Philipps Universität Marburg

# ON METRIC CONNECTIONS WITH TOTALLY SKEW-SYMMETRIC TORSION TENSOR

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**Abstract**

We consider a 1-parameter family of metric connections $\nabla^s$ with totally skew-symmetric torsion tensors on a Riemannian manifold and derive a Weitzenböck formula for the Laplace operator, arising from such a connection. Various notions related to $\nabla^s$ are defined and developed in the process, mimicking what is normally done with the Levi-Civita connection $\nabla^g$. We investigate the matter of skew torsion further by introducing weakly non-degenerate and non-degenerate split torsion and show examples of manifolds, admitting such connections.

**Zusammenfassung**

Wir betrachten eine 1-parametrische Schar metrischer Zusammenhänge $\nabla^s$ mit schiefsymmetrischen Torsiontensoren auf einer riemmanschen Mannigfaltigkeit und leiten eine Weitzenböck-Identität für den Laplace Operator ab, der aus einem solchen Zusammenhang stammt. Verschiedene Begriffe, die mit $\nabla^s$ verbunden sind, werden in diesem Prozess definiert und entfaltet, indem man dem folgt, was normalerweise mit dem Levi-Civita Zusammenhang $\nabla^g$ getan wird. Wir untersuchen die schiefsymmetrische Torsion weiter durch die Einführung von schwach nicht entartete bzw. nicht entartete aufgespaltete Torsion und zeigen Beispiele von Mannigfaltigkeiten, die solche Zusammenhänge zulassen.

# Contents

# Introduction

A huge portion of the fundamentals in differential geometry is centered around the notions of a differentiable manifold, metrics, connections, geodesics, torsion, and curvature. Classically, much attention is drawn by the Levi-Civita connection $\nabla^g$ on a manifold, which is the unique torsionless metric connection of the Riemannian $(M, g)$. While the metric condition appears to be quite natural, one may ask himself whether we really need to impose a condition on the torsion to vanish, and indeed metric connections with non-vanishing torsion have been studied and used widely in the literature.

The extra freedom given by the presence of torsion enables us to devise more sophisticated models and describe more phenomena. This is particularly important in the field of mathematical physics, where manifolds with torsion have been extensively used as the torsion tensor is thought to couple with spin just as curvature couples with energy. This lies in the heart of Einstein-Cartan theory. Thus, torsion-gravity with spinning matter is a complete and self-consistent setting for modern physics, with potential applications wherever spin effects may be important, stretching from quantum mechanics to the standard models of particle physics and early cosmology.

Another important appearance of metric connections with torsion, and more precisely totally skew-symmetric torsion, is in superstring theory. There, the number of preserved supersymmetries essentially depends on the number of parallel spinors. The existence of parallel spinors is, on the other hand, a severe holonomy condition, and in the case of the Levi-Civita connection the resulting holonomy groups are known to be $SU(n)$, $Sp(n)$, $G_2$, and Spin(7), corresponding to Calabi-Yau, hyperkähler, parallel $G_2$, and parallel Spin(7) manifolds. These geometries are of great interest, but are also very restrictive and have thus been generalized by the introduction of torsion into the picture. This is done by considering metric connections with totally skew torsion tensor, whose holonomy lies in one of the aforementioned groups. A torsion tensor is called *totally skew-symmetric*, or in short, *totally skew*, if considered as a type $(0, 3)$ tensor using the metric, it is a 3-form.

Keeping the physical background in mind, it is no surprise that metric connections with skew torsion have mainly been considered in the context of spin geometry. A focal object on a Riemannian spin manifold $(M, g)$ is the Dirac operator $D$–the famous square root of the Bochner Laplace operator $(\nabla^*\nabla)^g$ of the Levi-Civita connection $\nabla^g$. On the one hand, they are directly related by the Lichnerowicz formula

$$D^2\psi = (\nabla^*\nabla)^g\psi + \frac{1}{4}\mathrm{Scal}^g\psi$$

on spinors, where $\mathrm{Scal}^g$ is the scalar curvature of the Riemannian curvature tensor $R^g$. On the other hand, there is another famous identity–the Weitzenböck formula–which relates $(\nabla^*\nabla)^g$ and the Hodge Laplacian $\Delta = d\delta + \delta d$ via the curvature:

$$\Delta = (\nabla^*\nabla)^g + q(\mathcal{R}^g),$$

where $q(\mathcal{R}^g)$ is a zero order (no differentiation) curvature term. These formulas are of great use while estimating the operators' lowest eigenvalues and proving vanishing theorems.

Once we have allowed the presence of torsion, another useful tool for investigating the geometry of a given manifold is the consideration of a whole 1-parameter family of connections with skew-symmetric torsion. Let $(M,g)$ be a Riemannian manifold with a metric connection $\nabla$ with totally skew torsion tensor $T$. Then, the Levi-Civita connection $\nabla^g$ and $\nabla$ are connected via

$$\nabla_X Y = \nabla^g_X Y + \frac{1}{2}T(X,Y).$$

What one does is to consider the 1-parameter family of connections

$$\nabla^s_X Y = \nabla^g_X Y + \frac{s}{2}T(X,Y)$$

depending on $s \in \mathbb{R}$. Such a rescaled torsion is, of course, again totally skew. Moreover, particular scale factors give rise to interesting geometric quantities. For example, with $s = \frac{1}{3}$, the Dirac operator $D^{\frac{1}{3}}$ of a naturally reductive space with a spin structure coincides with Kostant's cubic Dirac operator. In some situations, introducing the scale factor in formulas like the Weitzenböck or the Lichnerowicz formula also allows for a better lowest eigenvalue estimate of the operators in question.

One of the main goals of this thesis is to go through some of the classical theory, allowing torsion and not assuming the existence of a spin structure on the manifold, and derive a general form for the Weitzenböck identity on $k$-forms. This is achieved in theorem 5.16:

Theorem 0.1. *Let $\nabla^s$ be the family of metric connections with totally skew-symmetric torsion defined by* (5.4)*. Then the Weitzenböck-type formula*

$$\Delta^{\frac{s}{2}}\omega = (\nabla^*\nabla)^s\omega + q(\mathcal{R}^g)\omega - 2sdT\Box\omega + 2s^2S(\omega) + 4s^2B(\omega)$$

*relates the Bochner-Laplace operator of $\nabla^s$ to the Laplacian $\Delta^{\frac{s}{2}}$ of $\nabla^{\frac{s}{2}}$, both acting on a $k$-form $\omega$.*

In [**ABBK13**], the skew torsion of the connection that's used has a special form–it is of *split type*. This notion is defined there for the first time and it is one of our objectives to elaborate on it a bit more in this thesis. This, it is further refined in *weakly non-degenerate* and *non-degenerate* split torsion and examples are presented.

There is a good amount of previous work done on these topics. Metric connections with skew torsion have been widely considered throughout Agricola, Friedrich, and Ivanov's articles: [**AF04a**], [**AF10**], [**AF14**], [**Agr03**], [**FI02**], [**Iv02**], [**Iv04**] to name a few. A good combined source is [**Agr06**]. We have used Semmelmann's lecture notes [**Sem11**] and habilitation thesis [**Sem01**] as guidelines in the first chapter. Most results on Killing forms, conformal Killing forms, and special Killing forms with torsion may be cross-referenced in [**HKWY10**], [**HKWY12**], and [**KKY09**]. The notion of split torsion originates from [**ABBK13**] and reappears in [**AK14**], and is tightly related to generalized Wallach spaces, which are classified by Nikonorov in [**Nik16**]. The description of the Wallach spaces is put together using results from [**B02**], [**BCCS09**], [**Dra08**], [**HBL71**], [**Ish99**], [**Kar88**], [**Mas74**], [**Mü80**], [**Mü81**], [**Wa72**], and [**Yo09**]. Further references are also given throughout the text.

**Structure of the thesis.** The text is divided in two chapters and an appendix in two parts.

The main object of the first chapter is to derive the general form of the Weitzenböck formula in Theorem 5.16, as described at its very beginning in a short introductory word. We start by setting up the basic algebraic facts, notions, and conventions. Apart from the standard algebra that is used, we introduce two new operators, *the diamond and box operator*, in Def. 1.6. We then proceed to transfer the algebraic picture to a manifold setting and introduce the $\nabla$-differential and $\nabla$-divergence in Def. 2.3 in the process. A study of $\nabla$-Killing and $\nabla$-conformal Killing forms ensues, adopting ideas of Semmelmann et. al., Houri et. al., and others. It is followed by examples of manifolds admitting such special forms. In the next section we define a number of Laplace operators which may be considered in the presence of torsion. Thus, we finally close in on our aim to derive an expression for the Weitzenböck formula in equation (5.2). At the end of the chapter, some known applications are presented to illustrate how such formulas are usually put into practice. Theorem 6.3 is an example of a vanishing theorem derived using equation (5.10).

The second chapter also starts with a short introductory word, followed by a section of preliminaries. There, the notions of *weakly non-degenerate* and *non-degenerate* split torsion are defined. We also cite the classification theorem for generalized Wallach spaces from [**Nik16**]. The following two sections are concerned with the two different non-degeneracy conditions, each of them containing a series of examples, tightly related to the normed division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$. In the case of weakly non-degenerate torsion, those are the Stiefel manifolds, while in the case of non-degenerate split torsion, they are the Wallach spaces. The Wallach spaces are then elaborated on in greater detail to show the richness of the structure they admit.

CHAPTER 1

# Metric connections acting on differential forms

In the first chapter, we go through a series of known results in Riemannian geometry, generalizing them from the point of view of metric connections with totally skew symmetric torsion. The goal is to obtain a systematic exposition of the results and to derive a general expression for the well-known Weitzenböck formula

$$\Delta\omega = (\nabla^*\nabla)\omega + \mathrm{Ric}(\omega)$$

This is achieved in theorem 5.16. A simple application in the spirit of the Bochner-type vanishing theorems is given in theorem 6.3.

We start by laying down the algebraic basis needed for this work whilst also setting the notational conventions. Apart from the standard algebra that is used, we introduce two new operators, *the diamond and box operator*, in Def. 1.6.

We then proceed to transfer the algebraic picture to a manifold setting and introduce the $\nabla$-differential and $\nabla$-divergence in Def. 2.3 in the process.

A study of $\nabla$-Killing and $\nabla$-conformal Killing forms ensues, adopting ideas of Semmelmann et. al., Houri et. al. and others. It is followed by examples of manifolds admitting such special forms.

In the next section we define a number of Laplace operators which may be considered in the presence of torsion. Thus, we finally close in on our aim to derive a more generalized expression for the Weitzenböck formula in equation (5.2). We also introduce a 1-parameter family of connections into the picture to generalize even further, inspired by results in [**ABBK13**], and finally arrive at the desired identity (5.10) in theorem 5.16.

At the end of the chapter, some known applications are presented to illustrate how such formulas are usually put into practice. Theorem 6.3 is an example of a vanishing theorem derived using equation (5.10).

## 1. Algebraic preliminaries

Given a Riemannian manifold $(M, g)$, our main goal is to describe the action of a metric connection $\nabla$ with skew torsion on differential forms. We start with the action of an arbitrary metric connection $\nabla$ on the tangent bundle, which is the standard covariant differentiation. Then one defines naturally covariant derivative on the cotangent bundle and extends this action to the whole tensor bundle over the manifold. Finally, we focus only on those connections, which have totally skew torsion tensor.

**1.1. Lie algebra actions on alternating forms.** We start with the algebraic picture. Let $\mathfrak{g}$ be a Lie algebra with elements $g, h, \ldots$ and $V$ be a vector space with elements $X, Y, \ldots$ We let $\mathfrak{g}$ act on $V$ via the representation $\varrho : \mathfrak{g} \to \mathrm{End}(V)$. The action of $\mathfrak{g}$ on $V^*$ is given by $(\varrho(g)(\eta))(X) := -\eta(\varrho(g)X)$ where $\eta \in V^*$. The representation $\varrho$ extends to the whole tensor algebra $T^\bullet V$ by imposing that it acts as a derivation with respect to the tensor product and commutes with contractions. We elaborate on this setting starting from an arbitrary endomorphism $\phi \in \mathrm{End}(V)$. Since $\phi$ acts as a derivation and commutes with contractions, considering $\mathbb{R} \otimes V = V$, we get that $\phi \equiv 0$ on $\mathbb{R}$. Then $\phi(\alpha(X)) = 0$ for $X \in V$ and $\alpha \in V^*$, leading to the following expression for the action of $\phi$ on $V^*$:

$$\phi(\alpha)(X) = -\alpha(\phi(X)). \tag{1.1}$$

Fix an orthonormal basis $\{e_i\}_1^n$ of the $n$-dimensional $V$. It determines a dual basis $\{e^j\}$ of $V^*$ via $e^j(e_i) = \delta_i^j$. Denote by $A = (a_{ij})_{n\times n} \in M^n(\mathbb{R})$ the matrix of $\phi$ with respect to $\{e_i\}$:

$$\phi(X) = \sum_{i,j}(a_{ij}X_j)e_i, \quad \text{where} \quad X = \sum_i X_i e_i \in V.$$

Then the matrix, corresponding to $\phi$ on $V^*$ is $-A^t$, i.e.

$$\phi(\alpha) = \sum_{i,j}(-a_{ji}\alpha_j)e^i, \quad \text{where} \quad \alpha = \sum_i \alpha_i e_i \in V^*.$$

We endow $V$ with the standard inner product $\langle e_i, e_j\rangle = \delta_{ij}$ and use it to identify $V$ and $V^*$ via

$$\begin{aligned} \flat : V &\longrightarrow V^* \\ X &\longmapsto X^\flat = \langle X, -\rangle \end{aligned}$$

and its inverse $\sharp : V^* \longrightarrow V$. *Flat* ($\flat$) and *sharp* ($\sharp$) are called *the musical isomorphisms*. Note that $e_i^\flat = e^i$. Hence, if $X = \sum_i X_i e_i$, then $X^\flat = \sum_i (X^\flat)_i e^i = \sum_i X_i e^i$, so that $(X^\flat)_i = X_i$. Now

$$\phi(X)^\flat = \sum_{i,j}(a_{ij}X_j)e^i \text{ while } \phi(X^\flat) = \sum_{i,j}(-a_{ji}X_j)e^i$$

so $\phi$ and $\flat$ commute if and only if $A = -A^t$, i.e. $\phi$ is skew-symmetric. On the other hand, if $\phi$ is symmetric, they anticommute. When we consider skew-symmetric endomorphisms $\phi$, like for example $\phi = \varrho(g) = g$ for $g \in \mathfrak{g} \subset \mathfrak{so}(n)$, we freely identify $X$ and $X^\flat$ without any risk of confusion.

REMARK 1.1. We shall use the same letter for a vector $\alpha^\sharp = X \in V$ and its dual covector $\alpha = X^\flat \in V^*$ whenever possible, identifying $V$ and $V^*$ freely. Other notable conventions are that our exterior product is defined so that $(\omega_1 \wedge \omega_2)(X,Y) := \omega_1(X)\omega_2(Y) - \omega_1(Y)\omega_2(X)$ and we interpret $A = \sum_{i,j} A_{ij}e^i \otimes e^j \in V^* \otimes V^*$ as $\sum_{i,j} A_{ij}e^i \otimes e_j \in V^* \otimes V$, that is $V^* \otimes V^*$ is identified with $V^* \otimes V = \mathrm{End}(V)$ via the isomorphism $id \otimes \sharp$.

Let us see how $\phi \in \mathrm{End}(V)$, respectively $g \in \mathfrak{g}$, acts on $\wedge^k V^*$:

PROPOSITION 1.2. *If $\omega \in \wedge^k V^*$, $\phi \in \mathrm{End}(V)$, we have:*

$$\phi(\omega) = \sum_i \phi(e^i) \wedge (e_i \lrcorner\ \omega) \tag{1.2}$$

*In particular, for $\phi = \varrho(g)$, $g \in \mathfrak{g}$, we get the action of $\mathfrak{g}$ on $\Lambda^k V^*$.*

PROOF. The statement is trivially true for $k = 1$. Assume that $\omega \in \wedge^k V^*$, $\eta \in \wedge^l V^*$ and the claim is true for all elements of $\wedge^p V^*$, $p < k + l$.

$$\begin{aligned} \phi(\omega\wedge\eta) = \phi(\omega)\wedge\eta + \omega\wedge\phi(\eta) &= \sum_i \big(\phi(e^i)\wedge(e_i \lrcorner\ \omega)\wedge\eta + \omega\wedge\phi(e^i)\wedge(e_i \lrcorner\ \eta)\big) \\ &= \sum_i \phi(e^i)\wedge\Big((e_i \lrcorner\ \omega)\wedge\eta + (-1)^k\omega\wedge(e_i \lrcorner\ \eta)\Big) = \sum_i \phi(e^i)\wedge(e_i \lrcorner\ (\omega\wedge\eta)) \end{aligned}$$

proves the proposition by induction. $\square$

PROPOSITION 1.3. *If $\omega \in \wedge^k V^*$, $\phi \in \mathrm{End}(V)$, and $X_1, X_2, \ldots, X_k \in V$, we have:*

$$\phi(\omega)(X_1,\ldots,X_k) = -\sum_{i=1}^k \omega(X_1,\ldots,\phi(X_i),\ldots,X_k). \tag{1.3}$$

PROOF. Substituting in the previous proposition we get

$$\begin{aligned}\phi(\omega)(X_1,\dots,X_k) &= \sum_{i,j}(-1)^{j+1}\phi(e^i)(X_j)\omega(e_i,X_1,\dots,\widehat{X_j},\dots,X_k)\\ &= \sum_{i,j}(-1)^j e^i(\phi(X_j))\omega(e_i,X_1,\dots,\widehat{X_j},\dots,X_k)\\ &= \sum_j(-1)^j\omega(\phi(X_j),X_1,\dots,\widehat{X_j},\dots,X_k) = -\sum_j \omega(X_1,\dots,\phi(X_j),\dots,X_k).\ \square\end{aligned}$$

From now on we work with $\mathfrak{g}=\mathfrak{so}(n)\cong\wedge^2V$, $V=\mathbb{R}^n$, and $\varrho(g)=g$. We identify $\mathfrak{so}(n)\cong\wedge^2V^*$ as follows:

$$\wedge^2V^*\ni e_i\wedge e_j\longleftrightarrow E_{ij}\in\mathfrak{so}(n)$$

where the matrices $E_{ij}$ denote the standard basis elements of the Lie algebra $\mathfrak{so}(n)$, i. e. the endomorphisms mapping $e_i$ to $e_j$, $e_j$ to $-e_i$, and everything else to zero. Note that since $E_{ij}$ has a $-1$ on position $(i,j)$ and a 1 on position $(j,i)$, if we write $\omega=\sum_{i,j}\omega_{ij}e_i\wedge e_j\in\wedge^2V^*$, its corresponding matrix in $\mathfrak{so}(n)$ will be $A=(-\omega_{ij})_{n\times n}=(\omega_{ji})_{n\times n}$.

PROPOSITION 1.4. *Let $\alpha\in\wedge^2V^*$ with corresponding matrix $A\in\mathfrak{so}(n)$ and $Z\in V$. Then*

$$\varrho(\alpha)Z=AZ=Z\lrcorner\ \alpha.$$

*If $\alpha$ is an element of the form $\alpha=X\wedge Y$, $X,Y\in V$, and $\omega\in\wedge^kV^*$, we have*

$$\varrho(X\wedge Y)Z=\langle Z,X\rangle\,Y-\langle Z,Y\rangle\,X=Y.(X\lrcorner\ Z^\flat)-X.(Y\lrcorner\ Z^\flat),$$
$$\varrho(X\wedge Y)\omega=Y\wedge(X\lrcorner\ \omega)-X\wedge(Y\lrcorner\ \omega).$$

PROOF. $A$ is given by $A=(-\alpha_{ij})_{n\times n}$. We compute:

$$\varrho(\alpha)Z=AZ=\sum_{i,j}(-\alpha_{ij}Z_j)e_i=\sum_{i,j}\alpha_{ji}Z_je_i=Z\lrcorner\ \alpha.$$

Taking $\alpha=X\wedge Y$, the computation for $\varrho(X\wedge Y)Z$ is trivial. The last line follows directly from Prop 1.2:

$$\sum_i(X\wedge Y)(e_i)\wedge(e_i\lrcorner\ \omega)=\sum_i(X_iY-Y_iX)\wedge(e_i\lrcorner\ \omega)=Y\wedge(X\lrcorner\ \omega)-X\wedge(Y\lrcorner\ \omega).\qquad\square$$

REMARK 1.5. Let $\alpha,\beta\in\wedge^2V^*$ with corresponding matrices $A,B\in\mathfrak{so}(n)$. One can compute

$$\begin{aligned}\varrho(\alpha)\beta &= \sum_i Ae_i\wedge(e_i\lrcorner\ \beta)=\sum_i Ae_i\wedge Be_i=\sum_{i,j,k}A_{ji}e_j\wedge B_{ki}e_k\\ &= \sum_{j<k}\sum_i -A_{ji}B_{ik}e_j\wedge e_k+\sum_{k<j}\sum_i -B_{ki}A_{ij}e_j\wedge e_k\\ &= \sum_{j<k}-(AB)_{jk}e_j\wedge e_k+\sum_{k<j}(BA)_{kj}e_k\wedge e_j=\sum_{j<k}-[A,B]_{jk}e_j\wedge e_k\end{aligned}$$

so the matrix corresponding to $\varrho(\alpha)\beta$ is $[A,B]$, meaning that the action of $\mathfrak{so}(n)$ on itself given by $\varrho$ coincides with the adjoint action.

We define two algebraic operations, generalizing the action of $\mathfrak{so}(n)$ on the tensor algebra over $V$.

DEFINITION 1.6. For $\omega\in\wedge^kV^*$ and $\eta\in\wedge^lV^*$ we define:

$$\eta\Diamond\omega:=\sum_i(e_i\lrcorner\ \eta)\wedge(e_i\lrcorner\ \omega)\ \in\wedge^{k+l-2}V^*\text{ for }k,l\geq1,$$
$$\eta\Box\omega:=\frac{1}{2}\sum_{i,j}(e_j\lrcorner\ (e_i\lrcorner\ \eta))\wedge(e_j\lrcorner\ (e_i\lrcorner\ \omega))\ \in\wedge^{k+l-4}V^*\text{ for }k,l\geq2,$$

The definition does not depend on the choice of an orthonormal basis. Because of their notation, we shall loosely refer to these operations as the *diamond operation* and the *box operation.* Our most important examples will be in the case $\eta = T$, where $T$ is the totally skew torsion of a metric connection (see def. 1.8). In this case we will refer to $T\Diamond$ and $T\Box$ as the *diamond and box operators.* A list of identities used for algebraic manipulations with $\Diamond$ and $\Box$ may be found in Appendix B.

We now show that the diamond operation is really a generalization of the action $\varrho$ of $\mathfrak{so}(n) \cong \wedge^2 V^*$.

LEMMA 1.7. *Let $\eta$ be an element of $\wedge^l V^*$ and $\omega$ of $\wedge^k V^*$. The following holds for any $\gamma \in \wedge^\bullet V^*$:*

$$\eta\Diamond(\omega \wedge \gamma) = (\eta\Diamond\omega) \wedge \gamma + (-1)^{kl}\omega \wedge (\eta\Diamond\gamma).$$

*In particular, if $l = 2$ or $l = n - 2$ in case $n$ is even, the identity above is just the Leibniz rule turning $\eta\Diamond$ into a derivation. Thus, $\varrho(\eta) = \eta\Diamond$ on $\wedge^\bullet V^*$.*

PROOF. We show the derivation rule directly:

$$\begin{aligned}\eta\Diamond(\omega \wedge \gamma) &= \sum_i (e_i \lrcorner\ \eta) \wedge (e_i \lrcorner\ (\omega \wedge \gamma))\\ &= \sum_i (e_i \lrcorner\ \eta) \wedge (e_i \lrcorner\ \omega) \wedge \gamma + (-1)^k \sum_i (e_i \lrcorner\ \eta) \wedge \omega \wedge (e_i \lrcorner\ \gamma)\\ &= (\eta\Diamond\omega) \wedge \gamma + (-1)^{kl}\omega \wedge (\eta\Diamond\gamma).\end{aligned}$$

For $\omega \in V^*$, the identity

$$\eta\Diamond\omega = \omega \lrcorner\ \eta = \varrho(\eta)\omega$$

is a trivial check, showing that $\eta\Diamond$ and $\varrho(\eta)$ coincide on 1-forms and hence on $\Lambda^\bullet V^*$. $\Box$

Further arithmetic identities concerning the diamond and box operations are given in the appendix.

**1.2. The torsion action on alternating forms.** Now, let $(M, g)$ be a Riemannian manifold, equipped with a metric connection $\nabla$:

$$\nabla_X Y = \nabla^g_X Y + A_X Y \tag{1.4}$$

where $X, Y \in TM$, $\nabla^g_X$ is the Levi-Civita connection, and $A_X : TM \to TM$ is an endomorphism field on $TM$. We may also consider $A_X$ as a 2-form $A_X : TM \times TM \to \mathbb{R}$ because $\nabla$ is metric. Note that $A_X(Y, Z) = \langle A_X Y, Z\rangle$ so $A_X Y = Y \lrcorner\ A_X = A_X\Diamond Y$. $\langle -, -\rangle$ is the scalar product, induced by the metric $g$ on $TM$ at a fixed point. We associate a torsion tensor $T : TM \times TM \to TM$ to $\nabla$:

$$T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y].$$

Due to its symmetries and using the metric, one can consider $T : \wedge^2 TM \otimes TM \to \mathbb{R}$. According to the splitting

$$\wedge^2 TM \otimes TM \cong \wedge^3 TM \oplus TM \oplus V$$

under $\mathfrak{so}(n)$, where $V$ is the Cartan product of $\wedge^2 TM$ and $TM$, the possible torsion tensors are classified (this is known as the Cartan classification):

DEFINITION 1.8. We call a torsion tensor $T$

- *vectorial* if $T : TM \longrightarrow \mathbb{R}$;
- *totally skew* or *skew-symmetric* if $T : \wedge^3 TM \longrightarrow \mathbb{R}$;
- *of Cartan type* if $T : V \longrightarrow \mathbb{R}$.

LEMMA 1.9. *Summarizing the results of the previous subsection, we extend* (1.4) *as a derivation to the whole tensor algebra over $M$. In particular, for a k-form $\omega$*

$$\nabla_X \omega = \nabla^g_X \omega + A_X\Diamond\omega. \tag{1.5}$$

COROLLARY 1.10. *If $\nabla$ is a connection with skew-symmetric torsion $T$, $A_X = \frac{1}{2}X \lrcorner\ T$ and*

$$\nabla_X\omega = \nabla^g_X\omega + \frac{1}{2}(X \lrcorner\ T)\Diamond\omega. \tag{1.6}$$

COROLLARY 1.11. *If $\nabla$ is a connection with vectorial torsion, $A_X = 2X \wedge V$ for some vector field $V$ and*

$$\nabla_X\omega = \nabla^g_X\omega + 2(X \wedge V)\Diamond\omega = \nabla^g_X\omega + 2(V \wedge (X \lrcorner\ \omega) - X \wedge (V \lrcorner\ \omega)).$$

We investigate connections with skew torsion in more detail in the following sections.

## 2. Connections with skew-symmetric torsion acting on differential forms

Let $\nabla$ be a metric connection with totally skew torsion tensor $T$ (see eqn. 1.6). We use the same letter for $T : TM \times TM \longrightarrow TM$ and $T : \wedge^3 TM \longrightarrow \mathbb{R}$. Geometric quantities defined with respect to the Levi-Civita connection will be marked with an upper index $g$ (e. g. $\mathcal{R}^g$), while quantities without upper index refer to the connection $\nabla$ (e. g. $\mathcal{R}$) –an upper index $\nabla$ may still be explicitly written in the latter case lest confusions occur. In any case, the metric $g$ will always be fixed, but the connection may vary.

DEFINITION 2.1. We define a $\nabla$-*adapted frame* as follows: at a point $p \in M$ we can fix an orthonormal frame $\{e_i\}$ such that the connection components vanish, that is $(\nabla_{e_i} e_j)(p) = 0$. We then use parallel transport with respect to $\nabla$ to define the orthonormal frame in a neighbourhood around $p$.

From now on we assume that $e_1, \ldots, e_n$ is an orthonormal frame. The formulas we obtain are generally invariant under orthogonal basis changes. Use of $\nabla$-adapted frames will be mentioned explicitly.

### 2.1. Basic identities.

Let's prove some useful formulas involving inner products and the defined diamond and box operations.

PROPOSITION 2.2. *The following equalities hold for any $T \in \Lambda^3 TM$ and any $\omega \in \Lambda^k TM$:*

$$-2X \lrcorner\ T = \sum_{i=1}^n T(X, e_i) \wedge e_i \tag{2.1}$$

$$(X \lrcorner\ T)\Diamond\omega = -\sum_{i=1}^n e_i \wedge (T(X, e_i) \lrcorner\ \omega) \tag{2.2}$$

$$((X \lrcorner\ T)\Diamond\omega)(X_1, .., X_k) = \sum_{j=1}^k (-1)^j \omega(T(X, X_j), X_1, .., \widehat{X_j}, .., X_k). \tag{2.3}$$

PROOF. We confirm the first equality:

$$\sum_i T(X, e_i) \wedge e_i = \sum_{i,j} T(X, e_i, e_j) e_j \wedge e_i = \sum_{i<j} -2T(X, e_i, e_j) e_i \wedge e_j = -2(X \lrcorner\ T).$$

The second one follows easily from Prop. 1.2:

$$(X \lrcorner\ T)\Diamond\omega = \sum_{i,j=1}^n T(X, e_i, e_j) e_j \wedge (e_i \lrcorner\ \omega) = -\sum_{j=1}^n e_j \wedge (T(X, e_j) \lrcorner\ \omega).$$

(2.3) is a coordinate-free alternative to (2.2), obtained by just plugging the arguments in. $\square$

**2.2. Divergence and differential.** We would like to define two new operators on $k$-forms, which in some sense generalize the exterior differentiation $d$ and the divergence $\delta$. These operators depend on a previously fixed metric connection $\nabla$ with skew torsion $T$.

Definition 2.3. The $\nabla$-*differential* and the $\nabla$-*divergence* of a $k$-form $\omega$ are defined as

$$d^\nabla \omega := \sum_i e_i \wedge \nabla_{e_i}\omega, \quad \delta^\nabla \omega := -\sum_i e_i \lrcorner\ \nabla_{e_i}\omega.$$

The definitions do not depend on the choice of orthonormal basis. Of course, these notions coincide with the usual definitions in case $T = 0$. These two operators have appeared previously in the mathematical physics literature (see [**HKWY10**], where the diamond and box operations also appear implicitly as special cases of the contracted wedge product) and are tightly related to massless Dirac equations. It is natural to define a corresponding Hodge-Laplace operator, as is again done in [**HKWY10**]:

Definition 2.4. The *Hodge Laplacian with torsion* is defined by

$$\Delta := d^\nabla \delta^\nabla + \delta^\nabla d^\nabla$$

We now recall the definition of the Hodge star operator and state some of its properties without proof.

Definition 2.5. Let $(M, g)$ be a closed, oriented, Riemannian manifold. We define the *Hodge star* operator $* : \wedge^k TM \longrightarrow \wedge^{n-k}TM$ assigning to each $\eta \in \wedge^k TM$ the form $*\eta$ such that for every $\omega \in \wedge^k TM$

$$\omega \wedge *\eta = \langle \omega, \eta \rangle \, dV$$

holds, where $dV$ is the volume form of $M$.

Lemma 2.6. *We state some properties of $*$ without proof (see* [**Sem11**] *- 7.3 on p.174):*

$$\begin{aligned} *1 &= dV & \langle X \lrcorner\ \omega, \eta\rangle &= \langle \omega, X \wedge \eta\rangle \\ *X &= X \lrcorner\ dV & *(X \lrcorner\ \omega) &= (-1)^{k-1} X \wedge *\omega \\ *^2 &= (-1)^{k(n-k)} & \langle \omega, *\eta\rangle &= (-1)^{k(n-k)} \langle *\omega, \eta\rangle \end{aligned}$$

*Here $\omega$ is a $k$-form, $X$ is a vector field or its dual 1-form, and $\eta$ is a form of appropriate degree.*

Proposition 2.7. *On a closed, oriented, Riemannian manifold $(M^n, g)$ the operators $d^\nabla$ and $\delta^\nabla$ are formally adjoint. Moreover, the Hodge star operator $*$ relates them by the formula*

$$\delta^\nabla \omega = -(-1)^{n(k+1)} * d^\nabla * \omega,$$

*where $\omega$ is a $k$-form.*

Proof. The proof of the adjointness is similar to the Levi-Civita case since we only need the connection to be metric. Let $\omega$ be a $k$-form and $\eta$ a $(k+1)$-form. First note that in the $\nabla$-adapted frame around a fixed point $p \in M$ the 1-form $\sum_i \nabla_{e_i} e_i$ vanishes at $p$, but it doesn't depend on the choice of frame, so it is identically zero. Then compute as follows:

$$\begin{aligned} \int_{M^n} \langle d^\nabla \omega, \eta \rangle \, dV &= \int_{M^n} \left\langle \sum_i e_i \wedge \nabla_{e_i}\omega, \eta \right\rangle dV = \int_{M^n} \sum_i \langle \nabla_{e_i}(e_i \wedge \omega), \eta\rangle \, dV \\ &= \int_{M^n} \sum_i e_i \langle e_i \wedge \omega, \eta \rangle \, dV - \int_{M^n} \sum_i \langle e_i \wedge \omega, \nabla_{e_i}\eta\rangle \, dV. \end{aligned}$$

Note that

$$\sum_i e_i \langle e_i \wedge \omega, \eta\rangle \, dV = d \sum_i (-1)^{i+1} \langle e_i \wedge \omega, \eta \rangle \, e_1 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_n$$

is exact, so

$$\begin{aligned}\int_{M^n} \langle d^\nabla \omega, \eta \rangle \, dV &= -\int_{M^n} \sum_i \langle e_i \wedge \omega, \nabla_{e_i}\eta \rangle \, dV = -\int_{M^n} \sum_i e_i \wedge \omega \wedge *(\nabla_{e_i}\eta) \\ &= (-1)^{k+1} \int_{M^n} \sum_i \omega \wedge e_i \wedge *(\nabla_{e_i}\eta) = -\int_{M^n} \sum_i \omega \wedge *(e_i \lrcorner \; \nabla_{e_i}\eta) \\ &= \int_{M^n} \omega \wedge *\delta^\nabla \eta = \int_{M^n} \langle \omega, \delta^\nabla \eta \rangle \, dV,\end{aligned}$$

proving the first assertion. Using that $*\nabla = \nabla *$ we further confirm

$$\begin{aligned}*d^\nabla * \omega &= * \sum_i (e_i \wedge \nabla_{e_i}(*\omega)) = * \sum_i (e_i \wedge *(\nabla_{e_i}\omega)) \\ &= (-1)^{k+1} *^2 \sum_i (e_i \lrcorner \; \nabla_{e_i}\omega) = -(-1)^{n(k+1)} \delta^\nabla \omega.\end{aligned}$$ □

We now prove some identities involving the newly defined diamond and box operators.

PROPOSITION 2.8. *The diamond operator and the box operator are related to $d^\nabla$ and $\delta^\nabla$ by*

$$T\Diamond\, \omega = d\omega - d^\nabla \omega, \quad T\Box\, \omega = \delta\omega - \delta^\nabla \omega.$$

PROOF. We use (2.2) and $e_k \lrcorner \; T = \frac{1}{2}\sum_{i,j} T(e_i, e_j, e_k) e_i \wedge e_j$ to compute

$$\begin{aligned}d\omega - d^\nabla \omega &= \sum_i e_i \wedge (\nabla^g_{e_i}\omega - \nabla_{e_i}\omega) = -\frac{1}{2}\sum_i e_i \wedge ((e_i \lrcorner \; T)\Diamond \omega) \\ &= \frac{1}{2}\sum_{i,j} e_i \wedge e_j \wedge (T(e_i, e_j) \lrcorner \; \omega) = \sum_k (e_k \lrcorner \; T) \wedge (e_k \lrcorner \; \omega) = T\Diamond\omega.\end{aligned}$$

Further, we find that

$$\delta\omega - \delta^\nabla\omega = \sum_i e_i \lrcorner \; (\nabla_{e_i}\omega - \nabla^g_{e_i}\omega) = \frac{1}{2}\sum_i e_i \lrcorner \; ((e_i \lrcorner \; T)\Diamond\omega) = \frac{1}{2}\sum_i (e_i \lrcorner \; T)\Diamond(e_i \lrcorner \; \omega) = T\Box\omega,$$

where the last step follows directly from the definitions of $\Diamond$ and $\Box$. □

COROLLARY 2.9. *$T\Diamond$ and $T\Box$ are formally adjoint, i.e. on a closed, oriented, Riemannian manifold $(M^n, g)$*

$$\int_{M^n} \langle T\Diamond\omega, \eta \rangle \, dV = \int_{M^n} \langle \omega, T\Box\eta \rangle \, dV$$

*holds for a $k$-form $\omega$ and a $(k+1)$-form $\eta$, $k \geq 1$.*

PROOF. This is true for both pairs $d$, $\delta$ and $d^\nabla$, $\delta^\nabla$ hence it follows for their differences $T\Diamond$ and $T\Box$. □

Recall a standard formula relating the exterior differential $d$ with a covariant differentiation $\nabla$.

LEMMA 2.10. *If $\nabla$ is a metric connection with torsion $T$, and $\omega$ is a $k$-form, then*

$$\begin{aligned}d\omega(X_0, \ldots, X_k) &= \sum_{i=0}^k (-1)^i (\nabla_{X_i}\omega)(X_0, \ldots, \widehat{X_i}, \ldots, X_k) \\ &\quad - \sum_{0 \leq i < j \leq k} (-1)^{i+j} \omega(T(X_i, X_j), X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k).\end{aligned}$$

Corollary 2.11. *We have the following basis-free expressions for $T\Diamond\omega$ and $d^\nabla\omega$:*

$$\left(d^\nabla\omega\right)(X_1,\dots,X_{k+1}) = \sum_j (-1)^{j+1}(\nabla_{X_j}\omega)(X_1,\dots,\widehat{X_j},\dots,X_{k+1}),$$

$$(T\Diamond\,\omega)(X_1,\dots,X_{k+1}) = \sum_{i<j}(-1)^{i+j+1}\omega(T(X_i,X_j),X_1,\dots,\widehat{X_i},\dots,\widehat{X_j},\dots,X_{k+1}).$$

Proof. The first identity is seen directly from the definition $d^\nabla = \sum_i e_i \wedge \nabla_{e_i}$, while the second one follows from Lemma 2.10 and $d - d^\nabla = T\Diamond$. $\square$

Corollary 2.12. *If $\nabla\omega = 0$, then $T\Diamond\,\omega = d\omega$ and $T\Box\,\omega = \delta\omega$. For $\omega = T$ we have $T\Diamond\,T = 2\sigma_T$ (see the following remark) and $T\Box\,T = 0$, so $\delta^\nabla T = \delta T$.*

Remark 2.13. This justifies that for any metric connection $\nabla$, we will just write $\delta T$ for the divergence of the torsion. Note also that using the first formula we easily get

$$\begin{aligned}\sigma_T(X,Y,Z,V) :&= \langle T(X,Y),T(Z,V)\rangle + \langle T(Y,Z),T(X,V)\rangle + \langle T(Z,X),T(Y,V)\rangle\\ &= \frac{1}{2}(T\Diamond T)(X,Y,Z,V),\end{aligned}$$

which is also shown in [**Agr06**]. The 4-form $\sigma_T$ appears in many special geometries and often carries significant information about the structure of the manifold. We will use the last expression, as well as the identity

$$d^gT(X,Y,Z,V) = \left(\sum_{XYZ}(\nabla_X T)(Y,Z,V)\right) - (\nabla_V T)(X,Y,Z) + 2\sigma_T(X,Y,Z,V)$$

from the same source later on. One can easily confirm it in our language. That is just $d^gT = d^\nabla T + T\Diamond T$, where we have taken into account Cor 2.11, $T\Diamond T = 2\sigma_T$, and the fact that $T$ is a 3-form.

Example 2.14. In the simple case of a Lie group we are aware of a series of examples of forms, which are not $\nabla$-parallel, but are still $d^\nabla$-closed. Let $\nabla$ be one of the canonical flat connections of the Lie group and take a basis $\{e_i\}$ of $\nabla$-parallel 1-forms. Their wedge products produce higher degree $\nabla$-parallel forms. Since

$$d^\nabla(f\omega) = df\wedge\omega + f.d^\nabla\omega,$$

we can take $\omega = e_{i_1}\wedge\ldots\wedge e_{i_k}$ and $f = x_{i_p}$, $1\le p\le k$, $\{i_j\}_{j=1}^k \subset \{1,\dots,n\}$, where $n$ is the dimension of the Lie group. The $k$-form $f\omega$ is not $\nabla$-parallel anymore, but it is $d^\nabla$-closed. In a similar manner we can produce $\delta^\nabla$-coclosed forms on Lie groups, leaning on the expression

$$\delta^\nabla(f\omega) = -df \lrcorner\ \omega + f.\delta^\nabla\omega.$$

The algebraic identities we've used are summed up in Prop. B.2.

We proceed to find an expression for $(d^\nabla)^2$.

Proposition 2.15. *On a $k$-form $\omega$*

$$(d^\nabla)^2\omega = dT\Diamond\omega + T\Diamond(T\Diamond\omega) - T\Diamond(\nabla^g\omega) - (\nabla^gT)\Diamond\omega.$$

Proof. We manipulate the expression

$$(d^\nabla)^2\omega = (d - T\Diamond)^2\omega = 0 - d(T\Diamond\omega) - T\Diamond(d\omega) + T\Diamond(T\Diamond\omega).$$

According to Prop. B.2 in the appendix, we have

$$d(T\Diamond\omega) = (\nabla^gT)\Diamond\omega - dT\Diamond\omega + T\Diamond(\nabla^g\omega) - T\Diamond(d\omega),$$

and we directly arrive at

$$(d^\nabla)^2\omega = -(\nabla^gT)\Diamond\omega + dT\Diamond\omega - T\Diamond(\nabla^g\omega) + T\Diamond(T\Diamond\omega).$$ $\square$

Thus we observe that $d^\nabla$ does not in general square to zero so it will not be a useful tool to investigate cohomology. However, there is a certain class of forms, called special ∇-Killing forms, on which $d^\nabla$ does square to zero. We introduce them in the following chapter.

## 3. ∇-parallel forms and ∇-Killing forms

Recall how a conformal Killing $k$-form is defined: for an arbitrary vector space $V$ we have a splitting

$$V \otimes \wedge^k V \cong \wedge^{k+1} V \oplus \wedge^{k-1} V \oplus \wedge^{k,1} V$$

into irreducible $\mathfrak{so}(\dim V)$-representations, where $\wedge^{k,1}V$ is just the Cartan product of $V$ and $\wedge^k V$. According to this splitting, applied to the tangent space at a point $T_pM = V$ of an $n$-dimensional manifold $M$, we may write the covariant derivative of a $k$-form $\omega$ with respect to any metric connection $\nabla$ as

$$\nabla_X \omega = \frac{1}{k+1} X \lrcorner\ d^\nabla \omega - \frac{1}{n-k+1} X \wedge \delta^\nabla \omega + P^\nabla_X(\omega). \tag{3.1}$$

Here $P^\nabla_X(\omega)$ is simply the projection of $\nabla_X\omega$ on the last summand of the splitting. Defined in this way, $P^\nabla$ is usually called a *twistor operator*. See [**Sem01**] for details in the case when $\nabla$ is the Levi-Civita connection. The same construction applies to any metric connection.

DEFINITION 3.1. Let $X \in TM$ be arbitrary. We call a differential $k$-form $\omega$:

(1) $\nabla$-*parallel*, if it satisfies $\nabla_X\omega = 0$;
(2) $\nabla$-*Killing*, if it satisfies $\nabla_X\omega = \frac{1}{k+1} X \lrcorner\ d^\nabla\omega$;
   – Further, if $\omega$ satisfies the equation
   $$\nabla_X(d^\nabla\omega) = cX \wedge \omega$$
   for some constant $c$, we call it *special* $\nabla$-*Killing* or just *special*;
(3) $\nabla$-$*$-*Killing*, if it satisfies $\nabla_X\omega = -\frac{1}{n-k+1} X \wedge \delta^\nabla\omega$;
   – Further, if $\omega$ satisfies the equation
   $$\nabla_X(\delta^\nabla\omega) = -cX \lrcorner\ \omega$$
   for some constant $c$, we call it *special* $\nabla$-$*$-*Killing* or just *special*;
(4) $\nabla$-*conformal Killing*, if it satisfies $P^\nabla_X(\omega) = 0$

When $\nabla$ is the Levi-Civita connection, we obtain the classical definitions.

Killing and conformal Killing forms have been studied by Bochner and Yano ([**YB54**], [**Y70**]), and more recently by Semmelmann in his habilitation thesis [**Sem01**], where one of the results is a classification of all manifolds admitting special Killing or $*$-Killing forms with respect ot the Levi-Civita connection. It turns out that a compact, simply connected manifold, admitting a $\nabla^g$-special Killing form, must be either isometric to $S^n$, or be Saskian, 3-Saskian, nearly Kähler, or weak $G_2$. Special forms have been introduced by Tachibana and Yu ([**TY70**]) and have also appeared in the context of torsion ([**HKWY12**]). They exhibit some interesting properties, motivating the definition of the Hodge Laplacian with torsion $\Delta := d^\nabla\delta^\nabla + \delta^\nabla d^\nabla$.

Recently, Killing and conformal Killing forms have also been used in the mathematical physics literature in the series of papers: [**E16a**], [**E16b**], [**E17**], [**KRT07**], and others. They appear naturally and define conserved quantities for geodesic motion, as well as symmetry operators of the massive and massless Dirac equations in curved background. They are then used to extend known symmetry algebras through the introduction of an appropriate Lie bracket to bigger Lie superalgebras. However, this Lie superalgebra structure appears only on spaces with constant curvature. The setting generally allows the presence of torsion and ∇-Killing and ∇-conformal Killing forms have indeed been considered ([**HKWY10**], [**HKWY12**]).

We will now investigate the action of $\nabla$ on low order forms, with special emphasis on parallel objects. To start with, we show that the Killing equation is the same for all connections with totally skew-symmetric torsion, in particular it coincides with the one for the Levi-Civita connection. This will also give us the opportunity to define a special torsion, hence connection, which makes a given Killing vector of constant length parallel.

PROPOSITION 3.2. *Let $\nabla$ be a metric connection with skew-symmetric torsion $T$. The vector field $X$ is Killing iff it is $\nabla$-Killing. Hence, any $\nabla$-parallel vector field is Killing. Conversely, for a Killing vector field $X$ of constant length, there exists a metric connection $\nabla$ with skew-symmetric torsion $T$, for which $X$ is $\nabla$-parallel. One such is given by $T = \frac{1}{||X||^2} X \wedge d^g X$.*

PROOF. Since classical Killing vector fields coincide with $\nabla$-Killing vector fields for connections with skew-symmetric torsion, we just use the term Killing vector field. Here is our justification:

$$\begin{aligned}\langle \nabla_Y X, Z\rangle = -\langle \nabla_Z X, Y\rangle &\iff \left\langle \nabla^g_Y X, Z\right\rangle + \frac{1}{2}T(Y,X,Z) = -\left\langle \nabla^g_Z X, Y\right\rangle - \frac{1}{2}T(Z,X,Y)\\ &\iff \left\langle \nabla^g_Y X, Z\right\rangle = -\left\langle \nabla^g_Z X, Y\right\rangle.\end{aligned}$$

Obviously, every $\nabla$-parallel vector field is $\nabla$-Killing, hence Killing. The converse is more interesting. Can we find an appropriate connection which turns any Killing vector parallel? Some special geometries lead us to the suspicion that if $X$ is Killing, then $\nabla$ with $T = \frac{1}{||X||^2} X \wedge d^g X$ will do the trick. This turns out to be true only if $X$ is of constant length. Let $X$ be such a vector field, $Y$ and $Z$ – arbitrary.

$$\begin{aligned}\langle \nabla_Y X, Z\rangle &= \left\langle \nabla^g_Y X, Z\right\rangle + \frac{1}{2||X||^2}(X \wedge d^g X)(Y,X,Z)\\ &= \left\langle \nabla^g_Y X, Z\right\rangle + \frac{1}{2||X||^2}\Big( \langle X, Y\rangle \left(\left\langle \nabla^g_X X, Z\right\rangle - \left\langle \nabla^g_Z X, X\right\rangle\right) +\\ &\qquad + ||X||^2\left(\left\langle \nabla^g_Z X, Y\right\rangle - \left\langle \nabla^g_Y X, Z\right\rangle\right) + \langle X, Z\rangle \left(\left\langle \nabla^g_Y X, X\right\rangle - \left\langle \nabla^g_X X, Y\right\rangle\right)\Big)\\ &= \left\langle \nabla^g_Y X, Z\right\rangle - \left\langle \nabla^g_Y X, Z\right\rangle + \frac{1}{2||X||^2}\Big( \langle X, Y\rangle \left\langle \nabla^g_X X, Z\right\rangle - \langle X, Z\rangle \left\langle \nabla^g_X X, Y\right\rangle \Big) = 0.\end{aligned}$$

This concludes the proof. Note that we needed to use $\nabla^g_X X = 0$, which is granted by the constant length condition, because otherwise we can't make the term $\left\langle \nabla^g_X X, \langle X, Y\rangle Z - \langle X, Z\rangle Y\right\rangle$ vanish, since $\langle X, Y\rangle Z - \langle X, Z\rangle Y = \varrho(Y \wedge Z)X$ runs through all possible vectors as $Y$ and $Z$ vary. □

REMARK 3.3. We saw that a $\nabla$-parallel vector field is Killing. Dually, a $\nabla$-parallel 1-form should also be Killing as we readily see from $\nabla\omega = 0$, which implies also $d^\nabla \omega = 0$ and $d\omega = T \Diamond \omega$. Then

$$\nabla^g_X \omega = \nabla_X \omega - \frac{1}{2}(X \lrcorner\ T)\Diamond\omega = -\frac{1}{2}T(X, \omega^\sharp) = \frac{1}{2}(\omega \lrcorner\ T)(X) = \frac{1}{2}(T\Diamond\omega)(X) = \frac{1}{2}X \lrcorner\ d\omega,$$

which is precisely the Killing condition. Such 1-forms exist in many geometric situations, for example on quasi-Sasakian manifolds or $G_2$-manifolds with non-vanishing Lee form.

We showed that a Killing vector field, or equivalently 1-form, is simultaneously Killing for all metric connections with totally skew torsion. This is also true for conformal Killing 1-forms:

PROPOSITION 3.4. *Every conformal Killing 1-form is also conformal Killing with respect to any metric connection with totally skew-symmetric torsion.*

PROOF. Let $\omega$ be a conformal Killing 1-form, and $\nabla$ - a metric connection with totally skew-symmetric torsion. We can write simultaneously

$$\begin{aligned}\nabla^g_X \omega &= \frac{1}{2}X \lrcorner\ d\omega - \frac{1}{n}X \wedge \delta\omega \quad \text{and}\\ \nabla_X \omega &= \frac{1}{2}X \lrcorner\ d^\nabla\omega - \frac{1}{n}X \wedge \delta^\nabla\omega + P^\nabla_X(\omega).\end{aligned}$$

Subtracting the first line from the second one and using the last lemma, we get

$$-\frac{1}{2}X \lrcorner\ (T\Diamond\omega) = -\frac{1}{2}X \lrcorner\ (T\Diamond\omega) + \frac{1}{n}X \wedge (T\Box\omega) + P^{\nabla}_X(\omega),$$

which simplifies to $P^{\nabla}_X(\omega) = 0$ right away ($T\Box\omega = 0$) and finishes the proof. □

PROPOSITION 3.5. *The torsion $T$ is (conformal) Killing if and only if it is $\nabla$-(conformal) Killing. In particular, if $T$ is Killing, one has*

$$\nabla_X T = X \lrcorner\ (\frac{1}{4}dT - \frac{1}{2}\sigma_T). \tag{3.2}$$

PROOF. Assume $T$ is conformal Killing:

$$\nabla^g_X T = \frac{1}{4}X \lrcorner\ dT - \frac{1}{n-2}X \wedge \delta T. \tag{3.3}$$

We show that this condition is equivalent to the $\nabla$-conformal Killing condition:

$$\nabla_X T = \frac{1}{4}X \lrcorner\ d^{\nabla}T - \frac{1}{n-2}X \wedge \delta^{\nabla}T. \tag{3.4}$$

We verify this, subtracting the two identities. The divergence terms cancel out, since $\delta T$ is the same for both connections. The difference (3.4) - (3.3) is

$$\frac{1}{2}(X \lrcorner\ T)\Diamond T = \frac{1}{4}X \lrcorner\ (-T\Diamond T),$$

which is easily seen to be true by $X \lrcorner\ (\omega\Diamond\eta) = -(X \lrcorner\ \omega)\Diamond\eta + (-1)^k\omega\Diamond(X \lrcorner\ \eta)$ from Prop. B.2. The explicit form (3.2) follows directly from (3.4) and $dT - d^{\nabla}T = 2\sigma_T$ (see corollary 2.12 and the following remark). □

We just saw that for some special forms: 1-forms and the 3-form, corresponding to the torsion, the $\nabla$-conformal Killing condition is preserved in the class of metric connections with totally skew torsion. This condition is also preserved in the conformal class of the metric $g$.

REMARK 3.6. Under the conformal change $\hat{g} = e^{2\lambda}g$, covariant differentiation on $k$-forms with respect to the Levi-Civita connections $\hat{\nabla}^g$ and $\nabla^g$ is related via ([**Sem01**], p.17)

$$\hat{\nabla}^g_X\omega = \nabla^g_X\omega - kd\lambda(X)\omega - d\lambda \wedge (X \lrcorner\ \omega) + X \wedge (\mathrm{grad}(\lambda) \lrcorner\ \omega).$$

If we consider the connection with torsion

$$\nabla_X\omega = \nabla^g_X\omega + \frac{1}{2}(X \lrcorner\ T)\Diamond\omega,$$

the appropriate connection to look at after the conformal change is

$$\hat{\nabla}_X\omega = \hat{\nabla}^g_X\omega + \frac{1}{2}(X \lrcorner\ T)\Diamond\omega.$$

In this case, we get the relation

$$\hat{\nabla}_X\omega = \nabla_X\omega - kd\lambda(X)\omega - d\lambda \wedge (X \lrcorner\ \omega) + X \wedge (\mathrm{grad}(\lambda) \lrcorner\ \omega). \tag{3.5}$$

PROPOSITION 3.7. *Let $\omega$ be a $\nabla$-conformal Killing $k$-form on the Riemannian manifold $(M, g)$ and consider the conformally equivalent metric $\hat{g} = e^{2\lambda}g$. Then $\hat{\omega} = e^{(k+1)\lambda}\omega$ satisfies*

$$\hat{\nabla}_X\hat{\omega} = \frac{1}{k+1}X \lrcorner\ d^{\hat{\nabla}}\hat{\omega} - \frac{1}{n-k+1}X^{\hat{\flat}} \wedge \delta^{\hat{\nabla}}\hat{\omega},$$

*where $\hat{\flat}$ is the dual with respect to $\hat{g}$.*

PROOF. The computation goes exactly the same way as in the case of the Levi-Civita connection, so we just highlight some key moments. Let $\{e_i\}$ be a $g$-orthonormal frame, whose dual is $\{\sigma_i\}$. Then $\{\hat{e}_i = e^{-\lambda}e_i\}$ is a $\hat{g}$-orthonormal frame and $\hat{\sigma_i} = e^{\lambda}\sigma_i$ is its dual. Then, with (3.5)

$$d^{\hat{\nabla}}\hat{\omega} = \sum_i \hat{\sigma_i} \wedge \hat{\nabla}_{\hat{e}_i}\hat{\omega} = \sum_i \sigma_i \wedge \hat{\nabla}_{e_i}\hat{\omega} = \sum_i \sigma_i \wedge \nabla_{e_i}\hat{\omega} + (k-k)d\lambda \wedge \hat{\omega} = d^{\nabla}\hat{\omega}.$$

Similarly, again using (3.5) one can compute

$$\delta^{\hat{\nabla}}\hat{\omega} = e^{-2\lambda}\left(\delta^{\nabla}\hat{\omega} + (2k-n)\mathrm{grad}(\lambda) \lrcorner \hat{\omega}\right).$$

One then substitutes $\hat{\omega} = e^{(k+1)\lambda}\omega$, obtaining

$$\begin{aligned} d^{\nabla}\hat{\omega} &= e^{(k+1)\lambda}\left((k+1)d\lambda \wedge \omega + d^{\nabla}\omega\right), \\ \delta^{\nabla}\hat{\omega} &= e^{(k+1)\lambda}\left(-(k+1)\mathrm{grad}(\lambda) \lrcorner\ \omega + \delta^{\nabla}\omega\right) \quad \text{and} \\ \hat{\nabla}_X\hat{\omega} &= e^{(k+1)\lambda}\Big((k+1)d\lambda(X)\omega + \hat{\nabla}_X\omega\Big). \end{aligned}$$

With these three identities, (3.5), and $e^{-2\lambda}X^{\hat{\flat}} = X^{\flat}$, the result follows. □

COROLLARY 3.8. *The $\nabla$-conformal Killing $k$-form $\omega$ is conformal to a $\nabla$-$*$-Killing form iff there exists a function $\lambda$, such that*

$$d^{\nabla}\omega = -(k+1)d\lambda \wedge \omega$$

*and conformal to a $\nabla$-Killing form iff there exists a function $\lambda$, such that*

$$\delta^{\nabla}\omega = (n-k+1)\mathrm{grad}(\lambda) \lrcorner\ \omega.$$

*If both conditions are satisfied for the same $\lambda$, $\omega$ is conformal to a parallel form.*

PROOF. From the intermediate computations in the last proof we have:

$$\begin{aligned} d^{\hat{\nabla}}\hat{\omega} &= e^{(k+1)\lambda}\left((k+1)d\lambda \wedge \omega + d^{\nabla}\omega\right), \\ \delta^{\hat{\nabla}}\hat{\omega} &= e^{-2\lambda}.e^{(k+1)\lambda}\left(-(n-k+1)\mathrm{grad}(\lambda) \lrcorner\ \omega + \delta^{\nabla}\omega\right). \end{aligned}$$

which show the claim. □

COROLLARY 3.9. *If a $k$-form is simultaneously $\nabla$-parallel for two different metrics $g$ and $\hat{g} = e^{2\lambda}g$ of the same conformal class, then $\lambda$ is constant.*

PROOF. In this case we have $d^{\nabla}\omega = \delta^{\nabla}\omega = 0$ so that

$$d\lambda \wedge \omega = 0 \quad \text{and} \quad \mathrm{grad}(\lambda) \lrcorner\ \omega = 0.$$

We can thus compute

$$0 = \mathrm{grad}(\lambda) \lrcorner\ (d\lambda \wedge \omega) = |\mathrm{grad}(\lambda)|^2\omega - d\lambda \wedge (\mathrm{grad}(\lambda) \lrcorner\ \omega) = |\mathrm{grad}(\lambda)|^2\omega,$$

proving the claim. □

The $\nabla$-conformal Killing condition is also invariant under the Hodge star operator.

PROPOSITION 3.10. *If $\omega$ is a $\nabla$-conformal Killing $k$-form, then $*\omega$ is a $\nabla$-conformal Killing $(n-k)$-form.*

PROOF. We apply $*$ to the defining equation

$$\nabla_X\omega = \frac{1}{k+1}X \lrcorner\ d^{\nabla}\omega - \frac{1}{n-k+1}X \wedge \delta^{\nabla}\omega.$$

Using $*\nabla = \nabla*$, Lemma 2.6, and Prop. 2.7, we can intertwine $d^{\nabla}$ and $\delta^{\nabla}$, interchanging $X \lrcorner\ .$ and $X \wedge .$ in the process, and then determine the signs, arriving at

$$\nabla_X(*\omega) = \frac{1}{n-k+1}X \lrcorner\ d^{\nabla}(*\omega) - \frac{1}{k+1}X \wedge \delta^{\nabla}(*\omega),$$

which is exactly the condition for the $(n-k)$-form $*\omega$ to be $\nabla$-conformal Killing. □

Observing that $*$ interchanges $d^\nabla$-closed and $\delta^\nabla$-closed, we directly arrive at:

COROLLARY 3.11. *The $k$-form $\omega$ is $\nabla$-Killing iff $*\omega$ is $\nabla$-$*$-Killing. Moreover, $\omega$ is special $\nabla$-Killing with constant $c$ iff $*\omega$ is special $\nabla$-$*$-Killing with the same constant $c$.*

PROOF. We apply the Hodge star $*$ to the definition $\nabla_X d^\nabla\omega = cX\wedge\omega$ and use Lemma 2.6 to determine the signs that occur when we swap $*$ and $d^\nabla$ and $*(X\wedge\omega)$ and $X \lrcorner *\omega$, arriving precisely at $\nabla_X\delta^\nabla\omega = -cX\lrcorner\,\omega$. □

PROPOSITION 3.12. *If $\omega$ and $\eta$ are $\nabla$-$*$-Killing $k$- and $l$-forms, then $\omega\wedge\eta$ is a $\nabla$-$*$-Killing $(k+l)$-form.*

PROOF. From B.2 we have

$$d^\nabla(\omega\wedge\eta) = d^\nabla\omega\wedge\eta + (-1)^k\omega\wedge d^\nabla\eta = 0,$$

directly showing the $d^\nabla$-closedness, as well as (see Prop. B.1)

$$\delta^\nabla(\omega\wedge\eta) = \delta^\nabla\omega\wedge\eta + (-1)^k\omega\wedge\delta^\nabla\eta - \omega\Diamond(\nabla\eta) - (-1)^k(\nabla\omega)\Diamond\eta.$$

This simplifies using that $\omega$ and $\eta$ are $\nabla$-conformal Killing. Namely, we compute

$$\omega\Diamond(\nabla\eta) = -\frac{1}{n-l+1}\sum_i (e_i\lrcorner\,\omega)\wedge e_i\wedge\delta^\nabla\eta = (-1)^k\frac{k}{n-l+1}\omega\wedge\delta^\nabla\eta$$

and similarly

$$(\nabla\omega)\Diamond\eta = (-1)^k\frac{l}{n-k+1}\delta^\nabla\omega\wedge\eta,$$

arriving at

$$\delta^\nabla(\omega\wedge\eta) = \frac{n-k-l+1}{n-k+1}\delta^\nabla\omega\wedge\eta + (-1)^k\frac{n-k-l+1}{n-l+1}\omega\wedge\delta^\nabla\eta$$

after grouping the similar summands. We can rewrite this last identity in the form

$$\frac{1}{n-k-l+1}\delta^\nabla(\omega\wedge\eta) = \frac{1}{n-k+1}\delta^\nabla\omega\wedge\eta + (-1)^k\frac{1}{n-l+1}\omega\wedge\delta^\nabla\eta,$$

We now multiply by $-1$ and wedge with $X$ on the left to obtain

$$-\frac{1}{n-k-l+1}X\wedge\delta^\nabla(\omega\wedge\eta) = (\nabla_X\omega)\wedge\eta + \omega\wedge(\nabla_X\eta) = \nabla_X(\omega\wedge\eta),$$

which is precisely the $\nabla$-conformal Killing condition for $\omega\wedge\eta$. □

There is also an equivalent definition for special $\nabla$-Killing, respectively $\nabla$-$*$-Killing forms.

PROPOSITION 3.13. *The $\nabla$-Killing $k$-form $\omega$ is special with constant $c$ iff for any $X, Y$*

$$\nabla^2_{XY}\omega = \frac{c}{k+1}(g(X,Y)\omega - X\wedge(Y\lrcorner\,\omega)).$$

*The corresponding condition for a $\nabla$-$*$-Killing $l$-form $\eta$ with constant $c$ is*

$$\nabla^2_{XY}\eta = \frac{c}{n-l+1}(g(X,Y)\eta - X\lrcorner\,(Y\wedge\eta)).$$

PROOF. For a special $\nabla$-Killing form $\omega$ we have

$$\nabla_X\omega = \frac{1}{k+1}X\lrcorner\, d^\nabla\omega.$$

We differentiate and use $\nabla_X(d^\nabla\omega) = cX\wedge\omega$ to obtain

$$\nabla_X\nabla_Y\omega = \frac{1}{k+1}(\nabla_X Y\lrcorner\, d^\nabla\omega + Y\lrcorner\,(cX\wedge\omega))$$

so that

$$\nabla^2_{XY}\omega = \frac{c}{k+1}Y\lrcorner\,(X\wedge\omega) = \frac{c}{k+1}\left(g(X,Y)\omega - X\wedge(Y\lrcorner\,\omega)\right).$$

Conversely, we can expand this last identity for a $\nabla$-Killing $k$-form $\omega$ to arrive at

$$Y \lrcorner\ (\nabla_X d^\nabla \omega) = cY \lrcorner\ (X \wedge \omega)$$

for any $Y$, meaning that $\nabla_X d^\nabla \omega = cX \wedge \omega$, so that $\omega$ is special. The corresponding statements for $\nabla$-$*$-Killing forms follow by just applying $*$ and using Lemma 2.6. $\square$

Corollary 3.14. *The following formula holds for the curvature on a special $\nabla$-Killing $k$-form $\omega$:*

$$R(X,Y)\omega = \frac{c}{k+1}(X \wedge Y)\Diamond\omega.$$

Proof. We just apply the definition $R(X,Y) = \nabla^2_{XY} - \nabla^2_{YX}$ and substitute the expression from the proposition. $\square$

Proposition 3.15. *If $\omega$ is a special $\nabla$-Killing $k$-form, it is an eigenform of the Hodge Laplacian with torsion $\Delta$ (see definition 2.4). More precisely,*

$$\Delta\omega = -c(n-k)\omega, \quad \textit{where} \quad \nabla_X(d^\nabla \omega) = cX \wedge \omega.$$

*Further, if $\eta$ is a special $\nabla$-$*$-Killing $l$-form, it is an eigenform of $\Delta$ satisfying*

$$\Delta\eta = -cl\eta, \quad \textit{where} \quad \nabla_X(\delta^\nabla \eta) = -cX \lrcorner\ \eta.$$

Proof. The proof is a direct computation. We will only do it for $\eta$ as the other case is similar.

$$\nabla_X(\delta^\nabla \eta) = -cX \lrcorner\ \eta \Longrightarrow d^\nabla \delta^\nabla \eta = -cl\eta,$$

but $\eta$ is $d^\nabla$-closed, so

$$\Delta\eta = d^\nabla \delta^\nabla \eta + \delta^\nabla d^\nabla \eta = d^\nabla \delta^\nabla \eta = -cl\eta. \qquad \square$$

Using the equivalent definition, we can also show the following:

Proposition 3.16. *If $\omega$ is a special $\nabla$-Killing $k$-form, it is an eigenform of the Bochner Laplacian with torsion $\nabla^*\nabla = -tr(\nabla^2)$. More precisely,*

$$\nabla^*\nabla\omega = -\frac{c(n-k)}{k+1}\omega, \quad \textit{where} \quad \nabla_X(d^\nabla \omega) = cX \wedge \omega.$$

*Further, if $\eta$ is a special $\nabla$-$*$-Killing $l$-form, it is an eigenform of $\nabla^*\nabla$ satisfying*

$$\nabla^*\nabla\eta = -\frac{cl}{n-l+1}\eta, \quad \textit{where} \quad \nabla_X(\delta^\nabla \eta) = -cX \lrcorner\ \eta.$$

Proof. We simply take the trace in

$$\nabla^2_{XY}\omega = \frac{c}{k+1}(g(X,Y)\omega - X \wedge (Y \lrcorner\ \omega))$$

or the corresponding $\nabla$-$*$-Killing identity and compute the coefficient. $\square$

Corollary 3.17. *Let $(M,g)$ be a compact manifold and $\omega$ a special $\nabla$-Killing $k$-form with constant $c$. Then $c < 0$.*

Proof. We integrate $\nabla^*\nabla\omega = -\frac{c(n-k)}{k+1}\omega$ over $M$:

$$||\nabla\omega||^2 = \int_M \langle \nabla^*\nabla\omega, \omega\rangle\, dV = -\frac{c(n-k)}{k+1}\int_M \langle \omega, \omega\rangle\, dV = -\frac{c(n-k)}{k+1}||\omega||^2 > 0,$$

hence $c < 0$. $\square$

Observe that although in general $(d^\nabla)^2 \neq 0$, for a special $\nabla$-Killing form $\omega$, $(d^\nabla)^2\omega = 0$ does hold. This allows us to prove the following:

Proposition 3.18. *If $\omega$ is a special $\nabla$-Killing $k$-form of odd degree $k$, then $\omega \wedge (d^\nabla\omega)^p$ is a special $\nabla$-Killing form of degree $p(k+1)+k$.*

PROOF. The degree of $d^\nabla\omega$ is $(k+1)$, which is even. Set $\eta = \omega\wedge(d^\nabla\omega)^p$. Then $d^\nabla\eta = (d^\nabla\omega)^{p+1}$ and

$$\nabla_X(d^\nabla\eta) = (p+1)(\nabla_X(d^\nabla\omega))\wedge(d^\nabla\omega)^p = (p+1)cX\wedge\omega\wedge(d^\nabla\omega)^p = c(p+1)X\wedge\eta,$$

so $\eta$ is special with constant $c(p+1)$. □

This shows that given a special $\nabla$-Killing 1-form, we can produce a whole series of special $\nabla$-Killing forms of odd degree.

One often considers the curvature of a manifold as an endomorphism on the space of 2-forms. The case when it is symmetric is usually of interest, since this implies the symmetry of the Ricci tensor. It has been previously observed, that this holds if the torsion $T$ is parallel, or even if $\nabla T$ is a 4-form, which in our current terminology is just $T$ being $\nabla$-Killing. In fact, this condition is also necessary, as shown in ([**Iv02**], Corollary 3.4.):

PROPOSITION 3.19. *([**Iv02**], Cor. 3.4.) Let $(M, g, \nabla, T)$ be a Riemannian manifold with a metric connection $\nabla$ with totally skew-symmetric torsion $T$. The following conditions are equivalent:*

(1) $\nabla^g T = \frac{1}{4}dT$;
(2) $\nabla T$ *is a 4-form;*
(3) $R(X, Y, Z, V) = R(Z, V, X, Y)$.

## 4. Examples

In this section we give a few examples of manifolds admitting interesting forms in terms of their behaviour with respect to a characteristic connection $\nabla$. We also find many eigenforms of the Hodge Laplacian with torsion $\Delta$ (see def. 2.4). We already encountered this operator in the previous section and we will further investigate its properties in the next one.

**4.1. 3-$(\alpha,\delta)$-Sasaki manifolds.** An almost 3-contact metric manifold $(M, \phi_i, \xi_i, \eta_i, g)$ is 3-$(\alpha,\delta)$-Sasaki if it additionally satisfies

$$d\eta_i = 2\alpha\Phi_i + 2(\alpha-\delta)\eta_j\wedge\eta_k.$$

Here and further on we assume that $(i, j, k)$ is an even permutation of $(1, 2, 3)$. We have defined $\Phi_i(X, Y) := g(X, \phi_i Y)$ and the real constant $\delta$ is such that it satisfies

$$\eta_k([\xi_i, \xi_j]) = 2\delta\epsilon_{ijk}.$$

The Killing vector fields $\xi_i$ are of constant length and $\alpha \neq 0$ is a real constant. Many further identities can be shown and we list some of them here:

LEMMA 4.1. *A 3-$(\alpha,\delta)$-Sasakian manifold $(M, \phi_i, \xi_i, \eta_i, g)$ satisfies*

$$N_{\phi_i} := [\phi_i, \phi_i] + d\eta_i\otimes\xi_i = 0 \quad i = 1, 2, 3$$
$$d\Phi_i = 2(\delta-\alpha)(\eta_k\wedge\Phi_j - \eta_j\wedge\Phi_k)$$
$$\xi_i \lrcorner\ \Phi_i = 0, \quad \xi_j \lrcorner\ \Phi_i = -\eta_k, \quad \xi_k \lrcorner\ \Phi_i = \eta_j.$$

PROOF. For a proof and further details see [**AD19**]. □

Furthermore, as any 3-$(\alpha,\delta)$-Sasaki manifold is a canonical almost 3-contact metric manifold, it admits an unique canonical connection $\nabla$ with totally skew torsion $T$.

LEMMA 4.2. *The canonical connection $\nabla$ has totally skew torsion*

$$T = \sum_{i=1}^{3}\eta_i\wedge d\eta_i + 8(\delta-\alpha)\eta_{123}.$$

*Furthermore, if we set $\beta = 2(\delta - 2\alpha)$, the following are satisfied:*

$$\nabla_X\xi_i = \beta(\eta_k(X)\xi_j - \eta_j(X)\xi_k),\ \nabla_X\eta_i = \beta(\eta_k(X)\eta_j - \eta_j(X)\eta_k),\ \nabla_X\phi_i = \beta(\eta_k(X)\phi_j - \eta_j(X)\phi_k).$$

PROOF. For a proof and further details see [**AD19**]. □

We investigate the behaviour of the defining forms of the structure and their products and derivatives with respect to $d^\nabla$, $\delta^\nabla$, and $\Delta$. Lengthy, but straightforward computations produce the following table:

| $\omega$ | $\nabla_X\omega$ | $\nabla$-closed | $\nabla$-cocl. | eigenv. | type |
|---|---|---|---|---|---|
| $\eta_i$ | $\beta(\eta_k(X)\eta_j - \eta_j(X)\eta_k)$ | no | yes | $4\beta^2$ | Killing |
| $\eta_{ij}$ | $\beta(\eta_i(X)\eta_i + \eta_j(X)\eta_j) \wedge \eta_k$ | yes | no | $4\beta^2$ | - |
| $\Phi_k + \eta_{ij}$ | $\beta(\eta_j(X)(\Phi_i + \eta_{jk}) - \eta_i(X)(\Phi_j + \eta_{ki}))$ | no | yes | $2\beta^2$ | - |
| $\eta_{123}$ | 0 | yes | yes | 0 | Parallel |
| $d^\nabla\Phi_i$ | $\beta^2(\eta_j(X)(\eta_i \wedge \Phi_j - \eta_j \wedge \Phi_i)$ $+\eta_k(X)(\eta_i \wedge \Phi_k - \eta_k \wedge \Phi_i))$ | no | no | $3\beta^2$ | - |
| $\sum_{i=1}^3 \eta_i \wedge \Phi_i$ | 0 | yes | yes | 0 | Parallel |
| $\eta_i \wedge \Phi_j + \eta_j \wedge \Phi_i$ | - | no | yes | $9\beta^2$ | - |
| $\eta_i \wedge \Phi_j - \eta_j \wedge \Phi_i$ | - | no | no | $3\beta^2$ | - |
| $\sigma_T$ | 0 | yes | yes | 0 | Parallel |
| $\sum_{i=1}^3 \Phi_i \wedge \Phi_i$ | 0 | yes | yes | 0 | Parallel |
| $\sum_{ijk} \Phi_i \wedge \eta_{jk}$ | 0 | yes | yes | 0 | Parallel |

TABLE 4.1. Notable forms

**4.2. The Stiefel manifold $V_{n,2} = SO(n)/SO(n-2)$.** Consider the $(2n-3)$-dimensional Stiefel manifold $V_{n,2} = SO(n)/SO(n-2)$, where $SO(n-2)$ is embedded as the upper left block of $SO(n) = \{A \in M^n(\mathbb{R}) | AA^t = Id\}$. On a Lie algebra level we have

$$\mathfrak{so}(n) = \left\{ \begin{bmatrix} h & \boldsymbol{x}_0 & \boldsymbol{x}_1 \\ -\boldsymbol{x}_0^t & 0 & x_{2n-3} \\ -\boldsymbol{x}_1^t & -x_{2n-3} & 0 \end{bmatrix} \in M^n(\mathbb{R}) \middle| h \in \mathfrak{so}(n-2), \boldsymbol{x}_{0,1} \in \mathbb{R}^{n-2}, x_{2n-3} \in \mathbb{R} \right\}.$$

We interpret $\boldsymbol{x}_{0,1}$ as column-vectors $\boldsymbol{x}_0 = (x_1, x_3, \ldots, x_{2n-5})^t$ and $\boldsymbol{x}_1 = (x_2, x_4, \ldots, x_{2n-4})^t$. Consider the splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with $\mathfrak{g} = \mathfrak{so}(n)$, $\mathfrak{h} = \mathfrak{so}(n-2)$, and

$$\mathfrak{m} = \mathbb{R}^{n-2} \oplus \mathbb{R}^{n-2} \oplus \mathbb{R} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3.$$

The tangent space to $V_{n,2}$ at the origin may be identified with $\mathfrak{m}$, which is split as a direct sum of irreducible $\mathfrak{h}$-modules. Here each of $\mathfrak{m}_1$ and $\mathfrak{m}_2$ is a standard representation of $\mathfrak{h} = \mathfrak{so}(n-2)$ and $\mathfrak{m}_3$ is a trivial representation. Moreover, for $\{i,j,k\} = \{1,2,3\}$ the following commutation relations are satisfied:

$$[\mathfrak{h}, \mathfrak{m}_i] \subset \mathfrak{m}_i, \quad [\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{h}, \quad [\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{m}_k.$$

Thus, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a reductive splitting. The direct sum is orthogonal with respect to any metric of the form

$$g_t = g \restriction_{\mathfrak{m}_1 \oplus \mathfrak{m}_2} + t.g \restriction_{\mathfrak{m}_3},$$

where $g$ is the metric, induced on $\mathfrak{m}$ by the Killing form of $\mathfrak{g}$ and $t > 0$ is a real parameter. Fix a basis of $\mathfrak{m}$ as follows: let $e_i \in \mathfrak{m}$, $i = 1, 2, \ldots, 2n-3$ be such that $x_i = 1$ and all other coordinates are zero. This basis is orthogonal with respect to $g_t$ and we make it orthonormal by taking $\xi = \frac{1}{\sqrt{t}} e_{2n-3}$ instead of $e_{2n-3}$.

Proposition 4.3. *The homogeneous space* $(V_{n,2} = SO(n)/SO(n-2), g_1)$*, where* $g_1$ *is induced by the Killing form of* $\mathfrak{so}(n)$ *is naturally reductive with canonical connection* $\nabla = \nabla^g + \frac{1}{2}T$ *whose totally skew torsion tensor* $T(X,Y) = -[X,Y]_{\mathfrak{m}}$ *is given by*

$$T = (e_{12} + e_{34} + \ldots + e_{2n-5,2n-4}) \wedge e_{2n-3}.$$

Any isotropy-invariant tensor will be parallel with respect to the canonical connection $\nabla$. We can use representation theory to obtain the following table of invariants:

| Tensor type | $V_{4,2}$ | $V_{5,2}$ | $V_{6,2}$ | $V_{n,2}, n \geq 7$ |
|---|---|---|---|---|
| Vector field | 1 | 1 | 1 | 1 |
| Symmetric (0,2) | 5 | 4 | 4 | 4 |
| Symmetric (0,3) | 5 | 4 | 4 | 4 |
| 2-form | 4 | 1 | 1 | 1 |
| 3-form | 4 | 5 | 1 | 1 |
| 4-form | 1 | 5 | 5 | 1 |

Table 4.2. Invariants on Stiefel manifolds

Additional interesting invariant tensors appear in lower dimensions due to the fact that $V_{n,2}$ carries some additional $(n-2)$- and $(n-1)$-forms. The invariant vector field is $\xi$ and the invariant 2-form

$$F = e_{12} + e_{34} + \ldots + e_{2n-5,2n-4}.$$

As always, we have identified $\mathfrak{m}$ and $\mathfrak{m}^*$ via $g_t$. The metric dual of $e_i$ is still denoted by $e_i$ for $1 \leq i \leq 2n-4$, while the dual of $\xi$ is $\eta$. We define an almost contact metric structure $(V_{n,2}, g_t, \phi, \eta, \xi)$, where

$$\phi = -E_{12} - E_{34} - \ldots - E_{2n-5,2n-4}$$

is the endomorphism defined by $F(X,Y) = g_t(X, \phi Y)$. One can check that $\phi^2 = -Id + \eta \otimes \xi$ and the structure is indeed almost contact metric. According to Wang's theorem (see [**KN69**], Chapter X), the Levi-Civita connection is given by the mapping $\Lambda^{g_t}_{\mathfrak{m}} : \mathfrak{m} \longrightarrow \mathfrak{so}(2n-3)$ with

$$\Lambda^{g_t}_{\mathfrak{m}}(X)Y = \frac{1}{2}[X,Y]_{\mathfrak{m}} + U(X,Y).$$

The tensor $U(X,Y)$ is defined by the formula

$$g_t(U(X,Y),Z) = \frac{1}{2}g_t([Z,X]_{\mathfrak{m}}, Y) + \frac{1}{2}g_t(X,[Z,Y]_{\mathfrak{m}}).$$

We can proceed to calculate

$$\Lambda^{g_t}_{\mathfrak{m}}(X)Y = \frac{1}{2}[X,Y]_{\mathfrak{m}_3}, \qquad \Lambda^{g_t}_{\mathfrak{m}}(X)\xi = \frac{t}{2}[X,\xi], \qquad \Lambda^{g_t}_{\mathfrak{m}}(\xi)Y = \left(1 - \frac{t}{2}\right)[\xi, Y], \qquad \Lambda^{g_t}_{\mathfrak{m}}(\xi)\xi = 0,$$

where $X, Y \in \mathfrak{m}_1 \oplus \mathfrak{m}_2$. Now we easily see that $\xi$ (or equivalently $\eta$) is Killing and the following holds:

$$\nabla^{g_t}_X \eta = \frac{\sqrt{t}}{2} X \lrcorner\ F = \frac{1}{2} X \lrcorner\ d\eta.$$

Thus $d\eta = \sqrt{t}F$, from which easily follows that the Nijenhuis tensor

$$N_\phi(X,Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2[X,Y] + d\eta(X,Y)\xi$$

of $\phi$ vanishes for all $t > 0$. We can finally state the following:

PROPOSITION 4.4. *The almost contact metric structure $(V_{n,2} = SO(n)/SO(n-2), g_t)$ is normal which makes it $\alpha$-Sasakian with $\alpha = \sqrt{t}$.*

We can also consider the family of connections $\nabla^{s,t}_X Y$ for $s \in \mathbb{R}$ such that

$$g_t(\nabla^{s,t}_X Y, Z) = g_t(\nabla^{g_t}_X Y, Z) + \frac{s}{2} T(X, Y, Z)$$

and describe $\nabla^{s,t}$ as a mapping $\Lambda^{s,t}_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{so}(2n-3)$. First we find $\Lambda^{g_t}_{\mathfrak{m}}(X)$ as elements of $\mathfrak{so}(2n-3) \cong \Lambda^2\mathfrak{m}$:

$$\Lambda^{g_t}_{\mathfrak{m}}(X) = -\frac{\sqrt{t}}{2}(X \lrcorner\ F) \wedge \eta, \qquad \Lambda^{g_t}_{\mathfrak{m}}(\xi) = \left(\frac{\sqrt{t}}{2} - \frac{1}{\sqrt{t}}\right) F.$$

Here $X \in \mathfrak{m}_1 \oplus \mathfrak{m}_2$. We thus obtain the action of $\Lambda^{g_t}_{\mathfrak{m}}$ on $\omega \in \Lambda^k\mathfrak{m}$ by just letting it act via $\Diamond$:

$$\Lambda^{g_t}_{\mathfrak{m}}(X)\omega = -\frac{\sqrt{t}}{2}\left((X \lrcorner\ F) \wedge \eta\right) \Diamond\omega, \qquad \Lambda^{g_t}_{\mathfrak{m}}(\xi)\omega = \left(\frac{\sqrt{t}}{2} - \frac{1}{\sqrt{t}}\right) F \Diamond\omega.$$

Then we deal with the torsion term finding $T_0$ such that $g_t(T_0(X,Y), Z) = T(X,Y,Z)$, namely

$$T_0(X,Y) = -[X,Y]_{\mathfrak{m}_1 \oplus \mathfrak{m}_2} - \frac{1}{t}[X,Y]_{\mathfrak{m}_3}.$$

Now $\nabla^{s,t}_X Y = \nabla^{g_t}_X Y + \frac{s}{2} T_0(X,Y)$ and we can use $T_0$ to obtain $\Lambda^{s,t}_{\mathfrak{m}}$ from $\Lambda^{g_t}_{\mathfrak{m}}$:

$$\Lambda^{s,t}_{\mathfrak{m}}(X) = \frac{s-t}{2\sqrt{t}}(X \lrcorner\ F) \wedge \eta, \qquad \Lambda^{s,t}_{\mathfrak{m}}(\xi) = \frac{s+t-2}{2\sqrt{t}} F.$$

Direct computations lead to the following:

PROPOSITION 4.5. *$\eta$ is a special $\nabla^{s,t}$-Killing 1-form, which is an eigenform of $\Delta^{s,t}$. More precisely:*

$$\nabla^{s,t}_X \eta = \frac{1}{2} X \lrcorner\ d^{s,t}\eta, \quad d^{s,t}\eta = \frac{t-s}{\sqrt{t}} F, \quad \delta^{s,t}\eta = 0, \quad \Delta^{s,t}\eta = \frac{(t-s)^2(n-2)}{t}\eta.$$

*Further, $F$ is special $\nabla^{s,t}$-closed and also an eigenform of $\Delta^{s,t}$:*

$$\nabla^{s,t}_X F = \frac{s-t}{2\sqrt{t}} X \wedge \eta, \quad d^{s,t}F = 0, \quad \delta^{s,t}F = \frac{(t-s)(n-2)}{\sqrt{t}}\eta, \quad \Delta^{s,t}F = \frac{(t-s)^2(n-2)}{t}F.$$

*In particular, none of those are parallel unless $s = t$.*

COROLLARY 4.6. *$T = \sqrt{t}F \wedge \eta$ is also a special $\nabla^{s,t}$-Killing form.*

PROOF. $\eta$ is a special Killing 1-form and $d^{s,t}\eta = \frac{t-s}{\sqrt{t}} F$, so $T = \sqrt{t} F \wedge \eta = \frac{t}{t-s} d^{s,t}\eta \wedge \eta$ is also special. $\square$

We have obtained an example of a metric connection $\nabla^{s,t}$ with totally skew torsion $sT \in \Lambda^3\mathfrak{m}$ which is $\nabla^{s,t}$-Killing, yet not $\nabla^{s,t}$-parallel.

PROPOSITION 4.7. *The torsion $sT$ of $\nabla^{s,t}$ is $\nabla^{s,t}$-Killing but not parallel unless $s = 0$ or $s = t$.*

PROOF. The torsion $sT$ of $\nabla^s$ equals

$$sT = s\sqrt{t} F \wedge \eta,$$

which is already special $\nabla^{s,t}$-Killing by the previous corollary. In general, it is not parallel:

$$\nabla^{s,t}_X sT = \frac{1}{4} X \lrcorner\ d^{s,t}(sT) = \frac{s\sqrt{t}}{4} X \lrcorner\ (F \wedge d^{s,t}\eta) = \frac{s(t-s)}{4} X \lrcorner\ (F \wedge F). \qquad \square$$

We go on to investigate the cases $n = 4$ and $n = 5$ in some more detail.

**4.3. The Stiefel manifold $V_{4,2} = SO(4)/SO(2)$.** We consider the 5-dimensional $V_{4,2} = SO(4)/SO(2)$. First, we give the additional 2-forms:

PROPOSITION 4.8. *The following 2-forms are isotropy-invariant:*

$$F_1 = e_{12} + e_{34}, \quad F_2 = e_{13} - e_{24}, \quad F_3 = e_{14} + e_{23}, \quad F_4 = e_{13} + e_{24}.$$

*$(V_{4,2}, g_t, \phi_i, \xi, \eta)$ with $\phi_i$ such that $F_i(X,Y) = g(X, \phi_i Y)$, where $1 \le i \le 4$, is an almost contact metric manifold. The Nijenhuis tensor $N_i$ of the structure $\phi_i$ is always totally skew. For $i = 1,4$ it is zero, hence the structure is normal, and for $i = 2,3$ it is non-zero. The structure $\phi_1$ is $\alpha$-Sasakian, $\phi_2$ and $\phi_3$ are semi-cosymplectic, while $\phi_4$ is quasi-Sasakian non-Sasakian.*

PROOF. The isotropy is one-dimensional and is represented by $h = E_{12} \in \mathfrak{h}$. One computes $ad(h) = E_{13} + E_{24}$ and checks that the four given forms are indeed isotropy-invariant. On the other hand, representation theory tells us that there are exactly four such invariants, so we are done. We can see directly from the expressions for $F_i$ that $\phi_i^2 = -Id + \eta \otimes \xi$ and $g(X, \phi_i Y) = -g(\phi_i X, Y)$, which is enough to conclude that the structures are almost contact metric. We omit the computations behind the statement about the Nijenhuis tensors. We know from the general case that $\phi_1$ is $\alpha$-Sasakian. The semi-cosymplectic condition is $\delta\eta = \delta F = 0$, which we can readily confirm using the formulas for $\Lambda^{g_t}_{\mathfrak{m}}$. The same formulas enable us to show also that $dF_4 = 0$, which is the quasi-Sasakian condition, along with being normal. For the precise expressions, see the following table 4.3 with $s = 0$. $\square$

Let's consider the family of connections

$$\nabla^{s,t}_X Y = \nabla^{g_t}_X Y + \frac{s}{2} T_0(X,Y)$$

and the associated operators $d^{s,t}, \delta^{s,t}$, and $\Delta^{s,t} =: \Delta$. We study the behaviour of $\eta, F_i$, and $T$ with respect to those operators, arriving at the following table 4.3.

| $\omega$ | $\nabla^{s,t}_X \omega$ | $d^{s,t}\omega$ | $\delta^{s,t}\omega$ | eigenvalue | type |
|---|---|---|---|---|---|
| $\eta$ | $\frac{1}{2} X \lrcorner\ d^{s,t}\eta$ | $\frac{t-s}{\sqrt{t}} F_1$ | 0 | $\frac{(t-s)^2(n-2)}{t}$ | Sp. K |
| $F_1$ | $\frac{s-t}{2\sqrt{t}} X \wedge \eta$ | 0 | $\frac{(t-s)(n-2)}{\sqrt{t}}\eta$ | $\frac{(t-s)^2(n-2)}{t}$ | Sp. CK |
| $F_2$ | $\frac{s-t}{2\sqrt{t}}(X \lrcorner\ F_3) \wedge \eta + \frac{s+t-2}{\sqrt{t}}\eta(X) F_3$ | $\frac{2(s-1)}{\sqrt{t}} F_3 \wedge \eta$ | 0 | $\frac{4(1-s)^2}{t}$ | - |
| $F_3$ | $\frac{t-s}{2\sqrt{t}}(X \lrcorner\ F_2) \wedge \eta - \frac{s+t-2}{\sqrt{t}}\eta(X) F_2$ | $\frac{2(1-s)}{\sqrt{t}} F_2 \wedge \eta$ | 0 | $\frac{4(1-s)^2}{t}$ | - |
| $F_4$ | $\frac{s-t}{2\sqrt{t}}(\phi_4\phi_1 X) \wedge \eta$ | 0 | 0 | 0 | Harm. |

TABLE 4.3. Notable forms

We do not investigate higher degree forms, since 3-forms are given by the Hodge star $*_t$ of 2-forms and our operators behave well with respect to $*_t$. Note that the operator depends on the metric $g_t$. Closed and coclosed are interchanged, and so are special Killing and special conformal Killing. An example of the last phenomenon is the following observation: we know from the general case that $T = \sqrt{t} F_1 \wedge \eta$ is special Killing. Its Hodge dual is proportional to $F_1$, which is special conformal Killing, as noted in the table.

Inspired by the observation that $T$ is the Hodge dual of $F_1$, we obtain an example of a metric connection with totally skew torsion which is $\nabla$-closed, yet not $\nabla$-parallel.

PROPOSITION 4.9. *Consider*

$$\nabla^{s,p} = \nabla^{g_1} + \frac{1}{2} T^{s,p} = \nabla^{g_t} + \frac{1}{2}(sT + p(*F_4)),$$

*where we have chosen $t = 1$ for the metric in the sake of simplicity and $*$ means $*_1$. For the pair $(s,p) = (\frac{4}{3}, \frac{2}{3})$, $d^{s,p} T^{s,p} = 0$ but $\nabla^{s,p} T^{s,p} \neq 0$.*

Proof. To ease notation, we skip all the occurrences of $t = 1$ in the formulas computed so far. Recall some of the properties of the Hodge star on $V_{4,2}$:

$$\nabla^{s,p}* = *\nabla^{s,p}; \quad *d^{s,p} = (-1)^k \delta^{s,p} * \text{ on a } k\text{-form}; \quad *^2 = 1.$$

We compute

$$\Lambda^{s,p}_{\mathfrak{m}}(X) = \Lambda^{s}_{\mathfrak{m}}(X) + \frac{p}{2}.(X \lrcorner *F_4)\lozenge.$$

To show that $\nabla^{s,p}T^{s,p} \neq 0$ and $d^{s,p}T^{s,p} = 0$ we can equivalently show $\nabla^{s,p}*T^{s,p} \neq 0$ and $\delta^{s,p}*T^{s,p} = 0$ so we work with the simpler $*T^{s,p} = sF_1 + pF_4$. Note that $*F_4 = -F_4 \wedge \eta$. First,

$$\begin{aligned}\nabla^{s,p}_X(sF_1) &= s\nabla^s_X F_1 - \frac{sp}{2}[X \lrcorner (F_4 \wedge \eta), F_1] = s\nabla^s_X F_1 - \frac{sp}{2}[(X \lrcorner F_4) \wedge \eta, F_1] - \frac{sp}{2}\eta(X)[F_4, F_1]\\ &= s\nabla^s_X F_1 - \frac{sp}{2}[(X \lrcorner F_4) \wedge \eta, F_1]\end{aligned}$$

We then find

$$\nabla^{s,p}_X(pF_4) = p\nabla^s_X F_4 - \frac{p^2}{2}[X \lrcorner (F_4 \wedge \eta), F_4] = p\nabla^s_X F_4 - \frac{p^2}{2}[(X \lrcorner F_4) \wedge \eta, F_4].$$

The codiffrential $\delta^{s,p} = -e_i \lrcorner \nabla^{s,p}_{e_i}$ is easy to compute:

$$\begin{aligned}\delta^{s,p}(sF_1 + pF_4) &= s\delta^s F_1 + p\delta^s F_4 + \frac{sp}{2}\sum_i e_i \lrcorner [(e_i \lrcorner F_4) \wedge \eta, F_1] + \frac{p^2}{2}\sum_i e_i \lrcorner [(e_i \lrcorner F_4) \wedge \eta, F_4]\\ &= 2s(1-s)\eta + \frac{sp}{2}\sum_i e_i \lrcorner [(e_i \lrcorner F_4) \wedge \eta, F_1] + \frac{p^2}{2}\sum_i e_i \lrcorner [(e_i \lrcorner F_4) \wedge \eta, F_4]\\ &= 2s(1-s)\eta + 2p^2\eta = 2(s(1-s) + p^2)\eta.\end{aligned}$$

Choosing $s = \frac{4}{3}$, $p = \frac{2}{3}$, we conclude that

$$\nabla^{\frac{4}{3},\frac{2}{3}}T^{\frac{4}{3},\frac{2}{3}} \neq 0 \quad \text{and} \quad d^{\frac{4}{3},\frac{2}{3}}T^{\frac{4}{3},\frac{2}{3}} = - * \delta^{\frac{4}{3},\frac{2}{3}} * T^{\frac{4}{3},\frac{2}{3}} = 0$$

by observing that $\nabla^{\frac{4}{3},\frac{2}{3}}_{e_1}T \neq 0$, for example. □

**4.4. The Stiefel manifold $V_{5,2} = SO(5)/SO(3)$.** Consider the 7-dimensional $V_{5,2}$. First, we give the additional 3-forms:

Proposition 4.10. *The following 3-forms are isotropy-invariant:*

$$F_1 = e_{127} + e_{347} + e_{567}, \quad F_2 = e_{135}, \quad F_3 = e_{246}, \quad F_4 = e_{136} + e_{145} + e_{235}, \quad F_5 = e_{146} + e_{236} + e_{245}.$$

*The $\alpha$-Sasakian $(V_{5,2}, g_t, \phi, \xi, \eta)$ with $\phi$ such that $F(X,Y) = g(X, \phi Y)$ (see prop 4.4) admits a nearly parallel $G_2$-structure for $t = \sqrt{\frac{3}{2}}$ with the defining 3-form given by*

$$\psi = F_1 + F_2 - F_5 = e_{127} + e_{347} + e_{567} + e_{135} - e_{146} - e_{236} - e_{245}.$$

Proof. It is known that $\psi$ has stabilizer $G_2$. We can compute its differential and compare it to the Hodge star to see that the structure is nearly parallel, seeing already that $\delta\psi = 0$. Look ahead to table 4.4 with $s = 0$ for the precise expressions. The Hodge star $*_t$ depends on the metric. One can compute

$$*_t F_1 = \frac{\sqrt{t}}{2}F \wedge F, \quad *_t F_2 = -F_3 \wedge \eta, \quad *_t F_3 = F_2 \wedge \eta, \quad *_t F_4 = F_5 \wedge \eta, \quad *_t F_5 = -F_4 \wedge \eta.$$

Therefore

$$*_t\psi = \frac{\sqrt{t}}{2}F \wedge F - F_3 \wedge \eta + F_4 \wedge \eta$$

and

$$d\psi = tF \wedge F + \frac{3}{\sqrt{t}}F_4 \wedge \eta - \frac{3}{\sqrt{t}}F_3 \wedge \eta = \frac{3}{\sqrt{t}}\left(\frac{t\sqrt{t}}{3}F \wedge F - F_3 \wedge \eta + F_4 \wedge \eta\right).$$

One now sees that $*_t\psi$ is proportional to $d\psi$ if and only if $\frac{t}{3} = \frac{1}{2} \iff t = \frac{3}{2}$. □

Again, proceed to consider the family of connections $\nabla^{s,t}_X Y = \nabla^{g_t}_X Y + \frac{s}{2} T_0(X,Y)$ and the associated operators $d^{s,t}, \delta^{s,t}$, and $\Delta^{s,t} =: \Delta$. We summarize the results regarding the forms $F_i$ in table 4.4. Higher degree forms are not investigated since they are given by the Hodge star $*_t$ of

| $\omega$ | $\nabla^{s,t}_X \omega$ | $d^{s,t}\omega$ | $\delta^{s,t}\omega$ | eigenv. | type |
|---|---|---|---|---|---|
| $F_1$ | $\frac{t-s}{2}(X \lrcorner\ F) \wedge F$ | $(t-s)F \wedge F$ | 0 | $\frac{4(s-t)^2}{t}$ | Sp. K |
| $F_2$ | $\frac{t-s}{2\sqrt{t}}(\phi X \lrcorner\ F_2) \wedge \eta + \frac{s+t-2}{2\sqrt{t}}\eta(X)F_4$ | $\frac{1-s}{\sqrt{t}} F_4 \wedge \eta$ | 0 | $-$ | - |
| $F_3$ | $\frac{t-s}{2\sqrt{t}}(\phi X \lrcorner\ F_3) \wedge \eta - \frac{s+t-2}{2\sqrt{t}}\eta(X)F_5$ | $\frac{s-1}{\sqrt{t}} F_5 \wedge \eta$ | 0 | $-$ | - |
| $F_4$ | $\frac{t-s}{2\sqrt{t}}(\phi X \lrcorner\ F_4) \wedge \eta + \frac{s+t-2}{2\sqrt{t}}\eta(X)(2F_5 - 3F_2)$ | $\frac{1-s}{\sqrt{t}}(2F_5 - 3F_2) \wedge \eta$ | 0 | $-$ | - |
| $F_5$ | $\frac{t-s}{2\sqrt{t}}(\phi X \lrcorner\ F_5) \wedge \eta + \frac{s+t-2}{2\sqrt{t}}\eta(X)(3F_3 - 2F_4)$ | $\frac{1-s}{\sqrt{t}}(3F_3 - 2F_4) \wedge \eta$ | 0 | $-$ | - |

Table 4.4. Notable forms

the ones we already know. In this example, the "generic" 3-forms that we investigate are not eigenforms of $\Delta$ (except for $F_1$, which we will show to be special Killing). However, there exist linear combinations, which are eigenforms of $\Delta$.

Proposition 4.11. *$F_1$ is a special $\nabla^{s,t}$-Killing 3-form. The following are eigenforms of $\Delta$:*

$$\Delta(F_2 - F_5) = \frac{9(s-1)^2}{t}(F_2 - F_5), \qquad \Delta(3F_2 + F_5) = \frac{(s-1)^2}{t}(3F_2 + F_5),$$
$$\Delta(F_3 - F_4) = \frac{9(s-1)^2}{t}(F_3 - F_4), \qquad \Delta(3F_3 + F_4) = \frac{(s-1)^2}{t}(3F_3 + F_4).$$

Proof. We know that $F_1 = \sqrt{t} F \wedge \eta$ is special from the general case. Using the identities in the table above, one can compute:

$$\Delta F_1 = \frac{4(s-t)^2}{t} F_1, \quad \Delta F_2 = \frac{(s-1)^2}{t}(3F_2 - 2F_5), \quad \Delta F_3 = \frac{(s-1)^2}{t}(3F_3 - 2F_4),$$
$$\Delta F_4 = \frac{(s-1)^2}{t}(7F_4 - 6F_3), \quad \Delta F_5 = \frac{(s-1)^2}{t}(7F_5 - 6F_2)$$

The linear combinations given in the proposition are now easily found. ☐

Remark 4.12. Observe that the $G_2$ structure $\psi$ should also be an eigenform of $\Delta$ for $s = 0, t = \frac{3}{2}$. Indeed, in this case the eigenvalues of $F_1$ and $F_2 - F_5$ coincide, equalling 6. This leads us to investigate another interesting value of the parameter $t$: $t = \frac{1}{2}$, for which $F_1$ and $3F_2 + F_5$ have coinciding eigenvalues.

## 5. Laplacians acting on forms

**5.1. Laplacians acting on $k$-forms.** We would like to continue our investigation on operators acting on forms by looking at different types of Laplacians. We have at our disposal the classical Hodge and Bochner-Laplace operators, as well as two analogously defined operators, using an arbitrary connection with skew-symmetric torsion. We can also investigate the Lichnerowicz Laplacian and its generalized version, both defined and studied in [**Die13**], and their skew-torsion counterparts. We define all of them presently.

(1) We start with the classical Hodge Laplacian. It is defined as

$$\Delta^g := d\delta + \delta d.$$

This operator is elliptic, formally self-adjoint, positive, and on a compact manifold a $k$-form $\omega$ satisfies $\Delta^g \omega = 0$ iff $d\omega = 0$ and $\delta\omega = 0$.

(2) In Definition 2.4 we already introduced the Hodge Laplacian with torsion

$$\Delta = d^\nabla \delta^\nabla + \delta^\nabla d^\nabla.$$

This is an analogue of the Hodge Laplacian, which is again formally self-adjoint and positive. The claim that on a compact manifold $\Delta\omega = 0$ iff $d^\nabla \omega = 0$ and $\delta^\nabla \omega = 0$ holds as well. Yet, since we showed that $(d^\nabla)^2 \neq 0$, $\Delta \neq (d^\nabla + \delta^\nabla)^2$. We also note that $\Delta$ and $\Delta^g$ coincide on functions: $\Delta f = \Delta^g f$. Indeed, using $df = d^\nabla f$ we easily see

$$\Delta f = d^\nabla \delta^\nabla f + \delta^\nabla d^\nabla f = \delta^\nabla df$$

and analogously $\Delta^g f = \delta df$. Now

$$(\Delta^g - \Delta) f = (\delta - \delta^\nabla)(df) = T \square (df) = 0.$$

(3) Next, we define the Bochner-Laplace operator. Whenever we have a connection $\nabla$, this is just the operator $\nabla^*\nabla$, acting on a $k$-form. In the case of the Levi-Civita connection we denote it by $(\nabla^*\nabla)^g$. The relation

$$(\nabla^*\nabla)\omega = -tr(\nabla^2 \omega)$$

holds whenever $\nabla$ is metric. We refer to [**Sem11**], p.193 for the proof. Observe also that if we have torsion and define the Hessian of a function as $\mathrm{Hess}^\nabla(f)(X,Y) := (\nabla df)(X,Y)$, it differs from the usual one, that is:

$$\begin{aligned} \mathrm{Hess}^\nabla(f)(X,Y) &= (\nabla df)(X,Y) = \nabla^2_{XY} f = X(Y(f)) - (\nabla_X Y) f \\ \mathrm{Hess}^g(f)(X,Y) &= (\nabla^g df)(X,Y) = (\nabla^g)^2_{XY} f = X(Y(f)) - (\nabla^g_X Y) f \\ \Longrightarrow \mathrm{Hess}^\nabla(f)(X,Y) &= \mathrm{Hess}^g(f)(X,Y) - \frac{1}{2} T(X,Y)(f). \end{aligned}$$

In particular, this shows that $\mathrm{Hess}^\nabla$ fails to be symmetric any more and this failure is measured by the torsion tensor, while its symmetric part is exactly $\mathrm{Hess}^g$. Both Hessians have equal traces, hence the Bochner-Laplace operators defined for $\nabla$ and $\nabla^g$ coincide on functions:

$$(\nabla^*\nabla) f = -tr(\mathrm{Hess}^\nabla(f)) = -tr(\mathrm{Hess}^g(f)) = (\nabla^*\nabla)^g f.$$

(4) The last operator we define is the Lichnerowicz Laplacian. First we need an additional quantity—the curvature endomorphism $q(\mathcal{R})$.

$$q(\mathcal{R})\omega := \sum_{i,j} e_j \wedge (e_i \lrcorner\ \mathcal{R}(e_i, e_j)\omega)$$

on a $k$-form $\omega$. This object is thoroughly studied by Semmelmann in [**Sem01**]. We can use it to define the Lichnerowicz Laplacian (adopting the terminology from [**Die13**]):

$$\Delta_L := \nabla^*\nabla + q(\mathcal{R}).$$

It is handled more easily by representation theory. In fact, it acts as a Casimir operator on certain homogeneous spaces ([**MS10**], [**AS12**]). The link with representation theory becomes more obvious when one rewrites $q(\mathcal{R})$ as follows:

$$q(\mathcal{R})\omega = \sum_{i<j} \varrho(e_i \wedge e_j)\mathcal{R}(e_i, e_j)\omega = \sum_{i<j} \varrho(e_i \wedge e_j)\varrho(\mathcal{R}(e_i \wedge e_j))\omega =: q(\mathcal{R})^\varrho \omega.$$

If we want to make the representation visible, we denote $\Delta_L^\varrho := \nabla^*\nabla + q(\mathcal{R})^\varrho$. This allows us to speak about $\Delta_L^\pi$ for representations $\pi$, different from the natural representation $\varrho$ (see Prop 1.2), which we have been using since Section 2. It is precisely how Moroianu and Semmelmann treat the subject. We cite the following theorem ([**MS10**], Lemma 5.2):

THEOREM 5.1. *Let $G$ be a compact semi-simple Lie group and $H \leq G$ a compact subgroup such that $M = G/H$ is the naturally reductive homogeneous space with Riemannian metric induced by $-B$, where $B$ is the Killing form of $G$. Let $\nabla$ denote the canonical connection of $M$. Let also $\varrho : H \longrightarrow \mathrm{End}(E)$ be a representation of the isotropy group and $EM := G \times_\varrho E$ be the associated vector bundle over $M$. Then the curvature endomorphism $q(\mathcal{R})^\varrho$ acts fibre-wise on $EM$ as $-Cas^H_\varrho$. Moreover, if we consider the space of sections $\Gamma(EM)$ as a $G$-representation via the left regular representation $l$, the differential operator $\Delta^\varrho_L$ acts on $\Gamma(EM)$ as $-Cas^G_l$.*

Since we only use the representation $\varrho$, we continue writing $\Delta_L$ without the superscript. We proceed by computing the expression for $\Delta$ on forms, but first we introduce a slight abuse of notation:

REMARK 5.2. We write $T\square(\nabla\omega) = \frac{1}{2}\sum_{i,j}(e_i \lrcorner\ (e_j \lrcorner\ T)) \wedge \big(e_i \lrcorner\ \nabla_{e_j}\omega\big)$ for conciseness.

THEOREM 5.3. *The Hodge Laplacian $\Delta = d^\nabla \delta^\nabla + \delta^\nabla d^\nabla$ on a $k$-form $\omega$ is given by*

$$\Delta\omega = (\nabla^*\nabla)\omega + q(\mathcal{R})\omega - 2T\square(\nabla\omega) = \Delta_L - 2T\square(\nabla\omega). \tag{5.1}$$

PROOF. Let $\omega$ be a $k$-form. We compute the Laplacian by the definitions of $d^\nabla$ and $\delta^\nabla$:

$$\begin{aligned}
\Delta\omega &= -\sum_{i,j} e_i \wedge \nabla_{e_i}(e_j \lrcorner\ \nabla_{e_j}\omega) - \sum_{i,j} e_i \lrcorner\ \nabla_{e_i}(e_j \wedge \nabla_{e_j}\omega)\\
&= -\sum_{i,j} e_i \wedge (e_j \lrcorner\ \nabla_{e_i}\nabla_{e_j}\omega) - \sum_{i,j} e_i \lrcorner\ (e_j \wedge \nabla_{e_i}\nabla_{e_j}\omega)\\
&= -\sum_{i,j} e_i \wedge (e_j \lrcorner\ \nabla_{e_i}\nabla_{e_j}\omega) - \sum_{i,j} \delta_{ij}\nabla_{e_i}\nabla_{e_j}\omega + \sum_{i,j} e_j \wedge (e_i \lrcorner\ \nabla_{e_i}\nabla_{e_j}\omega)\\
&= -\sum_{i} \nabla^2_{e_ie_i}\omega + \sum_{i,j} e_j \wedge (e_i \lrcorner\ (\nabla_{e_i}\nabla_{e_j}\omega - \nabla_{e_j}\nabla_{e_i}\omega))\\
&= (\nabla^*\nabla)\omega + \sum_{i,j} e_j \wedge (e_i \lrcorner\ \mathcal{R}(e_i,e_j)\omega) + \sum_{i,j} e_j \wedge (e_i \lrcorner\ \nabla_{[e_i,e_j]}\omega).
\end{aligned}$$

So far this was a straightforward computation, which led us to the expected Bochner-Laplace and curvature terms. We derive the last term exploiting the identity $[e_i, e_j] = -T(e_i, e_j)$ in the adapted frame. Now

$$\begin{aligned}
\sum_{i,j} e_j \wedge (e_i \lrcorner\ \nabla_{[e_i,e_j]}\omega) &= -\sum_{i,j,k} T(e_i,e_j,e_k) e_j \wedge (e_i \lrcorner\ \nabla_{e_k}\omega)\\
&= -\sum_{i,k} (e_i \lrcorner\ (e_k \lrcorner\ T)) \wedge (e_i \lrcorner\ \nabla_{e_k}\omega) = -2T\square(\nabla\omega)
\end{aligned}$$

concludes the proof. $\square$

Obviously, the formula we obtained for the Laplacian coincides with the well-known Weitzenböck type formula in the Riemannian case. Let's try to write down the term $q(\mathcal{R})$ more explicitly.

PROPOSITION 5.4. *The curvature endomorphisms of $\nabla$ and $\nabla^g$ are related by*

$$q(\mathcal{R})\omega = q(\mathcal{R}^g)\omega + \frac{1}{2}\delta T(\omega) + \frac{1}{4}S(\omega) - dT\square\omega + \frac{1}{2}\sigma_T\square\omega + \frac{1}{2}B(\omega),$$

*where $B(\omega) = \sum_k (e_k \lrcorner\ T) \wedge ((e_k \lrcorner\ T) \lrcorner\ \omega)$.*

PROOF. In the appendix we define the $(0,4)$-tensor $F$ by the identity

$$R(X,Y,Z,V) = R^g(X,Y,Z,V) + F(X,Y,Z,V),$$

as well as the 2-form $\mathcal{F}(X,Y) = F(X,Y,-,-)$. We then prove the identity

$$\mathcal{R}(X,Y)\omega = \mathcal{R}^g(X,Y)\omega + \varrho(\mathcal{F}(X,Y))\omega.$$

This means that $q(\mathcal{R})\omega - q(\mathcal{R}^g)\omega$ is nothing else, but

$$\begin{aligned}\sum_{i,j} e_j \wedge (e_i \lrcorner\ \varrho(\mathcal{F}(e_i,e_j))\omega) &= \frac{1}{2}\sum_{i,j,k,l} F_{ijkl} e_j \wedge (e_i \lrcorner\ \varrho(e_k \wedge e_l)\omega) \\ &= \frac{1}{2}\sum_{i,j,k,l} F_{ijkl} e_j \wedge (e_i \lrcorner\ (e_l \wedge (e_k \lrcorner\ \omega) - e_k \wedge (e_l \lrcorner\ \omega))) \\ &= \sum_{i,j,k,l} F_{ijkl} e_j \wedge (e_i \lrcorner\ (e_l \wedge (e_k \lrcorner\ \omega))) \\ &= \sum_{i,j,k,l} F_{ijkl} e_j \wedge (\delta_{il} e_k \lrcorner\ \omega - e_l \wedge (e_i \lrcorner\ (e_k \lrcorner\ \omega))) \\ &= \sum_{i,j,k} F_{ijki} e_j \wedge (e_k \lrcorner\ \omega) + \sum_{i,j,k,l} F_{ijkl} e_j \wedge e_l \wedge (e_k \lrcorner\ (e_i \lrcorner\ \omega)).\end{aligned}$$

We deal with the two summands independently. Let's write down $F_{ijkl}$ explicitly:

$$F_{ijkl} = \frac{1}{2}(\nabla_{e_i} T)(e_j,e_k,e_l) - \frac{1}{2}(\nabla_{e_j} T)(e_i,e_k,e_l) + \frac{1}{4}\langle T(e_i,e_j), T(e_k,e_l)\rangle + \frac{1}{4}\sigma_T(e_i,e_j,e_k,e_l).$$

First observe that $F_{ijki} = \mathrm{Ric}_{jk} - \mathrm{Ric}^g_{jk}$ is a 2-tensor, whose action on $\omega$ is exactly the first summand we acquired. By proposition A.1 we get

$$\sum_{i,j,k} F_{ijki} e_j \wedge (e_k \lrcorner\ \omega) = \frac{1}{2}\delta T(\omega) + \frac{1}{4} S(\omega).$$

To compute the second term note that $e_j \wedge e_l$ is antisymmetric in $j$ and $l$, as is $e_k \lrcorner\ (e_i \lrcorner\ \omega)$ in $k$ and $i$. This means that we can take the antisymmetric part of $F_{ijkl}$ according to those symmetries. Hence,

$$\begin{aligned}&\sum_{i,j,k,l} F_{ijkl} e_j \wedge e_l \wedge (e_k \lrcorner\ (e_i \lrcorner\ \omega)) = \\ &\sum_{i,j,k,l}\Big(\frac{1}{4}(\nabla_{e_i}T)(e_j,e_k,e_l) - \frac{1}{4}(\nabla_{e_k}T)(e_j,e_i,e_l) - \frac{1}{4}(\nabla_{e_j}T)(e_i,e_k,e_l) + \frac{1}{4}(\nabla_{e_l}T)(e_i,e_k,e_j) \\ &\quad + \frac{1}{4}\sigma_T(e_i,e_j,e_k,e_l) + \frac{1}{8}\langle T(e_i,e_j),T(e_k,e_l)\rangle - \frac{1}{8}\langle T(e_k,e_j),T(e_i,e_l)\rangle\Big) e_{jl} \wedge (e_k \lrcorner\ (e_i \lrcorner\ \omega)) \\ &= \sum_{i,j,k,l}\Big(\frac{1}{4} dT(e_i,e_j,e_k,e_l) - \frac{1}{2}\sigma_T(e_i,e_j,e_k,e_l) + \frac{1}{4}\sigma_T(e_i,e_j,e_k,e_l) + \frac{1}{8}\sigma_T(e_i,e_j,e_k,e_l) \\ &\quad - \frac{1}{8}\langle T(e_k,e_i),T(e_j,e_l)\rangle\Big) e_{jl} \wedge (e_k \lrcorner\ (e_i \lrcorner\ \omega)) \\ &= \sum_{i,k}\Big(-\frac{1}{2}(e_k \lrcorner\ (e_i \lrcorner\ dT)\Big) + \frac{1}{4}(e_k \lrcorner\ (e_i \lrcorner\ \sigma_T))) \wedge (e_k \lrcorner\ (e_i \lrcorner\ \omega)) + \frac{1}{2}\sum_k (e_k \lrcorner\ T) \wedge ((e_k \lrcorner\ T) \lrcorner\ \omega) \\ &= -dT\square\omega + \frac{1}{2}\sigma_T\square\omega + \frac{1}{2}B(\omega).\end{aligned}$$

Here we used the definition of $\sigma_T$ as well as the formula

$$dT(X,Y,Z,V) = \sum_{XYZ}((\nabla_X T)(Y,Z,V)) - (\nabla_V T)(X,Y,Z) + 2\sigma_T(X,Y,Z,V)$$

from Appendix A of [**Agr06**] (Corollary A.1). We also denoted the last term by $B(\omega)$ for brevity. □

As a consequence we get the following:

PROPOSITION 5.5. *If $\omega$ is a $k$-form, the Laplacian $\Delta\omega$ of $\omega$ is given by the formula*

$$\begin{aligned}\Delta\omega =&(\nabla^*\nabla)\omega + q(\mathcal{R}^g)\omega + \frac{1}{2}\delta T(\omega) + \frac{1}{4}S(\omega) - 2T\Box(\nabla\omega)\\ &-(dT)\Box\omega + \frac{1}{2}\sigma_T\Box\omega + \frac{1}{2}B(\omega)\end{aligned}\tag{5.2}$$

REMARK 5.6. Finding $q(\mathcal{R})$ by definition can sometimes be easy. In case $\mathcal{R} : \wedge^2 TM \longrightarrow \wedge^2 TM$ is a symmetric endomorphism, we have

$$\langle \mathcal{R}(\alpha), \beta\rangle = \langle \alpha, \mathcal{R}(\beta)\rangle\,,$$

so only the part of $\alpha$ which lies in $\mathrm{Im}(\mathcal{R}) = \mathfrak{hol}$ counts. Here $\beta \in \mathfrak{so}(n)$ is arbitrary. Recall that $q(\mathcal{R})$ can be written as

$$q(\mathcal{R}) = \sum_{i<j} \varrho(e_i\wedge e_j)\varrho(\mathcal{R}(e_i\wedge e_j)).$$

In this form we see that the summation just runs through an orthonormal basis of $\mathfrak{so}(n)$ (we scale the metric on $\Lambda^2 TM \cong \mathfrak{so}(n)$ so that $e_i \wedge e_j$ has unit length), and of course the sum doesn't depend on a change of the basis. We can take a basis of the holonomy algebra $\mathfrak{hol}$ and complete it to a basis of $\mathfrak{so}(n)$. Then

$$q(\mathcal{R}) = \sum_{\alpha} \varrho(\alpha)\varrho(\mathcal{R}(\alpha)),$$

where $\alpha$ runs over this part of the basis, which spans the holonomy algebra $\mathfrak{hol}$.

At this point we recall the splitting

$$V\otimes \wedge^k V \cong \wedge^{k+1}V \oplus \wedge^{k-1}V \oplus \wedge^{k,1}V$$

and consider the positive operator

$$(P^*P)(\omega) := \left((P^\nabla)^*\circ P^\nabla\right)(\omega) = -\sum_i \left(\nabla_{e_i}(P^\nabla_{e_i}(\omega)) - P^\nabla_{\nabla_{e_i}e_i}(\omega)\right)$$

PROPOSITION 5.7. *On a compact manifold, $(P^*P)\omega = 0 \Longleftrightarrow \omega$ is a $\nabla$-conformal Killing form.*

PROOF. The reverse direction is clear by definition. The forward direction is proven after we integrate over the manifold since $P^*P$ is a positive operator. □

PROPOSITION 5.8. *Let $\omega$ be a $k$-form. Then*

$$q(\mathcal{R})\omega - 2T\Box(\nabla\omega) = \frac{k}{k+1}\delta^\nabla d^\nabla\omega + \frac{n-k}{n-k+1}d^\nabla\delta^\nabla\omega - (P^*P)\omega.$$

PROOF. Differentiating equation (3.1), we get

$$(\nabla^*\nabla)(\omega) = \frac{1}{k+1}\delta^\nabla d^\nabla\omega + \frac{1}{n-k+1}d^\nabla\delta^\nabla\omega + (P^*P)(\omega).$$

Substituting this identity and $\Delta = d^\nabla\delta^\nabla + \delta^\nabla d^\nabla$ in (5.1), we obtain

$$q(\mathcal{R})\omega - 2T\Box(\nabla\omega) = \Delta\omega - (\nabla^*\nabla)\omega = \frac{k}{k+1}\delta^\nabla d^\nabla\omega + \frac{n-k}{n-k+1}d^\nabla\delta^\nabla\omega - (P^*P)(\omega),$$

proving the proposition. □

COROLLARY 5.9. *If $\omega$ is a $\nabla$-conformal Killing $k$-form on a compact manifold $(M, g)$, we have*

$$\int_M (q(\mathcal{R})\omega - 2T\Box(\nabla\omega), \omega) \geq 0.$$

PROOF. We have $(P^*P)\omega = 0$ for a $\nabla$-conformal Killing form $\omega$. What's left on the right hand side is obviously positive. □

Let's consider $B$, which appeared in the last proposition, as an operator $B : \wedge^k TM \longrightarrow \wedge^k TM$.
We derive some of its properties. First, a couple of definitions are needed:

DEFINITION 5.10. We call the torsion tensor $T$ of a metric connection *of Einstein type* if

$$\delta T = 0 \quad \text{and} \quad S(X,Y) = \lambda . g(X,Y),\ \lambda \in \mathbb{R}.$$

This definition appears first in [**AF14**] (Def 2.11). The assumption $S = \lambda . g$ is met in a few notable cases, such as nearly Kähler, nearly parallel $G_2$, and isotropy irreducible homogeneous spaces.

REMARK 5.11. Note that $\lambda$ is actually unique. It is determined by the torsion and one can easily compute it using Proposition A.1:

$$\mathrm{Ric}(X,Y) = \mathrm{Ric}^g(X,Y) - \frac{1}{4}S(X,Y) \quad \Longrightarrow \quad \frac{\mathrm{Scal}}{n}g = \frac{\mathrm{Scal}^g}{n}g - \frac{\lambda}{4}g \quad \Longrightarrow$$
$$\frac{3}{2n}||T||^2 = \frac{\lambda}{4} \quad \Longrightarrow \quad \lambda = \frac{6||T||^2}{n} > 0.$$

DEFINITION 5.12. Let $M = G/H$ be a homogeneous space with a metric connection $\nabla$ with skew torsion $T$. Define the Lie algebra

$$\mathfrak{g}_T := Lie\{X \lrcorner\ T | X \in TM\} \subset \mathfrak{so}(n),$$

generated by the elements $X \lrcorner\ T$, $X \in TM$.

This Lie algebra was first investigated by Agricola and Friedrich in [**AF04a**]. Let's define the mapping $\alpha : TM \to \Lambda^2 TM$ by

$$\alpha(X) = X \lrcorner\ T.$$

and its metric adjoint $\beta : \wedge^2 TM \longrightarrow TM$. We will contract a 2-form $\omega$ in a $k$-form $\eta$ by the rule

$$\omega \lrcorner\ \eta = \frac{1}{2}\sum_{i,j} \omega_{ij} e_j \lrcorner\ (e_i \lrcorner\ \eta).$$

Now we can write down an explicit expression for $\beta$, which is defined by the relation $\langle \eta, \alpha(X)\rangle = \langle \beta(\eta), X\rangle$:

$$\langle \eta, \alpha(X)\rangle = \langle \eta, X \lrcorner\ T\rangle = \eta \lrcorner\ (X \lrcorner\ T) = X \lrcorner\ (\eta \lrcorner\ T) = \langle X, \eta \lrcorner\ T\rangle\,,$$

arriving at $\beta(\eta) = \eta \lrcorner\ T$.

LEMMA 5.13. *It holds that* $\beta \circ \alpha(X) = \frac{1}{2}S(X)$.

PROOF. We compute directly:

$$\begin{aligned}(X \lrcorner\ T) \lrcorner\ T &= \frac{1}{2}\sum_{i,j}(X \lrcorner\ T)_{ij}(e_j \lrcorner\ (e_i \lrcorner\ T)) = \frac{1}{2}\sum_{i,j,k,p} T_{ijk} X_k T_{ijp} e_p \\ &= \frac{1}{2}\sum_{k,p} S_{kp} X_k e_p = \frac{1}{2}S(X).\end{aligned}$$

□

PROPOSITION 5.14. *The following hold for the operator* $B : \wedge^k TM \longrightarrow \wedge^k TM$*:*

(1) $B \equiv 0$ *for* $k = 1$,
(2) $B : \wedge^2 TM \longrightarrow \wedge^2 TM$ *is a symmetric operator,*
(3) *On a 2-form* $\omega$, $B(\omega) = \alpha \circ \beta(\omega)$.

*Moreover, if* $M$ *is homogeneous and* $T$ *is of Einstein type,* $B$ *is a multiple of a projection on the generators of* $\mathfrak{g}_T$.

PROOF. The first claim is obvious. For the second one, we see that for any $X, Y, Z, V \in TM$,

$$\langle B(X,Y), Z \wedge V\rangle = T(T(X,Y), Z, V) = \langle T(X,Y), T(Z,V)\rangle\,. \tag{5.3}$$

Even more, let's consider the set of $\frac{n(n-1)}{2}$ vectors $\{T(e_i,e_j)\}_{1 \le i < j \le n}$, where we embed $\mathbb{R}^n$ in $\mathbb{R}^{\frac{n(n-1)}{2}}$ by just adding enough zeroes on the end of the vectors. We must assume $n \geq 3$. Then their Gramian matrix is exactly the matrix of $B$ as an operator in $\wedge^2 TM$. As such, it is positive semi-definite and has rank equal to the number of linearly independent vectors in the considered set. This number equals $n - \dim(\mathrm{Ker}T)$, where

$$\mathrm{Ker}T := \{X \in TM | X \lrcorner\ T = 0\}.$$

Now $B$ has a kernel of dimension

$$\dim(\mathrm{Ker}B) = \frac{n(n-1)}{2} - n + \dim(\mathrm{Ker}T).$$

In particular, this number is strictly positive when $n > 3$. We verify the third claim directly:

$$\begin{aligned} B(\omega) &= \sum_k (e_k \lrcorner\ T) \wedge ((e_k \lrcorner\ T) \lrcorner\ \omega) = \sum_k \langle \omega, e_k \lrcorner\ T \rangle (e_k \lrcorner\ T) \\ &= \sum_k (\omega \lrcorner\ (e_k \lrcorner\ T)).(e_k \lrcorner\ T) = \sum_k (e_k \lrcorner\ (\omega \lrcorner\ T)).(e_k \lrcorner\ T) = (\omega \lrcorner\ T) \lrcorner\ T. \end{aligned}$$

Now suppose $T$ is of Einstein type with $S = \lambda.g$. Lemma 5.13 reads

$$\beta \circ \alpha(X) = \frac{1}{2} S(X) = \frac{\lambda}{2} X,$$

meaning that $\beta \circ \alpha = \frac{\lambda}{2} Id$. Compute

$$B^2(\omega) = \alpha \circ \beta \circ \alpha \circ \beta(\omega) = \frac{\lambda}{2} B(\omega).$$

If we define $B_\lambda := \frac{2}{\lambda} B$, we get $B_\lambda^2 = B_\lambda$. This means that $B_\lambda$ is a projection and its image is precisely the set of generators of $\mathfrak{g}_T$, when considered on a homogeneous space. $\square$

We will now try to simplify (5.2) by introducing a one-parameter family of connections and focusing on a specific value of the parameter.

**5.2. Laplacians for a one-parameter family of connections.** Let $\nabla$ be a metric connection with totally skew torsion $T$. Define a one-parameter family $\{\nabla^s\}_{s\in\mathbb{R}}$ in the space of metric connections by considering the line between $\nabla$ and $\nabla^g$:

$$\nabla^s_X Y = \nabla^g_X Y + 2sT(X,Y). \tag{5.4}$$

The connection $\nabla = \nabla^{\frac{1}{4}} =: \nabla^c$ that we used to define the family will be referred to as the *canonical* connection of the family. The torsion tensor of the new connection $\nabla^s$ is $T^s = 4sT$, so that $T^s$ is again totally skew, enabling us to use all formulas proven so far. Even more, we can express quantities and identities involving the pair $(\nabla^s, T^s)$ in terms of the canonical pair $(\nabla^c, T)$. For the curvature identities, this is done in the appendix.

This family has been successfully used in the literature to study rescaled differential operators, a central example being the Dirac operator on spin manifolds with the scaling factor $s = \frac{1}{12}$ for the connection (see [**AF04b**], [**Agr03**], [**Kos99**] ). One can obtain estimates of the lowest eigenvalue of the square of this rescaled Dirac operator ([**ABBK13**]).

We will make use of theorem B.1 of [**ABBK13**], according to which in the case when $\nabla^c T = 0$, the following identities hold:

$$(\nabla^s_X T)(U,V,W) = (2s - \frac{1}{2})\sigma_T(U,V,W,X),$$

$$\sum_{X,Y,Z} R^s(X,Y,Z,V) = s(6-8s)\sigma_T(X,Y,Z,V).$$

Here $X, Y, Z, U, V, W$ are vector fields. Moreover, $R^s(X,Y,Z,V) = R^s(Z,V,X,Y)$ and the fact that $\mathrm{Ric}^s$ is symmetric follow from the expression for the first Bianchi identity. In the appendix we give a formula for the general case and see that in the case of parallel torsion it reduces to the theorem we just cited.

REMARK 5.15. Observe that the first identity means that $T$ cannot simultaneously be parallel with respect to $\nabla^s$ for two different values of $s$ unless $\sigma_T = 0$, which is rarely the case.

Let's denote $d^s := d^{\nabla^s}$ and $\delta^s := \delta^{\nabla^s}$ for simplicity and form the corresponding Laplacians $\Delta^s = d^s\delta^s + \delta^s d^s$, etc. After $s$ is introduced, we see directly from the definitions that $\sigma_T$, $S$, and $B(\omega)$ transform to $16s^2\sigma_T$, $16s^2 S$, and $16s^2 B(\omega)$, as $T$ is replaced by $4sT$. Formula (5.2) then reads

$$\begin{aligned}\Delta^s\omega = {} & (\nabla^*\nabla)^s\omega + q(\mathcal{R}^g)\omega + 2s\ \delta T(\omega) + 4s^2 S(\omega) - 8sT\Box(\nabla^s\omega) \\ & - 4sdT\Box\omega + 8s^2\sigma_T\Box\omega + 8s^2 B(\omega).\end{aligned} \tag{5.5}$$

We want to derive separate formulas for $\Delta^s$ and $(\nabla^*\nabla)^s$ and see if we could rescale them in a clever way. Working by definition and using the identities from proposition B.1, we get

$$\begin{aligned}\Delta^s\omega &= d^s\delta^s\omega + \delta^s d^s\omega = (d - T^s\Diamond)(\delta - T^s\Box)\omega + (\delta - T^s\Box)(d - T^s\Diamond)\omega \\ &= \Delta^g\omega - 4s(d(T\Box\omega) + T\Box(d\omega) + \delta(T\Diamond\omega) + T\Diamond(\delta\omega)) + 16s^2(T\Box(T\Diamond\omega) + T\Diamond(T\Box\omega)) \\ &= \Delta^g\omega - 4s((dT)\Box\omega - \delta T\Diamond\omega + 4T\Box(\nabla^g\omega)) + 16s^2(2\sigma_T\Box\omega - B(\omega) - \frac{1}{2}S(\omega)) \\ &= \Delta^g\omega - 4s(dT)\Box\omega + 4s\delta T\Diamond\omega - 16sT\Box(\nabla^g\omega) + 32s^2\sigma_T\Box\omega - 16s^2 B(\omega) - 8s^2 S(\omega).\end{aligned}$$

Now, for the Bochner-Laplace operator $(\nabla^*\nabla)^s\omega = -\sum_i \nabla^s_{e_i}\nabla^s_{e_i}\omega + \sum_i \nabla^s_{\nabla^s_{e_i}e_i}\omega$ in the adapted frame $\nabla^s_{e_i}e_i = \nabla^g_{e_i}e_i + 2sT(e_i, e_i)$ implies $\nabla^g_{e_i}e_i = 0$, hence we have

$$(\nabla^*\nabla)^s\omega = -\sum_i \nabla^s_{e_i}\nabla^s_{e_i}\omega \quad \text{and} \quad (\nabla^*\nabla)^g\omega = -\sum_i \nabla^g_{e_i}\nabla^g_{e_i}\omega.$$

Again with the help of proposition B.1 we compute

$$\begin{aligned}(\nabla^*\nabla)^s\omega &= -\sum_i \nabla^s_{e_i}\nabla^s_{e_i}\omega = -\sum_i \nabla^s_{e_i}(\nabla^g_{e_i}\omega + 2s(e_i \lrcorner\ T)\Diamond\omega) \\ &= (\nabla^*\nabla)^g\omega - 2s(e_i \lrcorner\ T)\Diamond\nabla^g_{e_i}\omega - 2s\nabla^s_{e_i}(T(e_i, e_j) \wedge (e_j \lrcorner\ \omega)) \\ &= (\nabla^*\nabla)^g\omega - 4sT\Box(\nabla^g\omega) - 2s\nabla^s_{e_i}T(e_i, e_j, e_k)e_k \wedge (e_j \lrcorner\ \omega) - 2sT(e_i, e_j) \wedge (e_j \lrcorner\ \nabla^s_{e_i}\omega) \\ &= (\nabla^*\nabla)^g\omega - 4sT\Box(\nabla^g\omega) + 2s\delta T(e_j, e_k)e_k \wedge (e_j \lrcorner\ \omega) - 4sT\Box(\nabla^s\omega) \\ &= (\nabla^*\nabla)^g\omega - 8sT\Box(\nabla^g\omega) + 2s\delta T\Diamond\omega + 8s^2\sigma_T\Box\omega - 8s^2 B(\omega) - 4s^2 S(\omega).\end{aligned}$$

We arrive at the expressions

$$\Delta^s\omega = \Delta^g\omega - 4s(dT)\Box\omega + 4s\delta T\Diamond\omega - 16sT\Box(\nabla^g\omega) + 32s^2\sigma_T\Box\omega - 16s^2 B(\omega) - 8s^2 S(\omega), \tag{5.6}$$

$$(\nabla^*\nabla)^s\omega = (\nabla^*\nabla)^g\omega - 8sT\Box(\nabla^g\omega) + 2s\delta T\Diamond\omega + 8s^2\sigma_T\Box\omega - 8s^2 B(\omega) - 4s^2 S(\omega). \tag{5.7}$$

One can also combine (5.6), proposition B.1, and $\Delta^g = \Delta^g_L$ to see how the Lichnerowicz Laplacian rescales:

$$\Delta^s_L\omega = \Delta^g_L\omega - 4s(dT)\Box\omega + 4s\delta T\Diamond\omega - 8sT\Box(\nabla^g\omega) + 16s^2\sigma_T\Box\omega. \tag{5.8}$$

Proposition 5.4 gives us

$$q(\mathcal{R}^s)\omega = q(\mathcal{R}^g)\omega + 2s\delta T\Diamond\omega + 4s^2 S(\omega) - 4sdT\Box\omega + 8s^2\sigma_T\Box\omega + 8s^2 B(\omega). \tag{5.9}$$

Our goal is to derive a Weitzenböck-type formula where the first order derivative term $T\Box(\nabla^g\omega)$ vanishes and now we are finally in position to state the following:

THEOREM 5.16. *Let $\nabla^s$ be the family of metric connections with totally skew-symmetric torsion defined by* (5.4)*. Then the Weitzenböck-type formula*

$$\Delta^{\frac{s}{2}}\omega = (\nabla^*\nabla)^s\omega + q(\mathcal{R}^g)\omega - 2sdT\Box\omega + 2s^2 S(\omega) + 4s^2 B(\omega) \tag{5.10}$$

*relates the Bochner-Laplace operator of $\nabla^s$ to the Laplacian $\Delta^{\frac{s}{2}}$ of $\nabla^{\frac{s}{2}}$, both acting on a $k$-form $\omega$.*

PROOF. We want to find a combination of equations (5.6) and (5.7) that eliminates the first order term $T\Box(\nabla^g\omega)$. This means that the value of the parameter $s$ in (5.6) has to equal one half of the value of the parameter in (5.7). Having this and the classical relation $\Delta^g\omega = (\nabla^*\nabla)^g\omega + q(\mathcal{R}^g)\omega$ in mind, (5.10) follows directly. □

**5.3. Laplacians acting on 1-forms.** Here we will take a closer look at the case when $\omega$ is a 1-form. We recall some of the simplifications that arise:

- the $\Diamond$ operation is just contraction,
- the $\Box$ operation always returns zero,
- the operator $B$ is identically zero,
- $q(\mathcal{R})\omega = \mathrm{Ric}(\omega)$.

We work directly in the context of the 1-parameter family of connections defined in the previous subsection. When we want to emphasize a result about the canonical connection $\nabla^{\frac{1}{4}}$, we omit the superscript and write simply $\nabla$. Having said this, equation (5.5) takes the form

$$\Delta^s\omega = \Delta^s_L\omega - 8sT\Box(\nabla^s\omega) = (\nabla^*\nabla)^s\omega + \mathrm{Ric}^s(\omega) - 8sT\Box(\nabla^s\omega).$$

PROPOSITION 5.17. *For a 1-form $\omega$ the following formula holds:*

$$\Delta^s\omega = \Delta^s_L\omega - 4s\, d^s\omega \lrcorner\ T.$$

*We emphasize the relation*

$$2T\Box(\nabla^s\omega) = d^s\omega \lrcorner\ T. \tag{5.11}$$

PROOF. We just need to verify (5.11). Indeed, observe that

$$d^s\omega(e_i, e_j) = (\nabla^s_{e_i}\omega)(e_j) - (\nabla^s_{e_j}\omega)(e_i),$$

hence

$$2T\Box(\nabla^s\omega) = \sum_{i,j} e_j \lrcorner\ (e_i \lrcorner\ T).(\nabla^s_{e_i}\omega)(e_j) = \frac{1}{2}\sum_{i,j} d^s\omega(e_i, e_j).(e_j \lrcorner\ (e_i \lrcorner\ T)) = d^s\omega \lrcorner\ T,$$

having taken into account the anti-symmetry with respect to the indices $i, j$. □

We can also reduce equation (5.10) to the case of a 1-form. It now reads

$$\Delta^{\frac{s}{2}}\omega = (\nabla^*\nabla)^s\omega + \mathrm{Ric}^g(\omega) + 2s^2S(\omega) = (\nabla^*\nabla)^s\omega + \mathrm{Ric}^{\frac{s}{\sqrt{2}}}(\omega) - \sqrt{2}s\delta T\Diamond\omega.$$

Equation (5.6) will also come to use as soon as we try to compare our Hodgle Laplacian to the Riemannian one. It looks like

$$\Delta^s\omega = \Delta^g\omega + 4s\delta T\Diamond\omega - 16sT\Box(\nabla^g\omega) - 8s^2S(\omega).$$

Now, using (5.11) we can draw a conclusion under some natural assumptions, but first we prove a lemma.

PROPOSITION 5.18. *Let $(M^n, g)$ be a Riemannian manifold, endowed with a metric connection $\nabla^s$ with totally skew-symmetric torsion $T^s = 4sT$, which is of Einstein type, i.e. it satisfies*

$$S(X,Y) = \lambda g(X,Y),\ \lambda \in \mathbb{R}; \quad \delta T = 0.$$

*Then any closed or $d^s$-closed 1-form is an eigenform of the Hodge Laplacian $\Delta^g$ exactly when it is an eigenform of $\Delta^s$.*

PROOF. Under the assumptions of the proposition equation (5.6) takes the form

$$(\Delta^s - \Delta^g)\omega = 8s^2\lambda\omega - 8s\, d^g\omega \lrcorner\ T$$

for a 1-form $\omega$. This proves the proposition in case $\omega$ is closed. To confirm the other case we calculate the difference

$$d\omega \lrcorner\ T - d^s\omega \lrcorner\ T = (T^s\Diamond\omega) \lrcorner\ T = 4s\,(\omega \lrcorner\ T) \lrcorner\ T = 2s(\omega \lrcorner\ S) = 2s\lambda\,\omega,$$

according to Lemma 5.13. Now the proposition holds for $\omega$ $d^s$-closed as well. □

PROPOSITION 5.19. *Let $(M^6, J, g)$ be a nearly Kähler manifold with* $\mathrm{Scal}^g = 30$*. If $X$ is a Killing vector field, let $X^\flat$ and $(JX)^\flat$ be the respective dual 1-forms. They are eigenforms of the Laplacian $\Delta = d^\nabla\delta^\nabla + \delta^\nabla d^\nabla$ of the characteristic connection $\nabla$. More precisely,*

$$\Delta X^\flat = 16X^\flat, \quad \Delta(JX)^\flat = 8(JX)^\flat.$$

PROOF. Here we make extensive use of the results, obtained in [**MS10**]. There, the authors prove that on such a manifold every Killing vector field $X$ satisfies

$$\Delta^g X^\flat = 10X^\flat, \qquad \Delta^g(JX)^\flat = 18(JX)^\flat,$$
$$\Delta_L X^\flat = 12X^\flat, \qquad \Delta_L(JX)^\flat = 12(JX)^\flat.$$

In the current setting we have $\delta T = 0$ and $S = 4g$, hence (5.8) reads

$$\Delta_L\omega = \Delta^g\omega - 2T\square(\nabla^g\omega)$$

on a 1-form. This immediately implies $2T\square(\nabla^g X^\flat) = -2X^\flat$ and $2T\square(\nabla^g(JX)^\flat) = 6(JX)^\flat$ for a Killing vector field $X$. The first equation of proposition B.1 reads $T\square(\nabla\omega) = T\square(\nabla^g\omega) - \omega$, hence

$$\Delta X^\flat = \Delta_L X^\flat - 2T\square(\nabla X^\flat) = 12X^\flat + 4X^\flat = 16X^\flat,$$
$$\Delta(JX)^\flat = \Delta_L(JX)^\flat - 2T\square(\nabla(JX)^\flat) = 12(JX)^\flat - 4(JX)^\flat = 8(JX)^\flat.$$

Thus, the proposition holds. □

Consider a $d^\nabla$-closed conformal-Killing 1-form $\omega$. It satisfies $\nabla_X\omega = -\frac{1}{n}X \wedge \delta\omega$, but in this case $\delta\omega = f$ is just a function and the wedge product is redundant.

PROPOSITION 5.20. *Let $(M^n, g)$ be a Riemannian manifold, endowed with a metric connection $\nabla$ with totally skew-symmetric torsion $T$. Let $\omega$ be a 1-form, which satisfies $\nabla_X\omega = -\frac{1}{n}f.X$ for any vector field $X$ and function $f$. Then $f = \delta\omega$ and*

$$\Delta\omega = \frac{n}{n-1}\mathrm{Ric}(\omega)$$

*holds. In particular, $\omega$ is an eigenform of $\Delta$ if the manifold is $\nabla$-Einstein.*

PROOF. First note that since $\omega$ is a 1-form, $T\square\omega = 0$. Hence, $\delta\omega = \delta^\nabla\omega$. Also $d^\nabla\omega = 0$ holds. Now

$$\Delta\omega = df = (\nabla^*\nabla)(\omega) + \mathrm{Ric}(\omega).$$

We used the definition of $\Delta$ and the fact that for a function $f$ we have $df = d^\nabla f$. Further,

$$(\nabla^*\nabla)\omega = -\sum_i \nabla_{e_i}\nabla_{e_i}\omega = \frac{1}{n}\sum_i \nabla_{e_i}(f.e_i) = \frac{1}{n}\sum_i (\nabla_{e_i}f)e_i = \frac{1}{n}df.$$

Finally, $\frac{n-1}{n}df = \mathrm{Ric}(\omega)$, hence $\Delta\omega = \frac{n}{n-1}\mathrm{Ric}(\omega)$, concluding the proof. □

A family of forms of similar type are studied by Moroianu in [**Mor07**]. Take a Gradient Conformal Vector Field $X$. That is $X$, such that its dual 1-form $\omega$ is conformal and exact:

$$\nabla^g_Y\omega = \frac{1}{2}Y \lrcorner\ d\omega - \frac{1}{n}Y \wedge \delta\omega, \quad \omega = dh$$

for an arbitrary vector field $Y$ and a function $h$. We now easily simplify to $\nabla^g_Y\omega = -\frac{1}{n}f.Y$, where $f = \Delta^g h$.

EXAMPLE 5.21. One rather trivial example is the 1-form

$$\eta := \sum_i x_i.e_i$$

on a Lie group. Here $\{e_i\}$ is the dual to a $\nabla$-parallel basis with respect to a metric connection $\nabla$ with non-zero totally skew torsion $T$, and $\{x_i\}$ are just the coordinate functions. It is obvious that $\eta$ is not parallel itself, since $\nabla_{e_i}\eta = e_i$, but it fulfills $d^\nabla\eta = 0$. Moreover, we easily compute $\delta^\nabla\eta = -n$, where $n$ is the dimension of the Lie group, hence for an arbitrary vector field $X = \sum_i a_i.e_i$ it holds that:

$$\nabla_X \eta = \sum_i a_i.\nabla_{e_i}\eta = \sum_i a_i.e_i.\frac{-1}{n}.(-n) = -\frac{1}{n}\,\delta^{\nabla}\eta\,.X.$$

## 6. Applications

**6.1. Yano and Bochner's techniques.** In this section we give a short description of the general ideas underlying some of the previous research on Killing and conformal Killing forms.

First we refer to [**YB54**] and [**Y70**] for the following:

THEOREM 6.1. *Hopf principle: let* $L\phi := g^{ij}\frac{\partial^2\phi}{\partial x^i \partial x^j} + h^i \frac{\partial \phi}{\partial x^i} \geq 0$ *on a compact manifold* $(M^n, g)$. *Then* $L\phi = 0$.

This theorem may be applied to any operator whose symbol is the metric, such as the elliptic operator $\nabla^*\nabla$. The introduction of torsion only affects the first order part, so the Hopf principle actually applies to all Laplace operators we defined in section 4. The function that is usually of interest is the length function of an interesting object, e.g. a special form: Killing, conformal Killing, harmonic, etc. One combines this with the technique of integrating Weitzenböck type formulas (as in equation (5.1)) over the manifold. In our notation that reads:

$$\int_M \langle \Delta\omega - (\nabla^*\nabla)\omega - q(\mathcal{R})\omega + 2T\Box(\nabla\omega), \omega\rangle \, dV = 0 \Longleftrightarrow$$

$$\int_M \langle \Delta\omega, \omega\rangle - \langle(\nabla^*\nabla)\omega, \omega\rangle - \langle q(\mathcal{R})\omega, \omega\rangle + 2\,\langle T\Box(\nabla\omega), \omega\rangle \, dV = 0 \Longleftrightarrow$$

$$\int_M ||d^{\nabla}\omega||^2 + ||\delta^{\nabla}\omega||^2 - ||\nabla\omega||^2 - \frac{1}{2}\sum_{i,j}\langle \varrho(e_i\wedge e_j)\varrho(\mathcal{R}(e_i\wedge e_j))\omega, \omega\rangle + 2\,\langle T\Box(\nabla\omega), \omega\rangle \, dV = 0$$

Those techniques may be used to formulate existence and non-existence theorems about special forms after imposing conditions on the curvature and torsion terms, see [**YB54**]. To that end, we can further simplify the curvature expression as follows (we assume summation over repeated indices to simplify notation):

$$\int_M -\frac{1}{2}\langle \varrho(e_i\wedge e_j)\varrho(\mathcal{R}(e_i\wedge e_j))\omega, \omega\rangle \, dV = \int_M -\frac{1}{4}\langle R_{ijkl}\varrho(e_i\wedge e_j)\varrho(e_k\wedge e_l)\omega, \omega\rangle \, dV =$$

$$\int_M \frac{1}{4}R_{ijkl}\,\langle \varrho(e_k\wedge e_l)\omega, \varrho(e_i\wedge e_j)\omega\rangle \, dV = \int_M R_{ijkl}\,\langle e_l\wedge(e_k \lrcorner\ \omega), e_j\wedge(e_i\lrcorner\ \omega)\rangle \, dV =$$

$$\int_M R_{ijkl}\,\langle e_k\lrcorner\ \omega, \delta_{lj}(e_i\lrcorner\ \omega) - e_j\wedge(e_l\lrcorner\ e_i\lrcorner\ \omega)\rangle \, dV =$$

$$-\int_M \mathrm{Ric}_{ik}\,\langle e_k\lrcorner\ \omega, e_i\lrcorner\ \omega\rangle + R_{ijkl}\,\langle e_j\lrcorner\ e_k\lrcorner\ \omega, e_l\lrcorner\ e_i\lrcorner\ \omega\rangle \, dV =$$

$$-\int_M S(\mathrm{Ric})_{ik}\,\langle e_k\lrcorner\ \omega, e_i\lrcorner\ \omega\rangle + (R_{im})_{ijkl}\,\langle e_j\lrcorner\ e_k\lrcorner\ \omega, e_l\lrcorner\ e_i\lrcorner\ \omega\rangle \, dV.$$

The final equality holds because we can split the Ricci tensor into its symmetric and antisymmetric parts in the first summand and only the symmetric part survives, since it is multiplied by a symmetric expression in the inner product after it. Similarly, the symmetries of the second summand imply, that only the 4-form part of the curvature tensor will impact the result, i.e. the image $R_{im}$ of the Bianchi map, considered in Appendix A. We can write

$$\int_M \langle q(\mathcal{R})\omega, \omega\rangle \, dV = \int_M \left(\mathrm{Ric}^g - \frac{1}{4}S\right)_{ik}\langle e_i\lrcorner\ \omega, e_k\lrcorner\ \omega\rangle \, dV +$$

$$+\int_M \left(\frac{1}{4}d^{\nabla}T + \frac{1}{3}\sigma_T\right)_{ijkl}\langle e_j\lrcorner\ e_k\lrcorner\ \omega, e_l\lrcorner\ e_i\lrcorner\ \omega\rangle \, dV.$$

In the case of zero torsion only the Ricci tensor remains. We can also simplify the expression

$$\int_M 2\,\langle T\Box(\nabla\omega),\omega\rangle = \int_M \left\langle \varrho(e_j \lrcorner\ T)(\nabla_{e_j}\omega),\omega\right\rangle = -\int_M \left\langle(\nabla_{e_j}\omega),\varrho(e_j \lrcorner\ T)\omega\right\rangle$$
$$= \sum_j \int_M \left\langle \nabla_j\omega, \nabla^g_j\omega - \nabla_j\omega\right\rangle.$$

Now we can build on the available results using our setting and the idea of a 1-parameter family of connections.

We also cite a couple of observations available in [**Y70**] (p.68 Prop 2.1 & p.69 Prop 2.2):

PROPOSITION 6.2. *A $k$-form $\omega$ is $\nabla$-Killing exactly when $\delta^\nabla\omega = 0$ and $\Delta\omega = (k+1)\delta^\nabla(\nabla\omega)$. Further, a $d^\nabla$-exact, $\nabla$-Killing form on a compact manifold is zero.*

PROOF. One verifies the second claim as follows: $\omega = d^\nabla\eta$ and $\delta^\nabla\omega = 0$ hold by assumption. Now

$$\int_M \left\langle \delta^\nabla d^\nabla\eta,\eta\right\rangle dV = 0 \Longrightarrow d^\nabla\eta = 0 = \omega. \qquad \Box$$

Another interesting application of the introduced machinery is the following:

THEOREM 6.3. *Let $(M^n, g)$ be a compact Riemannian manifold endowed with a family of connections with totally skew-symmetric torsion $\nabla^s = \nabla^g + \frac{1}{2}T^s$. Assume that the Ricci tensor of $\nabla^s$ is symmetric, which is equivalent to $\delta T^s = 0$, and that we can fix $s$ such that $\mathrm{Ric}^{s\sqrt{2}} \geq 0$. Then for $t \in \mathbb{R}, |t| \leq |s|$, $M^n$ admits no non-trivial $\nabla^t$-harmonic 1-forms. In particular, $M^n$ admits no $\nabla^s$-harmonic 1-forms.*

PROOF. We use equation (5.3) with $\delta T^s = 0$ taken into account:

$$\Delta^s\omega = (\nabla^*\nabla)^{2s}\omega + \mathrm{Ric}^{s\sqrt{2}}(\omega).$$

The condition $\delta T^s = 0$ also implies that the endomorphism $\mathrm{Ric}^{s\sqrt{2}}$ is symmetric, so if $\omega$ is $\nabla^s$-harmonic, integrating the identity leads us directly to

$$0 = \int_{M^n}\left(\left\langle(\nabla^*\nabla)^{2s}\omega,\omega\right\rangle + \left\langle \mathrm{Ric}^{s\sqrt{2}}(\omega),\omega\right\rangle\right) = \int_{M^n}\left(||\nabla^{2s}\omega||^2 + \left\langle \mathrm{Ric}^{s\sqrt{2}}(\omega),\omega\right\rangle\right) \geq 0.$$

Equality is attained only when $\omega$ is identically zero, so there are no non-trivial $\nabla^s$-harmonic 1-forms. We have some freedom regarding the parameter $s$. Recall equation (A.6). In our case it leads to

$$\mathrm{Ric}^{s\sqrt{2}}(\omega) = \mathrm{Ric}^g(\omega) - 8s^2 S(\omega).$$

We know that $S$, viewed as an endomorphism, is positive-definite, so whenever $|t| \leq |s|$, we will have $\mathrm{Ric}^{t\sqrt{2}} \geq \mathrm{Ric}^{s\sqrt{2}} \geq 0$, hence the non-existence of $\nabla^t$-harmonic forms. $\Box$

REMARK 6.4. Again the main examples, where the conditions of the last propositions are fulfilled, are nearly Kähler manifolds, nearly parallel $G_2$-manifolds, and isotropy irreducible homogeneous spaces. Let $(M^n, g)$ be such a manifold. $M^n$ is Einstein, admits a characteristic connection with torsion $T$ such that $\delta T = 0$, and there is a positive constant $\lambda$ with $S(X,Y) = \lambda g(X,Y)$. Then:

$$\mathrm{Ric}^s(X,Y) = \mathrm{Ric}^g(X,Y) - 4s^2 S(X,Y) = \left(\frac{\mathrm{Scal}^g}{n} - 4\lambda s^2\right) g(X,Y).$$

Now the condition $\mathrm{Ric}^{s\sqrt{2}} \geq 0$ reads

$$\frac{\mathrm{Scal}^g}{n} - 8\lambda s^2 \geq 0, \quad \text{or equivalently} \quad |s| \leq \sqrt{\frac{\mathrm{Scal}^g}{8\lambda n}}.$$

We have previously determined $\lambda = \frac{6||T||^2}{n}$, arriving at

$$|s| \leq \sqrt{\frac{\mathrm{Scal}^g}{48||T||^2}}.$$

CHAPTER 2

# Manifolds with split torsion

In this chapter we consider reductive homogeneous spaces $M = G/H$ with decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{m} = \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_p$ splits into $p$ not necessarily inequivalent $\mathfrak{h}$-modules. The isotropy decomposition of $\mathfrak{m}$ into $p$ $\mathfrak{h}$-irreducible modules has naturally been an object of investigation in the mathematical literature for a long time. In the case $p = 1$, the resulting *isotropy irreducible homogeneous spaces* are very well understood and have already been classified (see [**Wo68**], [**WZ91**]). There has also been a great deal of interest in this kind of splitting for arbitrary $p$ in the search of Einstein metrics on homogeneous spaces. A number of authors, including Arvanitoyeorgos, Chrysikos, Nikonorov, Sakane and others, have contributed greatly to this topic (see, for example: [**AC09**], [**AC10**], [**ACS13**], [**Arv93**], [**CN19**], [**Sa99**], ...).

Apart from $p = 1$, the case $p = 3$ has probably been studied in most detail. It is precisely the setting in which locally 3-symmetric spaces, which are a natural generalization of locally symmetric spaces, arise. These spaces appear with different names in the literature and have recently been classified by Nikonorov in [**Nik16**], where they are called *generalized Wallach spaces*, which is also the name we are going to use for them. The notion of *split torsion* is defined in [**ABBK13**] and reappears in [**AK14**], where its presence enables the improvement of an eigenvalue estimate for the Dirac operator with torsion. Here, we use it as a starting point and build on it to define the notion of *manifolds admitting split torsion* and refine it further, introducing *non-degenerate* and *weakly non-degenerate* split torsion. In both cases, we present examples, which are taken from the classification of generalized Wallach spaces [**Nik16**]. These examples are tightly related to the normed division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$, which is made more evident at the end of the chapter, where we elaborate on the Wallach spaces in more detail.

## 1. Preliminaries

**1.1. Split torsion.** Let $(M = G/H, g)$ be a Riemannian homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, such that $\mathfrak{m} = \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_p$ splits into $p$ irreducible $\mathfrak{h}$-invariant modules and assume $p \geq 3$. Let $\nabla$ be an invariant metric connection on $M$ with totally skew torsion tensor $T$ (recall def. 1.8 and eqn. 1.6). We refine the notion of skew torsion in the following:

Definition 1.1. We call the totally skew torsion tensor $T$ *of split type* and refer to $T$ as *split torsion* if whenever $T(e_i, e_j, e_k) \neq 0$, the basis elements $e_{i,j,k}$ belong to different components of $\mathfrak{m}$.

Definition 1.2. We call the Riemannian homogeneous space $(M, g, \nabla)$ a *manifold with split torsion* if it admits an invariant metric connection $\nabla$ with totally skew torsion of split type.

A series of examples of manifolds with split torsion that we have already encountered are the Stiefel manifolds $V_{n,2}$.

In some cases, we can take advantage of the split condition and condense the information encoded in the torsion tensor in a simpler mapping $\gamma$. Assume that we can write

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3,$$

where $\mathfrak{m}_i$ are not necessarily irreducible and the split condition for the tensor is fulfilled with respect to this decomposition. If $p = 3$, this is exactly the irreducible decomposition and the assumption is empty, but in general it is not.

Definition 1.3. Let $X_i \in \mathfrak{m}_i$ and define $\gamma : \mathfrak{m}_1 \times \mathfrak{m}_2 \times \mathfrak{m}_3 \longrightarrow \mathbb{R}$ via

$$\gamma(X_1, X_2, X_3) := T(X_1, X_2, X_3).$$

The mapping $\gamma$ is sufficient to recover the full torsion tensor $T$, so we can translate algebraic properties of $\gamma$ to geometric properties. We introduce two non-degeneracy notions:

Definition 1.4. Let $X_i \in \mathfrak{m}_i$ and define a mapping $\Gamma_{X_1} : \mathfrak{m}_2 \longrightarrow \mathfrak{m}_3$ by

$$\langle \Gamma_{X_1}(X_2), X_3 \rangle := \gamma(X_1, X_2, X_3), \tag{1.1}$$

where $\langle -, - \rangle$ is an $\mathfrak{h}$-invariant scalar product on $\mathfrak{m}_3$. We call $\gamma$:

- *non-degenerate*, if for any fixed non-zero vector $X_1 \in \mathfrak{m}_1$, $\Gamma_{X_1}$ is an isomorphism. Such a trilinear map $\gamma$ is also called a *triality*. In this case we call the corresponding split torsion tensor $T$ *non-degenerate*;
- *weakly non-degenerate*, if there exists a non-zero vector $X_1 \in \mathfrak{m}_1$, such that $\Gamma_{X_1}$ is an isomorphism. In this case we call the corresponding split torsion tensor $T$ *weakly non-degenerate.*

Remark 1.5. We only chose $X_1 \in \mathfrak{m}_1$ in the definition for concreteness. One can always rename the modules from the very beginning, or equivalently use $X_2 \in \mathfrak{m}_2$ or $X_3 \in \mathfrak{m}_3$ in the definition.

Proposition 1.6. *The mapping $\Gamma_{X_1}$, defined in Def. 1.4, is $\mathfrak{h}$-equivariant iff $X_1$ is $\mathfrak{h}$-invariant. In this case, the following statements hold:*

- *If $\mathfrak{m}_2$ is irreducible, $\Gamma_{X_1}$ is injective;*
- *If $\mathfrak{m}_3$ is irreducible, $\Gamma_{X_1}$ is surjective.*

Proof. Confirming the necessary and sufficient condition is a standard check. The following statements are a direct consequence from the $\mathfrak{h}$-invariance of the kernel and image of $\Gamma_{X_1}$. □

**1.2. Naturally reductive homogeneous spaces.** Consider the homogeneous space $M = G/H$, where $G$ and $H$ are compact with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. We call $M$ reductive if $\mathfrak{h}$ admits a complement $\mathfrak{m}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. In this case, we identify $\mathfrak{m}$ with the tangent space $T_oM$ to $M$ at a fixed point $o \in M$, which we call the origin. We can make an arbitrary choice since the manifold is homogeneous, and one usually takes $o := eH$, the coset of $G/H$ containing the unit element $e \in G$. A metric $g$ on $M$ would thus correspond to a scalar product $\langle -, - \rangle$ on $\mathfrak{m}$, which is invariant under the isotropy action, i.e. which fulfills

$$\langle [Z, X]_{\mathfrak{m}}, Y \rangle + \langle X, [Z, Y]_{\mathfrak{m}} \rangle = 0 \tag{1.2}$$

for $X, Y \in \mathfrak{m}$ and $Z \in \mathfrak{h}$. Having $\mathfrak{m}$ reductive renders the projections of the commutators onto $\mathfrak{m}$ redundant while $Z \in \mathfrak{h}$. However, they are necessary if we let $Z$ vary in $\mathfrak{g}$, which is the definition of a naturally reductive metric.

Definition 1.7. A metric $g$ on $M$ such that (1.2) holds for any $X, Y \in \mathfrak{m}$ and $Z \in \mathfrak{g}$ is called *naturally reductive.* The Riemannian manifold $(M = G/H, g)$ is called a *naturally reductive homogeneous space.*

In other words, we require the metric to be not only $ad(\mathfrak{h})$-invariant, but also $ad(\mathfrak{g})$-invariant. The expression on the left hand side of (1.2) is the definition of the tensor $2U(X, Y, Z) = 2\langle U(X, Y), Z \rangle$ on $M$. Fixing the metric $g$ also fixes the Levi-Civita connection $\nabla^g$. Recall that if we have a metric connection $\nabla$ with totally skew torsion $T$, we can consider the 1-parameter family

$$\nabla^s := \nabla^g + 2sT, \quad s \in \mathbb{R}.$$

Assume that $g$ is naturally reductive. There is a natural candidate for a totally skew torsion tensor: $T^c(X, Y) := -[X, Y]_{\mathfrak{m}}$. This is the torsion of a special connection for the homogeneous space—the so-called canonical connection $\nabla^c$. It is totally skew, as the naturally reductive property (1.2) shows directly. Generating the 1-parameter family of connections $\nabla^s = \nabla^g + 2sT^c$, the canonical connection $\nabla^c$ corresponds to the value $s = \frac{1}{4}$, i.e. $\nabla^c = \nabla^{\frac{1}{4}}$. It satisfies the additional properties $\nabla^c T^c = 0$ and $\nabla^c R^c = 0$.

REMARK 1.8. One can equivalently define a naturally reductive homogeneous space as a connected, simply connected Riemannian manifold, admitting a metric connection with totally skew torsion, which renders its torsion and curvature tensors parallel. From this point of view, symmetric spaces are naturally reductive homogeneous spaces whose canonical connection coincides with their Levi-Civita connection.

REMARK 1.9. According to a theorem by Wang, connections on homogeneous spaces let themselves be described through maps $\Lambda : \mathfrak{m} \to \mathfrak{so}(\mathfrak{m})$. The canonical connection corresponds to the zero map and the so-called canonical affine connection—to the map $\Lambda(X)Y = \frac{1}{2}[X, Y]_{\mathfrak{m}}$. The second one is used to describe the Levi-Civita connection, which then corresponds to the map

$$LC(X)Y = \Lambda(X)Y + U(X, Y).$$

In [**Agr06**] we find another useful property of naturally reductive homogeneous spaces:

PROPOSITION 1.10. *Let $M = G/H$ be a naturally reductive homogeneous space with canonical connection $\nabla^c$. The holonomy algebra $\mathfrak{hol}^c$ of the canonical connection is a Lie subalgebra of the Lie algebra $\mathfrak{h}$ of the isotropy group $H$.*

### 1.3. Generalized Wallach spaces.

DEFINITION 1.11. A reductive homogeneous space $M = G/H$, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, with three isotropy summands $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$, satisfying the condition

$$[\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{h},$$

is called a *generalized Wallach space.*

REMARK 1.12. The generalized Wallach spaces are also known as *three-locally-symmetric spaces* or *locally 3-symmetric spaces.*

As a direct consequence of the definition, we have the following fundamental property:

COROLLARY 1.13. *On a generalized Wallach space, the relations*

$$[\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{m}_k$$

*for $\{i, j, k\} = \{1, 2, 3\}$ are satisfied.*

PROOF. This is easily seen using the Killing form of $\mathfrak{g}$. □

Generalized Wallach spaces are good candidates for manifolds with split torsion.

PROPOSITION 1.14. *A naturally reductive generalized Wallach space is a manifold with split torsion.*

PROOF. The canonical connection $\nabla^c$ of a naturally reductive homogeneous space $(M, g)$ has torsion, given by $T^c(X, Y) = -[X, Y]_{\mathfrak{m}}$, which is parallel, totally skew, and according to the last corollary—of split type when $M$ is generalized Wallach. Thus, $(M, g, \nabla)$ is manifold with split torsion. In particular, this is the case when $M = G/H$ with $G$ a compact, simple Lie group. □

Note that so far we haven't required the modules $\mathfrak{m}_i$ to be irreducible, so we are actually working in the case $p \geq 3$. In case they are irreducible, $p = 3$ and we can turn to the classification result of [**Nik16**]:

THEOREM 1.15. *Let $G/H$ be a connected and simply connected compact homogeneous space. Then $G/H$ is a generalized Wallach space if and only if it is one of the following types:*

1. *$G/H$ is a direct product of three irreducible symmetric spaces of compact type;*
2. *The group $G$ is simple and the pair $(\mathfrak{g}, \mathfrak{h})$ is one of the pairs in Table 1 (the embedding of $\mathfrak{h}$ in $\mathfrak{g}$ is determined by the following requirement: the corresponding pairs $(\mathfrak{g}, \mathfrak{k}_i)$ and $(\mathfrak{k}_i, \mathfrak{h})$, $i = 1, 2, 3$, in Table 2 are symmetric);*

(3) *$G = F \times F \times F \times F$ and $H = \operatorname{diag}(F) \subset G$ for some connected simply connected compact simple Lie group $F$, with the following description on the Lie algebra level:*

$$(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{f} \oplus \mathfrak{f} \oplus \mathfrak{f} \oplus \mathfrak{f}, \operatorname{diag}(\mathfrak{f}) = \{(X, X, X, X) | X \in \mathfrak{f}\}),$$

*where $\mathfrak{f}$ is the Lie algebra of $F$, and (up to permutation) $\mathfrak{p}_1 = \{(X, X, -X, -X)|X \in \mathfrak{f}\}$, $\mathfrak{p}_2 = \{(X, -X, X, -X)|X \in \mathfrak{f}\}$, $\mathfrak{p}_3 = \{(X, -X, -X, X)|X \in \mathfrak{f}\}$.*

The examples we are interested in are generalized Wallach spaces of types (2) and (3). We do not produce tables 1 and 2, that are referred to in the theorem. Table 1 consists of three 3-parameter and two 1-parameter families, as well as ten isolated pairs, where $\mathfrak{g}$ is an exceptional Lie algebra. Table 1 also contains the dimensions of the modules $\mathfrak{m}_i$ for each of the given pairs $(\mathfrak{g}, \mathfrak{h})$.

According to Prop. 1.6, $\Gamma_{X_1}$ is $\mathfrak{h}$-equivariant on a generalized Wallach space only if $\mathfrak{m}_1$ is 1-dimensional. A quick scan of table 1 tells us that for type (2) this is only possible on the Stiefel manifolds $V_{n,2}$, which appear as a subfamily of $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{so}(k + l + m), \mathfrak{so}(k) \oplus \mathfrak{so}(l) \oplus \mathfrak{so}(m))$. For generalized Wallach spaces of type (3), the dimensions of all modules $\mathfrak{m}_i$ equal $\dim F$, which cannot be 1.

## 2. Weakly non-degenerate split torsion

A simple consequence of prop. 1.6 is the following:

PROPOSITION 2.1. *Let $(M, g, \nabla)$ be a manifold with weakly non-degenerate split torsion $T$. Let $\xi \in \mathfrak{m}_1$ be such that $\Gamma_\xi : \mathfrak{m}_2 \longrightarrow \mathfrak{m}_3$ is an isomorphism and assume further that $\nabla\xi = 0$. Then:*

- *$\mathfrak{m}_2$ and $\mathfrak{m}_3$ are isomorphic as $\mathfrak{h}$ representations;*
- *$\xi \lrcorner\ T$ is a non-degenerate 2-form on $\mathfrak{m}_2 \oplus \mathfrak{m}_3$;*
- *$M$ possesses invariant $k$-forms in all degrees.*

PROOF. According to prop. 1.6, $\nabla\xi = 0$ means that the isomorphism $\Gamma_\xi$ is $\mathfrak{h}$-equivariant, hence an isomorphism of $\mathfrak{h}$ representations.

Both $T$ and $\xi$ are $\nabla$-parallel, so $\xi \lrcorner\ T$ is also $\nabla$-parallel. If we write $\mathfrak{m}_2 \cong \mathfrak{m}_3 \cong \mathfrak{m}'$ as $h$-modules, $\xi \lrcorner\ T$ is the element corresponding to $\mathbb{R}$ in the $\mathfrak{h}$-module splitting

$$\Lambda^2(\mathfrak{m}_2 \oplus \mathfrak{m}_3) \cong \Lambda^2(2\mathfrak{m}') \cong 3\Lambda^2(\mathfrak{m}') \oplus S^2_0(\mathfrak{m}') \oplus \mathbb{R}.$$

More generally, we can decompose

$$\Lambda^k\mathfrak{m} \cong \Lambda^k(\mathfrak{m}_1 \oplus 2\mathfrak{m}') \cong \bigoplus_{i=0}^{k}\bigoplus_{j=0}^{i} \Lambda^{k-i}\mathfrak{m}_1 \otimes \Lambda^{i-j}\mathfrak{m}' \otimes \Lambda^j\mathfrak{m}'.$$

For even $k$, consider the summand with $i = k$, $j = k/2$. That is

$$\mathbb{R} \otimes \Lambda^{k/2}\mathfrak{m}' \otimes \Lambda^{k/2}\mathfrak{m}' \cong \Lambda^2(\Lambda^{k/2}\mathfrak{m}') \oplus S^2_0(\Lambda^{k/2}\mathfrak{m}') \oplus \mathbb{R},$$

where we find the isotropy-invariant $k$-form, which we denote by $\Psi \in \Lambda^k\mathfrak{m}' \subset \Lambda^k\mathfrak{m}$. We now have an invariant form of each even degree $k$. Since $\xi \in \mathfrak{m}_1$, $0 \neq \xi \wedge \Psi \in \Lambda^{k+1}\mathfrak{m}$ will be an invariant form of odd degree $k + 1$. □

PROPOSITION 2.2. *Let $(G/H, g, \nabla^c)$ be a naturally reductive homogeneous space with weakly non-degenerate split torsion $T$, such that $\mathfrak{m}_1$ is 1-dimensional and $\mathfrak{m}_2 \cong \mathfrak{m}_3$. Then $(M, g, \nabla)$ admits an almost contact metric structure.*

PROOF. Let $\xi \in \mathfrak{m}_1$. Since $\dim(\mathfrak{m}_1) = 1$, it must be $\nabla$-parallel, hence Killing. If $\dim(\mathfrak{m}_2) = n$, the manifold is $(2n+1)$-dimensional and $\xi \lrcorner\ T \in \Lambda^2(\mathfrak{m}_2 \oplus \mathfrak{m}_3)$ being non-degenerate means that it gives rise to an almost complex structure $\phi : \mathfrak{m}_2 \oplus \mathfrak{m}_3 \longrightarrow \mathfrak{m}_2 \oplus \mathfrak{m}_3$. Define $F(X, Y) := g(X, \phi Y)$. Then $F$ is proportional to $\xi \lrcorner\ T$ and

$$\phi Y = \lambda T(Y, \xi) \tag{2.1}$$

for a constant $\lambda \in \mathbb{R}$. Moreover, we can use equation (2.1) to extend $\phi$ to $\phi : \mathfrak{m} \longrightarrow \mathfrak{m}$. Then $\phi\xi = 0$ and $\phi^2 = -Id$ on $\mathfrak{m}_2 \oplus \mathfrak{m}_3$, so that

$$\phi^2 = -Id + \eta \otimes \xi$$

and $(M, g, \phi, \xi, \eta)$ is an almost contact metric structure with $\xi$ Killing. □

**2.1. The Stiefel manifolds.** An example that we already encountered in Chapter 1 and in the classification of generalized Wallach spaces are the Stiefel manifolds $V_{n,2} = SO(n)/SO(n-2)$. We refer to section 4.2. for the explicit construction of the homogeneous space. The torsion of the canonical connection is weakly non-degenerate and we fall in the conditions of the last proposition.

In this case, the almost contact metric structure is $\alpha$-Sasakian, as proven in proposition 4.4.

**The complex case.** We can follow the same framework for the $(4n-6)$-dimensional manifold $V_{n,2}^{\mathbb{C}} = SU(n)/S(U(n-2) \times U(1) \times U(1))$, where $U(n-2)$ is embedded as the upper left block of $U(n) = \{A \in M^n(\mathbb{C}) | A\bar{A}^t = Id\}$. $V_{n,2}^{\mathbb{C}}$ also falls in the classification of generalized Wallach spaces.

The isotropy $H = S(U(n-2) \times U(1) \times U(1))$ is

$$H = \left\{ \begin{bmatrix} U & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & z & 0 \\ \mathbf{0} & 0 & \frac{1}{z\det(U)} \end{bmatrix} \in SU(n) \middle| U \in U(n-2), z \in \mathbb{C} \right\}.$$

On a Lie algebra level, we have

$$\mathfrak{su}(n) = \left\{ \begin{bmatrix} u & \boldsymbol{w}_0 & \boldsymbol{w}_1 \\ -\bar{\boldsymbol{w}}_0^t & -i\lambda & w_{2n-3} \\ -\bar{\boldsymbol{w}}_1^t & -\bar{w}_{2n-3} & i\lambda - tr(u) \end{bmatrix} \middle| u \in \mathfrak{u}(n-2), \lambda \in \mathbb{R}, \boldsymbol{w}_{0,1} \in \mathbb{C}^{n-2}, w_{2n-3} \in \mathbb{C} \right\}.$$

We interpret $\boldsymbol{w}_{0,1}$ as column-vectors $\boldsymbol{w}_0 = (w_1, w_3, \ldots, w_{2n-5})^t$ and $\boldsymbol{w}_1 = (w_2, w_4, \ldots, w_{2n-4})^t$.

Consider the splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with $\mathfrak{g} = \mathfrak{su}(n)$, $\mathfrak{h} = \mathfrak{u}(n-2) \oplus i\mathbb{R}$, and

$$\mathfrak{m} = \mathbb{C}^{n-2} \oplus \mathbb{C}^{n-2} \oplus \mathbb{C} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3.$$

The tangent space to $V_{n,2}^{\mathbb{C}}$ at the origin may be identified with $\mathfrak{m}$, which is split as a direct sum of $\mathfrak{h}$-modules. The metric $g$, induced by the Killing form of $\mathfrak{g}$, is naturally reductive. Since the homogeneous space is generalized Wallach, its canonical connection has split torsion.

PROPOSITION 2.3. *The split torsion tensor $T^c(X,Y) = -[X,Y]_{\mathfrak{m}}$ of the canonical connection of $V_{n,2}^{\mathbb{C}}$ is weakly non-degenerate.*

PROOF. Define $\gamma : \mathfrak{m}_1 \times \mathfrak{m}_2 \times \mathfrak{m}_3 \longrightarrow \mathbb{R}$ as in Dfn 1.3

$$\gamma(X_1, X_2, X_3) := g(T(X_1, X_2), X_3) = T(X_1, X_2, X_3)$$

and $\Gamma : \mathfrak{m}_1 \longrightarrow \mathfrak{m}_2$ via

$$g(\Gamma X_1, X_2) := \gamma(X_1, X_2, \xi),$$

where

$$\xi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \in \mathfrak{m}_3$$

One can confirm directly that $\Gamma$ is an isomorphism between $\mathfrak{m}_1$ and $\mathfrak{m}_2$. □

However, in this case $\mathfrak{m}_1$ and $\mathfrak{m}_2$ differ as isotropy representations. Indeed, one easily shows that an element $(u, i\lambda) \in \mathfrak{h}$ acts as $u + i\lambda Id$ on $\mathfrak{m}_1$ and as $u + (tr(u) - i\lambda)Id$ on $\mathfrak{m}_2$. In fact, we might have chosen any nonzero element of $\mathfrak{m}_3$ with the same success, which means that there is no $\mathfrak{h}$-invariant vector field in $\mathfrak{m}_3$. This could already be seen if we observe that the $\mathfrak{h}$-modules $\mathfrak{m}_i$ are actually irreducible, so that $\mathfrak{m}_3$ does not contain a trivial 1-dimensional submodule.

**The quaternionic case.** The classification of generalized Wallach spaces also contains the manifold $V^{\mathbb{H}}_{n,2} = Sp(n)/Sp(n-2) \times Sp(1) \times Sp(1)$. It has dimension $8n - 12$ and can be constructed in the same manner as $V_{n,2}$ and $V^{\mathbb{C}}_{n,2}$, only by working with symplectic matrices in $M^n(\mathbb{H})$. It is again a manifold, admitting weakly non-degenerate split torsion.

REMARK 2.4. A remarkable case is the above examples occurs when $n = 3$. Then $V^{\mathbb{C}}_{3,2} = W^6$ and $V^{\mathbb{H}}_{3,2} = W^{12}$ are two of the Wallach spaces, which will be considered in greater detail in the upcoming section. They are our main examples of manifolds admitting *non-degenerate* split torsion.

## 3. Non-degenerate split torsion

In this section we turn our attention to the notion of non-degenerate split torsion, quickly guiding us to the Wallach spaces, which are then described in more detail.

PROPOSITION 3.1. *If $\gamma$ is non-degenerate, $\mathfrak{m}_1 \cong \mathfrak{m}_2 \cong \mathfrak{m}_3 \cong V$ as vector spaces and $V$ is a division algebra.*

PROOF. The first statement is contained in the definition. We can now write

$$\gamma : V \times V \times V \longrightarrow \mathbb{R} \quad \text{or} \quad \gamma : V \times V \longrightarrow V,$$

which gives us a multiplication on $V$, where multiplication on the left or right by a non-zero element is an isomorphism, hence $V$ is a division algebra. □

REMARK 3.2. This is already quite restrictive since we know that a division algebra $\mathbb{K}$ over $\mathbb{R}$ may only have dimension 1, 2, 4, or 8. We state the following known results:

- if $\mathbb{K}$ is commutative and associative, it must be isomorphic to either $\mathbb{R}$ or $\mathbb{C}$;
- if $\mathbb{K}$ is non-commutative, but associative, it must be isomorphic to $\mathbb{H}$;
- if $\mathbb{K}$ is non-associative, but alternative, it must be isomorphic to $\mathbb{O}$.

REMARK 3.3. We previously saw that a triality defines a division algebra. The converse is also true – every division algebra gives rise to a triality.

DEFINITION 3.4. A *normed triality* is a triality $\gamma$, satisfying $|\gamma(X_1, X_2, X_3)| \leq ||X_1||.||X_2||.||X_3||$ and such that for all $X_{i,j} \in \mathfrak{m}_{i,j}$ there exists $X_k \in \mathfrak{m}_k$ for which equality is attained. Here $\{i, j, k\} = \{1, 2, 3\}$.

REMARK 3.5. Every normed triality corresponds to a normed division algebra.

We sketch one way to obtain the four normed trialities: let $n$ be 1, 2, 4, or 8. We can embed $\mathbb{R}^n$ and $\mathrm{Spin}(n)$ in $\mathrm{Cliff}(n)$ and view $\mathbb{R}^n$ as the vector representation of $\mathrm{Spin}(n)$. Here we understand $\mathrm{Spin}(n)$ as the Lie group generated by even number of multiplications with unit vectors in the Clifford algebra, so that $\mathrm{Spin}(1) = \mathbb{Z}_2$ and $\mathrm{Spin}(2) = U(1)$. Let $\Delta^{\pm}_n$ be the $+$ and $-$ spin representations. For $n = 1, 2$ these are $\Delta^+_1 \cong \Delta^-_1 \cong \mathbb{R}$ and $\Delta^+_2 \cong \Delta^-_2 \cong \mathbb{C}$, coinciding with the corresponding vector representations. Finally, Clifford multiplication is a map $\mu : \mathbb{R}^n \times \Delta^{\pm}_n \longrightarrow \Delta^{\mp}_n$, which can also be written as $\mu : \mathbb{R}^n \times \Delta^{\pm}_n \times \Delta^{\mp}_n \longrightarrow \mathbb{R}$ giving the desired normed triality.

### 3.1. Generalities on Wallach spaces.

Let $\mathbb{K}$ stand for one of the normed division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$. We denote by $W_{\mathbb{K}}$ the homogeneous spaces $W^3 = SU(2)$, $W^6 = U(3)/U(1)^3$, $W^{12} = Sp(3)/Sp(1)^3$, and $W^{24} = F_4/\mathrm{Spin}(8)$, respectively. These are the Wallach spaces (see [**Wa72**]), which are generalized Wallach spaces from type (2) in the sense of the classification theorem 1.15. Write the reductive decomposition as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with

$$\mathfrak{m} = \mathfrak{m}^{\mathbb{K}}_1 \oplus \mathfrak{m}^{\mathbb{K}}_2 \oplus \mathfrak{m}^{\mathbb{K}}_3.$$

PROPOSITION 3.6. *The $\mathfrak{h}$-modules $\mathfrak{m}^{\mathbb{K}}_i$ have the same dimension. For $\mathbb{K} = \mathbb{H}, \mathbb{O}$ they are not isomorphic as $\mathfrak{h}$-representations.*

PROOF. We can look up the table in [**Nik16**] to confirm that the dimensions are equal, but the deeper reason is that the spaces $W_{\mathbb{K}}$ are manifolds with non-degenerate split torsion, i.e. they are related to trialities, namely the normed trialities, described at the end of the previous section. We will also provide an explicit description of the spaces in the sequel, where this will become computationally obvious.

A remark following from Theorem 1 in [**Dra08**] states that if $\mathfrak{h}$ has maximal rank in $\mathfrak{g}$ and $\mathfrak{m}_i$ are irreducible, then none of them is trivial, no two are isomorphic as $\mathfrak{h}$-representations, yet all of them have the same dimension. We already saw a counterexample in the Stiefel manifolds $V_{n,2}$ if the rank condition is not satisfied. For the Wallach spaces with $\mathbb{K} = \mathbb{H}$ or $\mathbb{O}$, on the other hand, $rank(G) = rank(H)$, and we can apply the result, proving the second statement. □

As a homogeneous space, $W_{\mathbb{K}}$ arises in a quite particular context—it is a principal orbit of cohomogeneity 2, related to a 5-, 8-, 14-, or 26-dimensional representation of $SO(3), SU(3), Sp(3)$, or $F_4$, depending on $\mathbb{K}$ ([**HBL71**], Theorem 5, p.16). Those are actually the isotropy representations on the tangent space at the origin of the following symmetric spaces: $M^5 = SU(3)/SO(3)$, $M^8 = SU(3)$, $M^{14} = SU(6)/Sp(3)$, and $M^{26} = E_6/F_4$. They are orbit-equivalent to polar actions on $\mathbb{R}^n \cong \mathfrak{m}$, whose principal orbits are known to be isoparametric submanifolds. The codimension 2 action is isometric and may be restricted to the sphere, where the principal orbits will now be of codimension 1, i.e. hypersurfaces. These spaces are naturally related to the normed division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$, as we will see in the Jordan algebra construction. The above mentioned symmetric spaces admit an invariant symmetric trace-free $(0,3)$-tensor, closely related to their structure, whose existence translates to the split torsion on the Wallach spaces (see also [**AFH13**]). We look for other isotropy-invariant tensors using computer computations. From now on we will largely exclude $W^3 = SU(2)$ from our considerations, since it is a Lie group.

PROPOSITION 3.7. *The number of linearly independent isotropy-invariant tensors of the corresponding types on each Wallach space except for $W^3$ is given in Table* 3.1.

| Tensor type | $W^6$ | $W^{12}$ | $W^{24}$ |
|---|---|---|---|
| Symmetric (0,2) | 3 | 3 | 3 |
| Symmetric (0,3) | 2 | 1 | 1 |
| Type (1,2) | 6 | 3 | 3 |
| 2-form | 3 | 0 | 0 |
| 3-form | 2 | 1 | 1 |
| 4-form | 3 | 6 | 3 |
| 5-form | 0 | 3 | 3 |
| 6-form | 1 | 0 | 1 |

TABLE 3.1. Invariants on Wallach spaces

We have computed the dimensions of the spaces of invariants on the various symmetric or skew-symmetric powers of the tangent bundle to $W_{\mathbb{K}}$, obtaining the results given in table, by considering $\Lambda^k(\mathfrak{m})$ or $S^k(\mathfrak{m})$ as representation spaces of the isotropy action and decomposing them into irreducible modules. In the relevant cases we will state the full decompositions, while here we have just counted the number of trivial modules.

We shall elaborate some more on the structure of $\mathfrak{m} = \mathfrak{m}_1^{\mathbb{K}} \oplus \mathfrak{m}_2^{\mathbb{K}} \oplus \mathfrak{m}_3^{\mathbb{K}}$ with emphasis on the cases $\mathbb{K} = \mathbb{H}$ and $\mathbb{O}$. The blocks $\{\mathfrak{m}_i^{\mathbb{K}}\}_{i=1}^3$ are of dimension $\dim \mathbb{K}$, but we will not consider them as copies of $\mathbb{K}$, since they differ as isotropy representations. Let's fix an orthogonal (with respect to the negative of the Killing form of $\mathfrak{g}$) basis $\{e_i\}_{i=1}^{3\dim\mathbb{K}}$ of $\mathfrak{m}$ such that

$$\mathfrak{m}_1^{\mathbb{K}} = \operatorname{span}\{e_1, \dots, e_{\dim\mathbb{K}}\},$$
$$\mathfrak{m}_2^{\mathbb{K}} = \operatorname{span}\{e_{\dim\mathbb{K}+1}, \dots, e_{2\dim\mathbb{K}}\},$$
$$\mathfrak{m}_3^{\mathbb{K}} = \operatorname{span}\{e_{2\dim\mathbb{K}+1}, \dots, e_{3\dim\mathbb{K}}\}.$$

We denote the covectors dual to $e_i$ with $\sigma^i$. Keep in mind that each of the blocks $\mathfrak{m}_i^{\mathbb{K}}$ is isotropy-invariant. We can now directly detect some of the invariants in the table.

- **Symmetric $(0,2)$-tensors.**
  These are actually the standard metrics on the blocks $\mathfrak{m}_i^{\mathbb{K}}$. We introduce the following notation: let
  $$g_1 := \sigma^1 \odot \sigma^1 + \ldots + \sigma^{\dim\mathbb{K}} \odot \sigma^{\dim\mathbb{K}}$$
  be the standard metric on $\mathfrak{m}_1^{\mathbb{K}}$ and define $g_2, g_3$ on $\mathfrak{m}_2^{\mathbb{K}}, \mathfrak{m}_3^{\mathbb{K}}$ in a similar fashion. A general metric on $\mathfrak{m}$ will then be written as
  $$g_{\lambda,\mu,\nu} := \lambda g_1 + \mu g_2 + \nu g_3, \quad \lambda, \mu, \nu > 0.$$
  We set $g := g_{1,1,1}$ and assume to be working with this metric in case nothing else is mentioned. Then $e_i$ and $\sigma^i$ are also metric dual.
- **Torsion type tensors.**
  We are interested in metric connections with torsion, so we would like to know which the possible torsion tensors are. They are tensors of type $(1,2)$ with one skew-symmetry between the arguments because of the metricity. However, we would ideally like to work with connections with totally skew torsion, i.e. torsion tensors, which considered as type $(0,3)$ are actually 3-forms.
- **Symmetric $(0,3)$-tensors and 3-forms.**
  Moving down in the rows of the table, we come to the symmetric 3-tensors. The numbers on this row match those on the row with 3-forms for a reason. We explain the relation here and describe both cases simultaneously. Let's start with the 3-forms. There is always one of them, which we call $T$, anticipating the relation to torsion. Only $W^6$ exhibits a peculiarity here due to its dimension. The Hodge dual $*T$ is also an invariant 3-form and $T \neq *T$ since $T \wedge T = 0$, but $T \wedge *T = |T|^2\mathrm{vol}$. There is a correspondence between the symmetric $(0,3)$-tensors and the 3-forms. Namely, if
  $$T = \sum_{i<j<k} T_{ijk}(\sigma^i \wedge \sigma^j \wedge \sigma^k), \text{ then } \ s := \sum_{i<j<k} T_{ijk}(\sigma^i \odot \sigma^j \odot \sigma^k)$$
  will be an invariant, traceless, totally symmetric $(0,3)$-tensor, and the other way around. This explains the two-dimensional subspace of invariants in $S^3(W^6)$. We should note that this correspondence works only because the torsion is split.
- **Invariant 2-forms.**
  The existence of invariant 2-forms is of particular interest, since there must be a non-degenerate 2-form $\omega$ corresponding to every almost complex structure $J$ on the tangent space. The form and the almost complex structure are related through $g(X, JY) = \omega(X, Y)$. Since we are working with even dimensional manifolds, it is quite natural to raise the question whether they are almost complex or not. This, surprisingly, turns out not to be the case in general, despite the leading example of $W^6$, which admits both a Kähler and a nearly Kähler structure. The reason why it admits more invariants of this type than the other two Wallach spaces is because they are not related to the common underlying structure of the spaces, but to the dimension of the particular manifold. In this case $\mathfrak{m}$ splits into three 2-dimensional isotropy-invariant blocks, so their "volume forms" would be invariant 2-forms. This is exactly the same reason why $W^{12}$ admits three more invariant 4-forms than $W^{24}$.

Remark 3.8. On the last note we might try to relax the almost complex condition and look for other structures, like quaternion Kähler (qK), for example. Alekseevsky has shown that every compact homogeneous qK manifold is necessarily a symmetric space (these are also known as Wolf spaces). This allows us to discard directly the existence of qK structures on the Wallach spaces. There is an analogue to qK structures after the introduction of torsion, which is called qKT - quaternionic Kähler with torsion. Just as a qK structure is defined by a parallel (w.r.t. the Levi-Civita connection) rank three distribution in the bundle of almost complex structures over the manifold, a qKT structure is defined by such a distribution, which must be parallel with respect to a metric connection with (skew) torsion. Such manifolds have been of interest in the mathematical physics literature. We will try to answer the question whether the Wallach spaces admit qKT structures.

- **Invariant 4-forms.**
  The existence of 4-forms may also be related to important geometric data. The form $\sigma_T = \frac{1}{2}\sum_i (e_i \lrcorner\ T) \wedge (e_i \lrcorner\ T)$ appears in many non-integrable geometries, while the fundamental form of a qK or qKT structure is defined by $\Omega := I \wedge I + J \wedge J + K \wedge K$, where $I, J$, and $K$ form a local basis of the rank 3 parallel distribution in the bundle of 2-forms, defining the structure.

Viewed as reductive homogeneous spaces, we might ask if the Wallach spaces admit naturally reductive metrics.

Proposition 3.9. *The spaces $(W^6, g_{1,1,1})$, $(W^{12}, g_{1,1,1})$, and $(W^{24}, g_{1,4,4})$ are naturally reductive.*

Proof. The proof is a direct computation. We realize the homogeneous spaces explicitly as described in the following sections and see which metric is induced by the Killing form of the Lie algebra of the group of transitive actions. Only for $W^{24}$ does the resulting metric have different scaling factors on the blocks $\mathfrak{m}_i^{\mathbb{O}}$. $\square$

Knowing that $W^{12}$ and $W^{24}$ are both naturally reductive, and that they only admit a 1-dimensional space of isotropy-invariant 3-forms each, this space must be spanned by the totally skew torsion $T^c$ of the canonical connection, viewed as a type $(0,3)$ tensor. This would also mean that all metric connections with totally skew torsion will belong to the 1-parameter family $\nabla^t = \nabla^g + 2tT^c$ spanned by the canonical connection and the Levi-Civita connection.

**A unified description via Hermitian matrices over $\mathbb{K}$.** In this paragraph we present a unified description of the Wallach spaces following Massey [**Mas74**], Ishikawa [**Ish99**], and others. Let us consider the Jordan algebra

$$H^3(\mathbb{K}) \;=\; \{A \in M^3(\mathbb{K}) | A^* = A\}.$$

Multiplication is defined by the Jordan product $A \circ B = \frac{1}{2}(AB + BA)$. We refer to [**Yo09**], p.31 for a more detailed treatment of the Jordan product and its properties. Here we just briefly state the most relevant ones:

Proposition 3.10. *The following hold in $\big(H^3(\mathbb{K}), \circ\big)$:*

- *$\big(H^3(\mathbb{K}), \circ\big)$ is an algebra whose unit is the identity matrix;*
- *$\langle A, B\rangle := tr(A \circ B)$ is a positive-definite inner product.*

A typical element of $H^3(\mathbb{K})$ has the form

$$A = \begin{bmatrix} \xi_1 & u_{12} & u_{13} \\ \bar{u}_{12} & \xi_2 & u_{23} \\ \bar{u}_{13} & \bar{u}_{23} & \xi_3 \end{bmatrix}$$

with $\xi_{1,2,3} \in \mathbb{R}$ and $u_{ij} \in \mathbb{K}$, $1 \le i < j \le 3$. The corresponding dimensions equal $3 + 3\dim\mathbb{K} = 6, 9, 15, 27$. We wish to consider the automorphism groups $G_{\mathbb{K}} := \mathrm{Aut}(H^3(\mathbb{K}))$, namely: $G_{\mathbb{R}} = O(3)$, $G_{\mathbb{C}} = U(3)$, $G_{\mathbb{H}} = Sp(3)$, $G_{\mathbb{O}} = F_4$. In the first three cases, the action is given by conjugation $T \bullet A := TAT^{-1} = TAT^*$, while in the last one $F_4$ is defined abstractly as the automorphism group of the Jordan algebra. Details about this action and algebraic facts concerning $F_4$ can be readily found in [**Yo09**]. The following is well-known:

PROPOSITION 3.11. *The Lie algebras of the automorphism groups $G_{\mathbb{K}}$ are given by the skew-Hermitian matrices: $Lie(G_{\mathbb{K}}) = \mathfrak{h}^3(\mathbb{K}) = \{X \in M^3(\mathbb{K}) | X^* = -X\}$ in the cases $\mathbb{K} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, and $\mathfrak{f}_4$ when $\mathbb{K} = \mathbb{O}$.*

The space $H^3(\mathbb{K})$ decomposes to $H^3(\mathbb{K}) = H_0^3(\mathbb{K}) \oplus \mathbb{R}$ under the action of $G_{\mathbb{K}}$. The first factor is the traceless part, whereas the second contains multiples of the identity matrix. Next we endow $H^3(\mathbb{K})$ with the metric $\langle A, B\rangle := tr(A \circ B)$. With respect to this metric, the automorphism groups act isometrically. Now we can restrict the action on the irreducible representation $H_0^3(\mathbb{K})$ to its unit sphere $S_{\mathbb{K}} := S(H_0^3(\mathbb{K}))$. This last sphere has dimension $4, 7, 13$, or $25$ depending on $\mathbb{K}$. Understanding the action on the aforementioned spheres is the main object of the rest of the paragraph.

**Orbit structure.** We describe the orbit structure by using an algebraic fact, which is well known in the real and complex cases, and still holds for the quaternions and the octonions. Namely, we claim that each matrix from $H^3(\mathbb{K})$ may be brought to diagonal form with real entries on the diagonal under the action of the corresponding automorphism group. The resulting set of diagonal elements and their multiplicities are unique up to permutation of the entries. We call them eigenvalues. This lets us describe each orbit by a set $\{\xi_1, \xi_2, \xi_3\}$ of three real numbers (allowing multiplicities). Further, these real numbers must satisfy the conditions

$$\xi_1 + \xi_2 + \xi_3 = 0 \quad \text{and} \quad (\xi_1)^2 + (\xi_2)^2 + (\xi_3)^2 = 1, \tag{3.1}$$

characterizing $S_{\mathbb{K}}$. The case $\xi_1 = \xi_2 = \xi_3$ is immediately excluded. We obtain two solutions in the case when $\xi_1 = \xi_2 \neq \xi_3$, namely:

$$\{\xi_1, \xi_2, \xi_3\} = \{\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\} \quad \text{or} \quad \{\xi_1, \xi_2, \xi_3\} = \{-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\}.$$

They describe the exceptional orbits. The last remaining case $\xi_1 \neq \xi_2 \neq \xi_3 \neq \xi_1$ will be the generic case, corresponding to the principal orbits. From now on, let's denote by one symbol the orbits

$$\mathcal{O}_{\vec{\xi}} \;=\; \{A\vec{\xi}A^* \in S_{\mathbb{K}} | A \in G_{\mathbb{K}}\} \;\text{ for }\; \mathbb{K} \neq \mathbb{O}, \quad \mathcal{O}_{\vec{\xi}} \;=\; \{A \bullet \vec{\xi} \in S_{\mathbb{K}} | A \in G_{\mathbb{O}}\}$$

through $\vec{\xi} = \operatorname{diag}(\xi_1, \xi_2, \xi_3) \in S_{\mathbb{K}}$. Each orbit is a homogeneous space $\mathcal{O}_{\vec{\xi}} = G_{\mathbb{K}}/\mathrm{Stab}_{\vec{\xi}}(G_{\mathbb{K}})$.

PROPOSITION 3.12. *The stabilizer of a point $\vec{\xi} \in H_0^3(\mathbb{K})$ is given by:*

- $\mathrm{Stab}_{\vec{\xi}}(G_{\mathbb{R}}) = (\mathbb{Z}_2)^3$,
- $\mathrm{Stab}_{\vec{\xi}}(G_{\mathbb{C}}) = U(1)^3$,
- $\mathrm{Stab}_{\vec{\xi}}(G_{\mathbb{H}}) = Sp(1)^3$,
- $\mathrm{Stab}_{\vec{\xi}}(G_{\mathbb{O}}) = \mathrm{Spin}(8)$

*for a principal orbit and*

- $\mathrm{Stab}_{\vec{\xi}}(G_{\mathbb{R}}) = O(2) \times \mathbb{Z}_2$,
- $\mathrm{Stab}_{\vec{\xi}}(G_{\mathbb{C}}) = U(2) \times U(1)$,
- $\mathrm{Stab}_{\vec{\xi}}(G_{\mathbb{H}}) = Sp(2) \times Sp(1)$,
- $\mathrm{Stab}_{\vec{\xi}}(G_{\mathbb{O}}) = \mathrm{Spin}(9)$

*in the case of an exceptional orbit.*

Proof. Take $T = (t_{ij})_{3\times 3} \in G_{\mathbb{K}}$ for $\mathbb{K} \neq \mathbb{O}$. The condition $T.\vec{\xi}.T^* = \vec{\xi}$ is equivalent to $T.\vec{\xi} = \vec{\xi}.T$ or

$$\begin{bmatrix} t_{11}\xi_1 & t_{12}\xi_2 & t_{13}\xi_3 \\ t_{21}\xi_1 & t_{22}\xi_2 & t_{23}\xi_3 \\ t_{31}\xi_1 & t_{32}\xi_2 & t_{33}\xi_3 \end{bmatrix} = \begin{bmatrix} t_{11}\xi_1 & t_{12}\xi_1 & t_{13}\xi_1 \\ t_{21}\xi_2 & t_{22}\xi_2 & t_{23}\xi_2 \\ t_{31}\xi_3 & t_{32}\xi_3 & t_{33}\xi_3 \end{bmatrix},$$

which implies $t_{ij} = 0$ for $i \neq j$ in the case of a principal orbit. The condition $T.T^* = Id$ then imposes $|t_{ii}| = 1$ for $i = 1, 2, 3$, arriving at the result. Assume $\vec{\xi} = \mathrm{diag}\,(\xi_1, \xi_1, \xi_3)$ belongs to an exceptional orbit. Then we get $t_{13} = t_{23} = t_{31} = t_{32} = 0$ so that

$$T = \begin{bmatrix} t_{11} & t_{12} & 0 \\ t_{21} & t_{22} & 0 \\ 0 & 0 & t_{33} \end{bmatrix}.$$

Again, the claim follows directly from the condition $T.T^* = Id$.

We are now left with the case $\mathbb{K} = \mathbb{O}$, for which we refer to [**Yo09**], chapter II once again. Theorems 2.7.1 and 2.7.4 resolve the issue here. □

Corollary 3.13. *The principal orbits are the homogeneous spaces*

- $G_{\mathbb{R}}/\mathrm{Stab}_{\vec{\xi}}(G_{\mathbb{R}}) = O(3)/(\mathbb{Z}_2)^3 \subset S^4_{\mathbb{R}}$,
- $G_{\mathbb{C}}/\mathrm{Stab}_{\vec{\xi}}(G_{\mathbb{C}}) = U(3)/U(1)^3 = W^6_{\mathbb{C}} \subset S^7_{\mathbb{C}}$,
- $G_{\mathbb{H}}/\mathrm{Stab}_{\vec{\xi}}(G_{\mathbb{H}}) = Sp(3)/Sp(1)^3 = W^{12}_{\mathbb{H}} \subset S^{13}_{\mathbb{H}}$,
- $G_{\mathbb{O}}/\mathrm{Stab}_{\vec{\xi}}(G_{\mathbb{O}}) = F_4/\mathrm{Spin}(8) = W^{24}_{\mathbb{O}} \subset S^{25}_{\mathbb{O}}$

*while the exceptional orbits are the projective planes*

- $G_{\mathbb{R}}/\mathrm{Stab}_{\vec{\xi}}(G_{\mathbb{R}}) = O(3)/O(2) \times \mathbb{Z}_2 = \mathbb{RP}^2$,
- $G_{\mathbb{C}}/\mathrm{Stab}_{\vec{\xi}}(G_{\mathbb{C}}) = U(3)/U(2) \times U(1) = \mathbb{CP}^2$,
- $G_{\mathbb{H}}/\mathrm{Stab}_{\vec{\xi}}(G_{\mathbb{H}}) = Sp(3)/Sp(2) \times Sp(1) = \mathbb{HP}^2$,
- $G_{\mathbb{O}}/\mathrm{Stab}_{\vec{\xi}}(G_{\mathbb{O}}) = F_4/\mathrm{Spin}(9) = \mathbb{OP}^2$.

*Note that all principal orbits are of codimension one in the ambient spheres.*

**Algebraic description of the orbits.** We can describe the orbits not only as homogeneous spaces, but also algebraically. We've already mentioned that each orbit is defined by a set of three eigenvalues (possibly with repetitions for the exceptional orbits). These are obviously the eigenvalues of any matrix in the same orbit, i.e. we have the description

$$\mathcal{O}_{\vec{\xi}} = \{A \in S_{\mathbb{K}} | A \text{ has eigenvalues } \{\xi_1, \xi_2, \xi_3\}\}.$$

We now state the following:

Proposition 3.14. *The orbit through $\vec{\xi}$ is the set*

$$\begin{aligned} \mathcal{O}_{\vec{\xi}} &= \{A \in S_{\mathbb{K}} | (A - \xi_1.Id)(A - \xi_2.Id)(A - \xi_3.Id) = 0\} \\ &= \{A \in S_{\mathbb{K}} | A^3 - \frac{1}{2}A - \xi_1\xi_2\xi_3 Id = 0\}. \end{aligned}$$

Proof. Let $char_B(X)$ be the characteristic polynomial of a $3 \times 3$ matrix $B$, i.e.

$$char_B(X) = X^3 - \sigma_1(B)X^2 + \sigma_2(B)X - \sigma_3(B)Id,$$

where $\sigma_{1,2,3}(B)$ are the elementary symmetric polynomials of the eigenvalues of $B$. We want to show that an orbit $\mathcal{O}_{\vec{\xi}}$ is defined by the matrix equation $char_{\vec{\xi}}(A) = 0$ which reads

$$A^3 - \sigma_1(\vec{\xi})A^2 + \sigma_2(\vec{\xi})A - \sigma_3(\vec{\xi})Id = 0.$$

Observe that if $A$ solves the equation, then $T.A.T^*$ for $T \in G_{\mathbb{K}}$, $\mathbb{K} \neq \mathbb{O}$ does so too, due to the identity $T.A^k.T^* = (T.A.T^*)^k$. On the other hand, if $\alpha \in F_4$, $\alpha(X \circ Y) = \alpha(X) \circ \alpha(Y)$ for any

$X, Y \in H^3(\mathbb{O})$. One also easily sees that $\alpha(X^k) = \alpha(X \circ X \cdots \circ X) = \alpha(X)^k$. In particular, if $A$ annihilates $char_{\vec{\xi}}(X)$, so does $\alpha(A)$. Since we know that every matrix satisfies its characteristic equation, i.e. $char_{\vec{\xi}}(\vec{\xi}) = 0$ holds, we can conclude that $char_{\vec{\xi}}(A) = 0$ for every matrix $A$ in the orbit of $\vec{\xi}$.

Conversely, let a matrix $A \in S_{\mathbb{K}}$ annihilate $char_{\vec{\xi}}(X)$, i.e.

$$(A - \xi_1.Id)(A - \xi_2.Id)(A - \xi_3.Id) = 0.$$

We can once more apply an element $\alpha \in G_{\mathbb{K}}$ to this identity, diagonalizing $A$ to $D = \alpha(A) = \operatorname{diag}(d_1, d_2, d_3)$. Now we are multiplying diagonal matrices, so the matrix equation is equivalent to the system

$$(d_i - \xi_1)(d_i - \xi_2)(d_i - \xi_3) = 0, \quad i = 1, 2, 3.$$

Taking $D \in S_{\mathbb{K}}$ into account, this means $\{d_1, d_2, d_3\} = \{\xi_1, \xi_2, \xi_3\}$, hence $A = \alpha^{-1}(D) \in \mathcal{O}_{\vec{\xi}}$.

We obtain the final form of the expression by computing

$$\sigma_1(\vec{\xi}) = tr(\vec{\xi}) = 0, \quad \sigma_2(\vec{\xi}) = \frac{1}{2}(\sigma_1(\vec{\xi})^2 - \xi_1^2 - \xi_2^2 - \xi_3^2) = -\frac{1}{2}, \quad \sigma_3(\vec{\xi}) = \xi_1\xi_2\xi_3. \qquad \square$$

**Tangent space description.** In general, considering a homogeneous space $M = G/H$, one normally decomposes $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ and identifies $\mathfrak{m} \cong T_e(G/H)$. This gives a nice picture of the tangent space, but we are now interested in finding out how it looks inside the ambient space $H_0^3(\mathbb{K})$. Consider a curve $T(t) \subset G_{\mathbb{K}}$ with $T(0) = Id$ and the corresponding curve $A(t) = T(t).\vec{\xi}.T(t)^*$ in $S_{\mathbb{K}}$. Differentiating and evaluating at $t = 0$, we get

$$\dot{A}(0) = \dot{T}(0).\vec{\xi} - \vec{\xi}.\dot{T}(0) = [\dot{T}(0), \vec{\xi}] \in T_{\vec{\xi}}\mathcal{O}_{\vec{\xi}} \subset T_{\vec{\xi}}S_{\mathbb{K}} \subset H_0^3(\mathbb{K}).$$

We obtain every element of $T_{\vec{\xi}}\mathcal{O}_{\vec{\xi}}$ in $H_0^3(\mathbb{K})$ through the map

$$\mathfrak{h}^3(\mathbb{K}) \ni \dot{T}(0) \longmapsto [\dot{T}(0), \vec{\xi}] =: X \in H_0^3(\mathbb{K}).$$

Of course, this map has a kernel, which is the isotropy algebra at the point. A direct check shows that its image consists of matrices which have zeroes on the main diagonal. More precisely, matrices of the form

$$X = \begin{bmatrix} 0 & u_{12} & u_{13} \\ \bar{u}_{12} & 0 & u_{23} \\ \bar{u}_{13} & \bar{u}_{23} & 0 \end{bmatrix}, u_{ij} \in \mathbb{K}.$$

Considering $\mathcal{O}_{\vec{\xi}} \subset S_{\mathbb{K}}$ as a submanifold, we write for the tangent space

$$T_{\vec{\xi}}S_{\mathbb{K}} = T_{\vec{\xi}}\mathcal{O}_{\vec{\xi}} \oplus N_{\vec{\xi}}\mathcal{O}_{\vec{\xi}}.$$

PROPOSITION 3.15. *On a principal orbit $\mathcal{O}_{\vec{\xi}}$ we have:*

$$T_{\vec{\xi}}\mathcal{O}_{\vec{\xi}} = \begin{bmatrix} 0 & \mathbb{K} & \mathbb{K} \\ * & 0 & \mathbb{K} \\ * & * & 0 \end{bmatrix} \cong \mathbb{K} \oplus \mathbb{K} \oplus \mathbb{K} \quad \textit{and} \quad N_{\vec{\xi}}\mathcal{O}_{\vec{\xi}} = \vec{n}_{\xi}.\mathbb{R},$$

*where the elements in the lower left corner of $T_{\vec{\xi}}\mathcal{O}_{\vec{\xi}}$ are determined by those in the upper right by the Hermitian property and*

$$\vec{n}_{\xi} = \frac{1}{\sqrt{3}} \begin{bmatrix} \xi_2 - \xi_3 & 0 & 0 \\ 0 & \xi_3 - \xi_1 & 0 \\ 0 & 0 & \xi_1 - \xi_2 \end{bmatrix}.$$

Proof. We've already mentioned that the matrices in $T_{\vec{\xi}}\mathcal{O}_{\vec{\xi}}$ must have zeroes along the diagonal. The fact that all such appear follows from dimensional reasons. The principal orbit $\mathcal{O}_{\vec{\xi}}$ has codimension one, so $N_{\vec{\xi}}\mathcal{O}_{\vec{\xi}} = \mathbb{R}.\vec{n}_\xi$ is one-dimensional. According to the condition $\vec{n}_\xi \perp T_{\vec{\xi}}\mathcal{O}_{\vec{\xi}}$, we may take $\vec{n}_\xi = \mathrm{diag}\,(\zeta_1, \zeta_2, \zeta_3) \in H_0^3(\mathbb{K})$ and solve the system

$$\begin{cases} \vec{n}_\xi \perp \vec{\xi} \\ tr(\vec{n}_\xi) = 0 \\ |\vec{n}_\xi| = 1 \end{cases} \Leftrightarrow \begin{cases} \xi_1.\zeta_1 + \xi_2.\zeta_2 + \xi_3.\zeta_3 = 0 \\ \zeta_1 + \zeta_2 + \zeta_3 = 0 \\ \zeta_1^2 + \zeta_2^2 + \zeta_3^2 = 1. \end{cases}$$

Up to the choice of a sign for the normal, the solution reads $\vec{n}_\xi = \frac{1}{\sqrt{3}}\mathrm{diag}\,(\xi_2 - \xi_3, \xi_3 - \xi_1, \xi_1 - \xi_2)$. $\square$

Proposition 3.16. *On the exceptional orbit $\mathcal{O}_0 := \mathcal{O}_{p_0}$ through $p_0 = \mathrm{diag}\,(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}})$ we have:*

$$T_{p_0}\mathcal{O}_0 = \begin{bmatrix} 0 & 0 & \mathbb{K} \\ 0 & 0 & \mathbb{K} \\ * & * & 0 \end{bmatrix} \cong \mathbb{K} \oplus \mathbb{K} \quad \text{and} \quad N_{p_0}\mathcal{O}_0 = \begin{bmatrix} 0 & \mathbb{K} & 0 \\ * & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \oplus \vec{n}_\xi.\mathbb{R},$$

*where the elements in the lower left corner of $T_{p_0}\mathcal{O}_0$ and $N_{p_0}\mathcal{O}_0$ are determined by those in the upper right by the Hermitian property and*

$$\vec{n}_\xi = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

*An analogous result holds for the other exceptional orbit, defined by $-p_0$.*

Proof. The argument is the same as in the previous proposition, but this time the extra relation $\xi_1 = \xi_2$ causes the vanishing of an additional block in the tangent space, which now becomes a part of the normal space. The last normal direction is the same as before, only with the values for $\vec{\xi}$ being substituted in the expression. The fact that everything holds good for the second exceptional orbit as well is obvious. $\square$

**The normal sphere bundle to an exceptional orbit and preferred geodesics.** From the description of the tangent and normal spaces to $\mathcal{O}_{\vec{\xi}}$ viewed as a submanifold in $S_\mathbb{K}$, we obtain a description of the normal sphere bundle to an exceptional orbit. Let $p_0 = \mathrm{diag}\,(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}) \in S_\mathbb{K}$ define the exceptional orbit $\mathcal{O}_0$ as before. The normal sphere bundle to $\mathcal{O}_0$ is the set

$$N\mathcal{O}_0 = \{(p, n) | p \in \mathcal{O}_0, n \in S(N_p\mathcal{O}_0)\}.$$

Proposition 3.17. *The normal sphere bundle $N\mathcal{O}_0$ to an exceptional orbit is diffeomorphic to a principal orbit of the $G_\mathbb{K}$ action on $S_\mathbb{K}$. More precisely, we have a fibration $\mathbb{KP}^1 \longrightarrow W_\mathbb{K} \longrightarrow \mathbb{KP}^2$.*

Proof. We let $G_\mathbb{K}$ act on an element $(p, n) \in N\mathcal{O}_0$ by its action on $H_0^3(\mathbb{K})$ in both components. The action is transitive on the first component with stabilizer computed in the beginning of the section. The element $n$ has the form

$$n = \begin{bmatrix} t & u & 0 \\ \bar{u} & -t & 0 \\ 0 & 0 & 0 \end{bmatrix}, t \in \mathbb{R}, u \in \mathbb{K}$$

and has unit length. This in particular means that it lies in $S(H_0^2(\mathbb{K})) \subset H_0^3(\mathbb{K})$ embedded as the upper left $2 \times 2$ block. To see what the isotropy of $(p, n)$ is we should let the stabilizer of $p$ act on $n$. We argue as before: for the different division algebras the groups $O(2)$, $U(2)$, $Sp(2)$, or $\mathrm{Spin}(9)$ act on $S(H_0^2(\mathbb{K}))$, and the orbits are described by the eigenvalues of a diagonal matrix. In this case they are already uniquely determined by the traceless and unit length conditions, and indeed after computing the stabilizers $(\mathbb{Z}_2)^2$, $U(1)^2$, $Sp(1)^2$, and Spin(8), we see that there is just one orbit and the space is actually homogeneous. We finally have

$$S(H_0^2(\mathbb{K})) = \begin{cases} \mathbb{RP}^1 = O(2)/(\mathbb{Z}_2)^2, & \mathbb{K} = \mathbb{R} \\ \mathbb{CP}^1 = U(2)/U(1)^2, & \mathbb{K} = \mathbb{C} \\ \mathbb{HP}^1 = Sp(2)/Sp(1)^2, & \mathbb{K} = \mathbb{H} \\ \mathbb{OP}^1 = \mathrm{Spin}(9)/\mathrm{Spin}(8), & \mathbb{K} = \mathbb{O} \end{cases}$$

Now the full action on $N\mathcal{O}_0$ is transitive with stabilizer the intersection of the stabilizers of the two components, which are $(\mathbb{Z}_2)^3$, $U(1)^3$, $Sp(1)^3$, and Spin(8) in the respective cases for $\mathbb{K}$. The resulting homogeneous spaces coincide with the principal orbits of the action of $G_{\mathbb{K}}$ on $S_{\mathbb{K}}$. Moreover, we have seen that the fibre of the bundle

$$\pi : W_{\mathbb{K}} \longrightarrow \mathbb{KP}^2,$$

where $\pi(p, n) = p$ is the natural projection, is the projective line $\mathbb{KP}^1$. □

Remark 3.18. Note that we may consider the Jordan algebra construction on matrices from $H^k(\mathbb{K})$ ($k \geq 3$, $\mathbb{K} \neq \mathbb{O}$), write down the automorphism group $G_{\mathbb{K}}(k) = O(k)$, $U(k)$, or $Sp(k)$, which acts by conjugation, separate the traceless part, restrict to the sphere, and obtain similar results for $k \geq 4$ as we did for $k = 2, 3$. We saw that the cohomogeneity of the action is 0 or 1 for $k = 2$ or 3. Indeed, it equals $k - 2$ for any $k$. Due to the fact that we can diagonalize matrices, we will find that all orbits are of the type $G_{\mathbb{K}}(k)/G_{\mathbb{K}}(k_1) \times ... \times G_{\mathbb{K}}(k_s)$, where $k_1 + ... + k_s = k$ is any partition of $k$. The principal orbits will have $k_i = 1, i = 1, .., s$, while those of minimal dimension will equal $\mathbb{KP}^{k-1}$. Note that the families for $s = 3$ and $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ appear in the classification of generalized Wallach spaces.

Definition 3.19. We call *preferred geodesics* the geodesics obtained in the following way: consider a fixed point $\vec{\xi} \in \mathcal{O}_{\vec{\xi}} \subset S_{\mathbb{K}}$, at which we have our tangent space description and the 3-dimensional space $X$, consisting of diagonal matrices with real entries:

$$X = \{A = \mathrm{diag}\,(\lambda_1, \lambda_2, \lambda_3) | \lambda_{1,2,3} \in \mathbb{R}\}.$$

The condition $tr(A) = 0$ defines a hyperplane in $X$. The intersection of this hyperplane with the sphere $S(H^3(\mathbb{K}))$ is a great circle, i.e. a geodesic on one hand, and it equals $X \cap S_{\mathbb{K}}$ on the other. This is one preferred geodesic. All others are obtained by this one applying the isometric group action.

Proposition 3.20. *Each preferred geodesic intersects all orbits of the group action, always orthogonally, and it meets principal and exceptional orbits* 6 *or* 3 *times, respectively.*

Proof. We again work with the geodesic through $\vec{\xi}$. It consists only of diagonal matrices, while the diagonal of $T_{\vec{\xi}}\mathcal{O}_{\vec{\xi}}$ is empty. Thus they intersect orthogonally. There are $3! = 6$ points of intersection because we can permute the diagonal entries within each orbit. In the case of an exceptional orbit each is met twice due to the equal eigenvalues, so we get $3 = 3!/2$ points of intersection. Moreover, apart from the 5 (or 2) other points of intersection with $\mathcal{O}_{\vec{\xi}}$, each other point of the geodesic is again a diagonal matrix and belongs to a different orbit. Obviously, every possible set of eigenvalues may be acquired, so it intersects all orbits. Due to the diagonal form, we get similar tangent space descriptions at the other points and always come to the conclusion that the intersection is orthogonal. □

We can now interpret the result regarding the normal sphere bundle: if we fix a principal orbit, going from any point along the preferred geodesic intersecting it, we will eventually meet an exceptional orbit, and in this way we will be able to arrive at any point of the exceptional orbit, and more so coming from any possible normal direction. Having taken the normal directions into account, the correspondence will be bijective.

**The Gauss map and the second fundamental form.** Here we describe the embeddings of the orbits in the ambient sphere by means of the second fundamental form, or equivalently, the shape operator. We start by defining the Gauss map. The Gauss map $\gamma$ for a hypersurface is defined as

$$\gamma : \mathcal{O}_{\vec{\xi}} \longrightarrow S_{\mathbb{K}} : \quad \gamma(\vec{\zeta}) = \vec{n}_{\zeta},$$

assigning to each point $\vec{\zeta} \in \mathcal{O}_{\vec{\xi}}$ of the hypersurface its unit normal vector. Its differential

$$(\gamma_*)_{\vec{\zeta}} : T_{\vec{\zeta}}\mathcal{O}_{\vec{\xi}} \longrightarrow T_{\vec{\zeta}}S_{\mathbb{K}}$$

is the shape operator at $\vec{\zeta}$, which is the symmetric operator corresponding to the second fundamental form. Its eigenvectors and eigenvalues are of particular importance as those are the principal directions and the principal curvatures of the hypersurface. We proceed to compute them for a principal orbit $\mathcal{O}_{\vec{\xi}}$. We can restrict our attention to $\vec{\zeta} = \vec{\xi}$ due to the homogeneity of the orbit.

PROPOSITION 3.21. *The second fundamental form of the hypersurface $\mathcal{O}_{\vec{\xi}}$ for $\vec{\xi} = \operatorname{diag}(\xi_1, \xi_2, \xi_3)$ s.t. it is a principal orbit has three distinct eigenvalues, each of which with multiplicity* $\dim \mathbb{K}$*. They are determined by the entries of $\vec{\xi}$ and have values $\lambda_1 = \frac{\xi_3\sqrt{3}}{(\xi_2-\xi_1)}$, $\lambda_2 = \frac{\xi_2\sqrt{3}}{(\xi_1-\xi_3)}$, and $\lambda_3 = \frac{\xi_1\sqrt{3}}{(\xi_3-\xi_2)}$.*

PROOF. We will compute the action of the differential $\gamma_*$ of the Gauss map on a basis of $T_{\vec{\xi}}\mathcal{O}_{\vec{\xi}}$. We fix an orthonormal basis $\{e_k\}_{k=1}^{3\dim\mathbb{K}}$ of the tangent space $T_{\vec{\xi}}\mathcal{O}_{\vec{\xi}} \subset H^3(\mathbb{K})$. Let's illustrate the choice of basis for the field $\mathbb{K} = \mathbb{C}$. The other cases are analogous.

$$e_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$
$$e_4 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, \quad e_5 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad e_6 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{bmatrix}.$$

Let's consider a curve $g(t)$ in $G_{\mathbb{K}}$ s.t. $[\dot{g}(0), \vec{\xi}] = e_1$. This is possible because we showed that the elements of $\mathfrak{g}_{\mathbb{K}}$ map surjectively to the tangent space taking their commutator with the defining point $\vec{\xi}$. We obtain the vector $e_1$ as the image of

$$\dot{g}(0) = \frac{1}{(\xi_2 - \xi_1)\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The corresponding curve in $G_{\mathbb{K}}$ is then

$$\left\{ g(t) = \begin{bmatrix} \cos(u) & \sin(u) & 0 \\ -\sin(u) & \cos(u) & 0 \\ 0 & 0 & 1 \end{bmatrix} \middle| u = \frac{t}{(\xi_2 - \xi_1)\sqrt{2}}, \ t \in \mathbb{R} \right\} \subset G_{\mathbb{K}}.$$

The vector $e_1$ is now tangent to the curve $g(t).\vec{\xi}.g(t)^*$ in the orbit. To compute $\gamma_*(e_1)$ we consider the image

$$\gamma(g(t).\vec{\xi}.g(t)^*) = g(t).\vec{n}_{\xi}.g(t)^* =: n(t)$$

of the curve in $\mathcal{O}_{\vec{\xi}}$. Note that $\gamma$ commutes with the group action because the latter is isometric. We have computed $\vec{n}_{\xi}$ explicitly in Prop 3.15. All that's left now is to compute $n(t)$, differentiate, and evaluate at $t = 0$. We obtain:

$$n(t) = \frac{1}{\sqrt{3}} \begin{bmatrix} -\xi_3\cos(2u) + \xi_2\cos^2(u) - \xi_1\sin^2(u) & \frac{3}{2}\xi_3\sin(2u) & 0 \\ \frac{3}{2}\xi_3\sin(2u) & \xi_3\cos(2u) + \xi_2\sin^2(u) - \xi_1\cos^2(u) & 0 \\ 0 & 0 & \xi_1 - \xi_2 \end{bmatrix}$$

$$\dot{n}(0) = \sqrt{\frac{3}{2}}.\frac{\xi_3}{(\xi_2 - \xi_1)} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{\xi_3\sqrt{3}}{(\xi_2 - \xi_1)} e_1.$$

Hence

$$\gamma_*(e_1) = \dot{n}(0) = \frac{\xi_3\sqrt{3}}{(\xi_2 - \xi_1)} e_1 \text{ and } \lambda_1 = \frac{\xi_3\sqrt{3}}{(\xi_2 - \xi_1)}.$$

Thus we've found one eigenvector of the second fundamental form and its corresponding eigenvalue. For $e_2$ we proceed likewise with

$$\dot{g}(0) = \frac{1}{(\xi_2 - \xi_1)\sqrt{2}} \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\left\{ g(t) = \begin{bmatrix} \cos(u) & i\sin(u) & 0 \\ i\sin(u) & \cos(u) & 0 \\ 0 & 0 & 1 \end{bmatrix} \middle| u = \frac{t}{(\xi_2 - \xi_1)\sqrt{2}},\ t \in \mathbb{R} \right\} \subset G_{\mathbb{K}}.$$

The expression for $n(t)$ now reads

$$n(t) = \frac{1}{\sqrt{3}} \begin{bmatrix} -\xi_3\cos(2u) + \xi_2\cos^2(u) - \xi_1\sin^2(u) & \frac{3}{2}i\xi_3\sin(2u) & 0 \\ -\frac{3}{2}i\xi_3\sin(2u) & \xi_3\cos(2u) + \xi_2\sin^2(u) - \xi_1\cos^2(u) & 0 \\ 0 & 0 & \xi_1 - \xi_2 \end{bmatrix},$$

whose differential at $t = 0$ is

$$\dot{n}(0) = \sqrt{\frac{3}{2}}.\frac{\xi_3}{(\xi_2 - \xi_1)} \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{\xi_3\sqrt{3}}{(\xi_2 - \xi_1)} e_2.$$

Thus we've seen that $\lambda_1$ has multiplicity equal to $\dim\mathbb{C} = 2$ in this case. For the other division algebras, we do the same computation for all imaginary units, getting the required multiplicity. To work with the other blocks of the tangent space we only need to apply a cyclic permutation of the indices, which results in the given values for $\lambda_2$ and $\lambda_3$. □

Remark 3.22. Take the principal orbit through $\vec{\xi} = \mathrm{diag}\,(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$. This is in some sense the "middle" orbit. It is expected to be minimal by Wallach in [**Wa72**]. We can directly compute

$$\lambda_1 = \sqrt{3}, \quad \lambda_2 = 0, \quad \lambda_3 = -\sqrt{3} \Rightarrow H = \lambda_1 + \lambda_2 + \lambda_3 = 0$$

confirming this conjecture. It was also seen by Karcher in [**Kar88**], among other sources, where he proves that the largest volume hypersurface in a family of isoparametric hypersurfaces is minimal.

**The positive scalar curvature metrics.** Wallach [**Wa72**] classified all even-dimensional homogeneous spaces with strictly positive sectional curvature, producing a list containing all even-dimensional compact rank one symmetric spaces (CROSS's) and three exceptional examples—the three Wallach spaces $W_{\mathbb{K}}$, $\mathbb{K} = \mathbb{C}, \mathbb{H}, \mathbb{O}$. Using the results so far, we see how the positive scalar curvature metrics on these spaces arise in a natural manner. We use the same notation with $p_0$ defining the orbit $\mathcal{O}_0 \cong \mathbb{KP}^2$. Fix a principle orbit, defined by a generic $\vec{\xi} = \mathrm{diag}\,(\xi_1, \xi_2, \xi_3)$, and consider the fibre bundle

$$\pi : W_{\mathbb{K}} \longrightarrow \mathbb{KP}^2$$

with fibre $\mathbb{KP}^1$ established in Proposition 3.17. Choose metrics in the following way:

- $g_B$ is the Fubini-Study metric on the base;
- $g_F$ is any Riemannian metric on the fibre;
- $g_T := \pi_F^* g_F + \pi^* g_B$, $t > 0$ is a real parameter, $\pi_F$ is the projection on the fibre.

The following fact is well-known:

PROPOSITION 3.23. *The projection* $\pi : (W_{\mathbb{K}}, g_T) \longrightarrow (\mathbb{KP}^2, g_B)$ *is a smooth Riemannian submersion with kernel* $\mathrm{Ker}(\pi) = \mathbb{KP}^1$.

We can now make use of the O'Neill formulas for a Riemannian submersion.

PROPOSITION 3.24. $W_{\mathbb{K}}$ *admits a metric with strictly positive sectional curvature.*

PROOF. Quoting one of O'Neill's formulas we have

$$K_W(X,Y) = K_0(X,Y) - \frac{3}{4}|\mathcal{V}[X,Y]|^2,$$

where $K_W$ and $K_0$ are the sectional curvatures on $W_{\mathbb{K}}$ and $\mathbb{KP}^2$, and $\mathcal{V}[X,Y]$ denotes the vertical part of the commutator.

$$|\mathcal{V}[X,Y]|^2 = (\pi_F^* g_F)([X,Y],[X,Y]).$$

The fibre is compact, so $(\pi_F^* g_F)([X,Y],[X,Y]) \le C$ for some constant $C > 0$. It is known that the Fubini-Study metric $g_B$ has a pinching factor of $\frac{\min(K_0)}{\max(K_0)} = \frac{1}{4}$. Assume that we have scaled it in such a way that $min(K_0) = 1$. Moreover, we can do this for any metric on the fibre, in particular for any rescaling $\lambda g_F$ of $g_F$, $\lambda > 0$, obtaining

$$K_W(X,Y) = K_0(X,Y) - \frac{3}{4}\lambda^2 (\pi_F^* g_F)([X,Y],[X,Y]) \ge 1 - \frac{3}{4}\lambda^2 C.$$

Now

$$K_W(X,Y) > 0 \Longleftrightarrow 0 < \lambda < \sqrt{\frac{4}{3C}}.$$

□

**3.2. Explicit realizations of the Wallach spaces.** We proceed to study the Wallach spaces on a case by case basis. The flag manifold $W^6 = U(3)/U(1)^3$ has been extensively investigated and is very well understood by now. We refer to [**BFGK90**], Chapter 5 for further details on the space and the precise realization. Here we only give a short description, serving the purpose to illustrate the results in Table 3.1.

**The complex Wallach space $W^6$.** Consider $\mathfrak{g} = \mathfrak{u}(3) = \{A \in M^3(\mathbb{C}) \big| A^t = -\bar{A}\}$ and $\mathfrak{h} \subset \mathfrak{g}$ the subalgebra of diagonal matrices. The Killing form of $\mathfrak{g}$ is given by $B(X,Y) = tr(XY)$ and induces a metric $g = -\frac{1}{2}Re(B)$. We fix an orthonormal basis

$$e_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},$$
$$e_4 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, e_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, e_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}.$$

**The invariants.** An element $h \in \mathfrak{h}$ of the isotropy algebra acts on $X \in \mathfrak{m}$ by $ad(h)(X) = [h,X] \in \mathfrak{m}$. This action extends to arbitrary tensors requiring it to act as a derivation with respect to the tensor product (respectively wedge or symmetric product). The invariants from the table above are now explicitly given by:

$$S^2(\mathfrak{m}): g_1 = \sigma^{11} + \sigma^{22}, \quad g_2 = \sigma^{33} + \sigma^{44}, \quad g_3 = \sigma^{55} + \sigma^{66};$$
$$S^3(\mathfrak{m}): s_1 = \sigma^{135} + \sigma^{146} - \sigma^{236} + \sigma^{245},\ s_2 = -\sigma^{136} + \sigma^{145} - \sigma^{235} - \sigma^{246};$$
$$\Lambda^2(\mathfrak{m}): \omega_1 = \sigma^1 \wedge \sigma^2, \quad \omega_2 = \sigma^3 \wedge \sigma^4, \quad \omega_3 = \sigma^5 \wedge \sigma^6;$$
$$\Lambda^3(\mathfrak{m}): T = \sigma^{135} + \sigma^{146} - \sigma^{236} + \sigma^{245},\ *T = -\sigma^{136} + \sigma^{145} - \sigma^{235} - \sigma^{246};$$
$$\Lambda^4(\mathfrak{m}): \alpha_1 = \sigma^{1234}, \quad \alpha_2 = \sigma^{1256}, \quad \alpha_3 = \sigma^{3456};$$
$$\Lambda^5(\mathfrak{m}): \text{none};$$
$$\Lambda^6(\mathfrak{m}): \sigma^{123456}.$$

The nearly Kähler structure of the manifold is given by the form $\omega = \omega_1 - \omega_2 + \omega_3$. It defines the almost complex structure $J$ through $\omega(X,Y) = g(X, JY)$. Given a geometric structure defined by certain objects, we call a connection with respect to which those objects are parallel *characteristic*. In this case this would be a connection $\nabla$ with the properties $\nabla g = 0$ and $\nabla\omega = 0$ (or equivalently $\nabla J = 0$). It is given by

$$\nabla_X Y = \nabla^g_X Y + \frac{1}{2}(\nabla^g_X J)JY,$$

where $\nabla^g$ denotes the Levi-Civita connection. On the other hand, we have realized the manifold as a naturally reductive homogeneous space, and as such it admits a canonical connection $\nabla^c$. It coincides with $\nabla$. Thus the torsion of this connection equals $T$ which is totally skew, and more precisely, non-degenerate split.

**The quaternionic Wallach space $W^{12}$.** Consider $W^{12} = Sp(3)/Sp(1)^3$ with the reductive decomposition

$$\mathfrak{sp}(3) = \mathfrak{h} \oplus \mathfrak{m} = (\mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)) \oplus (\mathfrak{m}_1^{\mathbb{H}} \oplus \mathfrak{m}_2^{\mathbb{H}} \oplus \mathfrak{m}_3^{\mathbb{H}}),$$

where

$$\mathfrak{sp}(3) = \{A \in M^3(\mathbb{H}) \big| A^t = -\bar{A}\},$$

$$\mathfrak{m}_1^{\mathbb{H}} = \left\{ \begin{bmatrix} 0 & v & 0 \\ -\bar{v} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Big| v \in \mathbb{H} \right\}, \quad \mathfrak{m}_2^{\mathbb{H}} = \left\{ \begin{bmatrix} 0 & 0 & v \\ 0 & 0 & 0 \\ -\bar{v} & 0 & 0 \end{bmatrix} \Big| v \in \mathbb{H} \right\},$$

$$\mathfrak{m}_3^{\mathbb{H}} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & v \\ 0 & -\bar{v} & 0 \end{bmatrix} \Big| v \in \mathbb{H} \right\}, \quad \mathfrak{h} = \left\{ \begin{bmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & q_3 \end{bmatrix} \Big| q_{1,2,3} \in \mathrm{Im}(\mathbb{H}) \right\}.$$

One can directly check that this choice fulfills all abovementioned properties. We choose bases in the corresponding spaces: on $\mathfrak{h}$ take $\{h_p\}_{p=1}^9$ with

$$h_1 = \mathrm{diag}\,\{i,0,0\}, \quad h_2 = \mathrm{diag}\,\{j,0,0\}, \quad h_3 = \mathrm{diag}\,\{k,0,0\},$$
$$h_4 = \mathrm{diag}\,\{0,i,0\}, \quad h_5 = \mathrm{diag}\,\{0,j,0\}, \quad h_6 = \mathrm{diag}\,\{0,k,0\},$$
$$h_7 = \mathrm{diag}\,\{0,0,i\}, \quad h_8 = \mathrm{diag}\,\{0,0,j\}, \quad h_9 = \mathrm{diag}\,\{0,0,k\}.$$

Here $i, j, k$ denote the basic unit imaginary quaternions satisfying $i^2 = j^2 = k^2 = -1$ and $ij = k$. On $\mathfrak{m}$ take $\{e_p\}_{p=1}^{12}$ such that $e_{1,2,3,4}$ form a basis of $\mathfrak{m}_1^{\mathbb{H}}$, $e_{5,6,7,8}$ - of $\mathfrak{m}_2^{\mathbb{H}}$, and $e_{9,10,11,12}$ - of $\mathfrak{m}_3^{\mathbb{H}}$. More precisely,

$$e_1 = A \in \mathfrak{m}_1^{\mathbb{H}} \text{ with } v = 1, \qquad e_2 = A \in \mathfrak{m}_1^{\mathbb{H}} \text{ with } v = i$$
$$e_3 = A \in \mathfrak{m}_1^{\mathbb{H}} \text{ with } v = j, \qquad e_4 = A \in \mathfrak{m}_1^{\mathbb{H}} \text{ with } v = k$$

We fix the bases of $\mathfrak{m}_2^{\mathbb{H}}$ and $\mathfrak{m}_3^{\mathbb{H}}$ in a similar fashion. This allows us to implement all matrices in a computer algebra program and perform computations there. In particular, we compute the isotropy representation $iso : \mathfrak{h} \to \mathfrak{so}(\mathfrak{m}) = \mathfrak{so}(12)$ and use it to search for invariant objects.

**The invariants.** Explicit computations give us bases of the spaces from table 3.1:

$$\begin{aligned}
S^2(\mathfrak{m}) : {}& g_1 = \sigma^{1,1} + \sigma^{2,2} + \sigma^{3,3} + \sigma^{4,4}, \quad g_2 = \sigma^{5,5} + \sigma^{6,6} + \sigma^{7,7} + \sigma^{8,8},\\
& g_3 = \sigma^{9,9} + \sigma^{10,10} + \sigma^{11,11} + \sigma^{12,12};\\
S^3(\mathfrak{m}) : {}& s = \sigma^{1,5,9} + \sigma^{1,6,10} + \sigma^{1,7,11} + \sigma^{1,8,12} - \sigma^{2,5,10} + \sigma^{2,6,9} - \sigma^{2,7,12} + \sigma^{2,8,11} -\\
& \quad - \sigma^{3,5,11} + \sigma^{3,6,12} + \sigma^{3,7,9} - \sigma^{3,8,10} - \sigma^{4,5,12} - \sigma^{4,6,11} + \sigma^{4,7,10} + \sigma^{4,8,9};\\
\Lambda^2(\mathfrak{m}) : {}& \textbf{none};\\
\Lambda^3(\mathfrak{m}) : {}& T = \sigma^{1,5,9} + \sigma^{1,6,10} + \sigma^{1,7,11} + \sigma^{1,8,12} - \sigma^{2,5,10} + \sigma^{2,6,9} - \sigma^{2,7,12} + \sigma^{2,8,11} -\\
& \quad - \sigma^{3,5,11} + \sigma^{3,6,12} + \sigma^{3,7,9} - \sigma^{3,8,10} - \sigma^{4,5,12} - \sigma^{4,6,11} + \sigma^{4,7,10} + \sigma^{4,8,9};\\
\Lambda^4(\mathfrak{m}) : {}& \alpha_1 = \sigma^{1,2,3,4}, \qquad \alpha_2 = \sigma^{5,6,7,8}, \qquad \alpha_3 = \sigma^{9,10,11,12},\\
& \alpha_4 = \sigma^{1,2,9,10} + \sigma^{1,2,11,12} + \sigma^{1,3,9,11} - \sigma^{1,3,10,12} + \sigma^{1,4,9,12} + \sigma^{1,4,10,11} -\\
& \quad - \sigma^{2,3,9,12} - \sigma^{2,3,10,11} + \sigma^{2,4,9,11} - \sigma^{2,4,10,12} - \sigma^{3,4,9,10} - \sigma^{3,4,11,12},\\
& \alpha_5 = \sigma^{1,2,5,6} + \sigma^{1,2,7,8} + \sigma^{1,3,5,7} - \sigma^{1,3,6,8} + \sigma^{1,4,5,8} + \sigma^{1,4,6,7} +\\
& \quad + \sigma^{2,3,5,8} + \sigma^{2,3,6,7} - \sigma^{2,4,5,7} + \sigma^{2,4,6,8} + \sigma^{3,4,5,6} + \sigma^{3,4,7,8},\\
& \alpha_6 = \sigma^{5,6,9,10} - \sigma^{5,6,11,12} + \sigma^{5,7,9,11} + \sigma^{5,7,10,12} + \sigma^{5,8,9,12} - \sigma^{5,8,10,11} -\\
& \quad - \sigma^{6,7,9,12} + \sigma^{6,7,10,11} + \sigma^{6,8,9,11} + \sigma^{6,8,10,12} - \sigma^{7,8,9,10} + \sigma^{7,8,11,12};\\
\Lambda^5(\mathfrak{m}) : {}& \beta_1 = \sigma^{1,5,6,7,12} - \sigma^{1,5,6,8,11} + \sigma^{1,5,7,8,10} - \sigma^{1,6,7,8,9} + \sigma^{2,5,6,7,11} + \sigma^{2,5,6,8,12} +\\
& \quad + \sigma^{2,5,7,8,9} + \sigma^{2,6,7,8,10} - \sigma^{3,5,6,7,10} - \sigma^{3,5,6,8,9} + \sigma^{3,5,7,8,12} +\\
& \quad + \sigma^{3,6,7,8,11} + \sigma^{4,5,6,7,9} - \sigma^{4,5,6,8,10} - \sigma^{4,5,7,8,11} + \sigma^{4,6,7,8,12},\\
& \beta_2 = \sigma^{1,2,3,5,12} + \sigma^{1,2,3,6,11} - \sigma^{1,2,3,7,10} - \sigma^{1,2,3,8,9} - \sigma^{1,2,4,5,11} + \sigma^{1,2,4,6,12} +\\
& \quad + \sigma^{1,2,4,7,9} - \sigma^{1,2,4,8,10} + \sigma^{1,3,4,5,10} - \sigma^{1,3,4,6,9} + \sigma^{1,3,4,7,12} -\\
& \quad - \sigma^{1,3,4,8,11} + \sigma^{2,3,4,5,9} + \sigma^{2,3,4,6,10} + \sigma^{2,3,4,7,11} + \sigma^{2,3,4,8,12},\\
& \beta_3 = \sigma^{1,5,10,11,12} - \sigma^{1,6,9,11,12} + \sigma^{1,7,9,10,12} - \sigma^{1,8,9,10,11} + \sigma^{2,5,9,11,12} + \sigma^{2,6,10,11,12} +\\
& \quad + \sigma^{2,7,9,10,11} + \sigma^{2,8,9,10,12} - \sigma^{3,5,9,10,12} - \sigma^{3,6,9,10,11} + \sigma^{3,7,10,11,12} +\\
& \quad + \sigma^{3,8,9,11,12} + \sigma^{4,5,9,10,11} - \sigma^{4,6,9,10,12} - \sigma^{4,7,9,11,12} + \sigma^{4,8,10,11,12};\\
\Lambda^6(\mathfrak{m}) : {}& \text{none}.
\end{aligned}$$

We see that there is a unique invariant 3-form $T$ listed above. The naturally reductive metric on $W^{12}$ is given by $g := g_{1,1,1} = g_1 + g_2 + g_3$. In general, the torsion of the canonical connection is given by $T^c(X,Y) = -[X,Y]_{\mathfrak{m}}$ for $X, Y \in \mathfrak{m}$ and is totally skew. Since our computations show that there is only one invariant 3-form $T$ looking as listed above, $T$ *has* to coincide with $T^c$ up to a factor. This is indeed the case, and in fact more holds:

LEMMA 3.25. *The torsion $T^c$ of the canonical connection is $-T$. The coefficients of the invariant 3-form $T =: \frac{1}{3}\sum_{i,j,k} T_{ijk}\sigma^{i,j,k}$ have the property that to each pair $i, j$, there is at most one index $k$ such that $T_{ijk} \neq 0$. The commutator structure of $\mathfrak{m}$ is then given by*

$$[e_i, e_j] \;=\; T_{ijk}e_k \text{ for the unique } T_{ijk} \neq 0, \text{ and } \; [e_i, e_j] \;=\; 0 \text{ otherwise.}$$

The last lemma says, of course, that $T$ is split and that it determines the commutator structure of the manifold completely. Here we should add that $dT = 4(\alpha_4 - \alpha_5 - \alpha_6)$, so we again see that the

4-forms which are more naturally related to the general geometric structure are $\alpha_{4,5,6}$ while $\alpha_{1,2,3}$ are rather exceptional for the particular space.

**Geometric structures.** We point out that there is no invariant 2-form on $W^{12}$, so that the manifold cannot be almost complex.

REMARK 3.26. This contradicts a result in [**AM07**], which we show is based on an erroneous claim. Namely, the structures $J_l$, $l = 1, 2, 3$ defined on p. 7 of [**AM07**] are thought to be invariant based on the property

$$J_l[Y, X]_{\mathfrak{m}} = [Y, J_l X]_{\mathfrak{m}} \tag{3.2}$$

for any $X \in \mathfrak{m}$, $Y \in \mathfrak{h}$. Take, for example, $l = 3$, $X = e_1$ and $Y = h_1$. $J_3$ is defined via

$$J_3 \begin{bmatrix} 0 & a_{12} & a_{13} \\ -\bar{a_{12}} & 0 & a_{23} \\ -\bar{a_{13}} & -\bar{a_{23}} & 0 \end{bmatrix} = \begin{bmatrix} 0 & ka_{12} & ka_{13} \\ k\bar{a_{12}} & 0 & ka_{23} \\ k\bar{a_{13}} & k\bar{a_{23}} & 0 \end{bmatrix}$$

Then the left hand side of the invariance property (3.2) equals

$$J_3[h_1, e_1]_{\mathfrak{m}} = J_3 e_2 = e_3$$

while for the right hand side we get

$$[h_1, J_3 e_1]_{\mathfrak{m}} = [h_1, e_4]_{\mathfrak{m}} = -e_3,$$

so that $J_3$ cannot be invariant. Similar arguments show that $J_1$ and $J_2$ also aren't invariant.

We have already mentioned that $W^{12}$ cannot admit a quaternion Kähler structure since it isn't a symmetric space. We now try to answer the question as to whether there exists a quaternion Kähler structure with torsion (where the torsion tensor is assumed totally skew). This amounts to determining whether a $\nabla^t$-invariant rank 3 distribution in the bundle of antisymmetric endomorphisms of $TW^{12}$ exists for some value of $t$. First we work with the canonical connection, i.e. the case $t = 1/4$.

PROPOSITION 3.27. *The holonomy algebra $\mathfrak{hol}^c$ equals the isotropy algebra $\mathfrak{h} = \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$:*

$$\mathfrak{hol}^c = \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) = \mathfrak{h}.$$

PROOF. The holonomy algebra is generated by all curvature transformations, which we can explicitly compute using Wang's theorem as mentioned in Remark 3.3. Then the curvature endomorphism of any connection is given through its corresponding map $\Lambda : \mathfrak{m} \to \mathfrak{so}(\mathfrak{m})$ by

$$R(X, Y)Z = \Lambda(X)\Lambda(Y)Z - \Lambda(Y)\Lambda(X)Z - \Lambda([X, Y]_{\mathfrak{m}})Z - iso([X, Y]_{\mathfrak{h}})Z$$

for $X, Y, Z \in \mathfrak{m}$. In particular, for the canonical connection $\Lambda \equiv 0$, so only the final term of the expression remains. It is nonzero only when $[X, Y]_{\mathfrak{h}} \neq 0$, that is when $X, Y \in \mathfrak{m}_i^{\mathbb{H}}$ belong to the same block of $\mathfrak{m}$. We have these relations explicitly:

$$\begin{array}{lll}
[\mathfrak{m}_1^{\mathbb{H}}, \mathfrak{m}_1^{\mathbb{H}}]: & [\mathfrak{m}_2^{\mathbb{H}}, \mathfrak{m}_2^{\mathbb{H}}]: & [\mathfrak{m}_3^{\mathbb{H}}, \mathfrak{m}_3^{\mathbb{H}}]: \\
{[e_1, e_2] = 2h_1 - 2h_4} & [e_5, e_6] = 2h_1 - 2h_7 & [e_9, e_{10}] = 2h_4 - 2h_7 \\
{[e_1, e_3] = 2h_2 - 2h_5} & [e_5, e_7] = 2h_2 - 2h_8 & [e_9, e_{11}] = 2h_5 - 2h_8 \\
{[e_1, e_4] = 2h_3 - 2h_6} & [e_5, e_8] = 2h_3 - 2h_9 & [e_9, e_{12}] = 2h_6 - 2h_9 \\
{[e_2, e_3] = 2h_3 + 2h_6} & [e_6, e_7] = 2h_3 + 2h_9 & [e_{10}, e_{11}] = 2h_6 + 2h_9 \\
{[e_2, e_4] = -2h_2 - 2h_5} & [e_6, e_8] = -2h_1 - 2h_8 & [e_{10}, e_{12}] = -2h_5 - 2h_8 \\
{[e_3, e_4] = 2h_1 + 2h_4} & [e_7, e_8] = 2h_1 + 2h_7 & [e_{11}, e_{12}] = 2h_4 + 2h_7.
\end{array}$$

One now easily sees that all elements of the isotropy algebra appear as elements of the holonomy algebra, so $\mathfrak{hol}^c = \mathfrak{h} = \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$. □

We are interested in the representation theory of $\mathfrak{hol}^c = \mathfrak{h}$, which we view as the 9-dimensional Lie algebra with basis $\{h_p\}_{p=1}^{9}$. We choose the Cartan subalgebra $\mathfrak{c}$ spanned by $h_1, h_4$, and $h_7$.

Whenever we are given a complex representation $\varrho : \mathfrak{h} \to \mathrm{End}(V)$,

$$V_\alpha := \{v \in V | \varrho(h)v = i.\alpha(h)v \quad \forall h \in \mathfrak{c}\ \},$$

where $\alpha \in \mathfrak{h}^*$, is called a weight space of $V$ with weight $\alpha$. The weights of the adjoint representation are called roots. The set of all roots always contains an even number of elements and is symmetric with respect to multiplication by $-1$. Thus we can choose half of them, which we call positive, and the other half - negative, as long as the sums of positive roots, if still roots, are positive. With respect to this choice, one defines partial ordering in the set of weights. One can then characterize the finite-dimensional irreducible representations of $\mathfrak{h}$ using the notion of highest weight - that is a weight of a representation, which is higher than any other weight of the representation with respect to the introduced partial ordering. The theorem is that an irreducible representation always admits a highest weight and any two representations with the same highest weight are isomorphic to each other.

Let us consider the action of $\mathfrak{h}$ on the complexification $W_1 := \mathfrak{m}_1^{\mathbb{H}} \otimes \mathbb{C}$. Fixing the positive roots

$$(2,0,0),\ (0,2,0),\ (0,0,2) \in \mathfrak{h}^*,$$

we find that the highest weight of $W_1$ is $(1,1,0)$. If we define $W_2$ and $W_3$ analogously, they have highest weights $(1,0,1)$ and $(0,1,1)$, respectively. Moreover, we can use Weyl's dimension formula to determine the dimension of an $\mathfrak{h}$-representation with highest weight $\alpha = (a,b,c)$, $a,b,c \geq 0$. The formula simplifies to $\dim(V_\alpha) = (a+1)(b+1)(c+1)$ in our case. We conclude that the three modules $\mathfrak{m}_i^{\mathbb{H}}$ are irreducible since their complexifications are. Let's write once again the $\mathfrak{h}$-invariant splitting of $\mathfrak{m}$ into blocks and compute its second exterior power:

$$\mathfrak{m} = \mathfrak{m}_1^{\mathbb{H}} \oplus \mathfrak{m}_2^{\mathbb{H}} \oplus \mathfrak{m}_3^{\mathbb{H}}$$
$$\Lambda^2(\mathfrak{m}) = \Lambda^2(\mathfrak{m}_1^{\mathbb{H}}) \oplus \Lambda^2(\mathfrak{m}_2^{\mathbb{H}}) \oplus \Lambda^2(\mathfrak{m}_3^{\mathbb{H}}) \oplus \mathfrak{m}_1^{\mathbb{H}} \wedge \mathfrak{m}_2^{\mathbb{H}} \oplus \mathfrak{m}_1^{\mathbb{H}} \wedge \mathfrak{m}_3^{\mathbb{H}} \oplus \mathfrak{m}_2^{\mathbb{H}} \wedge \mathfrak{m}_3^{\mathbb{H}}.$$

The 6-dimensional representation $\Lambda^2(\mathfrak{m}_1^{\mathbb{H}})$ splits into two irreducible modules:

$$\Lambda^2(\mathfrak{m}_1^{\mathbb{H}}) = V^3_{1,(2,0,0)} \oplus V^3_{1,(0,2,0)},$$

which we have indexed by the $\mathfrak{m}_i^{\mathbb{H}}$ they arise from and their highest weights. They are both 3-dimensional by Weyl's formula, as the superscript indicates. Explicit bases are given by:

$$V^3_{1,(2,0,0)} = \mathrm{span}\{e_{12} + e_{34}, e_{13} - e_{24}, e_{14} + e_{23}\}$$
$$V^3_{1,(0,2,0)} = \mathrm{span}\{e_{12} - e_{34}, e_{13} + e_{24}, e_{14} - e_{23}\}.$$

We give the analogous results for $\Lambda^2(\mathfrak{m}_2^{\mathbb{H}})$ and $\Lambda^2(\mathfrak{m}_3^{\mathbb{H}})$:

$$V^3_{2,(2,0,0)} = \mathrm{span}\{e_{56} + e_{78}, e_{57} - e_{68}, e_{58} + e_{67}\}$$
$$V^3_{2,(0,0,2)} = \mathrm{span}\{e_{56} - e_{78}, e_{57} + e_{68}, e_{58} - e_{67}\}$$
$$V^3_{3,(0,2,0)} = \mathrm{span}\{e_{9,10} + e_{11,12}, e_{9,11} - e_{10,12}, e_{9,12} + e_{10,11}\}$$
$$V^3_{3,(0,0,2)} = \mathrm{span}\{e_{9,10} - e_{11,12}, e_{9,11} + e_{10,12}, e_{9,12} - e_{10,11}\}.$$

Of course, at the end we have said nothing else than that $\Lambda^2(\mathfrak{m})$ contains two isomorphic copies of $\mathfrak{h}$, on both of which it acts by its adjoint representation. Thus the splitting we've obtained is actually the splitting into root spaces, as also seen from the weights, which equal the positive roots we had fixed. Let's turn to the modules of type $\mathfrak{m}_i^{\mathbb{H}} \wedge \mathfrak{m}_j^{\mathbb{H}}$. The highest weight vectors there are obtained by wedging the corresponding highest weight vectors of $\mathfrak{m}_i^{\mathbb{H}}$ and $\mathfrak{m}_j^{\mathbb{H}}$. The weights are then added, so we conclude that $\mathfrak{m}_1^{\mathbb{H}} \wedge \mathfrak{m}_2^{\mathbb{H}}, \mathfrak{m}_1^{\mathbb{H}} \wedge \mathfrak{m}_3^{\mathbb{H}}$, and $\mathfrak{m}_2^{\mathbb{H}} \wedge \mathfrak{m}_3^{\mathbb{H}}$ admit highest weight vectors for the weights $(2,1,1), (1,2,1)$, and $(1,1,2)$, respectively. Thus each of them, being 16-dimensional, reduces to a sum of a $3.2.2 = 12$-dimensional irreducible module with the mentioned weight and a 4-dimensional module with weight $(0,1,1)$, $(1,0,1)$, or $(1,1,0)$, respectively. The last claim can also be seen by observing that

$$\begin{aligned}[.,.] : \mathfrak{m}_i^{\mathbb{H}} \wedge \mathfrak{m}_j^{\mathbb{H}} &\longrightarrow \mathfrak{m}_k^{\mathbb{H}} \\ e_i \wedge e_j &\longmapsto [e_i, e_j]\end{aligned}$$

is an $\mathfrak{h}$-equivariant map since $[h,[e_i,e_j]] = [[h,e_i],e_j] + [e_i,[h,e_j]]$ holds for any $h \in \mathfrak{h}$, $e_i \in \mathfrak{m}_i^{\mathbb{H}}$, $e_j \in \mathfrak{m}_j^{\mathbb{H}}$. The 4-dimensional module is now isomorphic to the image of the map, and the 12-dimensional one is its kernel. Now we finally have the irreducible decomposition

$$\begin{aligned}\Lambda^2(\mathfrak{m}) =& V^3_{1,(2,0,0)} \oplus V^3_{1,(0,2,0)} \oplus V^3_{2,(2,0,0)} \oplus V^3_{2,(0,0,2)} \oplus V^3_{3,(0,2,0)} \oplus V^3_{3,(0,0,2)} \\ &\oplus V^{12}_{(2,1,1)} \oplus V^{12}_{(1,2,1)} \oplus V^{12}_{(1,1,2)} \oplus V^4_{(0,1,1)} \oplus V^4_{(1,0,1)} \oplus V^4_{(1,1,0)}.\end{aligned} \tag{3.3}$$

Yet none of the $\mathfrak{h}$-invariant 3-dimensional modules in the splitting defines a qKT structure on $W^{12}$, because they do not contain almost complex structures. The property $J^2 = -Id$ fails in all cases.

**Structure group.** Consider the following setting: let $M = G/H$ be an $n$-dimensional Riemannian homogeneous space. It gives rise to the principle fibre bundle $G \to G/H$, whose fibre is isomorphic to $H$. $H$ being also the structure group of the principal bundle is in a natural way a subgroup of $SO(n)$. We now ask the question how $H$ sits in $SO(n)$.

Consider $W^{12} = Sp(3)/Sp(1)^3$. We are looking for an embedding of $Sp(1)^3$ in $SO(12)$. It will be easier to work on a Lie algebra level, where this translates to finding $\mathfrak{h} = \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ in $\mathfrak{so}(12)$. There are many such embeddings, but we can discard most of them by decomposing $\mathfrak{so}(12) \cong \Lambda^2(\mathfrak{m})$ in $\mathfrak{h}$-irreducible modules w.r.t. the embedding in question and comparing the result with the decomposition 3.3. Performing this with the help of the computer algebra program LiE results in the following:

PROPOSITION 3.28. *The Lie algebra $\mathfrak{h}$ lies in $\mathfrak{so}(12)$ in the following way:*
*Embed $SO(4)^3 \subset SO(12)$ diagonally which gives $3\mathfrak{so}(4) = 2\mathfrak{sp}(1) \oplus 2\mathfrak{sp}(1) \oplus 2\mathfrak{sp}(1)$ in $\mathfrak{so}(12)$. Group the six $\mathfrak{sp}(1)$ summands in three pairs, taking for each pair copies coming from different $\mathfrak{so}(4)$'s. Then take the diagonal in each of the three pairs, finding $\mathfrak{h} \subset 2\mathfrak{sp}(1) \oplus 2\mathfrak{sp}(1) \oplus 2\mathfrak{sp}(1) \subset \mathfrak{so}(12)$.*

PROOF. We are led to this construction by the computations done above. The tangent space $\mathfrak{m}$ is split into three different 4-dimensional blocks, each of which admits nontrivial actions from two of the $\mathfrak{sp}(1)$'s in $\mathfrak{h}$ and gets acted upon trivially by the third copy of $\mathfrak{sp}(1)$. We achieve this effect by taking the diagonals as described above. The choice of the pairing is not unique, but they all lead to equivalent results due to the available symmetries. □

**The octonionic Wallach space $W^{24}$.** In this section we focus our discussion on the Wallach space of biggest dimension: $W^{24} = F_4/\mathrm{Spin}(8)$. We do computations mainly using the predefined realization of $\mathfrak{f}(4)$ as a compact subgroup of all $26 \times 26$ matrices with real entries "f(4, Compact)" from Maple's Differential geometry package. We will thus not make it explicit here as we did in the $W^{12}$ case. A $27 \times 27$ matrix realization given in [**BCCS09**] is also used to cross-check some computations using MATLAB. The number of dimensions we are dealing with limits the amount of computations we can perform, but the simplicity of the isotropy algebra simplifies some other computations. We adopt the weights used for the simple Lie algebra $D_4$ in LiE. The three non-isomorphic irreducible 8-dimensional representations of $D_4 = \mathfrak{so}(8)$ have weights $[1,0,0,0]$, $[0,0,1,0]$, and $[0,0,0,1]$, where the first one is the standard representation, and the other two are the half-spin representations. The adjoint representation has weight $[0,1,0,0]$.

**The invariants.** The following decomposition of the Lie algebra $\mathfrak{f}(4)$ is well-known ([**B02**]):

$$\mathfrak{f}_4 = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{so}(8) \oplus (\mathfrak{m}_1^{\mathbb{O}} \oplus \mathfrak{m}_2^{\mathbb{O}} \oplus \mathfrak{m}_3^{\mathbb{O}}).$$

The 8-dimensional representations $\mathfrak{m}^{\mathbb{O}}_{1,2,3}$ of the isotropy algebra must either be reducible to 8-fold sums of the trivial representation, or be equal to some 8-dimensional $\mathfrak{so}(8)$-representation. A computer check shows that they are irreducible. Now they must be the three 8-dimensional representations mentioned above: $\mathfrak{m}^{\mathbb{O}}_1$ with weight $[1,0,0,0]$, $\mathfrak{m}^{\mathbb{O}}_2$ with weight $[0,0,1,0]$, and $\mathfrak{m}^{\mathbb{O}}_3$ with weight $[0,0,0,1]$. We decompose all spaces that appear in Table 3.1 with respect to the action of the isotropy algebra $\mathfrak{h} = \mathfrak{so}(8)$.

PROPOSITION 3.29. *One has the following decompositions in $\mathfrak{h}$-irreducible modules:*

$$S^2(\mathfrak{m}) = V^{56,1} \oplus V^{56,2} \oplus V^{56,3} \oplus V^{35,1} \oplus V^{35,2} \oplus V^{35,3} \oplus \mathfrak{m}^{\mathbb{O}}_1 \oplus \mathfrak{m}^{\mathbb{O}}_2 \oplus \mathfrak{m}^{\mathbb{O}}_3 \oplus 3\mathbb{R}.$$

*In particular, there are: three invariant symmetric $(0,2)$-tensors;*
*One invariant symmetric $(0,3)$-tensor of split type:*

$$\begin{aligned} S^3(\mathfrak{m}) =& V^{350} \oplus V^{224,1} \oplus V^{224,2} \oplus V^{224,3} \oplus V^{224,4} \oplus V^{224,5} \oplus V^{224,6} \oplus V^{112,1} \oplus V^{112,2} \oplus V^{112,3} \\ &\oplus 2V^{56,1} \oplus 2V^{56,2} \oplus 2V^{56,3} \oplus V^{35,1} \oplus V^{35,2} \oplus V^{35,3} \oplus 2\mathfrak{h} \oplus 3\mathfrak{m}^{\mathbb{O}}_1 \oplus 3\mathfrak{m}^{\mathbb{O}}_2 \oplus 3\mathfrak{m}^{\mathbb{O}}_3 \oplus \mathbb{R}; \end{aligned}$$

*No $\mathfrak{h}$-invariant 2-forms:*

$$\Lambda^2(\mathfrak{m}) = V^{56,1} \oplus V^{56,2} \oplus V^{56,3} \oplus 3\mathfrak{h} \oplus \mathfrak{m}^{\mathbb{O}}_1 \oplus \mathfrak{m}^{\mathbb{O}}_2 \oplus \mathfrak{m}^{\mathbb{O}}_3; \tag{3.4}$$

*One invariant 3-form of split type:*

$$\begin{aligned} \Lambda^3(\mathfrak{m}) =& V^{350} \oplus 2V^{160,1} \oplus 2V^{160,2} \oplus 2V^{160,3} \oplus 3V^{56,1} \oplus 3V^{56,2} \oplus 3V^{56,3} \\ &\oplus V^{35,1} \oplus V^{35,2} \oplus V^{35,3} \oplus 2\mathfrak{h} \oplus 2\mathfrak{m}^{\mathbb{O}}_1 \oplus 2\mathfrak{m}^{\mathbb{O}}_2 \oplus 2\mathfrak{m}^{\mathbb{O}}_3 \oplus \mathbb{R}; \end{aligned}$$

*Three invariant 4-forms;*
*Three invariant 5-forms;*
*One invariant 6-form;*
*Three invariant tensors of type $(1,2)$.*

PROOF. The proof is a direct computation using LiE. We've indexed all irreducible $\mathfrak{h}$-modules by their dimensions (in superscript). The isomorphic modules are grouped together, while the non-isomorphic ones are distinguished by the second superscript. The known modules $\mathfrak{m}^{\mathbb{O}}_i$, $\mathfrak{h}$, and $\mathbb{R}$ are mentioned explicitly. The statement that the symmetric $(0,3)$-tensor and the 3-form are of split type is seen directly in the computations. For example, the $\mathbb{R}$ module in $\Lambda^3(\mathfrak{m})$ arises in the following way: consider

$$\begin{aligned} \Lambda^3(\mathfrak{m}^{\mathbb{O}}_1 \oplus \mathfrak{m}^{\mathbb{O}}_2 \oplus \mathfrak{m}^{\mathbb{O}}_3) &= (\Lambda^3(\mathfrak{m}^{\mathbb{O}}_1 \oplus \mathfrak{m}^{\mathbb{O}}_2) \otimes \mathbb{R}) \oplus (\Lambda^2(\mathfrak{m}^{\mathbb{O}}_1 \oplus \mathfrak{m}^{\mathbb{O}}_2) \otimes \mathfrak{m}^{\mathbb{O}}_3) \oplus ... \\ &= ... \oplus (\Lambda^2(\mathfrak{m}^{\mathbb{O}}_1) \oplus \mathfrak{m}^{\mathbb{O}}_1 \otimes \mathfrak{m}^{\mathbb{O}}_2 \oplus \Lambda^2(\mathfrak{m}^{\mathbb{O}}_2)) \otimes \mathfrak{m}^{\mathbb{O}}_3 \oplus ... \\ &= ... \oplus \mathfrak{m}^{\mathbb{O}}_1 \otimes \mathfrak{m}^{\mathbb{O}}_2 \otimes \mathfrak{m}^{\mathbb{O}}_3 \oplus ... \end{aligned}$$

The three-fold tensor product on the last line occurs only once in the decomposition and is not $\mathfrak{h}$-irreducible. Its decomposition in irreducible modules contains the single copy of $\mathbb{R}$ that we find at the end. In this manner we see that the tensor spanning this 1-dimensional module is of split type. Exactly the same reasoning applies to $S^3(\mathfrak{m})$. □

As before, each of the symmetric $(0,2)$-tensors corresponds to the "standard metric" on $\mathfrak{m}^{\mathbb{O}}_i$. As we can see, all results match those of $W^{12}$ precisely, with the exception of the three additional four forms in the previous case, stemming from the fact that the tangent space splits into three four-dimensional blocks. This underlines the similarity of the geometry, and is most notably represented in the symmetric (0,3)-tensor and the 3-form, which are again of split type. Note that here the modules $V^p$ are already irreducible.

**Geometric structure.** The situation here is again much similar to what happens with $W^{12}$. The invariant torsion tensors have no vectorial components. They are also neither pure skew-symmetric, nor of pure Cartan type. Considered as $(0,3)$-tensors, they all project to the invariant 3-form when skew-symmetrized. Following this observation, we can consider the same family of connections as we did in the previous case.

**Structure group.** As in the $W^{12}$ case, we describe the structure group of $F_4 \to F_4/\mathrm{Spin}(8)$. We are looking for an embedding of Spin(8) in $SO(24)$. On a Lie algebra level this means finding $\mathfrak{so}(8)$ in $\mathfrak{so}(24)$. Again, there are many ways to do this, but we discard most of them by comparing with 3.4. This time the picture looks as follows:

PROPOSITION 3.30. *One embeds $SO(8) \times SO(8) \times SO(8) \subset SO(24)$ diagonally, obtaining $\mathfrak{so}(8) \oplus \mathfrak{so}(8) \oplus \mathfrak{so}(8)$ in $\mathfrak{so}(24)$ and takes an appropriate diagonal embedding of $\mathfrak{so}(8) \subset \mathfrak{so}(8) \oplus \mathfrak{so}(8) \oplus \mathfrak{so}(8)$.*

PROOF. The Lie algebra $\mathfrak{h} = \mathfrak{so}(8)$ acts on each 8-dimensional block of $\mathfrak{m}$ via a different irreducible representation. Namely, we see all three possible 8-dimensional irreducible representations [1,0,0,0], [0,0,1,0], and [0,0,0,1] of $\mathfrak{so}(8)$ occur once. We can obtain this effect by letting all copies of $\mathfrak{so}(8)$ in $\mathfrak{so}(8) \oplus \mathfrak{so}(8) \oplus \mathfrak{so}(8)$ act on every block of $\mathfrak{m}$, but in a different way. This is done by using the $S_3$-symmetry of the Dynkyn diagram of $D_4 = \mathfrak{so}(8)$ to twist the algebras while taking the diagonal. It is precisely what we mean by an "appropriate diagonal embedding". This construction may indeed be realized, as one can show using LiE. $\square$

# Appendix

## A. Curvature identities

Let's find an expression for the curvature operator $\mathcal{R}(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$ with $R(X,Y,Z,V) = \langle \mathcal{R}(X,Y)Z, V\rangle$. We know from [**Agr06**] the following:

PROPOSITION A.1. *If $e_1, \dots, e_n$ is an orthonormal frame and $S(X,Y) = \sum_i \langle T(e_i,X), T(e_i,Y)\rangle$, then the following identities hold:*

$$R(X,Y,Z,V) = R^g(X,Y,Z,V) + \frac{1}{2}(\nabla_X T)(Y,Z,V) - \frac{1}{2}(\nabla_Y T)(X,Z,V) + \frac{1}{4}\langle T(X,Y), T(Z,V)\rangle + \frac{1}{4}\sigma_T(X,Y,Z,V), \tag{A.1}$$

$$\mathrm{Ric}(X,Y) = \mathrm{Ric}^g(X,Y) - \frac{1}{2}(\delta T)(X,Y) - \frac{1}{4}S(X,Y), \tag{A.2}$$

$$\mathrm{Scal} = \mathrm{Scal}^g - \frac{3}{2}\|T\|^2 . \tag{A.3}$$

The condition that $\nabla$ is metric implies that each curvature transformation $\mathcal{R}(X,Y)$ is skew-adjoint, i. e. $\mathcal{R}$ can be interpreted as an endomorphism $\mathcal{R}: \wedge^2 TM \longrightarrow \mathfrak{hol}^\nabla \subset \wedge^2 TM$, where $\mathfrak{hol}^\nabla$ is the holonomy algebra of $\nabla$. As such, it is defined by

$$\mathcal{R}(e_i \wedge e_j) = \frac{1}{2}\sum_{k,l} R_{ijkl}(e_k \wedge e_l).$$

Here $\{e_i\}$ is an orthonormal basis as always. This definition can be seen to be linked to the first one by the relation

$$\mathcal{R}(X,Y)\omega = \varrho(\mathcal{R}(X \wedge Y))\omega$$

when applied to a $k$-form $\omega$. The same argument applies to the Levi-Civita connection $\nabla^g$. Denote the last four terms on right hand side of (A.1) by $F(X,Y,Z,V)$, i. e.

$$R(X,Y,Z,V) \ = \ R^g(X,Y,Z,V) + F(X,Y,Z,V).$$

$F(X,Y,-,-)$ may be considered as a 2-form, which we denote by $\mathcal{F}(X,Y)$. Note also that the 2-form $R(X,Y,-,-)$ is nothing else but $\mathcal{R}(X \wedge Y)$.

PROPOSITION A.2. *If $\omega$ is a $k$-form, then*

$$\mathcal{R}(X,Y)\omega = \mathcal{R}^g(X,Y)\omega + \varrho(\mathcal{F}(X,Y))\omega. \tag{A.4}$$

PROOF. It is obvious from the previous comments that

$$\mathcal{R}(X \wedge Y) = \mathcal{R}^g(X \wedge Y) + \mathcal{F}(X,Y).$$

Applying $\varrho$ to both sides we get the claim directly. □

We should mention that a Ricci-type decomposition of the curvature tensor is possible. For this purpose we need the block-interchanging symmetry of the curvature tensor, i. e. we require the torsion $T$ to be $\nabla$-Killing. In the case when it is actually parallel, the decomposition is obtained in [**AF14**]. There, the curvature is split with respect to the kernel and image of the Bianchi map

$b : R_{ijkl} \longmapsto R_{ijkl} + R_{jkil} + R_{kijl}$. We can still do the same, obtaining the same kernel part, and a slightly different expression for the term in the image. More precisely, $R = R_{ker} + R_{im}$ with

$$R_{ker}(X,Y,Z,V) = R^g(X,Y,Z,V) + \frac{1}{4}\langle T(X,Y),T(Z,V)\rangle - \frac{1}{12}\sigma_T(X,Y,Z,V)$$
$$R_{im}(X,Y,Z,V) = \frac{1}{4}d^\nabla T(X,Y,Z,V) + \frac{1}{3}\sigma_T(X,Y,Z,V).$$

It is important to note that $R_{im}$ is a 4-form so it does not affect the Ricci or scalar curvatures, and we may write the known decomposition

$$R_{ker} = W + \frac{1}{n-2}\left(\mathrm{Ric} - \frac{\mathrm{Scal}}{n}g\right)\owedge g + \frac{\mathrm{Scal}}{2n(n-1)}g\owedge g,$$

where

$(A\owedge B)(X,Y,Z,V) = A(X,Z)B(Y,V) + A(Y,V)B(X,Z) - A(X,V)B(Y,Z) - A(Y,Z)B(X,V)$ is the Kulkarni-Nomizu product, and $W$ is the Weyl curvature tensor. It is traceless and it is related to $W^g$ by

$$\begin{aligned} W(X,Y,Z,V) =& W^g(X,Y,Z,V) + \frac{1}{4}\langle T(X,Y),T(Z,V)\rangle - \frac{1}{12}\sigma_T(X,Y,Z,V) \\ &+ \frac{1}{4(n-2)}(S\owedge g)(X,Y,Z,V) - \frac{3\|T\|^2}{4(n-1)(n-2)}(g\owedge g)(X,Y,Z,V). \end{aligned}$$

We give the formulas in the case of a 1-parameter family of connections. From now on we denote the values, corresponding to the canonical connection with a superscript "c", while so far we have used no superscript for those quantities. According to (5.4) they correspond to the parameter value $s = \frac{1}{4}$. We still use "g" to indicate the Riemannian values, i.e. the ones for $s = 0$.

Proposition A.3. *If $\nabla^s$ is the 1-parameter family of metric connections, given by (5.4), then the following identities hold:*

$$\begin{aligned} R^s(X,Y,Z,V) = {}& R^g(X,Y,Z,V) + 2s((\nabla^c_X T)(Y,Z,V) - (\nabla^c_Y T)(X,Z,V)) \\ &+ 4s^2\langle T(X,Y),T(Z,V)\rangle + 2s(1-2s)\sigma_T(X,Y,Z,V), \end{aligned} \tag{A.5}$$

$$\begin{aligned} \mathrm{Ric}^s(X,Y) &= \mathrm{Ric}^g(X,Y) - 2s(\delta T)(X,Y) - 4s^2 S(X,Y), \\ \mathrm{Scal}^s &= \mathrm{Scal}^g - 24s^2\|T\|^2. \end{aligned} \tag{A.6}$$

*The first Bianchi identity is given by*

$$\begin{aligned} \sum_{X,Y,Z} R^s(X,Y,Z,V) = {}& dT(X,Y,Z,V) + (\nabla^c_V T)(X,Y,Z) + (4s-1)\sum_{X,Y,Z}(\nabla^c_X T)(Y,Z,V) \\ &+ (6s - 8s^2 - 2)\sigma_T(X,Y,Z,V). \end{aligned}$$

Proof. The first identity follows after a straightforward computation, substituting $\nabla^s$ and $T^s$ for $\nabla$ and $T$ in (A.1). The following equalities are easily obtained by contraction in (A.5), or again by substituting with $\nabla^s$ and $T^s$ in the corresponding equations from proposition A.1. Finally, the first Bianchi identity follows from the one we have in the canonical case in theorem 2.6., [**Agr06**], which reads

$$\sum_{X,Y,Z} R^c(X,Y,Z,V) = dT(X,Y,Z,V) - \sigma_T(X,Y,Z,V) + (\nabla^c_V T)(X,Y,Z),$$

and the observation that

$$\begin{aligned} R^s(X,Y,Z,V) = {}& R^c(X,Y,Z,V) + \left(2s - \frac{1}{2}\right)((\nabla^c_X T)(Y,Z,V) - (\nabla^c_Y T)(X,Z,V)) \\ &+ \left(4s^2 - \frac{1}{4}\right)\langle T(X,Y),T(Z,V)\rangle - \left(2s - \frac{1}{2}\right)^2\sigma_T(X,Y,Z,V). \square \end{aligned}$$

Remark A.4. We again write equation (A.5) as $R^s(X,Y,Z,V) = R^g(X,Y,Z,V) + F^s(X,Y,Z,V)$ and denote by $\mathcal{F}^s(X,Y)$ the 2-form $F^s(X,Y,-,-)$.

## B. Algebraic identities

Here are some helpful identities involving $\Diamond$ and $\Box$ that we use in the text:

Proposition B.1. *If $\nabla^s$ is a 1-parameter family of metric connections, given by* (5.4), *and $\omega$ is a $k$-form, then the following identities hold:*

$$\begin{aligned}
T\Box(\nabla^s\omega) &= T\Box(\nabla^g\omega) - 2s\sigma_T\Box\omega + 2sB(\omega) + s.S(\omega),\\
d^s(T\Box\omega) &= (d^sT)\Box\omega - 2(\nabla^sT)\Box\omega + 2T\Box(\nabla^s\omega) - T\Box(d^s\omega),\\
\delta^s(T\Diamond\omega) &= -(\delta^sT)\Diamond\omega + 2(\nabla^sT)\Box\omega + 2T\Box(\nabla^s\omega) - T\Diamond(\delta^s\omega),\\
d^s(T\Diamond\omega) &= -(d^sT)\Diamond\omega + (\nabla^sT)\Diamond\omega + T\Diamond(\nabla^s\omega) - T\Diamond(d^s\omega),\\
(d^sT)\Box\omega &= (d^gT)\Box\omega - 8s\sigma_T\Box\omega,\\
T\Diamond(T\Box\omega) + T\Box(T\Diamond\omega) &= 2\sigma_T\Box\omega - B(\omega) - \frac{1}{2}S(\omega).
\end{aligned}$$

Proof. The computations for the first four identities are involving but straightforward. We are only required to follow the definitions closely and manipulate forms in a standard way. Here we skip them. The fifth identity follows directly from $d - d^s = T^s\Diamond$ and the definition of $\sigma_T$, and the last one follows easily from

$$4sT\Box(T\Diamond\omega) = (\delta - \delta^s)(T\Diamond\omega)$$

and the previous formulas in the proposition. The only new identity that will have to be established is

$$(\nabla^sT)\Box\omega - (\nabla^gT)\Box\omega = -2s\sigma_T\Box\omega,$$

which is again a direct computation. $\Box$

We also give the relevant computation rules concerning interior product with a vector field and the derivation rules of $d^\nabla$ and $\delta^\nabla$:

Proposition B.2. *Let $X$ be a vector field and $\omega, \eta$ $k$- and $l$-forms. The following identities hold:*

$$\begin{aligned}
X \lrcorner\ (\omega\Diamond\eta) &= -(X \lrcorner\ \omega)\Diamond\eta + (-1)^k\omega\Diamond(X \lrcorner\ \eta),\\
X \lrcorner\ (\omega\Box\eta) &= (X \lrcorner\ \omega)\Box\eta + (-1)^k\omega\Box(X \lrcorner\ \eta),\\
d^\nabla(\omega\wedge\eta) &= d^\nabla\omega\wedge\eta + (-1)^k\omega\wedge d^\nabla\eta,\\
\delta^\nabla(\omega\wedge\eta) &= \delta^\nabla\omega\wedge\eta + (-1)^k\omega\wedge\delta^\nabla\eta - \omega\Diamond(\nabla\eta) - (-1)^k(\nabla\omega)\Diamond\eta.
\end{aligned}$$

*In particular, for a function $f$ and a form $\omega$, we obtain*

$$\begin{aligned}
d^\nabla(f\omega) &= df\wedge\omega + f.d^\nabla\omega,\\
\delta^\nabla(f\omega) &= -df \lrcorner\ \omega + f.\delta^\nabla\omega.
\end{aligned}$$

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