Gorenstein stable surfaces satisfying $K_X^2 = 2$ and $\chi(\mathcal{O}_X) = 4$

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Abstract

We define and study a concrete stratification of the moduli space of Gorenstein stable surfaces X satisfying $K_X^2 = 2$ and $\chi(\mathcal{O}_X) = 4$, by first establishing an isomorphism with the moduli space of plane octics with certain singularities, which is then easier to handle concretely. In total, there are 47 inhabited strata with altogether 78 components.

Kurzzusammenfassung

Wir definieren und untersuchen eine Stratifizierung des Modulraums der Gorensteinschen stablien Flächen X mit den numerischen Invarianten $K_X^2 = 2$ und $\chi(\mathcal{O}_X) = 4$. Dazu zeigen wir, dass dieser Modulraum zu einem Modulraum gewisser ebener Kurven vom Grad acht isomorph ist, was eine konkrete Untersuchung ermöglicht. Letztendlich zerlegen wir den Modulraum in 47 Straten mit insgesamt 78 Komponenten.

Contents

Introduction

One of the most important bounds on the geography of surfaces of general type is Noether's inequality $2\chi(\mathcal{O}_X) \leq K_X^2 + 6$. A minimal surface of general type satisfying equality here is said to be on the Noether line. Since they have been studied intensively by Horikawa [27], they are also called *Horikawa surfaces*. The smallest possible invariants on the Noether line are $K_X^2 = 2$ and $\chi(\mathcal{O}_X) = 4$ and it is a classical fact that (the canonical model of) the corresponding surfaces are double-covers of $\mathbb{P}^2$, branched over an octic curve with at worst simple singularities, via the morphism defined by the canonical linear system $|K_X|$. Conversely, a double-cover of the plane branched over an octic with at worst simple singularities gives an example of such a surface. Therefore, the Gieseker-moduli space $\mathfrak{M}_{2,4}$ of canonical models X of surfaces of general type with invariants $K_X^2 = 2$ and $\chi(\mathcal{O}_X) = 4$ is in bijection with (in fact, isomorphic to) the moduli space of plane curves of degree 8 with at worst simple singularities.

The subject of this thesis is the study of the modular compactification $\overline{\mathfrak{M}}_{2,4}$ of $\mathfrak{M}_{2,4}$ parametrising stable surfaces X with the same numerical invariants $K_X^2 = 2$ and $\chi(\mathcal{O}_X) = 4$. We refer to this as the *KSBA-compactification*, for Kollár and Shepherd-Barron [32] and Alexeev [1]. We thereby continue the series of works by Franciosi, Pardini and Rollenske [**?**, 16–18] who investigated the moduli spaces parametrising Gorenstein stable surfaces X with $K_X^2 = 1$ using similar methods.

We can only handle the *Gorenstein* stable surfaces since this allows us to consider the canonical map. As in the classical case, the canonical linear system $|K_X|$ defines a double-cover of the plane, branched over an octic curve; this is the content of Corollary 1.7. Conversely, if $B \subset \mathbb{P}^2$ is a curve of degree 8 such that the pair $(\mathbb{P}, \frac{1}{2}B)$ is log-canonical, then the double-cover X of $\mathbb{P}^2$ branched over B, which is essentially unique since $\mathrm{Pic}(\mathbb{P}^2)$ is torsion-free, is a Gorenstein stable surface satisfying $K_X^2 = 2$ and $\chi(\mathcal{O}_X) = 4$. This way, we obtain an isomorphism between the moduli spaces of those plane octics and the moduli space $\overline{\mathfrak{M}}_{2,4}^{\mathrm{Gor}} \subset \overline{\mathfrak{M}}_{2,4}$ of Gorenstein stable surfaces X satisfying $K_X^2 = 2$ and $\chi(\mathcal{O}_X) = 4$; see Theorem 2.3. This allows us to use the rich theory of plane curves and the computer algebra system Macaulay2 [19] to get some understanding of the boundary components of $\mathfrak{M}_{2,4}$ in $\overline{\mathfrak{M}}_{2,4}^{\mathrm{Gor}}$.

More precisely, we will define a stratification of $\overline{\mathfrak{M}}_{2,4}^{\mathrm{Gor}}$ by means of three indicators: the degree of non-normality, the number and degrees of isolated irrational singularities and whether the irrational singularities are simply elliptic or cusps. The interest in the third indicator, even though not necessary to understand the birational isomorphism type of the surface, comes from the relation with another stratification induced by the degeneration of mixed Hodge structures on $H^2(X)$ as defined by Green, Griffiths, Kerr, Laza and Robles [20, 21, 30, 45, 46].

We will see that all inhabited strata are of the expected dimension, but many of the numerically characterised strata decompose further into disjoint components; this is mostly reflected by the birational types of the minimal resolutions. On the level of curves, the different components correspond to special configurations of the non-simple singularities.

Moreover, we will define a *Hodge type* of our surfaces under investigation (Definition 1.18). For the stratification of the locus parametrising normal surfaces, the Hodge type is constant on the strata, as shown in Proposition 3.12. For the locus of non-normal surfaces, however, this is much more complicated, as we will indicate in Example 3.17 and Example 3.18. This is why on the locus of non-normal surfaces, the stratification will not be fine enough to control the Hodge type.

The thesis is organised follows: In Chapter 1, we investigate the geometry of the surfaces of interest, i.e., we prove that they are canonically double-covers of the plane, discuss the singularities they may have, prove some constraints on the possible birational isomorphism types and we study the mixed Hodge structure on second cohomology.

Going back and forth between curves and branched double-covers defines an isomorphism between our moduli space of interest $\overline{\mathfrak{M}}_{2,4}^{\mathrm{Gor}}$ and the moduli space of certain plane curves; this will be the subject of Chapter 2. This moduli space has at least one more notable compactification, due to Hacking [22]; in Chapter 4, we will present a few remarks and questions in this direction.

Before that, in Chapter 3, we will define and study the stratification; first, for the locus parametrising normal surfaces and then for the remaining part. The full degeneration diagram would be incomprehensible, which is why we restrict the presentation to two fragments which give a sufficiently good idea of the situation.

To ease the flow of presentation, we have two appendices, one about the possible configurations of certain singularities on curves of degree 8 or 6, Appendix A, and the explanation and listing of the Macaulay2-code, Appendix B. Together they constitute most of the proof of the main theorem about which strata are inhabited and how they decompose into different components.

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Einleitung

Die Geographie der algebraischen Flächen vom allgemeinen Typ wird unter anderem von der Noetherschen Ungleichung $2\,\chi(\mathcal{O}_X) \leq K_X^2 + 6$ beschränkt. Wir sagen über eine minimale Fläche vom allgemeinen Typ X, sie liege auf der *Noether-Gerade*, wenn hier Gleichheit gilt. Die Flächen auf der Noether-Geraden wurden insbesondere von Horikawa untersucht [27], weshalb man sie auch *Horikawa-Flächen* nennt. Die kleinstmöglichen Invarianten K_X^2 und $\chi(\mathcal{O}_X)$ auf der Noether-Geraden, die sich tatsächlich durch Flächen vom allgemeinen Typ realisieren lassen, sind $K_X^2 = 2$ und $\chi(\mathcal{O}_X) = 4$. Jene Flächen X (genauer, ihre kanonischen Modelle,) sind allesamt kanonische verzweigte doppelte Überlagerungen der projektiven Ebene $\mathbb{P}^2$. Die Verzweigungskurve in der Ebene ist dabei vom Grad 8 und hat höchstens einfache Singularitäten. Bildet man umgekehrt die doppelte Überlagerung von $\mathbb{P}^2$ verzweigt über einer solchen Oktik, so erhält man eine minimale Fläche vom allgemeinen Typ. Folglich ist der Gisekersche Modulraum $\mathfrak{M}_{2,4}$ der kanonischen Modelle von Flächen vom allgemeinen Typ X mit $K_X^2 = 2$ und $\chi(\mathcal{O}_X) = 4$ isomorph zum Modulraum der ebenen Kurven vom Grad 8 mit höchstens einfachen Singularitäten.

Hauptgegenstand der vorliegenden Dissertation ist die Untersuchung der modularen Kompaktifizierung $\overline{\mathfrak{M}}_{2,4} \supset \mathfrak{M}_{2,4}$, welche die sogenannten *stabilen Flächen* X mit numerischen Invarianten $K_X^2 = 2$ und $\chi(\mathcal{O}_X) = 4$ parametrisiert. Wir nennen $\overline{\mathfrak{M}}_{2,4}$ die *KSBA-Kompaktifizierung*, nach Kollár, Shepherd-Barron [32] und Alexeev [1]. Hiermit setzen wir die Arbeiten von Marco Franciosi, Rita Pardini und Sönke Rollenske [**?**, 15–18] über die Gorensteinschen stablien Flächen X mit $K_X^2 = 1$ fort.

Die angewandte Methode zwingt uns ebenfalls dazu, uns auf die Gorensteinschen Flächen $\overline{\mathfrak{M}}_{2,4}^{\mathrm{Gor}} \subset \overline{\mathfrak{M}}_{2,4}$ zu beschränken. Wie im klassischen Fall können wir nämlich dann die jeweiligen Flächen X mittels des kanonischen Linearsystems $|K_X|$ als doppelte Überlagerungen der Ebene realisieren (Korollar 1.7). Wieder sind die Verzweigungskurven vom Grad 8, jedoch mit komplizierteren Singularitäten. Genauer sind es die Kurven $B \subset \mathbb{P}^2$ vom Grad 8 mit der Eigenschaft, dass das Paar $(\mathbb{P}^2, \frac{1}{2}B)$ log-kanonisch ist. Wir werden zeigen, dass der Modulraum $\overline{\mathfrak{M}}_{2,4}^{\mathrm{Gor}}$ zum Modulraum jener ebener Kurven isomorph ist (Satz 2.3), was es uns schließlich ermöglicht, die Geometrie der ebenen Kurven, die Singularitätentheorie und letztlich das Computeralgebra-System Macaulay2 [19] zu benutzen, um die Randkomponenten von $\overline{\mathfrak{M}}_{2,4}$ in $\overline{\mathfrak{M}}_{2,4}^{\mathrm{Gor}}$ besser zu verstehen.

Dazu definieren wir eine Stratifizierung von $\overline{\mathfrak{M}}_{2,4}^{\mathrm{Gor}}$ gemäß folgender Anhaltspunkte: Der Grad der Nichtnormalität, die Anzahl und der Grad elliptischer Singularitäten, sowie ob es sich dabei um einfach-elliptische, oder kuspidale elliptische Singularitäten handelt. Während der Grad der Nichtnormalität und die Anzahl und der Grad der elliptischen Singularitäten ausreicht, um den birationalen Typ der jeweiligen Flächen zu identifizieren, interessieren wir uns zudem für den Unterschied zwischen einfach-ellitpischen und kuspidalen Singularitäten, weil dies im engen Zusammenhang mit einer anderen, Hodge-theoretischen Stratifizierung steht, wie sie aus (teilweise in Bearbeitung befindlichen) Arbeiten von Green, Griffiths, Kerr, Laza und Robles [20, 21, 30, 45, 46] hervorgeht. Zu diesem Zweck werden wir für die betrachteten Flächen das Konzept von *Hodge Typen* einführen (Definition 1.18). Aus unseren Untersuchungen ergibt sich, dass der Hodge Typ konstant auf den Straten im Ort der normalen Flächen ist (Proposition 3.12). Dies trifft allerdings nicht auf die Straten zu, die nicht-normale Flächen parametrisieren, wie die Beispiele 3.17 und 3.18 zeigen.

Als Hauptresultat lässt sich zusammenfassen, dass alle nicht-leeren Straten die erwartete Dimension haben und dass ihre Komponenten paarweise disjunkt sind (Satz 3.8). Zudem können wir wie bereits angedeutet den birationalen Typ einer stabilen Gorensteinschen Fläche X mit $K_X^2 = 2$ und $\chi(\mathcal{O}_X) = 4$ an der zugehörigen Komponente der Stratifizierung ablesen (Satz 3.11 und Proposition 3.15).

Diese Dissertation ist wie folgt strukturiert. In Kapitel 1 untersuchen wir die Geometrie der Gorensteinschen Flächen mit den oben genannden numerischen Invarianten und zeigen, dass sie allesamt als doppelte Überlagerungen der Ebene, verzweigt über einer Oktik, entstehen. Zudem diskutieren wir die möglichen Singularitäten und gehen erste Schritte hin zur Berechnung der gemischten Hodgestruktur auf der zweiten Kohomologie.

Dass die Zuweisung, welche einer Gorensteinschen stabilen Fläche mit den genannten Invarianten die Verzweigungskurve der kanonischen doppelten Überlagerung zuordnet, ein Isomorphismus ist, ist Hauptgegenstand des 2. Kapitels. Umgekehrt bedeutet dies, dass $\overline{\mathfrak{M}}_{2,4}$ eine Kompaktifizierung des Modulraums der glatten ebenen Kurven vom Grad 8 definiert; dazu präsentieren wir in Kapitel 4 einige Bemerkungen und Fragen.

Zuvor, in Kapitel 3, definieren und untersuchen wir die Stratifizierung, zunächst für den Ort der normalen Flächen, danach für den Ort der übrigen Flächen. Hier befinden sich die Hauptresultate der vorliegenden Arbeit. Ein Diagramm, welches alle Komponenten, oder auch nur alle Straten, gemeinsam mit den möglichen Degenerationen, abbilden würde, wäre zu kompliziert. Jedoch vermitteln zwei spezielle Fragmente einen hinreichend guten Eindruck von der Gesamtsituation.

Diverse Resultate über die möglichen Konfigurationen von Singularitäten auf Oktiken und Sextiken, sowie der Macaulay2-Code, welche den Hauptanteil der Beweise der Hauptresultate bilden, sind in die Anhänge A und B ausgelagert.

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0.1 Notations and conventions. We work with schemes over the complex numbers $\mathbb{C}$. *Varieties* are reduced and proper schemes of finite type over $\mathbb{C}$ and a *surface* is a purely two-dimensional variety in this sense. A *curve* is a possibly non-reduced projective scheme, purely of dimension 1. In some places, points (of schemes) are implicitly understood as $\mathbb{C}$-rational points, but this is always clear from the context.

0.1.1 Notation. Let X be a proper complex scheme of pure dimension n.

- $p_g(X) = h^n(\mathcal{O}_X)$.
- $p_a(X) = (-1)^n(\chi(\mathcal{O}_X) - 1)$.
- If $n = 1$, then $\deg(L) = \chi(L) - \chi(\mathcal{O}_X)$ for all $L \in \mathrm{Pic}(X)$.
- $q(X) = h^1(\mathcal{O}_X)$.
- To avoid confusion between topological and holomorphic Euler characteristics, we use $\chi(-)$ only for coherent $\mathcal{O}_X$-modules and $\chi_{\mathrm{top}}(X)$ for the topological Euler characteristic of the analytification X^{an}.
- To ease notation, we use $H^*(X;\mathbb{C}) := H^*(X^{\mathrm{an}};\mathbb{C})$ for the singular cohomology of the analytification.

0.1.2 Semi-log-canonical varieties and pairs. We briefly recall the definition of a semi-log-canonical pair (from Kollár's exposition [34, Chapter 5], see there for more details). If a finite type scheme X over $\mathbb{C}$ satisfies the following two conditions, it is called *demi-normal*.

1. X satisfies Serre's condition (S_2), i.e., for every $x \in X$ we have
$$\mathrm{depth}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}) \geq \min\{2, \dim(\mathcal{O}_{X,x})\}.$$
2. X is regular or double normal crossing in codimension 1, i.e., if $x \in X$ is the generic point of a sub-variety of codimension one, then either $\mathcal{O}_{X,x}$ is regular, or its completion $\mathcal{O}^{\wedge}_{X,x}$ with respect to the maximal ideal is isomorphic to the complete local ring $\mathbb{C}[[x,y]]/(xy)$.

For a demi-normal scheme X, with normalisation $\pi\colon \overline{X} \to X$, the *conductor locus* $F := \mathrm{supp}(\pi_*\mathcal{O}_{\overline{X}}/\mathcal{O}_X) \subset X$ is purely one-codimensional, reduced and π is generically a double-cover over F, as explained by Kollár [34, Section 5.1]. The ideal sheaf defining F, $\mathrm{ann}_{\mathcal{O}_X}(\pi_*\mathcal{O}_{\overline{X}}/\mathcal{O}_X) = \mathcal{H}om_{\mathcal{O}_X}(\pi_*\mathcal{O}_{\overline{X}}, \mathcal{O}_X)$, is also an ideal in $\pi_*\mathcal{O}_{\overline{X}}$ which defines the conductor locus $\overline{F} \subset \overline{X}$ in the normalisation. It is the reduced pre-image of F.

Note that a demi-normal scheme satisfies Serre's condition (S_2) and is Gorenstein at all points of codimension one, i.e., satisfies (G_1). Therefore, there is a canonical sheaf ω_X and it is a divisorial sheaf which is locally free in codimension one. In particular, we can choose a canonical Weil-divisor K_X which is Cartier in codimension one.

A pair (X, D) of a variety X and an effective $\mathbb{Q}$-divisor D on X is *semi-log-canonical* if X is demi-normal, F and the support of D have no component in common, the divisor $K_X + D$ is $\mathbb{Q}$-Cartier and the pair $(\overline{X}, \pi_*^{-1}D + \overline{F})$ is log-canonical (cf. Kollár [34, Definition 2.8]).

A variety X is *semi-log-canonical* if the pair $(X, 0)$ is semi-log-canonical. In particular, the canonical divisor K_X is $\mathbb{Q}$-Cartier then, i.e., X is $\mathbb{Q}$-Gorenstein. A *stable surface* is a projective, connected, semi-log-canonical surface X whose canonical divisor K_X is ample. More generally, a *stable log-surface* is a semi-log-canonical pair (X, Δ) where X is a connected and projective surface and such that $K_X + \Delta$ is ample.

We will also need the notion of Du Bois-singularities; see Kollár [34, Chapter 6] for a concise introduction. Since we will ultimately be concerned with Gorenstein stable surfaces, for our purposes, it is enough to note the following two facts. For one, semi-log-canonical singularities are Du Bois, see Kollár [34, Corollary 6.32] or Kovács, Schwede, Smith [37, Theorem 4.16]. Conversely, Du Bois-singularities which are demi-normal and Gorenstein are semi-log-canonical by Doherty [12, Theorem 4.2].

0.1.3 Stable surfaces and their normalisations—Kollár's glueing. Let X be a stable surface with normalisation $\pi\colon \overline{X} \to X$ and conductor loci $F \subset X$ and $\overline{F} \subset \overline{X}$. Then $K_{\overline{X}} + \overline{F}$ is an ample $\mathbb{Q}$-Cartier divisor. Moreover, the restriction of π to $\overline{F} \to F$ is generically a double-cover and after passing to normalisations $\overline{F}^\nu \to F^\nu$, it is the quotient of a Galois involution $\tau\colon \overline{F}^\nu \to \overline{F}^\nu$. The surface X can then be recovered from these data as the following diagram is a composition of push-outs:

$$\begin{array}{ccccc} \overline{F}^\nu & \xrightarrow{\nu} & \overline{F} & \longrightarrow & \overline{X} \\ \big\downarrow{\scriptstyle /\tau} & & \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi} \\ F^\nu & \xrightarrow{\nu} & F & \longrightarrow & X \end{array}$$

This follows from Kollár's Glueing Theorem [34, Theorem 5.13] and its proof there. More precisely, this theorem captures when exactly the data $(\overline{X}, \overline{F}, \tau)$ arise from a stable surface X (or a stable log-surface (X, Δ) in the presence of a boundary divisor $\overline{\Delta}$). Moreover, in this correspondence, X is Gorenstein if and only if the involution τ on $\overline{F}^\nu$ induces a fixed-point free involution on the pre-images of the nodes of $\overline{F}$, as shown by Franciosi, Pardini and Rollenske [17, Addendum to Theorem 3.2].

0.1.4 Divisors on demi-normal varieties. A ($\mathbb{Q}$-)*Weil divisor* D on X is a $\mathbb{Z}$- (respectively $\mathbb{Q}$-)linear combination of integral sub-varieties of codimension one in X. Its *support* is the reduced union of all sub-varieties with non-trivial coefficient. If X is demi-normal, a divisor D on X is said to be *well-behaved* if its support and the conductor locus $F \subset X$ do not share a common component. An effective *Cartier divisor* is a sub-scheme $D \subset X$ whose ideal sheaf $\mathcal{O}_X(-D)$ is invertible and D is said to be *almost-Cartier* (cf. Hartshorne [24, 25]), if the ideal sheaf is invertible at all points of codimension one, i.e., outside a closed sub-scheme of codimension at least two.

An effective almost-Cartier divisor $D \subset X$ gives rise to an effective Weil divisor through $\sum_{C \subset X} \operatorname{length}(\mathcal{O}_{D,\eta_C})C$, where the sum runs through the integral closed sub-schemes $C \subset X$ and $\eta_C \in C$ is the generic point, the length being computed as $\mathcal{O}_{C,\eta_C}$-module. We will, by abuse of language, call a Weil divisor *almost-Cartier*, if it is the difference of two effective divisors arising from almost-Cartier divisors; furthermore, it is called Cartier if the corresponding almost-Cartier generalised divisor is Cartier. As usual, a ($\mathbb{Q}$-)Weil divisor is called $\mathbb{Q}$-Cartier if it has an integral multiple which is Cartier. For example, if L is a divisorial sheaf on a demi-normal scheme, a regular section $s \in H^0(X, L)$ gives rise to an effective almost-Cartier divisor $Z(s)$ with corresponding ideal sheaf $\operatorname{im}(s^\vee\colon L^\vee \to \mathcal{O}_X)$.

0.1.5 Numerical connectedness. Recall that a Gorenstein curve C is said to be *numerically m-connected* if for every generically Gorenstein strict sub-curve $B \subset C$, $\deg_B(\omega_C|_B) - (2p_a(B) - 2) \geq m$ (cf. Catanese, Hulek, Franciosi, Reid [9]). This is a very useful generalisation of the classical notion numerical connectedness of curves on smooth surfaces.

1. The geometry of the surfaces

In this first chapter we investigate the geometry of Gorenstein stable surfaces X with $K_X^2 = 2$ and $\chi(\mathcal{O}_X) = 4$. At first, we show that they all arise as double-covers of the plane, branched over some curve of degree 8. Then we identify the birational isomorphism types of the minimal resolutions, give characterisations of the singularities and we conclude with some results about the mixed Hodge structures on their second cohomology.

Remark 1.1. Let X be a Gorenstein stable surface with $K_X^2 = 2$. If $\chi(\mathcal{O}_X) \geq 4$, then the stable Noether inequality due to Liu and Rollenske [40] implies $p_g(X) < K_X^2 + 2$. Thus, from

$$4 \leq \chi(\mathcal{O}_X) = 1 - q(X) + p_g(X) \leq 4 - q(X) \leq 4$$

we conclude $\chi(\mathcal{O}_X) = 4$, $q(X) = 0$ and $p_g(X) = 3$. Conversely, if $K_X^2 = 2$ and $p_g(X) = 3$, then $q(X) = 0$ and so $\chi(\mathcal{O}_X) = 4$. This will be shown in a manuscript in preparation by Franciosi, Pardini and Rollenske [15].

1.1 The canonical linear system. The aim of this section is to show that the canonical map of a Gorenstein stable surface X satisfying $K_X^2 = 2$ and $\chi(\mathcal{O}_X) = 4$ is a double-cover of $\mathbb{P}^2$ branched over an octic.

If X is a reducible Gorenstein stable surface, it may happen that some non-trivial section of the canonical bundle is not regular; that is, it could vanish on a component. In the case under consideration, however, this does not happen:

Lemma 1.2 — *Assume X is a Gorenstein stable surface with $K_X^2 = 2$. Then X has at most two components and every non-trivial section of ω_X is regular. If, in addition, $p_g(X) \geq 2$, then $|\omega_X|$ has no fixed part and a general effective canonical divisor is well-behaved and reduced.*

Proof. We consider the decomposition $X = \bigcup_{i=1}^s X_i$ into irreducible components and assume that $s \geq 2$. Then there is a corresponding decomposition of the normalisation into disjoint components $\overline{X} = \coprod_{i=1}^s \overline{X}_i$ where $\overline{X}_i$ is the component over X_i. The conductor locus $\overline{F} \subset \overline{X}$ accordingly decomposes as $\overline{F} = \bigcup_{i=1}^s \overline{F}_i$, $\overline{F}_i \subset \overline{X}_i$.

Since ω_X is ample, $2 = \omega_X^2 = \sum_{i=1}^s (\omega_X|_{X_i})^2$ is a sum of s positive integers, so that we have to have $s = 2$ and $\omega_X|_{X_1}^2 = \omega_X|_{X_2}^2 = 1$. In particular, the invertible sheaves $\pi^*\omega_X|_{\overline{X}_i} \cong \omega_{\overline{X}_i}(\overline{F}_i)$, $i = 1, 2$, are ample with $(\pi^*\omega_X|_{\overline{X}_i})^2 = 1$. This furthermore implies that every member of $|\omega_{\overline{X}_i}(\overline{F}_i)|$ is reduced and irreducible.

We now show that every non-trivial section of ω_X is regular, i.e., that the natural maps $p_i\colon H^0(X, \omega_X) \to H^0(\overline{X}_i, \omega_{\overline{X}_i}(\overline{F}_i))$, $i = 1, 2$, given by pull-back and restriction, are injective. Since $\ker(p_1)$ and $\ker(p_2)$ only have the trivial element in common, it suffices to show that they agree. To this end, note that the restriction of p_1, to $\ker(p_2)$ factors through the inclusion $H^0(\overline{X}_1, \omega_{\overline{X}_1}) \to H^0(\overline{X}_1, \omega_{\overline{X}_1}(\overline{F}_1))$ (and likewise for p_2 in place of p_1). Thus, it suffices to prove $H^0(\overline{X}_i, \omega_{\overline{X}_i}) = 0$ for both, $i = 1, 2$. But $(\overline{X}_i, \overline{F}_i)$ is a stable log-pair with $\omega_{\overline{X}_i}(\overline{F}_i)^2 = 1$ and these are classified by Franciosi, Pardini and Rollenske [17, Theorem 1.1]. From this result it follows that $p_g(\overline{X}_i) = 0$, as claimed. Alternatively, it can be shown that a non-trivial section of ω_X had to be nowhere vanishing, hence $\overline{F}_i^2 = \omega_{\overline{X}_i}(\overline{F}_i)^2 = 1$, in contradiction with Riemann-Roch.

It remains to show that if $p_g(X) \geq 2$, then $|\omega_X|$ has no fixed part and a general member is reduced and well-behaved. The generic fibre of a morphism with reduced source is reduced (see the Stacks Project [49, Tag 054Z]). Thus, if $p_g(X) \geq 2$, a general member of $|\omega_X|$ is generically reduced, hence reduced. Moreover, an effective and reduced member of the canonical linear system is well-behaved since a section of an invertible sheaf vanishes along the conductor with even multiplicity.

Since every member of $|\omega_X|_{X_i}|$ is irreducible, if the linear system $|\omega_X|$ would fix a curve $C \subset X_i$, then the restriction map $|\omega_X| \to |\omega_X|_{X_i}|$ could only be constant, mapping everything to C. But assuming $\dim(|\omega_X|) = p_g(X) - 1 \geq 1$, the injective map $|\omega_X| \to |\omega_X|_{X_i}|$ is not constant. Thus, $|\omega_X|$ cannot fix a curve. This completes the proof. □

The following examples show that a special canonical curve could very well be non-reduced or non-well-behaved.

Examples 1.3.

1. *(A canonical curve in the conductor)* When we obtain X as a union of two copies of $\mathbb{P}^2$ along a quartic $\overline{F} \in |\mathcal{O}_{\mathbb{P}^2}(4)|$ with at worst nodal singularities, then X is a Gorenstein stable surface with $K_X^2 = 2$ and $\chi(\mathcal{O}_X) = 4$ and the members of the canonical linear system are the unions of compatible lines in either plane. Therefore, if $\overline{F}$ is a union of a general line and a smooth cubic, then the corresponding line in the conductor is a member of the canonical linear system.
2. *(A well-behaved, non-reduced canonical curve)* Let $f\colon X \to \mathbb{P}^2$ be a double-cover branched along the union of a smooth septic and a general line (that is, meeting the septic transversely). Then X is a normal Gorenstein stable surface with $K_X^2 = 2$ and $\chi(\mathcal{O}_X) = 4$ and the (non-reduced) pre-image of the line occurs as a member of the canonical linear system on X.

In the following result we collect the most basic numerical properties of an arbitrary member of the canonical linear system.

Lemma 1.4 — *Assume that X is a Gorenstein stable surface satisfying $K_X^2 = 2$ and $\chi(\mathcal{O}_X) = 4$. Let $C \in |\omega_X|$ be a canonical curve. Then the following holds.*

a) The curve C is Gorenstein and has at most two components.

b) The identities $h^0(\mathcal{O}_C) = 1$ and $\chi(\mathcal{O}_C) = -2$ hold. In particular, C is connected and $h^0(\omega_C) = p_a(C) = 3$.

c) The invertible sheaf $L := \omega_X|_C \in \mathrm{Pic}(C)$ is a square root of ω_C, i.e., $L^2 \cong \omega_C$, and we have $\chi(L) = 0$ and $\deg(L) = h^0(L) = 2$.

Proof. a) Since X is Gorenstein, every canonical curve C on X is Gorenstein. Furthermore, $K_X C = 2$ and K_X has positive degree on each component of C, so that there can be at most two such.

b) By adjunction, ω_C is isomorphic to the cokernel of the inclusion $\omega_X^2(-C) \to \omega_X^2$. The relevant fragment of the associated exact cohomology sequence, together with the Kodaira Vanishing Theorem for semi-log-canonical surfaces (Liu and Rollenske [39, Proposition 3.1]) shows $h^1(\omega_C) = h^2(\omega_X)$. Serre duality implies $h^0(\mathcal{O}_C) = h^0(\mathcal{O}_X) = 1$ and applying the Riemann-Roch Formula for Cartier divisors on semi-log-canonical surfaces due to Liu and Rollenske [40, Theorem 3.1] gives $-\chi(\mathcal{O}_C) = \chi(\mathcal{O}_X(-C)) - \chi(\mathcal{O}_X) = \frac{1}{2}(-C)(-C - K_X) = K_X^2 = 2$. Therefore, $\chi(\mathcal{O}_C) = -2$ and $p_a(C) = 1 - \chi(\mathcal{O}_C) = 3$. Finally, from $h^0(\mathcal{O}_C) = 1$ and Serre duality we conclude $p_a(C) = h^1(\mathcal{O}_C) = h^0(\omega_C)$.

c) That L is a square-root of ω_C follows from adjunction. Since L is the cokernel of the inclusion $\mathcal{O}_X \cong \omega_X(-C) \to \omega_X$ we get $\chi(L) = \chi(\omega_X) - \chi(\mathcal{O}_X) = 0$ by Serre duality. This readily implies $\deg(L) = p_a(C) - 1 = 2$.

This completes the proof. □

We aim to show:

Proposition 1.5 — *Let C be a general, i.e., well-behaved and reduced, canonical curve on a Gorenstein stable surface X satisfying $K_X^2 = 2$ and $\chi(\mathcal{O}_X) = 4$. Then C is numerically 4-connected and honestly hyperelliptic, the double-cover $C \to \mathbb{P}^1$ being defined by the sections of $\omega_X|_C$. Moreover, if C is reducible, then its two components are smooth rational curves.*

In the proof, we will use the following result, which is well known in the smooth case. In the present version, it is presumably also well known to some experts. It follows from Rosenlicht's version of the Clifford Inequality for singular surfaces; see Rosenlicht [47, Theorem 16] for the original proof or Kleiman and Martins [31, Theorem 3.1] for a modern account and further references.

Lemma 1.6 — *Let C be a reduced and irreducible Cohen-Macaulay curve. An invertible sheaf $L \in \operatorname{Pic}(C)$ of degree one has at most two linearly independent global sections and if there are in fact two such, then L is globally generated and the associated morphism $\varphi_{|L|}\colon C \to \mathbb{P}^1$ is an isomorphism.*

Proof. Any L with $h^0(L) \geq 2$ and $\deg(L) = 1$ violates Clifford's inequality for singular curves $2(h^0(L)-1) \leq \deg(L)$; thus, anything but $h^1(L) = 0$ would lead to a contradiction. But then $\chi(L) = h^0(L) \geq 2$ and so $p_a(C) = 1-\chi(\mathcal{O}_C) = 1+\deg(L)-\chi(L) \leq 0$. Hence, $C \cong \mathbb{P}^1$ and $h^0(L) = 2$. □

Proof of Proposition 1.5. Recall from the earlier Lemmas 1.2 and 1.4 that a general member $C \in |K_X|$ is Gorenstein, reduced, well-behaved and has $p_a(C) = 3$. We showed, furthermore, that $L := \omega_X|_C$ is a square-root of ω_C with $\deg(L) = h^0(L) = 2$.

If C is irreducible, then it is clearly numerically 4-connected and if L were not globally generated, say at $x \in C$, then $L(-x)$ were of degree one with two linearly independent sections and so we had to have $C \cong \mathbb{P}^1$ by Lemma 1.6, in contradiction with $p_a(C) = 3$. Thus, if C is irreducible, $\varphi_{|L|}\colon C \to \mathbb{P}^1$ is a morphism of degree two, as claimed.

If C is reducible, then $C = C_1 \cup C_2$ with $\deg(L|_{C_1}) = \deg(L|_{C_2}) = 1$. Below, we will show that $h^0(L|_{C_1}), h^0(L|_{C_2}) \geq 2$. Assuming this for the moment, it follows that $(C_i, L|_{C_i}) \cong (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ for both $i = 1, 2$, by Lemma 1.6. In particular, $\deg(\omega_C|_{C_i}) - (2\,p_a(C_i)-2) = 4$ for both $i = 1, 2$. Thus, C is numerically 4-connected and the Curve Embedding Theorem due to Catanese, Franciosi, Hulek and Reid [9, Theorem 1.1] implies that L is globally generated.

It remains to show that $h^0(L|_{C_1}), h^0(L|_{C_2}) \geq 2$. Since $|K_X|$ has no fixed part, every component admits a non-trivial section of L; thus, $h^0(L|_{C_1}), h^0(L|_{C_2}) \geq 1$. If we had $h^0(L|_{C_1}) = 1$, then we had to have a non-trivial section of L vanishing on all of C_1. But then the restriction of this section to $H^0(L|_{C_2})$ were non-trivial and vanishing on the separating conductor[1], which had to have length $\deg(L|_{C_2}) = 1$ then,

[1] The *separating conductor* is the conductor locus of the partial normalisation $C_1 \amalg C_2 \to C$. Actually, in this particular case, where we know a-posteriori that $C_1 \cong C_2 \cong \mathbb{P}^1$, this partial normalisation is already the full normalisation and so we are talking about the usual conductor locus.

in contradiction with the fact that in our case the separating conductor has to have even length: Since C is Gorenstein, the length of the separating conductor on C_i is precisely $\deg(\omega_C|_{C_i}) - (2\,p_a(C_i) - 2) = 2 - (2\,p_a(C_i) - 2)$, an even number. Hence, $h^0(L|_{C_1}) \geq 2$ and by the same argument for C_2 we also get $h^0(L|_{C_2}) \geq 2$. This finishes the proof. □

Corollary 1.7 — *If X is a Gorenstein stable surface satisfying $K_X^2 = 2$ and $\chi(\mathcal{O}_X) = 4$, then the canonical linear system on X is base-point free and realises X as a double-cover of $\mathbb{P}^2$ which is branched over an octic.*

Proof. For a general canonical curve $C \in |K_X|$, the restriction $\omega_X|_C$ is base-point free by Proposition 1.5. Since $h^1(\mathcal{O}_X) = q(X) = 0$ by assumption (cf. Remark 1.1), the restriction map $H^0(\omega_X) \to H^0(\omega_X|_C)$ is surjective; hence, so is the evaluation map $H^0(\omega_X) \to H^0(\omega_X|_p)$ for every $p \in C$. This implies that $|K_X|$ is base-point free. Since K_X is ample, the canonical map $\varphi := \varphi_{|K_X|}\colon X \to \mathbb{P}^2$ is finite and of degree $K_X^2 = 2$. Finally, if $d \in \mathbb{N}$ is such that the branch divisor is of degree $2d$, then we have to have $\varphi_*\omega_X = \varphi_*\varphi^*(\omega_{\mathbb{P}^2}(d)) = \omega_{\mathbb{P}^2}(d) \oplus \omega_{\mathbb{P}^2}$. Thus, $3 = p_g(X) = h^0(\omega_{\mathbb{P}^2}(d) \oplus \omega_{\mathbb{P}^2}) = h^0(\mathcal{O}_{\mathbb{P}^2}(d-3))$, which is possible only if $d = 4$; hence, the branch divisor is an octic. □

The following corollary is equivalent to the former; we present a separate proof, though, because it shows how to compute the canonical ring from the canonical ring of a general canonical curve.

Corollary 1.8 — *The canonical ring $R(X, \omega_X)$ of a Gorenstein stable surface X with $\chi(\mathcal{O}_X) = 4$ and $K_X^2 = 2$ is isomorphic to $\mathbb{C}[x_0, x_1, x_2, z]/(z^2 - f_8)$, where x_0, x_1 and x_2 are of degree 1 and z is of degree 4 and where $f_8 \in \mathbb{C}[x_0, x_1, x_2]$ is a non-trivial homogeneous polynomial of degree 8.*

Proof. Let $x_0 \in H^0(\omega_X)$ be a general section, such that its associated canonical divisor $C = (x_0)_0 \in |K_X|$ is an honestly hyperelliptic curve of genus 3, as granted by Proposition 1.5. Then the section ring of the invertible sheaf $L = \omega_X|_C$ is isomorphic to $\mathbb{C}[y_1, y_2, z]/(z^2 - g_8)$ for some homogeneous $g_8 \in \mathbb{C}[x_1, x_2]$ of degree 8, where $\deg(y_1) = \deg(y_2) = 1$ and $\deg(z) = 4$, as shown by Catanese, Franciosi, Hulek and Reid [9, Lemma 3.5]. By Kodaira vanishing and since $q(X) = 0$, the restriction map $R(X, \omega_X) \to R(C, L)$ is surjective and the kernel is generated by x_0. This is easily seen to imply that the associated map $\mathbb{C}[x_0, x_1, x_2, z] \to R(X, \omega_X)$ is surjective with kernel generated by $z^2 - f_8$ for some homogeneous $f_8 \in \mathbb{C}[x_0, x_1, x_2]$ of degree 8 which lifts $g_8 \in \mathbb{C}[x_1, x_2]$. □

We conclude with the remark that conversely, a sufficiently nice plane octic gives rise to a Gorenstein stable surface with the desired invariants. Precisely, a double-cover $X \to \mathbb{P}^2$ branched over a divisor B is semi-log-canonical if and only of the pair $(\mathbb{P}^2, \frac{1}{2}B)$ is log-canonical, by Alexeev and Pardini [3, Lemma 2.3]. For later reference, we deal not just with a single curve, but with a family of such.

Proposition 1.9 — *If $B \subset \mathbb{P}^2_S$ is a flat family of octics, it is in particular a relative Cartier divisor. Thus, we can form the double-cover $\mathscr{X} \to \mathbb{P}_S$ branched over B. Assume that every fibre B_s, $s \in S(\mathbb{C})$, is such that the pair $(\mathbb{P}^2, \frac{1}{2}B_s)$ is log-canonical. Then the composition $f\colon \mathscr{X} \to \mathbb{P}^2_S \to S$ is a flat family of Gorenstein stable surfaces $\mathscr{X}_s$, $s \in S(\mathbb{C})$, such that $K^2_{\mathscr{X}_s} = 2$ and $\chi(\mathcal{O}_{\mathscr{X}_s}) = 4$. Furthermore, $f_*\omega_{\mathscr{X}/S}$ is free and $B \subset \mathbb{P}^2_S$ can be recovered up to isomorphism as the branch divisor of the double-cover $\mathscr{X} \to \mathbb{P}_S(f_*\omega_{\mathscr{X}/S})$.*

Proof. At first, suppose that we are dealing with a single double-cover $\varphi\colon X \to \mathbb{P}^2$, branched over an octic $B \in |\mathcal{O}_{\mathbb{P}^2}(8)|$ such that the pair $(\mathbb{P}^2, \frac{1}{2}B)$ is log-canonical, so that X is semi-log-canonical, as discussed above. Note that since φ is finite and $\omega_X = \varphi^*\omega_{\mathbb{P}^2}(4) = \varphi^*\mathcal{O}_{\mathbb{P}^2}(1)$, the canonical divisor on X is Cartier and ample, i.e., X is Gorenstein and stable. The invariants $K_X^2 = 2$ and $\chi(\mathcal{O}_X) = 4$ are computed as follows: K_X defines a double-cover onto $\mathbb{P}^2$, hence, $K_X^2 = 2$ and $\chi(\mathcal{O}_X) = \chi(\varphi_*\mathcal{O}_X) = \chi(\omega_{\mathbb{P}^2} \oplus \omega_{\mathbb{P}^2}(4)) = 1 + 3 = 4$.

Now let S be a scheme of finite type over $\mathbb{C}$ and let $B \subset \mathbb{P}^2_S$ be a relative Cartier divisor of degree 8. Let $\mathscr{X} \to \mathbb{P}^2_S$ be the double-cover branched over B; more precisely, the cover taking the square-root of the section of $\mathcal{O}_{\mathbb{P}^2_S/S}(8)$ defining B. Since $\varphi_*\mathcal{O}_{\mathscr{X}} = \mathcal{O}_{\mathbb{P}^2_S} \oplus \mathcal{O}_{\mathbb{P}^2_S}(-4)$ is locally free, the double-cover morphism is flat; thus, $f\colon \mathscr{X} \to S$ is flat.

For every $s \in S(\mathbb{C})$, the fibre $\mathscr{X}_s$ naturally identifies with the double-cover of $\mathbb{P}^2$ branched over B_s. Thus, $\mathscr{X}_s$ is a Gorenstein stable surface if and only if $(\mathbb{P}^2, \frac{1}{2}B_s)$ is log-canonical. Let us suppose that this is indeed the case for all $s \in S(\mathbb{C})$. Then all fibres $(f_*\omega_{\mathscr{X}/S})(s) = H^0(\mathscr{X}_s, \omega_{\mathscr{X}_s})$, $s \in S(\mathbb{C})$, are 3-dimensional and by naturality we observe $\mathscr{X}$ as another double-cover $g\colon \mathscr{X} \to \mathbb{P}_S(f_*\omega_{\mathscr{X}/S})$. On the other hand, $f_*\omega_{\mathscr{X}/S}$ is the direct image of $\varphi_*\omega_{\mathscr{X}/S} = \omega_{\mathbb{P}^2_S/S} \oplus \omega_{\mathbb{P}^2_S/S}(4) = \mathcal{O}_{\mathbb{P}^2_S/S}(-3) \oplus \mathcal{O}_{\mathbb{P}^2_S/S}(1)$ along the projection $\mathbb{P}^2_S \to S$; thus, $f_*\omega_{\mathscr{X}/S} \cong \mathcal{O}_S \otimes H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \cong \mathcal{O}_S^3$. Tracing through the sequence of morphisms involved shows that the induced isomorphism $\mathbb{P}^2_S \cong \mathbb{P}_S(f_*\omega_{\mathscr{X}/S})$ identifies the double-covers f and g. This proves the claim. □

Recall that the *log-canonical threshold* of an effective divisor $D \subset X$ on a variety X is the number $\operatorname{lcth}(X, D) = \sup\{t \geq 0 \mid (X, tD) \text{ is log-canonical}\}$, see Kollár [34, Section 8.2]. Thus, the a pair $(\mathbb{P}^2, \frac{1}{2}B)$ is log-canonical if and only if the log-canonical threshold of B (in $\mathbb{P}^2$) is at least $\frac{1}{2}$. For brevity, we introduce a term for the plane curves with this property.

Definition 1.10 — *A plane curve $C \subset \mathbb{P}^2$ is said to be* half-log-canonical *if the pair $(\mathbb{P}^2, \frac{1}{2}C)$ is log-canonical; equivalently, if* $\operatorname{lcth}(\mathbb{P}^2, C) \geq \frac{1}{2}$.

This definition is independent of the embedding since the condition on the singularities is (analytically) local.

1.2 The normalisation and the minimal resolution. As we have shown above, every Gorenstein stable surface X with $K_X^2 = 2$ and $\chi(\mathcal{O}_X) = 4$ arises as a double-cover of the plane, branched over an octic curve $B \in |\mathcal{O}_{\mathbb{P}^2}(8)|$ such that the pair $(\mathbb{P}^2, \frac{1}{2}B)$ is log-canonical. For such an octic, the ceiling $\lceil \frac{1}{2}B \rceil$ is supposed to be reduced, by definition. That is, all integral components of B have to appear with coefficient ≤ 2. In particular, every admissible B decomposes as a sum $B = B' + 2B''$ of (possibly trivial) reduced effective divisors. Moreover, B'' can have at worst nodes and is smooth at the points of intersection with B'; a proof can be found in Kollár [34, Corollary 2.32], but this also follows from the classification of semi-log-canonical hypersurface singularities discussed in Proposition 1.15 below.

If $\pi\colon \overline{X} \to X$ denotes the normalisation, then by a result of Pardini [43, Proposition 3.2], the composition $\overline{\varphi} = \varphi \circ \pi\colon \overline{X} \to \mathbb{P}^2$ is the double-cover branched over B' and the conductor loci $F \subset X$ and $\overline{F} \subset \overline{X}$ are the reduced pre-images of B'' under φ and $\overline{\varphi}$, respectively.

This proves most of the following statement which we state for later reference.

Proposition 1.11 — *Let X be a Gorenstein stable surface with numerical invariants $K_X^2 = 2$ and $\chi(\mathcal{O}_X) = 4$, let $\varphi\colon X \to \mathbb{P}^2$ be the canonical double-cover and let $B \subset \mathbb{P}^2$ be the branch divisor. Then the following holds:*

a) The pair $(\mathbb{P}^2, \frac{1}{2}B)$ is log-canonical; in particular, there is a unique decomposition $B = B' + 2B''$ with B', B'' effective and reduced; B'' is nodal.

b) The composition of $\varphi\colon X \to \mathbb{P}^2$ with the normalisation $\pi\colon \overline{X} \to X$ is the double-cover branched over B' and the reduced pre-images of B'' in $\overline{X}$ and X are the conductor loci $\overline{F} \subset \overline{X}$ and $F \subset X$, respectively.

c) The morphism $(\varphi \circ \pi)|_{\overline{F}}\colon \overline{F} \to B''$ is a double-cover branched over the Cartier divisor $B'|_{B''}$ and it factors through the isomorphism $\varphi|_F\colon F \to B''$.

Proof. We have discussed *a)* and *b)* right above the statement of the proposition.

Regarding *c)*: That $(\varphi \circ \pi)|_{\overline{F}}\colon \overline{F} \to B''$ is a double-cover branched over the Cartier divisor $B'|_{B''}$ follows from b). It also follows that $\varphi|_F\colon F \to B''$ is of degree one, hence, generically an isomorphism. It remains to show that it is an isomorphism everywhere. Since B'' has only nodes and $\varphi|_F$ is finite, it suffices to observe that $\varphi|_F$ is bijective, which holds by construction. □

1.2.1 The birational geometry of the minimal resolutions. The birational geometry of the surfaces under investigation strongly depends on the number and degrees of the irrational singularities. Recall that an isolated surface singularity (X, x) is called *irrational* if it is not rational; i.e., if the exceptional divisor E in the minimal resolution $(Y, E) \to (X, x)$ has strictly positive arithmetic genus. Its *degree* is the negative of the self-intersection number $-E^2$. From the classification of semi-log-canonical hypersurface singularities in dimension two, it follows that if (X, x) is irrational and semi-log-canonical, then $p_a(E) = 1$ (see Proposition 1.15 below), i.e., (X, x) is *elliptic*. It also follows that the only elliptic semi-log-canonical singularities occurring on double-covers of a smooth surface have to have degree 1 or 2. We distinguish two cases: If E is a smooth elliptic curve, (X, x) is said to be *simply elliptic* and otherwise *cuspidal* (or *a cusp*). The latter may happen if E is a cycle of rational curves.

The holomorphic Euler characteristic of the resolution is easy to compute:

Lemma 1.12 — *Let X be a log-canonical surface with k irrational singularities. For any resolution of singularities Y of X, we have $\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_X) - k$.*

Proof. Since the holomorphic Euler characteristic is a birational invariant of smooth surfaces, we can suppose that $f\colon Y \to X$ is the minimal resolution. Then f has connected fibres since X is normal, $\chi(R^1 f_*\mathcal{O}_Y) = h^0(R^1 f_*\mathcal{O}_Y) = k$ by Liu, Rollenske [39, Lemma A.6] and $R^i f_*\mathcal{O}_Y = 0$ for all $i \geq 2$ for dimension reasons. Thus, $\chi(\mathcal{O}_Y) = \chi(f_*\mathcal{O}_Y) - \chi(R^1 f_*\mathcal{O}_Y) = \chi(\mathcal{O}_X) - k$, as claimed. □

The following two results are part of a manuscript in preparation by Franciosi, Pardini and Rollenske [15]. For simplicity, we restrict both to the relevant case.

Proposition 1.13 (Franciosi, Pardini, Rollenske) — *Assume that X is a normal Gorenstein stable surface satisfying $K_X^2 = 2$ and $\chi(\mathcal{O}_X) = 4$. Let k be the number of elliptic singularities of X and suppose that the minimal resolution Y of X satisfies $\kappa(Y) = -\infty$. Then either Y is rational with $\chi(\mathcal{O}_Y) = 1$ and $k = 3$, or $\chi(\mathcal{O}_Y) = 0$ and $k = 4$; in the latter case, the four elliptic singularities are simple.*

Proof after Franciosi, Pardini and Rollenske [15]. With the same proof as in Franciosi, Pardini, Rollenske [17, Lemma 4.5] we get that either Y is rational ($\chi(\mathcal{O}_Y) = 1$), or $Y_{\min}$ is ruled of genus 1 ($\chi(\mathcal{O}_Y) = 0$) and that in the latter case all elliptic singularities are simple. Application of Lemma 1.12 yields $\chi(\mathcal{O}_Y) = 4-k$, which completes the proof. □

Theorem 1.14 (Franciosi, Pardini, Rollenske) — *Let X be a normal Gorenstein stable surface satisfying $K_X^2 = 2$. Denote by $f\colon Y \to X$ its minimal resolution and by $\sigma\colon Y \to Y_{\min}$ a minimal model of Y. Furthermore, let d be the sum of the degrees of the elliptic singularities. If $\kappa(Y) \geq 0$, then there are only the following possibilities:*

i) $Y = Y_{\min}$ is of general type, $K_Y^2 = 2$ and X is its canonical model ($d = 0$).
ii) $Y_{\min}$ is of general type with $K_{Y_{\min}}^2 = 1$, $\sigma\colon Y \to Y_{\min}$ is the blow up in one point and X has a unique elliptic singularity of degree 1 ($d = 1$).
iii) $Y = Y_{\min}$ is properly elliptic ($\kappa(Y) = 1$); in this case, $d = 2$.
iv) $\kappa(Y) = 0$ or 1, $\sigma\colon Y \to Y_{\min}$ is a blow-up in one point and $d = 3$.
v) $\kappa(Y) = 0$, $\sigma\colon Y \to Y_{\min}$ is a sequence of two blow-ups and $d = 4$.

Proof after Franciosi, Pardini and Rollenske [15]. Let $E_i \subset Y$, $i = 1, \ldots, n$, be the exceptional curves over the elliptic singularities and let $G = \sum_{i=1}^m G_i \subset Y$ be the exceptional divisor contracted by $\sigma\colon Y \to Y_{\min}$. Then the canonical divisor can be written in two different ways as $K_Y = f^*K_X - \sum_{i=1}^n E_i$ and $K_Y = \sigma^*K_{Y_{\min}} + G$. The sum of degrees can be written as $d = K_Y \sum_{i=1}^n E_i = -\sum_{i=1}^n E_i^2 \geq n$. We have $G_i f^*K_X \geq 1$ for each component $G_i \subset G$ since K_X is ample and since no component of G is contracted by f. Moreover, every (-1)-curve $G_i \subset G$ satisfies $G_iE \geq 2$, for $-1 = G_iK_Y = G_if^*K_X - G_iE \geq 1 - G_iE$. In particular, $GE > m$ unless $m = 0$. We introduce the two central (in-)equalities:

$$d = K_YE = K_Y(f^*K_Y - K_Y) = f^*K_X^2 - K_Y^2 = 2 - K_{Y_{\min}}^2 + m$$
$$2 = K_Yf^*K_X = \sigma^*K_{Y_{\min}}f^*K_X + Gf^*K_X \geq \sigma^*K_{Y_{\min}}f^*K_X + m \geq m$$

Using $K_{Y_{\min}}^2 \geq 0$ (from $\kappa(Y) \geq 0$), they yield $d \leq m + 2 \leq 4$. The cases i)–v) correspond to the possible values for $d = 0, \ldots, 5$, respectively:

Let $d = 0$. Then $K_Y = f^*K_X$ is big and nef; this is case i).

Let $d = 1$. Then $K_{Y_{\min}}^2 = 1 + m \geq 1$, so that $K_{Y_{\min}}$ is minimal of general type. Moreover, $1 = d = GE > m$; thus, $m = 0$. This is case ii).

Let $d = 2$. Then $K_{Y_{\min}}^2 = m$. But $m \geq 1$ is impossible, for $1 = d = K_YE = GE + \sigma^*K_{Y_{\min}}E > m$. Hence, we have to have $Y = Y_{\min}$ and $K_Y^2 = m = 0$. On the other hand, $\kappa(Y) \neq 0$, since $K_Yf^*K_X = 2 > 0$; thus, $\kappa(Y) = 1$, as in case iii).

Let $d = 3$. Then $K_{Y_{\min}}^2 = m - 1$, hence $m \geq 1$. In fact, $m = 1$: If we had $m \geq 2$, then $Y_{\min}$ would be minimal of general type and by $3 = K_YE = K_{Y_{\min}}\sigma_*E + GE$ and $GE > m \geq 2$ would imply $K_{Y_{\min}}\sigma_*E = 0$. In particular, we would have to have $K_{Y_{\min}}\sigma_*E_i = 0$ for each component E_i of E. But since $Y_{\min}$ had to be minimal of general type, the components E_i had to be rational then, which is impossible. Therefore, $m = 1$ and $K_{Y_{\min}}^2 = 0$, corresponding to case iv).

Finally, let $d = 4$. Then $m = 2$ and $K_{Y_{\min}}^2 = 0$, for $0 \leq K_{Y_{\min}}^2 = m-2$ and $m \leq 2$. To complete the proof, it is left to show that $\kappa(Y) = 0$. Indeed, for $2 = \sigma^*K_{Y_{\min}}f^*K_X + m$ to hold, we have to have $\sigma^*K_{Y_{\min}}f^*K_X = 0$; together with $K_{Y_{\min}}^2 = 0$ this implies that $\kappa(Y) = 0$ and we arrive at case v). □

The birational classification in the non-normal case will be established as needed later. It will turn out that the only reducible normalisation is $\mathbb{P}^2 \amalg \mathbb{P}^2$ and the possible irreducible normalisations are K3-surfaces, rational, or ruled over a curve of genus 1.

1.3 The singularities. Since double-covers of smooth varieties branched over Cartier divisors are sub-varieties of the total space of a line bundle, defined by a single regular equation, the surfaces under investigation have to have hypersurface singularities, if any. Therefore, the classification of semi-log-canonical hypersurface singularities (see Liu, Rollenske [38]) gives a complete list of analytic germs of singular points we might get. Since we are dealing with double-cover singularities, we only need to consider those of multiplicity two. For simplicity, we restrict our attention to the singularities of the branch curves.

Symbol	Equation in $\mathbb{C}[x,y]$	Conditions	μ
A_n	$x^2 + y^{n+1}$	$n \geq 1$	n
D_n	$y(x^2 + y^{n-2})$	$n \geq 4$	n
E_6	$x^3 + y^4$		6
E_7	$x^3 + xy^3$		7
E_8	$x^3 + y^5$		8
X_9	$x^4 + \lambda(xy)^2 + y^4$	$\lambda^2 \neq 4$	9
J_{10}	$x^3 + \lambda(xy)^2 + y^6$	$4\lambda^3 + 27 \neq 0$	10
$X_p = T_{2,4,p-5}$	$x^4 + (xy)^2 + y^{4+p-9}$	$p \geq 10$	p
$Y_{r,s} = T_{2,4+r,4+s}$	$x^{4+r} + (xy)^2 + y^{4+s}$	$r, s \geq 1$	$9 + r + s$
$J_{2,p} = T_{2,3,p+6}$	$x^3 + (xy)^2 + y^{6+p}$	$p \geq 1$	$10 + p$
A_∞	x^2		0
D_∞	$x^2 y$		1
$J_{2,\infty}$	$x^3 + (xy)^2$		4
X_∞	$x^4 + (xy)^2$		5
$Y_{r,\infty}$	$x^{r+4} + (xy)^2$	$r \geq 1$	$r + 5$
$Y_{\infty,\infty}$	$(xy)^2$		4

Table 1: The classification of half-log-canonical curve singularities. We refer to Arnold, Gusein-Zade, Varchenko [5, Chapter 15] for the notation.

Proposition 1.15 — *Let $C \subset \mathbb{P}^2$ be a plane curve of degree 8. Then the double-cover of $\mathbb{P}^2$ branched over C is a Gorenstein stable surface X if and only if the analytic germs of the singular points of C are among those listed in Table 1.*

Proof. This is just the relevant part (multiplicity 2) of the list given by Liu and Rollenske [38] after appropriate transformations where necessary and with the refinement of the series $T_{2,\bullet,\bullet}$ into $X_\bullet$, $J_{2,\bullet}$ and $Y_{\bullet,\bullet}$. □

Remark 1.16. Of course, du Val-singularities $A_\bullet$, $D_\bullet$, $E_\bullet$ on the branch curve correspond to canonical singularities on the surface. Table 2 provides a correspondence for the remaining types of singularities.

Symbol	Branch curve singularity	Double-cover singularity
X_9	ordinary quadruple-point	simply elliptic of degree 2
$X_\bullet$ $Y_{\bullet,\bullet}$	degenerate quadruple-point	cuspidal elliptic of degree 2
J_{10}	non-degenerate $[3;3]$-point	simply elliptic of degree 1
$J_{2,\bullet}$	degenerate $[3;3]$-point	cuspidal elliptic of degree 1
A_∞	double-line	double normal crossing
D_∞	double-line + transversal line	pinch point
$J_{2,\infty}$	double-line + tangential line	
$X_\infty, Y_{\bullet,\infty}$	double-line + double-point	degenerate cusps
$Y_{\infty,\infty}$	transversely meeting double-lines	

Table 2: Dictionary: branch curve singularities $\leftrightarrow$ surface singularities

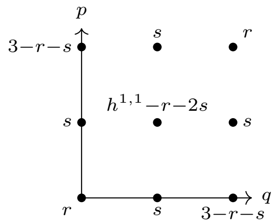
Remark 1.17. If a Gorenstein stable surface X with $K_X^2 = 2$ and $\chi(\mathcal{O}_X) = 4$ has an A_n- or D_n-singularity, then $n \leq 48$, since the maximal Milnor number of a singular point of the branch curve of the canonical double-cover, which has degree 8, does not exceed 49 and this maximal number is attained only by the union of eight concurrent lines (cf. Lemma A.5), which is of course neither A_n nor D_n. (Similar bounds exist for the other families listed above.) However, also the upper bound $n \leq 48$ is most probably not sharp and determining the maximal n such that there exists a plane curve C of fixed degree with an A_n-singularity, e.g., is a hard question[2]. For example, the maximal A_n-singularity on a quintic is for $n = 12$, e.g., by Wall [52], and on a sextic, the maximal A_n is for $n = 19$, which follows from Yang's classification [54]. This seems to be about everything that is known in this direction, at least according to a related discussion on MathOverflow answered by user JNS [51].

1.4 The mixed Hodge structure on $H^2(X)$. The stratification we will define and study later is motivated by recent work (partially in progress) of Green, Griffiths, Kerr, Laza and Robles [20, 21, 30, 45, 46] about degenerations of Hodge structures. Roughly, there should be a stratification of the moduli space of our surfaces under investigation, according to the type of polarised mixed Hodge structure on $H^2(X)$. It should be noted that the details about this Hodge-theoretic stratification are subject to work in progress. Therefore, we can only give an informal description. We refer to Robles' exposition [46] and the references therein for more details; for the basic theory of mixed Hodge structures see Durfee's short introduction [14] and the comprehensive account by Peters and Steenbrink [44].

Given a flat Gorenstein degeneration $\mathscr{X} \to S$, $\mathscr{X}_s$ smooth projective, we can associate a *limiting polarised mixed Hodge structure* with the family of Hodge structures $H^2(\mathscr{X}_s;\mathbb{C})$. The Deligne splitting gives an $\mathbb{R}$-split polarised mixed Hodge structure. Furthermore, representation theory gives rise to a relation among these Hodge diamonds, called the *polarised relations*. They reflect which Hodge structures are more degenerate than others.

[2]This was pointed out to the author by Michael Lönne.

Since $h^{2,0}(X) = h^{0,2}(X) = p_g(X) = 3$ for a smooth surface X of general type satisfying $K_X^2 = 2$ and $\chi(\mathcal{O}_X) = 4$, our case of interest corresponds to the Hodge numbers $h = (3, h^{1,1}, 3)$. That is, the Deligne splitting $H^2(X;\mathbb{C}) = \bigoplus_{p,q} I^{p,q}$ of the mixed Hodge structures has a Hodge diamond (indicating $\dim(I^{p,q})$) of the form



where $r, s \geq 0$, $r + s \leq 3$ and $r + 2s \leq h^{1,1}$. With this diamond denoted by $\Diamond_{r,s}$, the polarised relation is defined as $\Diamond_{r,s} \leq \Diamond_{t,u}$ if and only if $r \leq t$ and $r + s \leq t + u$, cf. Robles [46, Example 4.22]. This is illustrated in the degeneration diagram Figure 1. We will ignore $h^{1,1}$, just as we will ignore canonical surface singularities.

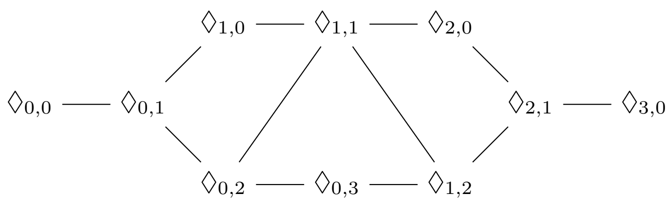


Figure 1: Degeneration diagram for the Hodge types.

Dolgachev [13] has introduced the notion of *cohomologically insignificant degenerations*: If $\mathscr{X} \to \Delta$ is a flat and projective family of varieties, where $\Delta \subset \mathbb{C}$ is the unit disc and where all fibres $\mathscr{X}_s$, $s \in \Delta^* = \Delta - 0$ are smooth, we can compare Deligne's natural mixed Hodge structure on the cohomology of the special fibre $H^n(\mathscr{X}_0;\mathbb{R})$ and the limiting mixed Hodge structure on $H^n(\mathscr{X}_s;\mathbb{R})$, $s \neq 0$, via the specialisation map. The variety $\mathscr{X}_0$ is said to be *cohomologically n-insignificant* if these mixed Hodge structures on H^n agree on (p,q)-components where $pq = 0$ for all such families with $\mathscr{X}_0$ as special fibre. By a result of Steenbrink [48, Theorem 2], a projective variety with at worst du Bois singularities is cohomologically insignificant, that is, cohomologically n-insignificant for all n. This applies in particular to semi-log-canonical surfaces since they are Du Bois (cf. Kollár [34, Corollary 6.32] or Kovács, Schwede, Smith [37, Theorem 4.16]). For our purposes, it is therefore enough to work with Deligne's mixed Hodge structure.

Definition 1.18 — *Let X be a Gorenstein stable surface satisfying $K_X^2 = 2$ and $\chi(\mathcal{O}_X) = 4$. Then X is said to be of* Hodge type $\Diamond_{r,s}$ *if $r = \dim(H^2(X;\mathbb{C}))^{(0,0)}$ and $s = \dim(H^2(X;\mathbb{C}))^{(1,0)}$, where $(H^2(X;\mathbb{C}))^{(p,q)}$ is the (p,q)-component of Deligne's mixed Hodge structure on $H^2(X;\mathbb{C})$.*

It can be shown that the moduli space under investigation in the forthcoming sections is stratified according to the Hodge type; in fact, the irrationality stratification defined below gives a refinement of this stratification, as follows from Proposition 3.12.

For the computations of Hodge types of a Gorenstein stable surface X, we have to know the Hodge structure on its minimal resolution. We will do the computation for our surfaces of interest as soon as we know their minimal resolutions. To get an idea of how this is done, we show how to read off the number of cuspidal elliptic singularities from the Hodge type.

Let X be a normal Gorenstein stable surface satisfying $p_g(X) = 3$ and with irrational singularities $p_1, \dots, p_n$, of which precisely $0 \leq m \leq n$ are cusps. Let $f\colon Y \to X$ be the resolution at the p_i, so that Y has only rational singularities. We denote the exceptional curve over p_i by E_i and their disjoint unions by $D = \coprod_{i=1}^n \{p_i\}$ and $E = \coprod_{i=1}^n E_i$. Moreover, we let $i\colon D^{\mathrm{an}} \to X^{\mathrm{an}}$ and $j\colon E^{\mathrm{an}} \to Y^{\mathrm{an}}$ denote the inclusion maps. Then we get a Mayer–Vietoris exact sequence of mixed Hodge structures, by Peters and Steenbrink [44, Corollary-Definition 5.37]

$$H^k(X;\mathbb{C}) \xrightarrow{(f^*,i^*)} H^k(Y;\mathbb{C}) \oplus H^k(D;\mathbb{C}) \xrightarrow{j^*-f|_E^*} H^k(E;\mathbb{C}) \to H^{k+1}(X;\mathbb{C}).$$

Focussing on $H^2(X;\mathbb{C})$ and using that $H^1(D) = H^2(D) = 0$ for dimension reasons, we get the following exact sequence:

$$H^1(Y;\mathbb{C}) \to \bigoplus_{i=1}^n H^1(E_i;\mathbb{C}) \to H^2(X;\mathbb{C}) \to H^2(Y;\mathbb{C}).$$

If p_i is simply elliptic, then E_i is an elliptic curve and $H^1(E_i)$ carries a pure Hodge structure of weight 1 with $h^{1,0}(E_i) = 1$. If p_i is a cusp, so that E_i is a cycle of rational curves, then $H^1(E_i)$ is one-dimensional and its mixed Hodge structure is concentrated in weight 0. Concerning the mixed Hodge structure on $H^2(Y)$, since we are only interested in the $(p,0)$-components, we can pretend that Y is regular since the rational singularities do not contribute.

Since $(H^1(Y))^{(0,0)} = (H^2(Y))^{(0,0)} = 0$, we conclude that $(H^2(X))^{(0,0)}$ is isomorphic to the $(0,0)$-component of $\bigoplus_{i=1}^n H^1(E_i)$. That is, $\dim(I^{(0,0)}) = m$, the number of cusps. Likewise, since $H^2(Y)$ has no part of weight 1, the part of weight 1 in $H^2(X)$ entirely comes from $\bigoplus_{i=1}^n H^1(E_i)$, so that $\dim(I^{(1,0)}) \leq n - m$ is at most the number of simply elliptic singularities. To actually compute the dimension of the $(1,0)$-component, we have to know more about the map $H^1(Y) \to \bigoplus_{i=1}^n H^1(E_i)$. So far, this discussion shows:

Lemma 1.19 — *Let X be a normal Gorenstein stable surface with $p_g(X) = 3$ having exactly r cuspidal elliptic singularities. Then X is of Hodge type $\Diamond_{r,s}$ for a certain $0 \leq s \leq 3 - r$. In this case the number of simply elliptic singularities is at least s.*

2. Remarks about the moduli space

Our moduli space of interest is the KSBA-compactification of the Gieseker moduli space $\mathfrak{M}_{2,4}$ of canonical models of surfaces of general type with invariants $K_X^2 = 2$ and $\chi(\mathcal{O}_X) = 4$. For our techniques to apply, we restrict to the open locus of Gorenstein surfaces $\overline{\mathfrak{M}}_{2,4}^{\mathrm{Gor}} \subset \overline{\mathfrak{M}}_{2,4}$. Since this extra condition happens to simplify the definition of the moduli problem, we recall the details only for the space of Gorenstein stable surfaces.

Let $\mathcal{M}_{2,4}^{\mathrm{Gor}}$ be the category whose objects are pairs $(T, f\colon \mathscr{X} \to T)$ consisting of a scheme T of finite type over $\mathbb{C}$ and a flat family $f\colon \mathscr{X} \to T$ of Gorenstein stable surfaces $\mathscr{X}_t$, $t \in T(\mathbb{C})$, all satisfying $K_{\mathscr{X}_t}^2 = 2$ and $\chi(\mathcal{O}_{\mathscr{X}_t}) = 4$. The morphisms are fibre squares, as usual, so that the codomain fibration exhibits $\mathcal{M}_{2,4}^{\text{Gor}}$ as a category fibred in groupoids over the category of complex schemes of finite type. According to the seminal works of Kollár, Shepherd-Barron [32, 33] and Alexeev [1, 2], $\mathcal{M}_{2,4}^{\text{Gor}}$ is a separated Deligne–Mumford stack (in the étale topology), coarsely represented by a quasi-projective scheme $\overline{\mathfrak{M}}_{2,4}^{\text{Gor}}$.

Remark 2.1. If we consider only smoothable surfaces in $\overline{\mathfrak{M}}_{2,4}$, then $\overline{\mathfrak{M}}_{2,4}^{\text{Gor}}$ is dense, since every Gorenstein stable surface X with $K_X^2 = 2$ and $\chi(\mathcal{O}_X) = 4$ is smoothable by Proposition 1.7.

We will now explain how the results of the previous chapter relate $\overline{\mathfrak{M}}_{2,4}^{\text{Gor}}$ to the following moduli space of polarised curves. The linear system of plane octic curves $H := |\mathcal{O}_{\mathbb{P}^2}(8)| (= \text{Hilb}^{8n-20}(\mathbb{P}^2))$ comes with its universal family $\mathscr{B} \subset H \times \mathbb{P}^2$, which is a relative Cartier divisor with respect to the projection $p_1 \colon H \times \mathbb{P}^2 \to H$. Performing the relative double-cover branched over $\mathscr{B}$ yields a flat, projective family $\pi \colon \mathscr{X} \to H$ of two-dimensional schemes. The locus $U \subset H$ parametrising curves B such that the fibre $\mathscr{X}_B \subset \mathscr{X}$ is semi-log-canonical is precisely the locus of half-log-canonical curves.

Lemma 2.2 — *In the notation of the preceding paragraph, the locus $U \subset H$ parametrising curves $B \subset \mathbb{P}^2$ such that the fibre $\mathscr{X}_B$ is semi-log-canonical is open.*

Proof. We will use a proof strategy outlined by Kovács [35]. By construction, every fibre $\mathscr{X}_B$ of $\pi \colon \mathscr{X} \to H$ is a double-cover of the plane, hence Gorenstein. A Gorenstein singularity is semi-log-canonical if and only if it is Du Bois; thus, the locus $U \subset H$ with semi-log-canonical fibres equals the locus with Du Bois-fibres. Since H is smooth, the Du Bois-locus is open, by Kovács' and Schwede's inversion of adjunction for Du Bois pairs [36, Theorem A; Lemma 4.5]. ☐

Restricting the family $\mathscr{X}$ to U defines a morphism $U \to \overline{\mathfrak{M}}_{2,4}^{\text{Gor}}$, $B \mapsto [\mathscr{X}_B]$, which is surjective by Corollary 1.7. Moreover, it induces an isomorphism of stacks:

Theorem 2.3 — *As above, let $U \subset |\mathcal{O}_{\mathbb{P}^2}(8)|$ be the space of half-log-canonical plane curves of degree* 8*. Taking the double-cover branched over the curves induces an isomorphism of algebraic stacks*

$$[U/\operatorname{PGL}(3,\mathbb{C})] \to \mathcal{M}_{2,4}^{\text{Gor}}.$$

In particular, $\mathcal{M}_{2,4}^{\text{Gor}}$ is smooth.

Proof. Using the notations introduced above, we will construct the inverse morphism and show that it is an isomorphism. By Proposition 1.9, any $(f \colon \mathscr{X} \to T) \in (\mathcal{M}_{2,4}^{\text{Gor}})_T$ is a relative double-cover of a projective bundle $\mathbb{P}_T(f_*\omega_{\mathscr{X}/T})$ with a relative branch divisor $\mathscr{B} \subset \mathbb{P}_T(f_*\omega_{\mathscr{X}/T})$. Thus, the associated $\operatorname{PGL}(3,\mathbb{C}) = \operatorname{Aut}(\mathbb{P}^2)$-torsor P comes with an induced $\operatorname{PGL}(3,\mathbb{C})$-equivariant morphism $P \to U$ corresponding to the family of octics. This defines an element of $[U/\operatorname{PGL}(3,\mathbb{C})]_T$. It is straightforward to check that this assignment is functorial for pull-backs along morphisms $T' \to T$, thus defining a morphism $\mathcal{M}_{2,4}^{\text{Gor}} \to [U/\operatorname{PGL}(3,\mathbb{C})]$. It is fully faithful and its essential image consists of those objects whose underlying $\operatorname{PGL}(3,\mathbb{C})$-torsors are locally trivial in the Zariski topology, yet again by Proposition 1.9. Thus, it suffices to show that this is the case for all $\operatorname{PGL}(3,\mathbb{C})$-torsors P with a $\operatorname{PGL}(3,\mathbb{C})$-equivariant morphism $P \to U$. In other words, what we have to show is that if $\mathbb{P} \to T$ is an étale-locally trivial $\mathbb{P}^2$-bundle with a relative divisor $\mathscr{B} \subset \mathbb{P}$ which is fibre-wise an octic, then $\mathbb{P}$ is locally trivial in the Zariski topology. But since $\mathcal{O}_{\mathbb{P}}(-3K_{\mathbb{P}/T} - \mathscr{B})$ restricts to $\mathcal{O}_{\mathbb{P}^2}(1)$ on the geometric fibres, this is indeed the case. ☐

For later reference, we observe that the classifying morphism $U \to \overline{\mathfrak{M}}_{2,4}^{\mathrm{Gor}}$ maps $\mathrm{PGL}(3,\mathbb{C})$-invariant locally closed sets to locally closed sets, hence, also $\mathrm{PGL}(3,\mathbb{C})$-invariant stratifications to stratifications. For this, it is enough to show that it is a geometric quotient in the sense of Mumford [42, Definition 0.6], for then $\overline{\mathfrak{M}}_{2,4}^{\mathrm{Gor}} \cong U/G$ carries the quotient topology.

Corollary 2.4 — *The coarse moduli space* $U/\mathrm{PGL}(3,\mathbb{C}) \cong \overline{\mathfrak{M}}_{2,4}^{\mathrm{Gor}}$ *is a quasi-projective scheme and the classifying morphism* $U \to U/\mathrm{PGL}(3,\mathbb{C})$ *is a geometric quotient. In particular, it maps* $\mathrm{PGL}(3,\mathbb{C})$*-invariant locally closed sets to locally closed sets.*

Proof. Since $\mathcal{M}_{2,4}^{\mathrm{Gor}}$ has a quasi-projective moduli space $\overline{\mathfrak{M}}_{2,4}^{\mathrm{Gor}}$ by the works of Kollár, Shepherd-Barron [32] and Alexeev [1], we conclude that so does $[U/\mathrm{PGL}(3,\mathbb{C})]$. This space, $U/\mathrm{PGL}(3,\mathbb{C})$, is then the categorical quotient in the category of schemes. On the other hand, since half-log-canonical plane octics can be shown to be GIT-stable, cf. Remark 4.2, the classifying map $U \to U/\mathrm{PGL}(3,\mathbb{C})$ equivariantly factors through the GIT-quotient $H^s \to H^s/\!/\mathrm{PGL}(3,\mathbb{C})$, which is geometric by Mumford [42, Theorem 1.10, cf. Chapter 1 §4]. (Here, $H^s \subset |\mathcal{O}_{\mathbb{P}^2}(8)|$ is the locus of GIT-stable points with respect to the $\mathrm{PGL}(3,\mathbb{C})$-action.) But then the classifying morphism $U \to U/G$ has to be geometric as well, as claimed. □

Remark 2.5. The moduli space under consideration is thus birationally equivalent to the moduli space of plane curves of degree 8, for which, according to Böhning, Graf von Bothmer and Kröker [7, p. 506], it is not known whether it is rational or not.

Question 2.6. It is tempting to call $U/\mathrm{PGL}(3,\mathbb{C})$ the moduli space of half-log-canonical octics, but it is not obvious whether $U/\mathrm{PGL}(3,\mathbb{C})$ is really a moduli space of curves, or one of polarised curves. *Are there pairs of abstractly, but not projectively isomorphic half-log-canonical plane octics?* Note that by Hassett [26, Proposition 2.1], two abstractly isomorphic nodal plane octics are indeed projectively isomorphic.

For further questions and comparisons between $\mathcal{M}_{2,4}^{\mathrm{Gor}}$ and different moduli spaces of curves, see Chapter 4.

3. A Stratification of the moduli space

As is well known, normalising usually does not work well in flat families. For the moduli space at hand, we can get hands on this quite explicitly, in that we can cover it by strata on which the normalisation can be performed in families. Recall from Proposition 1.11 that if $f\colon X \to \mathbb{P}^2$ is the canonical double-cover with branch curve B, then $B = B' + 2B''$ for reduced effective divisors B', B'' and the composition $\overline{f} = f \circ \pi\colon \overline{X} \to \mathbb{P}^2$ is the double-cover branched over B'; furthermore, the conductor on $\overline{X}$ is the pull-back of B''. In other words, the non-reduced part of the branch curve controls the non-normal locus of X. This motivates the first approximation to a stratification.

Definition 3.1 (The (non-)normality stratification) — *For a non-negative integer* $0 \le a \le 4$ *we let* $\mathfrak{M}^{(a)} \subset \overline{\mathfrak{M}}_{2,4}^{\mathrm{Gor}}$ *be the locus of those surfaces whose branch divisor* $B = B' + 2B'' \subset \mathbb{P}^2$ *for the canonical double-cover is such that* $B'' = \lfloor \frac{1}{2} B \rfloor$ *is of degree* a*. The open and dense subset consisting of the normal surfaces is denoted by* $\mathfrak{N} := \mathfrak{M}^{(0)}$.

Since the degree of the branch divisors B is eight, we clearly only need to consider $a = 0, \dots, 4$ to cover $\overline{\mathfrak{M}}_{2,4}^{\mathrm{Gor}}$ as $\bigcup_{a=0}^{4} \mathfrak{M}^{(a)}$.

Proposition 3.2 — *For each $a = 0, \dots, 4$, the subset $\mathfrak{M}^{(a)} \subset \overline{\mathfrak{M}}^{\mathrm{Gor}}_{2,4}$ is locally closed and its closure in $\overline{\mathfrak{M}}^{\mathrm{Gor}}_{2,4}$ is $\overline{\mathfrak{M}}^{(a)} = \bigcup_{a \le b \le 4} \mathfrak{M}^{(b)}$.*

Proof. As before, we let $U \subset |\mathcal{O}_{\mathbb{P}^2}(8)|$ be the open sub-scheme parametrising half-log-canonical plane octic curves and let $f\colon U \to \overline{\mathfrak{M}}^{\mathrm{Gor}}_{2,4}$ be the classifying map. By Corollary 2.4, it suffices to show that the pre-images $f^{-1}(\mathfrak{M}^{(a)})$ define such a stratification of U. The locus $U_a \subset U$ of half-log-canonical curves $B = B' + 2B''$ with $\deg(B'') \ge a$ may alternatively be characterised as the locus of $B \in U$ such that $\deg(B_{\mathrm{red}}) \le 8 - a$.

Note that quite generally, the space $V_n \subset |\mathcal{O}_{\mathbb{P}^2}(8)|$ consisting of the divisors B such that $\deg(B_{\mathrm{red}}) \le n$ is closed, being a union of closed sub-spaces

$$V_n = \bigcup_{\substack{\sum_i a_i n_i = 8,\\ \sum_i n_i \le n}} \sum_i a_i |\mathcal{O}_{\mathbb{P}^2}(n_i)|,$$

where $\sum_i a_i |\mathcal{O}_{\mathbb{P}^2}(n_i)|$ is shorthand notation for the image of the morphism

$$\prod_i |\mathcal{O}_{\mathbb{P}^2}(n_i)| \to |\mathcal{O}_{\mathbb{P}^2}(8)|,\ (B_i)_i \mapsto \sum_i a_i B_i.$$

Therefore, $U_a = U \cap V_{8-a}$ is closed, $f^{-1}(\mathfrak{M}^{(a)}) = U_a \setminus U_{a+1}$ is locally closed and from the above presentation it easily follows that $\overline{f^{-1}(\mathfrak{M}^{(a)})} = U_a$. This completes the proof. ☐

This very rough stratification will be refined in the following sections.

3.1 The locus of normal surfaces. We now turn to the stratification of the moduli space $\mathfrak{N} = \mathfrak{M}^{(0)}$ of normal Gorenstein stable surfaces with $K_X^2 = 2$ and $\chi(\mathcal{O}_X) = 4$. We stratify $\mathfrak{N}$ according to the number of irrational singularities, their degree and whether they are simply elliptic or cusps.

Definition 3.3 (The irrationality stratification) — *Given non-negative integers a, b, c and d satisfying $a + b + c + d \le 4$ we define the subset $\mathfrak{N}_{1^a \overline{1}^b 2^c \overline{2}^d}$ of the stratum of normal surfaces $\mathfrak{N}$ consisting of those surfaces having precisely a simply elliptic singularities of degree 1, b cusps of degree 1, c simply elliptic singularities of degree 2 and d cusps of degree 2. To ease notation, indices with exponent 0 are omitted, an exponent of 1 will be omitted and so on, e.g., $2^1 = 2$, $1^2 = 11$, etc.*

For example, an $X \in \mathfrak{N}_{122\overline{2}} = \mathfrak{N}_{1^1 \overline{1}^0 2^2 \overline{2}^1}$ has exactly one simply elliptic singularity of degree one, two simply elliptic singularities of degree two and one cusps of degree two. The empty list of degrees corresponds to the stable surfaces with only canonical singularities, i.e., $\mathfrak{N}_\emptyset = \mathfrak{N}_{1^0 \overline{1}^0 2^0 \overline{2}^0} = \mathfrak{M}_{2,4}$ is the dense open of canonical surfaces.

Remark 3.4. By definition, the loci $\mathfrak{N}_{1^a \overline{1}^i 2^b \overline{2}^j} \subset \mathfrak{N}$ are pair-wise disjoint and since the normal surfaces under investigation have at most four irrational singularities (Theorem 1.14 and Proposition 1.13), $\mathfrak{N}$ is indeed covered by the loci $\mathfrak{N}_{1^a \overline{1}^i 2^b \overline{2}^j}$, as $a + b + i + j \le 4$.

Remark 3.5. Local singularity theory, most notably Brieskorn's result [8], implies that that a singularity of type X_p may degenerate to a singularity of type X_q with $q \ge p$ or certain singularities of type $Y_{\bullet,\bullet}$, but none of them can degenerate to a triple-point or to a milder quadruple-point. Similarly, a [3; 3]-point may degenerate more and more, or it may even degenerate to a quadruple-point, but none of the series $X_\bullet$ or $Y_{\bullet,\bullet}$. This prevents certain strata to appear at the boundary of other strata. For example, the boundary of $\mathfrak{N}_2$ is covered by all strata parametrising surfaces with at least one (possibly degenerate) quadruple-point. More generally, the closure of $\mathfrak{N}_{1^a\overline{1}^b2^c\overline{2}^d}$ in $\mathfrak{N}$ is contained in the union of the strata $\mathfrak{N}_{1^{a'}\overline{1}^{b'}2^{c'}\overline{2}^{d'}}$ where $a'+b' \geq a+b$ and $b' \geq b$, as well as $c'+d' \geq c+d$ and $d' \geq d$.

Proposition 3.6 — *The strata $\mathfrak{N}_{1^a\overline{1}^b2^c\overline{2}^d} \subset \mathfrak{N}$ are locally closed and the closure of a stratum is contained in a union of strata.*

Proof. Let $U \subset |\mathcal{O}_{\mathbb{P}^2}|$ be the locus parametrising half-log-canonical plane octics and let $V \subset U$ be the open sub-space parametrising reduced curves. Then the pre-images of the strata under the classifying morphism $V \to \mathfrak{N} \subset \overline{\mathfrak{M}}_{2,4}^{\mathrm{Gor}}$ are disjoint and $\mathrm{PGL}(3,\mathbb{C})$-invariant by construction. By Corollary 2.4, it suffices to show that these pre-images are locally closed. This can be shown for each case separately by elementary plane curve geometry. We omit the details and conclude the proof. $\square$

Remark 3.7. The motivation to consider not just the stratification according the number and degree of irrational singularities, which is usually enough to get control over the birational geometry of the minimal resolution, but to also distinguish between simply elliptic and cuspidal singularities, comes from the relevance to the mixed Hodge structure discussed in Section 1.4.

3.1.1 The strata of the irrationality stratification. We are ready to state and prove the main results about the irrationality stratification.

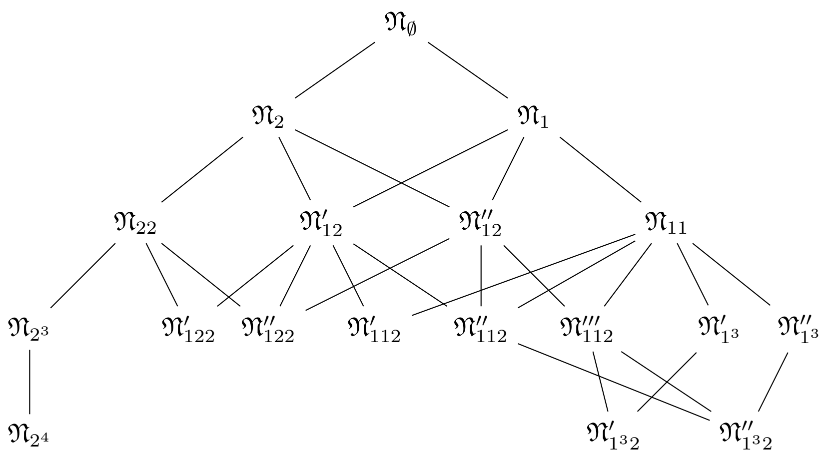


Figure 2: The degeneration diagram showing the components of the strata parametrising surfaces with only simply elliptic singularities

Theorem 3.8 — *All strata $\mathfrak{N}_{1^a\overline{1}^b2^c\overline{2}^d}$ with $a+b+c+d \leq 3$ and the two strata $\mathfrak{N}_{1^32}$ and $\mathfrak{N}_{2^4}$ are equidimensional of expected dimension $36-9a-10b-8c-9d$. The remaining strata are empty. Furthermore, the irreducible components of the strata are pair-wise disjoint.*

Script reference Strata/Components	Section	Script reference Strata/Components	Section
Listing 4		Listing 10	
$\mathfrak{N}_2, \mathfrak{N}_{22}, \mathfrak{N}_{2^3}, \mathfrak{N}_{2^4}$	I	$\mathfrak{N}'_{12\overline{2}}, \mathfrak{N}'_{\overline{1}22}$	I.1
$\mathfrak{N}_1, \mathfrak{N}_{11}$	II	$\mathfrak{N}'_{1\overline{22}}, \mathfrak{N}'_{\overline{1}2\overline{2}}$	I.2
$\mathfrak{N}_{12}, \mathfrak{N}_{122}, \mathfrak{N}_{112}$(*)	III	$\mathfrak{N}'_{\overline{122}}$	I.3
Listing 5		$\mathfrak{N}''_{12\overline{2}}, \mathfrak{N}''_{\overline{1}22}$	II.1
$\mathfrak{N}_{\overline{1}}, \mathfrak{N}_{1\overline{1}}, \mathfrak{N}_{\overline{11}}$	I	$\mathfrak{N}''_{1\overline{22}}, \mathfrak{N}''_{\overline{1}2\overline{2}}$	II.2
$\mathfrak{N}_{\overline{2}}, \mathfrak{N}_{2\overline{2}}, \mathfrak{N}_{\overline{22}}$	II	$\mathfrak{N}''_{\overline{122}}$	II.3
$\mathfrak{N}_{12}, \mathfrak{N}_{1\overline{2}}, \mathfrak{N}_{\overline{1}2}, \mathfrak{N}_{\overline{12}}$	III	Listing 11	
Listing 6		$\mathfrak{N}'_{11\overline{2}}, \mathfrak{N}'_{1\overline{1}2}$	I.1
$\mathfrak{N}_{122}$	I	$\mathfrak{N}'_{1\overline{12}}, \mathfrak{N}'_{\overline{11}2}$	I.2
$\mathfrak{N}_{112}$	II	$\mathfrak{N}'_{\overline{112}}$	I.3
$\mathfrak{N}_{1^3}$	III	$\mathfrak{N}''_{11\overline{2}}, \mathfrak{N}''_{1\overline{1}2}$	II.1
Listing 7		$\mathfrak{N}''_{1\overline{12}}, \mathfrak{N}''_{\overline{11}2}$	II.2
$\mathfrak{N}_{1^32}$	I, II	$\mathfrak{N}''_{\overline{112}}$	II.3
Listing 8		$\mathfrak{N}'''_{11\overline{2}}, \mathfrak{N}'''_{1\overline{1}2}$	III.1
$\mathfrak{N}_{22\overline{2}}, \mathfrak{N}_{2\overline{22}}, \mathfrak{N}_{\overline{2}^3}$	—	$\mathfrak{N}'''_{1\overline{12}}, \mathfrak{N}'''_{\overline{11}2}$	III.2
Listing 9		$\mathfrak{N}'''_{\overline{112}}$	III.3
$\mathfrak{N}'_{11\overline{1}}, \mathfrak{N}'_{1\overline{11}}, \mathfrak{N}'_{\overline{1}^3}$	I.1–I.3	Listing 12	
$\mathfrak{N}''_{11\overline{1}}, \mathfrak{N}''_{1\overline{11}}, \mathfrak{N}''_{\overline{1}^3}$	II.1–II.3	$\mathfrak{N}_{1^4} = \emptyset$	—

(*) In Listing 4 III.1 & III.2, only the dimensions are computed; the rest about $\mathfrak{N}_{12}$ is in Listing 5 III.1 and for $\mathfrak{N}_{122}$ and $\mathfrak{N}_{112}$ see Listing 6 I and II.

Table 3: The catalogue of scripts and strata

Proof. It follows from Proposition 1.13 and Theorem 1.14 that $a + b + c + d \leq 4$ and if $a + b + c + d = 4$, then $b = d = 0$. Moreover, Proposition A.10 shows that $\mathfrak{N}_{1122}$ and $\mathfrak{N}_{12^3}$ are empty, as is $\mathfrak{N}_{1^4}$, by Proposition A.6. It remains to show that all other strata are equidimensional, with all components pair-wise disjoint and of expected dimension. Since we have translated the problem into plane curve geometry, we can systematically use the computer algebra system Macaulay2 [19] to

1. check which strata are inhabited (by producing elements explicitly),
2. find all irreducible components and
3. compute their dimension.

The scripts and explanations about how they work are the content of Appendix B.

Which stratum is dealt with where is listed in Table 3. □

Remark 3.9. If a stratum $\mathfrak{N}_{1^a\overline{1}^b2^c\overline{2}^d}$ decomposes into the union of multiple components, we mostly use the following ad-hoc notation: we decorate the components with primes, i.e.,

$$\mathfrak{N}_{1^a\overline{1}^b2^c\overline{2}^d} = \mathfrak{N}'_{1^a\overline{1}^b2^c\overline{2}^d} \cup \mathfrak{N}''_{1^a\overline{1}^b2^c\overline{2}^d} \, (\cup \mathfrak{N}'''_{1^a\overline{1}^b2^c\overline{2}^d}) \, .$$

There are four exceptions, $\mathfrak{N}_{1\bar{1}2}$, $\mathfrak{N}_{1\overline{12}}$, $\mathfrak{N}_{12\bar{2}}$ and $\mathfrak{N}_{\bar{1}2\bar{2}}$, where a refined notation is explained (and used only) in the Macaulay2-code. The choices for the order are somewhat arbitrary. As a rule of thumb, more primes indicate that the configuration of singularities of the branch curve is more special. We give a few examples; see Definition A.1 for the notions:

If $X \in \mathfrak{N}_{12} = \mathfrak{N}'_{12} \cup \mathfrak{N}''_{12}$, then the branch curve of the canonical double cover $X \to \mathbb{P}^2$ has exactly one (non-degenerate) [3; 3]-point and one (ordinary) quadruple-point and up to automorphisms, there are two possibilities. Namely, either the distinguished tangent line of the [3; 3]-point misses the quadruple-point ($X \in \mathfrak{N}'_{12}$), or it passes through it ($X \in \mathfrak{N}''_{12}$). That this indeed splits $\mathfrak{N}_{12}$ into two components is non-trivial but follows from the Macaulay2-code Listing 5 III.1. Likewise, $\mathfrak{N}_{122}$ has two components, but no more since there can be only one quadruple-point on the distinguished tangent line of the [3; 3]-point (Lemma A.8).

In a different flavour, $\mathfrak{N}_{111} = \mathfrak{N}'_{111} \cup \mathfrak{N}''_{111}$ where the [3; 3]-points of the branch curve of $X \in \mathfrak{N}''_{111}$ are with tangents along a conic.

The decomposition $\mathfrak{N}_{1^32} = \mathfrak{N}'_{1^32} \cup \mathfrak{N}''_{1^32}$ comes from the two cases described in Proposition A.11.

Remark 3.10. In total, $\mathfrak{N}$ is covered by two strata with four components, six strata with three components, 13 strata with two components and 16 irreducible strata. Hence, the number of inhabited strata of the irrationality stratification on $\mathfrak{N}$ is 37 and there are 68 pair-wise disjoint components. The degeneration diagram for the components of all strata parametrising surfaces with simply elliptic singularities is shown in Figure 2. The complete degeneration diagram showing all strata of $\mathfrak{N}$ would be incomprehensibly complicated.

Theorem 3.11 — *Table 4 lists the components of the strata of the irrationality stratification and the birational isomorphism type of their members.*

Proof. Let X be a normal Gorenstein stable surface with $K_X^2 = 2$ and $\chi(\mathcal{O}_X) = 4$, let $f\colon Y \to X$ be the minimal resolution and let $\sigma\colon Y \to Y_{\min}$ be a minimal model. By Lemma 1.12, $\chi(\mathcal{O}_{Y_{\min}}) = \chi(\mathcal{O}_Y) = 4 - k$, where k is the number of elliptic singularities of X. Furthermore, Theorem 1.14 and Proposition 1.13 constrain the possible Kodaira dimensions $\kappa(Y_{\min}) = \kappa(Y)$ in such a way that from the Enriques–Kodaira classification of algebraic surfaces (cf. Barth, Hulek, Peters, Van de Ven [6, VI Theorem 1.1]), the claim follows for $\mathfrak{N}_\emptyset$, $\mathfrak{N}_1$, $\mathfrak{N}_2$, $\mathfrak{N}_{11}$, $\mathfrak{N}_{22}$, $\mathfrak{N}_{122}$, $\mathfrak{N}_{222}$, $\mathfrak{N}_{1^32}$ and $\mathfrak{N}_{2^4}$ and the cuspidal versions $\mathfrak{N}_{\bar{1}}$, $\mathfrak{N}_{\bar{2}}$ etc. In other words, the only strata which need extra care are $\mathfrak{N}_{12}$, $\mathfrak{N}_{1^3}$ and $\mathfrak{N}_{112}$ and their versions with cusps.

Before we deal with these cases, we introduce some more notation. Since the rational singularities of X admit a crepant resolution, $K_Y = f^*K_X - E$, where $E \subset Y$ is the sum of the exceptional curves $E_i \subset Y$ over the elliptic singularities $p_i \in X$, $i = 1, \ldots, k$. Likewise, let $G \subset Y$ be the divisor such that $K_Y = \sigma^* K_{Y_{\min}} + G$.

If X has two elliptic singularities, one of degree one and one of degree two, then σ is a single blow-up in a smooth point and either $\kappa(Y) = 0$ or $\kappa(Y) = 1$, again by Theorem 1.14. Since $\chi(\mathcal{O}_{Y_{\min}}) = 2$ and by the classification of algebraic surfaces, $\kappa(Y) = 0$ if and only if $Y_{\min}$ is a K3 surface. In the above notation, this is the case if and only if $K_Y = G$, where G is the (-1)-curve contracted by σ. But $K_Y = f^*K_X - E$, where E is the sum of two disjoint curves E_i, $i = 1, 2$, with $p_a(E_i) = 1$ and $E_i^2 = -i$. Therefore, $\kappa(Y) = 0$ if and only if $f^*K_X = E + G$. But then $K_X = f_*f^*K_X = f_*G$ has to be a rational curve passing through the two irrational singularities. On the other hand, $K_X = \varphi^*\mathcal{O}_{\mathbb{P}^2}(1)$, so that φ_*G has to be the line joining the quadruple-and the $[3;3]$-point of the branch curve, passing through the $[3;3]$-point in special tangent direction. Thus, if $\kappa(Y) = 0$, then X is a member of either $\mathfrak{N}''_{12}$, $\mathfrak{N}''_{\bar{1}2}$, $\mathfrak{N}''_{1\bar{2}}$ or $\mathfrak{N}''_{\bar{1}\bar{2}}$. Conversely, if X is a member of any of those components, then the distinguished tangent line of the $[3;3]$-point is contained in the branch curve (Lemma A.8), hence gives rise to a rational canonical curve on X passing through the elliptic singularities in such a way that $f^*K_X = E + G$ for a (-1)-curve G. Hence, X is a member of either $\mathfrak{N}''_{12}$, $\mathfrak{N}''_{\bar{1}2}$, $\mathfrak{N}''_{1\bar{2}}$ or $\mathfrak{N}''_{\bar{1}\bar{2}}$ if and only if $Y_{\min}$ is a K3 surface.

For $X \in \mathfrak{N}_{1^3}$, $\chi(\mathcal{O}_Y) = 1$ and we have two possibilities, namely, either $Y_{\min}$ is Enriques (Theorem 1.14) or rational (Proposition 1.13). In any case, $q(Y) = 0$, as we will show below. Therefore, Castelnuovo's Rationality Criterion implies that Y is rational if and only if $P_2(Y) = 0$. The branch curve $B \subset \mathbb{P}^2$ of the canonical double cover has three $[3;3]$-points and from Corollary A.4, it follows that the they are not collinear and that none of them lies on a distinguished tangent line of another $[3;3]$-point of B. Thus, they either align along a smooth conic, which has to be contained in the octic then, or they do not. But the sections of ω_Y^2 correspond exactly to the conics passing through all three points in distinguished tangent directions, so that $P_2(Y) \neq 0$ if and only if $X \in \mathfrak{N}''_{1^3}$. In other words, if $X \in \mathfrak{N}''_{1^3}$, then $Y_{\min}$ is an Enriques surface and if $X \in \mathfrak{N}'_{1^3}$, then Y is rational. The same argument applies to the cuspidal versions $\mathfrak{N}_{11\bar{1}}$, $\mathfrak{N}_{1\bar{1}\bar{1}}$ and $\mathfrak{N}_{\bar{1}^3}$.

The last case we have to consider is that X has two elliptic singularities of degree one and one of degree two. Since X has exactly three elliptic singularities, $\chi(\mathcal{O}_Y) = 1$ and from Theorem 1.14 and Proposition 1.13 we conclude that either $Y_{\min}$ is an Enriques surface and $\sigma\colon Y \to Y_{\min}$ is a blow up in two points (possibly infinitely close), or Y is rational. If $X \in \mathfrak{N}'''_{112}$ or any of the cuspidal versions, i.e., if the branch curve of the canonical double-cover $X \to \mathbb{P}^2$ contains the two distinguished tangents, which meet in the quadruple-point, then the union of those lines defines a trivialisation of $2K_{Y_{\min}}$; more precisely, the corresponding section of $\mathcal{O}_{\mathbb{P}^2}(2)$ lifts to a section of $2K_X - 2E$ with vanishing locus twice the disjoint union of two disjoint (-1)-curves, which constitute the exceptional locus of $\sigma\colon Y \to Y_{\min}$. If X is a member of either $\mathfrak{N}'_{112}$ or $\mathfrak{N}''_{112}$, however, this does not work and X is rational. An alternative way to see this is as follows. Let $B \subset \mathbb{P}^2$ be the branch curve of the canonical double-cover. Let $p_1, p_2 \in B$ be the $[3;3]$-points and let $p_3 \in \mathbb{P}^2$ be the quadruple-point. Consider the Cremona-transformation $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with centres p_1, p_2, p_3. Let $B' \subset \mathbb{P}^2$ be the reduced curve supported on the pull-back $(\varphi^{-1})^*B$, but neglecting the components with even multiplicity. Then the double-covers X and X' branched over B and B', respectively, are birational. In fact, the double cover branched over B' is the normalisation of the double-cover branched over $(\varphi^{-1})^*B$, which is birational to X. If $X \in \mathfrak{N}'_{112}$, then $X' \in \mathfrak{N}''_{222}$, which is rational and if $X \in \mathfrak{N}''_{112}$, then $X' \in \mathfrak{N}''_{122}$, which is rational as well. Furthermore, as the $[3;3]$- or quadruple-points of X degenerate, those of X' degenerate as well, but again this does not affect the birational isomorphism type.

Finally, we compute $p_g(Y)$ and $q(Y)$. If X has a single elliptic singularity, then the canonical linear system $|K_Y|$ is one-dimensional, corresponding to the pencil of lines through the non-simple singularity of the branch curve. Hence, $p_g(Y) = 2$ and $q(Y) = 0$. If X has two elliptic singularities, then $|K_Y|$ is a single point, corresponding to the line joining the two non-simple singularities of the branch curve. If X has at least three elliptic singularities, then there is no line through all the corresponding singularities of the branch curve, by Lemma A.3. Thus, $p_g(Y) = 0$. Since $4 - \chi(\mathcal{O}_Y)$ is the number of elliptic singularities, this implies $q(Y) = 0$ if X has at most three, and $q(Y) = 1$ if X has four elliptic singularities. $\square$

Components	Minimal model of the resolution $Y_{\min}$
$\mathfrak{N}_{\emptyset}$	General type, $K^2_{Y_{\min}} = 2$, $\chi(\mathcal{O}_{Y_{\min}}) = 4$
$\mathfrak{N}_1, \mathfrak{N}_{\overline{1}}$	General type, $K^2_{Y_{\min}} = 1$, $\chi(\mathcal{O}_{Y_{\min}}) = 3$
$\mathfrak{N}_2, \mathfrak{N}_{\overline{2}}$	Properly elliptic, $\chi(\mathcal{O}_{Y_{\min}}) = 3$, $p_g(Y_{\min}) = 2$
$\mathfrak{N}_{11}, \mathfrak{N}_{1\overline{1}}, \mathfrak{N}_{\overline{11}}$	Properly elliptic, $\chi(\mathcal{O}_{Y_{\min}}) = 2$, $p_g(Y_{\min}) = 1$
$\mathfrak{N}'_{12}, \mathfrak{N}'_{1\overline{2}}, \mathfrak{N}'_{\overline{1}2}, \mathfrak{N}'_{\overline{12}}$	Properly elliptic, $\chi(\mathcal{O}_{Y_{\min}}) = 2$, $p_g(Y_{\min}) = 1$
$\mathfrak{N}''_{12}, \mathfrak{N}''_{1\overline{2}}, \mathfrak{N}''_{\overline{1}2}, \mathfrak{N}''_{\overline{12}}$	K3
$\mathfrak{N}_{22}, \mathfrak{N}_{2\overline{2}}, \mathfrak{N}_{\overline{22}}$	K3
$\mathfrak{N}'_{1^3}, \mathfrak{N}'_{11\overline{1}}, \mathfrak{N}'_{1\overline{11}}, \mathfrak{N}'_{\overline{1}^3}$	Rational
$\mathfrak{N}''_{1^3}, \mathfrak{N}''_{11\overline{1}}, \mathfrak{N}''_{1\overline{11}}, \mathfrak{N}''_{\overline{1}^3}$	Enriques
$\mathfrak{N}'_{112}, \mathfrak{N}'_{11\overline{2}}, \dots, \mathfrak{N}'_{\overline{112}}$	Rational
$\mathfrak{N}''_{112}, \mathfrak{N}''_{11\overline{2}}, \dots, \mathfrak{N}''_{\overline{112}}$	Rational
$\mathfrak{N}'''_{112}, \mathfrak{N}'''_{11\overline{2}}, \dots, \mathfrak{N}'''_{\overline{112}}$	Enriques
$\mathfrak{N}'_{122}, \mathfrak{N}'_{12\overline{2}}, \dots, \mathfrak{N}'_{\overline{122}}$	Rational
$\mathfrak{N}''_{122}, \mathfrak{N}''_{12\overline{2}}, \dots, \mathfrak{N}''_{\overline{122}}$	Rational
$\mathfrak{N}_{2^3}, \mathfrak{N}_{22\overline{2}}, \mathfrak{N}_{2\overline{22}}, \mathfrak{N}_{\overline{2}^3}$	Rational
$\mathfrak{N}'_{1^32}, \mathfrak{N}''_{1^32}$	Ruled of genus 1
$\mathfrak{N}_{2^4}$	Ruled of genus 1

Table 4: The birational types of the normalisations.

The Hodge type $\lozenge_{r,s}$ of $X \in \mathfrak{N}$ roughly behaves as follows: As we introduce a simply elliptic singularity, s increases by one and as a simply elliptic singularity degenerates to a cusp, s decreases by one and r increases by one. In fact, this only fails in the case where it is numerically impossible since there are four elliptic singularities. In this case, the $(1,0)$-classes become linearly dependent.

Proposition 3.12 — *The Hodge type is constant on every stratum of the irrationality stratification of* $\mathfrak{N}$ *and they are given as in Figure 3.*

Proof. Recall from Lemma 1.19 that if X has exactly r cusps, then it is of Hodge type $\lozenge_{r,s}$ for some $0 \leq s \leq 3 - r$. It remains to compute s in each possible case. Recall the set-up in which we proved the lemma. We let $Y \to X$ be the resolution of the elliptic singularities, with exceptional arithmetically elliptic curves E_i, $i = 1, \dots, n$, and considered the Mayer–Vietoris exact sequence

$$H^1(Y) \to \bigoplus_{i=1}^{n} H^1(E_i) \to H^2(X) \to H^2(Y).$$

In this set-up, renumbering if necessary, we can suppose that the curves $E_1, \dots, E_k$ are smooth elliptic and that the remaining ones, $E_{k+1}, \dots, E_n$, are cycles of rational curves. Then the induced exact sequence of $(1,0)$-parts becomes:

$$H^{1,0}(Y) \to \bigoplus_{i=1}^{k} H^1(E_i)^{1,0} \to (H^2(X))^{1,0} \to 0.$$

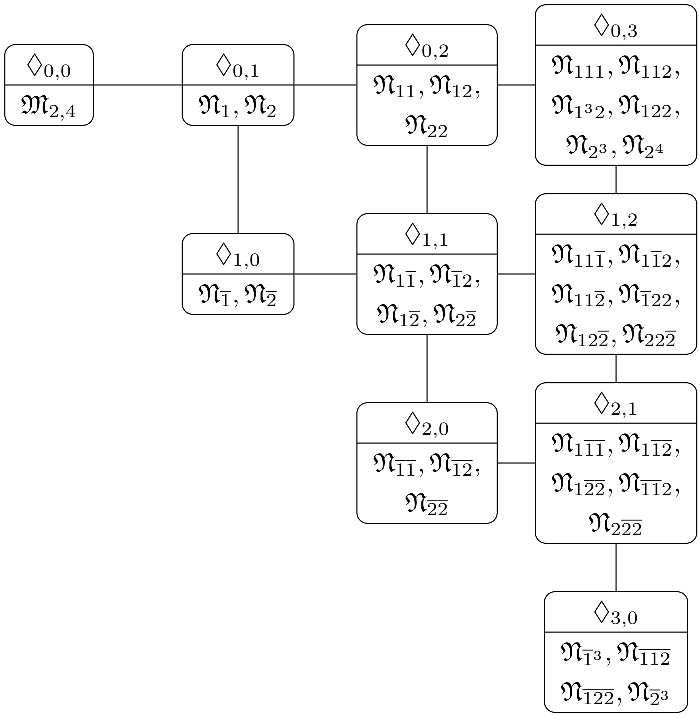


Figure 3: Degeneration diagram for Hodge types, cf. Proposition 3.12.

Thus, $s = \dim(H^2(X))^{1,0} = k - \dim \operatorname{im}(H^{1,0}(Y) \to \bigoplus_{i=1}^k H^1(E_i))$. In particular, if $k = 0$, then $s = 0$. In what follows, we assume $k \geq 1$.

Our claims only concern the dimensions in degree $(0,0)$, $(1,0)$ and $(2,0)$; for this reason and since the remaining singularities of Y are rational, we can assume without loss of generality that Y is minimal.

It follows from the proof of Theorem 3.11 that $q(Y) = 0$, unless X is ruled of genus 1. Clearly, if $q(Y) = 0$, then $s = k$.

If Y is ruled of genus 1, then $q(Y) = 1$ and the curves E_i, $i \leq k$, are multi-sections of the ruling $Y \to C$. On the one hand, the pull-back morphism $H^1(C) \to H^1(E_i)$ is multiplication with the degree, hence injective. On the other hand, it factors through $H^1(Y) \to H^1(E_i)$, which is injective as well then. Therefore, $s = k - 1$. Note that the strata where Y is ruled are those with $n = k = 4$ and $r = 0$. In conclusion, the members of $\mathfrak{N}_{1^3 2}$ or $\mathfrak{N}_{2^4}$ have Hodge type $\Diamond_{0,3}$. □

3.2 The loci of non-normal surfaces. We define a stratification of $\mathfrak{M}^{(n)}$ analogously to the irrationality stratification of $\mathfrak{N}$:

Definition 3.13 — *Let $1 \leq n \leq 4$. Given non-negative integers $a, b, c, d \geq 0$, we let $\mathfrak{M}_{n;1^a\overline{1}^b 2^c \overline{2}^d} \subset \mathfrak{M}^{(n)}$ be the locus parametrising surfaces $X \in \mathfrak{M}^{(n)}$ with exactly a simply elliptic and exactly b cuspidal singularities of degree 1 and exactly c simply elliptic and exactly d cuspidal singularities of degree 2. We apply the analogous abbreviation-conventions as before.*

The arguments used to prove Proposition 3.2 and Proposition 3.6 also show:

Proposition 3.14 — *The strata* $\mathfrak{M}_{n;1^a\bar{1}^b2^c\bar{2}^d} \subset \mathfrak{M}^{(n)}$ *are locally closed and the closure of one stratum is contained in a union of strata.*

Unlike the irrationality stratification of the locus of normal surfaces, this stratification is not finer than the Hodge type stratification. Namely, singularities of type $J_{2,\infty}$, X_∞ or $Y_{r,\infty}$ strongly affect the Hodge structure in a way that is hard to control. This is why we concentrate on isolated irrational singularities in the irrationality stratification of the locus of non-normal surfaces, even though it its insufficient for reading off the Hodge type. The Examples 3.17 and 3.18 below illustrate this.

We proceed by investigating which strata are inhabited. Afterwards, we compute their dimensions and the birational types of their members.

The stratum $\mathfrak{M}^{(4)}$: The members of $\mathfrak{M}^{(4)}$ are the double-covers of $\mathbb{P}^2$ branched over the double-quartics with at worst nodes. Thus, there is only one inhabited stratum $\mathfrak{M}_{4;\emptyset} = \mathfrak{M}^{(4)}$. It is isomorphic to the moduli space of nodal plane quartics (cf. Hassett [26]) which is rational (as shown by Katsylo [29]).

The stratum $\mathfrak{M}^{(3)}$: The members of $\mathfrak{M}^{(3)}$ are double-covers of $\mathbb{P}^2$ branched over a reduced conic and a double-cubic with at worst nodes. Since a reduced conic has at worst a node, the members of $\mathfrak{M}^{(3)}$ do not have isolated irrational singularities. Hence, yet again, only $\mathfrak{M}_{3;\emptyset} = \mathfrak{M}^{(3)}$ is inhabited.

The stratum $\mathfrak{M}^{(2)}$: Since a member $X \in \mathfrak{M}^{(2)}$ is a double-cover of $\mathbb{P}^2$ branched over $B = B' + 2B''$, where B' is a reduced quartic and B'' is a reduced conic, the isolated elliptic singularities come from the non-simple singularities of the quartic, of which only one is possible, namely, an ordinary quadruple-point, arising only as the union of four concurrent lines by Hui's classification [28]. Therefore, $\mathfrak{M}^{(2)} = \mathfrak{M}_{2;\emptyset} \cup \mathfrak{M}_{2;2}$.

The stratum $\mathfrak{M}^{(1)}$: A member of $\mathfrak{M}^{(1)}$ has a branch divisor of the form $B' + 2B''$ where B'' is a line and B' is a reduced sextic. By Proposition A.12, the only possible non-simple singularities B' might have are either a (possibly degenerate) quadruple-point, or a (possibly degenerate) $[3;3]$-point, or two non-degenerate $[3;3]$-points (with distinct distinguished tangent lines). Furthermore, in the last case, B' is the union of three conics meeting in the two $[3;3]$-points. Thus, the inhabited strata of $\mathfrak{M}^{(1)}$ are $\mathfrak{M}_{1;\emptyset}$, $\mathfrak{M}_{1;1}$, $\mathfrak{M}_{1;2}$, $\mathfrak{M}_{1;\bar{1}}$, $\mathfrak{M}_{1;\bar{2}}$ and $\mathfrak{M}_{1;11}$.

Strata	Dimension	Birational type of normalisation
$\mathfrak{M}^{(4)} = \mathfrak{M}_{4;\emptyset}$	6	$\mathbb{P}^2 \amalg \mathbb{P}^2$
$\mathfrak{M}^{(3)} = \mathfrak{M}_{3;\emptyset}$	6	Rational
$\mathfrak{M}_{2;\emptyset}$	11	Weak del Pezzo of degree 2
$\mathfrak{M}_{2;2}$	3	Ruled of genus 1
$\mathfrak{M}_{1;\emptyset}$	21	K3-Surface
$\mathfrak{M}_{1;1}$, $\mathfrak{M}_{1;\bar{1}}$	12,11	Rational
$\mathfrak{M}_{1;2}$, $\mathfrak{M}_{1;\bar{2}}$	13,12	Rational
$\mathfrak{M}_{1;11}$	3	Ruled of genus 1

Table 5: The strata for non-normal surfaces

Proposition 3.15 — *All the strata $\mathfrak{M}_{1;\emptyset}$, $\mathfrak{M}_{1;1}$, $\mathfrak{M}_{1;\bar{1}}$, $\mathfrak{M}_{1;2}$, $\mathfrak{M}_{1;\bar{2}}$, $\mathfrak{M}_{1;11}$, $\mathfrak{M}_{2;\emptyset}$, $\mathfrak{M}_{2;2}$, $\mathfrak{M}_{3;\emptyset}$, $\mathfrak{M}_{4;\emptyset}$ are irreducible and of dimension as indicated in Table 5.*

Proof. We first show that $\mathfrak{M}^{(n)}$ is irreducible for all $n = 1, \ldots, 4$. As in Theorem 2.3, we denote by $U \subset |\mathcal{O}_{\mathbb{P}^2}(8)|$ the locus of half-log-canonical plane octics. The pre-image $U_n \subset |\mathcal{O}_{\mathbb{P}^2}(8-2n)| \times |\mathcal{O}_{\mathbb{P}^2}(n)|$ of U under the closed embedding $|\mathcal{O}_{\mathbb{P}^2}(8-2n)| \times |\mathcal{O}_{\mathbb{P}^2}(n)| \to |\mathcal{O}_{\mathbb{P}^2}(8)|$, $(B', B'') \mapsto B' + 2B''$ is open, hence smooth and irreducible. By construction, the composition $U_n \hookrightarrow U \to U/\operatorname{PGL}(3,\mathbb{C}) \cong \overline{\mathfrak{M}}^{\mathrm{Gor}}_{2,4}$ identifies $U_n/\operatorname{PGL}(3,\mathbb{C})$ with $\mathfrak{M}^{(n)}$. Since U_n is irreducible, so is $\mathfrak{M}^{(n)}$, as claimed. In addition, this proves

$$\dim \mathfrak{M}^{(n)} = \dim |\mathcal{O}_{\mathbb{P}^2}(8-2n)| + \dim |\mathcal{O}_{\mathbb{P}^2}(n)| - 8,$$

which gives $\dim \mathfrak{M}^{(1)} = 21$, $\dim \mathfrak{M}^{(2)} = 11$ and $\dim \mathfrak{M}^{(3)} = \dim \mathfrak{M}^{(4)} = 6$. (The stabiliser of a general plane curve of degree ≥ 3 is discrete and in the cases under consideration we have either $n \geq 3$ or $8 - 2n \geq 4$.) Since $\mathfrak{M}_{n;\emptyset}$ is open in $\mathfrak{M}^{(n)}$ we get the claimed results for these strata.

In the remaining cases, we argue similarly. Let $\mathfrak{M} \subset \mathfrak{M}^{(n)}$ be any of the strata. We let $V \subset U_n \subset |\mathcal{O}_{\mathbb{P}^2}(8-2n)| \times |\mathcal{O}_{\mathbb{P}^2}(n)|$ be the pre-image of $\mathfrak{M}$ under the restricted classifying map $U_n \to \mathfrak{M}^{(n)}$. Then V dominates $\mathfrak{M}$, so that it would be enough to show that V is irreducible. However, it will be customary to restrict to certain sub-spaces in order to gain more control.

For the strata $\mathfrak{M} \subset \mathfrak{M}^{(1)}$, where B'' is a line, we can fix this line; then the condition for B' along B'' is that their local intersection multiplicities are at most 2 everywhere. That this is an open condition follows as in the proof of Proposition 3.2.

The easiest case is $\mathfrak{M} = \mathfrak{M}_{1;2}$. For every member $B' + 2B'' \in V$, where $p \in \mathbb{P}^2$ is the quadruple-point of B', there is a plane automorphism mapping p to the point $(0;0;1)$ and the line B'' to the line at infinity $L = \{z = 0\}$, where we are using homogeneous coordinates $(x;y;z) \in \mathbb{P}^2$. (We could have used any pair of a point and a line missing the point, of course.) The linear system of sextics with multiplicity at least 4 in $(0;0;1)$ is of dimension 17 (Listing 13 I.1). Let $V' \subset |\mathcal{O}_{\mathbb{P}^2}(6)|$ be the locus of sextics B' such that $B' + 2L$ is a member of V. Then the $\operatorname{PGL}(3,\mathbb{C})$-orbit of $V' + 2L \subset V$ is all of V and so $V'/\operatorname{PGL}(3,\mathbb{C}) \cong \mathfrak{M}$. Note that a sextic B' with a quadruple-point at $(0;0;1)$ lies in V' if and only if it is reduced, the quadruple-point is non-degenerate, all remaining singularities are simple and every intersection with B'' has multiplicity at most 2. All these conditions are open in the linear system of sextics with multiplicity ≥ 4 in $(0;0;1)$. Therefore, V' is irreducible and, hence, so is $V'/\operatorname{PGL}(3,\mathbb{C}) \cong \mathfrak{M}_{1;2}$. Since the group of automorphisms fixing a point and a line missing the point is of dimension 4, we conclude that $\mathfrak{M}_{1;2}$ is irreducible and of dimension $17 - 4 = 13$.

The stratum $\mathfrak{M}_{1;\bar{2}}$ is handled similarly; the difference is that we also have to fix the special tangent direction, which we can still do using automorphisms. This way, we get an open sub-set of a linear sub-space of dimension 15 (Listing 13 I.2), with stabiliser of dimension 3, so that this stratum is irreducible of dimension 12.

Let us turn to $\mathfrak{M}_{1;1}$, where we argue similarly. Again, we can fix the singular point, which we want to be a $[3;3]$-point, so we should also fix the distinguished tangent line and we still have automorphisms left to fix the line B''. The linear systems of sextics with at least a $[3;3]$-point in a fixed point and with fixed special tangent direction is of dimension 15 (Listing 13 II.1). By Lemma A.3 b), the only other singularities a reduced sextic can have besides a $[3;3]$-point are at most triple-points and they have to be off the distinguished tangent line. Furthermore, if a reduced sextic has a $[3;3]$-point, then it has at most one other non-simple singularity, which is another $[3;3]$-point; a closed condition. Since the stabiliser is of dimension 3, again we conclude that the stratum under consideration is irreducible and 12-dimensional.

The same argument shows that $\mathfrak{M}_{1;\bar{1}}$ is irreducible and of dimension $14-3=11$ since the sub-space of $|\mathcal{O}_{\mathbb{P}^2}(6)|$ parametrising sextics with a degenerate $[3;3]$-point at a fixed point and a fixed tangent (but variable second order direction) is irreducible and of dimension 14 as computed in Listing 13 II.2.

We argue a little differently for $\mathfrak{M}_{1;11}$. Note that after fixing the locus of $[3;3]$-points with their distinguished tangent directions, the locus of admissible lines B'' is independent of the sextic, for the sextic is a union of three distinct conics passing through the points in distinguished tangent direction, it meets lines with multiplicity 3 only in the $[3;3]$-points and so the double-line may be any line missing those two points. Since the corresponding space of sextics is irreducible and one-dimensional Listing 13 II.3, $\mathfrak{M}_{1;11}$ is irreducible and of dimension 3.

The last case we have to work out is $\mathfrak{M}_{2;2}$. The inverse image $V \subset U_2$ of $\mathfrak{M}_{2;2}$ consists of the octics decomposing as $B' + 2B''$ with a reduced quartic B' and a reduced conic B'', where B' has a quadruple-point, hence, is a union of four concurrent lines, and B'' has at worst nodes. Furthermore, the nodes of B'' have to be off B'. Since there is a 1-parameter family of analytically distinct quadruple-points, we can neither fix the quartic, nor the conic, which could be smooth or a union of two lines. However, we can consider the linear system in $|\mathcal{O}_{\mathbb{P}^2}(4)| \times |\mathcal{O}_{\mathbb{P}^2}(2)|$ given by pairs (B', B'') where B' is a quartic with a quadruple-point in $(1;1;1)$ and B'' is a conic in the pencil $\{\lambda xy + \mu z^2 = 0\}_{(\lambda;\mu)\in\mathbb{P}^1}$. In addition, we ask that the quartic contains the lines $\{x = z\}$ and $\{y = z\}$. Since for any $B = B' + 2B'' \in V$, at most two lines in B' can be tangent to B'', we find for at least two of the lines in B' a transversal intersection point with B''. Therefore, B is projectively equivalent to a member of this 3-dimensional linear system, up to finitely many choices of parameters. The only exceptional parameters are either $(\lambda;\mu) = (0;1)$, or those where the quadruple-point is degenerate, or where $(\lambda;\mu) = (1;0)$ and where the quartic passes through the point $(0;0;1)$. These conditions are clearly closed, so that we have an open, irreducible sub-scheme which is a finite cover of $\mathfrak{M}_{2;2}$. □

Finally, we discuss the birational geometry of the non-normal surfaces:

Proposition 3.16 — *The minimal models of the minimal resolution of the possible non-normal Gorenstein stable surfaces X satisfying $K_X^2 = 2$ and $\chi(\mathcal{O}_X) = 4$ are as listed in Table 5 above.*

Proof. Let X be a non-normal Gorenstein stable surface satisfying $K_X^2 = 2$ and $\chi(\mathcal{O}_X) = 4$, let $\overline{X}$ be its normalisation, denote its minimal resolution by $g\colon Y \to \overline{X}$ and let $\sigma\colon Y \to Y_{\min}$ be a minimal model of Y. By our description of the normalisation (Proposition 1.11), the branch curve $B \subset \mathbb{P}^2$ of the canonical double-cover decomposes as $B = B' + 2B''$ for two reduced effective divisors B', B'' and $\overline{X}$ is the double-cover of $\mathbb{P}^2$ branched over B'. We have to have $\deg(B') \in \{0,2,4,6\}$, where $X \in \mathfrak{M}^{(i)}$ if and only if $\deg(B') = 2i$.

In case $B' = 0$, observe that $X = \mathbb{P}^2 \amalg_{B''} \mathbb{P}^2$, as the double-cover branched over $2B''$ and $Y = \overline{X} = \mathbb{P}^2 \amalg \mathbb{P}^2$ is the unbranched double cover of the plane.

For the remaining cases, we make use of formulas and basic facts concerning double-covers which can be found in Barth, Hulek, Peters, Van de Ven [6, V 22].

If $\deg(B') = 2$, we have two cases: either B' is a smooth conic, or the union of two lines. In the first case, $\overline{X}$ is $\mathbb{P}^1 \times \mathbb{P}^1$ and in the latter, it is the quadric cone, which is resolved by the Hirzebruch surface $\mathbb{F}_2$. That is, the minimal models are all rational.

If $X \in \mathfrak{M}^{(2)}$, then B' is a quartic with only simple singularities, unless it is a union of four concurrent lines (cf. Hui's classification [28]). If $X \in \mathfrak{M}_{2;\emptyset}$, i.e., B' has only simple singularities, $\overline{X}$ is a del Pezzo surface of degree 2 (possibly singular, with ADE-singularities corresponding to those of B'), the double-cover being defined by the anti-canonical linear system. In fact, $-K_Y = -g^*K_{\overline{X}}$ is ample, as the pull-back of an ample bundle along a finite morphism and $K_Y^2 = K_{\overline{X}}^2 = 2$ since the degree is 2.

If $X \in \mathfrak{M}_{2;2}$, i.e., B' is the union of four concurrent lines, then the pencil of lines through their common intersection point gives rise to a ruling of Y over a curve of genus 1. Explicitly, the blow up of $\mathbb{P}^2$ in the quadruple-point is the Hirzebruch surface $\mathbb{F}_1$ and the double-cover over the four fibres of the ruling $\mathbb{F}_1 \to \mathbb{P}^1$ coming from the four branches of C induces a ruling $Y \to E$, where the elliptic curve E is the double-cover of $\mathbb{P}^1$ branched over the four points corresponding to the lines in question.

Finally, in case $X \in \mathfrak{M}^{(1)}$, where B' is a sextic, the only non-simple singularities B' can have are: (a) none; (b) a (not necessarily ordinary) quadruple-point, (c) a (possibly degenerate) [3;3]-point (see A.1 for the definition), (d) a pair of non-degenerate [3; 3]-points. In case (d), the sextic decomposes as the union of three conics (of which at least two are smooth), passing through the two [3, 3]-points. Here, the cases (a), (b), (c) and (d) correspond to the cases that $X \in \mathfrak{M}_{1;\emptyset}$, or $X \in \mathfrak{M}_{1;2} \cup \mathfrak{M}_{1;\bar{2}}$, or $X \in \mathfrak{M}_{1;1} \cup \mathfrak{M}_{1;\bar{1}}$, or $X \in \mathfrak{M}_{1;11}$, respectively.

That Y is a K3-surface in case (a) is well-known. Furthermore, by the canonical bundle formula, in all cases, $\omega_{\overline{X}} = \mathcal{O}_{\overline{X}}$. In the remaining cases (b)–(d), this implies $\kappa(Y) = -\infty$, for, $-K_Y$ is the sum of the exceptional divisors over the elliptic singularities. Therefore, $Y_{\min}$ is either rational (if $\chi(\mathcal{O}_Y) = 1$), or ruled of genus $1 - \chi(\mathcal{O}_Y) \geq 1$. Thus, we only have to compute the holomorphic Euler characteristic of Y. Since $\overline{X}$ is a flat degeneration of a K3-surface, $\chi(\mathcal{O}_{\overline{X}}) = 2$. From this we finally conclude $\chi(\mathcal{O}_Y) = 1$ in case (b) or (c) and $\chi(\mathcal{O}_Y) = 0$ in case (d), as claimed. ☐

3.2.1 The Hodge type in the non-normal case. For a double-cover of the plane $X \to \mathbb{P}^2$ branched over a half-log-canonical curve $B = B' + 2B''$ where B' and B'' are reduced, the Hodge type of X depends not just on the irrational singularities of B', but also on the nodes of B'' and the way how B' and B'' meet. In particular, the Hodge type is not constant on the strata of the irrationality stratification of the locus of non-normal surfaces. The list of possibilities gets quite complicated for the cases we would have to consider here. By way of example, we indicate the possible Hodge types on $\mathfrak{M}_{2;\emptyset}$ and $\mathfrak{M}^{(4)} = \mathfrak{M}_{4;\emptyset}$. The remaining strata can be dealt with analogously.

Example 3.17. Given $X \in \mathfrak{M}_{2;\emptyset}$, we let $\overline{X} \to X$ be the normalisation and denote the conductor loci by $F \subset X$ and $\overline{F} \subset \overline{X}$. As explained in the introduction, X is the push-out of the diagram $F \leftarrow \overline{F} \to \overline{X}$ and by Peters and Steenbrink [44, Corollary-Definition 5.37] we get the associated Mayer–Vietoris exact sequence

$$0 \to H^1(\overline{F};\mathbb{C}) \to H^2(X;\mathbb{C}) \to H^2(\overline{X};\mathbb{C}) \oplus H^2(F;\mathbb{C}) \to H^2(\overline{F};\mathbb{C}).$$

In fact, the left-most term is $H^1(\overline{X};\mathbb{C}) \oplus H^1(F;\mathbb{C}) = 0$, which can be seen as follows: Recall from Proposition 1.11 that if $B = B' + 2B''$ is the branch curve of X, where B' is a reduced quartic and B'' is a reduced conic, then $\overline{X}$ is the double-cover branched over B' and $F \cong B''$. In particular, $H^1(F;\mathbb{C}) = 0$. Since $\overline{X}$ is rational, $H^1(\overline{X};\mathbb{C}) = 0$ as well.

If, in addition, $\overline{F}$ is connected, then $H^2(F;\mathbb{C}) \to H^2(\overline{F};\mathbb{C})$ is an isomorphism. Thus, $\dim(H^2(X;\mathbb{C}))^{2,0} = h^{2,0}(\overline{X}) = 0$, $\dim(H^2(X;\mathbb{C}))^{1,0} = \dim(H^1(\overline{F};\mathbb{C}))^{1,0}$ and $\dim(H^2(X;\mathbb{C}))^{0,0} = \dim(H^1(\overline{F};\mathbb{C}))^{0,0}$. Now recall that $\overline{F}$ is a double-cover of B'' branched over $B''|_{B'}$. Thus, if B' and B'' meet transversely, then $\overline{F}$ is a smooth curve of genus 3, hence X has Hodge type $\lozenge_{0,3}$. As $B''|_{B'}$ gets doubled points, either due to tangency or due to double-points of B' along B'', $\overline{F}$ acquires nodes. This results either in a nodal curve of genus 2, a curve of genus 1 with 2 nodes, a curve of genus 0 with 3 nodes, or the union of 2 rational curves meeting transversely in 4 points. The first two cases give Hodge types $\lozenge_{1,2}$, $\lozenge_{2,1}$, the latter two have Hodge type $\lozenge_{3,0}$.

If B'' is the union of two lines, then by the same argument as above, the $(2,0)$-part is trivial, so that there are no more possible Hodge types than those above. The same applies if we pass to $\mathfrak{M}_{2;2}$, where B' has a quadruple-point.

Example 3.18. Let B'' be a smooth or nodal but reduced quartic in $\mathbb{P}^2$. With branch curve $B = 2B''$, we get that $X = \mathbb{P}^2 \amalg_{B''} \mathbb{P}^2$ and $\overline{X} = \mathbb{P}^2 \amalg \mathbb{P}^2$ with $\overline{F} = B'' \amalg B''$ and $F \cong B''$ in such a way that $\pi|_{\overline{F}} \colon \overline{F} \to F$ is the trivial double-cover. Tracing through the maps in the Mayer–Vietoris sequence for X as the push-out of the diagram $F \leftarrow \overline{F} \to \overline{X}$, one quickly finds an isomorphism of Deligne's mixed Hodge structures $H^2(X;\mathbb{C}) \cong H^1(B'';\mathbb{C})$. Therefore, the surfaces parametrised by the irreducible stratum $\mathfrak{M}^{(4)} = \mathfrak{M}_{4;\emptyset}$ realise all Hodge types $\lozenge_{r,s}$ with $r + s = 3$.

4. Further remarks and questions

4.1 Comparison with known compact moduli spaces of curves. There are at least three related compactifications of the moduli space of smooth plane curves of degree 8, namely, the GIT-quotient of $|\mathcal{O}_{\mathbb{P}^2}(8)|$ under the action of $\mathrm{PGL}(3,\mathbb{C})$, Hassett's moduli space of stable log-surfaces which admit a smoothing to $(\mathbb{P}^2, C)$ with C a curve of degree 8 [26] and Hacking's moduli space $\mathcal{M}_8$ of so-called *stable pairs of degree* 8, namely, pairs (X, D) consisting of a surface X and an effective $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor D on X, where $\mathcal{O}_X(3D + 8K_X) \cong \mathcal{O}_X$ and such that $(X, (\frac{3}{8} + \varepsilon)D)$ is a stable log-surface for some $\varepsilon > 0$, subject to a smoothability condition that makes sure that the plane octics are dense [22].

Question 4.1. Perhaps, a $\mathbb{Q}$-Gorenstein degeneration of a smooth (or Gorenstein) stable surface $X \in \mathfrak{M}_{2,4}$, say canonically the double-covers of $\mathbb{P}^2$ branched over $B \in |\mathcal{O}_{\mathbb{P}^2}(8)|$, will itself be a double-cover of a surface X' branched over some curve B', where $(X', \frac{1}{2}B')$ is semi-log-canonical. This raises the question whether (the closure of $\overline{\mathfrak{M}}^{\mathrm{Gor}}_{2,4}$ in) $\overline{\mathfrak{M}}_{2,4}$ is isomorphic to some projective moduli space of semi-log-canonical pairs (X, D) which are in some sense degenerations of log-canonical pairs of the form $(\mathbb{P}^2, \frac{1}{2}B)$ with an octic B. (For an example, see 4.5 below.)

More concretely, let $\mathcal{M}_8$ be Hacking's moduli stack of $\mathbb{Q}$-Gorenstein smoothable families of stable pairs of degree 8, which is a separated, proper and smooth Deligne–Mumford stack (Hacking [22, Theorem 4.4 & 7.2]), with coarse moduli space denoted by M_8.

Each half-log-canonical octic $B \in U$, gives rise to a stable pair of degree 8, $(\mathbb{P}^2, B)$, for, $(\mathbb{P}^2, (\frac{3}{8} + \varepsilon)B)$ is a stable log-surface for all $0 < \varepsilon \leq \frac{1}{8}$. This induces a morphism $\mathcal{M}^{\mathrm{Gor}}_{2,4} \to \mathcal{M}_8$ which seems worthwhile to study. *Does it extend to a morphism* $\overline{\mathcal{M}}_{2,4} \to \mathcal{M}_8$*?*

In that case, one naive hope would be that the locus of stable pairs (X, D) of degree 8 such that $(X, \frac{1}{2}D)$ is semi-log-canonical is closed and isomorphic to $\overline{\mathfrak{M}}_{2,4}$, but this locus is not closed in M_8 (see Example 4.6 below).

Remark 4.2. For a stable pair $(\mathbb{P}^2, D)$ of degree d in the sense of Hacking, the curve D is GIT-stable, see Hacking [22, Section 10]. In particular, we get a morphism from $\overline{\mathfrak{M}}^{\mathrm{Gor}}_{2,4}$ to the GIT quotient $|\mathcal{O}_{\mathbb{P}^2}(8)|^{\mathrm{ss}}/\operatorname{PGL}(3,\mathbb{C})$ and, thus, yet another possible compactification which could be studied.

Remark 4.3. While Hacking's moduli space M_8 properly contains $\overline{\mathfrak{M}}^{\mathrm{Gor}}_{2,4}$, Hassett's space $\overline{\mathcal{P}}_8$ is too small; the plane curves it parametrises have to have log-canonical threshold at least 1.

Question 4.4. Recall that the stratum $\mathfrak{M}^{(4)} = \mathfrak{M}_{4;\emptyset}$ is isomorphic to the moduli space of nodal plane quartics. Is the closure of $\mathfrak{M}^{(4)}$ in $\overline{\mathfrak{M}}_{2,4}$ isomorphic to Hassett's compactification of the space of smooth plane quartics [26]?

4.2 Beyond the Gorenstein locus. We briefly demonstrate that the Gorenstein locus $\overline{\mathfrak{M}}^{\mathrm{Gor}}_{2,4}$ is properly contained and not closed in $\overline{\mathfrak{M}}_{2,4}$.

Example 4.5. The log-canonical surface $\mathbb{P}(1,1,4)$ has an essentially unique 1-parameter $\mathbb{Q}$-Gorenstein smoothing $\mathscr{Z} \to \mathbb{A}^1$, where $\mathscr{Z} \subset \mathbb{P}(2,2,2,4) \times \mathbb{A}^1$ is given as the vanishing locus of the polynomial $x_1^2 + ty - x_0x_2$, where t is the coordinate of $\mathbb{A}^1$ and where x_0, x_1, x_2 and y are the coordinates of $\mathbb{P}(2,2,2,4)$, cf. Hacking's exposition [23, p. 52 f]. For all $t \neq 0$, the fibre $\mathscr{Z}_t$ is isomorphic to $\mathbb{P}(2,2,2) \cong \mathbb{P}^2$ and $\mathscr{Z}_0$ is isomorphic to $\mathbb{P}(1,1,4)$. Let $Q \subset \mathbb{P}(2,2,2,4)$ be a sufficiently general hypersurface of degree 16, defining a relative Cartier divisor $\mathscr{Z} \cap Q$ missing the singular point in the special fibre. Assume furthermore that each pair $(\mathscr{Z}_t, \frac{1}{2}\mathscr{Z}_t \cap Q)$ is log-canonical, at least for all t sufficiently close to 0. For example, we may assume that each of the curves $Q_t := \mathscr{Z}_t \cap Q$ is smooth. Let $\mathscr{X} \to \mathscr{Z}$ be the double-cover branched over $Q \cap \mathscr{Z}$; this defines a $\mathbb{Q}$-Gorenstein family over $\mathbb{A}^1$ where each fibre $\mathscr{X}_t$, $t \neq 0$, is a double-cover of $\mathbb{P}^2$ branched over an octic, hence, a Gorenstein stable surface with $K^2_{\mathscr{X}_t} = 2$ and $\chi(\mathcal{O}_{\mathscr{X}_t}) = 4$, whilst the central fibre $\mathscr{X}_0$ is a double-cover of $\mathbb{P}(1,1,4)$, branched over a curve Q_0 of degree 16 missing the singular point and such that the pair $(\mathbb{P}(1,1,4), \frac{1}{2}Q_0)$ is log-canonical. Hence, $\mathscr{X}_0$ is semi-log-canonical and $K_{\mathscr{X}_0}$ is the pull-back of the $\mathbb{Q}$-Cartier divisor $K_{\mathbb{P}(1,1,4)} + \frac{1}{2}Q_0 \in |\mathcal{O}_{\mathbb{P}(1,1,4)}(2)|$, so that $\mathscr{X}_0$ is stable of Gorenstein-index 2. Since the family is $\mathbb{Q}$-Gorenstein, we conclude that $\mathscr{X}_0$ is in the closure of $\overline{\mathfrak{M}}^{\mathrm{Gor}}_{2,4}$ in $\overline{\mathfrak{M}}_{2,4}$.

Note that the pair $(\mathscr{Z}_0, Q_0)$ of the above example is an element of Hacking's moduli space $\mathcal{M}_8$ at the boundary of the image of $\overline{\mathfrak{M}}^{\mathrm{Gor}}_{2,4}$. There are, however, many elements in $\mathcal{M}_8$ which are not contained in the image of $\overline{\mathfrak{M}}^{\mathrm{Gor}}_{2,4}$, despite the fact that $\overline{\mathfrak{M}}^{\mathrm{Gor}}_{2,4}$ is dense in $\mathcal{M}_8$. There is also something to say about the boundary of (the image of) $\overline{\mathfrak{M}}^{\mathrm{Gor}}_{2,4}$ in Hacking's moduli space $\mathcal{M}_8$ of stable pairs of degree 8 (and, therefore, also in the GIT-quotient). Namely, there are curves $C \subset \mathbb{P}^2$ such that $(\mathbb{P}^2, C)$ is a stable pair of degree 8, but where $(\mathbb{P}^2, \frac{1}{2}C)$ is not log-canonical; since the image of $\overline{\mathfrak{M}}^{\mathrm{Gor}}_{2,4}$ in $\mathcal{M}_8$ is dense, they occur as limits of classes of half-log-canonical curves, though.

Example 4.6. Every octic $C \subset \mathbb{P}^2$ with global log-canonical threshold between $\frac{3}{8}$ and $\frac{1}{2}$ gives rise to a pair in Hacking's moduli space $\mathcal{M}_8$ which is not contained in the image of $\overline{\mathfrak{M}}^{\mathrm{Gor}}_{2,4}$.

One kind of example is given by general octics with a singular point of type Z_{11}, whose log-canonical threshold is $\frac{7}{15}$. (A curve singularity of type Z_{11} is analytically locally the union of an E_6-singularity and a general line passing through it. To get an explicit example, we can just take a quartic with an E_6-singularity and a general quartic passing through the E_6-singularity, resulting in an octic with one singularity of type Z_{11} and 13 ordinary double-points.) This is by far not the only type of singularity that can occur on an octic which has no (or only admissible) other singularities.

A. Half-log-canonical plane curves of small degree

In this appendix, we prove the results about plane curves of degree at most eight with $[3;3]$- and quadruple-points which were used in the earlier chapters. We obstruct the existence of certain configurations using basic intersection theory and well-known results about the Milnor number.

The classification of possible (configurations of) singularities on plane conics or cubics is easy. In the case of quartics, a complete classification is known; it can be found in Hui's thesis [28]. It turns out that the only reduced quartics with a non-simple singularity are the unions of four concurrent lines, admitting a unique ordinary quadruple-point, also called singularity of type X_9. Degtyarev [10] has classified all plane quintics up to rigid isotopy and all the possible singularities on quintics. The last case where the complete classification is known, the sextics, is mostly due to Urabe [50], Yang [54] and Degtyarev, see [11, Section 7.2.3] and the references therein.

Already the list provided by Yang [54] (even though restricted to the sextics with maximal total Milnor number 19) is so long that for certain questions, it is not easy to read off the relevant informations from the data. There are 128 irreducible maximising sextics and many more reducible ones and *"[t]he list [of the remaining reduced sextics] is too long to be printed [in an article]"* [54, Remark 4.1]. In conclusion, the classification of possible configurations of singularities on octic curves is clearly out of reach. Therefore, we study here just as much as we need to understand the strata; that is, we ignore simple singularities and concentrate only on those with log-canonical threshold exactly $\frac{1}{2}$.

For a start, we recall the notion of an n-fold-point with an infinitely near n-fold-point, an $[n;n]$-point, for short. A non-degenerate $[n;n]$-point should be pictured as n-fold-points with n local branches with a common tangent direction; however, degenerate $[n;n]$-points may have less branches.

Definition A.1 — *Let $0 \in C \subset \mathbb{C}^2$ be the germ of an isolated curve singularity, defined by a convergent power series $f \in \mathbb{C}\{x,y\}$.*

1. *The germ (or the point, slightly abusively) is said to be a* n-fold-point*, if f has multiplicity n, i.e., $f \in (x,y)^n - (x,y)^{n+1}$.*
2. *If, moreover, the degree n-part of f is the product of n pairwise distinct linear factors, then the germ is said to be an* ordinary n-fold-point*. That is, C is a union of n smooth curves meeting transversely in 0.*
3. *An n-fold-point $0 \in C \subset \mathbb{C}^2$ is said to have an* infinitely near n-fold-point *if the strict transform $C' \subset X$ of C in the blow-up $(X,E) \to (\mathbb{C}^2, 0)$ has an n-fold-point along E. In this case, the germ is called an $[n;n]$-point. It is called* non-degenerate *if the n-fold-point of C' is ordinary.*
4. *If $C \subset \mathbb{P}^2$ is a plane curve with an $[n;n]$-point $p \in C$, the point on the exceptional line $E \subset \mathrm{Bl}_p\mathbb{P}^2$ where the strict transform C' has its n-fold-point corresponds to a tangent direction at p in $\mathbb{P}^2$; we refer to this as the* distinguished tangent *of the $[n;n]$-point. The unique line in $\mathbb{P}^2$ passing through p in this direction is called the* distinguished tangent line.

Remark A.2. Note that the distinguished tangent line $\ell \subset \mathbb{P}^2$ of an $[n;n]$-point $p \in C \subset \mathbb{P}^2$ is determined by the property that the intersection multiplicity of ℓ and C at p exceeds n. In particular, if C contains a line through p, then this must be its distinguished tangent line.

We will mostly be concerned with certain $[2;2]$-points, $[3;3]$-points and 4-fold-points, also known as *quadruple-points*. We give a quick overview:

A $[2;2]$-point is a double-point whose strict transform in the blow-up has a double-point along the exceptional line. That is, the $[2;2]$-points are the singularities of type A_n as $n \geq 3$, where A_3 is the non-degenerate $[2;2]$-point.

The half-log-canonical $[3;3]$-points are the singularities of type J_{10} (the non-degenerate $[3;3]$-point) and $J_{2,p}$ for $p \geq 1$. Blowing up once, the strict transform of a J_{10} has a non-degenerate triple-point (a D_4) along the exceptional line and the strict transform of a $J_{2,p}$ has a D_{4+p} along the exceptional line. Moreover, the branches are transversal to the exceptional line, for otherwise it would be a quadruple-point (or worse). In particular, no component of a $[3;3]$-point is an ordinary cusp A_2.

Likewise, the ordinary quadruple-points are the singularities of type X_9, whereas the half-log-canonical degenerate quadruple-points split up into the two families X_p, $p \geq 10$ and $Y_{r,s}$ for $r, s \geq 1$. The singularities of type X_p are, locally analytically, the union of a degenerate double-point of type A_{p-8} and a non-degenerate double-point and those of type $Y_{r,s}$ are unions of two degenerate double-points of type A_{r+1} and A_{s+1}. In particular, an X_p, $p \geq 10$, has a single special tangent direction (the distinguished tangent direction of the underlying degenerate double-point) and a $Y_{r,s}$ has two such.

Lemma A.3 — *Let C be a plane curve of degree d. Then the following hold:*

a) If C has an n-fold-point, then $d \geq n$ and $d = n$ if and only if C is a union of n concurrent lines, the intersection-point being the n-fold-point.

b) If C has an m-fold-point and an n-fold-point, then $d \geq m + n - 1$ and if $d = m + n - 1$, then C contains the line joining those two points.

c) More generally, if C has s collinear singular points of multiplicity n_i, $i = 1, \ldots, s$, then $d \geq 1 - s + \sum_{i=1}^{s} n_i$ and if $d = 1 - s + \sum_{i=1}^{s} n_i$, then the line joining them is contained in C.

d) If C has an $[n;n]$-point, then $d \geq 2n - 1$ and if $d = 2n - 1$, then C contains the distinguished tangent line.

e) If C has an $[n;n]$-point and an m-fold point on the distinguished tangent line of the $[n;n]$-point, then $d \geq 2n + m$, unless C contains the distinguished tangent, in which case $d \geq 2n + m - 2$.

f) If C has an $[m;m]$- and an $[n;n]$-point with a common distinguished tangent line, then $d \geq 2n + 2m - 3$ and if $d < 2n + 2m$, then C contains the distinguished tangent line.

Proof. The proofs of those statements are very similar; by way of example, we only prove a few of them. Note that since we assumed n-fold-points and $[n;n]$-points to be isolated singularities, if C contains a line L through such a point, is does so with multiplicity 1. In particular, the residual curve $C - L$ (in divisor-notation) has degree $d - 1$ and does not contain L.

c) If C is as claimed, then the line L joining the s singular points witnesses $d = CL \geq \sum_{i=1}^{s} n_i$, unless $L \subset C$, in which case the same argument applied to the residual curve $C' = C - L$ yields $d - 1 = C'L \geq \sum_{i=1}^{s}(n_i - 1)$, hence the claim.

d) If C has an $[n; n]$-point, then the distinguished tangent line L witnesses that $d = CL \geq 2n$, unless C contains L. In this case, the residual curve $C' = C - L$ has an $[n-1; n-1]$-point with distinguished tangent direction L; thus, we get $d - 1 = C'L \geq 2(n-1)$, hence, $d \geq 2n - 1$, as claimed.

□

The following is an immediate corollary.

Corollary A.4 — *Let C be a plane octic curve. Then the following holds:*
a) Any two $[3, 3]$-points on C have distinct distinguished tangent lines.
b) No three $[3; 3]$-points on C are collinear.

This implies that two $[3; 3]$-points on a conic are in general position such that there is exactly a pencil of conics joining both points, passing through them in distinguished tangent direction. Three $[3; 3]$-points on a conic can be in special position in the sense that the tangents may align along a conic. It turns out that there are at most three $[3; 3]$-points on a plane octic, but before we can prove this, we have to recall a few basic facts from singularity theory. We refer to Milnor's seminal book [41] or Wall [53, Chapter 6] for the local theory.

Recall that with a holomorphic function germ $f \in \mathbb{C}\{x, y\}$ we can associate the *Milnor number* $\mu(f) = \dim_{\mathbb{C}}(\mathbb{C}\{x, y\}/J_f)$ where $J_f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ is the Jacobian ideal of f generated by the partial derivatives. Let $C \subset \mathbb{P}^2$ be a plane curve passing through a point $p \in C$ and choose local holomorphic coordinates x, y at $p \in \mathbb{P}^2$. Then the curve C is the vanishing locus of a function germ $f \in \mathbb{C}\{x, y\}$ and it makes sense to define the *Milnor number of C at p*, $\mu_p(C) := \mu(f)$. If C is reduced, we define its *total Milnor number* $\mu(C) = \sum_{p \in C_{\text{sing}}} \mu_p(C)$.

We recall from Wall's exposition [53, Sections 7.1 & 7.5]:

Lemma A.5 — *Let $C \subset \mathbb{P}^2$ be a reduced plane curve of degree d.*
a) The total Milnor number of C is bounded by $\mu(C) \leq (d-1)^2$ and the maximum $\mu(C) = (d-1)^2$ is attained only by the union of d concurrent lines with its $(d-1)$-fold-point.
b) Let C^ν be the normalisation of C. Then

$$\chi_{\text{top}}(C^\nu) = (3-d)d + \sum_{p \in C_{\text{sing}}} (\mu_p(C) + r_p(C) - 1),$$

where $r_p(C)$ is the number of analytically local branches of C through p.
In particular, if C is a reduced octic with four $[3; 3]$- or quadruple-points, then C has at least four rational components, $\chi_{\text{top}}(C^\nu) \geq 8$, and if C has exactly four components, then there are no additional singular points.

Proof. For a) and b) see Wall [53, Section 7.5, p. 177 & Corollary 7.1.3]. The *in particular*-part can be derived from b) as follows: It suffices to show that for a $[3; 3]$- or quadruple-point $p \in C$, we have $\mu_p(C) + r_p(C) - 1 \geq 12$, since then, if there are at least four such, $\chi_{\text{top}}(C^\nu) \geq 8$ and equality implies that there are no more singular points and that $\mu_p(C) + r_p(C) - 1 = 12$ for all four $p \in C_{\text{sing}}$. As the normalisation decomposes into disjoint components $C^\nu = \coprod_{i=1}^s C_i^\nu$, we get $\sum_{i=1}^s 2 - 2g(C_i^\nu) = \chi_{\text{top}}(C^\nu) \geq 8$, hence $g(C_i^\nu) = 0$ at least four times, which yields four rational components. Therefore, if C has exactly four components, then all of them are rational and $\chi_{\text{top}}(C^\nu) = 8$.

To complete the proof, we have to show that $[3;3]$- and quadruple-points $p \in C$ indeed satisfy $\mu_p(C) + r_p(C) - 1 \geq 12$. For the half-log-canonical singularities this follows from the classification (Proposition 1.15) since $\mu(X_p) = p$ as $p \geq 9$, $\mu(J_{10}) = 10$, $\mu(Y_{r,s}) = 9 + r + s$ as $r, s \geq 1$ and $\mu(J_{2,p}) = 10 + p$ for $p \geq 1$; cf. Arnold, Gusein-Zade, Varchenko [5, Chapter 15, p. 246 ff]. The more general case follows from Wall [53, Theorem 6.5.9] using the concept of infinitely near points; we omit the details. □

Using this, we can prove that an octic has at most three $[3;3]$-points:

Proposition A.6 —
- *a) If a plane curve has a* $[3;3]$*-point, then its degree is at least* 5.
- *b) If a plane curve has three* $[3;3]$*-points, then its degree is at least* 8.
- *c) If a plane octic curve has three* $[3;3]$*-points with distinguished tangents along a conic, then the octic contains the conic.*
- *d) There does not exist a plane octic curve with four* $[3;3]$*-points.*

Proof. Statement a) follows immediately from Lemma A.3 d). To prove b), let C be a plane curve of degree d with three $[3;3]$-points p_i, $i = 1, 2, 3$. By Corollary A.4, they are in general position insofar as that there exists a smooth conic $D \subset \mathbb{P}^2$ through p_1, p_2 and p_3 which passes through p_1 and p_2 in distinguished tangent direction. If D is not contained in C, then $2d = DC \geq 2 \cdot 6 + 3$, hence, $d \geq 8$. If D is contained in C, then the residual curve $C' = C - D$ of degree $d - 2$ has three $[2;2]$-points along D; hence, $2(d-2) = DC' \geq 12$, which yields $d \geq 8$. We analogously conclude part c) since if D also passes through p_3 in distinguished tangent direction but is not contained in C, then $2d \geq 3 \cdot 6 = 18$, hence $d \geq 9$.

To prove d), we will derive a contradiction from the assumption that there exists such a curve C. It follows from part b) above that such a C has to be reduced: since all four $[3;3]$-points are, by assumption, isolated singularities, they have to lie on the reduced part, which has to have degree at least 7 then, which is impossible unless C is reduced.

Lemma A.5 implies that C has at least four rational components and that if C has only 4 components, it has no more singularities than the four $[3;3]$-points. To prove the claim, we have to rule out all possible cases. We distinguish the cases according to the number of lines in C.

If C would not contain any line, it had to be a union of four smooth conics. Since a sextic has at most two $[3;3]$-points by part b), we had to have three of the four $[3;3]$-points on each of the four conics. Thus, if there were such a conic C, it had to be given as follows: Suppose $p_i \in C$, $i = 1, \ldots, 4$, are pairwise distinct $[3;3]$-points. By Corollary A.4, those points and their distinguished tangents are sufficiently general such that there are four conics C_i, $i = 0, \ldots, 3$, uniquely determined by the following properties:

- C_0 contains p_1, p_2 and p_3, passing through p_1 and p_2 in distinguished tangent direction.
- C_1 contains p_2, p_3 and p_4, passing through p_2 and p_3 in distinguished tangent direction.
- C_2 contains p_1, p_3 and p_4, passing through p_1 and p_3 in distinguished tangent direction.
- C_3 contains p_1, p_2 and p_4, passing through p_1 and p_2 in distinguished tangent direction.

Then C is the union of those four. In particular, C is uniquely determined by the four points and two of the distinguished tangents. This is the key to the proof that no such C can exist. With the help of projective automorphism, we can fix three points p_i, $i = 1, 2, 3$, and two tangent directions in p_1 and p_2, as long as they do not point towards any of the remaining two points. From this, we can compute C_0 and derive the distinguished tangent at p_3. Then we compute C_1, C_2 and C_3 in dependence of a variable fourth point $p_4 \in \mathbb{P}^2$ and consider their tangent lines at p_4. If an octic C as desired existed, then for at least one point p_4, all three tangent lines would agree. A computation shows that this happens if and only if p_4 lies on C_0, but then $C_0 = C_1 = C_2 = C_3$, which is an irrelevant degenerate case. An explicit calculation in Macaulay2 can be found in Listing 12.

Now suppose that C contains exactly one line $L \subset C$. Then the only possibility is that $C = E + D_1 + D_2 + L$, where E is an irreducible cubic and D_1, D_2 are irreducible conics. Since C has only four components, all four of them are rational and C has no extra singularities. In particular, E must be rational, hence either nodal or with an ordinary cusp. But both are impossible since neither nodes nor ordinary cusps can contribute to $[3; 3]$-points. Thus, C cannot contain just one line.

It remains to consider the possibility that C decomposes into the union of two lines and a possibly reducible sextic $D \subset C$. By Lemma A.5 a), $\mu(D) \leq 25$. From this or Proposition A.6 b) we conclude that D can have at most two $[3; 3]$-points. On the other hand, by Corollary A.4 a), two lines on an octic can give rise to branches of at most two $[3; 3]$-points of C, so that the residual sextic D had to have at least two $[3; 3]$-points. Thus, D had to have exactly two such and along the lines it had to have two $[2; 2]$-points, which have Milnor number at least 3, so that D had to have total Milnor number $\mu(D) \geq 26 > 25$, yet again a contradiction.

This completes the proof. □

Proposition A.7 — *If $C \subset \mathbb{P}^2$ is a plane octic, then the following holds:*

a) No three quadruple-points on C are collinear.

b) If C has four quadruple-points, then C is a union of four (possibly reducible) conics, meeting precisely in the (necessarily ordinary) quadruple-points.

Proof. Statement a) is a special case of Lemma A.3 c).

To prove b), let C be a plane octic with four quadruple-points. By part a), the quadruple-points are in general position so that there is a pencil of conics through those four points which spans the tangent spaces. First, observe that if a conic D of this pencil is tangent to a local analytic branch of one of the quadruple-points, then D has to be contained in C, for otherwise we had to have $16 = DC \geq 3 \cdot 4 + 5 = 17$. Thus, to conclude that C is a union of four members of the pencil, it suffices to show that at least one of the quadruple-points is non-degenerate. In fact, they all are: If one of them were degenerate, then there would exist a conic D of the pencil passing through it in the corresponding tangent direction, so that $D \subset C$ as above and $D(C - D) \geq 3 \cdot 3 + 4 = 13 > 12$, unless $2D \subset C$, which is excluded since the quadruple-points are assumed to be isolated singularities. □

Lemma A.8 — *Let C be a plane octic curve. Suppose that C has a $[3; 3]$-point and a quadruple-point along the distinguished tangent line L of the $[3; 3]$-point. Then C contains L and the residual septic $C - L$ meets L only in those two points. In particular, there is no second quadruple-point along L.*

Proof. Let p_1 and p_2 be the $[3;3]$- and quadruple-point in question. Then C contains L since $CL = 8 \leq 6{+}4 \leq \mathrm{I}_{p_1}(C,L){+}\mathrm{I}_{p_2}(C,L)$, where $I_p(-,-)$ denotes the intersection multiplicity at p. Since C is reduced at p_1, L is not contained in the residual septic $D = C - L$ and so $7 = DL \geq \mathrm{I}_{p_1}(D,L) + \mathrm{I}_{p_2}(D,L) \geq 4+3 = 7$; thus, D and L meet only in p_0 and p_1 and $\mathrm{I}_{p_1}(D,L) = 4$, $\mathrm{I}_{p_2}(D,L) = 3$. □

Remark A.9. The proof of Lemma A.8 furthermore shows that if an octic has a quadruple-point on the distinguished tangent line of a $[3;3]$-point, then it is not a special tangent line of the quadruple-point as well. Nonetheless, the quadruple-point could be degenerate.

Proposition A.10 — *There exists no plane octic curve admitting two* $[3;3]$*-points and two quadruple-points, or with one* $[3;3]$*-point and three quadruple-points.*

Proof. By Proposition A.4 a), b) and Lemma A.8, in both cases, the four points in question are general enough so that there exists a pencil of conics through all four points in question. In particular, there exists a conic through all four points, passing through one of the $[3;3]$-points in distinguished tangent direction. In both cases, this easily gives a contradiction comparing intersection numbers and local intersection multiplicities as before. We omit the details. □

Proposition A.11 — *Let* $C \subset \mathbb{P}^2$ *be a plane octic with three* $[3;3]$*-points and a quadruple-point. Then* C *decomposes into four rational components; more precisely:*

Let $p_1, p_2, p_3 \in C$ *be the* $[3;3]$*-points with distinguished tangent lines* L_1, L_2, L_3*, respectively. Then only the following two configurations are possible.*

i) The three distinguished tangent lines L_1, L_2, L_3 *meet in a point* $p_4 \in \bigcap_{i=1}^{3} L_i$ *and* $C = D_5 + L_1 + L_2 + L_3$*, where* D_5 *is a rational quintic which passes through* p_4 *and has three* A_3*-singularities (non-degenerate tac-nodes) at* p_1, p_2, p_3 *with distinguished tangent lines* L_1, L_2, L_3*, respectively.*

ii) There exists a conic D_2 *passing through the three* $[3;3]$*-points in distinguished tangent direction and* $C = D_4 + D_2 + L_1 + L_2$*, where* D_4 *is a rational quartic with an ordinary double-point in the intersection* $p_4 \in L_1 \cap L_2$*. Furthermore,* D_4 *is tangent to* D_2 *in* p_1 *and* p_2 *and has an* A_3*-singularity at* p_3 *with distinguished tangent line* L_3*.*

In either case, the $[3;3]$*-points at* p_1, p_2, p_3 *and the quadruple-point at* p_4 *are non-degenerate and* C *has no further singularities.*

Proof. Let C be a plane octic with three $[3;3]$-points and one quadruple-point. Then, as in the proof of Proposition A.6 above, we conclude that C is reduced. By Lemma A.5 b), C has at least four rational components and if C has exactly four, then there are no extra singularities.

Let $C = C_0 + C_1 + C_2 + C_3 + C_4$ with $d_i := \deg(C_i)$, $d_1 \geq d_2 \geq d_3 \geq d_4$, and $C_1, \ldots, C_4$ rational. (Note that C_0 could be empty or reducible).

Clearly, $d_4 \leq 2$ and if $d_4 = 2$, then $d_0 = 0$ and $d_1 = d_2 = d_3 = 2$ as well. That is, C would be a union of four smooth conics. Going through the list of possible intersections of pairs of conics shows that there is no way for their union to have a quadruple- and three $[3;3]$-points. Hence, we conclude $d_4 = 1$, i.e., C contains at least one line.

Before we continue, note that if C has only four components and contains a cubic, then the cubic is rational and so has a node or ordinary cusp. Either singularity does not contribute to a $[3;3]$-point and so has to be part of the quadruple-point, since C has no extra singularities.

Now assume that C has no more than the four singularities and contains a line. Then the line must meet the residual septic in the singular points of C. For the intersection multiplicities to add up to 7, this has to be the quadruple- and one of the $[3;3]$-points. This shows that $d_\bullet = (0,3,3,1,1)$ is impossible since the quadruple-point of C had to be the union of the double-points of the two rational cubics, not allowing either line to pass through the quadruple-point as well. Similarly, this excludes $d_\bullet = (0,3,2,2,1)$, for, the cubic C_1 had to have a double-point at, say, $p \in C_1$ and the line C_4 would have to pass through it. But in order not to introduce an extra singularity of C, they must not meet anywhere else, so that the intersection at p had to be with multiplicity 3, which implies $\mu_p(C)+r_p(C)-1 \geq 13$, a contradiction to (the proof of) Lemma A.5. The only possibilities with $d_0 = 0$ which remain are those corresponding to the claim and since every line has to pass through one of the $[3;3]$-points and the quadruple-point, the configurations have to be as claimed.

It remains to show that C cannot have five or more components. Note that in this case, C had to contain at least two lines.

We first exclude the case that C contains at least four lines, i.e., $d_\bullet = (4,1,1,1,1)$ or $(1,4,1,1,1)$. Suppose C were the union of four lines and a possibly reducible quartic. Then the quartic had to have at least three non-collinear $[2;2]$-points, which is impossible, e.g., by Hui's classification [28].

If C had at least five components but at most three lines, then C had to contain at least two lines, L_1, L_2, and a smooth conic D. The residual quartic then had to have exactly two components, of which at least one had to be rational, hence, either $C = L_1 + L_2 + L_3 + D + D'$ for a third line L_3 and an irreducible cubic D', or $C = L_1 + L_2 + D + D' + D''$ for two more irreducible conics D', D''. Thus, we are left with these two cases.

If we had $C = L_1+L_2+L_3+D+D'$ with D' an irreducible cubic, then $\chi_{\text{top}}(C^\nu) = 8$ or 10, so that C could have at most one additional singularity, necessarily of type A_1 or A_2. Therefore, the lines had to be concurrent, for otherwise C had to have at least three singularities with local branches having different tangent directions. In particular, the conic D had to meet at least one of the lines transversely. But in that case, one of the intersection points had to be the quadruple-point of C; hence, D would pass through the common intersection point of the three lines. Therefore, D could be tangent to at most one of them, which would result in too many singularities for C, a contradiction.

Finally, if $C = L_1 + L_2 + D + D' + D''$ for two more irreducible conics D', D'', then $\chi_{\text{top}}(C^\nu) = 10$, so as before, C could have at most one additional singularity, of type A_1 (since lines and conics have no cusps). Therefore, the intersection point of L_1 and L_2 had to be an extra double-point or the quadruple-point of C. If it were an extra double-point, then the residual sextic $D + D' + D''$ had to have two $[2;2]$-points, a $[3;3]$-point and a quadruple-point, resulting in a total Milnor number of at least 25. But the only sextic with this (maximal) Milnor number is a union of six concurrent lines by Lemma A.5 a), so this is impossible. Hence, the two lines had to meet in the quadruple-point of C, so that two of the conics would necessarily pass through this point as well, say D and D'. But then they could not be tangent to the lines in the $[3;3]$-points, as would be necessary, a contradiction.

This rules out all cases as claimed and completes the proof. □

We also have to study the possible configurations of non-simple singularities a reduced sextic can have. Note that the upper bounds we give below are most likely far from optimal.

Proposition A.12 — *Let C be a reduced half-log-canonical plane sextic curve (cf. Table 1). Then the only possible non-simple singularities C can have are:*

i) One X_p, $9 \leq p \leq 24$.
ii) One $Y_{r,s}$, $r, s \geq 1$, $r+s \leq 15$.
iii) One J_{10}.
iv) One $J_{2,p}$, $1 \leq p \leq 14$.
v) Two J_{10}.

In the last case, C decomposes as a union of three conics, at most one of which degenerates into a union of two lines.

Proof. The maximal total Milnor number $\mu(C)$ of a reduced sextic $C \subset \mathbb{P}^2$ without a 6-fold-point is 24. Since all reduced non-simple plane curve singularities with log-canonical threshold at least $\frac{1}{2}$ have Milnor number greater or equal than 9, we conclude that C can have at most two such. Moreover, this explains the given upper bounds since the Milnor number of a singularity of type X_p, $Y_{r,s}$ and $J_{2,p}$ is p, $9+r+s$ and $10+p$, respectively.

If C has a quadruple-point, then C cannot have a second non-simple singularity: By Lemma A.3 b), it must contain the line L joining them. But then the residual quintic $C' = C - L$ has to have either two triple-points, or a triple-point and a $[2;2]$-point with L as distinguished; both options are impossible by Lemma A.3 b) and e), respectively.

Thus, it remains to show that if C has two $[3;3]$-points, then they are both non-degenerate and that C is a union of three conics then, meeting tangentially in both points. In fact, as in the proof of Proposition A.7, we can conclude that if C has two $[3;3]$-points, then it has at least three rational components and that if it has exactly three, then there are no further singular points. Then either C is the union of three rational conics, or C contains a line. The only possibility for C to contain exactly one line is that $C = C_1 + C_2 + C_3$, with C_1 a line, C_2 a rational conic and C_3 a rational cubic. But a rational cubic has an A_1 or A_2-singularity, which does not contribute to a $[3;3]$-point, so that C had to have an extra singular point, a contradiction. Thus, C contains a conic D (which may be the union of two lines). Since the residual quartic $C' = C - D$ cannot have a $[3;3]$-point, the conic D must pass through both $[3;3]$-points in distinguished tangent direction and C' has to have two $[2;2]$-points where C has its $[3;3]$-points. But then C' decomposes as a union of two conics meeting tangentially in those two points. Since the intersection number of any pair of these three conics has to be $4 = 2+2$, the $[3;3]$-points have to be non-degenerate. □

The quartics are easy to deal with: There is only one quartic with a non-simple singularity, namely, the union of four concurrent lines, giving rise to a single X_9-singularity, see, e.g., Hui [28]. Alternatively, this also follows from Lemma A.5 since the Milnor number of a non-simple singularity is at least $9 = (4-1)^2$.

Since this is used in the Macaulay2-code Listing 13 II.2, note that from Hui's classification we also know that there are quartics with globally two different kinds of A_5-singularities, namely, some where the higher order directions are non-trivial, and some where they are not.

B. The Macaulay2-code

We conclude this thesis with a quick tour through the arguments used in the Macaulay2-code which computes the dimension of the various strata and shows that the components are disjoint. All scripts are included below a quick introduction to the line of arguments; moreover, they can be obtained from a GitLab repository [4]. They are inspired by a similar script by Sönke Rollenske.

If the singularities we want a plane curve to have are controlled by a configuration which can be fixed by a suitable automorphism, then the dimension of this component is easy to compute using Macaulay2. One puts all constraints in an ideal and asks the system for a minimal generating set of the module of octics satisfying these equations. An example illustrating this is given in Listing 1.

Listing 1: Example without parameters

```
S = QQ[x,y,z]; -- Homog. coordinate ring of PP^2
Point = ideal(x,y); -- Homog. ideal of (0;0;1) in PP^2
QuadruplePoint = Point^4;
m = super basis(8,QuadruplePoint) -- outputs:
--  | x8 x7y x7z x6y2 x6yz x6z2 x5y3 x5y2z x5yz2 x5z3 x4y4 x4y3z
--   ----------------------------------------------------------------
--   x4y2z2 x4yz3 x4z4 x3y5 x3y4z x3y3z2 x3y2z3 x3yz4 x2y6 x2y5z
--   ----------------------------------------------------------------
--   x2y4z2 x2y3z3 x2y2z4 xy7 xy6z xy5z2 xy4z3 xy3z4 y8 y7z y6z2 y5z3
--   ----------------------------------------------------------------
--   y4z4 |
assert(numgens source m == 35);
```

It shows that the sub-space of the vector space of octic forms in x, y, z whose associated plane curve has multiplicity at least four in $(0;0;1) \in \mathbb{P}^2$ is of dimension 35. Thus, their linear system is of dimension 34 and there is an open sub-space V of octics where the quadruple-point is non-degenerate and which has no further non-simple singularities. Every plane octic curve with a quadruple-point is projectively equivalent to one of those with a quadruple-point in $(0;0;1)$. Therefore, the space of octic curves with exactly one quadruple-point and no other non-simple singularities is the quotient V/G, where $G \subset \mathrm{PGL}(3,\mathbb{C}) = \mathrm{Aut}(\mathbb{P}^2)$ is the stabiliser of the point $(0;0;1)$. Since the dimension of G is 6, we conclude $\dim(\mathfrak{N}_2) = 34 - 6 = 28$.

When the configuration cannot be fixed by an automorphism, then we have to consider parameters. As an example Listing 2, we consider the case of a $[3;3]$-point and a quadruple-point. Up to projective automorphism, there are two distinct configurations; one where the distinguished tangent of the $[3;3]$-point points towards the quadruple-point and one where it does not.

Listing 2: Example with a parameter

```
A = QQ[t]; -- Affine coordinate ring of parameter space
S = A[x,y,z]; -- Homog. coordinate ring of trivial PP^2-family
P = ideal(x,y); -- Homog. ideal of (0;0;1) in PP^2
PwT = ideal(x^2,y-t*x); -- (0;0;1) with tangent direction y-tx
P33 = PwT^3; -- Corresponding [3;3]-point constraints
Q = ideal(y,z); -- Homog ideal of (1;0;0)
I0 = intersect(Q^4,sub(P33,{t=>0}));
I1 = intersect(Q^4,sub(P33,{t=>1}));
m0 = super basis(8,I0);
m1 = super basis(8,I1);
assert(numgens source m0 > numgens source m1);
```

```
-- Thus, something special is going on if t = 0. In fact, that is where y-t*x
     lies in Q. Another component?
-- We consider the universal octic with a [3;3]-point as prescribed:
m = super basis(8,P33);
n = numgens source m;
RA = A[a_0..a_(n-1)];
params = gens RA;
RS = RA[gens S];
inc = map(RS,S);
f = sum for i from 0 to (n-1) list a_i*inc(m_(0,i));
-- The conditions that it has a quadruple-point at Q:
toBeZero = f%inc(Q^4);
toBeZeroCoefficients =
for term in terms toBeZero list leadCoefficient(term);
M = matrix for eq in toBeZeroCoefficients list
for g in gens RA list sub(leadCoefficient(eq//g),A);
-- For every t = t_0, the kernel of sub(M,{t=>t_0}) corresponds to the space
    of octic forms with a quadruple-point in Q and a [3;3]-point in P with
    distinguished tangent direction y-t_0*x. Thus, the rank of M drops where
     something interesting is happening:
droppingRankConditions = minors(numgens target M, mingens image M);
assert(droppingRankConditions == ideal(t));
-- Thus, generically, the rank of M is maximal (10, in fact) and it drops if
    and only if t = 0. Furthermore, the difference between the octic forms
    obtained for t = 0 and those arising as limits t --> 0, t != 0,
    corresponds to the difference between the kernels.
Kspecial = mingens kernel sub(M,{t=>0});
Kgeneral = mingens sub(kernel M, {t=>0});
assert isSubset(image Kgeneral, image Kspecial);
assert(image Kspecial != image Kgeneral);
-- They give rise to the following octics:
special = sub(matrix{ for j from 0 to numgens source Kspecial-1 list
sub(f,for i from 0 to numgens target Kspecial-1 list
params_i=>Kspecial_(i,j))},{t=>0});
special = sub(special, S);
general = sub(matrix{ for j from 0 to numgens source Kgeneral-1 list
sub(f,for i from 0 to numgens target Kgeneral-1 list
params_i=>Kgeneral_(i,j))},{t=>0});
general = sub(general, S);
-- The line y is contained once in every member of the special locus, but it
    is contained twice in every member of the general locus:
use S;
assert( special%y == 0 and not special%y^2 == 0 );
assert( general%y^2 == 0 );
-- Since non-reduced octics are not allowed in this stratum, the components
    are disjoint.
```

Since jobs like creating the ideals containing the constraints or building a universal family etc. have to be done multiple times, they are provided as functions in the file *octicsFunctions.m2*; it will be included at the very end, see Listing 14. Explanations how they work can be found in the comments there. For example, there is also a function checking whether a quadruple-point is ordinary. (It blows up once and checks that the discriminant is non-trivial. Therefore, if applied over a coefficient ring which is not a field, it only means that it is generically non-degenerate.) Similarly, there is a function checking if a $[3;3]$-point is non-degenerate. Listing 3 is an example-application, proving that there exist admissible octics with three $[3;3]$-points with distinguished tangents in general directions not along a conic, i.e., an inhabitant of $\mathfrak{N}'_{13}$. Note that we know no easy geometric description of such an octic.

Listing 3: The stratum $\mathfrak{N}'_{13}$ is inhabited

```
load "octicsFunctions.m2";
S = QQ[x,y,z]; -- Homog. coordinate ring of PP^2
L = {x,y,z}; -- Lines
T = {x-y,y-z,x-z}; -- Tangent lines
F = {z,x,y}; -- Aux. localisation variables
assert all(#L,i-> ideal(L_i,T_i,F_i) == ideal gens S);
P = apply(L,T,pair->trim ideal pair);
PwT = apply(L,T,(l,t)->trim ideal(l^2,t));
P33 = apply(#L,i->get33Ideal(L_i,T_i,F_i));
-- P_0 = (0;0;1), P_1 = (1;0;0), P_2 = (0;1;0).
-- They are not(!) with tangents along a conic:
assert(basis(2,intersect PwT) == 0);
m = super basis(8,intersect P33);
-- f = randomValue(m); -- resulted in:
f =  -(7/20)*x^5*y^3 -(7/3)*x^4*y^4 +3*x^3*y^5 +(21/20)*x^5*y^2*z +(18/5)*x
    ^4*y^3*z +(5/3)*x^3*y^4*z -9*x^2*y^5*z -(21/20)*x^5*y*z^2 +10*x^4*y^2*z
    ^2 +7*x^3*y^3*z^2 -(27/4)*x^2*y^4*z^2 +9*x*y^5*z^2 +(7/20)*x^5*z^3
    -(322/15)*x^4*y*z^3 +(1/18)*x^3*y^2*z^3 +(1121/180)*x^2*y^3*z^3 +(107/6)
    *x*y^4*z^3 -3*y^5*z^3 +(51/5)*x^4*z^4 -(2083/60)*x^3*y*z^4 +(341/12)*x
    ^2*y^2*z^4 +(391/60)*x*y^3*z^4 -(125/12)*y^4*z^4 +(4139/180)*x^3*z^5
    -(4139/60)*x^2*y*z^5 +(4139/60)*x*y^2*z^5 -(4139/180)*y^3*z^5;
sing = radical ideal singularLocus ideal f;
-- The only singularities are at P_0, P_1 and P_2:
assert(sing == intersect P);
-- And they are at least of the desired type:
assert all(P33, i-> f%i == 0);
-- But they are also non-degenerate:
assert isOrdinary33Point(f,T_0,x,y,z);
assert isOrdinary33Point(f,T_1,z,y,x);
assert isOrdinary33Point(f,T_2,z,x,y);
-- As claimed.
```

Finally, we include the actual Macaulay2-scripts, followed by the file providing the functions.

Listing 4: Computing the dimension of most strata without parameters

```
----------------------------------------------------------------------
-- M2-script studying (most of) the strata of the stratification which can be
     handled without parameters.
-- The strata are: N_2, N_{22}, N_{222}, N_{2^4}, N_1 and N_{11}. For later
    reference we also compute the dimensions of the components of the strata
     N_{12}, N_{122}, N_{112}, despite the fact that the proof that these
    components are actually distinct will be handled in other files.
--
-- File name: ParameterFreeCases.m2
----------------------------------------------------------------------
--
clearAll();
load "octicsFunctions.m2";
--
-- I.  Ordinary quadruple points only
--
-- Notation:
-- S = QQ[x,y,z] is the homogeneous coordinate ring of the projective plane (
    over the rationals). We consider the following plane points: P_0 =
    (0;0;1), P_1 = (1;0;0), P_2 = (0;1;0), P_3 = (1;1;1) and P_4 = (1;1;0).
S = QQ[x,y,z];
P = { idealFromCoords(0,0,1), idealFromCoords(1,0,0),
        idealFromCoords(0,1,0), idealFromCoords(1,1,1),
        idealFromCoords(1,1,0) };
```

```
-- Ideals of quadruple-points:
P4 = apply(P,i->i^4);
--
-- In each case, it is easy to see that the strata are inhabited, so we omit
    the constructions od explicit constructions.
--
-- I.1  One quadruple-point (N_2) --- at P_0:
assert(35 == rank source super basis(8,P4_0));
-- Since we fixed a point, the stabiliser has dimension 6; thus, the stratum
    is of dimension 35 - 6 - 1 = 28.
--
-- I.2  Two quadruple-points (N_{22}) --- at P_0 and P_1:
assert(25 == rank source super basis(8,intersect(P4_{0,1})));
-- Since we fixed two points, the dimension of the stabiliser is 4 and the
    dimension of stratum is 25 - 4 - 1 = 20.
--
-- I.3  Three non-collinear quadruple-points (N_{222}) at P_0, P_1 and P_2:
assert(15 == rank source basis(8,intersect(P4_{0,1,2})));
-- The stabiliser of three non-collinear points is two-dimensional; hence,
    the dimension of the stratum is 15 - 2 - 1 = 12.
--
-- I.4  Four quadruple-points (N_{2^4}) in general position, i.e., no three
    collinear --- at P_0,P_1,P_2 and P_3:
assert(5 == rank source super basis(8,intersect(P4_{0..3})));
-- There are only finitely many automorphisms fixing four points of which no
    three are collinear and so the stratum is of dimension 5 - 1 = 4.
--
-- Since on a reduced octic there are no three quadruple-points on a line,
    this covers all cases we have to consider.
--
-- II.   Non-degenerate [3;3]-Points only
--
-- Similar notation as before, but now we need to take of the distinguished
    tangent directions.
L = {y,z};
T = {x-y,y-z};
F = {z,x}; -- (aux. localisation variables)
Q = apply(2,i->trim ideal(L_i,T_i));
--
-- We will consider octics with [3;3]-points in Q_i = (L_i,T_i), i=0,1 with
    distinguished tangent line T_i, i=0,1 respectively. Explicitly, Q_i =
    P_i, i=0,1, with tangent lines defined by the linear forms T_0 = x - y
    and T_1 = y - z, respectively.
assert(Q == P_{0,1});
-- Ideals for [3;3]-points with prescribed tangent directions:
Q33 = apply(2,i->get33Ideal(L_i,T_i,F_i));
--
-- II.1  One [3;3]-point (N_{1}) --- at Q_0 = P_0
assert(33 == rank source super basis(8,Q33_0));
-- The stabiliser of a line and a point on the line is 5; therefore, the
    stratum has dimension 33 - 5 - 1 = 27.
--
-- II.2  Two [3;3]-points (N_{11}) in general position, i.e., none of the
    points lies on the other's distinguished tangent line:
assert(21 == rank source super basis(8,intersect(Q33)));
-- An automorphism preserving this configuration preserves exactly the line
    joining the Q_i and the two distinguished tangent lines; dually, this
    amounts to fixing three points. Thus, the stabiliser is of dimension 2;
    thus, as expected, the dimension of the stratum equals 21 - 2 - 1 = 18.
--
-- Since on a reduced octic the distinguished tangent of a [3;3]-point does
    not point towards another [3;3]-point, this is the only case we have to
```

```
    consider.
--
-- II.3  More [3;3]-points
--         ... need parameters!
-- The cases of three or four [3;3]-points will be considered in separate
    files. Concretely, 111.m2 deals with the case of three [3;3]-points and
    shows that there are two disjoint components, both of dimension 9. One
    corresponds to the case where the tangents are in general direction and
    the second is when they all lie on a conic. The file 1111.m2 helps
    proving that there is no plane octic with more than three [3;3]-points.
    The degenerate cases will be content of the files 1[bar]1[bar]1bar.m2
    and upToTwoDegCases.m2.
--
-- III.   Mixed (non-degenerate) cases.
--
-- It will turn out that parameters are needed to understand the components
    of these strata, but we include a first approximation for sake of
    exposition.
--
-- Same notation as above. Note that (only) P_0, P_3 and P_4 are on the line
    T_0 and on the line T_1 there lie P_1 and P_3 (only):
assert all({0,3,4}, i->(T_0%P_i) == 0); -- => P_0,P_3,P_4 on T_0
assert all({1,3}, i->(T_1%P_i) == 0); -- => P_1,P_2 on T_1
assert all({1,2}, i->(T_0%P_i) != 0); -- => P_1,P_2 not on T_0
assert all({0,2,4}, i->(T_1%P_i) != 0); -- => P_0,P_2,P_4 not on T_1
--
-- III.1  Two mixed singularities.
--
-- III.1.1 One quadruple- and one [3;3]-point (N_{12})
--
-- In 12.m2, we will see that there are two disjoint components which we
    denote by N'_{12} and N''_{12}.
--
-- In general position (N'_{12}), i.e., the distinguished tangent of the
    [3;3]-point (at Q_1) misses the quadruple point (at P_0):
m0 = super basis(8,intersect(P4_0,Q33_1));
assert(23 == rank source m0);
-- Preserving a line, a point on, and a point off the line, we get a
    stabiliser sub-group of dimension 3; hence, the dimension of (this
    component of) the stratum is computed as 23 - 3 - 1 = 19.
--
-- In special position (N''_{12}), that is, the distinguished tangent [3;3]-
    point (at Q_1) points towards the quadruple-point (at P_3):
m1 = super basis(8,intersect(P4_3,Q33_1));
assert(24 == rank source m1);
-- The dimension of this extra component is 24 - 4 - 1 = 19 (as well) since
    preserving a line through a pair of points is equivalent to fixing the
    two points, resulting in a stabilising projective subgroup of dimension
    4.
--
-- III.2  Three mixed singularities.
--
-- III.2.1 Two quadruple- and one [3;3]-point (N_{122})
--
-- In 122.m2 we will see that there are two disjoint components which we
    denote by N'_{122} and N''_{122}. They arise as follows:
--
-- In general position (N'_{122}), i.e., the distinguished tangent of the
    [3;3]-point points towards neither of the quadruple-points:
assert(13 == rank source super basis(8,intersect(Q33_0,P4_1,P4_2)));
-- The configuration under consideration is fixed by precisely those
    automorphisms preserving the three concurrent lines meeting in Q_0 and
```

```
    the line joining P_1 and P_2; thus, the stabiliser sub-group is of
    dimension 1 and so (this component of) the stratum is 11-dimensional.
--
-- In special position (N''_{122}), that is, the quadruple-point at P_3 lies
    on the distinguished tangent line of the [3;3]-point, but the other
    quadruple-point (at P_1) does not:
assert(14 == rank source super basis(8,intersect(Q33_0,P4_1,P4_3)));
-- Again, we can already compute the dimension of this extra component as 14
    - 2 - 1 = 11 since the automorphisms fixing the configuration are those
    fixing the three points, which has two-dimensional stabiliser sub-group
    (in PGL(3,CC)).
--
-- Since there are no two quadruple-points and a [3;3]-point of an octic
    curve on one line, there are no more cases to consider.
--
-- III.2.2 Two [3;3]- and one quadruple-point (N_{112})
--
-- As we will see in 112.m2, this stratum has three pair-wise disjoint
    components N'_{112}, N''{112} and N'''_{112} arising as follows.
--
-- N'_{112}: In general position, i.e., the distinguished tangents of the
    [3;3]-points (Q_0, Q_1) both miss the quadruple-point (at P_2):
assert(11 == rank source super basis(8,intersect(Q33_0,Q33_1,P4_2)));
-- Again, we compute the dimension of (this component of) the stratum: The
    stabiliser of this configuration is finite since besides the three
    points fixed already, we automatically fix a fourth (in general position
    ) as the intersection point of the two distinguished tangent lines. Thus
    , the dimension is 11 - 1 = 10.
--
-- N''_{112}: The quadruple-point (P_4) lies on the distinguished tangent
    line of only one of the [3;3]-points (at Q_0):
assert(12 == rank source super basis(8,intersect(Q33_0,Q33_1,P4_4)));
--
-- N'''_{112}: The quadruple-point (at P_3) lies on the distinguished tangent
     lines of both [3;3]-points (at Q_0, Q_1):
assert(13 == rank source super basis(8,intersect(Q33_0,Q33_1,P4_3)));
--
-- Both extra components have dimension 10 as well: The dimension of the
    stabiliser of the configuration is 1 or 2, resp., since the
    distinguished tangent line(s) at Q_0 (and Q_1) are given by the line
    joining P_4 and Q_0 (and Q_1, respectively). Therefore we get 12 - 1 - 1
     = 13 - 2 - 1 = 10 as the dimension.
--
-- Since it cannot happen that two [3;3]- and one quadruple-point are
    collinear, there is nothing more to check.
--
-- III.3  Four mixed singularities.
--          ... need parameters!
--
-- The only possible case (one quadruple- and three [3;3]-points) will be
    considered in the separate file 1112.m2. We will see that the stratum
    has two disjoint components of dimension one.
--
-- IV.   Degenerate singularities
--
-- The degenerate cases (with or without) parameters will be handled in other
     files.
--
-- EOF parameterFreeCases.m2 -----------------------------------------
```

Listing 5: The remaining strata parametrising curves with at most two non-simple singularities

```
-----------------------------------------------------------------------
-- M2-script studying the components of the strata parametrising octic curves
     with at most two non-simple singularities which were not already
    handled in parameterFreeCases.m2. I.e., N_{2bar}, N_{1bar}, N_{12}, N_
    {12bar}, N_{1bar2}, N_{1bar2bar}, N_{22bar}, N_{2bar^2}, N_{11bar} and
    N_{1bar^2}.
--
-- Filename upToTwoSingularities.m2
-----------------------------------------------------------------------
--
clearAll();
load "octicsFunctions.m2";
--
-- Notation:  (will be extended later)
-- A0 is the affine coordinate ring of rational AA^3 with coordinates s_0,
    s_1 and t.
-- S0 = A0[x,y,z] is the homogeneous coordinate ring of the trivial PP^2-
    bundle over AA^3.
--
-- We consider the points P_0 = (0;0;1) and P_1 = (1;0;0) and single out
    special lines {y = tx} and {z = 0} through these points. These
    distinguished lines are the vanishing loci of the linear forms L_0 and
    L_1, respectively.
--
-- Further below, we will consider various linear affine sub-spaces of AA^3 (
    t = 0 or 1, for example), with affine coordinate ring A and S = A[x,y,z]
     will be the homogeneous coordinate ring of the restricted PP^2-bundle.
--
A0 = QQ[s_0,s_1,t];
S0 = A0[x,y,z];
L = {x,y};
T = {y-t*x,z};
F = {z,x};
P = apply(L,T,pair->trim ideal pair);
P33 = apply(2,i->get33Ideal(L_i,T_i,F_i));
Pd4 = apply(2,i->getDegenerateQuadrupleIdeal(L_i,T_i,F_i));
--
-- I.  Degenerate [3;3]-points
--
-- I.1  One degenerate [3;3]-point (N_{1bar})
-- We put a [3;3]-point at P_1 with distinguished tangent line {z = 0}
(RS,inc,params,f) = universalFamily(8,P33_1);
assert(#params == 33);
M1 = directedDegConditions33Point(f,inc L_1, inc T_1, inc F_1, s_1);
assert(numgens target M1 == 2 and isSurjective M1);
-- The kernel of M defines the linear system of octics with a degenerate
     [3;3]-point at P_1 with distinguished tangent line {z = 0} and second
    order direction parametrised by s_1, we conclude that this linear space
    has dimension 33 - 2 - 1 = 30; since the plane configuration of a point
    and a tangent direction at that point has stabiliser of dimension 5 (in
    PGL(3,CC)), we conclude that the stratum is of dimension 30 + 1 - 5 =
    26.
-- It is easy to construct examples where s = 0 or s != 0 geometrically, so
    we omit the construction of an explicit inhabitant here.
--
-- I.2  A degenerate and a non-degenerate [3;3]-point (N_{11bar})
-- Recall that the distinguished tangents must not point towards the other
     [3;3]-point. Thus, we might as well choose t = 1.
A = A0/(t-1);
```

```
pr = map(A,A0);
S = A[gens S0];
prS = map(S,S0);
I33s = intersect apply(P33, i-> prS i);
(RS,inc,params,f) = universalFamily(8,I33s);
assert(#params == 21);
-- We impose degeneracy at P_1 as above:
M1 = directedDegConditions33Point(f,
    inc prS L_1, inc prS T_1, inc prS F_1, s_1);
assert(numgens target M1 == 2 and isSurjective M1);
-- Thus, the kernel has dimension 21 - 2 = 19 everywhere and so the stratum (
    non-empty by I.4 below) is irreducible of dimension 19 + 1 - 3 = 17 (+1
    from s_1 and -3 from the stabiliser group).
--
-- I.3  Two degenerate [3;3]-points (N_{1bar^2})
-- Now we also let the [3;3]-point at P_0 degenerate:
M0 = directedDegConditions33Point(f,
    inc prS L_0, inc prS T_0, inc prS F_0, s_0);
N = M0 || M1;
assert(numgens target N == 4 and isSurjective N);
-- Since N is of maximal everywhere, the corresponding stratum  (inhabited by
     I.4 below) is of dimension 21 - 4 + 2 - 3 = 16; here 4 is the rank of N
    , +2 from s_0 and s_1 and -3 from the stabiliser group in GL(3,CC).
--
-- I.4  The strata are indeed inhabited
-- To show that N_{11bar} and N_{1bar^2} are actually inhabited, it is
    sufficient to construct octics in the family constrained by N of I.3 for
     any t != 0 and any s_i != 0. So that we can be sure that s_i = 0 does
    not cause problems, we also construct curves with this property:
S' = QQ[gens S];
special00 = map(S',S, gens(S') | {0,0,1});
special01 = map(S',S, gens(S') | {0,1,1});
special11 = map(S',S, gens(S') | {1,1,1});
-- The following four lines produce pseudo-random members which were hard-
    coded below.
--J = sub(idealFromKernel(N,f,params),S);
--fRand00 = randomElement(special00 J);
--fRand01 = randomElement(special01 J);
--fRand11 = randomElement(special11 J);
fRand00 = (3/4)*x*y^7-(45/64)*y^8+(5/18)*x^2*y^5*z+(9/8)*x*y^6*z-(3/10)*y^7*z
    +(2/9)*x^4*y^2*z^2+(3/2)*x^3*y^3*z^2-(1/2)*x^2*y^4*z^2+(28/9)*x*y^5*z
    ^2-(13/3)*y^6*z^2+(5/3)*x^5*z^3+(10/3)*x^4*y*z^3-(1/2)*x^3*y^2*z
    ^3+(12/5)*x^2*y^3*z^3-(749/30)*x*y^4*z^3+(271/15)*y^5*z^3-(9/80)*x^4*z
    ^4-(5/28)*x^3*y*z^4-(9/8)*x^2*y^2*z^4+(453/140)*x*y^3*z^4-(1019/560)*y
    ^4*z^4-x^3*z^5+3*x^2*y*z^5-3*x*y^2*z^5+y^3*z^5;
fRand01 = (188/105)*x^2*y^6-(7/4)*x*y^7+(36/25)*y^8-(102/35)*x^3*y^4*z-(1/10)
    *x^2*y^5*z+(1/70)*x*y^6*z-(10/27)*y^7*z+(16/35)*x^4*y^2*z^2-(2/3)*x^3*y
    ^3*z^2-(7/54)*x^2*y^4*z^2+(1/2)*x*y^5*z^2-(152/945)*y^6*z^2+(2/3)*x^5*z
    ^3+3*x^4*y*z^3-(2/15)*x^3*y^2*z^3-(35/18)*x^2*y^3*z^3-(497/45)*x*y^4*z
    ^3+(851/90)*y^5*z^3-(5/4)*x^4*z^4-(9/35)*x^3*y*z^4-14*x^2*y^2*z
    ^4+(1182/35)*x*y^3*z^4-(2557/140)*y^4*z^4-(14/15)*x^3*z^5+(14/5)*x^2*y*z
    ^5-(14/5)*x*y^2*z^5+(14/15)*y^3*z^5;
fRand11 = (29/6)*x^2*y^6+(2/27)*x*y^7-(7/8)*y^8-(31/6)*x^3*y^4*z-(35/8)*x^2*y
    ^5*z-(9/5)*x*y^6*z-(3/2)*y^7*z-(25/6)*x^4*y^2*z^2-(9/5)*x^3*y^3*z^2-5*x
    ^2*y^4*z^2+(1/3)*x*y^5*z^2+(7937/210)*y^6*z^2+(9/2)*x^5*z^3-(14/3)*x^4*y
    *z^3+(4/7)*x^3*y^2*z^3-x^2*y^3*z^3+(10313/210)*x*y^4*z^3-(1698/35)*y^5*z
    ^3+(10/7)*x^4*z^4+5*x^3*y*z^4+(9/35)*x^2*y^2*z^4-(743/35)*x*y^3*z
    ^4+(509/35)*y^4*z^4-(5/3)*x^3*z^5+5*x^2*y*z^5-5*x*y^2*z^5+(5/3)*y^3*z^5;
-- First of all, the singular loci are supported at P_0 and P_1 only:
expected = sub(intersect(prS P_0, prS P_1),S');
sing00 = ideal jacobian matrix fRand00;
sing01 = ideal jacobian matrix fRand01;
```

```
sing11 = ideal jacobian matrix fRand11;
assert(radical sing00 == expected);
assert(radical sing01 == expected);
assert(radical sing11 == expected);
-- On the other hand, we have there the mildest possible degenerate [3;3]-
    points, namely, singularities of type J_{2,1}:
assert(isJ21(fRand00,special00 prS T_0,x,y,z));
assert(isJ21(fRand00,special00 prS T_1,y,z,x));
assert(isJ21(fRand01,special01 prS T_0,x,y,z));
assert(isJ21(fRand01,special01 prS T_1,y,z,x));
assert(isJ21(fRand11,special11 prS T_0,x,y,z));
assert(isJ21(fRand11,special11 prS T_1,y,z,x));
--
-- A suitable deformation will have the same singularity at P_0, but a non-
    degenerate [3;3]-point at P_1, say, so that we also conclude that the
    other stratum is non-empty.
--
-- II.  Degenerate quadruple-points
--
use S0;
(RS,inc,params,f) = universalFamily(8,P_1^4);
assert(#params == 35);
-- This is the universal family of octic forms whose associated curves have a
     quadruple-point at P_1.
--
-- II.1  A single degenerate quadruple-point (N_{2bar})
-- Imposing degeneracy:
M1 = tangencyCond(f,
    inc L_1, inc T_1, inc F_1, Multiplicity=>4, Degree=>2);
assert(numgens target M1 == 2 and isSurjective M1);
-- Hence, the moduli space of plane octics with a degenerate quadruple-point,
     which is clearly inhabited, is of dimension 35 - 2 - 6 = 27. (Here, 6
    is the dimension of the stabiliser sub-group of GL(3,CC)).
--
-- II.2  Two quadruple-points, one degenerate (N_{22bar})
-- As before, we can take t = 1 since (on a reduced octic) there is no second
     quadruple-point on a special tangent line of a degenerate quadruple-
    point.
use S;
Iquads = intersect(prS P_0^4, prS P_1^4);
(RS,inc,params,f) = universalFamily(8, Iquads);
assert(#params == 25);
-- The degeneracy conditions are encoded in the vanishing of the following
    matrices:
M0 = tangencyCond(f,
    inc prS L_0, inc prS T_0, inc prS F_0,
    Multiplicity=>4, Degree=>2);
M1 = tangencyCond(f,
    inc prS L_1, inc prS T_1, inc prS F_1,
    Multiplicity=>4, Degree=>2);
assert(numgens target M0 == 2 and numgens target M1 == 2);
-- Imposing degeneracy at P_0 only:
assert isSurjective M0;
-- Thus, the rank of M0 is 2 and so the dimension of the corresponding
    stratum is 25 - 2 - 4 = 19; it is easy to see that it is actually
    inhabited.
--
-- II.3  Two quadruple-points, both degenerate (N_{2bar^2})
-- Imposing degeneracy at P_1 as well:
N = M0 || M1;
assert isSurjective N;
-- Thus, the rank of N is 4 and so the corresponding stratum is irreducible
```

```
    of dimension 25 - 4 - 3 = 18; again, it is easy to see that it is
    inhabited. For convenience, we construct a member:
use S';
-- The following two lines in comments were used to produce the hardcoded
    example fRand below.
--J = sub(idealFromKernel(N,f,params),S');
--fRand = randomElement(J);
fRand = -(3/8)*x^3*y^5-7*x^2*y^6+2*x*y^7-(15/7)*y^8-(10/9)*x^3*y^4*z+(49/20)*
    x^2*y^5*z-(3/5)*x*y^6*z+y^7*z+(35/4)*x^4*y^2*z^2-9*x^3*y^3*z^2+(35/27)*x
    ^2*y^4*z^2-(18/7)*x*y^5*z^2-(7/9)*y^6*z^2+(7/4)*x^4*y*z^3-(10/9)*x^3*y
    ^2*z^3-(1/4)*x^2*y^3*z^3+(18/5)*x*y^4*z^3-(15/16)*y^5*z^3+(3/50)*x^4*z
    ^4-(10/3)*x^3*y*z^4+4*x^2*y^2*z^4+(44/25)*x*y^3*z^4-(373/150)*y^4*z^4;
-- First of all, its singular locus is supported at P_0 and P_1:
expected = sub(intersect(prS P_0, prS P_1),S');
sing = ideal jacobian matrix fRand;
assert(radical sing == expected);
-- On the other hand, at these points, we have the mildest possible
    degenerate quadruple-points singularities:
assert(isX10(fRand,x,y));
assert(isX10(fRand,z,y));
-- Thus, the curve defined by fRand has only two singular points, which are
    quadruple-points of type X_{10}.
--
--
-- III.  A quadruple- and a [3;3]-point
-- Putting the [3;3]-point at P_0 and the quadruple-point at P_1, we have to
    distinguish the two cases whether the distinguished tangent at P_0
    points towards the quadruple-point (t = 0) or not.
--
-- III.1  The non-degenerate case (N_{12})
-- From parameterFreeCases.m2 III.1.1 we know that something special is
    happening if t = 0. Here we show that there are two disjoint components.
use A0;
-- The universal family with a [3;3]-point at P_0 with varying tagnent
(RS,inc,params,f) = universalFamily(8, P33_0);
assert(#params == 33);
-- Asking for a quadruple-point at P_1:
toBeZero = f%inc P_1^4;
toBeZeroCoeff = for Term in terms toBeZero list leadCoefficient(Term);
M = matrix for eq in toBeZeroCoeff list
               for g in params list leadCoefficient(eq//g);
-- M is a matrix A0^10 <-- A0^33. Its kernel encodes the parameters for which
     f has a quadruple-point at P_1. Thus, we have to see how the kernel
    changes as t varies. We investigate where the rank (generically 10)
    drops:
assert(numgens target M == 10);
droppingRankCond = minors(10, mingens image M);
assert(radical droppingRankCond == ideal(t));
-- Thus, the rank drops only if t = 0 and from parameterFreeCases.m2
-- III.1.1 we know that it drops exactly by 1 there.
--
-- For t != 0, this is the ideal containing the octic forms whose associated
    curves have a quadruple-point at P_1 and a [3;3]-point at P_0 with
    distinguished tangent line {y = tx}:
J = sub(idealFromKernel(M,f,params), S0);
-- Those with distinguished tangent line {y = 0} are obtained from the kernel
     if we substitute t -> 0 first:
Jspecial = sub(
    idealFromKernel(sub(M, {t=>0}), sub(f, {t=>0}), params), S0);
-- Substituting t -> 0 in J we get those which have distinguished tangent
    line {y = 0} and which are degenerations of such with t != 0:
Jintersection = sub(J,{t=>0});
```

```
-- Sanity check:
assert isSubset(Jintersection, Jspecial);
-- However:
assert not isSubset(Jspecial, Jintersection);
-- Thus, we get two distinct components. Their intersection consists of non-
    reduced curves only:
assert((gens Jintersection)%(sub(T_0,{t=>0}))^2 == 0);-- but
assert not ((gens Jspecial)%(sub(T_0,{t=>0}))^2 == 0);-- whereas
assert((gens Jspecial)%(sub(T_0,{t=>0})) == 0);
-- That is, every member of the component with t = 0 contains the line {y =
    0} at least once, but not all do so twice, whereas those which are
    limits as t -> 0 contain this line at least twice. Since we neglect non-
    reduced curves in N_{12}, once we have shown -- that both are inhabited,
     we get two irreducible components N'_{12} and N''_{12}, both of
    dimension 19 as computed in III.1.1 of parameterFreeCases.m2. But since
    it is easy to construct examples by elementary plane curve-geometry, we
    omit this here.
--
-- III.2  A [3;3]-point and a quadruple-point, at least one degenerate
-- Note that III.1 shows that also in the degenerate cases, the two
    components distinguishing t != 0 and t = 0 are disjoint. Therefore, we
    don't need to let t vary but consider the cases t = 1 and t = 0.
--
-- First we let t = 1 (N'_{12bar}, N'_{1bar2}, N'_{1bar2bar}).
use S;
Imixed = intersect(prS P33_0, prS P_1^4);
(RS,inc,params,f) = universalFamily(8, Imixed);
assert(#params == 23);
-- The degeneracy conditions:
M0 = directedDegConditions33Point(f,
    inc prS L_0, inc prS T_0, inc prS F_0, s_0);
M1 = tangencyCond(f,
    inc prS L_1, inc prS T_1, inc prS F_1,
    Multiplicity=>4, Degree=>2);
assert(numgens target M0 == 2 and isSurjective M0);
assert(numgens target M1 == 2 and isSurjective M1);
-- Thus, the two components where either the [3;3]-point (N'_{1bar2}) or the
    quadruple-point (N'_{12bar}) is degenerate are irreducible and of
    dimension 23 - 2 + 1 - 4 = 23 - 2 - 3 = 18.
--
-- Both degenerate:
N = M0 || M1;
assert isSurjective N;
-- Thus, also the component N'{1bar2bar} is irreducible and of dimension 23 -
     4 + 2 - 4 = 17. That these strata are inhabited is easy to see
    geometrically, so we omit constructions of explicit polynomials.
--
-- Now we let t = 0 (N''_{12bar}, N''_{1bar2}, N''_{1bar2bar}).
use A0;
B = A0/t;
prB = map(B,A0);
BS = B[gens S0];
prBS = map(BS,S0);
Imixed' = intersect(prBS P33_0, prBS P_1^4);
(RS,inc,params,f) = universalFamily(8, Imixed');
assert(#params == 24);
-- The degeneracy conditions:
M0 = directedDegConditions33Point(f,
    inc prBS L_0, inc prBS T_0, inc prBS F_0, s_0);
M1 = tangencyCond(f,
    inc prBS L_1, inc prBS T_1, inc prBS F_1,
    Multiplicity=>4, Degree=>2);
```

```
assert(numgens target M0 == 2 and numgens target M1 == 2);
-- Imposing degeneracy at the quadruple-point:
assert isSurjective M1;
-- Thus, the rank of M1 is constant and so the component N''_{12bar} is
    irreducible and of dimension 24 - 2 - 4 = 18.
-- Imposing degeneracy at the [3;3]-point:
droppingRankCond = minors(2,mingens image M0);
assert(radical droppingRankCond == ideal(s_0));
-- M0 is not surjective if and only if s_0 = 0. However, in this case, all
    octics will contain the distinguished tangent line twice:
Jspecial = sub(idealFromKernel(sub(M0,{s_0=>0}),sub(f,{s_0=>0}),params),BS);
assert((gens Jspecial)%(prBS T_0)^2 == 0);
-- Therefore, this case is irrelevant and for the remaining part, we see that
     we get an irreducible component N''_{1bar2} of dimension 24 - 2 + 1 - 5
     = 18 as well.
--
-- Imposing degeneracy at both points:
N = M0 || M1;
droppingRankCond = minors(4,mingens image N);
assert(radical droppingRankCond == ideal(s_0));
-- The same as before is happening; again in this case all octics will be non
    -reduced if s_0 = 0, so that neglecting this part, we get an irreducible
     component N''_{1bar2bar} of dimension 24 - 4 + 2 - 5 = 17. Again,
    producing inhabitants is easy, hence omitted.
--
-- EOF upToTwoSingularities.m2 ---------------------------------------
```

Listing 6: The strata $\mathfrak{N}_{112}$, $\mathfrak{N}_{122}$ and $\mathfrak{N}_{1^3}$

```
----------------------------------------------------------------------
-- M2-script to study the strata N_{122}, N_{112} and N_{111}
--
-- Filename threeNonDeg.m2
----------------------------------------------------------------------
--
-- We will deal with the three strata in three different sections, each with
    its own notation.
--
-- I  One [3;3]- and two quadruple-points (N_{122})
--
clearAll();
load "octicsFunctions.m2";
--
-- Notation:
-- A = QQ[t] is the coordinate ring of the rational line AA^1.
-- S = A[x,y,z] is the homogeneous coordinate ring of the trivial PP^2-family
     over AA^1.
-- We will consider octics with quadruple-points in P_0 = (0;0;1) and P_1 =
    (1;0;0) and a [3;3]-point in P_2 = (0,1,0) with varying distinguished
    tangent line defined by T_2 = x-tz.
--
-- We know that there will be one extra modulus of octics when t = 0. We aim
    to prove that this is the only instance where something unexpected is
    happening and that the two resulting families have no reduced octic in
    common. More precisely, the intersection corresponds to the octics
    containing the line defined by x twice.
--
-- Each component will be of dimension 11; this was computed in Sec. III.2.1
    of parameterFreeCases.m2.
--
A = QQ[t];
S = A[x,y,z];
```

```
L = {x,y,z};
T = {y,z,x-t*z};
F = { , ,y}; -- aux. localisation variables
-- The ideals of the reduced points:
P = apply(3,i->trim ideal(L_i,T_i));
-- The ideals with the constraints for the desired singularities:
P4 = apply(2,i->P_i^4);
P33 = get33Ideal(L_2,T_2,F_2);
-- The relevant ideals and the universal family of octics with a [3;3]-point
    in P_2 with distinguished tangent line T_2:
Iconstant = intersect(P4);
Ivariable = P33;
(RS,inc,params,f) = universalFamily(8,Ivariable);
assert(#params == 33);
use RS;
-- Reduction of f modulo the remaining conditions:
toBeZero = f%(inc Iconstant);
toBeZeroCoeff = for Term in terms toBeZero list leadCoefficient(Term);
M = matrix for eq in toBeZeroCoeff
                    list for g in params list leadCoefficient(eq//g);
-- The kernel of M is the space of coefficients for which f is as desired for
     general t; its dimension is 13 = 33 - 20, as expected:
assert(numgens target M == 20);
droppingRankCond = minors(20,mingens image M);
assert(radical droppingRankCond == ideal(t));
-- Thus, the rank only drops for t = 0; indeed:
assert(numgens kernel sub(M,{t=>0}) == 14);
--
-- It remains to study the intersection to see that we have two disjoint
    components. The argument is analogous so 111.m2, e.g.
Q = S/(t);
pr = map(Q,S);
use S;
J = ideal super basis(8,intersect(P4_0,P4_1,P33));
use Q;
J' = pr J;
Jspecial = ideal super basis(8,intersect(pr(P4_0),pr(P4_1),pr(P33)));
-- Sanity check:
assert isSubset(J',Jspecial);
-- However:
assert((gens J')%(pr T_2)^2 == 0);-- but
assert((gens Jspecial)%(pr T_2)^2 != 0);
-- This means that the degenerations of curves in N'_{122} towards N''_{122}
    contain the distinguished tangent line twice. Therefore, the components
    are disjoint, as claimed.
--
-- That both components are inhabited can be seen by means of basic geometric
     constructions. For example, four conics through P_0 and P_1, of which
    three also pass through P_2, all three with the same tangent direction
    at P_2, for the general component, and for the special component, take a
     nodal cubic with the node in P_0, let P_2 be a general point of the
    cubic, T_2 its tangent line, and P_1 the other intersection point of the
     line with the cubic; then consider the union of the nodal cubic, the
    tangent line, the line joining P_0 and P_1, and a general cubic through
    the configuration.
--
-- II  Two [3;3]-points and one quadruple-point
clearAll();
load "octicsFunctions.m2";
--
-- From the calculations in parameterFreeCases.m2 III.2.2, we know that there
     are possibly three components, depending on how many of the two
```

```
    distinguished tangent lines pass through the quadruple-point. This
    script shows that the three components N'_{112} (where both lines miss
    the quadruple-point), N''_{112} (where exactly one of the lines passes
    through the quadruple-point) and N'''_{112} (where the quadruple-point
    sits at the intersection of both lines) are irreducible and pair-wise
    disjoint. They are of dimension 10 by parameterFreeCases.m2 III.2.2.
--
-- Notation:
-- A = QQ[t_0,t_1] is the coordinate ring of the rational plane AA^2.
-- S = A[x,y,z] is the homogeneous coordinate ring of the trivial PP^2-family
     over AA^2.
-- We will consider octics with two [3;3]-points in P_0 = (0;0;1) and P_1 =
    (1;0;0), with varying distinguished tangent lines defined by the linear
    forms T_0 = x - t_0 y and T_1 = y - t_1 z, and a quadruple-point in P_2
    = (1;1;1).
--
A = QQ[t_0,t_1];
S = A[x,y,z];
L = {y,z,x-y};
T = {x-y*t_0,y-z*t_1,y-z};
F = {z,x,}; -- aux. localisation variables
-- The ideals of the reduced points:
P = apply(3,i->trim ideal(L_i,T_i));
-- The ideals with the constraints for the desired singularities:
P33 = apply(2,i->get33Ideal(L_i,T_i,F_i));
P4 = P_2^4;
Iconstant = P4;
Ivariable = intersect P33;
-- Building the universal family of octics with two [3;3]-points with
    distinguished tangents given by the linear forms T_0, T_1.
(RS,inc,params,f) = universalFamily(8, Ivariable);
assert(#params == 21);
-- Reduction of f modulo the quadruple-point conditions:
toBeZero = f%inc(Iconstant);
toBeZeroCoeff = for Term in terms toBeZero list leadCoefficient(Term);
M = matrix for eq in toBeZeroCoeff
                list for g in params list leadCoefficient(eq//g);
-- Generically, the rank of M is 10, but it drops if t_i = 1:
assert(numgens target M == 10);
droppingRankCond = minors(10,mingens image M);
assert(radical droppingRankCond == ideal((t_0-1)*(t_1-1)));
M1 = sub(M,{t_1=>1});
M2 = sub(M,{t_0=>1,t_1=>1});
assert(numgens kernel M1 == 12);
assert(numgens kernel M2 == 13);
--
-- It remains to compare the different components. For this, we need
    parametrisations of the different families. The kernels of the matrices
    above define different instances of f, and those give rise to the
    desired parametrisations:
use A;
A' = A/(t_1-1);
pr' = map(A',A);
S' = A'[gens S];
prS' = map(S',S);
RS' = S'[params];
prRS' = map(RS',RS);
A'' = A'/(t_0-1);
pr'' = map(A'',A');
S'' = A''[gens S'];
prS'' = map(S'',S');
RS'' = S''[params];
```

```
prRS'' = map(RS'',RS');
-- The general component and its degenerations as t_1 -> 1:
J = sub(idealFromKernel(M,f,params),S);
J' = prS' J;
-- The first special component and its degenerations as t_0 -> 1:
Jsp1 = sub(idealFromKernel(pr' M, prRS' f, gens RS'),S');
Jsp1' = prS'' Jsp1;
-- The last special component:
Jsp2 =sub(idealFromKernel(pr'' pr' M, prRS'' prRS' f, gens RS''),S'');
-- Sanity check:
assert isSubset(J',Jsp1);
assert isSubset(Jsp1',Jsp2);
-- Every limit of the general component along the first special component
     contains the line T_1 twice, but the elements of the first special
     component do not:
assert((gens J')%(prS' T_1)^2 == 0);
assert((gens Jsp1)%(prS' T_1)^2 != 0);
--
-- Similarly, the limits of the first special component along the second
     special component contain the line T_0 twice, but the actual elements of
      the second special component do not:
assert((gens Jsp1')%(prS'' prS' T_0)^2 == 0);
assert((gens Jsp2)%(prS'' prS' T_0)^2 != 0);
--
-- As claimed. Since when passing to N_{112} we neglect non-reduced curves,
     the pair-wise intersections are not contained and so the three
     components of the stratum are pair-wise disjoint.
--
-- It is easy to come up with geometric constructions of inhabitants for the
     general component (t_0 != 1 != t_1) or the second special component (t_0
      = t_1 = 1). Since this is a bit harder for the first special component,
      (t_1 != 1 and t_0 general), we show that the following form defines an
     inhabitant in that case.
use A';
R = A'/(t_0-1/2);-- isomorphic to QQ
q = map(R,A);
Q = R[gens S];
qQ = map(Q,S);
qQ' = map(Q,S');
expected = qQ intersect(P);
f = 2*x^5*y^3-(24/5)*x^4*y^4-(20/27)*x^3*y^5+(4/9)*x^2*y^6-(87497/840)*x*y
     ^7+(1179841/7560)*y^8-6*x^5*y^2*z+(159/35)*x^4*y^3*z+(28/135)*x^3*y^4*z
     +(12272/45)*x^2*y^5*z-(987109/2520)*x*y^6*z-(1150217/15120)*y^7*z+6*x^5*
     y*z^2-(52/5)*x^4*y^2*z^2+(21667/630)*x^3*y^3*z^2-(105043/315)*x^2*y^4*z
     ^2+(530869/630)*x*y^5*z^2-(304831/1260)*y^6*z^2-2*x^5*z^3+(923/35)*x^4*y
     *z^3-(201223/1890)*x^3*y^2*z^3-(923/20)*x^2*y^3*z^3-(122039/630)*x*y^4*z
     ^3+(908081/7560)*y^5*z^3-(110/7)*x^4*z^4+(30785/378)*x^3*y*z^4+(1305/14)
     *x^2*y^2*z^4-(73805/504)*x*y^3*z^4+(8795/216)*y^4*z^4-(53/6)*x^3*z
     ^5+(53/4)*x^2*y*z^5-(53/8)*x*y^2*z^5+(53/48)*y^3*z^5;
--
-- Claim: The octic curve defined by f is a member of the component indicated
      above.
-- Since
assert(f%qQ'(Jsp1) == 0);--, the curve has at least the desired singularities
     ; in fact, it has no more than the three singularities that we imposed:
sing = ideal jacobian matrix f;
assert(radical sing == expected);
-- and the three are ordinary, as claimed:
assert(isOrdinaryQuadruplePoint(sub(f,{x=>x+z,y=>y+z}),x,y,z));
assert(isOrdinary33Point(f,qQ T_0,y,x,z));
assert(isOrdinary33Point(f,qQ T_1,z,y,x));
-- This proves the claim and concludes this section.
```

```
--
-- III  Three [3;3]-points
--
clearAll();
load "octicsFunctions.m2";
--
-- Notation:
-- A = QQ[t] is the coordinate ring of the rational line AA^1.
-- S = A[x,y,z] is the homogeneous coordinate ring of the trivial PP^2-family
     over AA^1.
-- We will consider octics with [3;3]-points in P_i = V(L_i,T_i), i = 0,1,2,
    with distinguished tangent line V(T_i), respectively.
-- Concretely, P_0 = (0;0;1), P_1 = (1;0;0), P_2 = (0;1;0) and
--             T_0 = x - ty,  T_1 = y - z,   T_2 = x - z
--
-- The line we miss (T_0 = y) is irrelevant since it passes through P_1 and
    on a reduced octic, the distinguished tangent line of a [3;3]-point does
     not pass through another [3;3]-point.
--
A = QQ[t];
S = A[x,y,z];
L = {y,z,x};
T = {x-t*y,y-z,x-z};
F = {z,x,y}; -- aux. localisation variables
P = apply(3,i->trim ideal(L_i,T_i));
-- The ideals containing the [3;3]-point-constraints:
P33 = apply(3,i->get33Ideal(L_i,T_i,F_i));
-- The following ideal gives the constraints for [3;3]-points in P_1 and P_2
    with distinguished tangents as above.
Iconstant = intersect(P33_1,P33_2);
-- The following is the ideal for a [3;3]-point in P_0 whose distinguished
    tangent along the line {x = ty}, thus depending on t.
Ivariable = P33_0;
-- The family of octics fulfilling the variable constraints:
(RS,inc,params,f) = universalFamily(8,Ivariable);
assert(#params == 33);
-- Reduction of f modulo the remaining conditions:
toBeZero = f%(inc Iconstant);
toBeZeroCoeff = for Term in terms toBeZero list leadCoefficient(Term);
M = matrix for eq in toBeZeroCoeff
                     list for g in params list leadCoefficient(eq//g);
-- M is a Matrix A^24 <-- A^33, generically of maximal rank, leaving 33 - 24
    = 9 generic dimensions:
assert(numgens target M == 24);
droppingRankCond = minors(24, mingens image M);
assert(radical droppingRankCond == ideal(t+1));
-- Thus, the only instance where the rank drops is where t = -1, in which
    case the three tangents are on the conic V(xy-xz-yz).
--
-- If t != -1, we get an irreducible component N'_{111} (once we have shown
    that there indeed exist admissible octics so that it is not empty) and
    it is of dimension 9 + 1 - 1 = 9.
--
-- In this case, the kernel has dimension 10:
prA = map(QQ,A,{t=>-1});
Q = QQ[gens S];
pr = map(Q,S, (gens Q) | {-1});
assert(numgens kernel prA M == 10);
--
-- We now show that the intersection of the two components parametrises the
    octics containing the conic V(xy-xz-yz) twice. The notation is the same
    as before, but we specialise to t = -1.
```

```
use S;
-- The ideal of octics for t != -1:
J = ideal super basis(8,intersect P33);
-- The ideal of their limits as t -> -1:
J' = pr J;
use Q;
-- The ideal of octics for t = -1:
Jspecial = ideal super basis(8,intersect apply(P33,i->pr i));
-- Sanity check:
assert isSubset(J',Jspecial);
-- However:
conic = x*y - x*z - y*z;
assert((gens J')%conic^2 == 0);-- but
assert((gens Jspecial)%conic^2 != 0);
-- Hence, the intersection of the closures of the strata consist of non-
    reduced curves only. In particular, if inhabited, we get an extra
    component N''_{11} which is disjoint from the other component but of the
     same dimension 10 + 0 - 1 = 9.
--
-- Finally, we have to show that both components contain admissible curves
    with three non-degenerate [3;3]-points.
-- We begin with the component of general distinguished tangents. The
    notation is the same as before, but with a fixed value for t.
use S;
rand = 7/10;
QRand = QQ[gens S];
prRand = map(QRand,S,append(gens QRand,rand));
sing = intersect(apply(P,i->prRand(i)));
f = -15*x^5*y^3 + 3*x^4*y^4 - (119/50)*x^3*y^5 + 45*x^5*y^2*z - (46/35)*x^4*y
    ^3*z - (223/1000)*x^3*y^4*z + (357/50)*x^2*y^5*z - 45*x^5*y*z^2 -
    (162/35)*x^4*y^2*z^2 + (137353/21000)*x^3*y^3*z^2 - (18613/1500)*x^2*y
    ^4*z^2 - (357/50)*x*y^5*z^2 + 15*x^5*z^3 + (6/5)*x^4*y*z^3 - (3803/140)*
    x^3*y^2*z^3 - (8131/750)*x^2*y^3*z^3 + (40459/3000)*x*y^4*z^3 + (119/50)
    *y^5*z^3 + (61/35)*x^4*z^4 + (10319/525)*x^3*y*z^4 - (18973/500)*x^2*y
    ^2*z^4 + (546/25)*x*y^3*z^4 - (2891/750)*y^4*z^4 + (25/7)*x^3*z^5 -
    (15/2)*x^2*y*z^5 + (21/4)*x*y^2*z^5 - (49/40)*y^3*z^5;
--
-- Claim: The octic defined by f has only non-degenerate [3;3]-points,
    located at P_0, P_1 and P_2.
--
assert(f%prRand J == 0);
assert(radical ideal singularLocus ideal f == sing);
-- Therefore, the singular locus is supported exactly at the [3;3]-points we
    imposed.
--
-- We have to check that they are all non-degenerate [3;3]-points.
assert(isOrdinary33Point(f,prRand(T_0),x,y,z));
assert(isOrdinary33Point(f,prRand(T_1),z,y,x));
assert(isOrdinary33Point(f,prRand(T_2),x,z,y));
--
-- This proves the claim. In particular, N'_{111} is inhabited.
--
-- We now complete the discussion of three [3;3]-points by considering the
    extra component corresponding to octics where the distinguished tangents
     of the [3;3]-points are on a conic. We have to show that there exists
    such an octic with non-degenerate [3;3]-points.
--
-- Instead of the following 'random' element, we could also take the union of
     the conic and two general cubics tangent to the conic in P_0, P_1 and
    P_2.
--
-- The notation is the same as earlier, specialised to t = -1.
```

```
use Q;
sing = intersect(apply(P,i->pr(i)));
f = -(9/20)*x^5*y^3 - (8/3)*x^4*y^4 - (4/3)*x^3*y^5 + (27/20)*x^5*y^2*z +
    (149/15)*x^4*y^3*z - 8*x^3*y^4*z + 4*x^2*y^5*z - (27/20)*x^5*y*z^2 -
    (333/20)*x^4*y^2*z^2 + (18023/540)*x^3*y^3*z^2 + (1186/27)*x^2*y^4*z^2 -
     4*x*y^5*z^2 + (9/20)*x^5*z^3 + (85/6)*x^4*y*z^3 - (9077/270)*x^3*y^2*z
    ^3 - (55001/540)*x^2*y^3*z^3 - (1436/27)*x*y^4*z^3 + (4/3)*y^5*z^3 -
    (287/60)*x^4*z^4 - (3017/270)*x^3*y*z^4 + (2147/180)*x^2*y^2*z^4 +
    (1721/45)*x*y^3*z^4 + (538/27)*y^4*z^4 + (83/4)*x^3*z^5 + (249/4)*x^2*y*
    z^5 + (249/4)*x*y^2*z^5 + (83/4)*y^3*z^5;
--
-- Claim: The octic V(f) has only non-degenerate [3;3]-singularities, located
     in the points P_0, P_1 and P_2.
--
assert(f%Jspecial == 0);
assert(radical ideal singularLocus ideal f == sing)
-- Thus, the singular locus consists precisely of the [3;3]-points we
-- imposed. By the following three lines, they are non-degenerate:
assert(isOrdinary33Point(f,pr(T_0),x,y,z));
assert(isOrdinary33Point(f,pr(T_1),z,y,x));
assert(isOrdinary33Point(f,pr(T_2),x,z,y));
--
-- This proves the claim. In particular, N''_{111} is inhabited, too.
--
-- EOF threeNonDeg.m2 ------------------------------------------------
```

Listing 7: The components of $\mathfrak{N}_{13^2}$

```
----------------------------------------------------------------------
-- M2-script to study octics curves in PP^2 with a quadruple-point and three
    [3;3]-points, i.e., the stratum N_{1112}.
--
-- Filename 1112.m2
----------------------------------------------------------------------
--
-- If an octic has three [3;3]-points and a quadruple-point, then either the
    three distinguished lines are concurrent with intersection point the
    quadruple-point and the octic contains the lines (and the residual
    quintic is rational and irreducible), or the octic is the union of a
    conic, two lines tangent to the conic, say at P_0 and P_1, and a quartic
     which is tangent to the conic at P_0 and P_1 as well and which has a
    tac-node in a third point P_2 on the conic and an ordinary double-point
    at the intersection of the two lines. We will compute the dimensions of
    these two components.
--
-- I  Concurrent distinguished tangent lines (N'_{1112}).
--
clearAll();
load "octicsFunctions.m2"
--
-- The configuration of three concurrent lines and three non-collinear points
     on those lines is fixed by finitely many automorphisms and we can,
    without loss of generality, fix those four points.
--
-- Notation:
-- S = QQ[a_0,a_1][x,y,z] is the homogeneous coordinate ring of the trivial
    PP^2-family over rational AA^2.
-- We will consider all octic forms with [3;3]-points in the points P_0 =
    (0;0;1), P_1 = (1;0;0) and P_2 = (0;1;0) and a quadruple-point in P_3 =
    (1;1;1), where the distinguished tangent at P_i is the line joining P_i
    and P_3. (There is a pencil of such octics, which we will parametrise
    using the extra parameters a_0, a_1.)
```

```
--
use QQ[x,y,z];
P = {idealFromCoords(0,0,1), idealFromCoords(1,0,0),
     idealFromCoords(0,1,0), idealFromCoords(1,1,1)};
L = {y,z,z};
T = {x-y,y-z,x-z};
F = {z,x,y}; -- aux. localisation variables
scan(3,i-> assert( P_i == ideal(L_i,T_i) ) );
P33 = apply(3,i->get33Ideal(L_i,T_i,F_i));
-- The lines are chosen as explained above:
assert all(3, i->(T_i)%intersect(P_i,P_3) == 0);
-- The ideal containing the constraints:
I = intersect(intersect(P33),P_3^4);
-- the universal family:
(S,inc,params,f) = universalFamily(8,I);
assert(#params == 2);
use S;
T = apply(T,i->inc(i));
-- Sanity check: Every octic of the family contains all tree distinguished
    tangent lines:
assert( f%(T_0*T_1*T_2) == 0 );
-- It remains to show that the generic element has ordinary [3;3]-points at
    P_0, P_1 and P_2, an ordinary quadruple-point at P_3, and is singular
    only in those four points.
use S;
badParams = degeneracyConditions33Point(f,T_0,x,y,z);
badParams = badParams + degeneracyConditions33Point(f,T_1,z,y,x);
badParams = badParams + degeneracyConditions33Point(f,T_2,z,x,y);
-- Translate to move quadruple-point to (0;0;1):
g = sub(f,{x=>x+z,y=>y+z,z=>z});
badParams = badParams + degeneracyConditionsQuadruplePoint(g,x,y,z);
assert(codim badParams == 1);
-- Outside the bad locus, the singular locus is as expected:
g = (radical badParams)_*;
Sgood = S[u]/(u*product(g)-1);
J =  sub(radical(ideal(jacobian ideal f)+f),Sgood);
assert(J == sub(intersect P,Sgood));
-- As claimed.
--
-- Thus, the pencil given by f generically defines admissible octics and so
    the dimension of this component is one.
--
-- A note aside for the record: The (reduced) bad locus is given by the three
     points (1;0),(2;-1),(1;-1) (in PP^1 = Proj QQ[t_0,t_1]) and the octics
    we obtain for these parameters are the union of twice the conic through
    P_0, P_1 and P_2 tangent to two of them, the two corresponding
    distinguished lines, and twice the remaining distinguished line (-->
    members of M^3 with 2T_{2,3,infty} +  T_{2,4,infty} + 2T_{2,infty,infty
    }).
--
-- II  Distinguished tangents along a conic (N'_{1112}).
--
clearAll();
-- Requires:
load "octicsFunctions.m2";
--
-- The configuration is determined by two lines, each one of them with a
    point (not the intersection point), and a third point in general
    position. The automorphism group of the plane acts transitively on those
     configurations, with finite stabilisers. Therefore, we may fix any. The
     remaining data are determined by this configuration: The (to become
    quadruple-)point is the intersection of the lines and there is a unique
```

```
    conic through the other three points which is tangent to the lines, thus
     determining the third distinguished tangent direction.
--
-- Notation as in I
--
use QQ[x,y,z];
P = {idealFromCoords(0,0,1),idealFromCoords(1,0,0),
     idealFromCoords(0,1,0),idealFromCoords(1,1,1)};
L = {y,z,z};
T = {x-y,y-z,x+z};
F = {z,x,y}; -- aux. localisation variables
P33 = apply(3,i->get33Ideal(L_i,T_i,F_i));
PwT = apply(3,i->ideal(L_i^2,T_i));
conic = x*y-x*z+y*z;
scan(3,i->assert(P_i == ideal(L_i,T_i)));
assert(ideal super basis(2,intersect(PwT)) == ideal conic);
assert(ideal(T_0,T_1) == P_3);
I = intersect(intersect(P33),P_3^4);
(S,inc,params,f) = universalFamily(8,I);
assert(#params == 2);
T = apply(T,i->inc(i));
conic = inc(conic);
use S;
-- Sanity check: Every octic of the family contains the two distinguished
    tangents at P_0 and P_1 and the conic:
assert(f%(T_0*T_1*conic) == 0);
-- The generic octic of the system has non-degenerate [3;3]-points at P_0,
    P_1 and P_2 and a non-degenerate quadruple-point at P_3 and no other
    singularities:
badParams = degeneracyConditions33Point(f,T_0,x,y,z);
badParams = badParams + degeneracyConditions33Point(f,T_1,z,y,x);
badParams = badParams + degeneracyConditions33Point(f,T_2,z,x,y);
-- Translate to move quadruple-point to (0;0;1):
g = sub(f,{x=>x+z,y=>y+z,z=>z});
badParams = badParams + degeneracyConditionsQuadruplePoint(g,x,y,z);
assert(codim badParams == 1);
-- Outside the bad locus, the singular locus is as expected:
g = (radical badParams)_*;
Sgood = S[u]/(u*product(g)-1);
J = radical sub((ideal(jacobian ideal f)+f),Sgood);
assert(J == sub(intersect P,Sgood));
-- As claimed.
--
-- Thus, the pencil given by f generically defines admissible octics and so
    the dimension of this component is 1.
--
-- A note aside for the record: The (reduced) bad locus is given by the three
     points (1;0),(2;-1),(1;-1) (in PP^1 = Proj QQ[a_0,a_1]). (This is not a
     copy-paste mistake, the points are really the same in both cases.)
-- The degenerate conics are given by:
--     1. The two lines, the conic, and twice a conic through all four
--        points, tangent at P_2 (--> member of M^2 with T_{2,3,infty}
--        + T_{2,4,infty} + 2T_{2,6,infty});
--     2. The conic once and all three lines twice (--> member of M^3
--        with 3T_{2,3,infty}+3T_{2,infty,infty});
--     3. Both lines (once each), the conic twice and the line joining
--        P_2 and P_3 twice (--> member of M^3 with T_{2,3,infty} +
--        T_{2,4,infty} + T_{2,infty,infty}).
-- Observe that 3. is exactly the kind of degeneration that occurred in II
    above. Thus, in the closures of the components in the full moduli space
    meet in the stratum M^(2).
--
```

```
-- EOF 1112.m2 -------------------------------------------------------
```

Listing 8: The strata parametrising the degenerate versions of $\mathfrak{N}_{222}$

```
----------------------------------------------------------------------
-- M2-script to study octic curves in PP^2 with 3 quadruple-points, at least
    one of which is degenerate. Explicitly: N_{222bar}, N_{22bar^2} and N_{2
    bar^3}
--
-- Filename degenerate222.m2
----------------------------------------------------------------------
--
clearAll();
load "octicsFunctions.m2";
--
-- Notation:
-- A = QQ[t_0,t_1,t_2] is the coordinate ring of rational AA^3,
-- S = A[x,y,z] is the homogeneous coordinate ring of the trivial PP^2-family
     over AA^2. We will consider octics with certain singularities in the
    points P_i = V(L_i,T_i), i = 0,1,2, sometimes with a special or
    distinguished tangent line defined by T_i, where P_0 = (0;0;1), P_1 =
    (1;0;0), P_2 = (0;1;0) and T_0 = y - t_0 x, T_1 = z - t_1 y, T_2 = x -
    t_2 z.
--
-- It is enough to consider those tangents since up to automorphisms, the
    only configurations we miss are those where two degenerate quadruple-
    poins share a special tangent line, which is impossible on a reduced
    octic.
--
A = QQ[t_0,t_1,t_2];
S = A[x,y,z];
L = {x,y,z};
T = {y-t_0*L_0, z-t_1*L_1, x-t_2*L_2};
F = {z,x,y}; -- aux. localisation variables
P = apply(L,T,(f,g)->trim ideal(f,g));
P4 = apply(P,i->i^4);
Pd4 = apply(3,i->getDegenerateQuadrupleIdeal(L_i,T_i,F_i));
--
-- Preparation: The universal octic form with quadruple-points at P_0, P_1
    and P_2:
(RS,inc,params,f) = universalFamily(8,intersect P4);
assert(#params == 15);
-- And the degeneracy conditions:
M = apply(3, i-> tangencyCond(f, inc L_i, inc T_i, inc F_i,
        Multiplicity=>4, Degree=>2));
assert all(M,m->numgens target m == 2);
--
-- I.1  Two ordinary and one degenerate quadruple-point (N_{222bar})
-- Imposing degeneracy at P_0 means taking the kernel of M_0.
assert isSurjective M_0;
-- Thus, the stratum N_{222bar} is irreducible and its dimension is computed
    as 15 - 2 + 1 - 3 = 11. (Here, -2 comes from the rank of M_0, +1 is
    since t_0 is a parameter and -3 is the dimension of the stabiliser sub-
    group of GL(3,CC) acting on the kernel.) It is easy to see that there
    actually exist such curves, but it also follows from I.4 below.
--
-- I.2  One ordinary and two degenerate quadruple-points (N_{22bar^2})
-- Imposing degeneracy at P_0 and P_1 means taking the kernel of
N = M_0 || M_1;
assert isSurjective N;
-- Therefore, N_{22bar^2} is irreducible as well and of dimension 15 - 4 + 2
    - 3 = 10. (Non-empty by I.4 below.)
```

```
--
-- I.3  Three degenerate quadruple-points (N_{2bar^3})
N = M_0 || M_1 || M_2;
assert isSurjective N;
-- Therefore, N_{2bar^3} is irreducible as well and of dimension 15 - 6 + 3 -
     3 = 9. (Inhabited by I.4 below.)
--
-- I.4  The strata are non-empty
-- From the line of arguments above it follows that appropriate deformations
    of a member of N_{2bar^3} will define members of the other two strata.
    We set t_0 = 1, t_1 = 2 and t_2 = 3; the choice is arbitrary.
S' = QQ[gens S];
special = map(S',S,gens(S') | {1,2,3});
-- The next three (commented) lines were used to produce the 'random' element
     fRand hard-coded below.
--use S;
--J = sub(idealFromKernel(N,f,params),S);
--J' = special J;
--fRand = randomElement(J');
use S';
fRand = -(992/3)*x^4*y^4+(9364/15)*x^4*y^3*z-(10/9)*x^3*y^4*z-(4564/15)*x^4*y
    ^2*z^2-(3/5)*x^3*y^3*z^2+(26800/3)*x^2*y^4*z^2+(7/5)*x^4*y*z^3+(3/5)*x
    ^3*y^2*z^3+(2/15)*x^2*y^3*z^3-17858*x*y^4*z^3+18*x^4*z^4-(2692/45)*x^3*y
    *z^4+(2414/45)*x^2*y^2*z^4+(8/45)*x*y^3*z^4-12*y^4*z^4;
-- First of all, it is singular only at P_0, P_1 and P_2:
expected = intersect apply(P, i-> special i);
sing = ideal jacobian matrix fRand;
assert(radical sing == expected);
-- On the other hand, the quadruple-points at these points are degenerate,
    but as mild as possible, i.e., singularities of type X_{10}:
assert(isX10(fRand,x,y));
assert(isX10(fRand,z,y));
assert(isX10(fRand,z,x));
-- Thus, fRand defines an admissible member of N_{2bar^3}.
--
-- EOF degenerate222.m2 ------------------------------------------------
```

Listing 9: The strata parametrising the degenerate versions of $\mathfrak{N}_{111}$

```
----------------------------------------------------------------------------
-- M2-script to study octics in PP^2 with three [3;3]-points, at least one
    degenerate, i.e., the strata N_{111bar}, N_{11bar^2} and N_{1bar^3}
--
-- Filename degenerate111.m2
----------------------------------------------------------------------------
--
clearAll();
load "octicsFunctions.m2";
--
-- We will have to run through this twice, since the two different
    configurations, the special case with tangents along a conic and the
    general case, work better on different base rings.
--
-- Notation:
-- A0 = QQ[s_0,s_1,s_2,t_2] is the affine coordinate ring of AA^4 over the
    rational numbers,
-- S0 = A0[x,y,z] is the homogeneous coordinate ring of the projective plane
    over AA^4 and we consider the points P_0 = (0;0;1), P_1 = (1;0;0) and
    P_2 = (0;1;0) on the distinguished lines {y = -x}, {z = -y} and {x = t_2
     z}, respectively.
--
-- The common setup:
```

```
A0 = QQ[s_0,s_1,s_2,t_2];
S0 = A0[x,y,z];
L = {x,y,z};
T = {y+L_0,z+L_1,x-t_2*L_2};
F = {z,x,y};
P = apply(L,T,pair->trim ideal pair);
P33 = apply(3,i->get33Ideal(L_i,T_i,F_i));
PwT = apply(T,P,(l,p)->trim(p^2+l));
conic = super basis(2,intersect(PwT_0,PwT_1,P_2));
assert(conic == matrix(x*y+x*z+y*z));
conic = conic_(0,0);
assert(conic%PwT_2 == (t_2+1)*y*z);
-- Thus, the special case is t_2 = -1.
--
-- I  The general case t_2 != -1.
-- We also assume t_2 != 0, which is valid since it corresponds to the case
    where the distinguished tangent points towards another [3;3]-point, but
    then any octic contains the line twice.
A = A0[r,Degrees=>{ -1 }]/((t_2+1)*t_2*r-1);
locA = map(A,A0);
use S0;
--
-- Preparation, generating the universal family with three [3;3]-points in
    general position and the degeneracy conditions:
(RS,inc,params,f) = universalFamily(8,intersect P33);
assert(#params == 9);
M0 = apply(3,i->
  directedDegConditions33Point(f,inc L_i, inc T_i, inc F_i, s_i));
M = apply(M0,m->locA m);
--
-- I.1  One of the [3;3]-points degenerate (N'_{111bar})
--
assert(isSurjective M_0 and numgens target M_0 == 2);
-- Hence, the stratum is irreducible. The kernel of M_0 has dimension  9 - 2
    = 7; thus, the stratum is of dimension 7 + 2 - 1 = 8. (+2 from t_2 and
    s_0, -1 from rescaling.) It is easy to see that this stratum is
    inhabited, but it also follows from I.4 below.
--
-- I.2  Two of the [3;3]-points degenerate (N'_{11bar^2})
--
N = M_0 || M_1;
assert(isSurjective N and numgens target N == 4);
-- Thus, the stratum is irreducible and of dimension 5 + 3 - 1 = 7, since the
     kernel dimension is 9 - 4 = 5. It is non-empty by I.4.
--
-- I.3  Three degenerate [3;3]-points (N'_{1bar^3})
-- Sanity check: Since we have excluded t_2 = 0, -1, M_2 is surjective
assert isSurjective(M_2);
--
N = M_0 || M_1 || M_2;
-- The rank of N drops for one set of parameters:
--droppingRankCond = minors(numgens target N,N);
--droppingRankCond = ideal gens gb droppingRankCond;
-- took more than a day. Therefore, hard-coded:
droppingRankCond = ideal(
    s_0-(-t_2-19)/2,  s_1-(t_2-1)/2,  s_2-(19*t_2+1)/2,
    t_2^2+18*t_2+1,  16*r+17*t_2+305);
-- Note: t_2^2+18*t_2+1 = (t_2-(4*sqrt(5)-9))*(t_2-(-4*sqrt(5)-9)); moreover,
     the last generator is superfluous.
-- We investigate what happens for those special parameters.
B = A/droppingRankCond;
pr = map(B,A);
```

```
N' = pr N;
-- Since mingens fails to find the minimal number of generators even though B
     is a field, we use basis, which is ok since B is a field:
K' = super basis kernel N';
BS = B[gens S0];
BRS = BS[params];
f' = sub(f,BRS);
I = sub(idealFromImage(K',f',gens BRS),BS);
-- It turns out that all members of I contain a certain quintic:
quintic = 8*x^3*y^2+(t_2+21)*x^2*y^3+16*x^3*y*z+(-11*t_2+33)*x^2*y^2*z+(-6*
    t_2+2)*x*y^3*z+8*x^3*z^2+(-33*t_2+11)*x^2*y*z^2-88*t_2*x*y^2*z^2+(-55*
    t_2-3)*y^3*z^2+(-21*t_2-1)*x^2*z^3+(-42*t_2-2)*x*y*z^3+(-21*t_2-1)*y^2*z
    ^3,8*t_2*x^3*y^2+(3*t_2-1)*x^2*y^3+16*t_2*x^3*y*z+(231*t_2+11)*x^2*y^2*z
    +(110*t_2+6)*x*y^3*z+8*t_2*x^3*z^2+(605*t_2+33)*x^2*y*z^2+(1584*t_2+88)*
    x*y^2*z^2+(987*t_2+55)*y^3*z^2+(377*t_2+21)*x^2*z^3+(754*t_2+42)*x*y*z
    ^3+(377*t_2+21)*y^2*z^3;
assert((gens I)%quintic == 0);
-- I claim that this quintic has singularities of type A_4 at the point P_0,
    P_1 and P_2. Thus, the union with a general cubic passing through the
    P_i in prescribed tangent direction is a singularity of type J_{2,1}.
    Since there is a four-dimensional space of such cubics, this is what is
    going on geometrically.
-- Since the base field B is not QQ, we have to do this by hand.
D = B[u,v];-- The affine plane over B
sigma = map(D,D,{u,u*v});-- Blowing up the origin
-- At P_0:
l = map(D,BS,{u,v,1});-- A naive localisation away from {z = 0}
-- The distinguished tangent line localised:
tgt = (sigma l sub(T_0,BS))//u;
-- Obtaining the strict transform of the quintic:
totalTrans = sigma l(quintic);
strictTrans = totalTrans//u^2;-- Indeed:
assert(strictTrans*u^2 == totalTrans);
-- And it has an infinitely near double-point:
assert(ideal sub(strictTrans,{u=>0}) == ideal(tgt^2));
-- Moving the double-point to the origin:
tmp = sub(strictTrans,{v=>v-sub(tgt,{v=>0})});
assert(tmp%(ideal(u,v))^2 == 0);
totalTrans' = sigma tmp;
strictTrans' = totalTrans'//u^2;-- Indeed:
assert(strictTrans'*u^2 == totalTrans');
sndTgt = v-l(s_0);
assert(ideal sub(strictTrans',{u=>0}) == ideal(sndTgt^2));
-- However, the strict transform is smooth and tangent along the exceptional
     line:
sing = ideal(jacobian matrix strictTrans')+ideal(strictTrans');
assert(sing == ideal(1_D));
assert(strictTrans'%ideal(u,sndTgt^2) == 0);
--
-- Now the same at P_1:
l = map(D,BS,{1,u,v});-- A naive localisation away from {x = 0}
-- The distinguished tangent line localised:
tgt = (sigma l sub(T_1,BS))//u;
-- Obtaining the strict transform of the quintic:
totalTrans = sigma l(quintic);
strictTrans = totalTrans//u^2;-- Indeed:
assert(strictTrans*u^2 == totalTrans);
-- And it has an infinitely near double-point:
assert(ideal sub(strictTrans,{u=>0}) == ideal(tgt^2));
-- Moving the double-point to the origin:
tmp = sub(strictTrans,{v=>v-sub(tgt,{v=>0})});
assert(tmp%(ideal(u,v))^2 == 0);
```

```
totalTrans' = sigma tmp;
strictTrans' = totalTrans'//u^2;-- Indeed:
assert(strictTrans'*u^2 == totalTrans');
sndTgt = v-l(s_1);
assert(ideal sub(strictTrans',{u=>0}) == ideal(sndTgt^2));
-- However, the strict transform is smooth and tangent along the exceptional
    line:
sing = ideal(jacobian matrix strictTrans')+ideal(strictTrans');
assert(sing == ideal(1_D));
assert(strictTrans'%ideal(u,sndTgt^2) == 0);
--
-- And a last time, at P_2:
l = map(D,BS,{v,1,u});-- A naive localisation away from {y = 0}
-- The distinguished tangent line localised:
tgt = (sigma l sub(T_2,BS))//u;
-- Obtaining the strict transform of the quintic:
totalTrans = sigma l(quintic);
strictTrans = totalTrans//u^2;-- Indeed:
assert(strictTrans*u^2 == totalTrans);
-- And it has an infinitely near double-point:
assert(ideal sub(strictTrans,{u=>0}) == ideal(tgt^2));
-- Moving the double-point to the origin:
tmp = sub(strictTrans,{v=>v-sub(tgt,{v=>0})});
assert(tmp%(ideal(u,v))^2 == 0);
totalTrans' = sigma tmp;
strictTrans' = totalTrans'//u^2;-- Indeed:
assert(strictTrans'*u^2 == totalTrans');
sndTgt = v-l(s_2);
assert(ideal sub(strictTrans',{u=>0}) == ideal(sndTgt^2));
-- However, the strict transform is smooth and tangent along the exceptional
    line:
sing = ideal(jacobian matrix strictTrans')+ideal(strictTrans');
assert(sing == ideal(1_D));
assert(strictTrans'%ideal(u,sndTgt^2) == 0);
-- As claimed.
--
-- This means that we could have extra components where the degeneracy
    conditions are satisfied. But below we show that all elements of these
    potential extra components are limits of elements of the main component.
     Thus, there is really just one component.
--
-- We have to investigate which of the curves parametrised by the kernel of N
    ' are degenerations of curves parametrised by the kernel of N. For this,
     we consider the four 1-parameter degenerations where three of the four
    parameters t_2, s_0, s_1, s_2 are fixed.
use A;
conditions = (droppingRankCond_*)_{0..3};
parameterIdeals = apply(4,i->sub(ideal(drop(conditions,{i,i})),A));
rings = apply(parameterIdeals,quotient);
mapsRA = apply(rings,R->map(R,A));
mapsBR = apply(rings,R->map(B,R));
assert all(mapsBR,mapsRA,m->m_0 * m_1 === pr);-- (sanity check)
matrices = apply(mapsRA,m->m N);
kernels = apply(matrices,kernel);
limits = apply(kernels,mapsBR,(i,p)->p(i));
degenerations = sum limits;
assert(degenerations == kernel N');
-- Thus, the potential extra components are actually contained in the main
    component, which is, therefore, still irreducible.
--
-- We compute the dimension of N'_{1bar^3} as 3 + 4 - 1 = 6; the locus
    corresponding to the special parameters is of dimension 4 - 1 = 3.
```

```
--
-- I.4  The strata are not empty
-- It follows from our line of arguments so show that N'_{1bar^3} is
    inhabited, since appropriate degenerations will become members of the
    other strata. In fact, we have seen in I.3 that this component is indeed
     non-empty, but to see that the dimension is really 6 (as opposed to 3),
     we also need a curve for parameters where N has maximal rank, e.g., t_2
     = 2 and s_0 = s_1 s_2 = -1.
-- The following commented lines show how the hard-coded element fRand was
    obtained:
--B = symbol B;
--B = toField(A/(t_2-2,s_0+1,s_1+1,s_2+1));
--prB = map(B,A);
--N' = prB N;
--BS = B[gens S0];
--BRS = BS[params];
--f' = sub(f,BRS);
--I = sub(idealFromKernel(N',f',gens BRS),BS);
--fRand = randomElement(I);
-- This resulted in the following:
BS = QQ[gens S0];
specialise = map(BS, S0, gens(BS) | {-1,-1,-1,2});
use BS;
fRand = (3942/7)*x^5*y^3-(2336/7)*x^4*y^4+2628*x^3*y^5+(11826/7)*x^5*y^2*z
    +(6679/21)*x^4*y^3*z+(21884/21)*x^3*y^4*z-15768*x^2*y^5*z+(11826/7)*x^5*
    y*z^2-(22663/14)*x^4*y^2*z^2-(97338/7)*x^3*y^3*z^2-(16846/7)*x^2*y^4*z
    ^2+31536*x*y^5*z^2+(3942/7)*x^5*z^3-(38686/7)*x^4*y*z^3-(63821/7)*x^3*y
    ^2*z^3+(180575/7)*x^2*y^3*z^3+7800*x*y^4*z^3-21024*y^5*z^3-(136753/42)*x
    ^4*z^4+(77075/21)*x^3*y*z^4+(16275/2)*x^2*y^2*z^4-(162956/21)*x*y^3*z
    ^4-(188392/21)*y^4*z^4+468*x^3*z^5+1404*x^2*y*z^5+1404*x*y^2*z^5+468*y
    ^3*z^5;
-- First of all, the singular locus is supported at P_0, P_1 and P_2:
expected = intersect apply(P,i->specialise i);
sing = ideal( jacobian matrix fRand )+fRand;
assert(radical sing == expected);
-- Furthermore, at all three points, we have singularites of type J_{2,1}, i.
    e., the mildest form of degenerate [3;3]-points.
assert(isJ21(fRand, specialise T_0, x, y, z));
assert(isJ21(fRand, specialise T_1, y, z, x));
assert(isJ21(fRand, specialise T_2, x, z, y));
-- This concludes the discussion about the case of general distinguished
    tangent directions.
--
-- II  The special case t_2 = -1.
-- That is, all three tangents are along the conic. This conic is then
    contained in the octic and the residual septic meets it in three points
    of multiplicity at least four, but 2*6 = 12 = 3*4, so that the
    intersection multiplicity has to be exactly four at all three points,
    assuming the octic is reduced. But if s_0, or s_1 or s_2 equals 1, then
    the intersection multiplicity is at least 5 at some point and the octic
    is not admissible. Therefore, we can exclude these values right from the
     start.
use A0;
A = A0[r,Degrees=>{ -1 }]/(r*(s_0-1)*(s_1-1)*(s_2-1)+1,t_2+1);
prA = map(A,A0);
S = A[gens S0];
prS = map(S,S0);
--
-- Preparation, generating the universal family with three [3;3]-points in
    special position and the degeneracy conditions:
I = intersect apply(P33,i-> prS i);
(RS,inc,params,f) = universalFamily(8, I);
```

```
assert(#params == 10);
M = apply(3,i->directedDegConditions33Point(f,
        inc prS L_i, inc prS T_i, inc prS F_i, s_i));
assert all(M,m->numgens target m == 2);
assert(
    isSurjective(M_0) and
    isSurjective(M_0 || M_1) and
    isSurjective(M_0 || M_1 || M_2)
);
-- Therefore, the conditions, two for each point, are independent and we get
    linear spaces of dimension 10 - 2 = 8, 10 - 4 = 6 and 10 - 6 = 4,
    respectively. Since the stabiliser group of the configuration in the
    plane is finite, but s_0, s_1 and s_2 contribute one dimension each, the
     corresponding strata of the configuration are irreducible and of
    dimension 8 + 1 - 1 = 8, 6 + 2 - 1 = 7 and 4 + 3 - 1 = 6, respectively.
--
-- As in I.4, once we have constructed an inhabitant for the smallest stratum
    , it follows that each stratum is inhabited. Specialising now means
    choosing values for s_0, s_1 and s_2, different from 1, say s_0 = s_1 =
    s_2 = -1. Then r = 1/8.
BS = QQ[gens S0];
specialise = map(BS, S, gens(BS) | {1/8,-1,-1,-1,-1});
-- Again, we hard-coded an octic form obtained 'randomly':
--I = sub(idealFromKernel(M_0 || M_1 || M_2,f,params),S);
--I' = specialise I;
--fRand = randomElement(I');
-- This resulted in:
fRand = -(59/3)*x^5*y^3-(1/3)*x^4*y^4-(59/3)*x^3*y^5-59*x^5*y^2*z-(278/9)*x
    ^4*y^3*z+9*x^3*y^4*z-59*x^2*y^5*z-59*x^5*y*z^2-(331/3)*x^4*y^2*z^2-(8/3)
    *x^3*y^3*z^2+(28/3)*x^2*y^4*z^2-59*x*y^5*z^2-(59/3)*x^5*z^3-(388/3)*x^4*
    y*z^3-(451/3)*x^3*y^2*z^3-(92/3)*x^2*y^3*z^3-(29/3)*x*y^4*z^3-(59/3)*y
    ^5*z^3-(446/9)*x^4*z^4-(416/3)*x^3*y*z^4-(415/3)*x^2*y^2*z^4-(530/9)*x*y
    ^3*z^4-(29/3)*y^4*z^4-(59/3)*x^3*z^5-59*x^2*y*z^5-59*x*y^2*z^5-(59/3)*y
    ^3*z^5;
-- First of all, the singular locus is supported at P_0, P_1 and P_2:
expected = intersect apply(P,i->specialise prS i);
sing = ideal( jacobian matrix fRand )+fRand;
assert(radical sing == expected);
-- Furthermore, at all three points, we have singularities of type J_{2,1}, i
    .e., the mildest degenerate [3;3]-points possible:
assert(isJ21(fRand, specialise prS T_0, x, y, z));
assert(isJ21(fRand, specialise prS T_1, y, z, x));
assert(isJ21(fRand, specialise prS T_2, x, z, y));
-- As claimed.
--
-- EOF degenerate111.m2 ---------------------------------------------
```

Listing 10: The strata parametrising the degenerate versions of $\mathfrak{N}_{122}$

```
--------------------------------------------------------------------
-- M2-script studying plane octics with a [3;3]-point and two quadruple-
    points, at least one degenerate. That is, the following strata: N_{122
    bar}, N_{12bar^2}, N_{1bar22}, N_{1bar22bar}, N_{1bar2bar^2}.
--
-- Filename degenerate122.m2
--------------------------------------------------------------------
--
clearAll();
load "octicsFunctions.m2";
--
-- From 122.m2 we know that in any case, there will be disjoint components
    corresponding to whether one of the quadruple-point lies on the
```

```
    distinguished tangent line of the [3;3]-point. To limit the number of
    parameters, we go through this twice, once for each case.
--
-- The general set-up:
-- A0 = QQ[s,t_0,t_1,t_2] is the affine coordinate ring of rational affine 4-
    space AA^4,
-- S0 = A0[x,y,z] is the homogeneous coordinate ring of the trivial
    projective plane bundle over AA^4.
-- We consider the plane points P_0 = (0;0;1), P_1 = (1;0;0) and P_2 =
    (0;1;0) with distinguished tangent lines {y = t_0 x}, {z = t_1 y} and {x
     - t_2 z}, respectively.
--
-- In either case (quadruple-point on or off distinguished tangent), we can
    fix at least three points and two tangents. In each case, we put the
    [3;3]-point at P_2. Then t_2 != 0 gives to the component N' and t_2 = 0
    gives rise to N''.
--
A0 = QQ[s,t_0,t_1,t_2];
S0 = A0[x,y,z];
L = {x,y,z};
T = {y-t_0*L_0,z-t_1*L_1,x-t_2*L_2};
F = {z,x,y};
P = apply(L,T,pair->trim ideal pair);
P33 = get33Ideal(L_2,T_2,F_2);
--
-- I  General distinguished tangent
-- First, we walk through this in general position, e.g., t_2 = 1, so that
    there is no quadruple-point on the distinguished tangent line of the
    [3;3]-point.
--
A = A0/(t_2-1);
pr = map(A,A0);
S = A[gens S0];
prS = map(S,S0);
--
I = intersect(prS P33, prS P_0^4, prS P_1^4);
(RS,inc,params,f) = universalFamily(8, I);
assert(#params == 13);
-- This is the universal family of octic forms where the associated curves
    have a [3;3]-point at P_2 with distinguished tangent line {x = z} and
    two quadruple-points off this line (at P_1 and P_2).
--
-- I.1  One degenerate singularity
-- We have to understand the kernels of the following matrices encoding the
    degeneracy conditions.
M0 = tangencyCond(f, inc prS L_0,inc prS T_0, inc prS F_0,
    Multiplicity=>4, Degree=>2);
M1 = tangencyCond(f, inc prS L_1,inc prS T_1, inc prS F_1,
    Multiplicity=>4, Degree=>2);
M2 = directedDegConditions33Point(f,
    inc prS L_2, inc prS T_2, inc prS F_2, s);
-- All three are of maximal rank 2 everywhere:
assert(numgens target M0 == 2 and isSurjective M0);
assert(numgens target M1 == 2 and isSurjective M1);
assert(numgens target M2 == 2 and isSurjective M2);
-- This implies that N'_{122bar} and N'_{1bar22} are irreducible of dimension
     13 - 2 + 1 - 2 = 10. (Here, 13 - 2 is the rank of the kernel, +1 is for
     the parameter, either s or t_0, and 2 is the dimension of the
    stabiliser sub-group of GL(3,CC)). That it is indeed not empty will be
    shown in I.4 below.
--
-- I.2  Two degenerate singularities
```

```
-- Now we have to consider the common kernels of pairs of the above matrices:
M01 = M0 || M1;-- (both quadruple-points)
M02 = M0 || M2;-- (one quadruple-point and the [3;3]-point)
-- Again, they are of maximal rank 4 everywhere:
assert(isSurjective M01 and isSurjective M02);
-- Thus, N'_{12bar^2} and N'_{1bar22bar} are irreducible and of dimension 13
    - 4 + 2 - 2 = 9. (See I.4 below for non-emptiness.)
--
-- I.3  All three degenerate (N'_{1bar2bar^2})
-- We consider the kernel of the common kernel of all three matrices:
N = M0 || M1 || M2;
assert(isSurjective N);
-- It has maximal rank 6 everywhere and so the stratum is irreducible of
    dimension 13 - 6 + 3 - 2 = 8. That it is indeed inhabited will be the
    subject of the next section.
--
-- I.4  The strata are non-empty
-- It remains to produce inhabitants. From our line of arguments, it follows
    that it is enough to construct an inhabitant in the case that all three
    points are degenerate and for fixed parameters, say s = 1, t_0 = -1 and
    t_1 = -1:
use A;
S' = QQ[gens S];
pr' = map(QQ, A, {1,-1,-1,1});
prS' = map(S', S, gens(S') | {1,-1,-1,1});
-- The following commented lines were used to obtain the hard-coded example
    fRand below:
--J = prS' sub(idealFromKernel(N,f,params),S);
--fRand = randomElement(J);
use S';
fRand = (16/45)*x^4*y^4-(21/4)*x^3*y^5+(2291/315)*x^4*y^3*z-(15/7)*x^3*y^4*z
    +(63/4)*x^2*y^5*z+(1304/63)*x^4*y^2*z^2-(2/15)*x^3*y^3*z^2-(5/3)*x^2*y
    ^4*z^2-(63/4)*x*y^5*z^2+21*x^4*y*z^3-(1138/35)*x^3*y^2*z^3+(857/140)*x
    ^2*y^3*z^3+(2627/315)*x*y^4*z^3+(21/4)*y^5*z^3+(758/105)*x^4*z
    ^4+(2759/252)*x^3*y*z^4-(2927/630)*x^2*y^2*z^4-(2387/180)*x*y^3*z
    ^4-(171/35)*y^4*z^4;
-- First of all, the singular locus is as small as possible:
expected = prS' prS intersect P;
sing = ideal jacobian matrix fRand;
assert(radical sing == expected);
-- Moreover, the singularities are degenerate, but in the mildest way they
    can, i.e., of type J_{2,1} or X_{10}, respectively.
assert(isJ21(fRand, prS' prS T_2, x, z, y));
assert(isX10(fRand, z, y));
assert(isX10(fRand, y, x));
-- Thus, the octic defined by fRand defines a member of the most degenerate
    stratum and an appropriate deformation defines a member of the other
    strata.
--
-- II  With special distinguished tangent
-- Setting t_2 = 0, the distinguished tangent direction at the [3;3]-point at
     P_2 points towards the quadruple-point at P_0.
use A0;
A = symbol A; S = symbol S; RS = symbol RS;
A = A0/(t_2);
pr = map(A,A0);
S = A[gens S0];
prS = map(S,S0);
--
I = intersect(prS P33, prS P_0^4, prS P_1^4);
(RS,inc,params,f) = universalFamily(8, I);
assert(#params == 14);
```

```
--
-- II.1  One degenerate singularity
-- If we let one quadruple-point degenerate, we have two cases to consider,
    namely, whether it is the one on the distinguished tangent or the other
    one. We denote by N''a_{122bar} be the component for the latter case (
    P_1 degenerate) and N''b_{122bar} for the former case (P_0 degenerate);
    analogously for N''_{1bar22bar} in II.2.
--
-- We have to understand the kernels of the following matrices encoding the
    degeneracy conditions.
M0 = tangencyCond(f, inc prS L_0,inc prS T_0, inc prS F_0,
    Multiplicity=>4, Degree=>2);
M1 = tangencyCond(f, inc prS L_1,inc prS T_1, inc prS F_1,
    Multiplicity=>4, Degree=>2);
M2 = directedDegConditions33Point(f,
    inc prS L_2, inc prS T_2, inc prS F_2, s);
-- Imposing degeneracy is independent for all parameters at the quadruple-
    points, but not at the [3;3]-point:
assert(numgens target M0 == 2 and isSurjective M0);
assert(numgens target M1 == 2 and isSurjective M1);
assert(numgens target M2 == 2 and not isSurjective M2);
-- This already shows that N''a_{122bar} and N''b_{122bar} are irreducible
    and of dimension 14 - 2 + 1 - 3 = 10. For N''_{1bar22}, we have to see
    what happens if the rank drops:
droppingRankCond = minors(2, mingens image M2);
assert(radical droppingRankCond == ideal s);
-- That is, the rank drops if and only if the second order direction of the
    degenerate [3;3]-point is trivial. In this case, the line has to be
    contained twice:
J = sub(idealFromKernel(sub(M2,{s=>0}),sub(f,{s=>0}),params),S);
assert((gens J)%(prS T_2)^2 == 0);
-- Thus, this case is neglected in the stratum anyways, It follows that the
    component N''_{1bar22} is irreducible and 10-dimensional as well.
--
-- That these three components are actually inhabited is content of Section
    II.4 below.
--
-- II.2  Two degenerate singularities
-- We have to investigate the kernels of the common kernels of pairs of the
    above matrices.
M01 = M0 || M1;--> N''_{12bar^2}
M02 = M0 || M2;--> N''b_{1bar22bar}
M12 = M1 || M2;--> N''a_{1bar22bar}
assert isSurjective M01;
-- Therefore, N''_{12bar^2} is irreducible and its dimension is 14 - 4 + 2 -
    3 = 9. We will conclude the same for N''a_{1bar22bar} and N''b_{1
    bar22bar} once we have seen that their rank-dropping parameters are
    neglected anyways, for the same reason as in II.1:
droppingRankCond = minors(4,mingens image M02);
assert(radical droppingRankCond == ideal s);
droppingRankCond = minors(4,mingens image M12);
assert(radical droppingRankCond == ideal s);
--
-- That these three components are actually inhabited is content of Section
    II.4 below.
--
-- II.3  All three degenerate
-- We consider the kernel of the common kernel of all three matrices:
N = M0 || M1 || M2;
-- Of course, it is not surjective, but the locus where it is not is
    neglected for the same reason as in II.1:
droppingRankCond = minors(6,mingens image N);
```

```
assert(radical droppingRankCond == ideal s);
-- It follows that the component N''_{1bar2bar^2} is irreducible and of
    dimension 14 - 6 + 3 - 3 = 8.
--
-- II.4  The strata are non-empty
-- It remains to produce inhabitants. From our line of arguments, it follows
    that it is enough to construct an inhabitant in the case that all three
    points are degenerate and for fixed parameters, say s = 1, t_0 = -1 and
    t_1 = -1:
use A;
S' = QQ[gens S];
pr' = map(QQ, A, {1,-1,-1,0});
prS' = map(S', S, gens(S') | {1,-1,-1,0});
-- The following commented lines were used to obtain the hard-coded example
    fRand below:
--J = prS' sub(idealFromKernel(N,f,params),S);
--fRand = randomElement(J);
use S';
fRand = -(21/80)*x^4*y^4-(35/24)*x^3*y^5+(19/480)*x^4*y^3*z+(1/36)*x^3*y^4*z
    +(3/5)*x^4*y^2*z^2+(3/2)*x^3*y^3*z^2+(35/12)*x^2*y^4*z^2+(1/32)*x^4*y*z
    ^3-(2/15)*x^3*y^2*z^3+(40/27)*x^2*y^3*z^3-(4/15)*x^4*z^4-(239/120)*x^3*y
    *z^4-(191/60)*x^2*y^2*z^4-(35/24)*x*y^3*z^4;
-- First of all, the singular locus is as small as possible:
expected = prS' prS intersect P;
sing = ideal jacobian matrix fRand;
assert(radical sing == expected);
-- Moreover, the singularities are degenerate, but in the mildest way they
    can, i.e., of type J_{2,1} or X_{10}, respectively.
assert(isJ21(fRand, prS' prS T_2, z, x, y));
assert(isX10(fRand, z, y));
assert(isX10(fRand, y, x));
-- Thus, the octic defined by fRand defines a member of the most degenerate
    stratum and an appropriate deformation defines a member of the other
    strata.
--
-- EOF degenerate122.m2 ----------------------------------------------
```

Listing 11: The strata parametrising the degenerate versions of $\mathfrak{N}_{112}$

```
----------------------------------------------------------------------
-- M2-script to study octics in PP^2 with two [3;3]-points and one quadruple-
    point, at least one degenerate, i.e., the strata N_{112bar}, N_{11bar2},
     N_{11bar2bar}, N_{1bar1bar2} and N_{1bar1bar2bar}.
--
-- Filename degenerate112.m2
----------------------------------------------------------------------
--
clearAll();
load "octicsFunctions.m2";
--
-- Notation:
-- A0 is the affine coordinate ring of AA^5 with coordinates s_0, s_1, t_0,
    t_1 and t_2,
-- S0 = A0[x,y,z] is the homogeneous coordinate ring of the projective plane
    over AA^5. We consider three points with distinguished lines:
-- P_0 = (0;0;1), T_0 = y - t_0 x
-- P_1 = (1;0;0), T_1 = z - t_1 y
-- P_2 = (1;1;1), T_3 = y - x - t_3 (z - x)
-- We will consider quadruple-points and [3;3]-points, both ordinary and
    degenerate, supported at those points and with distinguished tangent
    direction spanned by their distinguished lines as indicated above. The
    variables s_0 and s_1 will be there to parametrise the higher degeneracy
```

```
     directions of the [3;3]-points.
--
-- Recall that there are three disjoint components in N_{112} which
    correspond to the cases that none, one ot both distinguished tangent
    directions at the [3;3]-points points towards the quadruple-point. We
    will run through this three times since each case requires a different
    base ring.
--
-- The common setup:
A0 = QQ[s_0,s_1,t_0,t_1,t_2];
S0 = A0[x,y,z];
L = {x,z,z-x};
T = {y-t_0*L_0,y-t_1*L_1,y-x-t_2*L_2};
F = {z,x,z};
P = apply(L,T,pair->trim ideal pair);
P33 = apply(2,i->get33Ideal(L_i,T_i,F_i));
-- Q = getDegenerateQuadrupleIdeal(L_2,T_2,F_2);
--
-- I  Distinguished tangents in general direction (t_0, t_1 != 1).
-- If we let t_2 vary (if necessary), we have enough automorphisms left to
    fix t_0 and t_1 to, say, t_0 = t_1 = -1.
A = A0/(t_0+1,t_1+1);
pr = map(A,A0);
S = A[gens S0];
prS = map(S,S0);
I = intersect(prS P33_0, prS P33_1, prS P_2^4);
(RS,inc,params,f) = universalFamily(8,I);
assert(#params == 11);
--
M0 = directedDegConditions33Point(f,
    inc prS L_0, inc prS T_0, inc prS F_0, s_0);
M1 = directedDegConditions33Point(f,
    inc prS L_1, inc prS T_1, inc prS F_1, s_1);
M2 = tangencyCond(f,
    inc prS L_2,inc prS T_2,inc prS F_2,
    Multiplicity=>4, Degree=>2);
-- I.1  One of the points degenerate
-- Imposing degeneracy at P_i means taking the linear system defined by the
    kernel of Mi.
assert(numgens target M0 == 2 and isSurjective M0);-- N'_{11bar2}
assert(numgens target M1 == 2 and isSurjective M1);-- N'_{11bar2}
assert(numgens target M2 == 2 and isSurjective M2);-- N'_{112bar}
-- Since the rank drops nowhere, both components, N'_{11bar2} and N'_{112bar
    }, are irreducible and of dimension 11 - 2 + 1 - 1 = 9. Here, 2 is the
    rank of the matrices, there is 1 extra parameter in each case (s_0, s_1,
     t_2) and we subtract 1 for rescaling.
--
-- I.2  Two of the points degenerate
-- Now we have to consider the common kernels of pairs of the matrices
M01 = M0 || M1;-- N'_{1bar1bar2}
M02 = M0 || M2;-- N'_{11bar2bar}
-- They are not surjective everywhere; where and why?
-- I.2.1 Both [3;3]-points degenerate
droppingRankCond = minors(4,mingens image M01);
assert(radical droppingRankCond == ideal(s_1-4,s_0-4));
-- Thus, the rank of M01 drops for s_0 = s_1 = 4 and we have to investigate
    what happens there. This case is neglectable since every octic
    corresponding to those parameters contains a special conic twice. In
    fact, it's the unique conic passing through the three points in
    prescribed tangent direction at P_0 and P_1 and this follows from
    intersection-theory, but we can also see it here:
M01special = sub(M01,{s_0=>4,s_1=>4});
```

```
Jspecial = sub(idealFromKernel(M01special,
        sub(f,{s_0=>4,s_1=>4}), params), S);
specialConic = sub(x*y-3*y^2+x*z+y*z,S);
assert((gens Jspecial)%specialConic^2 == 0);
-- For all other parameters, N has maximal rank 4, so that the component N'_
    {1bar1bar2} (non-empty by I.4 below) is irreducible of dimension 11 - 4
    + 2 - 1 = 8.
--
-- I.2.2 A [3;3]-point and a quadruple-point degenerate
droppingRankCond = minors(4,mingens image M02);
assert(radical droppingRankCond == ideal(2*t_2-1,s_0-4));
-- Thus, the rank of M02 drops for s_0 = 4, t_1 = 1/2. This case is
    neglectable as well since as in I.2.1, every octic satisfying the
    constraints for to these parameters contains the very same special conic
     twice:
M02special = sub(M02,{s_0=>4,t_2=>1/2});
Jspecial = sub(idealFromKernel(M02special,
        sub(f,{s_0=>4,t_2=>1/2}), params), S);
assert((gens Jspecial)%specialConic^2 == 0);
-- As in I.2.1 we conclude that N'_{11bar2bar} is irreducible and 8-
    dimensional; that it is not empty will be part of I.4 below.
--
-- I.3  All three singularities degenerate
-- We have to look at the common kernel of all three matrices.
N = M0 || M1 || M2;
-- Again, N is not surjective, but the parameters where it is not are
    irrelevant for us:
droppingRankCond = minors(6,mingens image N);
assert(radical droppingRankCond == intersect(
        ideal(s_1-4,s_0-4),
        ideal(2*t_2-1,s_0-4),
        ideal(2*t_2-1,s_1-4)));
-- These are the cases we have seen in I.2 (up to symmetry), where we already
     know that there is a conic contained twice. We conclude that the
    corresponding component is irreducible and has dimension 11 - 6 + 3 - 1
    = 7, once we have seen that it is not empty.
--
-- I.4  Non-emptiness of the strata
-- By our line of arguments, it suffices to produce an inhabitant of N'_{1
    bar1bar2bar} since appropriate deformations of this member will be
    elements of the other strata.
--
-- For this, we fix more or less arbitrary values for the parameters.
special = map(QQ, A, {1,1,-1,-1,1});
S' = QQ[gens S];
specialS = map(S', S, gens(S') | {1,1,-1,-1,1});
-- We 'randomly' obtained the following octic form for parameters s_0 = s_1 =
     t_2 = 1:
--J = specialS sub(idealFromKernel(N,f,params),S);
--fRand = randomElement J;
use S';
fRand = -10782*x^5*y^3-14169600*x^4*y^4+(29881520009/210)*x^3*y
    ^5-(7179779171/14)*x^2*y^6+(53866091961/70)*x*y^7-(26927310393/70)*y
    ^8-32346*x^5*y^2*z-(141546708/5)*x^4*y^3*z+(23847941291/210)*x^3*y^4*z
    +(12022783447/70)*x^2*y^5*z-(10269121787/10)*x*y^6*z+(53856117729/70)*y
    ^7*z-32346*x^5*y*z^2+(688392/5)*x^4*y^2*z^2-(7192780709/42)*x^3*y^3*z
    ^2+(35916966831/70)*x^2*y^4*z^2+(12028972583/70)*x*y^5*z
    ^2-(7176955691/14)*y^6*z^2-10782*x^5*z^3+(142624908/5)*x^4*y*z
    ^3-(23935325167/210)*x^3*y^2*z^3-(2398067559/14)*x^2*y^3*z
    ^3+(1587637157/14)*x*y^4*z^3+(29857735217/210)*y^5*z^3+(71237808/5)*x^4*
    z^4+(2995907368/105)*x^3*y*z^4+(931964/5)*x^2*y^2*z^4-(988201248/35)*x*y
    ^3*z^4-(1484261588/105)*y^4*z^4-(1592462/105)*x^3*z^5-(1592462/35)*x^2*y
```

```
    *z^5-(1592462/35)*x*y^2*z^5-(1592462/105)*y^3*z^5;
-- First of all, it is singular only at P_0, P_1 and P_2:
expected = intersect apply(P,i->specialS prS i);
sing = ideal(jacobian matrix fRand)+fRand;
assert(radical sing == expected);
-- On the other hand, the singularities there are as desired:
assert isJ21(fRand, specialS prS T_0, x, y, z);
assert isJ21(fRand, specialS prS T_1, z, y, x);
-- Moving P_2 = (1;1;1) to (0;0;1):
fRand' = sub(fRand, {x=>x+z,y=>y+z});
assert isX10(fRand', y, x);
-- Therefore, fRand indeed defines a member of N'_{1bar1bar2bar}. This
    finishes the discussion of the general configuration.
--
-- II  One tangent general, the other towards the quadruple-point
--     (t_0 = 1, t_1 != 1)
-- We have enough automorphisms left to fix t_1 = -1.
use A0;
A = A0/(t_0-1,t_1+1);
pr = map(A,A0);
S = A[gens S0];
prS = map(S,S0);
I = intersect(prS P33_0, prS P33_1, prS P_2^4);
(RS,inc,params,f) = universalFamily(8,I);
assert(#params == 12);
--
-- II.1  One of the points degenerate
-- This time, when imposing the degeneracy at a [3;3]-point, we have to
    distinguish whether we impose it at the [3;3]-point whose distinguished
    tangent points towards the quadruple-point (P_0) or the other one (P_1).
     The components are denoted by N''a_{11bar2} (if the degeneracy is at
    P_1) and N''b_{11bar2} (if the degeneracy is at P_0).
M0 = directedDegConditions33Point(f,
    inc prS L_0, inc prS T_0, inc prS F_0, s_0);
M1 = directedDegConditions33Point(f,
    inc prS L_1, inc prS T_1, inc prS F_1, s_1);
M2 = tangencyCond(f,
    inc prS L_2,inc prS T_2,inc prS F_2,
    Multiplicity=>4, Degree=>2);
-- Imposing degeneracy at P_i means taking the linear system defined by the
    kernel of Mi.
assert(numgens target M0 == 2);-- N''b_{11bar2}
assert(numgens target M1 == 2 and isSurjective M1);-- N''a_{11bar2}
assert(numgens target M2 == 2);-- N''_{112bar}
-- Since the rank of M1 drops nowhere, N''a_{11bar2} is irreducible and of
    dimension 12 - 2 + 1 - 2 = 9. Here, 2 is the rank of the matrices, there
     is 1 extra parameter in each case (s_0, s_1, t_2) and we subtract 2 for
     the stabiliser.
--
-- What happens at P_0 and P_2?
-- II.1.1 N''b_{11bar2}
droppingRankCond = minors(2, mingens image M0);
assert(radical droppingRankCond == ideal(s_0));
-- That is, the rank drops if and only if the second order direction of the
    degenerate [3;3]-point pointing towards the quadruple-point is trivial.
    In this case, the line has to be contained twice:
J = sub(idealFromKernel(sub(M0,{s_0=>0}),sub(f,{s_0=>0}),params),S);
assert((gens J)%(prS T_0)^2 == 0);
-- Thus, this case is neglected in the stratum anyways. It follows that the
    component N''b_{1bar22} is irreducible and 9-dimensional as well.
--
-- II.1.2 N''_{112bar}
```

```
droppingRankCond = minors(2, mingens image M2);
assert(radical droppingRankCond == ideal(t_2));
-- That is, the rank drops if and only if the quadruple-point's special
    tangent direction points towards a [3;3]-point, which as we know is
    impossible on a reduced octic. Thus, this case is irrelevant as well and
     so N''_{112bar} is irreducible and has dimension 9.
--
-- That these components are indeed not empty follows from II.4 below.
--
-- II.2  Two of the points degenerate
-- Again, we have two components N''a_{11bar2bar} and N''b_{11bar2bar}
    defined analogously as in II.1 above.
M01 = M0 || M1;-- N''_{1bar1bar2}
M02 = M0 || M2;-- N''b_{11bar2bar}
M12 = M1 || M2;-- N''a_{11bar2bar}
-- None of them is surjective everywhere.
--
-- II.2.1 N''_{1bar1bar2}
droppingRankCond = minors(4, mingens image M01);
assert(radical droppingRankCond == ideal(s_0));
-- For the same reason as in II.1.1, the corresponding parameters are
    neglected; thus, N''_{1bar1bar2} is irreducible and of dimension 12 - 4
    + 2 - 2 = 8.
--
-- II.2.2 N''b_{11bar2bar}
droppingRankCond = minors(4, mingens image M02);
assert(radical droppingRankCond == ideal(s_0*t_2));
-- The rank of M02 drops if s_0 = 0 or t_2 = 0; the former case is irrelevant
     by II.1.1 and the latter case is irrelevant by II.1.2. Therefore, N''b_
    {1bar12bar} is irreducible and 8-dimensional.
--
-- II.2.3 N''a_{11bar2bar}
droppingRankCond = minors(4, mingens image M12);
assert(radical droppingRankCond == ideal(t_2));
-- The rank of M12 drops if t_2 = 0, which is is irrelevant by II.1.2.
    Therefore, N''a_{1bar12bar} is irreducible and 8-dimensional, too.
--
-- That these components are indeed not empty follows from II.4 below.
--
-- II.3  All three singularities degenerate
N = M0 || M1 || M2;
droppingRankCond = minors(6, mingens image N);
assert(radical droppingRankCond == ideal(s_0*t_2));
-- Since the common rank of all three matrices drops only for s_0 = 0
-- or t_2 = 0, which is irrelevant (cf. II.2.2), we conclude that (if
    inhabited, see II.4 below), the component N''_{1bar1bar2bar} is
    irreducible and of dimension 12 - 6 + 3 - 2 = 7.
--
-- II.4  Non-emptiness of the strata
-- It suffices to construct a member of N''_{1bar1bar2bar} since appropriate
    deformations will define elements of the other strata. We are free to
    choose values for s_0, s_1 != 0 and t_2 != 0, e.g., s_0 = s_1 = t_2 = 1
    and, of course, t_0 = 1 and t_1 = -1.
special = map(QQ, A, {1,1,1,-1,1});
S' = QQ[gens S];
specialS = map(S', S, gens(S') | {1,1,1,-1,1});
-- We 'randomly' obtained the following octic form:
--J = specialS sub(idealFromKernel(N,f,params),S);
--fRand = randomElement J;
use S';
fRand = (81/8)*x^5*y^3-36*x^4*y^4-10*x^3*y^5+(59583/224)*x^2*y^6-(39045/112)*
    x*y^7+(26543/224)*y^8+(243/8)*x^5*y^2*z-(3327/280)*x^4*y^3*z-(93939/224)
```

```
    *x^3*y^4*z+(233721/280)*x^2*y^5*z-(779151/1120)*x*y^6*z+(29325/112)*y^7*
    z+(243/8)*x^5*y*z^2-(11517/280)*x^4*y^2*z^2+(77759/1120)*x^3*y^3*z
    ^2-(102789/280)*x^2*y^4*z^2+(804807/1120)*x*y^5*z^2-(229681/560)*y^6*z
    ^2+(81/8)*x^5*z^3-(53373/280)*x^4*y*z^3+(198099/224)*x^3*y^2*z
    ^3-(13191/10)*x^2*y^3*z^3+(713367/1120)*x*y^4*z^3-(1737/80)*y^5*z
    ^3-(35103/280)*x^4*z^4+(387041/1120)*x^3*y*z^4-(453051/1120)*x^2*y^2*z
    ^4+(306627/1120)*x*y^3*z^4-(2863/32)*y^4*z^4-60*x^3*z^5+180*x^2*y*z
    ^5-180*x*y^2*z^5+60*y^3*z^5;
-- First of all, it is singular only at P_0, P_1 and P_2:
expected = intersect apply(P,i->specialS prS i);
sing = ideal(jacobian matrix fRand)+fRand;
assert(radical sing == expected);
-- On the other hand, the singularities there are as desired:
assert isJ21(fRand, specialS prS T_0, x, y, z);
assert isJ21(fRand, specialS prS T_1, z, y, x);
-- Moving P_2 = (1;1;1) to (0;0;1):
fRand' = sub(fRand, {x=>x+z,y=>y+z});
assert isX10(fRand', y, x);
-- Therefore, fRand indeed defines a member of N''_{1bar1bar2bar}. This
    finishes the discussion about this configuration.
--
-- III  Both tangents pointing towards the quadruple-point
--      (t_0 = t_1 = 1)
use A0;
A = A0/(t_0-1,t_1-1);
pr = map(A,A0);
S = A[gens S0];
prS = map(S,S0);
I = intersect(prS P33_0, prS P33_1, prS P_2^4);
(RS,inc,params,f) = universalFamily(8,I);
assert(#params == 13);
--
-- The degeneracy conditions:
M0 = directedDegConditions33Point(f,
    inc prS L_0, inc prS T_0, inc prS F_0, s_0);
M1 = directedDegConditions33Point(f,
    inc prS L_1, inc prS T_1, inc prS F_1, s_1);
M2 = tangencyCond(f,
    inc prS L_2,inc prS T_2,inc prS F_2,
    Multiplicity=>4, Degree=>2);
-- III.1  One degenerate singularity
assert(
    numgens target M0 == 2 and
    numgens target M1 == 2 and
    numgens target M2 == 2);
-- Investigating the loci where they are not surjective. By symmetry, it
    suffices to look at the matrices M0 (--> N'''_{11bar2}) and M2 (--> N'''
    _{112bar}).
--
-- III.1.1 N'''_{11bar2}
droppingRankCond = minors(2,mingens image M0);
assert(radical droppingRankCond == ideal(s_0));
-- The rank of M0 drops only if the second order vanishing condition on the
    [3;3]-point vanishes; but the distinguished tangent line also passes
    through a quadruple-point, which is impossible on a reduced octic.
    Therefore, N'''_{11bar2} is irreducible and of dimension 13 - 2 + 1 - 3
    = 9.
--
-- III.1.2 N'''_{112bar}
droppingRankCond = minors(2,mingens image M2);
assert(radical droppingRankCond == ideal(t_2 * (t_2-1)));
-- The rank of M2 drops only if the special tangent of the quadruple-point
```

```
    points towards a [3;3]-point, which is impossible on a plane octic as
    long as it is reduced. Thus, also N'''_{112bar} is irreducible and 9-
    dimensional.
--
-- That both components are actually not empty will be part of III.4.
--
-- III.2  Two of the singularities degenerate
-- Again by symmetry, there are only two cases to consider.
M01 = M0 || M1;--> N'''_{1bar1bar2}
M02 = M0 || M2;--> N'''_{11bar2bar}
--
-- III.2.1 N'''_{1bar1bar2}
droppingRankCond = minors(4,mingens image M01);
assert(radical droppingRankCond == ideal(s_0 * s_1));
-- It is irrelevant that the rank drops, since it does so only if s_0 = 0 or
    s_1 = 0 and as explained in III.1.1, this requires the curve to be non-
    reduced. This implies that N'''_{1bar1bar2} is irreducible (if non-empty
    , which follows from III.4 below). Its dimension is 13 - 4 + 2 - 3 = 8.
--
-- III.2.2 N'''_{1bar1bar2}
droppingRankCond = minors(4,mingens image M02);
assert(radical droppingRankCond == ideal(s_0 * t_2 * (t_2-1)));
-- Comparing with III.1.1 and III.1.2, we see that here the locus where the
    rank drops is neglected and so this component (if not empty, cf. III.4
    below) is irreducible and of dimension 8.
--
-- III.3  All three points degenerate
N = M0 || M1 || M2;
droppingRankCond = minors(6,mingens image N);
assert(radical droppingRankCond == ideal(s_0 * s_1 * t_2 * (t_2-1)));
-- Analogously as in III.2, we see that the locus where N is not of maximal
    rank is neglected in the stratum and so N'''_{1bar1bar2bar} is
    irreducible and of dimension 13 - 6 + 3 - 3 = 7; that it is in fact
    inhabited will be shown in the next section.
--
-- III.4  The strata are not empty
-- As before, it suffices to construct an element of the component N'''_{1
    bar1bar2bar}.
-- We may choose any parameters such that s_0,s_1,t_2,t_2-1 != 0. For example
    , s_0 = s_1 = 1, t_2 = -1; recall: t_0 = t_1 = 1.
special = map(QQ, A, {1,1,1,1,-1});
S' = QQ[gens S];
specialS = map(S', S, gens(S') | {1,1,1,1,-1});
-- We 'randomly' obtained the following octic form:
--J = specialS sub(idealFromKernel(N,f,params),S);
--fRand = randomElement J;
use S';
fRand = -(24/35)*x^5*y^3-9*x^4*y^4-(40/27)*x^3*y^5+(7/3)*x^2*y^6+(12296/315)*
    x*y^7-(5708/189)*y^8+(72/35)*x^5*y^2*z+(3/5)*x^4*y^3*z+(5/4)*x^3*y^4*z
    +(67181/420)*x^2*y^5*z-(141049/420)*x*y^6*z+(72227/420)*y^7*z-(72/35)*x
    ^5*y*z^2+(375/7)*x^4*y^2*z^2-(17191/105)*x^3*y^3*z^2+(80/21)*x^2*y^4*z
    ^2+(29299/105)*x*y^5*z^2-(17917/105)*y^6*z^2+(24/35)*x^5*z^3-(2553/35)*x
    ^4*y*z^3+(177719/540)*x^3*y^2*z^3-(41603/84)*x^2*y^3*z^3+(73517/252)*x*y
    ^4*z^3-(201521/3780)*y^5*z^3+(972/35)*x^4*z^4-(20831/126)*x^3*y*z
    ^4+(13847/42)*x^2*y^2*z^4-(57643/210)*x*y^3*z^4+(51883/630)*y^4*z
    ^4+(6/35)*x^3*z^5-(18/35)*x^2*y*z^5+(18/35)*x*y^2*z^5-(6/35)*y^3*z^5;
-- First of all, it is singular only at P_0, P_1 and P_2:
expected = intersect apply(P,i->specialS prS i);
sing = ideal(jacobian matrix fRand)+fRand;
assert(radical sing == expected);
-- On the other hand, the singularities there are as desired:
assert isJ21(fRand, specialS prS T_0, x, y, z);
```

```
assert isJ21(fRand, specialS prS T_1, z, y, x);
-- Moving P_2 = (1;1;1) to (0;0;1):
fRand' = sub(fRand, {x=>x+z,y=>y+z});
assert isX10(fRand', y, x);
-- Therefore, fRand indeed defines a member of N'''_{1bar1bar2bar}. This
    finishes the discussion about this last configuration.
--
-- EOF degenerate112.m2 ---------------------------------------------
```

Listing 12: Excluding a certain configuration of conics

```
-------------------------------------------------------------------
-- M2-script accompanying the proof that there exists no plane octic curve
    with four (or more) [3;3]-points, by excluding a certain configuration
    of conics. That is, showing that N_{1111} is empty.
--
-- Filename 1111.m2
--
-------------------------------------------------------------------
--
clearAll();
load "octicsFunctions.m2";
--
-- This script excludes the configuration of four conics with four
    intersection points, where at each intersection point, three of the
    conics come together, and all three being tangent there.
-- Using plane automorphisms, we can fix three of the points and the conic,
    hence the tangents at the three points. We then consider a variable
    fourth point, and (depending on it) the remaining three conics uniquely
    determined by the condition that they pass through two of the three
    points in prescribed tangent direction, and through the fourth point (in
     any direction). If there were a configuration as above, then for some
    choice of the fourth point, all three tangents at the point would be
    tangent, but this does not happen, as we show below.
--
-- Notation:
-- A = QQ[s_0,s_1,s_2,u]/s_0*s_1*s_2*u-1 is the affine space of parameters we
     have to consider,
-- S = A[x,y,z] is the homogeneous coordinate ring of the trivial PP^2 bundle
     over A, and we consider the points P_0  = (0;0;1), P_1 = (1;0;0), P_2 =
     (0;1;0) and the variable point Q = (s_0;s_1;s_2). Furthermore, we have
    the conic defined by xy+xz+yz, passing through P_0, P_1 and P_2 with
    tangent line T_0 = x + y, T_1 = y + z, and T_2 = x + z, respectively. By
     construction, Q is off the coordinate axes, which is okay since we want
     our conics to be irreducible.
--
A = QQ[s_0,s_1,s_2,u]/ideal(s_0*s_1*s_2*u-1);
S = A[x,y,z];
Q = idealFromCoords(s_0,s_1,s_2);
L = {y,z,x};
T = {x+y,y+z,x+z};
P = apply(L,T,(l,t)->ideal(l,t));
PwT = apply(L,T,(l,t)->ideal(l^2,t));
conic = x*y+x*z+y*z;
assert(ideal super basis(2,intersect PwT) == ideal conic);
-- The ideals for the configurations of two of P_0, P_1, P_2 with tangent T_0
     and the point Q, to find the conics through those configurations.
groups = { intersect(PwT_0,PwT_1,Q),
        intersect(PwT_0,PwT_2,Q),
        intersect(PwT_1,PwT_2,Q) };
families = apply(groups,i->universalFamily(2,i));
-- In each case, there is only one conic:
```

```
assert all(families,i->#(i_2)==1);
conics = apply(families,i->i_3);
-- The conics have an unnecessary parameter, as a by-product of the
    universalFamily function, which we set to 1 for all of these:
tmp = apply(3,i->map(S,ring (conics_i),gens(S)|{1}));
conics = apply(3,i->(tmp_i)(conics_i));
use S;
-- Getting tangents we want to be multiples of another:
DQconics= apply(conics,
        i->((vars S)*sub(jacobian(ideal i),{x=>s_0,y=>s_1,z=>1}))_(0,0)
        );
m = matrix apply(DQconics,c->apply(gens S,w->c//w));
-- We want m to be of rank 1:
conditions = sub(trim minors(2,m),A);
assert(codim conditions == 1);
reducedConditions = radical conditions;
assert(reducedConditions == ideal sub(conic,{x=>s_0,y=>s_1,z=>s_2}));
--
-- Hence, the three conics fulfill the [3;3]-point-condition at Q only if Q
    lies on the other conic, which means that all three conics agree.
--
-- EOF 1111.m2 -------------------------------------------------------
```

The next file is used in the study of the components parametrising non-normal surfaces, hence, non-reduced curves.

Listing 13: Remarks about sextics

```
----------------------------------------------------------------------
-- M2-script to study sextic curves in PP^2 with quadruple- and [3;3]-points
--
-- Filename sextics.m2
----------------------------------------------------------------------
--
clearAll();
load "octicsFunctions.m2";
--
S = QQ[s][x,y,z];
--
L = {x,x};
T = {y,z};
F = {z,y}; -- (aux. localisation variables)
P = apply(2,i->trim ideal(L_i,T_i));
P33 = apply(2,i->get33Ideal(L_i,T_i,F_i));
--
-- I  Quadruple-points
--
-- I.1  One ordinary quadruple-point
assert(numgens source super basis(6,P_0^4) == 18);
-- The stabiliser group of a point on the plane is of dimension 6. Therefore,
     the dimension of the space of sextics with a  quadruple-point is 18 - 6
     - 1 = 11. Clearly, there exist admissible sextics with an ordinary
    quadruple-point, e.g., a union of four concurrent, and two general lines
    .
--
-- I.2  One degenerate quadruple-point
I = getDegenerateQuadrupleIdeal(L_0,T_0,F_0);
assert(numgens source super basis(6,I) == 16);
-- Since the group of automorphisms preserving the point and a tangent
    direction is of dimension 5, we conclude that the dimension of the space
     of sextics with a degenerate quadruple-point is 16 - 5 - 1 = 10.
-- Admissible sextics with a degenerate quadruple-point are easily
```

```
    constructed; e.g., the union of a cuspidal cubic, two general lines
    through the cusp, and a general line missing the cusp.
--
-- Since on a sextic there are no two quadruple-points, this is all we have
    to consider here.
--
-- II  [3;3]-points
--
-- II.1  One non-degenerate [3;3]-point
assert(numgens source super basis(6,P33_0) == 16);
-- Since the group of automorphisms preserving the point and a tangent
    direction is of dimension 5, we conclude that the dimension of the space
     of sextics with an ordinary [3;3]-point is 16 - 5 - 1 = 10. It is easy
    to construct an admissible sextic with an ordinary [3;3]-point; e.g.,
    the union of three general conics with a common point and common tangent
     in this point.
--
-- II.2  One degenerate [3;3]-point
(RS,inc,params,f) = universalFamily(6,P33_0);
assert(#params == 16);
M = directedDegConditions33Point(f,inc L_0, inc T_0, inc F_0, s);
assert(isSurjective M and numgens target M == 2);
-- Thus, the space of sextics with a degenerate [3;3]-point in P_0 has
    dimension 14 + 2 - 1 = 15 and so the dimension of the moduli space of
    sextics with a degenerate [3;3]-point is irreducible and of dimension 15
     - 6 = 9.
-- Constructing examples:
-- For every possible second order direction, there exists a (possibly
    reducible) quadric with an according A5-singularity; take the union of
    such quartics with a general conic passing through the point in
    prescribed tangent direction.
-- We have to produce two examples; one, where the second order information
    is trivial along the tangent, and one where it is not. For the former,
    take a smooth cubic and the tangent line of a flex point, as well as a
    general conic through the same point, being tangent there as well. For
    the general case, we can take the union of three conics with the same
    tangent in a single point, but where (at least) two of them also agree
    up to second order.
--
-- II.3  Two non-degenerate [3;3]-points
-- This case can be dealt with abstractly: We know that the two [3;3]-points
    have to be in general position so that there is  (exactly) a pencil of
    conics through the configuration and every sextic is a union of three
    pair-wise distinct such conics. Therefore, space of such sextics is 3-
    dimensional, but the group of stabilising automorphisms has dimension 2,
     leaving one dimension for the space of such sextics. Let us quickly
    confirm this:
assert(numgens source super basis(6,intersect(P33)) == 4);
-- Hence, the dimension is 4 - 2 - 1 = 1.
--
-- There are no more cases we have to consider.
--
-- EOF sextics.m2 ---------------------------------------------------
```

Finally, the file providing the procedures used in the main scripts, concluding this thesis.

Listing 14: The functions used in the remaining scripts

```
-----------------------------------------------------------------------
-- M2-script supplying functions to investigate certain linear systems of
    curves in the plane.
```

```
--
-- Filename octicsFunctions.m2
--
-- Written by Ben Anthes
--
-----------------------------------------------------------------------
--
--  Ideals
--
-- Here are some basic ideals containing the constraints for affine plane
    singularities:
pointWithTangentAffine = (pt,l) -> (pt^2)+ideal(l);
getA1Affine = (pt) -> pt^2;
getA2Affine = (pt,t) -> pt^2+ideal(t);
getA3Affine = (pt,t) -> (pointWithTangentAffine(pt,t))^2;
getX9Affine = (pt) -> pt^4;
getX10Affine = (pt,t) -> pt^2*getA2Affine(pt,t);
getX11Affine = (pt,t) -> pt^2*getA3Affine(pt,t);
getJ10Affine = (pt,t) -> (pointWithTangentAffine(pt,t))^3;
--
-- Once we fix a pair of generators of the maximal ideal of a point, the
    number of equations can be reduced by the following explicit
    presentations, where {t = 0} is the distinguished tangent line.
pointWithTangentInAffineCoords = (l,t) -> ideal(t,l^2);
getA2InAffineCoords = (l,t) -> ideal(t^2,t*l^2,l^3);
getA3InAffineCoords = (l,t) -> ideal(t^2,t*l^2,l^4);
getTacNodeInAffineCoords = (l,t) -> getA3InAffineCoords(l,t);
getJ10InAffineCoords = (l,t) -> ideal(t^3,t^2*l^2,t*l^4,l^6);
get33InAffineCoords = (l,t) -> ideal(t^3,t^2*l^2,t*l^4,l^6);
getX9InAffineCoords = (x,y) -> (ideal(x,y))^2;
getX10InAffineCoords = (l,t)->getA2InAffineCoords(l,t)*(ideal(t,l))^2;
getX11InAffineCoords = (l,t)->getA3InAffineCoords(l,t)*(ideal(t,l))^2;
--
-- When we pass to the projective plane, we have to localise at a parameter (
    f, in the following) to use the above affine ideals:
getPointWithTangentIdeal = (l,t,f) -> (
-- We suppose that in the ring
    S := ring t;
    K := coefficientRing S;
-- l, t and f are forms generating the irrelevant ideal.
    assert((degree f)_0 == 1);
    assert(ideal(l,t,f) == ideal(basis(1,S)));
-- We localise away from f,
    Sloc := S[local u,Degrees=>{-1}]/(u*f-1);
    loc := map(Sloc,S);
-- resulting in affine coordinates t' = ut, l' = ul:
    A := K[local l',local t'];
    use Sloc;
    r := map(Sloc,A,{l*u,t*u});
    use A;
    localIdeal := pointWithTangentInAffineCoords(l',t');
    use S;
    return trim preimage(loc,r(localIdeal));
);
get33Ideal = (l,t,f) -> ( -- same as for getPointWithTangentIdeal
    S := ring t;
    assert((degree f)_0 == 1);
    assert(ideal(l,t,f) == ideal(basis(1,S)));
    Sloc := S[local u,Degrees=>{-1}]/(u*f-1);
    loc := map(Sloc,S);
    use Sloc;
    localIdeal := getJ10InAffineCoords(u*l,u*t);
```

```
    return trim preimage(loc,localIdeal);
);
getTacNode = (l,t,f) -> ( -- same as for getPointWithTangentIdeal
    S := ring t;
    assert((degree f)_0 == 1);
    assert(ideal(l,t,f) == ideal(basis(1,S)));
    Sloc := S[local u,Degrees=>{-1}]/(u*f-1);
    loc := map(Sloc,S);
    use Sloc;
    localIdeal := getA3InAffineCoords(u*l,u*t);
    return trim preimage(loc,localIdeal);
);
getDegenerateQuadrupleIdeal = (l,t,f) -> (
-- same as for getPointWithTangentIdeal
    S := ring t;
    assert((degree f)_0 == 1);
    assert(ideal(l,t,f) == ideal(basis(1,S)));
    Sloc := S[local u,Degrees=>{-1}]/(u*f-1);
    loc := map(Sloc,S);
    use Sloc;
    localIdeal := getX10InAffineCoords(u*l,u*t);
    return trim preimage(loc,localIdeal);
);
--
--  Functions handling families
--
-- The next function takes a list of elements of the same ring S and returns
    the free linear combination of the elements of that list. That is
    encoded in a quadruple consisting of a ring RS isomorphic to S[a_0,...,
    a_n] with the inclusion map S --> RS, the list of generators params = {
    a_0,...,a_n} and the element which is the linear combination of the list
     elements with coefficients a_i.
opts = {Variable => a};
familyFromList = opts >> o -> (l) -> (
    if (#l == 0) then return (ZZ,map(ZZ,ZZ),{},0);
    S := ring (l_0);
    A := coefficientRing S;
    RA := A[o.Variable_0..o.Variable_(#l-1)];
    params := gens RA;
    RS := RA[gens S];
    inc := map(RS,S);
    l' := apply(l,p->inc p);
    f := sum(apply(gens RA,l',(p,q)->p*q));
    return (RS,inc,params,f);
);
-- The above function will be used to generate the universal families of
    plane curves of a certain degree satisfying certain constraints which
    are encoded in terms of an ideal. Caution: it implicitly assumes that S
    is free over its coefficient ring!
opts = {Variable => a};
universalFamily = opts >> o -> (d,i) -> (
    S := ring i;
    A := coefficientRing S;
    mm := mingens trim ideal super basis(d,trim i);
    m := apply(rank source mm,j->mm_(0,j));
    if not isWeaklyHomogeneous(mm,d) then (
        mm = super basis(d,trim i);
        m = apply(rank source mm,j->mm_(0,j));
    );
    assert(isWeaklyHomogeneous(mm,d));
    return familyFromList(m);
);
```

```
--
-- If K is a matrix with values in a free A-module A^n and if f is a free
    linear combination with n summands, then we can substitute the entries
    of the columns of K for the coefficients and get a ring element. The
    following function takes those elements and returns the ideal they
    generate.
idealFromImage = (K,f,params) -> (
    assert(#params == numgens target K);
    return ideal apply(numgens source K, j->
        sub(f,apply(numgens target K,i->params_i=>K_(i,j))));
);
-- This is just the application of the former, choosing a generating set (
    matrix) for the kernel of M instead of M itself.
opts = {Mingens => false};
-- The boolean option Mingens forces the application of mingens instead of
    gens to get a generating set of the kernel.
idealFromKernel = opts >> o -> (M,f,params) -> (
    K := gens kernel M;
    if(o.Mingens) then K = mingens image K;
    return idealFromImage(K,f,params);
);
--
--  Classifying plane curve singularities:
--
-- Check if f has an ordinary n-fold point in the affine origin.
-- Caution: If the base ring is not a field, this function is insufficient to
     conclude non-degeneracy; if the ring is a domain, then it shows generic
     non-degeneracy (if true is returned).
isOrdinaryNFoldSingularityAffine = (n,f,u,v) -> (
-- We assume that f is a polynomial in a ring generated by u and v
    A := ring f;
    assert(set gens A === set {u,v});
-- First of all, we want f to be a proper n-fold point:
    if (f%(ideal(u,v))^n != 0) then return false;
    if (f%(ideal(u,v))^(n+1) == 0) then return false;
-- Finally, we check in both blow up charts
    sigma := map(A,A,{u=>u,v=>u*v});
    tau := map(A,A,{u=>u*v,v=>v});
-- that the strict transforms are reduced along the exceptional lines:
    tmp0 := sigma(f)//u^n;
    tmp0 = sub(tmp0,{u=>0});
    if discriminant(tmp0,v) == 0 then return false;
    tmp1 := tau(f)//v^n;
    tmp1 = sub(tmp1,{v=>0});
    if discriminant(tmp1,u) != 0 then return true;
    return false;
);
--
-- The following function returns the ideal containing the conditions that an
     affine n-fold singularity in the origin is degenerate.
degeneracyConditionsNFoldSingularity = (n,f,u,v) -> (
-- We assume that f is a polynomial in a ring generated by u and v
    A := ring f;
    assert(set gens A === set {u,v});
-- First of all, we want f to be a proper n-fold point:
    if (f%(ideal(u,v))^n != 0) then (error "1"; return false);
    if (f%(ideal(u,v))^(n+1) == 0) then (error "2"; return false);
-- Finally, we consider in both blow up charts
    sigma := map(A,A,{u=>u,v=>u*v});
    tau := map(A,A,{u=>u*v,v=>v});
-- the strict transforms and their discriminants along the exceptional line:
    tmp0 := sigma(f)//u^n;
```

```
    tmp0 = sub(tmp0,{u=>0});
    tmp0 = discriminant(tmp0,v);
    tmp1 := tau(f)//v^n;
    tmp1 = sub(tmp1,{v=>0});
    tmp1 = discriminant(tmp1,u);
    return sub(intersect(ideal(tmp0),ideal(tmp1)),coefficientRing A);
);
--
-- The next few functions are either applications of the above, or work
    analogously.
isOrdinaryQuadruplePointAffine = (f,u,v) ->
    isOrdinaryNFoldSingularityAffine(4,f,u,v);
isOrdinaryQuadruplePoint = (f,x,y,z) -> (
-- Preparing to use isOrdinaryQuadruplePointAffine
-- We assume that
    S := ring f;-- is a polynomial ring with generators x,y,z
    assert(set gens S === set {x,y,z});
-- We ask whether f has an ordinary quadruple-point in the point x = y = 0:
    K := coefficientRing S;
    A := K[u,v];
    tmpMap := map(A,S,{x=>u,y=>v,z=>1}|apply(gens K,w->(w=>w)));
-- (A naive localisation map.)
    return isOrdinaryQuadruplePointAffine(tmpMap(f),u,v);
);
degeneracyConditionsQuadruplePoint = (f,x,y,z) -> (
-- Similar to isOrdinaryQuadruplePoint
    S := ring f;
    assert(set gens S === set {x,y,z});
    K := coefficientRing S;
    (KK,F) := flattenRing K;
    SS := K[x,y,z];
    ff := sub(f,SS);
    A := KK[u,v];
    tmpMap := map(A,SS,{u,v,1}); -- a naive localisation map
    use S;
    return preimage(F,
        degeneracyConditionsNFoldSingularity(4,tmpMap(ff),u,v));
);
degeneracyConditionsTriplePoint = (f,x,y,z) -> (
-- Similarly to isOrdinaryQuadruplePoint
    S := ring f;
    assert(set gens S === set {x,y,z});
    K := coefficientRing S;
    (KK,F) := flattenRing K;
    SS := K[x,y,z];
    ff := sub(f,SS);
    A := KK[u,v];
    tmpMap := map(A,SS,{u,v,1}); -- a naive localisation map
    use S;
    return preimage(F,
        degeneracyConditionsNFoldSingularity(3,tmpMap(ff),u,v));
);
isOrdinaryTriplePointAffine = (f,u,v) ->
    isOrdinaryNFoldSingularityAffine(3,f,u,v);
isOrdinaryTriplePoint = (f,x,y,z) -> (
-- As isOrdinaryQuadruplePoint
    S := ring f;
    assert(set gens S === set {x,y,z});
    K := coefficientRing S;
    A := K[u,v];
    tmpMap := map(A,S,{x=>u,y=>v,z=>1}); -- a naive localisation map
    return isOrdinaryTriplePointAffine(tmpMap(f),u,v);
```

```
);
isOrdinaryDoublePointAffine = (f,u,v) ->
    isOrdinaryNFoldSingularityAffine(2,f,u,v);
isOrdinaryDoublePoint = (f,x,y,z) -> (
-- As isOrdinaryQuadruplePoint
    S := ring f;
    assert(set gens S === set {x,y,z});
    K := coefficientRing S;
    A := K[u,v];
    tmpMap := map(A,S,{x=>u,y=>v,z=>1}|apply(gens K,w->(w=>w)));
-- (A naive localisation map.)
    return isOrdinaryDoublePointAffine(tmpMap(f),u,v);
);
isOrdinary33Point = (f,t,x,y,z) -> (
-- Similar to isOrdinaryQuadruplePoint
    S := ring f;
    assert(not t%x%z == 0);
    f0 := sub(f,{z=>1});
    t0 := sub(t,{z=>1});
    sigma := map(S,S,{y=>x*y});
    f0 = sigma(f0)//x^2;
-- We keep one copy of the exceptional line and ask for a non-degenerate
    quadruple-point.
    t0 = sigma(t0)//x; -- => critical point = ideal(x,t0)
    tmp = inverse(map(S,S,{y=>t0}));
    f0 = tmp(f0);
    return isOrdinaryQuadruplePoint(f0,x,y,z);
);
degeneracyConditions33Point = (f,t,x,y,z) -> (
-- Similar to isOrdinary33Point
    S := ring f;
    assert(not t%x%z == 0);
    f0 := sub(f,{z=>1});
    t0 := sub(t,{z=>1});
    sigma := map(S,S,{y=>x*y});
    f0 = sigma(f0)//x^3;
    t0 = sigma(t0)//x; -- => critical point = ideal(x,t0)
    tmp = inverse(map(S,S,{y=>t0}));
    f0 = tmp(f0);
    return degeneracyConditionsTriplePoint(f0,x,y,z);
);
--
isD5Affine = (f) -> (
-- This works only if the strict transform splits over QQ.
-- We assume that f is a polynomial in a ring generated by two elements u and
     v:
    A := ring f;
    assert(#(gens A) == 2);
    (u,v) := toSequence gens A;
-- First of all, we want f to be a proper triple-point:
    if (f%(ideal(u,v))^3 != 0) then return false;
    if (f%(ideal(u,v))^4 == 0) then return false;
-- Next, we blow up.
    sigma := map(A,A,{u=>u,v=>u*v});
-- By construction, f' is divisible by u^3 (but not by u^4).
    strictTrans := (sigma f)//u^3;
-- What we have to show is that the strict transform meets {u = 0} in two
    different points, one tangentially and one transversely.
    exc := sub(strictTrans,{u=>0});
-- Reparametrise if the presentation is inappropriate.
    if degree(v,exc) != 3 then return isD5Affine(sub(f,{u=>u+v}));
    if discriminant(exc,v) != 0 then (
```

```
        return false;
    );-- else:
    factors := factor exc;
-- We have to assume that exc splits as h*g^2 with g,h of degree 1, possibly
    with a factor of degree 0
    degs := set select(apply(#factors,i->factors#i#1),i->(i!=0));
    if (degs === set {1, 2}) then (
-- Find f_2:
        g := (select(factors,i->i#1 == 2))#0#0;
-- We want simple tangentiality to {u = 0} where g = 0:
        if strictTrans%ideal(u,g^2) != 0 then (
            return false;
        ) else (
            if strictTrans%ideal(u^2,g) != 0 then (
                return true;
            ) else (
                return false;
            );
        );
    ) else (
                error("I can't decide, pardon me.");
                return false;
    );
    return false;
);
isJ21 = (f,t,x,y,z) -> (
-- We blow up once, localise and ask for a D_5-singularity.
    S := ring f;
    assert(set gens S === set {x,y,z});
    assert(not t%x%z == 0);
    f0 := sub(f,{z=>1});
    t0 := sub(t,{z=>1});
    sigma := map(S,S,{y=>x*y});
    f0 = sigma(f0)//x^3;
    t0 = sigma(t0)//x;
-- We move the critical point (x,t0) = 0 to (x,y) = 0:
    tmp = inverse(map(S,S,{y=>t0}));
    f0 = tmp(f0);
    A := (coefficientRing S)[local u,local v];
    loc := map(A,S,{x=>u,y=>v,z=>1});
    return isD5Affine(loc f0);
);
isX10Affine = (f) -> (
-- This works only if the quadratic part of the strict transform splits over
    QQ.
-- The procedure is similar to isJ21Affine.
    A := ring f;
    assert(#(gens A) == 2);
    (u,v) := toSequence gens A;
    if (f%(ideal(u,v))^4 != 0) then return false;
    if (f%(ideal(u,v))^5 == 0) then return false;
    sigma := map(A,A,{u=>u,v=>u*v});
    strictTrans := (sigma f)//u^4;
    exc := sub(strictTrans,{u=>0});
    if degree(v,exc) != 4 then return isX10Affine(sub(f,{u=>u+v}));
    if discriminant(exc,v) != 0 then (
        return false;
    );-- else:
    factors := factor exc;
    admissibleFactors := select(factors,i->(degree(v,i#0),i#1)==(1,2));
    if #admissibleFactors == 1 then (
        g := admissibleFactors#0#0;
```

```
        h := exc//(g^2);
-- f = (cst)*g^2*h where h has degree 2 in v.
        if discriminant(h,v) == 0 then return false;-- else:
        if strictTrans%ideal(u,g^2) != 0 then (
            return false;
        ) else (
            if strictTrans%ideal(u^2,g) != 0 then (
                return true;
            ) else (
                return false;
            );
        );
    ) else (
        print("I can't decide, pardon me.");
        return false;
    );
);
isX10 = (f,x,y) -> (
-- Similar to isOrdinaryQuadruplePoint
    S := ring f;
    z := sum((set gens S) - (set {x,y}));
    assert(set gens S === set {x,y,z});
    A := (coefficientRing S)[local u,local v];
    loc := map(A,S,{x=>u,y=>v,z=>1});
    return isX10Affine(loc f);
);
--
--  Procedures to derive certain degeneracy conditions:
--
opts = {Multiplicity=>1,Degree => 1};
tangencyCond = opts >> o -> (f,l,t,z) -> (
-- We assume that f is an element of a ring RS = A[params][l,t,z] where l, t
    and z are homogeneous of degree 1 (implicit assumption) and that f has
    multiplicity o.Multiplicity in (l,t) = 0.
    RS := ring f;
    params := gens coefficientRing RS;
    A := coefficientRing coefficientRing RS;
-- We localise away from z = 0 and identify the result with AA^2 such that (l
    ,t) = 0 becomes the origin. By blowing up, we find the conditions that f
     is tangent to {t = 0} with multiplicity o.Degree.
    RSloc := RS[u,Degrees=>{ -1 }]/(u*z-1);
    loc := map(RSloc,RS);
    R := A[params][local t',local l'];
    r := map(RSloc,R,{u*(loc t),u*(loc l)});
    f' := (gens preimage(r,ideal loc f));
    assert(numgens source f' == 1 and numgens target f' == 1);
    f' = f'_(0,0);
    sigma := map(R,R,{t'*l',l'});
    strict := sigma(f')//(l'^(o.Multiplicity));
    exc := sub(strict,{l'=>0});
    toBeZero := exc%(t'^(o.Degree));
    toBeZeroCoeff := for t in terms toBeZero
        list sub(leadCoefficient(t),RS);
-- The kernel of the following matrix parametrises the possible values for
    params so that if they get substituted into f, the resulting element
    satisfies the constraints discussed above.
    M := matrix for eq in toBeZeroCoeff list
               for g in params list sub(leadCoefficient(eq//g),A);
    return M;
);
--
opts = {Multiplicity => 1,Degree => 1};
```

```
secondOrderCond = opts >> o -> (f,l,t,z,s) -> (
-- Similar to tangencyCond, but blowing up once more.
    RS := ring f;
    params := gens coefficientRing RS;
    A := coefficientRing coefficientRing RS;
    RSloc := RS[u,Degrees=>{ -1 }]/(u*z-1);
    loc := map(RSloc,RS);
    R := A[params][local t',local l'];
    r := map(RSloc,R,{u*(loc t),u*(loc l)});
    f' := (gens preimage(r,ideal loc f));
    assert(numgens source f' == 1 and numgens target f' == 1);
    f' = f'_(0,0);
    sigma := map(R,R,{t'*l',l'});
    strict := sigma(f')//(l'^(o.Multiplicity));
    strict = sigma(strict)//(l'^(o.Multiplicity));
    exc := sub(strict,{l'=>0});
    toBeZero := exc%((t'-s)^(o.Degree));
    toBeZeroCoeff := for t in terms toBeZero
        list sub(leadCoefficient(t),RS);
    M := matrix for eq in toBeZeroCoeff list
               for g in params list sub(leadCoefficient(eq//g),A);
    return M;
);
-- An immediate application: if f has a [3;3]-point in (l,t) = 0 with
    distinguished tangent line {t = 0}, then the following procedure returns
     the matrix encoding the conditions that the [3;3]-point is degenerate
    with specified second order information s.
directedDegConditions33Point = (f,l,t,z,s) ->
    secondOrderCond(f,l,t,z,s,Multiplicity=>3,Degree=>2);
--
--  A few auxiliary functions:
--
-- The following should be used with care since it assumes the  coordinates
    of the ring are called x,y and z.
idealFromCoords = (a,b,c) -> trim minors(2,matrix({{x,y,z},{a,b,c}}));
-- A function checking if a matrix is homogeneous of degree d in the first
    set of variables.
isWeaklyHomogeneous = (m,d) -> (
    e := listDeepSplice entries m;
    T := listDeepSplice(apply(e,i-> terms i));
    return all(T,t->(degree t)_0 == d);
);
-- A function flattening a list L.
listDeepSpliceAcc = (L,A) -> (
    if (not instance(L,List)) then return {L};
    if (#L == 0) then return A;
    return listDeepSpliceAcc(drop(L,1),A | listDeepSpliceAcc(L_0,{}));
);
listDeepSplice = (L) -> return listDeepSpliceAcc(L,{});
--
-- Some randomisation functions which are used to construct examples:
randomList = (n) -> apply(n,i->
    (2*random(0,1)-1)*(random(QQ^1,QQ^1))_(0,0));
randomVector = (n) -> diagonalMatrix(randomList(n))*random(QQ^n,QQ^1);
randomValue = (m) -> (m*randomVector(rank source m))_(0,0);
randomElement = (i) -> randomValue(matrix({i_*}));
randomLine = (x,y,z) -> (matrix({{x,y,z}})*randomVector(3))_(0,0);
--
-- EOF octicsFunctions.m2 --------------------------------------------
```

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