alpha_1)}s_{(n+\alpha_2)}s_{(n)}))$$

$$-\mathrm{mult}((2n-3+|\alpha|,n), \Delta^{(1^3)}(s_{(n-1+\alpha_1)}s_{(n-1+\alpha_2)}s_{(n-1)}))$$

$$-\mathrm{mult}((2n-3+|\alpha|,n-3), \Delta^{(1^3)}(s_{(n-2+\alpha_1)}s_{(n-2+\alpha_2)}s_{(n-2)}))$$

$$+\mathrm{mult}((2n-6+|\alpha|,n-3), \Delta^{(1^3)}(s_{(n-3+\alpha_1)}s_{(n-3+\alpha_2)}s_{(n-3)})).$$

The four involved numbers count in this order the number of matrices in $\Delta^{(1^4)}P^{\mathbb{Z}}_{4,n,\alpha}$ of the form

$$\begin{pmatrix} n+\alpha_1 & n+\alpha_2-a_{21} & a_{21} \\ a_{21} & n-a_{21} & \\ 0 & & \end{pmatrix} \text{ for all } a_{21} \in \{0,...,n\},$$

$$\begin{pmatrix} n-1+\alpha_1 & n-1+\alpha_2-a_{21} & a_{21}-1 \\ a_{21} & n-a_{21} & \\ 0 & & \end{pmatrix} \text{ for all } a_{21} \in \{1,...,n-1+\min\{1,\alpha_2\}\},$$

$$\begin{pmatrix} n-2+\alpha_1 & n-2+\alpha_2-a_{21} & a_{21}+1 \\ a_{21} & n-3-a_{21} & \\ 0 & & \end{pmatrix} \text{ for all } a_{21} \in \{0,...,n-3\},$$

$$\begin{pmatrix} n-3+\alpha_1 & n-3+\alpha_2-a_{21} & a_{21} \\ a_{21} & n-3-a_{21} & \\ 0 & & \end{pmatrix} \text{ for all } a_{21} \in \{0,...,n-3\}.$$

It follows

$$\begin{aligned} &\operatorname{mult}((2n+|\alpha|,n), \Delta_2^{(3,3)}\Delta^{(3)}\Delta^{(1^3)}(s_{(n+\alpha_1)}s_{(n+\alpha_2)}s_{(n)})) \\ &= n+1-(n-1+\min\{1,\alpha_2\})-(n-2)+(n-2) \geq 1. \end{aligned}$$

∎

The next lemmas are dedicated to the statement Conjecture 3.16 about fourfold products.

Lemma 3.23. *We have*

$$\begin{aligned} &(\Delta^{(2,1,1)})^2\Delta^{(1^4)}(s_{(n+\alpha_1)}s_{(n+\alpha_2)}s_{(n+\alpha_3)}s_{(n)}) \\ &= \sum_{A\in\Delta^{(2,1,1)}\Delta^{(1^4)}P^{\mathbb{Z}}_{4,n,\alpha}:\ a_{13}=0\vee a_{32}=0\vee a_{31}=a_{21}\vee a_{21}+a_{22}+a_{23}=a_{11}+a_{12}+a_{13}} s_{(|a_1|,|a_2|,|a_3|)}. \end{aligned}$$

Proof. Lemma 3.21 yields

$$\Delta^{(2,1,1)}\Delta^{(1^4)}(s_{(n+\alpha_1)}s_{(n+\alpha_2)}s_{(n+\alpha_3)}s_{(n)}) = \sum_{A\in\Delta^{(2,1,1)}\Delta^{(1^4)}P^{\mathbb{Z}}_{4,n,\alpha}} s_{(|a_1|,|a_2|,|a_3|)}.$$

There is an injective map $\Delta^{(2,1,1)}\Delta^{(1^4)}P^{\mathbb{Z}}_{4,n-1,\alpha} \to \Delta^{(2,1,1)}\Delta^{(1^4)}P^{\mathbb{Z}}_{4,n,\alpha}$ given by

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & \\ a_{31} & a_{32} & & \\ 0 & & & \end{pmatrix} \mapsto \begin{pmatrix} a_{11}+1 & a_{12} & a_{13}+1 & a_{14} \\ a_{21}+1 & a_{22} & a_{23} & \\ a_{31} & a_{32}+1 & & \\ 0 & & & \end{pmatrix}.$$

The matrices that are not hit are those with $a_{13}=0\vee a_{21}=0\vee a_{32}=0\vee a_{31}=a_{21}\vee a_{21}+a_{22}+a_{23}=a_{11}+a_{12}+a_{13}$. $a_{21}=0$ implies $a_{31}=a_{21}$ because of $0\leq a_{31}\leq a_{21}$. ∎

Let $(\Delta^{(2,1,1)})^2\Delta^{(1^4)}P_{4,n,\alpha} := \{A \in \Delta^{(2,1,1)}\Delta^{(1^4)}P_{4,n,\alpha} :\ a_{13}=0\vee a_{32}=0\vee a_{31}=a_{21}\vee a_{21}+a_{22}+a_{23}=a_{11}+a_{12}+a_{13}\}$. We know so far that

$$P_{4,n,\alpha} \supseteq \Delta^{(1^4)}P_{4,n,\alpha} \supseteq \Delta^{(2,1,1)}\Delta^{(1^4)}P_{4,n,\alpha}$$

is a sequence of unions of faces of $P_{4,n,\alpha}$ with corresponding dimensions

$$6 > 5 > 4$$

where we say that the dimension of a union of polytopes is the maximum of the dimensions of the polytopes. Here, we have again that the subset $(\Delta^{(2,1,1)})^2\Delta^{(1^4)}P_{4,n,\alpha} \subseteq$

$\Delta^{(2,1,1)}\Delta^{(1^4)}P_{4,n,\alpha}$ is a union of faces. Its dimension is 3 because the face $\{A \in P_{4,n,\alpha} : a_{41} = a_{14} = a_{13} = 0\} \subseteq (\Delta^{(2,1,1)})^2\Delta^{(1^4)}P_{4,n,\alpha}$ contains the 4 affine independent points

$$\begin{pmatrix} n+\alpha_1 & n+\alpha_2 & 0 & 0 \\ 0 & n+\alpha_3 & n & \\ 0 & 0 & & \\ 0 & & & \end{pmatrix}, \begin{pmatrix} n+\alpha_1 & n+\alpha_2 & 0 & 0 \\ 0 & n+\alpha_3 & 0 & \\ 0 & n & & \\ 0 & & & \end{pmatrix},$$

$$\begin{pmatrix} n+\alpha_1 & n+\alpha_2-1 & 0 & 0 \\ 1 & n+\alpha_3 & 0 & \\ 0 & n & & \\ 0 & & & \end{pmatrix}, \begin{pmatrix} n+\alpha_1 & n+\alpha_2-1 & 0 & 0 \\ 1 & n+\alpha_3-1 & n & \\ 1 & 0 & & \\ 0 & & & \end{pmatrix}.$$

Lemma 3.24. *We have*

$$\Delta^{(2,2)}(\Delta^{(2,1,1)})^2\Delta^{(1^4)}\big(s_{(n+\alpha_1)}s_{(n+\alpha_2)}s_{(n+\alpha_3)}s_{(n)}\big)$$
$$= \sum_{A\in(\Delta^{(2,1,1)})^2\Delta^{(1^4)}P^{\mathbb{Z}}_{4,n,\alpha}:\ a_{12}=0\vee a_{22}=0\vee a_{23}=0\vee a_{21}+a_{22}=a_{11}+a_{12}\vee a_{31}+a_{32}=a_{21}+a_{22}} s_{(|a_1|,|a_2|,|a_3|)}.$$

Proof. There is an injection $(\Delta^{(2,1,1)})^2\Delta^{(1^4)}P^{\mathbb{Z}}_{4,n-1,\alpha} \to (\Delta^{(2,1,1)})^2\Delta^{(1^4)}P^{\mathbb{Z}}_{4,n,\alpha}$ given by

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & \\ a_{31} & a_{32} & & \\ a_{41} & & & \end{pmatrix} \mapsto \begin{pmatrix} a_{11}+1 & a_{12}+1 & a_{13} & a_{14} \\ a_{21} & a_{22}+1 & a_{23}+1 & \\ a_{31} & a_{32} & & \\ a_{41} & & & \end{pmatrix}.$$

The matrices that are not hit are those with $a_{12} = 0$ or $a_{22} = 0$ or $a_{23} = 0$ or $a_{21} + a_{22} = a_{11} + a_{12}$ or $a_{31} + a_{32} = a_{21} + a_{22}$. ■

Let $\Delta^{(2,2)}(\Delta^{(2,1,1)})^2\Delta^{(1^4)}P_{4,n,\alpha} := \{A \in (\Delta^{(2,1,1)})^2\Delta^{(1^4)}P_{4,n,\alpha} : a_{12} = 0 \vee a_{22} = 0 \vee a_{23} = 0 \vee a_{21} + a_{22} = a_{11} + a_{12} \vee a_{31} + a_{32} = a_{21} + a_{22}\}$. The face $\{A \in P_{4,n,\alpha} : a_{41} = a_{14} = a_{13} = a_{23} = 0\} \subseteq \Delta^{(2,2)}(\Delta^{(2,1,1)})^2\Delta^{(1^4)}P_{4,n,\alpha}$ has dimension 2 because it contains the 3 affine independent points

$$\begin{pmatrix} n+\alpha_1 & n+\alpha_2 & 0 & 0 \\ 0 & n+\alpha_3 & 0 & \\ 0 & n & & \\ 0 & & & \end{pmatrix}, \begin{pmatrix} n+\alpha_1 & n+\alpha_2-1 & 0 & 0 \\ 1 & n+\alpha_3 & 0 & \\ 0 & n & & \\ 0 & & & \end{pmatrix},$$

$$\begin{pmatrix} n+\alpha_1 & n+\alpha_2-1 & 0 & 0 \\ 1 & n+\alpha_3-1 & 0 & \\ 1 & n & & \\ 0 & & & \end{pmatrix}.$$

We summarize that

$$P_{4,n,\alpha} \supseteq \Delta^{(1^4)}P_{4,n,\alpha} \supseteq \Delta^{(2,1,1)}\Delta^{(1^4)}P_{4,n,\alpha} \supseteq (\Delta^{(2,1,1)})^2\Delta^{(1^4)}P_{4,n,\alpha} \supseteq \Delta^{(2,2)}(\Delta^{(2,1,1)})^2\Delta^{(1^4)}P_{4,n,\alpha}$$

is a sequence of unions of faces of $P_{4,n,\alpha}$ with corresponding dimensions

$$6 > 5 > 4 > 3 > 2.$$

4. Relative arrangements of linear subspaces

4.1. **Introduction.** In this chapter we prove a Goresky-MacPherson type formula for the complement of two arrangements of linear subspaces sitting set theoretically inside another. The union $U_{\mathcal{A}}$ of a subspace arrangement $\mathcal{A}$ in $\mathbb{R}^d$ is the set theoretic union $\bigcup_{V\in\mathcal{A}} V \subseteq \mathbb{R}^d$ equipped with the subspace topology. Relative arrangements are a concept introduced in [21]. We say that a pair $(\mathcal{A}, \mathcal{B})$ of arrangements of subspaces in $\mathbb{R}^d$ is a *relative arrangement of subspaces* if $(U_{\mathcal{A}}, U_{\mathcal{B}})$ is a pair of topological spaces; i.e. $U_{\mathcal{B}} \subseteq U_{\mathcal{A}}$. Our main goal is to prove a formula for the cohomology of the complement $M_{\mathcal{A},\mathcal{B}} := U_{\mathcal{A}} \setminus U_{\mathcal{B}}$ in terms of combinatorial and dimension data. We first recall the classical Goresky-MacPherson formula.

For later use for a subspace V of $\mathbb{R}^d$ and a subspace arrangement $\mathcal{B}$ we write $\mathcal{M}_{V,\mathcal{B}}$ for $V \setminus U_{\mathcal{B}}$. In particular, $\mathcal{M}_{\mathbb{R}^d,\mathcal{B}} = \mathbb{R}^d \setminus U_{\mathcal{B}}$. For an arrangement of subspaces $\mathcal{B}$ we write $L_{\mathcal{B}}$ for the set of intersections $\bigcap_{V\in\mathcal{B}'} V$ for all subsets $\mathcal{B}' \subseteq \mathcal{B}$ ordered by reversed inclusion and enlarged by a unique minimal element $\hat{0}$ and a unique maximal element $\hat{1}$. We set $\mathring{L}_{\mathcal{B}} := L_{\mathcal{B}} \setminus \{\hat{0}, \hat{1}\}$. As usual we consider the intersection $\bigcap_{V\in\emptyset} V$ as $\mathbb{R}^d$, but deviate from the convention in the literature to identify $\mathbb{R}^d$ with the unique minimal element of $L_{\mathcal{B}}$. Also by our convention if the intersection of all subspaces in $\mathcal{B}$ is not empty then the intersection will not be identified with the unique maximal element $\hat{1}$. The partially ordered set $L_{\mathcal{B}}$ is indeed a lattice and is referred to as the *intersection lattice* of $\mathcal{B}$. For $V \in L_{\mathcal{B}}$ we write $(\mathbb{R}^d, V)$ for the interval of all subspaces in $L_{\mathcal{B}}$ strictly between $\mathbb{R}^d$ and V. We denote by $\tilde{H}_i((\mathbb{R}^d, V), R)$ the reduced homology of the simplicial complex of all linearly ordered subsets of $(\mathbb{R}^d, V)$ with coefficients in a ring R. Recall the formula by Goresky and MacPherson.

Theorem 4.1 (Goresky-MacPherson Formula)**.** [9] *Let $\mathcal{B} \not\ni \mathbb{R}^d$ be an arrangement of subspaces in $\mathbb{R}^d$. Then*

$$\tilde{H}^i(\mathcal{M}_{\mathbb{R}^d,\mathcal{B}}, \mathbb{Z}) \cong \bigoplus_{V\in\mathring{L}_{\mathcal{A}}\setminus\{\mathbb{R}^d\}} \tilde{H}_{d-\dim(V)-i-2}((\mathbb{R}^d, V), \mathbb{Z}).$$

If $(\mathcal{A}, \mathcal{B})$ is a relative arrangement then without changing the topology of $U_{\mathcal{A}}$, $U_{\mathcal{B}}$ and $M_{\mathcal{A},\mathcal{B}}$ we may assume that $\mathcal{B} \subseteq \mathcal{A}$. In this case $L_{\mathcal{B}} \subseteq L_{\mathcal{A}}$ and indeed, without changing the topology we may assume that $L_{\mathcal{B}}$ is an upper order ideal in $L_{\mathcal{A}}$. Thus $(L_{\mathcal{A}}, L_{\mathcal{B}})$ is a pair of lattices. We set $L_{\mathcal{A},\mathcal{B}} = (L_{\mathcal{A}} \setminus L_{\mathcal{B}}) \cup \{\hat{0}, \hat{1}\}$ and consider it partially ordered with the order inherited from $L_{\mathcal{A}}$. In particular, $\hat{0}$ is the unique least and $\hat{1}$ the unique maximal element in $L_{\mathcal{A},\mathcal{B}}$. Our assumptions imply that $L_{\mathcal{A},\mathcal{B}}$ is again a lattice and we call $L_{\mathcal{A},\mathcal{B}}$ the *intersection lattice* of $(L_{\mathcal{A}}, L_{\mathcal{B}})$.

For $V \in L_{\mathcal{A},\mathcal{B}}$ we also write $(\hat{0}, V)$ for the interval of all subspaces $W \in L_{\mathcal{A},\mathcal{B}}$ lying strictly between $\hat{0}$ and V.

Theorem 4.2. *Let $(\mathcal{A}, \mathcal{B})$ be a relative arrangement of linear subspaces such that $U_{\mathcal{A}} \neq U_{\mathcal{B}}$. Assume that for all $V, W \in \overset{\circ}{L}_{\mathcal{A},\mathcal{B}}$ with $V \subsetneq W$ we have* $\operatorname{codim}(W, V + \sum_{U \in \mathcal{B}} U \cap W) \geq 1$. *Then for all $i \geq 0$:*

$$\tilde{H}^i(\mathcal{M}_{\mathcal{A},\mathcal{B}}, \mathbb{Z}) \cong \bigoplus_{V \in \overset{\circ}{L}_{\mathcal{A},\mathcal{B}}} \bigoplus_{\ell+k=i-1} \tilde{H}^\ell((\hat{0}, V), \mathbb{Z}) \otimes \tilde{H}^k(\mathcal{M}_{V,\mathcal{B}}, \mathbb{Z}).$$

Note that $\mathcal{M}_{V,\mathcal{B}}$ is the complement $V \setminus U_{\mathcal{B}'}$ of the arrangements $\mathcal{B}' = \{U \cap V \mid U \in \mathcal{B}\}$ of subspaces of V. In particular, $\tilde{H}^k(\mathcal{M}_{V,\mathcal{B}}, \mathbb{Z})$ can be computed with the usual Goresky-MacPherson formula.

Now consider the following equivariant situation. If a finite subgroup G of $\mathrm{Gl}_n(\mathbb{R})$ leaves $U_{\mathcal{A}}$ invariant then we say that $\mathcal{A}$ is a G-arrangement. If $(\mathcal{A}, \mathcal{B})$ is a relative arrangement and $\mathcal{A}$, $\mathcal{B}$ are G-arrangements then we say that $(\mathcal{A}, \mathcal{B})$ is a relative G-arrangement. For a subspace V we write G_V for the stabilizer of V in G. The action of G on $\mathcal{M}_{\mathcal{A},\mathcal{B}}$ determines a representation on $\tilde{H}^i(\mathcal{M}_{\mathcal{A},\mathcal{B}}, \mathbb{C})$.

Theorem 4.3. *Let $(\mathcal{A}, \mathcal{B})$ be a relative G-arrangement of linear subspaces in $\mathbb{R}^n$ for a finite subgroup G of* $\mathrm{Gl}_n(\mathbb{R})$ *such that $U_{\mathcal{A}} \neq U_{\mathcal{B}}$. Assume that for all $V, W \in L_{\mathcal{A},\mathcal{B}}$ with $V \subsetneq W$ we have* $\operatorname{codim}(W, V + \sum_{U \in \mathcal{B}} U \cap W) \geq 1$. *Then for all $i \geq 0$:*

$$\tilde{H}^i(\mathcal{M}_{\mathcal{A},\mathcal{B}}, \mathbb{C}) \cong_G \bigoplus_{V \in \overset{\circ}{L}_{\mathcal{A},\mathcal{B}}} \mathrm{Ind}_{G_V}^G \bigoplus_{\ell+k=i-1} \tilde{H}^\ell((\hat{0}, V), \mathbb{C}) \otimes \tilde{H}^k(\mathcal{M}_{V,\mathcal{B}}, \mathbb{C}).$$

4.2. **Basics of diagrams of spaces.** In this section we describe the basic constructions and results from the theory of diagrams of spaces and homotopy colimits needed for our purposes. In our description we do not consider the full generality of the theory but a version tailored for the forthcoming applications. We refer to [3] and [20] for background on diagrams of spaces.

Let P be a finite partially ordered set, poset for short. We consider P as the small category whose objects are the poset elements and whose morphisms $p \to q$ are the order relations $p \geq q$ in P.

A diagram (of spaces) over P is a covariant functor $D : P \to \mathrm{Top}$ from the small category P with morphisms $p \to q$ whenever $p \geq q \in P$ to the category of CW-complexes and continuous functions. For every $p \in P$ we write D_p for the image of p under D and $d_{p,q} : D_p \to D_q$ for the image of $p \to q$ under D. We write U_D for $\dot{\cup}_{p \in P} D_p$. Let G be a group. We say that a topological space X is a G-space if G acts on X as a group of homeomorphisms. We call a diagram D over a poset P a G-diagram if G acts on P in an order preserving manner, D_p is a G_p-space for all $p \in P$, $\dot{\cup}_{p \in P} D_p$ is a G-space with $g \cdot D_p = D_{g \cdot p}$ for all $g \in G$, $p \in P$, and $g \cdot d_{p,q}(x) = d_{g \cdot p, g \cdot q}(g \cdot x)$ for all $g \in G$, $p \geq q \in P$ and $x \in D_p$. For every $p \in P$ we write G_p for the stabilizer of p in G. For a group G we denote $\simeq_G$ for G-homotopy equivalence of topological spaces and $\cong_G$ for G-isomorphie of modules.

Let P be a poset and D a P-diagram. We introduce two limit constructions for diagrams. The colimit $\mathrm{colim}(D)$ is $\dot{\cup}_{p \in P} D_p$ modulo the relation generated by $v \equiv d_{q,p}(v)$ for all $q \geq p \in P$ and $v \in D_q$. As a topological space, $\mathrm{colim}(D)$ is the quotient space of

$\dot{\cup}_{p\in P} D_p$ modulo this relation. The second one is called the homotopy colimit and denoted by hocolim(D) for any diagram D. We write $\hat{P}$ for the poset $P \cup \{\hat{0}, \hat{1}\}$ obtained from P by adding a least element $\hat{0}$ and a greatest element $\hat{1}$. For every $p \in P$ we write $P_{\leq p}$ for the poset $\{q \in P \mid q \leq p\}$. Let D be a diagram over the poset P. The homotopy colimit hocolim(D) of D is defined as the disjoint union $\dot{\bigcup}_{p\in P} \Delta(\widehat{P_{\leq p}}) \times D_p$ modulo the equivalence relation generated by

$$(u, v) \equiv (u, d_{q,p}(v)) \text{ for all } p \leq q \in P,\ u \in \Delta(\widehat{P_{\leq p}}) \subseteq \Delta(\widehat{P_{\leq q}}),\ v \in D_q.$$

Its topology also is the quotient topology.

Now we provide an elementary proof of a G-equivariant version of the Projection Lemma by a sequence of lemmas. We write $[n]$ for the set $\{1, ..., n\}$ and $2^{[n]}$ for its set of subsets. This is a poset by the inclusion relation. We use partitions of unity. We refer to [10] for background on paracompact spaces and partitions of unity. A partition of unity subordinate to an open covering $D_1, ..., D_n$ of a space X is a sequence $\{\phi_i\}_{i\in[n]}$ of continuous functions $\phi_i : X \to [0, 1]$ with the properties

- $\mathrm{supp}(\phi_i) \subseteq D_i$ for every $i \in [n]$.
- $\sum_{i=1}^{n} \phi_i(x) = 1$ for every $x \in X$.

If $D_1, ..., D_n$ is an open covering of a paracompact space X then there is a sequence $\{\phi_j\}_{j\in J}$ of continuous functions $\phi_j : X \to [0, 1]$ indexed over some set J with the properties

- For every $j \in J$ there is an $i \in [n]$ with $\mathrm{supp}(\phi_j) \subseteq D_i$.
- For every $x \in X$ there is a neighborhood $U(x)$ such that all but finitely many of the functions ϕ_j are zero on $U(x)$ and the sum of those that are not identically zero is 1 on $U(x)$.

Let $M_i = \{j \in J \mid \mathrm{supp}(\phi_j) \subseteq D_i\}$ for all $i \in [n]$. We set $M'_1 = M_1$ and $M'_i = M_i \setminus (M_1 \cup ... \cup M_{i-1})$ for all $i \geq 2$. Now we define functions $\phi'_i : X \to [0, 1]$ by $\phi'_i(x) = \sum_{j\in M'_i} \phi_j(x)$ for every $x \in X$. Then $\{\phi'_i\}_{i\in[n]}$ is a partition of unity subordinate to $D_1, ..., D_n$ as we defined it above.

Lemma 4.4. *Let $\{\phi'_i\}_{i\in[n]}$ be a partition of unity subordinate to an open covering $D_1, ..., D_n$ of a G-space X where G permutes $D_1, ..., D_n$. Then $\{\phi_i\}_{i\in[n]}$ given by*

$$\phi_i(x) = \frac{1}{|G|} \sum_{g\in G} \phi'_{gi}(gx)$$

is a partition of unity subordinate to the same covering with $\phi_i(gx) = \phi_{g^{-1}i}(x)$.

Proof. If $\phi_i(x) \neq 0$ then there is a $g \in G$ with $\phi'_{gi}(gx) \neq 0$. Then it follows $gx \in D_{gi} = gD_i$. Applying g^{-1} yields $x \in D_i$. Similarly, in a convergent sequence $(x_k)_k$ of points $x_k \in X$ with $\phi_i(x_k) \neq 0$, for every k there is a $g_k \in G$ with $\phi'_{g_k i}(g_k x_k) \neq 0$. We can divide the sequence $(g_k x_k)_k$ into the subsequences $(g x_{k_m})_m$ where all the elements g_{k_m} are a constant element g. Every of these subsequences $(g x_{k_m})_m$ converges to a point in $D_{g_i} = gD_i$ and then $(x_{k_m})_m$ converges to a point in D_i. Then the whole sequence $(x_k)_k$ converges to a point in D_i. It follows $\operatorname{supp}(\phi_i) \subseteq D_i$. Furthermore we have

$$\sum_{i=1}^{n} \phi_i(x) = \frac{1}{|G|} \sum_{g \in G} \sum_{i=1}^{n} \phi'_{gi}(gx)$$
$$= \frac{1}{|G|} \sum_{g \in G} \sum_{i=1}^{n} \phi'_i(gx) = \frac{1}{|G|} \sum_{g \in G} 1 = 1$$

and

$$\phi_i(gx) = \frac{1}{|G|} \sum_{h \in G} \phi'_{hi}(hgx) = \frac{1}{|G|} \sum_{h \in G} \phi'_{hg^{-1}i}(hx) = \phi_{g^{-1}i}(x).$$

■

Lemma 4.5. *Let X be a convex subspace of $\mathbb{R}^d$ that is also a G-space and $\phi, \psi : X \to X$ be two G-equivariant continuous maps. Let $H : X \times [0,1] \to X$ be a homotopy between ϕ and ψ. Then $H' : X \times [0,1] \to X$ defined by*

$$H'(x,t) = \frac{1}{|G|} \sum_{g \in G} g^{-1} H(gx, t)$$

is a G-equivariant homotopy between ϕ and ψ.

Proof. We have

$$H'(x,0) = \frac{1}{|G|} \sum_{g \in G} g^{-1} H(gx, 0) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \phi(gx) = \frac{1}{|G|} \sum_{g \in G} g^{-1} g \phi(x) = \phi(x),$$

analogously $H'(x,1) = \psi(x)$ and

$$H'(gx,t) = \frac{1}{|G|} \sum_{h \in G} h^{-1} H(hgx, t) = \frac{1}{|G|} \sum_{h \in G} g h^{-1} H(hx, t) = gH'(x,t).$$

■

Lemma 4.6. *Let D be a G-diagram over a poset P with n atoms $1, ..., n$ such that P is the intersection poset of $D_1, ..., D_n$. Let $f : 2^{[n]} \to P$, $I \mapsto \cap_{i \in I} D_i$. We define the $2^{[n]}$-diagram E by $E_I = f(I)$ and $e_{p,q} = d_{f(p),f(q)}$. G acts on $2^{[n]}$ by acting on the elements of subsets. In this way, E becomes a G-diagram. Then*

$$\operatorname{hocolim}(D) \simeq_G \operatorname{hocolim}(E).$$

Proof. We define $\psi : P \to 2^{[n]}$ by $\psi(p) = \{i \in [n] \mid D_p \subseteq D_i\}$ for all $p \in P$. We extend f and ψ on $\Delta(2^{[n]})$ and $\Delta(P)$ in the obvious way. Then $f \circ \psi = id_{\Delta(P)}$. f and ψ are G-equivariant. There is a linear homotopy H' between $id_{\Delta(2^{[n]})}$ and $\psi \circ f$ because $\Delta(2^{[n]})$ is convex. It follows from Lemma 4.5 that there is a G-equivariant homotopy H between $id_{\Delta(2^{[n]})}$ and $\psi \circ f$. The induced maps $f : \operatorname{hocolim}(E) \to \operatorname{hocolim}(D)$ and $\psi : \operatorname{hocolim}(D) \to \operatorname{hocolim}(E)$ are also G-equivariant and $f \circ \psi = id_{\operatorname{hocolim}(D)}$. We have $(\psi \circ f)(I) \supseteq I$ and $E_I = E_{(\psi \circ f)(I)}$ for every $I \in 2^{[n]}$. It follows that if $x \in E_I$ and $u \in \Delta(2^{[n]}_{\leq I})$ for some $I \in 2^{[n]}$ then

$(\psi \circ f)(u) \in \Delta(2^{[n]}_{\leq(\psi\circ f)(I)})$ and $x \in E_{(\psi\circ f)(I)}$. It follows that H induces a G-equivariant homotopy between $id_{\text{hocolim}(E)}$ and $\psi \circ f$ by

$$H((u,x),t) = (H(u,t),x)$$

for all $u \in \Delta(2^{[n]})$, $x \in \dot{\cup}_{I\in 2^{[n]}} E_I$ and $t \in [0,1]$. ■

Lemma 4.7 (Projection Lemma)**.** [17, Proposition 4.1][3, XII,3.1(iv)][23, Lemma 1.6][22, Proposition 3.1][7, Theorem 7.41] *Let X be a paracompact Hausdorff space covered by open spaces $D_1, ..., D_n$ and let G be a group of homeomorphisms on X permuting $D_1, ..., D_n$. Let P be the intersection poset of $D_1, ..., D_n$ with corresponding diagram D. Then*

$$\text{colim}(D) \simeq_G \text{hocolim}(D).$$

Proof. Let E be the G-diagram over $2^{[n]}$ from the previous lemma. Let pr be the projection

$$pr : \text{hocolim}(E) \to \text{colim}(E),\ (u,x) \mapsto x.$$

It follows from the paracompactness of X and from Lemma 4.4 that there is a partition of unity $\{\phi_i\}_{i\in[n]}$ for the covering with the property $\phi_i(gx) = \phi_{g^{-1}i}(x)$ for all $i \in [n]$ and $g \in G$. Let $e_1, ..., e_n$ be the vertices of $\Delta(2^{[n]})$. We define

$$\psi : \text{colim}(E) \to \text{hocolim}(E),\ x \mapsto \left(\sum_{i=1}^{n} \phi_i(x)e_i, x\right).$$

$\phi_i(x) \neq 0$ implies $x \in D_i$. It follows that $\psi(x) \in \Delta(2^{[n]}_{\leq I}) \times E_I$ for all $x \in \text{colim}(E)$ and $I = \{i \in [n] \mid \phi_i(x) \neq 0\}$. We have

$$\begin{aligned}\psi(gx) &= \left(\sum_{i=1}^{n} \phi_i(gx)e_i, gx\right) = \left(\sum_{i=1}^{n} \phi_{g^{-1}i}(x)e_i, gx\right) \\ &= \left(\sum_{i=1}^{n} \phi_i(x)e_{gi}, gx\right) = g\left(\sum_{i=1}^{n} \phi_i(x)e_i, x\right)\end{aligned}$$

for all $x \in \text{colim}(E)$ and $g \in G$. So ψ is G-equivariant. We have $pr \circ \psi = id_{\text{colim}(D)}$ while $\psi \circ pr$ and $id_{\text{hocolim}(E)}$ are homotopic by a linear homotopy. ■

Lemma 4.8. [19, Proposition 2.3] *Let D be a G-diagram over a poset P such that for all $p > q$ the map $d_{p,q}$ is trivial in homology. Then for all $i \geq 0$:*

$$\tilde{H}^i(\text{hocolim}(D), \mathbb{C}) \cong_G \bigoplus_{p\in P/G} \text{Ind}_{G_p}^{G} \bigoplus_{l+k=i-1} (\tilde{H}^l(\Delta(\widehat{P_{<p}}), \mathbb{C}) \otimes \tilde{H}^k(D_p, \mathbb{C})).$$

4.3. **Goresky-MacPherson formula for relative arrangements.** In order to apply Lemma 4.8, we have to make further assumptions on our relative arrangement. The following lemma provides a condition in terms of codimensions.

Lemma 4.9. *Let $(\mathcal{A}, \mathcal{B})$ be a relative G-arrangement of linear subspaces in $\mathbb{R}^n$ for a finite subgroup G of $\text{Gl}_n(\mathbb{R})$. Let $V, W \in \mathring{L}_{\mathcal{A},\mathcal{B}}$ with $V \subsetneq W$ such that $\text{codim}(W, V + \sum_{U\in\mathcal{B}} U \cap W) \geq 1$. Then the inclusion $\mathcal{M}_{V,\mathcal{B}} \to \mathcal{M}_{W,\mathcal{B}}$ is homotopically trivial.*

Proof. If $\operatorname{codim}(W, V+\sum_{U\in\mathcal{B}} U\cap W)\geq 1$ then there is a point $c\in W\setminus(V+\sum_{U\in\mathcal{B}} U\cap W)$. Let $v\in V$. Suppose there is a $t\in[0,1)$ such that $u:=tv+(1-t)c\in U_{\mathcal{B}}\cap W$. Then $c=\frac{1}{1-t}(u-tv)\in V+\sum_{U\in\mathcal{B}} U\cap W$, a contradiction. It follows that there is the null homotopy $H:\mathcal{M}_{V,\mathcal{B}}\times[0,1]\to\mathcal{M}_{W,\mathcal{B}}$ given by $H(v,t)=tv+(1-t)c$. ■

Example 4.10. (i) Let $n\in\mathbb{N}$, $k\in\{1,...,n-1\}$ and $\mathcal{A}^d_{n,k}$ be the arrangement of subspaces $\{(x_1,...,x_n)\in(\mathbb{R}^d)^n \mid x_i=0 \text{ for all } i\in I\}$ for all $I\subseteq\{1,...,n\}$ with $\#I=k$. We consider the relative arrangement $(\mathcal{A},\mathcal{B})$ with $\mathcal{A}=\mathcal{A}^d_{n,k}$ and $\mathcal{B}=\mathcal{A}^d_{n,k+1}$. In this case, the lattice $\mathring{L}_{\mathcal{A},\mathcal{B}}$ is an antichain and the assumption in the previous lemma holds vacuously.

(ii) Let $\mathcal{A}^d_{n,k}$ be the arrangement from the previous example for $k\geq 1$ or the k-equal arrangement in $(\mathbb{R}^d)^n$ for $k\geq 2$. We consider the relative arrangement $(\mathcal{A}^d_{n,k},\mathcal{A}^d_{n,n})$ for $k<n$. Then the assumption of the previous lemma holds, because the arrangement $\mathcal{A}^d_{n,n}$ consists of only the space of points with $x_1=...=x_n$ which is the intersection of all spaces of $\mathcal{A}^d_{n,k}$.

Theorem 4.11. *Let $(\mathcal{A},\mathcal{B})$ be a relative G-arrangement of linear subspaces in $\mathbb{R}^n$ for a finite subgroup G of $\mathrm{Gl}_n(\mathbb{R})$ such that $U_{\mathcal{A}}\neq U_{\mathcal{B}}$. Assume that for all $V,W\in\mathring{L}_{\mathcal{A},\mathcal{B}}$ with $V\subsetneq W$ we have $\operatorname{codim}(W,V+\sum_{U\in\mathcal{B}} U\cap W)\geq 1$. Then for all $i\geq 0$:*

$$\tilde{H}^i(\mathcal{M}_{\mathcal{A},\mathcal{B}},\mathbb{C})\cong_G \tilde{H}^i(L_{\mathcal{A},\mathcal{B}},\mathbb{C})\oplus\bigoplus_{V\in\mathring{L}_{\mathcal{A},\mathcal{B}}/G}\operatorname{Ind}^G_{G_V}\bigoplus_{\ell+k=i-1}\tilde{H}^\ell((\hat{0},V),\mathbb{C})\otimes\tilde{H}^k(\mathcal{M}_{V,\mathcal{B}},\mathbb{C}).$$

Proof. Let D be the G-diagram over $L_{\mathcal{A},\mathcal{B}}\setminus\{\hat{0}\}$ with $D_V=\mathcal{M}_{V,\mathcal{B}}=V\setminus U_{\mathcal{B}}$ for all $V\in\mathring{L}_{\mathcal{A},\mathcal{B}}$, $D_{\hat{1}}=\emptyset$ and the morphisms $d_{V,W}$ being the inclusions. Then $\operatorname{colim}(D)=\mathcal{M}_{\mathcal{A},\mathcal{B}}$ is a paracompact space with the open covering $\{V\setminus U_{\mathcal{B}}\mid V\in\mathcal{A}\}$. We have $\tilde{H}^i(\operatorname{colim}(D),\mathbb{C})\cong_G\tilde{H}^i(\operatorname{hocolim}(D),\mathbb{C})$ by the Projection Lemma. Lemma 4.8 together with Lemma 4.9 yield

$$\tilde{H}^i(\operatorname{hocolim}(D),\mathbb{C})\cong_G \tilde{H}^i(L_{\mathcal{A},\mathcal{B}},\mathbb{C})\oplus\bigoplus_{V\in\mathring{L}_{\mathcal{A},\mathcal{B}}/G}\operatorname{Ind}^G_{G_V}\bigoplus_{\ell+k=i-1}\tilde{H}^\ell((\hat{0},V),\mathbb{C})\otimes\tilde{H}^k(\mathcal{M}_{V,\mathcal{B}},\mathbb{C}).$$

■

Example 4.12. For $k\geq 1$ we consider the arrangement $\mathcal{A}_{n,k}$ of spaces $V_I=\{(x_1,...,x_n)\in\mathbb{R}^n\mid x_j=0 \text{ for all } j\in I\}$ for all $I\subseteq[n]$ and the relative arrangement $(\mathcal{A}_{n,k},\mathcal{A}_{n,k+1})$. The set $\mathring{L}_{\mathcal{A}_{n,k},\mathcal{A}_{n,k+1}}$ consists of all the spaces V_I with $\#I=k$, they are pairwise not comparable. Theorem 4.11 implies

$$\tilde{H}^i(\mathcal{M}_{\mathcal{A}_{n,k},\mathcal{A}_{n,k+1}},\mathbb{C})\cong_{S_n}\tilde{H}^i(L_{\mathcal{M}_{\mathcal{A}_{n,k},\mathcal{A}_{n,k+1}}},\mathbb{C})\oplus\operatorname{Ind}^{S_n}_{S_k\times S_{n-k}}\tilde{H}^i(\mathcal{M}_{V_{[k]},\mathcal{A}_{n,k+1}},\mathbb{C})$$

for all $i\geq 0$. For further considerations one has to look at the $(S_k\times S_{n-k})$-action on the poset $\{W\cap V_{[k]}\mid W\in\mathcal{A}_{n,k+1}\}$ and determine the $(S_k\times S_{n-k})$-representation $\tilde{H}^i(\mathcal{M}_{V_{[k]},\mathcal{A}_{n,k+1}},\mathbb{C})$ using the Goresky-MacPherson formula ([9],[19, Theorem 2.5]). We look at the case $i=0$. The S_n-representation $\tilde{H}^0(L_{\mathcal{A}_{n,k},\mathcal{A}_{n,k+1}},\mathbb{C})$ is $(\binom{n}{k}-1)$-dimensional. The space $\mathcal{M}_{V_{[k]},\mathcal{A}_{n,k+1}}$

is homeomorphic to $\{(x_1, ..., x_{n-k}) \in \mathbb{R}^{n-k} \mid x_i \neq 0 \text{ for all } i \in [n-k]\}$ which has 2^{n-k} connected components. We get the following formula for the dimension of $\tilde{H}^0(\mathcal{M}_{\mathcal{A}_{n,k},\mathcal{A}_{n,k+1}}, \mathbb{C})$:

$$\binom{n}{k} - 1 + \binom{n}{k}(2^{n-k} - 1) = \binom{n}{k}2^{n-k} - 1.$$

This implies that the sequence $\{\tilde{H}^0(\mathcal{M}_{\mathcal{A}_{n,k},\mathcal{A}_{n,k+1}}, \mathbb{C})\}_n$ does not stabilize, because the dimensions in a stabilizing sequence must grow polynomially (see [6]).

There is no known counterexample for relative arrangements of diagonal subspaces. More precisely, we mean the following: For two integer partitions μ and λ of $n_0 \in \mathbb{N}$ we say that μ is finer than λ if there are number partitions $\nu_i \vdash \lambda_i$ such that by concatenating them and then sorting this list we get μ. Let $\mathcal{A}^d_\lambda$ be the subspace arrangement defined as in Chapter 2 for every integer partition λ of n_0 and for every $n \geq n_0$ let $\lambda^{(n)}$ be the integer partition obtained from λ by adding $n - n_0$ parts of size 1. If μ and λ are integer partitions of n_0 such that μ is finer than λ then $\mu^{(n)}$ is finer than $\lambda^{(n)}$ for every $n \geq n_0$ and $\{\mathcal{A}^d_{\lambda^{(n)}}, \mathcal{A}^d_{\mu^{(n)}}\}_n$ is a sequence of relative arrangements.

Question 4.13. Let μ and λ be number partitions of $n_0 \in \mathbb{N}$ such that μ is finer than λ and $i \geq 0$. Does the sequence of S_n-representations $\{\tilde{H}^i(\mathcal{M}_{\mathcal{A}^d_{\lambda^{(n)}},\mathcal{A}^d_{\mu^{(n)}}}, \mathbb{C})\}_n$ stabilize?

References

[1] A. Björner, G. M. Ziegler, Combinatorial stratification of complex arrangements, *J. Amer. Math. Soc.*, **5**, (1992), 105–149

[2] P. V. M. Blagojević, W. Lück, G. M. Ziegler, Equivariant topology of configuration spaces, *Journal of Topology*, **8**, (2015), 414–456

[3] A. K. Bousfield, D. M. Kan, Homotopy limits, completions and localizations, *Lecture Notes in Mathematics*, **304**. Springer-Verlag, Berlin-New York, (1972), v+348

[4] T. Church, Homological stability for configuration spaces of manifolds, *Invent. Math.* **188** (2012), no. 2, 465–504

[5] T. Church, B. Farb, Representation theory and homological stability, *Advances in Mathematics* **245** (2013) 250–314

[6] T. Church, J. S. Ellenberg, B. Farb, FI-modules and stability for representations of symmetric groups, *Duke Math. J.* **164** (2015), no. 9, 1833–1910

[7] L. Fajstrup, E. Goubault, E. Haucourt, S. Mimram, M. Raussen, Directed algebraic topology and concurrency, *Springer, [Cham]*, (2016)

[8] N. Gadish, Representation stability for families of linear subspace arrangements, *Advances in Mathematics* **322** (2017), 341–377

[9] M. Goresky, R. MacPherson, Stratified Morse Theory. *Ergebnisse der Mathematik und ihrer Grenzgebiete*, **14**, Berlin, Heidelberg, New York: Springer 1988

[10] A. Hatcher, Vector Bundles and K-Theory, http://pi.math.cornell.edu/ hatcher/VBKT/VBpage.html

[11] P. Hersh, V. Reiner, Representation stability for cohomology of configuration spaces in $\mathbb{R}^d$, *Int. Math. Res. Not. IMRN* **2017** (2017), no. 5, 1433–1486

[12] I.G. Macdonald, Symmetric Functions and Hall Polynomials, 2. ed., Clarendon Press, Oxford, 1995

[13] D. Petersen, A spectral sequence for stratified spaces and configuration spaces of points, *Geometry & Topology* **21** (2017), 2527–2555

[14] A. Rapp, Representation stability on the cohomology of complements of subspace arrangements, *Algebraic Combinatorics* **2** (2019) no. 4, 603–611

[15] A. Rapp, Products of stabilizing representations, arXiv:1904.11743

[16] B. E. Sagan, The symmetric group, Representations, combinatorial algorithms, and symmetric functions, 2. ed., *Graduate Texts in Mathematics* **203** (2001) Springer-Verlag, New York

[17] G. Segal, Classifying spaces and spectral sequences, *Inst. Hautes Études Sci. Publ. Math.*, **34**, (1968)

[18] S. Sundaram and M. Wachs, The homology representations of the k-equal partition lattice, *Trans. Amer. Math. Soc.* **349** (1997), no. 3, 935–954.

[19] S. Sundaram and V. Welker, Group actions on arrangements of linear subspaces and applications to configuration spaces, *Trans. Amer. Math. Soc.* **349** (1997), no. 4, 1389–1420.

[20] R. M. Vogt, Homotopy limits and colimits, *Math. Z.*, **134**, (1973), 11–52,

[21] V. Welker, Partition Lattices, Group Actions on Subspace Arrangements & Combinatorics of Discriminants, Habilitationsschrift, Department of Mathematics and Computer Science, GH. Universität Essen (1996)

[22] V. Welker, G. M. Ziegler, R. T. Živaljević, Homotopy colimits—comparison lemmas for combinatorial applications, *J. Reine Angew. Math.*, **509**, (1999), 117–149

[23] G. M. Ziegler, R. T. Živaljević, Homotopy types of subspace arrangements via diagrams of spaces *Math. Ann.*, **295** (1993), no. 3, 527–548

Erklärung

Hiermit versichere ich, dass ich meine Dissertation mit dem Titel

Some Representation stability results and generalizations

selbständig und ohne fremde Hilfe verfasst, nicht andere als die in ihr angegebenen Quellen oder Hilfs-mittel benutzt, alle vollständig oder sinngemäß übernommenen Zitate als solche gekennzeichnet sowie die Dissertation in der vorliegenden oder einer ähnlichen Form noch bei keiner anderen in- oder ausländischen Hochschule anlässlich eines Promotionsgesuchs oder zu anderen Prüfungszwecken eingereicht habe. Dies ist mein erster Versuch einer Promotion.

Artur Rapp, Januar 2020

Deutsche Zusammenfassung

Darstellungsstabilität im Sinne von Church und Farb (siehe [5] und [4]) ist eine Eigenschaft von Folgen von Darstellungen von symmetrischen Gruppen. Für jede Zahl $n \in \mathbb{N}$ schreiben wir S_n für die symmetrische Gruppe auf $\{1, 2, ..., n\}$. Die irreduziblen Darstellungen der S_n sind durch Zahlpartitionen indiziert. Eine Zahlpartition $\lambda \vdash n$ ist eine endliche Folge $\lambda = (\lambda_1, ..., \lambda_l)$ positiver ganzer Zahlen mit $\lambda_1 \geq ... \geq \lambda_l$ und $\sum_{i=1}^{l} \lambda_i = n$. Wir schreiben S^λ für die irreduzible Darstellung, die zu λ korrespondiert und s_λ für dessen Frobenius-Charakteristik. Für jedes $n \in \mathbb{N}$ heißen die Funktionen s_λ mit $\lambda \vdash n$ Schur-Funktionen und bilden eine $\mathbb{Z}$-Basis von Λ_n, dem Ring der symmetrischen Funktionen mit Koeffizienten in $\mathbb{Z}$, deren Monome alle den Grad n haben. Sei $\lambda = (\lambda_1, ..., \lambda_l) \vdash n$. Dann definieren wir $\lambda + (1) := (\lambda_1 + 1, \lambda_2, ..., \lambda_l) \vdash n + 1$. Wenn ein S_n-Modul V die Zerlegung

$$V = \bigoplus_{\lambda \vdash n} a_\lambda S^\lambda$$

hat, dann definieren wir

$$V + (1) = \bigoplus_{\lambda \vdash n} a_\lambda S^{\lambda+(1)}.$$

Entsprechend definieren wir für eine symmetrische Funktion f mit Zerlegung

$$f = \sum_{\lambda \vdash n} a_\lambda s_\lambda$$

dann

$$f + (1) = \sum_{\lambda \vdash n} a_\lambda s_{\lambda+(1)}.$$

Als nächstes betrachten wir Folgen $\{V_n\}_{n\geq 0}$ von S_{n+n_0}-Darstellungen oder Folgen ihrer Charakteristiken. Eine solche Folge stabilisiert bei $m \geq n_0$, wenn

$$V_n = V_{n-1} + (1) \text{ für alle } n > m.$$

Die Folge stabilisiert scharf bei $m \geq n_0$, wenn m die kleinste Zahl ist mit

$$V_n = V_{n-1} + (1) \text{ für alle } n > m.$$

In Kapitel 2 betrachten wir Arrangements von diagonalen Unterräumen von $(\mathbb{R}^d)^n$ für natürliche Zahlen d und n. Für ein endliches Arrangement $\mathcal{A}$ von linearen Unterräumen von $(\mathbb{R}^d)^n$ definieren wir die Vereinigung $U_\mathcal{A} = \cup_{V \in \mathcal{A}} V$ und das Komlement $\mathcal{M}_\mathcal{A} = (\mathbb{R}^d)^n \setminus U_\mathcal{A}$. Der Schnittverband $L_\mathcal{A}$ ist die Menge aller Schnitte beliebig vieler Elemente von $\mathcal{A}$ sortiert durch umgekehrte Inklusion. Das kleinste Element $\hat{0}$ ist $(\mathbb{R}^d)^n$, der leere Schnitt, und das größte Element $\hat{1}$ ist der Schnitt aller Elemente von $\mathcal{A}$. Für eine Mengenpartition π von $\{1, ..., n\}$ sei W_π^d der lineare Unterraum aller n-Tupel $(w_1, ..., w_n)$ von Punkten aus $\mathbb{R}^d$, sodass $w_i = w_j$, wenn i und j im selben Block der Partition π sind. Für eine Zahlpartition λ schreiben wir $\mathcal{A}_\lambda^d$ für das Arrangement aller Unterr äume W_π^d, sodass π vom Typ λ ist, das heißt, dass die Teile von λ den Mächtigkeiten der Blöcke von π entsprechen. Allgemeiner setzen wir $\mathcal{A}_\Lambda^d = \cup_{\lambda \in \Lambda} \mathcal{A}_\lambda^d$ für jede endliche Menge Λ von Zahlpartitionen von n. Das Komplement $\mathcal{M}_\Lambda^d = (\mathbb{R}^d)^n \setminus \cup_{W \in \mathcal{A}_\Lambda^d} W$ ist eine reelle Mannigfaltigkeit. Falls $\Lambda = \{\lambda\}$, schreiben wir $\mathcal{M}_\lambda^d$ für $\mathcal{M}_\Lambda^d$. Die Wirkung der symmetrischen Gruppe S_n auf n-Tupeln von Punkten aus $\mathbb{R}^d$ durch Permutieren der Koordinaten induziert eine S_n-Darstellung auf der reduzierten singulären Kohomologie $\tilde{H}^i(\mathcal{M}_\Lambda^d, \mathbb{C})$. Formeln für diese S_n-Darstellung wurden von Sundaram und Welker in [19] hergeleitet. Wir untersuchen diese Darstellungen auf Darstellungsstabilität.

Das Hauptziel in Kapitel ist zu beweisen, dass Folgen dieser Darstellungen stabilisieren und Stabilitätsschranken herzuleiten. Dies ist der Inhalt von Theorem 2.1. Die Tatsache, dass diese Folgen stabilisieren, können auch aus Ergebnissen von Gadish ([8, Theorem A]) und Petersen ([13, Theorem 4.15]) impliziert werden. Deren Sätze beinhalten keine Aussagen über Schranken. Der Fall $\Lambda = \{(2, 1^{n-2})\}$ wurde von Church bewiesen ([4, Theorem 1]) und für diesen Fall bestimmten Hersh and Reiner die exakten Stabilitätsschranken ([11, Theorem 1.1]). Die Ergebnisse von Kapitel 2 sind in [14] veröffentlicht.

In Kapitel 3 definieren wir eine verallgemeinerte Art von Stabilität für Folgen von Darstellungen. Motiviert durch Lemma 2.2, wo wir zeigen, dass das Produkt einer stabilisierenden Folge mit einer konstanten Folge auch stabilisiert, zeigen wir in Kapitel 3, dass Produkte mehrerer stabilisierender Folgen gewisse rekursive Relationen erfüllen, die den Stabilitätsbegriff verallgemeinern. Wir verwenden Methoden aus der Theorie der symmetrischen Funktionen und Polytope.

Das Ziel von Kapitel 4 ist, weitere Beispiele zu suchen, wo Darstellungsstabilität gilt. Wir untersuchen relative Arrangements. Das sind Paare von Arrangements $(\mathcal{A}, \mathcal{B})$, sodass $U_\mathcal{B} \subseteq U_\mathcal{A}$. Für Folgen $\{(\mathcal{A}_n, \mathcal{B}_n)\}_n$ von relativen Arrangements kann man die Folge $\{\tilde{H}^i(U_{\mathcal{A}_n} \backslash U_{\mathcal{B}_n}, \mathbb{C})\}_n$ auf Darstellungsstabilität untersuchen. Wir gehen einen Schritt in diese Richtung, indem wir eine der Goresky-MacPherson-Formel ([9],[19, Theorem 2.5(ii)],[2, Theorem 2.1]) ähnliche Formel für relative Arrangements beweisen. Dazu verwenden wir Methoden von Ziegler und Živaljević ([23]). Genauer verwenden wir Homotopie-Kolimiten. An einem Beispiel sehen wir, dass Stabilität nicht immer gilt. Wir vermuten aber, dass für bestimmte Klassen relativer Arrangements Stabilität gilt.

Lebenslauf

Diese Seite enthält persönliche Daten. Sie ist deshalb nicht Bestandteil der Online-Veröffentlichung.