Supersymmetry in Conformal Geometric and Number-Theoretical Quantum Mechanics

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Zusammenfassung

In dieser Dissertation wird eine supersymmetrische Formulierung der Quantenmechanik auf konformen Mannigfaltigkeiten entwickelt, die auf Ergebnissen meiner Masterarbeit [77] aufbaut. Supersymmetrie stellt auf diesem Wege einen Zusammenhang zwischen Quantenmechanik auf konformen Mannigfaltigkeiten, der Spektralgeometrie von Schrödinger-Operatoren und Topologie her. Ein physikalisch motivierter Beweis der Abschätzung von Yang und Yau für den ersten Eigenwert des Laplace-Operators auf kompakten Riemannschen Flächen [80] wird, basierend auf diesen Überlegungen, vorgestellt. Die Beweisidee kann schließlich auf Schrödinger-Operatoren übertragen werden. Eine Anwendung der Eigenwertabschätzung für Schrödinger-Operatoren auf das Coulomb-Problem und den harmonischen Oszillator wird vorgestellt.

Weiterhin wird die Anwendung von Supersymmetrie auf Spin-Ketten motiviert, indem explizit das eindimensionale Ising-Modell mit Nächster-Nachbar-Wechselwirkung supersymmetrisch interpretiert wird. Motiviert durch Ref. [53], wird der Witten-Index [74] für Spin-Ketten eingeführt, welcher das zu Boltzmann-Faktoren korrespondierende Objekt auf dem dualen Konfigurationsraum ist. Dadurch wird ein Zusammenhang zwischen Witten-Indizes und n-Punkt-Korrelationsfunktionen hergestellt, so dass die Spin-Spin-Wechselwirkung durch die Betrachtung von Witten-Indizes auf Spin-Ketten interpretiert werden kann. Durch Anwendung der Ergebnisse auf entsprechende Unterräume des Konfigurationsraumes, wird ein rigoroser Zugang zum Vakuumerwartungswert für den Dichteoperator einer Spin-Kette erarbeitet, indem der Vakuumerwartungswert über n-Punkt-Korrelationsfunktionen ausgedrückt wird. Der Spezialfall supersymmetrischer Spin-Ketten wird behandelt, ferner wird gezeigt, dass in solchen Systemen keine Phasenübergänge auftreten können.

Es existieren zahlreiche Zugänge zur Riemannschen Zetafunktion und der Riemannschen Vermutung durch Verwendung von Konzepten aus der Physik, siehe, z.B., Ref. [69]. Einen

vielversprechenden Ansatz stellt das Primonen-Gas, auch Riemann-Gas genannt, dar, welches Zahlentheorie mit Quantenfeldtheorie und Statistischer Physik verbindet und von Julia [49] und Spector [70] unabhängig voneinander eingeführt wurde. Das Primonen-Gas beschreibt ein kanonisches Ensemble, welches die Riemannsche Zetafunktion $\zeta(\beta)$ als Zustandssumme hat, wobei $\beta = T^{-1}$ die inverse Temperatur ist. Da die Riemannsche Zetafunktion eine Singularität bei $\beta = 1$ besitzt, siehe, z.B., Ref. [6], erreicht das Primonen-Gas seine Hagedorn-Temperatur [36–40] an diesem Punkt, siehe Refn. [49, 70]. Das Verhalten des Primonen-Gases im Bereich jenseits der Hagedorn-Temperatur ist bisher nicht zufriedenstellend bekannt, allerdings gibt es hierzu Ansätze [23, 50]. In der Physik kondensierter Materie ist bekannt, dass hadronische Materie beim Erreichen der Hagedorn-Temperatur instabil wird [36–40]. Eine ähnliche Situation liegt in der Stringtheorie [7] und im Kontext von zahlentheoretisch motivierten Modellen [50] vor. Aufbauend auf der supersymmetrischen Erweiterung des Primonen-Gases [70, 71] wird ein Modell untersucht, das eine enge Verwandtschaft zum Primonen-Gas aufweist. Der Phasenübergang beim Erreichen der Hagedorn-Temperatur wird als Kopplung der Fermionen des supersymmetrischen Primonen-Gases und denen eines Gases von harmonischen Oszillatoren zu Bosonen-artigen Paaren erklärt. Hierbei liegt eine konzeptionelle Vergleichbarkeit mit den Cooper-Paaren der BCS-Theorie [9, 10, 21] vor. Darauf basierend wird ein neuartiger Zugang zur Riemannschen Vermutung vorgestellt.

Abstract

In this dissertation I work out a supersymmetric formulation of conformal geometric quantum mechanics, which based on ideas I started to develop in my Master's thesis [77]. In this approach, supersymmetry provides a fundamental connection between conformal geometric quantum mechanics, the spectral geometry of Schrödinger operators and topology. I use these links to give a physics proof of the famous Yang-Yau estimate for the first eigenvalue of the Laplacian on compact Riemann surfaces [80] and to generalize this physics-based proof to Schrödinger operators. Furthermore, I apply the derived eigenvalue estimate to the Coulomb problem and the harmonic oscillator.

Moreover, I motivate the application of supersymmetry to spin chain models by describing some properties of the 1D nearest neighbor Ising model in terms of supercharges [74]. By doing so, some important concepts are explained, which are necessary for the further work. Motivated by Ref. [53], I introduce the Witten index [74] for spin chains, which is an object on dual configuration spaces corresponding to Boltzmann weights. I establish a connection between Witten indices and n-point correlation functions. Thus, the spin-spin interactions can be interpreted by considering the Witten index of spin chains. Finally, by transferring the results to subspaces I obtain a rigorous expression of the vacuum expectation value for the density matrix of an arbitrary spin chain model in terms of correlation functions. Moreover, the special case of supersymmetric theories is analyzed and it is shown that no phase transitions can occur in spin chain models with supersymmetry. Furthermore, it is shown that my results are invariant under unitary transformations.

There exist numerous approaches to the Riemann zeta function and the Riemann hypothesis using different concepts from physics, see, e.g., Ref. [69]. A promising and well-known approach is the primon gas, also called Riemann gas, which is a toy model combining concepts of number theory, quantum field theory and statistical physics, introduced by Julia [49] and Spector [70] .

More precisely, the primon gas describes a canonical ensemble with the Riemann zeta function $\zeta(\beta)$ as partition function, where $\beta = T^{-1}$ is the inverse temperature. Since the Riemann zeta function has a singularity at $\beta = 1$, see, e.g., Ref. [6], the primon gas reaches its Hagedorn temperature [36–40] at this point, see Refs. [49, 70]. The behavior of the primon gas beyond the Hagdorn temperature is still not clear, but there are investigations concerning this point [23, 50]. Generally, it is well-known in condensed matter physics that hadronic matter becomes unstable at the Hagedorn temperature [36–40]. A similar situation exists in string theory [7] and there are observations in this direction in the context of number-theoretical gases [50]. Here, I use Spector's theory of the supersymmetric primon gas [70, 71] to analyze the behavior of a canonical ensemble, which is closely related to the primon gas. By doing so, I interpret the transition at the Hagedorn temperature as a coupling of the fermions of the supersymmetric primon gas and the fermions of an ensemble of harmonic oscillator states to boson-like pairs comparable with the formation of Cooper pairs in the BCS theory [9, 10, 21]. Based on this, I work out a novel link to the Riemann hypothesis.

Contents

Chapter 1

Introduction

In my Master's thesis [77] I developed a geometric formulation of quantum mechanics by generalizing the classical Maupertuis principle (see, e.g., Refs. [2, 59]). The Maupertuis principle provides a geometric formulation of classical mechanics by stating that the motion of particles in a potential (or under a force) is equivalent to the free motion of particles on a suitably curved surface. Or somewhat more mathematically [77]: it is always possible to rescale the metric of the configuration space (or more generally of spacetime) such that the resulting equation of motion is a geodesic equation for the transformed metric.

Quantum generalizations of the Maupertuis principle were already investigated by Collas [20] and by Karamatskou and Kleinert [52]. Their findings indicate that there is a connection between the presence of a potential and a formulation of quantum mechanics on curved spaces. I chose a somewhat different and new approach by studying quantum mechanics on conformal manifolds (cf. Ref. [14]). By using these concepts of conformal geometry I was able to find a purely geometric description of quantum systems and to show that different aspects of quantum mechanics have the same geometrical origin. In particular, my approach provided a mathematically profound connection between quantum dynamics and curved spaces. Moreover, I extended the Yang-Yau eigenvalue estimate for Laplacians [80] to Schrödinger operators by using the quantum mechanical Maupertuis principle, see Ref. [77].

In the first part of this dissertation I work out a supersymmetric formulation of quantum mechanics on conformal manifolds, which can be considered as a supersymmetric analogue of the conformal geometric quantum mechanics I started to develop in my Master's thesis [77]. But instead of studying properties induced by the intrinsically defined curvature, the supersymmetric formulation is a topological approach, i.e. that quantum states are characterized by topological invariants of the underlying configuration manifold. Therefore, the supersymmetric formulation on conformal manifolds is compatible with many different curvatures. The fact that the characterization of certain quantized supersymmetric models is closely related with the topology of the underlying configuration manifold is well-known and has been extensively studied, see, e.g., Refs. [4, 28, 29, 31, 32, 74, 75]. In this work, I connect the genus of compact surfaces to the spectral geometry of Schrödinger operators by means of the supersymmetric formulation of conformal geometric quantum mechanics. Therefore, in Chapter 3, I introduce a revised version of the results of my Master's thesis [77] on the conformal geometric formulation of quantum mechanics and the quantum mechanical formulation of the Maupertuis principle. By doing so, I introduce some novel formulations, which are useful for the further work. Moreover, I give an extension of the quantum mechanical Maupertuis principle to higher dimensions, where I show that the equivalence between the free motion of an uncharged particle on a curved manifold with the motion of a charged particle in flat space. Furthermore, I work out an explicit realization of the conformal geometric Fock space representation on a curved space. In Chapter 4, I give a physics proof of the Yang-Yau eigenvalue estimate [80] and its extension to Schrödinger operators using the supersymmetric approach to the conformal geometric formulation of quantum mechanics. By doing so, I show that the supersymmetric approach gives an ultimate link between the loose ends of conformal geometric quantum mechanics, the quantum mechanical Maupertuis principle in higher dimensions, the spectral geometry of Schrödinger operators and the topology of the underlying configuration space. Moreover, I provide applications of the eigenvalue estimate to the Coulomb problem and the harmonic oscillator.

In the second part, I focus on the application of supersymmetric concepts, especially the Witten index [74], to physical models motivated from number theory. In contrast to the first part, there are no geometrical and topological informations about the underlying configuration space. This is for so far interesting, since, as I show in the first part for a supersymmetric two-level system, the topology of the underlying configuration space of a quantum mechanical theory leads to a strict determination of the behavior of supersymmetric quantum system. In the discrete setting that is considered in the second part, standard methods of supersymmetric quantum mechanics are not practicable. In Ref. [53], Knauf introduced a general framework to describe the dynamics of spin chains with methods of finite Fourier analysis and number theory. More precisely, the connection between the Riemann hypothesis and the theory of phase transitions motivated Knauf [53–56] to construct and analyze a spin chain model with $\frac{\zeta(s-1)}{\zeta(s)}$ as the partition function (see Appendix). His approach is a well-understood microscopic realization of a number-theoretical model, which yields interesting insights to the theory of phase transitions. I use his algebraic framework in order to define the Witten index for spin chain models.

Many concepts and methods of particle physics can be transferred to condensed matter physics [3]. In a certain sense, one can interpret condensed matter physics as a low energy and non-relativistic limit of particle physics. Spin chain models, like the 1D nearest neighbor Ising model, are among the most studied systems in condensed matter physics and statistical physics. Many phenomena like collective magnetism can be understood with spin models and there are many ways to analyze important properties of spin chain models in an analytical manner (see, e.g., Ref. [11]). More recently, there have been investigations, which show that supersymmetry is a generic phenomenon in spin chain models, see, e.g., Refs. [26, 41–43]. Moreover, it is shown in Ref. [15] that there is a deep connection between spin-correlations in the massive Ising model and the geometry of ground states in two-dimensional supersymmetric QFT's.

In Chapter 5, I first motivate the application of supersymmetry to spin chains and provide a formulation of the 1D nearest neighbor Ising model in terms of supercharges. Then, I use methods motivated from Ref. [53] to introduce the Witten index [74] for spin chain models. It turns out that the Witten index itself is closely related to n-point correlation functions and yields insights into the supersymmetry of spin chains. By transferring my approach to subspaces of the spin chain configuration space, which here can be interpreted as spin chains with a reduced length, I get a complete description of the spin-spin interactions in terms of n-point correlation functions. Finally, by connecting Boltzmann weights on the configuration space with Witten indices on subspaces, which are the transformed observables on the dual configuration spaces, I derive an expression of the vacuum expectation value of the density matrix in terms of n-point correlation functions for an arbitrary spin chain by using the Poisson summation formula (see, e.g., Ref. [68]) for observables on finite groups. Moreover, I apply these results to supersymmetric spin chain models and get novel insights to phase transitions. Finally, I show that the results are invariant under unitary transformations, i.e. they are independent from the choice of a basis.

Nowadays numerous physics-based approaches to the Riemann zeta function and the Riemann hypothesis are known, see, e.g., Ref. [69]. A well-understood model is the primon gas, which combines number theory and statistical physics. Essential ideas about primon gas were first developed by Julia [49] and Spector [70]. According to Refs. [70, 71], the basic idea behind primon gas is that one considers a system described by a Hamiltonian H_{prim}, which has eigenstate states $|p\rangle$ with

$$H_{prim}|p\rangle = E_p|p\rangle, \tag{1.1}$$

where $E_p = E_0 \ln p$ for a prime number p and a fixed ground state energy E_0. The state $|p\rangle$ is called primon. Furthermore, one can go over to many-particle states [70]

$$|n\rangle = |\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots, \alpha_k, \ldots\rangle, \tag{1.2}$$

where the arbitrary non-negative integer α_k is the occupation number for the k-th single-particle state $|p_k\rangle$ with energy $E_{p_k} = E_0 \ln p_k$. Here, p_k is the k-th prime number, i.e. $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, \ldots$.

The primon gas is bosonic, since a state can be multiply occupied. A many-particle state $|n\rangle$ corresponds one on one to the unique prime factorization [70]

$$n = \prod_k p_k^{\alpha_k}. \tag{1.3}$$

The energy of a many-particle state $|n\rangle$ is [70, 71]

$$E_n = E_0 \ln n = E_0 \sum_k \alpha_k \ln p_k = \sum_k \alpha_k E_{p_k}. \tag{1.4}$$

The partition function for the primon gas becomes [70, 71]

$$Z(\beta) = \sum_n e^{-\beta E_n} = \sum_n \frac{1}{n^s} = \zeta(s), \tag{1.5}$$

with the inverse temperature $\beta = T^{-1}$ and $s = \beta E_0$. For $s = 1$ the Riemann zeta function $\zeta(s)$ diverges to $+\infty$, see, e.g., Ref. [6]. Therefore, the Hagedorn temperature is reached at $s = 1$ [49, 70].

Based on the statistical bootstrap model, which was first discussed by Hagedorn [36] and independently discovered by Frautschi [27], one can show [36–40] that hadronic matter becomes unstable at the Hagedorn temperature. Moreover, in string theory an analogous temperature can be defined [7], where as well as in the hadronic picture the number of degrees of freedom growths exponentially during the transition [12, 27, 36–40, 61].

One the other hand, the partition function of the primon gas has an interesting interpretation as the Riemann zeta function, one of the most important objects in number theory. The Riemann zeta function, given by Eq. (1.5), is defined for every complex number s with $\Re(s) > 1$, but there exists an unique analytic continuation, which is holomorphic for every $s \in \mathbb{C}$ except for the simple pole at $s = 1$, see, e.g., Ref. [6].

The Riemann hypothesis, a conjecture about the distribution of the zeros of the Riemann zeta function, is one of the most important unsolved problems in number theory. One assumes that every non-trivial zero, which are the zeros in the critical strip $\{\beta \in \mathbb{C} \mid 0 < \Re(\beta) < 1\}$, of the Riemann zeta function has real part one half.

In statistical physics, the distribution of the zeros of partition functions is closely related to the existence and classification of phase transitions. If we consider the partition function of a grand-canonical ensemble for a complex-valued fugacity, then the description of phase transitions, based on works of Yang and Lee [60, 78], explains the appearance of a phase transition in terms of an accumulation of the zeros of the partition function in the vicinity of a positive real number. Moreover, investigations of Grossmann, Rosenhauer and Lehmann [33–35] show the possibility of the classification of phase transitions for canonical ensembles in terms of the distribution of zeros of the partition function with a complex-valued inverse temperature.

There are promising applications of the primon gas to different problems in statistical physics and number theory. Dueñas and Svaiter [23] demonstrated a connection to the distribution of the zeros of the Riemann zeta function by considering a randomized primon gas. More precisely, they showed that the mean energy density of a randomized primon gas depends strongly on the zeros of the Riemann zeta function. They obtained a definition of the mean energy density in terms of $\frac{\zeta'}{\zeta}(s)$ by using a regularization procedure. Moreover, there are interesting applications to harmonic oscillators [76] and the Goldbach conjecture [67]. Julia [50] considers a grand-canonical version of the primon gas, where he studies the Hagedorn transition and speculates on applications to string theory and QCD. In Ref. [8] Bakas and Bowick interpret a gas of parafermions arithmetically. By doing so, they give an explanation of a boson-fermion transformation of the primon gas by using some number-theoretical ideas. Another approach to the Riemann zeta function based on a C^*-dynamical system was introduced by Bost and Connes [13].

Based on the work of Spector [70, 71], who formulated a supersymmetric extension of the primon gas by considering the Möbius function as an operator $(-1)^F$, my aim in Chapter 6 is to understand the Hagedorn transition of an canonical ensemble, which is closely related to the primon gas. By doing so, I also work out a novel connection to the Riemann hypothesis by giving an explicit physical interpretation of the analytic continuation of the canonical ensemble for inverse temperatures beyond the Hagedorn temperature.

Chapter 2

Supersymmetric Quantum Mechanics

Following Witten's groundbreaking achievements [74, 75], we briefly introduce some concepts of supersymmetric quantum mechanics that are used in this work.

2.1 Fundamentals of Supersymmetric Quantum Mechanics

In this section, we follow for the most part Refs. [48, 74, 75].

The Hamiltonian of a supersymmetric theory is given by $H = Q^2$, where Q is called a supercharge, see, e.g., Ref. [74]. According to Ref. [74], it is useful to consider multiple supercharges $Q_1, \ldots, Q_N$ with

$$Q_1^2 = \cdots = Q_N^2 = H \tag{2.1}$$

and thus

$$H = \frac{1}{N}(Q_1^2 + \cdots + Q_N^2). \tag{2.2}$$

Furthermore, one requires [74]

$$Q_i Q_j + Q_j Q_i = 0. \tag{2.3}$$

The case $N = 4$ corresponds to supersymmetry in four dimensions [74]. The Hamiltonian defines an isomorphism

$$H : \mathcal{H} \to \mathcal{H} \tag{2.4}$$

where $\mathcal{H}$ is the Hilbert space of the supersymmetric theory.

For an eigenstate $\psi \in \mathcal{H}$ with energy $E \neq 0$, due to the Schrödinger equation,

$$H\psi = E\psi \tag{2.5}$$

one has obviously

$$H(Q\psi) = Q(H\psi) \tag{2.6}$$

and

$$E \geq 0, \tag{2.7}$$

since $H = Q^2$ is positive semidefinite. Therefore, $Q\psi$ is also an eigenstate of H with energy E. Moreover, it follows that eigenstates with energy $E \neq 0$ occur in pairs $(\psi, Q\psi)$, see Ref. [74]. Or in other words, Q let all eigenspaces

$$\mathcal{H}^E = \{\psi \in \mathcal{H} \mid H\psi = E\psi\} \tag{2.8}$$

to an energy E invariant, i.e.

$$Q\left(\mathcal{H}^E\right) = \mathcal{H}^E. \tag{2.9}$$

Witten [74] introduced the operator $(-1)^F$ with

$$(-1)^F\chi = +\chi, \tag{2.10}$$

if χ describes a boson and

$$(-1)^F\chi = -\chi, \tag{2.11}$$

if χ describes a fermion.

Moreover, one denote $\mathcal{H}_+$ for the space of bosonic states and $\mathcal{H}_-$ for the space of fermionic states. Therefore, one has the decomposition [48, 74, 75]

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \tag{2.12}$$

with

$$Q(\mathcal{H}_\pm) = \mathcal{H}_\mp \tag{2.13}$$

by requiring

$$(-1)^F Q + Q(-1)^F = 0. \tag{2.14}$$

For a vacuum state ψ, i.e.

$$H\psi = 0 \tag{2.15}$$

with energy $E = 0$ one has

$$\langle\psi|H|\psi\rangle = \|Q\psi\|^2. \tag{2.16}$$

As a direct consequence of the definiteness of the norm $\|\cdot\|$ one has $Q\psi = 0$. Therefore, vacuum states do not necessarily occur pairwise [74].

Moreover, an important object in supersymmetric theories is the Witten index [74]

$$\mathrm{Tr}(-1)^F, \tag{2.17}$$

which can be regularized by the expression [74]

$$\mathrm{Tr}(-1)^F e^{-\beta H}. \tag{2.18}$$

The Witten index is the difference of the number of bosonic vacuum states $n_B^{E=0}$ and the number of fermionic vacuum states $n_F^{E=0}$ [74]

$$\mathrm{Tr}(-1)^F e^{-\beta H} = n_B^{E=0} - n_F^{E=0}. \tag{2.19}$$

Obviously, the Witten index is independent of the inverse temperature $\beta = T^{-1}$.

If there are at least two ($N = 2$) supercharges Q_1 and Q_2 one can define operators [74]

$$Q_\pm = \frac{1}{\sqrt{2}}(Q_1 \pm iQ_2). \tag{2.20}$$

Then, one has

$$Q_\pm^2 = 0, \tag{2.21}$$

since

$$Q_\pm^2\psi = \frac{1}{2}(Q_1^2 - Q_2^2)\psi = \frac{1}{2}(H\psi - H\psi) = 0. \tag{2.22}$$

Moreover, the Hamiltonian (2.2) becomes

$$H = \frac{1}{2}(Q_1^2 + Q_2^2) = Q_+Q_- + Q_-Q_+. \tag{2.23}$$

One call $\{Q_+, Q_-\} = Q_+Q_- + Q_+Q_- = H$ a supersymmetry algebra [74].

2.2 Cohomological Aspects of Supersymmetric Quantum Mechanics

Moreover, Witten [74, 75] analyzed the cohomology of supersymmetry algebras, see Fig. 2.1. Here, we denote $\mathcal{H}^k$ for the k-particle Hilbert space and the Fock space is

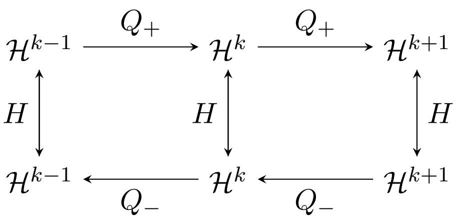

Figure 2.1: The top sequence in the diagram defines a cochain complex, while the bottom sequence defines a chain complex .

$$\mathcal{H} = \bigoplus_{k=0}^{\infty} \mathcal{H}^k. \tag{2.24}$$

A supersymmetry algebra fulfills the following properties, see Ref. [74]: if $\psi \in \mathcal{H}$ is a state with energy $E \neq 0$, then $Q_+\psi \in \mathcal{H}$ has also energy E. Also $Q_-\psi \in \mathcal{H}$ has energy E as H commutes with Q_+ and Q_-. Furthermore, for $E \neq 0$ the equation $Q_+\psi = 0$ has a solution $\psi = Q_+\varphi$ with $\varphi = \frac{1}{E}Q_-\psi$ because

$$\frac{1}{E}Q_+Q_-\psi = \frac{1}{E}(Q_+Q_- + Q_-Q_+)\psi = \frac{1}{E}H\psi = \psi,$$

where we used $Q_+\psi = 0$. For vacuum states with $E = 0$ we have $H\psi = 0$ and therefore $\langle\psi|H|\psi\rangle = \langle\psi|(Q_+Q_- + Q_-Q_+)|\psi\rangle = 2\|Q_+\psi\| = 0$. Consequently, vacuums states ψ fulfill $Q_+\psi = 0$.

On the other hand, a vacuum state ψ cannot be written as $Q_+\varphi$, because then φ would have the same energy $E = 0$ and therefore one would have $Q_+\varphi = 0 \neq \psi$. Thus, vacuum states are exactly those states, which vanish under Q_+ but cannot be written as $Q_+\varphi$.

As a direct consequence it follows that [74]

$$\dim\ker(H) = \dim(\ker Q_+/\mathrm{im}\ Q_+). \tag{2.25}$$

Now, let M be a compact and orientable Riemannian manifold of dimension n. The exterior derivative d and its adjoint δ, which acts on differential forms (see Appendix) fulfill

$$d^2 = \delta^2 = 0 \tag{2.26}$$

and the Hodge Laplacian is given by [48]

$$\Delta = d\delta + \delta d. \tag{2.27}$$

Following Ref. [75], by formally setting

$$Q_+ = d \tag{2.28}$$

and

$$Q_- = \delta \tag{2.29}$$

one obtains the supersymmetry algebra $\{d, \delta\} = \Delta$ and the Hamiltonian becomes

$$H = \Delta. \tag{2.30}$$

In this case the supercharges are [75]

$$Q_1 = d + \delta \tag{2.31}$$

and

$$Q_2 = i(d - \delta). \tag{2.32}$$

Therefore, one has [75]

$$\mathcal{H}^k = A^k, \tag{2.33}$$

where A^k is the space of k-forms on M (see Appendix). Then, the Fock space becomes the exterior algebra

$$\mathcal{H} = \bigoplus_{k=0}^{n} A^k, \tag{2.34}$$

since $A^k = 0$ for $k > n$. One can derive [74] this special realization of the supersymmetry algebra by considering the supersymmetric nonlinear sigma model [5, 22, 73, 82]. Then, by interpreting k-forms as bosons, if k is even and as fermions, if k is odd, one obtains [74]

$$\mathrm{Tr}(-1)^F e^{-\beta H} = \chi(M), \tag{2.35}$$

where $\chi(M)$ is the Euler characteristic, a topological invariant of M (see Appendix). One can connect supersymmetric quantum theory with topology in a more general context. Witten [75] interpreted the Morse inequalities in terms of supersymmetric quantum mechanics. Another deep insight was the derivation of the Atiyah-Singer index theorem by using supersymmetric quantum mechanics [4, 28, 29, 31, 32].

2.3 Example: Harmonic Oscillator

In this section, we summarize some facts about the supersymmetric harmonic oscillator from Ref. [51].

2.3.1 Bosonic and Fermionic Harmonic Oscillators

First the bosonic creation and annihilation operators are given by

$$\hat{b}^{\pm} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} \mp \frac{i\hat{p}}{m\omega} \right) \tag{2.36}$$

with the mass m, the frequency ω, the position operator $\hat{x}$ and the momentum operator $\hat{p}$. One has the commutation relation

$$\hat{b}^{-}\hat{b}^{+} - \hat{b}^{+}\hat{b}^{-} = 1. \tag{2.37}$$

For the bosonic Hamiltonian

$$H_B = \hbar\omega \left(\hat{b}^{+}\hat{b}^{-} + \frac{1}{2} \right) \tag{2.38}$$

the energy eigenvalues are

$$E_{n_B} = \hbar\omega \left(n_B + \frac{1}{2} \right) \tag{2.39}$$

with integers $n_B \geq 0$.

Moreover, for the fermionic creation and annihilation operators one has the commutation relation

$$\hat{f}^{+}\hat{f}^{-} + \hat{f}^{-}\hat{f}^{+} = 1. \tag{2.40}$$

For the fermionic Hamiltonian

$$H_F = \hbar\omega \left(\hat{f}^{+}\hat{f}^{-} - \frac{1}{2} \right) \tag{2.41}$$

the energy eigenvalues are

$$E_{n_F} = \hbar\omega \left(n_F - \frac{1}{2} \right) \tag{2.42}$$

with integers $n_F \in \{0, 1\}$.

2.3.2 The Supersymmetric Harmonic Oscillator

The supersymmetric creation and annihilation operators are given by

$$Q_{+} = \sqrt{\hbar\omega}\hat{b}^{-}\hat{f}^{+} \tag{2.43}$$

and

$$Q_- = \sqrt{\hbar\omega}\hat{b}^+ \hat{f}^-. \tag{2.44}$$

The supersymmetric Hamiltonian is [see Eq. (2.23)]

$$H_{SUSY} = Q_+ Q_- + Q_- Q_+ = \hbar\omega(\hat{b}^+ \hat{b}^- + \hat{f}^+ \hat{f}^-). \tag{2.45}$$

With Eqs. (2.41) and (2.38) one has

$$H_{SUSY} = H_B + H_F \tag{2.46}$$

and the energy spectrum becomes

$$E = \hbar\omega(n_B + n_F). \tag{2.47}$$

Since n_F can only be 0 or 1, there are exactly two representations for every energy E, one bosonic ($n_F = 0$) and one fermionic ($n_F = 1$).

Chapter 3

Quantum Mechanics on Conformal Manifolds

This chapter presents a revised version of Chapter 3 and Sect. 4.1 of my Master's thesis [77]. By doing so, some novel notations are introduced, which are useful for the further work. In order to motivate the supersymmetric formulation of quantum mechanics on conformal manifolds this chapter outlines a first conformal geometric approach to ordinary quantum mechanics. In Subsect. 3.1.1 and Subsect. 3.2.5 we present completely new results.

3.1 The Quantum Maupertuis Principle

In the following let (M, g) be a n-dimensional Riemannian manifold with metric g. On this manifold we consider a motion $q(t)$ in a potential V with energy E_0 fulfilling $\sup\{V(q) \mid q \in M\} < E_0$. Then,

$$E_0 = T + V = \frac{1}{2} g_{ij} \dot{q}^i \dot{q}^j + V(q), \tag{3.1}$$

where $1 \leq i, j \leq n$ and the dot denotes derivatives with respect to t. Here and in the sequel we make use of Einstein's sum convention that repeated indices have to be summed over.

The classical Maupertuis principle states the following, see Ref. [2]: upon introducing the new metric $\tilde{g} = 2(E_0 - V)g$ and parameterizing the motion with τ, where $d\tau = 2\sqrt{(E_0 - V)}dt$, the motion $\tilde{q}(\tau)$ becomes a geodesic on $(M, \tilde{g})$, i.e.

$$\tilde{\Gamma}^k_{ij} \frac{\partial \tilde{q}^i}{\partial \tau} \frac{\partial \tilde{q}^j}{\partial \tau} + \frac{\partial^2 \tilde{q}^k}{\partial \tau^2} = 0. \tag{3.2}$$

To generalize this picture to a quantum mechanical particle, we follow Refs. [20, 52] by considering a special conformally deformed metric on the configuration manifold. By doing so, we introduce some modifications of the findings in Refs. [20, 52] in order to introduce a formalism, which is needed for further investigations. Therefore, we explicitly distinguish the analysis for repulsive and attractive potentials, what allows us to consider more realistic applications, see, in particular, our findings in Subsect. 4.3.2. Moreover, our approach leads to an interesting generalization of the quantum Maupertuis principle to higher dimensions.

We consider a compact and orientable Riemannian manifold (M, g) and a smooth potential $V : M \to \mathbb{R}$ with $V \geq 0$ and $V \to \infty$ as $|x| \to \infty$. Here, we exclude oscillating potentials, as then $\psi \neq 0$ on a non-compact manifold.

Then, the Hamiltonian is (for $\hbar = 1$) given by

$$H = -\frac{1}{2m}\Delta + V, \tag{3.3}$$

with the Laplacian

$$\Delta = \frac{1}{\sqrt{\det g}}\,\partial_i\left(\sqrt{\det g}\;g^{ij}\partial_j\right). \tag{3.4}$$

Thus, the stationary Schrödinger equation for an eigenfunction from Sobolev space $\psi_i \in \mathcal{H}^2(M)$ with eigenvalue E_i (conceivably with degeneracy) becomes

$$H\psi_i = E_i\psi_i. \tag{3.5}$$

Now we consider for a fixed $E > 0$

$$M^E = \{x \in M \mid V(x) < E\}. \tag{3.6}$$

Then, any eigenfunction of H decreases exponentially on $M \setminus M^E$ and it is sufficient to consider only the motion of the particle on M^E.

In analogy to Ref. [20], we now perform a Weyl transformation of the metric $g \to \widetilde{g}$ with

$$\widetilde{g} = \left(1 - \frac{V}{\alpha}\right) g \tag{3.7}$$

with $\alpha \in [E, \infty)$ guaranteeing that $\widetilde{g} = \widetilde{g}(\alpha)$ is positive definite on M^E. Under this transformation the Laplacian $\widetilde{\Delta}$ on $(M^E, \widetilde{g})$ becomes

$$\begin{aligned}
\widetilde{\Delta} &= \frac{1}{\sqrt{\det \widetilde{g}}}\,\partial_i\left(\sqrt{\det \widetilde{g}}\;\widetilde{g}^{ij}\partial_j\right) \\
&= \frac{1}{\sqrt{\det \widetilde{g}}}\,\partial_i\left(\left(1-\frac{V}{\alpha}\right)^{\frac{n}{2}-1}\sqrt{\det g}\;g^{ij}\partial_j\right) \\
&= \frac{1}{\alpha}\left(\frac{n}{2}-1\right)\left(1-\frac{V}{\alpha}\right)^{-2} g^{ij}\partial_i V\partial_j + \left(1-\frac{V}{\alpha}\right)^{-1}\frac{1}{\sqrt{\det g}}\,\partial_i\left(\sqrt{\det g}\;g^{ij}\partial_j\right) \\
&= \frac{1}{\alpha}\left(\frac{n}{2}-1\right)\left(1-\frac{V}{\alpha}\right)^{-2} g^{ij}\partial_i V\partial_j + \left(1-\frac{V}{\alpha}\right)^{-1}\Delta,
\end{aligned} \tag{3.8}$$

with

$$\sqrt{\det \widetilde{g}} = \left(1-\frac{V}{\alpha}\right)^{\frac{n}{2}}\sqrt{\det g}. \tag{3.9}$$

We see that only for a constant potential V or $n = \dim(M) = 2$

$$\widetilde{\Delta} = \left(1-\frac{V}{\alpha}\right)^{-1}\Delta \tag{3.10}$$

applies on M^E.

Upon setting $\alpha = E$ we obtain the following equivalence:

$$-\frac{1}{2m}\widetilde{\Delta}\psi = E\psi \iff \left(-\frac{1}{2m}\Delta + V\right)\psi = E\psi. \tag{3.11}$$

Thus, the eigenfunctions ψ of H are the eigenfunctions of the free Hamiltonian $-\frac{1}{2m}\widetilde{\Delta}$ with eigenvalues $E \in \frac{1}{2m}\sigma(\widetilde{\Delta})$, where $\sigma(\widetilde{\Delta})$ denotes the spectrum of $\widetilde{\Delta}$ (see Appendix). This is the sought quantum mechanical generalization of the Maupertuis principle, cf. Refs. [20, 52]: 2-dimensional motions in a repulsive potential and n-dimensional motions in constant potentials on (M, g) are given by free motions on $(M, \widetilde{g})$. The energy eigenvalues are given by the spectrum of $-\frac{1}{2m}\widetilde{\Delta}$. This can be easily generalized to the case of an attractive potential $V = -|V|$, i.e. for a Hamiltonian

$$H = -\frac{1}{2m}\Delta - |V|. \tag{3.12}$$

Again, we restrict the motion onto M^E. The following analysis holds for both unbound ($E \geq 0$) and bound states ($E < 0$). Then, the Weyl transformation

$$\widetilde{g} = \left(1+\frac{|V|}{\beta}\right)g, \tag{3.13}$$

with $\beta \in [E, \infty)$ gives for $n = 2$ analogously to the repulsive case

$$\widetilde{\Delta} = \left(1+\frac{|V|}{\beta}\right)^{-1}\Delta. \tag{3.14}$$

Upon setting $\beta = E$ we get $E \in -\frac{1}{2m}\sigma(\widetilde{\Delta})$.

Thus, this somewhat naive approach of deriving a quantum version of the Maupertuis principle only works in two dimensions while the classical principle is valid for arbitrary dimension.

3.1.1 The Quantum Maupertuis Principle in Higher Dimensions

However, for arbitrary dimension $\dim(M) = n$ one can choose a somewhat different approach. By again first considering the case $V \geq 0$ Eq. (3.8) implies (with the above notation)

$$-\frac{1}{2m}\widetilde{\Delta}\psi = E\psi \iff -\frac{1}{2mE}\left(\frac{n}{2}-1\right)\widetilde{g}^{ij}\partial_i V\partial_j\psi - \frac{1}{2m}\Delta\psi + V\psi = E\psi. \tag{3.15}$$

For $n \neq 2$ we can choose the gauge field

$$A^j = -i\hbar\widetilde{g}^{ij}\partial_i V = \widetilde{g}^{ij}\widehat{p}_i V = \widehat{p}^j V, \tag{3.16}$$

with the momentum operator

$$\widehat{p}^j = -i\hbar\widetilde{g}^{ij}\partial_i \tag{3.17}$$

and the coupling constant

$$q = \frac{1}{2E}\left(\frac{n}{2}-1\right), \tag{3.18}$$

leading to

$$-\frac{1}{2m}\widetilde{\Delta}\psi = E\psi \iff \frac{1}{2m}(\widehat{p}-qA)^2\psi + \widetilde{V}\psi = E\psi, \tag{3.19}$$

with

$$\widetilde{V} = V - \frac{q^2}{2m}A^2. \tag{3.20}$$

The case of an attractive potential can be obtained from the last equations by replacing $V \to -|V|$.

Thus, in dimensions $n \neq 2$ the quantum Maupertuis principle states the equivalence between the free motion of an uncharged particle on a curved manifold with the motion of a charged particle in flat space. For large energies, the coupling constant vanishes and the charged particle behaves like an uncharged one. For $n = 2$ we get the result (3.11). Moreover, we see that the coupling constant increases linearly with the dimension n.

3.2 Conformal Geometry and Quantum Mechanics

The above analysis shows that the derivation of the quantum Maupertuis principle in two dimensions is based on a special conformal transformation. On the other hand, for higher dimensions we have to introduce a gauge field to obtain the equivalence between the free motion of an uncharged particle on a curved manifold with the motion of a charged particle in flat space. In fact, there is a close relation between these two approaches. This becomes already evident in conventional quantum mechanics, where one makes use of the invariance of the Schrödinger equation under $U(1)$-transformations of the wave function $\psi \to \lambda\psi$ [with $\lambda \in U(1)$] to describe a particle coupled to an electromagnetic field. Thus, the description of the dynamics of the particle, formally given by a gauge field, is based on the conformal invariance of the Schrödinger equation. This connection between conformal geometry and interactions is so far not very profound, since only a simple rescaling of the metric tensor is required.

Moreover, it is not obvious if a Schrödinger operator of the form $-D_iD^i+V$, as encountered in Eqs. (3.11) and (3.15), can be linked with conformal geometry in a natural way. Insights could be gained if D and V had a direct geometric interpretation and could describe real physical systems. Also, the connection between the more sophisticated formulation of quantum mechanics by the Fock space representation, which leads to many-particle quantum mechanics and quantum field theory, has no interpretation in the naive approach of the quantum Maupertuis principle. To address these issues, we take in this section a different route and make use of the generic conformal invariance of quantum mechanics, namely the invariance of the probability density under conformal transformations of the amplitude of the wave function. In more geometrical terms, wave functions are sections of a line bundle (see Appendix) and for a rigorous and more general formulation of the quantum Maupertuis principle it is necessary to consider such a line bundle over the conformal class. To elucidate this connection further we use here the concepts of conformal geometry to introduce a geometric formulation of quantum mechanics. Later we will see that this approach has a supersymmetric analogue, which itself yields novel insights into the topological characterization of quantum systems. The quantum Maupertuis principle for higher dimensions will be a special case of this supersymmetric formulation.

3.2.1 Geometric Hamiltonians

In conformal geometry one considers on an arbitrary Riemannian manifold (M, g) of dimension n the conformal class $[g]$ given by

$$[g] \equiv \{\lambda g \mid \lambda \in C^{\infty}(M, \mathbb{R}^{+})\}. \tag{3.21}$$

The pair $(M, [g])$ is called a conformal manifold. Here, two metrics $\widetilde{g}$ and g are equivalent if and only if there is a positive and smooth function λ with $\widetilde{g} = \lambda g$. Thus, the metric on a conformal manifold is only defined up to a scale factor and it is not possible to define the length of tangent vectors. However, the angle between two tangent vectors is well-defined since for all $v, w \in T_pM$

$$\frac{g_p(v,w)}{\sqrt{g_p(v,v)}\sqrt{g_p(w,w)}} = \frac{\widetilde{g}_p(v,w)}{\sqrt{\widetilde{g}_p(v,v)}\sqrt{\widetilde{g}_p(w,w)}} = \cos(\measuredangle(v,w)). \tag{3.22}$$

In this sense $[g]$ induces a well-defined conformal structure on the tangent bundle TM of M. Furthermore, the positivity and smoothness of $\lambda : M \to \mathbb{R}^{+}$ can be implemented by setting $\lambda = e^{\phi}$ for a smooth function ϕ on M.

The Christoffel symbols [given by Eq. (3.36)] vary under a Weyl transformation $g \to \widetilde{g} = \lambda g$. A simple computation shows that for given $\lambda = e^{\phi}$ one has

$$\widetilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + \frac{1}{2}(\delta^k_i \partial_j \phi + \delta^k_j \partial_i \phi - \delta_{ij} \partial^k \phi). \tag{3.23}$$

Here and in the sequel quantities with a tilde are associated with metric $\widetilde{g}$. Similarly, for the Ricci tensor

$$R_{ij} = \partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik} + \Gamma^l_{ij}\Gamma^k_{kl} - \Gamma^l_{ik}\Gamma^k_{lj}, \tag{3.24}$$

one gets

$$\widetilde{R}_{ij} = R_{ij} - \frac{n-2}{2}\nabla_i \nabla_j \phi + \frac{n-2}{4}(\nabla_i \phi)(\nabla_j \phi) - \frac{1}{2}\left(\Delta\phi + \frac{n-2}{4}\|\nabla\phi\|^2\right) g_{ij}. \tag{3.25}$$

Furthermore,

$$\widetilde{R} = e^{-\phi}\left(R - (n-1)\Delta\phi - \frac{(n-1)(n-2)}{4}\|\nabla\phi\|^2\right), \tag{3.26}$$

for the scalar curvature of the conformally transformed metric $\widetilde{g}$. Note, in the last two equations all quantities on the right hand side are associated with the initial metric g.

Here, we start with an arbitrary pseudo-Riemannian manifold (M, g) of dimension n and a conformal class defined by smooth functions $\lambda \in C^\infty(M, \mathbb{R}^+)$. We beginning with the Hilbert space

$$\mathcal{H} = \{\psi \in C^\infty(M, \mathbb{C}) \mid \langle\psi|\psi\rangle < \infty\} \tag{3.27}$$

where

$$\langle\psi|\varphi\rangle = \int_M dM\, \overline{\psi}\varphi \tag{3.28}$$

is the inner product.

Moreover, on the conformal class we introduce

$$\widetilde{\mathcal{H}}_\lambda = \{\lambda\psi \mid \psi \in \mathcal{H}\} \tag{3.29}$$

and the inner product

$$\langle\cdot|\cdot\rangle_\lambda : \ \widetilde{\mathcal{H}}_\lambda \times \widetilde{\mathcal{H}}_\lambda \to \mathbb{R} \tag{3.30}$$

give by

$$\langle\psi_\lambda|\varphi_\lambda\rangle = \int_M dM\, \lambda^{\frac{n}{2}}\overline{\psi}_\lambda\varphi_\lambda \tag{3.31}$$

for $\psi_\lambda, \varphi_\lambda \in \widetilde{\mathcal{H}}_\lambda$.

Now, we define the Hilbert space on the conformal class as

$$\mathcal{H}_\lambda = \{\psi_\lambda \in \widetilde{\mathcal{H}}_\lambda \mid \langle\psi_\lambda|\psi_\lambda\rangle < \infty\}. \tag{3.32}$$

The covariant derivative ∇ can be extended to a conformally invariant covariant derivative by

$$D_i = \nabla_i - \partial_i \ln\lambda = \nabla_i - \frac{\lambda_i}{\lambda}. \tag{3.33}$$

Here, D acts on the functions of $\mathcal{H}_\lambda$ and $\lambda_i = \partial_i\lambda = \frac{\partial\lambda}{\partial x^i}$. D is also called a Weyl connection as Weyl [72] introduced such a conformally invariant covariant derivative in the context of general relativity, see also Refs. [64, 65]. A simple calculation shows that D_i is indeed conformally invariant because for any $\psi_\lambda \in \mathcal{H}_\lambda$ there exists a $\psi \in C^\infty(M, \mathbb{C})$ with $\psi_\lambda = \lambda\psi$ and we obtain

$$D_i(\psi_\lambda) = D_i(\lambda\psi) = \lambda(\nabla_i\psi). \tag{3.34}$$

Similarly, for any covariant vector field $V : M \to TM$ with $\widetilde{V} = \lambda V$ one has

$$D_i\widetilde{V}_j = \lambda\nabla_i V_j, \tag{3.35}$$

since the Christoffel symbols

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}), \tag{3.36}$$

act as multiplication operators.

Because the gradient of a scalar field transforms like a tensor, D is indeed conformally invariant and covariant. Moreover, we can define a dual conformally invariant covariant derivative by

$$D^i = g^{ij}\nabla_j - g^{ij}\partial_j \ln(\lambda^{-1}) = \nabla^i + \frac{\partial^i \lambda}{\lambda}. \tag{3.37}$$

which acts on $\mathcal{H}_{\lambda^{-1}}$.

For any $\psi_{\lambda^{-1}} \in \mathcal{H}_{\lambda^{-1}}$ that can be written as $\psi_{\lambda^{-1}} = \lambda^{-1}\psi$ with $\psi \in C^\infty(M, \mathbb{C})$ one has then

$$D^i(\lambda^{-1}\psi) = \left(\nabla^i + \frac{1}{\lambda}\partial^i\lambda\right)(\lambda^{-1}\psi) = \lambda^{-1}(\nabla^i\psi). \tag{3.38}$$

The curvature tensor Ω_{ij} is given by

$$\Omega_{ij}\varphi \equiv (D_j D_i - D_i D_j)\varphi, \tag{3.39}$$

for any $\psi_\lambda \in \mathcal{H}_\lambda$. By using the fact that the Levi-Civita connection is torsion-free, i.e. $\Gamma^k_{ij} = \Gamma^k_{ji}$, we get $\nabla_i\nabla_j f = \nabla_j\nabla_i f$ for every smooth scalar function f on M and therefore

$$\begin{aligned}\Omega_{ij}\psi_\lambda &= \nabla_i[\partial_j \ln(\lambda)\psi_\lambda] - \nabla_j[\partial_i \ln(\lambda)\psi_\lambda] + \lambda^{-1}\partial_i\lambda\partial_j\psi_\lambda - \lambda^{-1}\partial_j\lambda\partial_i\psi_\lambda \\ &= [\nabla_i(\partial_j \ln(\lambda)) - \nabla_j(\partial_i \ln(\lambda))]\psi_\lambda = 0.\end{aligned} \tag{3.40}$$

Because $\nabla_i g_{jk} = 0$ one has $\nabla^j\nabla_i f = \nabla_i\nabla^j f$ for any differentiable function f on M and thus

$$\Omega^j_i\psi_\lambda = (D^j D_i - D_i D^j)\psi_\lambda = -2\nabla_i(\partial^j \ln(\lambda))\psi_\lambda. \tag{3.41}$$

Note, Ω_{ij} does transform as a tensor, but indices cannot be raised or lowered by the metric tensor g_{ij} but with regard to the conformal structure, i.e. with $\lambda^{-1}g^{ij}$ and by replacing λ by λ^{-1}.

Moreover, we get

$$\Omega^{ij}\psi_\lambda = (D^j D^i - D^i D^j)\psi_\lambda = 0. \tag{3.42}$$

Thus, associated with the curvature tensor (3.39) is a scalar curvature S with

$$S\psi_\lambda = (D^i D_i - D_i D^i)\psi_\lambda = -2[\Delta \ln(\lambda)]\psi_\lambda, \tag{3.43}$$

where $\Delta = \nabla_i \nabla^i$ is the Laplacian for the metric g. Thus, S is constant on $\mathcal{H}_\lambda$ but generally not on M.

In terms of the conformally invariant covariant derivative, $H = -D^i D_i$ is a natural choice for a free Hamiltonian. To allow for an interaction with the curvature of the underlying manifold one can consider the geometric Hamiltonian

$$H = -D^i D_i + K. \tag{3.44}$$

Here, the Gaussian curvature K (associated with the metric g) acts as a geometric potential.

Since

$$S = -2\frac{1}{\lambda}\Delta \ \ln(\lambda) = 4K \tag{3.45}$$

for $g_{ij} = \lambda \delta_{ij}$ we can interpret

$$H = -D^i D_i + \frac{1}{4} S \tag{3.46}$$

as a generalization of (3.44) acting on $\mathcal{H}_\lambda$. Therefore, the conformally invariant covariant derivative and its scalar curvature induce a symmetric differential operator of order two.

3.2.2 The Energy Spectrum

To understand the impact of the geometry of (M, g) we have to analyze the energy spectrum of H. This cannot be done for general λ and general Eq. (3.46). However, quite general results can be obtained in the case where D^i and D_i can be interpreted as creation and annihilation operators.

Vacuum States

To see when this is the case, we introduce vacuum states as states which vanish under the action of the conformally invariant covariant derivative. Thus, D_i, as defined by Eq. (3.33), can be seen as a generalization of the bosonic annihilation operator of the harmonic oscillator

$$a_i = \frac{1}{\sqrt{2}}(\partial_i + x_i). \tag{3.47}$$

First, we show that λ is a non-degenerate vacuum state of H. Because λ is square-integrable we have $\lambda \in \mathcal{H}_\lambda$ and from relation (3.34) we get $D_i \lambda = \lambda(\nabla_i 1_M) = 0$. Let $\varphi \in \mathcal{H}_\lambda$ be another vacuum state. Then, there exists a function $\psi \in C^\infty(M, \mathbb{C})$ with $\varphi = \lambda\psi$ and $D_i(\lambda\psi) = \lambda(\nabla_i \psi) = 0$ implying that ψ is constant. Because φ is not zero, ψ is also not zero. Thus, φ and

λ belong to the equivalence class $[\lambda] = \{\alpha\lambda \mid \alpha \in \mathbb{C} \setminus \{0\}\}$ describing the same quantum state and λ is a non-degenerate vacuum state of H. Because of $H\lambda = \frac{1}{4}S\lambda$, the vacuum energy is $\frac{1}{4}S$. And therefore λ is a non-degenerate vacuum state of H with energy $\frac{1}{4}S$.

Characterization of the Energy Spectrum

We can characterize the spectrum of H explicitly by

$$\begin{aligned} H(-D^j\lambda) &= -D^iD_i(-D^j\lambda) + \frac{1}{4}S(-D^j\lambda) = D^i[D^jD_i - \Omega_i^j]\lambda + \frac{1}{4}S(-D^j\lambda) \\ &= -D^i\Omega_i^j\lambda + \frac{1}{4}S(-D^j\lambda), \end{aligned} \tag{3.48}$$

where in the last step we have used $D_i\lambda = 0$. Therefore, $-D^j\lambda$ is an eigenfunction of H, if and only if $-D^i\Omega_i^j\lambda$ is proportional to $D^j\lambda$ or $-D^i\Omega_i^j\lambda = 0$, i.e. if and only if $\Omega_i^j = \omega\delta_i^j$ with $\omega \in \mathbb{R}$. For $\Omega_i^j = \omega\delta_i^j$, with a constant $\omega \in \mathbb{R}$, the scalar curvature is constant with $S = n\omega$. Furthermore, if $\Omega_i^j = \omega\delta_i^j$ with $\omega \in \mathbb{R}$ and k is a non-negative integer, then $(-D^j)^k\lambda$ is an eigenfunction of H with eigenvalue $k\omega + \frac{1}{4}S$.

We give a proof by induction. For $k = 1$ we have

$$H(-D^j\lambda) = \left(\omega + \frac{1}{4}S\right)(-D^j\lambda). \tag{3.49}$$

If $H((-D^j)^k\lambda) = \left(k\omega + \frac{1}{4}S\right)((-D^j)^k\lambda)$ holds for $n > 1$, then

$$\begin{aligned} -D^iD_i(-D^j)^{k+1} &= -D^iD_i(-D^j)(-D^j)^k\lambda = D^jD^iD_i(-D^j)^k\lambda - D^i\Omega_i^j(-D^j)^k\lambda \\ &= (-D^j)(k\omega)(-D^j)^k\lambda - \omega\delta_i^jD^i(-D^j)^k\lambda \qquad (3.50) \\ &= (k+1)\omega(-D^j)^{k+1}\lambda, \qquad (3.51) \end{aligned}$$

as stated.

As a consequence, we obtain a complete classification of the energy spectrum of H (for $\Omega_i^j = \omega\delta_i^j$ with $\omega \in \mathbb{R}$): for any non-negative integers $k_1, k_2, \ldots, k_n$

$$\left(\prod_{i=1}^n(-D^i)^{k_i}\right)\lambda, \tag{3.52}$$

is an eigenfunction of H with eigenvalue $(k_1 + k_2 + \cdots + k_n)\omega + \frac{1}{4}S$. Moreover, let $\psi_\lambda \in \mathcal{H}_\lambda$ be an eigenfunction of H with eigenvalue γ, then $D_j\psi_\lambda$ is an eigenfunction of H with eigenvalue $\gamma - \omega$ since

$$-D^iD_i(D_j\psi_\lambda) = -[D_jD^i + \Omega_j^i]D_i\psi_\lambda = (\gamma - \omega)D_j\psi_\lambda. \tag{3.53}$$

3.2.3 Fock Space Representations

The above analysis shows that the conformally invariant covariant derivative and its dual analogue define generalized ladder operators.

Upon using the standard notation

$$a_i \equiv \frac{1}{\sqrt{2}} D_i, \tag{3.54}$$

for the annihilation operator and

$$a_i^\dagger \equiv -\frac{1}{\sqrt{2}} D^i, \tag{3.55}$$

for the creation operator we obtain

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \tag{3.56}$$

and

$$[a_i, a_j^\dagger] = -\partial_i \partial^j \ln(\lambda) - \Gamma^j_{ik} \partial^k \ln(\lambda). \tag{3.57}$$

Thus, the commutator given by Eq. (3.57) depends via the Christoffel symbols on the metric emphasizing the geometrical meaning of our Fock space representation. Only if the curvature tensor Ω_i^j is equal to δ_i^j we get the usual commutator algebra for bosons.

3.2.4 The Harmonic Oscillator

Thus, the Fock space representation of quantum mechanics is only obtained for the case of constant scalar curvature $S = \omega n$ for a manifold M of dimension n. Of course, given the arbitrariness of M there are many ways how this can be realized. The simplest case is the Euclidean space, i.e. $M = \mathbb{R}^n$ and $g_{ij} = \delta_{ij}$. Then,

$$\Omega_i^j \psi_\lambda = -2\nabla_i(\partial^j \ln(\lambda))\psi_\lambda = \omega \delta_i^j \varphi, \tag{3.58}$$

for an arbitrary coordinate function x^i, is solved by

$$\lambda = e^{-\phi}, \tag{3.59}$$

with the quadratic form

$$\phi = \frac{1}{2} \sum_{k=1}^{n} x_k^2. \tag{3.60}$$

Here, we choose $\omega = 2$ and therefore $S = 2\dim(M) = 2n$ yielding

$$D_i = \partial_i + x_i, \tag{3.61}$$

and

$$D^i = \partial^i - x^i. \tag{3.62}$$

Therefore, D_i and $-D^i$ are the ordinary ladder operators of the harmonic oscillator.

Because of

$$L^2(\mathbb{R}, d\mu) \otimes L^2(\mathbb{R}, d\mu) \otimes \cdots \otimes L^2(\mathbb{R}, d\mu) \cong L^2(\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}, d\mu \times d\mu \times \cdots \times d\mu), \tag{3.63}$$

the functions

$$\lambda^{-1}(-D^n)^{k_n} \ldots (-D^2)^{k_2}(-D^1)^{k_1}\lambda, \tag{3.64}$$

with any non-negative integers $k_1, k_2, \ldots, k_n$ build an orthogonal basis of $L^2(\mathbb{R}^n, \mu(\mathbb{R}^n))$ with the n-dimensional Gauss measure $d\mu = \lambda \cdot dx^1 \ldots dx^n$. Here, the Fock states

$$(-D^n)^{k_n} \ldots (-D^2)^{k_2}(-D^1)^{k_1}\lambda, \tag{3.65}$$

are eigenstates of H. The L^2-spectrum of H is given by

$$(k_1 + k_2 + \cdots + k_n)\omega + \frac{n}{2}, \tag{3.66}$$

where we can interpret the numbers $k_1, k_2, \ldots, k_n$ as elementary modes of the oscillator.

3.2.5 A Representation in Curved Space

We now derive a Fock space representation for quantum mechanics with a non-trivial geometry. There are two ways to generalize from the Euclidean space to a curved space: (i) Intrinsically, for a harmonic potential that depends on a tangent vector q^i, i.e.

$$V = g_{ij}q^i q^j. \tag{3.67}$$

Then the associated motion is that of a free particle on a curved manifold. (ii) By coupling it to an external field $\phi(x)$ on M. Here, we employ (ii) and therefore consider a pseudo-Riemannian manifold (M, g) of dimension n with local coordinates $\{x_m\}$ and

$$g = \mathrm{diag}(x_1, \ldots, x_n). \tag{3.68}$$

As a specific example, we want to find a representation of the Euclidean harmonic oscillator on M. As we will see now, for this purpose it is sufficient to consider a linear field $\phi = -2\omega\sum_{m=1}^{n} x_m$ on M. As the following analysis shows the coupling occurs via $\lambda = \exp(\phi)$. To see this, we first calculate the curvature tensor. Because of

$$\partial_i\partial^j \ln(\lambda) = \partial_i\partial^j(-2\sum_{m=1}^{n} x_m) = 0, \tag{3.69}$$

one has

$$\Omega_i^j = \Gamma_{ik}^j \partial^k \ln(\lambda) = -2\omega \sum_{k=1}^{n} \Gamma_{ik}^j = -\omega \sum_{k=1}^{n} g^{jl}(\partial_i g_{kl} + \partial_k g_{il} - \partial_l g_{ik}). \tag{3.70}$$

Because g^{jl} is diagonal, only the term with $j = l$ contributes. Furthermore, for $i \neq j$ all three terms in the sum are zero. Thus, the only contribution comes from $i = j$ and thus,

$$\Omega_i^j = \omega\delta_i^j g^{jj}\partial_j g_{jj} = \omega\delta_i^j,$$

where in the second identity j is not summed over. Therefore,

$$\Omega_i^j = [a_i, a_j^\dagger] = \delta_i^j, \tag{3.71}$$

yielding the Fock space representation of quantum mechanics on a pseudo-Riemannian manifold with non-trivial geometry. Therefore, $(-D^n)^{k_n}\dots(-D^2)^{k_2}(-D^1)^{k_1}\lambda$ are eigenstates of the Hamiltonian (3.46) with energy $(k_1 + k_2 + \dots + k_n)\omega + \frac{n}{2}$. Therefore, we see that ϕ is the potential of the harmonic oscillator in a curved space and λ is the generic vacuum state. Thus, we have obtained the equivalence between the motion in a harmonic oscillator potential in Euclidean space with the motion in a linear potential in curved space.

3.2.6 $U(1)$-Transformations

A generic class of conformal transformations is defined by the Lie group $U(1)$, because all representatives of the equivalence class

$$[\psi] = \{e^{i\omega}\psi \mid \omega \in C^\infty(M, \mathbb{R})\}, \tag{3.72}$$

describe the same quantum state from the probabilistic point of view. We now consider over $\mathbb{C}$

$$\widetilde{\mathcal{H}}_\lambda^{\mathbb{C}} = \{\lambda\psi \mid \psi \in \mathcal{H}\}, \tag{3.73}$$

with $\lambda = e^{\phi}$ and $\phi \in C^{\infty}(M, \mathbb{C})$ yielding the complex-valued version of our quantum Hilbert space

$$\mathcal{H}_{\lambda}^{\mathbb{C}} = \{\psi_{\lambda} \in \widetilde{\mathcal{H}}_{\lambda}^{\mathbb{C}} \mid \langle \psi_{\lambda} | \psi_{\lambda} \rangle_{\lambda} < \infty\}. \tag{3.74}$$

In the following we choose $\phi = i\omega$ with $\omega \in C^{\infty}(M, \mathbb{R})$ and the conformally invariant covariant derivate (3.33) becomes

$$D = \nabla - i\nabla\omega. \tag{3.75}$$

Moreover, we have

$$D(e^{i\omega}\psi) = e^{i\omega}\nabla\psi, \tag{3.76}$$

and

$$\begin{aligned} D^2(e^{i\omega}\psi) &= D(e^{i\omega}(\nabla\psi)) \\ &= e^{i\omega}\nabla^2\psi. \end{aligned}$$

If we now consider $\nabla\omega$ as a gauge field $A = \nabla\omega$, we see, that in this framework the gauge invariance of the Schrödinger equation appears in a natural way. Here, the complex-valued scalar curvature is [see Eq. (3.43)]

$$S = -2i\Delta\omega. \tag{3.77}$$

Thus,

$$S = \mathrm{div}A, \tag{3.78}$$

which provides a geometrical connection between curvature and gauge potential: the conformal curvature is the origin of the sources and sinks of the gauge field A.

Chapter 4

Supersymmetric Quantum Mechanics on Conformal Manifolds

So far, we have two loose ends: the quantum Maupertuis principle and the conformal geometric formulation of quantum mechanics. As it turns out there is a fundamental connection between these approaches that gets revealed by supersymmetry. It is well-established that the characterization of suitable supersymmetric quantum theories is closely related to the topology of the underlying configuration manifold, see Refs. [4, 28, 29, 31, 32, 74, 75]. Here, we use the connection between conformal geometry and supersymmetry together to work out properties induced by the topology of the underlying configuration manifold. To get a physical link, we discuss the dependence of the energy gap between vacuum and first excited state on the topology of the underlying configuration manifold. By doing so, we introduce a formulation of quantum mechanics on conformal manifolds by constructing a supersymmetric Fock space representation and obtain a physics proof of the famous eigenvalue estimate by Yang and Yau [80]. In Ref. [77] we have applied this estimate to Schrödinger operators by using the quantum Maupertuis principle. In this work, we are able to extend our physics-based proof to Schrödinger operators in this way providing a non-trivial generalization of the result of Ref. [80].

4.1 Conformal Geometry and Supersymmetric Quantum Mechanics

Now, let $\mathcal{A}^k$ be the space of complex-valued k-forms on an open subset U of an arbitrary compact conformal manifold $(M,[g])$ with the real-valued realization $\lambda^{-1} \in C^\infty(M,\mathbb{R}^+)$. Moreover, we consider the Hilbert space of k-forms

$$\mathcal{H}^k = \{\psi \in \mathcal{A}^k \mid \langle\psi|\psi\rangle < \infty\} \tag{4.1}$$

with the inner product

$$\langle\psi|\varphi\rangle \equiv \int_M \overline{\psi} \wedge *\varphi. \tag{4.2}$$

The Hilbert space of our theory consists of differential forms, which are smooth sections of the exterior power of the cotangent bundle. In analogy to definition (3.37) of the dual conformally invariant covariant derivative in Subsect. 3.2.1, quantum states on the conformal class have to be transformed in the way $\psi \to \lambda^{-1}\psi$ for a conformal transformation g to $\widetilde{g} = \lambda g$.

Therefore, with the notation

$$\widetilde{\mathcal{H}}^k_{\lambda^{-1}} = \{\lambda^{-1}\psi \mid \psi \in \mathcal{H}^k\} \tag{4.3}$$

we define the Hilbert space over the conformal class $[g]$ by

$$\mathcal{H}^k_{\lambda^{-1}} = \{\psi_{\lambda^{-1}} \in \widetilde{\mathcal{H}}^k_{\lambda^{-1}} \mid \langle\psi_{\lambda^{-1}}|\psi_{\lambda^{-1}}\rangle < \infty\}. \tag{4.4}$$

Furthermore, we consider the direct sum

$$\mathcal{H}_{\lambda^{-1}} = \bigoplus_{k=0}^{n} \mathcal{H}^k_{\lambda^{-1}}. \tag{4.5}$$

In supersymmetric quantum field theory one considers particles described by k-forms as bosons, if k is even and as fermions, if k is odd [74, 75].

Let $d : \mathcal{A}^k \to \mathcal{A}^{k+1}$ be the exterior derivative and $\delta : \mathcal{A}^k \to \mathcal{A}^{k-1}$ be the adjoint of the exterior derivative. Then, in analogy to the conformal geometric approach in chapter 3, for $\psi_{\lambda^{-1}} \in \mathcal{H}_{\lambda^{-1}}$ we define raising and lowering operators

$$D_+(\lambda^{-1}\psi) = d(\lambda^{-1}\psi) - (d(\lambda^{-1}))\psi = \lambda^{-1} d\psi, \tag{4.6}$$

and

$$D_-(\lambda^{-1}\psi) = \delta(\lambda^{-1}\psi) - (\delta(\lambda^{-1}))\psi = \lambda^{-1}\delta\psi, \tag{4.7}$$

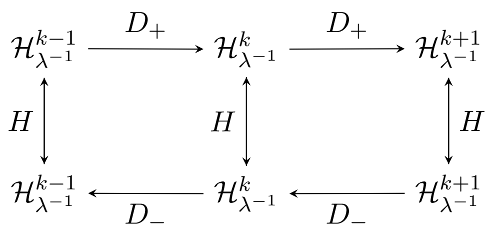

Figure 4.1: The operators D_+ and D_- forming a supersymmetry algebra. D_+ transforms a n-form to a $(n+1)$-form and D_- transforms a n-form to a $(n-1)$-form.

see Fig. 4.1.

They form a $(N = 2)$ supersymmetry algebra on the conformal class by

1. $D_- = D_+^\dagger$,
2. $D_+^2 = D_-^2 = 0$,
3. $-(D_+D_- + D_-D_+) = 2mH$,

where H is a free Hamiltonian for a particle with mass m. This can be seen as follows:

1. For $\varphi = \lambda^{-1}\psi$ and $\theta = \lambda^{-1}\xi$ in $\mathcal{H}_{\lambda^{-1}}$ one has

$$\langle D_+\varphi|\theta\rangle = \int_M |\lambda|^{-2} \cdot d\overline{\psi} \wedge *\xi = \int_M |\lambda|^{-2} \cdot \overline{\psi} \wedge \delta * \xi = \langle\varphi|D_-\theta\rangle. \tag{4.8}$$

2. Because $d^2 = 0$ one has

$$D_+^2(\lambda^{-1}\psi) = D_+(\lambda^{-1}d\psi) = \lambda^{-1}d^2\psi = 0. \tag{4.9}$$

   Similarly, $D_-^2(\lambda^{-1}\psi) = 0$, since $\delta^2 = 0$.

3. One has

$$-(D_+D_-+D_-D_+)(\lambda^{-1}\psi) = -D_+(\lambda^{-1}\delta\psi)-D_-(\lambda^{-1}d\psi) = -\lambda^{-1}(d\delta\psi+\delta d\psi) = -\lambda^{-1}\Delta\psi, \tag{4.10}$$

   with the Hodge Laplacian $\Delta : A^k \to A^k$.

We will see that, especially from the physics point of view, the supersymmetry algebra $\{D_+, D_-\} = 2mH$ is a non-trivial generalization of the supersymmetry algebra $\{d, \delta\} = \Delta$, which is analyzed, for instance, in Refs. [74, 75].

In physical terms:

$$\begin{aligned} D_+|\text{boson}\rangle &= |\text{fermion}\rangle, \\ D_+|\text{fermion}\rangle &= |\text{boson}\rangle, \\ D_-|\text{boson}\rangle &= |\text{fermion}\rangle, \\ D_-|\text{fermion}\rangle &= |\text{boson}\rangle, \end{aligned} \tag{4.11}$$

where the pairs of bosons and fermions have the same energy.

4.1.1 Vacuum States and Topology

Now, we characterize the energy spectrum of our theory. The cohomology of supersymmetry algebras is analyzed in Ref. [74]. Based on this cohomological analysis, it can be shown [4, 28, 29, 31, 32, 74, 75] that the description of the vacuum states of suitable supersymmetry models yields important results from differential geometry and topology. For instance, one can derive [4, 28, 29, 31, 32] a supersymmetry proof of the Atiyah-Singer index theorem, which connects topological invariants with the vacuum states of elliptic differential operators on compact manifolds.

The space of vacuum states is

$$\ker(H) = \{\psi_{\lambda^{-1}} \in \mathcal{H}_{\lambda^{-1}} \mid H\psi_{\lambda^{-1}} = 0\}. \tag{4.12}$$

In analogy to Subsect. 3.2.2, we first we show that $\lambda^{-1} \in \mathcal{H}^0_{\lambda^{-1}}$ is a non-degenerate vacuum state of H. From Ref. [74], one knows that the vacuum states of $\mathcal{H}^0_{\lambda^{-1}}$ are exactly the states $\psi_{\lambda^{-1}} \in \mathcal{H}^0_{\lambda^{-1}}$ with $D_+\psi_{\lambda^{-1}} = 0$. We have $D_+\lambda^{-1} = \lambda^{-1}d1_M = 0$. Moreover, for another vacuum state $\varphi \in \mathcal{H}^0_{\lambda^{-1}}$ there would be a function ψ with $\varphi = \lambda^{-1}\psi$ and $D_+(\lambda^{-1}\psi) = \lambda^{-1}d\psi = 0$ implying that ψ is constant. Because φ is not zero ψ is also not zero. Thus, φ and λ^{-1} belong to the equivalence class $[\lambda^{-1}] = \{\alpha\lambda^{-1} \mid \alpha \in \mathbb{C} \setminus \{0\}\}$ describing the same quantum state and λ^{-1} is a non-degenerate vacuum state of H.

Now, one could ask how to describe the vacuum states of $\mathcal{H}^1_{\lambda^{-1}}$. The answer is much more complicated than for the case of functions. Generally, the description of differential forms contains the informations about the topology of the underlying configuration space. For the case of a two-dimensional compact and orientable manifold with at most one boundary component we can characterize the number of vacuum states in $\mathcal{H}^1_{\lambda^{-1}}$ explicitly. Here, $\dim\ker(\Delta)$, where

Δ acting on one-forms, is the Betti number b_1, which is a topological invariant, see, e.g., Refs. [45, 62]. Furthermore, from algebraic topology [44] it is known that $b_1 = 2\gamma$ for a two-dimensional manifold of genus γ. Therefore, the number of vacuum states in $\mathcal{H}^1_{\lambda^{-1}}$ is 2γ. In higher dimensions no equivalent statement exists. In terms of supersymmetric quantum mechanics we can interpret λ^{-1} as a superpotential [51].

Furthermore, the vacuum states of $\mathcal{H}^1_{\lambda^{-1}}$ are characterized by λ^{-1}. If $\varphi \in \ker(\Delta)$, then, by Eq. (4.10), we have $H(\lambda^{-1}\varphi) = \lambda^{-1}\Delta\varphi = 0$. Therefore, the vacuum states of $\ker(\Delta)$ differ from the vacuum states of $\ker(H)$ only by the factor λ^{-1} . Therefore, we obtain the isomorphy

$$\ker(H) \simeq \ker(\Delta). \tag{4.13}$$

Interestingly, the conformal class $[g]$ have no influence on the topology of M, since Eq. (4.13) implies that the dimension of $\ker(H)$ is independent of λ^{-1}, while the dynamics is strong influenced by λ^{-1}. Later we will see that there is a connection between the conformal invariant formulation of quantum mechanics, supersymmetry and the topology of the underlying space M.

4.2 The Eigenvalue Estimate of Yang and Yau

Now, we consider the case $\lambda = 1$ and (M, g) being an arbitrary compact and orientable Riemannian manifold of dimension two with at most one boundary component with $\widetilde{g} = g$. Then, we have the ordinary free Schrödinger equation for 0-forms, i.e.

$$H\psi = -\frac{1}{2m}\delta d\psi = -\frac{1}{2m}\Delta\psi, \tag{4.14}$$

with $\psi \in \mathcal{H}^0_1$. Moreover, for 1-forms $\varphi \in \mathcal{H}^1_1$ we have

$$H\varphi = -\frac{1}{2m}(d\delta + \delta d)\varphi = -\frac{1}{2m}\Delta\varphi. \tag{4.15}$$

For an arbitrary eigenstate $\psi \in \mathcal{H}^0_1$ with eigenvalue $E \neq 0$ there is a corresponding superpartner $d\psi \in \mathcal{H}^1_1$ with the same energy E, since

$$H(d\psi) = -\frac{1}{2m}(d\delta + \delta d)d\psi = -\frac{1}{2m}d\delta d\psi - \frac{1}{2m}\delta d^2\psi = -\frac{1}{2m}d\delta d\psi = dH\psi = Ed\psi. \tag{4.16}$$

This is a generic property of supersymmetric theories and can analogously be shown [74] for any supersymmetry algebra.

For clarity, we denote the Hodge Laplacian acting on 0-forms and on 1-forms by Δ_0 and Δ_1, respectively. The pairs of superpartners consisting of 0-forms (bosons) and 1-forms (fermions) are elements of the Hilbert space given by $\mathcal{H}_1^0 \oplus \mathcal{H}_1^1$.

Furthermore, the Hilbert space for bosonic many-particle states is

$$\underbrace{\mathcal{H}_1^0 \otimes \cdots \otimes \mathcal{H}_1^0}_{l \text{ times}}, \tag{4.17}$$

where l is the number of bosons. For fermionic many-particle states we have

$$\underbrace{\mathcal{H}_1^1 \otimes \cdots \otimes \mathcal{H}_1^1}_{r \text{ times}}, \tag{4.18}$$

with r denoting the number of fermions. On these spaces there exists a natural extension of the Hodge Laplacian to separable many-particle states given by

$$\Delta(\omega_1 \otimes \cdots \otimes \omega_i \otimes \cdots \otimes \omega_p) = \sum_i \omega_1 \otimes \cdots \otimes \Delta\omega_i \otimes \cdots \otimes \omega_p, \tag{4.19}$$

for arbitrary differential forms ω_i.

Now, we define a supersymmetric Hamiltonian

$$H_{SUSY} = -\frac{1}{2m}\begin{pmatrix} \Delta_0 & 0 \\ 0 & \Delta_1 \end{pmatrix}, \tag{4.20}$$

and the Schrödinger equation becomes

$$-\frac{1}{2m}\begin{pmatrix} \Delta_0 & 0 \\ 0 & \Delta_1 \end{pmatrix}\begin{pmatrix} \psi \\ \varphi \end{pmatrix} = E\begin{pmatrix} \psi \\ \varphi \end{pmatrix}. \tag{4.21}$$

The pair of superpartners ψ and $\varphi = d\psi$ is a solution of equation (4.21) with $-\frac{1}{2m}\Delta_0\psi = E\psi$.

It is a generic property of supersymmetric theories that particles with energy $E \neq 0$ only occur in pairs of superpartners, while only the number of bosonic and fermionic vacuum states can be different [74]. Following Refs. [74, 75], we will distinguish bosons (0-forms) from fermions (1-forms) by the degree of the differential form.

Now, let E_1 denote the smallest eigenvalue greater than zero of the free Hamiltonian $-\frac{1}{2m}\Delta$. For 0-forms we denote by

$$\mathcal{H}_1^{0,E_1} = \left\{\psi \in \mathcal{H}_1^0 \mid -\frac{1}{2m}\Delta_0\psi = E_1\psi\right\} \tag{4.22}$$

the eigenspace to E_1. Furthermore, we define the space

$$\mathcal{H}_1^{1,E_1} = \left\{ \varphi \in \mathcal{H}_1^1 \mid \exists \psi \in \mathcal{H}_1^{0,E_1} \text{ with } \varphi = d\psi \right\} \tag{4.23}$$

for 1-forms, thus d is an isomorphism between $\mathcal{H}_1^{0,E_1}$ and $\mathcal{H}_1^{1,E_1}$.

As we already mentioned, on (M, g) with genus γ there is a single eigenstate (vacuum) with energy $E_0 = 0$ for 0-forms and there are 2γ linearly independent vacuum states for 1-forms. Thus, we have

$$\dim \ker(\Delta_0) = 1 \tag{4.24}$$

and

$$\dim \ker(\Delta_1) = 2\gamma. \tag{4.25}$$

Furthermore, we denote

$$\dim \mathcal{H}_1^{0,E_1} = D \tag{4.26}$$

for the order of degeneracy of $\mathcal{H}_1^{0,E_1}$. Then, by definition (4.23) we also have

$$\dim \mathcal{H}_1^{1,E_1} = D. \tag{4.27}$$

Furthermore, we introduce the sublevel sets

$$\mathcal{H}_1^{0,\leq E_1} = \left\{ \psi \in \mathcal{H}_1^0 \mid -\frac{1}{2m}\Delta_0\psi = E\psi \ \wedge \ E \leq E_1 \right\} \tag{4.28}$$

and

$$\mathcal{H}_1^{1,\leq E_1} = \left\{ \varphi \in \mathcal{H}_1^1 \mid -\frac{1}{2m}\Delta_1\varphi = E\varphi \ \wedge \ E \leq E_1 \right\}. \tag{4.29}$$

Since the Hodge Laplacian on M has a discrete spectrum, we get

$$\mathcal{H}_1^{0,\leq E_1} = \ker(\Delta_0) \oplus \mathcal{H}_1^{0,E_1} \tag{4.30}$$

and

$$\mathcal{H}_1^{1,\leq E_1} = \ker(\Delta_1) \oplus \mathcal{H}_1^{1,E_1}. \tag{4.31}$$

Now, we consider a supersymmetric two-level system spanned by

$$\mathcal{H}_{SUSY} = \mathcal{H}_1^{0,\leq E_1} \oplus \mathcal{H}_1^{1,\leq E_1}, \tag{4.32}$$

where a boson, described by a 0-form ψ, either can occupy a state with energy E_1, i.e. $\psi \in \mathcal{H}_1^{0,E_1}$ or can be a vacuum, i.e. $\psi \in \ker(\Delta_0)$. Analogously, the superpartner, described by the 1-form

$d\psi$, can be element of $\mathcal{H}_1^{1,E_1}$ or be a vacuum, i.e. $d\psi \in \ker(\Delta_1)$. In other words, particles can only occur in pairs of superpartners to the energy eigenvalue E_1 or can occupy a vacuum state. Moreover, the occupation number for the energy levels $E_0 = 0$ and E_1 have to be compatible with the order of degeneracy of the corresponding eigenspaces, i.e. that the energy level E_1 can not be occupied more than $2D$ times, thus there occur at the most D pairs of superpartners. Moreover, there are maximally $2\gamma + 1$ vacua.

In order to take regard to the concrete geometrical structure of M, we denote by $E_1(M)$ the smallest eigenvalue greater than zero of the free Hamiltonian $-\frac{1}{2m}\Delta_0$ on M. For the special case of the 2-sphere S^2 we obviously have

$$E_1(\mathrm{S}^2) = \frac{1}{2m}\frac{8\pi}{\mathrm{vol}(\mathrm{S}^2)}, \tag{4.33}$$

where $\mathrm{vol}(\mathrm{S}^2)$ is the 2-dimensional volume (i.e. area) of S^2. We first consider a many-particle state of $-\frac{1}{2m}\Delta_0$ given by

$$\psi = \psi_1^{E_1(M)} \otimes \cdots \otimes \psi_k^{E_1(M)} \otimes \cdots \otimes \psi_l^{E_1(M)} \otimes \psi^0, \tag{4.34}$$

where $\psi_k^{E_1(M)} \in \mathcal{H}_1^{0,E_1(M)}$ for $1 \leq k \leq l$ denotes an eigenstate of $-\frac{1}{2m}\Delta_0$ with energy $E_1(M)$ and ψ^0 is the single vacuum state with energy $E_0 = 0$.
Analogously we consider the superpartner of ψ given by the many-particle eigenstate of $-\frac{1}{2m}\Delta_1$

$$\varphi = d\psi_1^{E_1(M)} \otimes \cdots \otimes d\psi_k^{E_1(M)} \otimes \cdots \otimes d\psi_l^{E_1(M)} \otimes \varphi_1^0 \otimes \cdots \otimes \varphi_l^0 \otimes \cdots \otimes \varphi_s^0 \tag{4.35}$$

where φ_l^0 for $1 \leq l \leq s \leq 2\gamma$ is a vacuum of $-\frac{1}{2m}\Delta_1$.

Then, a many-particle state of our two-level system is given by

$$\begin{pmatrix} \psi_1^{E_1(M)} \otimes \cdots \otimes \psi_l^{E_1(M)} \otimes \psi^0 \\ d\psi_1^{E_1(M)} \otimes \cdots \otimes d\psi_l^{E_1(M)} \otimes \varphi_1^0 \otimes \cdots \otimes \varphi_s^0 \end{pmatrix}, \tag{4.36}$$

which is an eigenstate of H_{SUSY} described by pairs of superpartners $\begin{pmatrix} \psi_k^{E_1(M)} & d\psi_k^{E_1(M)} \end{pmatrix}^T$.

Furthermore, for the many-particle states (4.34) and (4.35) [see Eq. (4.19)] we have

$$-\frac{1}{2m}\Delta_0\psi = lE_1\psi, \tag{4.37}$$

and

$$-\frac{1}{2m}\Delta_1\varphi = lE_1\varphi. \tag{4.38}$$

Therefore, the expectation value of the total energy for the state $\begin{pmatrix}\psi & \varphi\end{pmatrix}^T$ is given by

$$\begin{aligned}\langle\begin{pmatrix}\psi & \varphi\end{pmatrix}|H_{SUSY}|\begin{pmatrix}\psi \\ \varphi\end{pmatrix}\rangle &= \langle\psi| - \frac{1}{2m}\Delta_0|\psi\rangle + \langle\varphi| - \frac{1}{2m}\Delta_1|\varphi\rangle \\ &= lE_1(M) + lE_1(M) = 2lE_1(M). \end{aligned} \tag{4.39}$$

Now, we fix both, the expectation value of the total energy and the number of particles. Moreover, we considerer the genus γ of M as a variable. Therefore, many-particle states (4.36) of our two-level system describe a microcanonical ensemble, where microscopic realizations are determined by γ. From the macroscopic point of view all microscopic realization are equiprobable, independently of γ, since they have the same energy. By considering different microscopic realization of our microcanonical ensemble we will figure out how the energy eigenvalue E_1 is transformed by varying γ. By doing so, we obtain a fundamental estimate of E_1 depending on γ, which leads us to the estimate of Yang and Yau [80].

Let us begin with $\gamma = 0$. Then, a state of the two-level system has the form

$$\begin{pmatrix}\psi_1^{E_1(M)} \otimes \cdots \otimes \psi_l^{E_1(M)} \otimes \psi^0 \\ d\psi_1^{E_1(M)} \otimes \cdots \otimes d\psi_l^{E_1(M)}\end{pmatrix}, \tag{4.40}$$

where the expectation value of the total energy is $2lE_1$ and the number of particles is $2l+1$, thus we have l pairs of superpartners and only one single vacuum state, since $\dim\ker(\Delta_1) = 2\gamma = 0$. The supersymmetric state (4.40) should not only describe a vacuum, thus we require $l \geq 1$. Now, we consider another microscopic realization on $\widetilde{M}$, where $\widetilde{M}$ has arbitrary genus γ. Moreover, we choose $\widetilde{g}$ in such a way, that $\mathrm{vol}(\widetilde{M}, \widetilde{g}) = \mathrm{vol}(M, g)$.

Because supersymmetry must be present, the new many-particle state over the configuration space $\widetilde{M}$ has the form

$$\begin{pmatrix}\psi_1^{E_1(\widetilde{M})} \otimes \cdots \otimes \psi_{l'}^{E_1(\widetilde{M})} \otimes \psi^0 \\ d\psi_1^{E_1(\widetilde{M})} \otimes \cdots \otimes d\psi_{l'}^{E_1(\widetilde{M})} \otimes \varphi_1^0 \otimes \cdots \otimes \varphi_n^0\end{pmatrix}, \tag{4.41}$$

where $n \leq 2\gamma$ is the number of occupied vacuum states in $\ker(\Delta_1)$. Since we fix the expectation value of the total energy and the number of particles, the following two equations hold:

$$2lE_1(M) = 2l'E_1(\widetilde{M}), \tag{4.42}$$

which states that the total energy of the two microcanonical realization is the same, and

$$2l' + n + 1 = 2l + 1, \tag{4.43}$$

which states that the number of particles is the same. Since $l \geq 1$, it follows from Eq. (4.42) that $l' \geq 1$. Therefore, we have

$$E_1(\widetilde{M}) = \frac{l}{l'} E_1(M). \tag{4.44}$$

Furthermore, Eq. (4.43) implies that

$$n = 2(l - l') = 2m \tag{4.45}$$

with $m = l - l' \leq \gamma$. Thus, we can write

$$l = l' + m. \tag{4.46}$$

Therefore, we obtain

$$E_1(\widetilde{M}) = \frac{l}{l'} E_1(M) = \frac{l' + m}{l'} E_1(M) \leq \left(1 + \frac{\gamma}{l'}\right) E_1(M) \leq (1 + \gamma) E_1(M). \tag{4.47}$$

There exist several estimates for upper and lower bounds for $E_1(M)$, also in the context of conformal geometry, see, e.g., Refs. [16–19, 57, 79, 81]. It can be shown that for a compact and orientable two-dimensional Riemannian manifold M of genus 0 with $M = \phi(\mathrm{S}^2)$, for an immersion ϕ, one has the estimate [46]

$$E_1(M) \leq \frac{1}{2m} \frac{8\pi}{\mathrm{vol}(M, g)}. \tag{4.48}$$

Upon using this, estimate Eq. (4.47) yields

$$E_1(\widetilde{M}) \leq \frac{1}{2m} \frac{8\pi(1 + \gamma)}{\mathrm{vol}(M, g)}, \tag{4.49}$$

which is exactly the famous estimate of Yang and Yau [80] for a compact and orientable two-dimensional Riemannian manifold with arbitrary genus γ. As mentioned, the above estimate holds for arbitrary compact and orientable Riemannian manifolds with at most one boundary component. The topological term γ is the most interesting part of the estimate, since on the one hand it is independent of the concrete geometry of M and on the other it could lead to a sudden increase of the upper bound energy, if the number of vacuum states increases.

Thus, our approach yields a physics proof of an important result of Riemannian geometry using supersymmetric quantum mechanics. Our analysis provides another illustration of the fundamental connection between supersymmetry and topology.

4.3 Applications to Schrödinger Operators

The eigenvalue estimate of Yang and Yau [80], which we derived here in a novel way, makes only statements about Laplacians. However, by making use of the quantum Maupertuis principle we can extend it to Schrödinger operators, in this way opening it to applications in quantum mechanics. To do so, we first show that the quantum Maupertuis principle is a special case of our supersymmetric formulation of quantum mechanics on conformal manifolds.

4.3.1 The Quantum Maupertuis Principle Revisited

For a compact manifold M of arbitrary dimension and a fixed energy $E > 0$, we again restrict the motion of a particle to

$$M^E = \{x \in M \mid V(x) < E\}, \tag{4.50}$$

since any eigenfunction of H decreases exponentially on $M \setminus M^E$.

Now, we choose

$$\lambda = \left(1 - \frac{V}{E}\right) \tag{4.51}$$

and obtain

$$\widetilde{g} = \left(1 - \frac{V}{E}\right) g. \tag{4.52}$$

Then, we have from Eq. (4.10) the eigenvalue equation

$$H\left(\left(1 - \frac{V}{E}\right)^{-1} \psi\right) = -\left(1 - \frac{V}{E}\right)^{-1} \Delta\psi = E\psi. \tag{4.53}$$

Furthermore, we obtain the equivalence:

$$-\frac{1}{2m}\left(1 - \frac{V}{E}\right)^{-1} \Delta\psi = E\psi \iff -\frac{1}{2m}\Delta\psi + V\psi = E\psi, \tag{4.54}$$

for every state $\psi \in \mathcal{H}_{\left(1+\frac{V}{E}\right)^{-1}}$ with the Hodge Laplacian Δ. Thus, we see that the Schrödinger equation for a particle in a potential is equivalent to a free particle described on a conformally deformed configuration space. This quantum version of the classical Maupertuis principle follows directly from our supersymmetric formalism.

In order to obtain a suitable formulation of quantum mechanics on conformally deformed configuration spaces, in Chapter 3, we had to take curvature terms into account, which arise from the conformal deformations. In Subsect. 3.1.1, we introduced the quantum Maupertuis principle in arbitrary dimension. There, we interpreted the curvature term, which is a result of the conformal deformation of the metric tensor, as the coupling of a charged particle to a gauge field. Only for the two-dimensional case the gauge field does not appear. In Sect. 3.2, where we worked out a more profound approach to quantum mechanics on conformally deformed spaces, we interpreted the curvature term as an external scalar-valued potential. The supersymmetric formulation yields on the one hand a profound formulation of quantum mechanics on conformal manifolds and on the other hand it yields a proper formulation of the quantum Maupertuis principle in arbitrary dimension, where no curvature terms occur. From an abstract point of view, one can argue that the non-appearance of curvature terms in the Hamiltonian of our supersymmetric formalism is a consequence of the topological nature of supersymmetry. If conformal transformations would lead to additional curvature terms in our supersymmetric theory, then the topological invariance might be violated.

4.3.2 The Eigenvalue Estimate for Schrödinger Operators

A generalized Hamiltonian for our two-level system, see Eq. (4.20), is given by

$$H_{SUSY} = -\frac{1}{2m}\begin{pmatrix} D_-D_+ & 0 \\ 0 & D_+D_- + D_-D_+ \end{pmatrix} \tag{4.55}$$

for arbitrary λ. For the special case of $\lambda = \left(1 + \frac{V}{E}\right)$ we obtain with Eq. (4.53):

$$H_{SUSY}\left(\left(1 - \frac{V}{E}\right)^{-1}\begin{pmatrix}\psi \\ \varphi\end{pmatrix}\right) = -\frac{1}{2m}\left(1 - \frac{V}{E}\right)^{-1}\begin{pmatrix}\Delta_0 & 0 \\ 0 & \Delta_1\end{pmatrix}\begin{pmatrix}\psi \\ \varphi\end{pmatrix}. \tag{4.56}$$

This leads to equivalence between the two eigenvalue equations

$$H_{SUSY}\left(\left(1 - \frac{V}{E}\right)^{-1}\begin{pmatrix}\psi \\ \varphi\end{pmatrix}\right) = E\begin{pmatrix}\psi \\ \varphi\end{pmatrix}, \tag{4.57}$$

and

$$-\frac{1}{2m}\begin{pmatrix}\Delta_0 & 0 \\ 0 & \Delta_1\end{pmatrix}\begin{pmatrix}\psi \\ \varphi\end{pmatrix} + V\begin{pmatrix}\psi \\ \varphi\end{pmatrix} = E\begin{pmatrix}\psi \\ \varphi\end{pmatrix}. \tag{4.58}$$

In analogy to the analysis in Chapter 3, for the two-dimensional case we have

$$H_{SUSY}\left(\left(1 - \frac{V}{E}\right)^{-1}\begin{pmatrix}\psi \\ \varphi\end{pmatrix}\right) = -\frac{1}{2m}\begin{pmatrix}\widetilde{\Delta}_0 & 0 \\ 0 & \widetilde{\Delta}_1\end{pmatrix}\begin{pmatrix}\psi \\ \varphi\end{pmatrix}, \tag{4.59}$$

where in the last step we consider H_{SUSY} on a conformally deformed configuration space $(M, \widetilde{g})$ with $\widetilde{g} = \left(1 - \frac{V}{E}\right) g$. Thus, for a two-dimensional manifold the action of the generalized Hamiltonian (4.55) on states defined on the configuration space $(M, \widetilde{g})$ is equivalent to the action of the supersymmetric Schrödinger equation, see Eq. (4.58), on states defined on the configuration space (M, g).

Now, we can extend the eigenvalue estimate for Laplacians to arbitrary Schrödinger operators. First, we specify $E = E_1$, thus the conformal factor becomes

$$\lambda_1 = \left(1 - \frac{V}{E_1}\right). \tag{4.60}$$

If we replace in the analysis of Sect. 4.2 the metric g by $\lambda_1 g$, then we obtain an estimate for the energy E_1 of the first exited state of the Schrödinger operator $H = -\frac{1}{2m}\Delta + V$ on (M, g). Thus, the inequality

$$E_1 \leq \frac{1}{2m} \frac{8\pi(1+\gamma)}{\mathrm{vol}(M, \widetilde{g})}, \tag{4.61}$$

holds with $\widetilde{g} = \left(1 - \frac{V}{E_1}\right) g$. Therefore, our analysis yields a physics proof of an explicit eigenvalue estimate for Schrödinger operators. We illustrate this by two examples.

Example: Coulomb Potential

We first consider a 3-dimensional Coulomb system in Euclidean space with metric g. To reduce the problem to two dimensions we introduce spherical coordinates and consider wave functions with

$$\psi \sim e^{im\varphi}.$$

Then, the effective potential is given by

$$V_{eff} = V_C - \frac{m^2}{r^2 \sin\theta}, \tag{4.62}$$

with the Coulomb potential $V_C = -\frac{e^2}{4\pi\epsilon_0 r}$. As the effective potential is attractive, Eq. (3.13) implies that the Schrödinger equation is equivalent to

$$-\frac{1}{2m}\widetilde{\Delta}\psi = |E|\psi, \tag{4.63}$$

where $\widetilde{\Delta}$ is the Laplacian of the metric

$$\widetilde{g}(E) = \left(\frac{|V_{eff}|}{|E|} - 1\right) g. \tag{4.64}$$

Eq. (4.61) then implies

$$|E_1| \leq \frac{1}{2m} \frac{8\pi}{\mathrm{vol}(M, \widetilde{g}(E_1))}. \tag{4.65}$$

By setting $M = \mathrm{S}^2$ with radius r_0, we have

$$\mathrm{vol}(\mathrm{S}^2, \widetilde{g}(E_1)) = 2\pi \int_0^{r_0} drr \int_0^{\pi} d\theta \sin\theta \left(\frac{|V_{eff}|}{|E_1|} - 1 \right). \tag{4.66}$$

The right hand side of Eq. (4.65) becomes smaller for larger volume. Therefore, we can choose $m = 0$ and get

$$\mathrm{vol}(\mathrm{S}^2, \widetilde{g}(E_1)) = 2\pi \int_0^{r_0} drr \int_0^{\pi} d\theta \sin\theta \left(\frac{e^2}{4\pi\epsilon_0 r |E_1|} - 1 \right) = \frac{e^2}{\epsilon_0 |E_1|} r_0 - 2\pi r_0^2. \tag{4.67}$$

By maximizing the volume we get

$$r_0 = \frac{e^2}{4\pi\epsilon_0 |E_1|}, \tag{4.68}$$

with

$$\mathrm{vol}_{max} = \frac{e^4}{8\pi\epsilon_0^2} \frac{1}{|E_1|^2}. \tag{4.69}$$

Finally, since the genus of a sphere (with no boundary components) is zero, we obtain

$$|E_1| \leq \frac{1}{2m} \frac{(8\pi\epsilon_0)^2}{e^4} |E_1|^2, \tag{4.70}$$

and therefore

$$|E_1| \geq \frac{e^4 m}{32\pi^2\epsilon_0^2\hbar^2}. \tag{4.71}$$

The bound is in fact the true value.

Example: 2D Harmonic Oscillator

Now we consider the 2-dimensional harmonic oscillator with $V = \frac{m\omega^2}{2} r^2$ and $M = D_{r_0}$, where D_{r_0} is the disk with radius r_0. For the conformally deformed metric

$$\widetilde{g} = \left(1 - \frac{m\omega^2 r^2}{2E_1} \right) g, \tag{4.72}$$

where g is the canonical metric on D_{r_0} we obtain

$$\mathrm{vol}(D_{r_0}, \widetilde{g}(E_1)) = 2\pi \int_{D_{r_0}} drr \left(1 - \frac{m\omega^2 r^2}{2E_1} \right) = 2\pi \left(\frac{r^2}{2} - \frac{m\omega^2 r^4}{8E_1} \right). \tag{4.73}$$

The volume has a maximum value

$$\mathrm{vol}_{max} = \frac{\pi E_1}{m\omega^2}, \tag{4.74}$$

at $r_0^2 = \frac{2E_1}{m\omega^2}$. Thus, since the genus of a disk (one boundary component) is zero, we obtain

$$E_1 \leq \frac{4\omega^2}{E_1}, \tag{4.75}$$

yielding the optimal estimate

$$E_1 \leq 2\omega. \tag{4.76}$$

This result differs from the exact value only by a factor 2.

Chapter 5

Supersymmetry of Spin Chains

5.1 A Supersymmetric Formulation of the Two-Particle Interactions of the Ising Spin Chain

To get started, we first introduce some elementary notions that allow us to describe the supersymmetry of spin chains. For simplicity, we restrict the analysis here to the 1D nearest neighbor Ising model [47]. However, the generalization to more complicated models is straight forward. Nevertheless, Ising spin chains are very complex from a pure analytical point of view. The 1D model is special as for this case spin interactions can be separated into two-particle interactions.

To be more specific, we consider the 1D Ising model with the energy

$$E = \langle \sigma_1, \ldots, \sigma_N | H_N | \sigma_1, \ldots, \sigma_N \rangle = -J \sum_{i=1}^{N-1} \sigma_i \sigma_{i+1} \tag{5.1}$$

with the Hamiltonian H_N and the coupling constant J, which describes the interaction of a spin chain of the length N represented by a N-particle state $|\sigma_1, \ldots, \sigma_N\rangle$. Here $\sigma_i \in \{-1, 1\}$ for every $i \in \{1, \ldots, N\}$, where $\sigma_i = 1$ means that the i-th particle has an up-spin and $\sigma_i = -1$ means that the i-th particle has a down-spin.

Obviously, the 1D nearest neighbor Ising model can be separated into two-particle interactions

$$E = \sum_{i=1}^{N-1} \langle \sigma_i, \sigma_{i+1} | H_2 | \sigma_i, \sigma_{i+1} \rangle, \tag{5.2}$$

where H_2 describes the two-particle interaction between the states $|\sigma_i, \sigma_{i+1}\rangle$.

The Ising spin chain can be represented supersymmetrically if there exists an operator Q with $H_2 = Q^2$. Then Q is called the supercharge [74]. If we represent a state $|\sigma_i, \sigma_{i+1}\rangle$ as a vector

$$|\sigma_i, \sigma_{i+1}\rangle \equiv \begin{pmatrix} \sigma_i \\ \sigma_{i+1} \end{pmatrix}, \tag{5.3}$$

then H_2 has the matrix representation

$$H_2 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5.4}$$

Therefore, we obtain the eigenvalue equation

$$\begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} \begin{pmatrix} \sigma_i \\ \sigma_{i+1} \end{pmatrix} = E \begin{pmatrix} \sigma_i \\ \sigma_{i+1} \end{pmatrix} \tag{5.5}$$

with the energy eigenvalues

$$E = \begin{cases} J & \text{for } \sigma_i = \sigma_{i+1}, \\ -J & \text{for } \sigma_i \neq \sigma_{i+1}. \end{cases} \tag{5.6}$$

A supersymmetric description requires $Q|\sigma_i, \sigma_{i+1}\rangle = \sqrt{E}|\sigma_i, \sigma_{i+1}\rangle$. Since E can be negative, Q need to have complex-valued eigenvalues. Moreover, $H_2 Q|\sigma_i, \sigma_{i+1}\rangle = Q H_2 |\sigma_i, \sigma_{i+1}\rangle = EQ|\sigma_i, \sigma_{i+1}\rangle$ implies that the state $Q|\sigma_i, \sigma_{i+1}\rangle$ has the same energy as $|\sigma_i, \sigma_{i+1}\rangle$. Since supersymmetry requires that $H = Q^2$, we have to solve the equation

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^2 \tag{5.7}$$

with

$$Q = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{5.8}$$

which leads to the system of equations

$$\begin{aligned} A^2 + BC &= 0, \\ B(A + D) &= 1, \\ C(A + D) &= 1, \\ D^2 + BC &= 0. \end{aligned} \tag{5.9}$$

This system of equations has a solution and therefore the Ising spin chain has a supersymmetric representation. Obviously, we obtain

$$B = C = \frac{1}{A + D}. \tag{5.10}$$

Now, $A^2 = D^2$ implies $A = \pm D$ and therefore the supercharge Q is represented by

$$Q \equiv \sqrt{J} \begin{pmatrix} A & \frac{1}{2A} \\ \frac{1}{2A} & A \end{pmatrix}. \tag{5.11}$$

with $A^2 + \frac{1}{4A^2} = 0$. We obtain four solutions

$$Q_1 \equiv -\frac{\sqrt{J}}{2} \begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}, \tag{5.12}$$

and

$$\begin{aligned} Q_2 &= Q_1^\dagger, \\ Q_3 &= -Q_1, \\ Q_4 &= -Q_1^\dagger. \end{aligned} \tag{5.13}$$

Furthermore, we have the eigenvalue equation

$$Q_1 \begin{pmatrix} \sigma_i \\ \sigma_{i+1} \end{pmatrix} = \lambda \begin{pmatrix} \sigma_i \\ \sigma_{i+1} \end{pmatrix}, \tag{5.14}$$

with

$$\lambda = \begin{cases} -\sqrt{J} & \text{for } \sigma_i = \sigma_{i+1}, \\ -i\sqrt{J} & \text{for } \sigma_i \neq \sigma_{i+1}. \end{cases} \tag{5.15}$$

With $H_2 = Q_i^2$ for $i \in \{1, 2, 3, 4\}$, we obtain for the energy eigenvalues $E = \lambda^2$. This implies exactly the result (5.6). For the total energy of a spin chain $|\sigma_1, \ldots, \sigma_i, \sigma_{i+1}, \ldots, \sigma_N\rangle$ with length N we get

$$E = \sum_{i=1}^{N-1} \langle \sigma_i, \sigma_{i+1} | Q_i^2 | \sigma_i, \sigma_{i+1} \rangle = -J \sum_{i=1}^{N-1} \sigma_i \sigma_{i+1}. \tag{5.16}$$

Thus, we see that the 1D Ising model with a constant length has a simple supersymmetry algebra, Q with $Q^2 = H_2$. Moreover, it becomes clear that concepts and theoretical ideas of supersymmetric quantum mechanics can be naturally applied to spin chains.

5.2 The Witten Index of Spin Chain Models

In Sect. 5.1, we worked out a representation of the supersymmetry algebra of the two-particle interaction, which occurs in the 1D nearest neighbor Ising model. There, we found an explicit matrix representation of Q with $Q^2 = \hat{H}$, where H is the two-particle interaction Hamiltonian. In theories with supersymmetry particles occur in pairs of bosons and fermions, thus one splits the Hilbert space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ into a bosonic subspace $\mathcal{H}_+$ and a fermionic subspace $\mathcal{H}_-$ [74]. Following Ref. [74], we consider the operator $(-1)^F$ with

$$(-1)^F \psi = \begin{cases} \psi & \text{for } \psi \in H_+ \\ -\psi & \text{for } \psi \in H_-. \end{cases} \tag{5.17}$$

In Sec. 5.1, spin chain configurations were represented by states $\boldsymbol{\sigma} = |\sigma_1, \ldots, \sigma_i, \ldots, \sigma_N\rangle$ with $\sigma_i \in \{-1, 1\}$. In Ref. [53], Knauf introduced a spin chain of length of N (see Appendix), where he transformed to a new spin configurations $\boldsymbol{\sigma} \to \tilde{\boldsymbol{\sigma}}$ with

$$\tilde{\sigma}_i = \begin{cases} 0 & \text{for } \sigma_i = 1, \\ 1 & \text{for } \sigma_i = -1. \end{cases} \tag{5.18}$$

Thus, following Ref. [53], a state of the spin chain is represented by $|\tilde{\sigma}_1, \ldots, \tilde{\sigma}_N\rangle \in \mathbb{Z}_2^N = G$ with the quotient ring $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}_2^N = \underbrace{\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2}_{N \text{ times}}$.

The partition function becomes [53]

$$Z(\beta) = \sum_{\tilde{\boldsymbol{\sigma}} \in G} e^{-\beta E(\tilde{\boldsymbol{\sigma}})}, \tag{5.19}$$

where the energy $E(\boldsymbol{\sigma})\boldsymbol{\sigma} = H\boldsymbol{\sigma}$ has also to be transformed to $E(\tilde{\boldsymbol{\sigma}})$. For example, the energy of the 1D nearest neighbor Ising spin chain is $E(\boldsymbol{\sigma}) = -J\sum_{i=1}^{N-1} \sigma_i \sigma_{i+1}$. Then, we have

$$E(\boldsymbol{\sigma}) = E(\tilde{\boldsymbol{\sigma}}) = -J \sum_{i=1}^{N-1} 4\tilde{\sigma}_i \tilde{\sigma}_{i+1} - 2(\tilde{\sigma}_i + \tilde{\sigma}_{i+1}) + 1. \tag{5.20}$$

Moreover, in Ref. [53], a general framework is introduced in order to consider observables on a dual spin chain configuration space $\widehat{G}$, which is defined as the dual group of G

$$\widehat{G} = \{\chi : G \to \mathbb{C}^\times |\ \chi \text{ homomorphism}\} \tag{5.21}$$

with $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ and the requirement $|\chi(g)| = 1$. χ is called a character of $\widehat{G}$, thus, as one can see, e.g., in Ref. [68]:

for $\tilde{\boldsymbol{\sigma}} = |\tilde{\sigma}_1, \ldots, \tilde{\sigma}_N\rangle \in G$ the characters are given by $\chi_{\boldsymbol{t}}(\tilde{\boldsymbol{\sigma}}) = e^{i\pi\langle \boldsymbol{t}|\tilde{\boldsymbol{\sigma}}\rangle} = e^{i\pi(t_1\tilde{\sigma}_1+\cdots+t_N\tilde{\sigma}_N)} = (-1)^{t_1\tilde{\sigma}_1+\cdots+t_N\tilde{\sigma}_N}$ with $\boldsymbol{t} = \langle t_1, \ldots, t_N| \in G^*$, where G^* denotes the dual space of G as a vector space. Every $\chi_t \in \widehat{G}$ is one to one represented by a $t \in G^*$ since the map $t \mapsto \chi_t = (-1)^{\langle \boldsymbol{t}|\tilde{\boldsymbol{\sigma}}\rangle}$ is bijective. The Fourier analysis based on this representation of characters is also relevant for the investigations in Refs. [53–56].

The measurable objects of a physical theory are its observables. In Ref. [53], an observable f of a spin chain is defined as a map $f : G \to \mathbb{R}$. Examples are the energy of a spin chain configuration $E(\tilde{\boldsymbol{\sigma}})$ or the Boltzmann weight $e^{-\beta E(\tilde{\boldsymbol{\sigma}})}$. In order to map an observable on defined configuration space G onto the dual configuration space $\widehat{G}$, one introduces the Fourier transform $\hat{f}$ of f, see Ref. [53], which is a map $\hat{f} : \widehat{G} \to \mathbb{C}$ given by

$$\hat{f}(\chi) = \sum_{\tilde{\boldsymbol{\sigma}} \in G} \overline{\chi}(\tilde{\boldsymbol{\sigma}}) f(\tilde{\boldsymbol{\sigma}}). \tag{5.22}$$

One has two different pictures for describing properties of spin chain models. On the one hand, we can consider observables on the ordinary spin chain configuration space. On the other hand, we are able to study dual observables on the space of characters, which are the coordinates of the dual configuration space. Now, we consider a situation were both pictures are closely related to each other by the Witten index, which yields a connection to n-point correlation functions. However, we first focus on the Boltzmann weight, which is an observable measuring the probability of occupancy.

For the Boltzmann weight $w(\tilde{\boldsymbol{\sigma}}) = e^{-\beta E(\tilde{\boldsymbol{\sigma}})}$ we obtain

$$\hat{w}(\chi_{\boldsymbol{t}}) = \hat{w}(\boldsymbol{t}) = \sum_{\tilde{\boldsymbol{\sigma}}} (-1)^{\langle \boldsymbol{t}|\tilde{\boldsymbol{\sigma}}\rangle} e^{-\beta E(\tilde{\boldsymbol{\sigma}})}. \tag{5.23}$$

The Fourier transform of the Boltzmann weights has some interesting interpretations and leads to novel insights to the spin-spin interaction. First, for $\boldsymbol{t} = \boldsymbol{0}$ we get by

$$\hat{w}(\boldsymbol{0}) = \sum_{\tilde{\boldsymbol{\sigma}}} e^{-\beta E(\tilde{\boldsymbol{\sigma}})} = Z(\beta) \tag{5.24}$$

the ordinary partition function of a canonical ensemble. For $\boldsymbol{t} = \boldsymbol{1}$ we obtain

$$\hat{w}(\boldsymbol{1}) = \sum_{\tilde{\boldsymbol{\sigma}}} (-1)^{\tilde{\sigma}_1+\cdots+\tilde{\sigma}_N} e^{-\beta E(\tilde{\boldsymbol{\sigma}})}. \tag{5.25}$$

Moreover, with

$$S = \sum_{i=1}^{N} \tilde{\sigma}_i \tag{5.26}$$

denoting the total spin of the state $|\tilde{\sigma}_1, \ldots, \tilde{\sigma}_N\rangle$ we get

$$(-1)^S = \begin{cases} +1 & \text{for } S \text{ even} \\ -1 & \text{for } S \text{ odd.} \end{cases} \tag{5.27}$$

If we interpret states with even total spin as bosons and states with odd total spin as fermions, then the character $\chi_{\mathbf{0}}$ introduce an operator $(-1)^F$ by

$$(-1)^F \tilde{\boldsymbol{\sigma}} = \chi_{\mathbf{1}}(\tilde{\boldsymbol{\sigma}})\tilde{\boldsymbol{\sigma}} = (-1)^S \tilde{\boldsymbol{\sigma}}, \tag{5.28}$$

which separates the space of spin chain configurations into a bosonic and a fermionic part. The total spin of the configuration $\tilde{\boldsymbol{\sigma}} \in G$ is additive, as one expects through the physics point of view. Thus, we see that the Witten index is given through $\hat{w}(\mathbf{1})$. Here, we see, that the transformed Boltzmann weights are the Witten index. Later we will see that other Fourier transforms of Boltzmann weights are the Witten indices on subspaces of G.

On the other hand, since $(-1)^{\tilde{\sigma}_1 + \cdots + \tilde{\sigma}_N} = (-1)^{\tilde{\sigma}_1} \cdots (-1)^{\tilde{\sigma}_N}$ and $\sigma_i = (-1)^{\tilde{\sigma}_i}$, we have together with Eq. (5.25)

$$\frac{\hat{w}(\mathbf{1})}{Z(\beta)} = \langle \sigma_1 \ldots \sigma_N \rangle, \tag{5.29}$$

with the N-point correlation function

$$\langle \sigma_1 \cdots \sigma_N \rangle = \frac{\sum_{\tilde{\boldsymbol{\sigma}} \in G} \sigma_1 \cdots \sigma_N e^{-\beta E(\tilde{\boldsymbol{\sigma}})}}{\sum_{\tilde{\boldsymbol{\sigma}} \in G} e^{-\beta E(\tilde{\boldsymbol{\sigma}})}}. \tag{5.30}$$

Moreover, with Eq. (5.29), we obtained a fundamental result connecting the spin-spin interaction in terms of the N-point correlation function with the Witten index of an arbitrary spin chain. If the spin chain is perfectly correlated, i.e. $\langle \sigma_1 \cdots \sigma_N \rangle = 1$, then the Witten index is equal to the canonical partition function. For the case $\langle \sigma_1 \cdots \sigma_N \rangle = 0$ the Witten index have to be 0, independently of the value of β. This means that all bosonic and fermionic states cancel each other, as one has in systems with broken supersymmetry [74].

5.2.1 The Witten Index on Subspaces

One can ask how the Witten index is to defined, if we fix some spin degrees of freedom, e.g. $\sigma_k = 0$ for $k \in \mathcal{K}$ and $\mathcal{K} \subseteq \mathcal{N}$, with $\mathcal{N} = \{1, \ldots, N\}$. Therefore, we have to consider the Witten index on suitable subspaces of the space of all spin configurations. Later we will see that the Witten index on subspaces is related with n-point correlation functions for spin-spin interactions between a subset of the N particles. Moreover, we will obtain an expression of the vacuum expectation value in terms of Witten indices using finite Fourier analysis.

As we already mentioned, every character $\chi_t \in \widehat{G}$ is characterized by a $\boldsymbol{t} \in G^*$, where $\boldsymbol{t}$ is given by its coordinates t_j for $j \in \mathcal{N}$. Moreover, we can represent any $\boldsymbol{t} \in G^*$ by the map $\mathcal{P}(\mathcal{N}) \to G^*$ with the power set $\mathcal{P}(\mathcal{N})$ of $\mathcal{N}$ and $\mathcal{K} \mapsto \boldsymbol{t}_{\mathcal{K}}$ by setting the coordinates of $\boldsymbol{t}_{\mathcal{K}}$ to

$$t_j = \begin{cases} 1 & \text{if } j \in \mathcal{K} \\ 0 & \text{if } j \in \mathcal{N} \setminus \mathcal{K}. \end{cases} \tag{5.31}$$

As a direct consequence, we get $\boldsymbol{t}_{\mathcal{K}} = \boldsymbol{t}_{\mathcal{K}'} \Leftrightarrow \mathcal{K} = \mathcal{K}'$ and therefore

$$G^* = \{\boldsymbol{t}_{\mathcal{K}} \mid \mathcal{K} \in \mathcal{P}(\mathcal{N})\}. \tag{5.32}$$

For a given $\mathcal{K} = \{i_1, \ldots, i_k\}$ we have

$$\chi_{\boldsymbol{t}_{\mathcal{K}}}(\tilde{\boldsymbol{\sigma}}) = (-1)^{\tilde{\sigma}_{i_1} + \cdots + \tilde{\sigma}_{i_k}}, \tag{5.33}$$

which defines an operator $(-1)^F$ on the subspace $G_{\mathcal{K}}$, where $G_{\mathcal{K}}$ is spanned by $\{\sigma_{i_1}, \ldots, \sigma_{i_k}\}$. Moreover, the total spin is $S = \sigma_{i_1} + \cdots + \sigma_{i_k}$. We obtain

$$\hat{w}(\boldsymbol{t}_{\mathcal{K}}) = \sum_{\tilde{\boldsymbol{\sigma}}} \sigma_{i_1} \cdot \cdots \cdot \sigma_{i_k} e^{-\beta E(\tilde{\boldsymbol{\sigma}})} = Z(\beta) \langle \sigma_{i_1} \cdot \cdots \cdot \sigma_{i_k} \rangle, \tag{5.34}$$

which is analogous to Eq. (5.29), but this times connecting the Witten index and n-point correlation functions on subspaces. For example, if we consider $\boldsymbol{t}' = \langle 1, \ldots, 1, 0|$, then $\chi_{\boldsymbol{t}'} = (-1)^{\tilde{\sigma}_1 + \cdots + \tilde{\sigma}_{N-1}}$ is the operator $(-1)^F$ for the sub spin-chain of length $N-1$ and the total spin $S' = \tilde{\sigma}_1 + \cdots + \tilde{\sigma}_{N-1}$. On the other hand, the Fourier transform $\hat{w}(\boldsymbol{t}')$ becomes

$$\hat{w}(\boldsymbol{t}') = \sum_{\tilde{\boldsymbol{\sigma}} \in G} \sigma_1 \cdots \sigma_{N-1} e^{-\beta E(\tilde{\boldsymbol{\sigma}})} = Z(\beta) \langle \sigma_1 \cdots \sigma_{N-1} \rangle, \tag{5.35}$$

which is a $(N-1)$-point correlation function.

With regard to Eq.(5.34), we denote for a given $\mathcal{K} \in \mathcal{P}(\mathcal{N})$

$$\frac{\hat{w}(\boldsymbol{t}_{\mathcal{K}})}{Z(\beta)} = \langle \sigma_{i_1} \cdot \cdots \cdot \sigma_{i_k} \rangle = C_{\mathcal{K}}. \tag{5.36}$$

Now, we introduce the annihilator (see, e.g., Ref. [68]) for a subgroup $A \subseteq G$ by

$$\widehat{G}^A = \{\chi \in \widehat{G} \mid \chi|_A = 1\}, \tag{5.37}$$

where $\chi|_A$ denote the restriction of χ to A.

An important result in finite Fourier analysis is the Poisson summation formula (see, e.g., Ref. [68]), which states that

$$\frac{1}{|A|}\sum_{\tilde{\boldsymbol{\sigma}}\in A} f(\tilde{\boldsymbol{\sigma}}) = \frac{1}{|G|}\sum_{\chi\in\widehat{G}^A} \hat{f}(\chi), \tag{5.38}$$

for any observable f, where $|\cdot|$ denotes the cardinality of a set.

Now, we consider again the Boltzmann weights $f(\tilde{\boldsymbol{\sigma}}) = e^{-\beta E(\tilde{\boldsymbol{\sigma}})}$ and fix $A = \{\mathbf{0}\}$ as the trivial subgroup of G. Then, we obtain $\widehat{G}^A = \widehat{G}$ and together with the Poisson summation formula we have

$$e^{-\beta E(\mathbf{0})} = \frac{1}{2^N}\sum_{\boldsymbol{t}\in G^*}\sum_{\tilde{\boldsymbol{\sigma}}\in G}(-1)^{\langle \boldsymbol{t}|\tilde{\boldsymbol{\sigma}}\rangle} e^{-\beta E(\tilde{\boldsymbol{\sigma}})} \tag{5.39}$$

by considering Eq. (5.38) for $A = \{\mathbf{0}\}$. This leads directly to

$$e^{-\beta E(\mathbf{0})} = \frac{1}{2^N}\sum_{\boldsymbol{t}\in G^*}\widehat{w}(\boldsymbol{t}) = \frac{1}{2^N}\sum_{\mathcal{K}\in\mathcal{P}(\mathcal{N})}\widehat{w}(\boldsymbol{t}_{\mathcal{K}}), \tag{5.40}$$

and therefore, with Eq. 5.34, we get

$$\frac{e^{-\beta E(\mathbf{0})}}{Z(\beta)} = \frac{1}{2^N}\sum_{\mathcal{K}\in\mathcal{P}(\mathcal{N})} C_{\mathcal{K}}. \tag{5.41}$$

The left side of Eq. (5.41) is the probability of occupancy for the state $|0\rangle = \mathbf{0}$

$$\langle\mathbf{0}|\frac{e^{-\beta H}}{Z(\beta)}|\mathbf{0}\rangle, \tag{5.42}$$

since

$$\rho = \frac{e^{-\beta H}}{Z(\beta)}, \tag{5.43}$$

is the density matrix of the systems described by the Hamiltonian H on the Hilbert space G. If $\mathbf{0}$ is a vacuum state, e.g. $E(\mathbf{0}) = \min_{\tilde{\boldsymbol{\sigma}}\in G}\{E(\tilde{\boldsymbol{\sigma}})\}$, then $\langle\mathbf{0}|\rho|\mathbf{0}\rangle$ is called the vacuum expectation value of ρ.

With the notation

$$\mathcal{K}_n = \{\mathcal{K}\in\mathcal{P}(\mathcal{N}) \mid |\mathcal{K}| = n\}, \tag{5.44}$$

we have the sum over all n- point correlation functions

$$\mathcal{C}_n = \sum_{\mathcal{K}\in\mathcal{K}_n} C_{\mathcal{K}}, \tag{5.45}$$

with $\mathcal{C}_0 = \frac{Z(\beta)}{Z(\beta)} = 1$.

This term describes the n-particle interactions of the system. Then, Eq. (5.41) becomes

$$\langle \mathbf{0}|\rho|\mathbf{0}\rangle = \frac{1}{2^N} \sum_{0\leq n\leq N} \mathcal{C}_n. \tag{5.46}$$

This in a remarkable result, since we rigorously derived an expression for the vacuum expectation value only in terms of n-point correlation functions for an arbitrary spin chain by using algebraic relations induced from the analysis of the Witten index of spin chain models.

For a spin chain model, where the Witten index vanishes on every subspace, we directly get $\langle \mathbf{0}|\rho|\mathbf{0}\rangle = 0$. In other words, the probability for the occupation of the vacuum state is zero. Therefore, a system, which has broken supersymmetry on every length scale of the regarded spin chain, never reaches its ground state.

5.2.2 Supersymmetric Theories and Phase Transitions

We call a theory supersymmetric, if there exist an operator Q with $Q^2 = H$. Following Ref. [74]: Q has to subdivide $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ into a bosonic subspace $\mathcal{H}_+$ and a fermionic subspace $\mathcal{H}_-$, where the subspaces are defined in relation to an operator $(-1)^F$ with $(-1)^F Q\psi = -(-1)^F\psi$. For every state $\tilde{\boldsymbol{\sigma}} \in G$ with energy $E \neq 0$ exists another state $Q\tilde{\boldsymbol{\sigma}}$ with the same energy. For $E = 0$ we have $H\tilde{\boldsymbol{\sigma}} = 0$ and therefore $Q\tilde{\boldsymbol{\sigma}} = 0$. Therefore, vacuum states ($E = 0$) do not occur in pairs, but all other bosons and fermions with non-zero energy do. As a direct consequence, the difference $\nu_B - \nu_F$ of the number of bosonic vacuum states ν_B and fermionic vacuum states ν_F is invariant, if particles with energy $E \neq 0$ condense into a vacuum state or if vacuum states excited into states with higher energy. In our approach, we obtain for the Witten index of a supersymmetric theory on $\mathcal{H} = G$:

$$\hat{w}(\mathbf{1}) = \sum_{\tilde{\boldsymbol{\sigma}}} (-1)^{\tilde{\sigma}_1+\cdots+\tilde{\sigma}_N} e^{-\beta E(\tilde{\boldsymbol{\sigma}})} = \nu_B - \nu_F \tag{5.47}$$

Moreover, Eq. (5.29) implies

$$\langle \sigma_1 \cdots \sigma_N \rangle = \frac{\nu_B - \nu_F}{Z(\beta)}. \tag{5.48}$$

For $\nu_B - \nu_F = 0$, we see that $\langle \sigma_1 \cdots \sigma_N \rangle = 0$. Analogously for $\mathcal{K} \in \mathcal{P}(\mathcal{N})$ we obtain

$$C_{\mathcal{K}} = \frac{\nu_B - \nu_F}{Z(\beta)}, \tag{5.49}$$

where ν_B, ν_F and $Z(\beta)$ are the number of bosonic vacuum states, the number of fermionic vacuum states and the partition function with respect to the configuration space $G_{\mathcal{K}}$.

By looking more closer to Eq. (5.46), we have

$$\langle \mathbf{0}|\rho|\mathbf{0}\rangle = \frac{1}{2^N} + \frac{1}{2^N}\sum_{1\leq n\leq N} \mathcal{C}_n \tag{5.50}$$

and see that for an uncorrelated system, i.e. $\mathcal{C}_{\mathcal{K}} = 0$ for every $\mathcal{K} \in \mathcal{P}(\mathcal{N})$, the probability of occupancy for the state $\mathbf{0}$ is $\frac{1}{2^N}$. In other words, Eq. (5.36) implies, for a spin chain with broken supersymmetry, i.e. on every subspace there are no vacuum states with zero energy ($\nu_B = \nu_F = 0$) that the expectation value $\langle \mathbf{0}|\rho|\mathbf{0}\rangle$ is exactly $\frac{1}{2^N}$.

Now, let us consider the situation

$$\langle \mathbf{0}|\rho|\mathbf{0}\rangle = \begin{cases} 1 & \text{for } T < T_c \\ < 1 & \text{for } T > T_c, \end{cases} \tag{5.51}$$

where T_c is a critical temperature. Moreover, let $\mathbf{0}$ be a state with minimal energy. Then, the system has to be in a totally ordered phase for $T < T_c$. As a consequence a phase transition occurs at T_c. From Eqs. (5.45) and (5.46), we can conclude that $\mathcal{C}_{\mathcal{K}} = 1$ for every $\mathcal{K} \in \mathcal{P}(\mathcal{N})$. But this means that all Witten indices (5.34) are equal to the canonical partition function $Z(\beta)$, independently of β, in the regime $T < T_c$. Since the Witten index has to be an integer for a supersymmetric theory, the partition function $Z(\beta)$ has to be identical with this integer, which is constant by varying β. For $T \to 0$, i.e. that β tends to infinity, $Z(\beta)$ tends to zero. Since $Z(\beta)$ is a continuous function, it follows that $Z(\beta) = 0$ for $T < T_c$. But this does not make sense from a physics point of view. As a consequence, we get the result that a supersymmetric spin chain with no external field cannot perform a phase transition into a totally ordered phase. Thus, the state of minimal energy cannot be occupied.

5.2.3 Unitary Transformations

Now, we have to show that the relations between n-point correlation functions and the Witten index $\nu_B - \nu_F$ is independent from the choice of the basis of G. For an unitary transformation $M : G \to G$ the transformed supercharge becomes, cf. Ref. [74, 75]

$$Q^* = MQM^{-1}, \tag{5.52}$$

and the Hamiltonian

$$H^* = (Q^*)^2 = MQ^2M^{-1} = MHM^{-1}, \tag{5.53}$$

for states $\tilde{\boldsymbol{\sigma}}^* = M\tilde{\boldsymbol{\sigma}}$.

The energy E of a state $\tilde{\boldsymbol{\sigma}} \in G$ is invariant under arbitrary invertible transformations M, especially for unitary transformations. This becomes clear, since for any state $\tilde{\boldsymbol{\sigma}} \in G$ with $H\tilde{\boldsymbol{\sigma}} = E\tilde{\boldsymbol{\sigma}}$, we have $H^*\tilde{\boldsymbol{\sigma}}^* = MH\tilde{\boldsymbol{\sigma}} = E\tilde{\boldsymbol{\sigma}}^*$ by using Eq.(5.53). Especially, vacuum states are mapped onto vacuum states under the action of M, since $Q^*\tilde{\boldsymbol{\sigma}}^* = 0$, if and only if, $Q\tilde{\boldsymbol{\sigma}} = 0$. As a consequence, we have

$$Z^*(\beta) = \sum_{\tilde{\boldsymbol{\sigma}}^*} e^{-\beta E(\tilde{\boldsymbol{\sigma}}^*)} = Z(\beta), \tag{5.54}$$

and

$$\chi^*_{\boldsymbol{t}}(\tilde{\boldsymbol{\sigma}}) = (-1)^{\langle \boldsymbol{t}^* | \tilde{\boldsymbol{\sigma}}^* \rangle} = (-1)^{\langle \boldsymbol{t} | M^{-1} M | \tilde{\boldsymbol{\sigma}} \rangle} = \chi_{\boldsymbol{t}}(\tilde{\boldsymbol{\sigma}}) \tag{5.55}$$

using that M is unitary.

In other words, Witten indices and n-point correlation functions are invariant under unitary transformations. This is consistent with the transformation properties of supersymmetry algebras in the general theory of supersymmetric quantum mechanics, see Ref. [74].

Chapter 6

Supersymmetry and the Hagedorn Transition in a Number-Theoretical Model

First, we summarize some results of Spector's supersymmetric extension of the primon gas [70, 71], on which our further work based on.

6.1 Spector's Supersymmetric Extension of the Primon Gas

In Refs. [70, 71], Spector studies the connection between the primon gas, number theory and applications in string theory intensively. By doing so, the primon gas is introduced by a generic supersymmetric approach.

A bosonic primon is generated by bosonic creation operators $b_1^\dagger, b_2^\dagger, b_3^\dagger, \dots$, thus, considering a vacuum state $|\text{vacuum}\rangle$, a primon with energy $E_0 \ln n$ can be interpreted as an excitation of bosonic elementary one-particle states [70]

$$|n\rangle = |\alpha_1, \alpha_2, \alpha_3, \alpha_4, \dots, \alpha_k, \dots\rangle = \prod_k (b_k^\dagger)^{\alpha_k} |\text{vacuum}\rangle, \tag{6.1}$$

where n is associated with the unique prime factorization

$$n = \prod_k p_k^{\alpha_k}. \tag{6.2}$$

Since $|1\rangle$ is the only state with energy zero, one has a single vacuum state [70]

$$|\text{vacuum}\rangle = |1\rangle. \tag{6.3}$$

Moreover, one introduces fermionic creation operators $f_1^\dagger, f_2^\dagger, f_3^\dagger, \ldots$ and obtains states with bosonic and fermionic excitations [70]

$$|n\rangle = \prod_k (b_k^\dagger)^{\alpha_k} (f_k^\dagger)^{\beta_k} |1\rangle, \tag{6.4}$$

where we have $\beta_k \in \{0, 1\}$, due to Fermi-Dirac statistics. The systems described by the states (6.4) is a free supersymmetric theory [70]. Furthermore, from Refs. [70, 71], it follows that the state $|n, d\rangle$ with

$$n = \prod_k p_k^{\alpha_k + \beta_k} = \prod_k p_k^{\alpha_k} \prod_k p_k^{\beta_k} = \frac{n}{d} d \tag{6.5}$$

and

$$d = \prod_k p_k^{\beta_k} \tag{6.6}$$

has the energy

$$E_n = E_0 \ln n = E_0 (\ln \frac{n}{d} + \ln d) \tag{6.7}$$

with a bosonic contribution $E_{n,b} = E_0 \ln \frac{n}{d}$ and a fermionic contribution $E_{n,f} = E_0 \ln d$. One can separate the supersymmetric system in terms of [70]

$$H_{prim} = H_{prim,+} + H_{prim,-}, \tag{6.8}$$

where $H_{prim,+} (H_{prim,-})$ describes the bosonic (fermionic) part.

Now, a central idea in Refs. [70, 71] is to introduce an operator $(-1)^F$ by

$$(-1)^F |n, d\rangle = \mu(d) |n, d\rangle \tag{6.9}$$

with the Möbius function

$$\mu(d) = \begin{cases} +1 & \text{if } d \text{ is square-free with an even number of prime factors,} \\ -1 & \text{if } d \text{ is square-free with an odd number of prime factors,} \\ 0 & \text{otherwise,} \end{cases} \tag{6.10}$$

where one defines bosons as states with $\mu(d) = 1$ and fermions as states with $\mu(d) = -1$. Following Ref. [70], one can make the following observation: if one consider an arbitrary many-particle state $|n, d\rangle$ with $n = \prod_k p_k^{\alpha_k + \beta_k}$ and $d = \prod_k p_k^{\beta_k}$, the total energy is $E_n = E_0 \ln n$. For

$\alpha_k > 1$ and $\beta_k = 0$, i.e. that the bosonic one-particle state with energy $E_{p_k} = E_0 \ln p_k$ is exited, but the fermionic counterpart not, there exists a state $|n, \tilde{d}\rangle$ with a $\tilde{d}$ obtained by replacing α_k and β_k by $\tilde{\alpha}_k = \alpha_k - 1$ and $\tilde{\beta}_k = 1$. This state has the same total energy $E_n = E_0 \ln n$ but we have $\mu(\tilde{d}) = -\mu(d)$. Since, the prime factorization of n is unique, for every bosonic (fermionic) state exists exactly one fermionic (bosonic) superpartner with the same energy. The only state, which has no superpartner is the vacuum $|1\rangle$, because we have $1 = \prod_k p_k^{\alpha_k}$ with $\alpha_k = 0$ for all k. The Hilbert space of the supersymmetric primon gas $\mathcal{H}_{prim}$ consists of all states $|n, d\rangle$ with $n \in \mathbb{N}$, where d is a factor of n, i.e. $d|n$. Therefore, the Witten index is [70]

$$\mathrm{Tr}\left[(-1)^F e^{-\beta H_{prim}}\right] = \sum_n e^{-\beta E_n} \sum_{d|n} \mu(d) = 1. \tag{6.11}$$

In Ref. [70], by using this result and the fact that the supersymmetric primon gas is entirely non-interacting, which is a direct consequence of Eq. (6.7), it is introduced a novel physics proof of the famous Möbuis inversion formula, which states that for any function $g : \mathbb{N} \to \mathbb{C}$ one has

$$g(n) = \sum_{d|n} h(d) \tag{6.12}$$

with

$$h(n) = \sum_{d|n} \mu(d) g\left(\frac{n}{d}\right). \tag{6.13}$$

Moreover, Spector's supersymmetric extension of the primon gas leads to substantial physical interpretations of fundamental results about Dirichlet series and dualities among partition functions of arithmetic quantum theories [71].

6.2 The Hagedorn Transition of a Harmonic Oscillator Gas

Since the primon gas reaches his Hagedorn temperature at $\beta = 1$, the behavior beyond this critical temperature and in the critical strip $\{\beta \in \mathbb{C} \mid 0 < \Re(\beta) < 1\}$ is unknown. There exist many expressions for the analytic continuation of the Riemann zeta function into the critical strip, i.e. that one has an expression of the Riemann zeta function, which is an analytic function on $\{\beta \in \mathbb{C} \mid 0 < \Re(\beta) < 1\}$ and identical to

$$\zeta(\beta) = \sum_n n^{-\beta} \tag{6.14}$$

for all $\beta \in \mathbb{C}$ with $\Re(\beta) > 1$.

One well-know analytic continuation, which can be found in most standard literature, see, e.g., Ref. [1], is

$$\zeta(\beta) = 1 + \frac{1}{\beta - 1} - \sum_{n=1}^{\infty} \int_0^1 dt \frac{t\beta}{(n+t)^{\beta+1}}, \tag{6.15}$$

which is defined for every $\beta \in \mathbb{C}$ with $\Re(\beta) > 0$. In Ref. [1], a smart way in order to derive Eq. (6.15) is introduced by using the identity

$$1 = \sum_{n=1}^{\infty} \frac{1}{n^{\beta}} - \sum_{n=1}^{\infty} \frac{1}{(n+1)^{\beta}}. \tag{6.16}$$

One aim of our work is to interpret the expression (6.15) of the Riemann zeta function in terms of Spector's supersymmetric formulation of the primon gas. By doing so, we obtain an interesting insight to the relation between the analytic continuation of the Riemann zeta function and the critical behavior at the Hagedorn temperature of a system, which is closely related to the primon gas.

More precisely, we consider a gas of supersymmetric harmonic oscillator states, where the vacuum energy is gauged to be zero. By the way, one obtain this gauging rigorously by using coherent states quantization in the Klauder–Berezin approach [30]. In particular, $|n,+\rangle$ ($|n,-\rangle$) should label bosonic (fermionic) states and H_{harm} the Hamiltonian with

$$H_{harm}|n\rangle_B = E_n|n\rangle_B \tag{6.17}$$

and

$$H_{harm}|n\rangle_F = E_n|n\rangle_F \tag{6.18}$$

with discrete energies

$$E_n = \omega n \tag{6.19}$$

for $n \in \mathbb{N}$. The frequency ω should be directly coupled to a heat bath, thus it depends on the inverse temperature

$$\omega = \omega(\beta) = -\frac{\ln \beta}{\beta} \tag{6.20}$$

with $0 < \beta < 1$ and a strict positive ω.

The Hilbert space can be written as

$$\mathcal{H}_{harm} = \mathcal{H}_{harm,+} \oplus \mathcal{H}_{harm,-}, \tag{6.21}$$

where $\mathcal{H}_{harm,+}$ contains all bosonic states $|n,+\rangle$ and $\mathcal{H}_{harm,-}$ contains all fermionic states $|n,-\rangle$. The Hamiltonian H_{harm} can be represented as $H_{harm} = H_{harm,+} + H_{harm,-}$ acting on $\mathcal{H}_{harm,+} \oplus \mathcal{H}_{harm,-}$ via

$$\mathcal{H}_{harm}|\psi\rangle = \mathcal{H}_{harm,+}|\psi\rangle + \mathcal{H}_{harm,-}|\psi\rangle \tag{6.22}$$

for any $|\psi\rangle \in \mathcal{H}_{harm}$. The following conditions have to be fulfilled:

$$H_{harm,+}|\psi\rangle = \begin{cases} H_{harm,+}|\psi\rangle = E|\psi\rangle & \text{for } |\psi\rangle \in \mathcal{H}_{harm,+}, \\ H_{harm,+}|\psi\rangle = 0 & \text{for } |\psi\rangle \in \mathcal{H}_{harm,-} \end{cases} \tag{6.23}$$

and

$$H_{harm,-}|\psi\rangle = \begin{cases} H_{harm,-}|\psi\rangle = E|\psi\rangle & \text{for } |\psi\rangle \in \mathcal{H}_{harm,-}, \\ H_{harm,-}|\psi\rangle = 0 & \text{for } |\psi\rangle \in \mathcal{H}_{harm,+}. \end{cases} \tag{6.24}$$

As shown in Ref. [74], every supersymmetric theory with Hamiltonian $H = H_+ + H_-$ over a Hilbert space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ has such a representation. Thus, we have for the Witten index [74]

$$\mathrm{Tr}\left[(-1)^F e^{-\beta H}\right] = \mathrm{Tr}\left[(-1)^F e^{-\beta H_+}\right] + \mathrm{Tr}\left[(-1)^F e^{-\beta H_-}\right] = \mathrm{Tr}\left[e^{-\beta H_+}\right] - \mathrm{Tr}\left[e^{-\beta H_-}\right], \tag{6.25}$$

which can be considered as partition function for supersymmetric systems. Following Ref. [74], the operator $(-1)^F$ is defined by

$$(-1)^F|\text{state}\rangle = \begin{cases} |\text{state}\rangle & \text{for } |\text{state}\rangle \in \mathcal{H}_+, \\ -|\text{state}\rangle & \text{for } |\text{state}\rangle \in \mathcal{H}_-. \end{cases} \tag{6.26}$$

For the harmonic oscillator gas we obtain

$$\mathrm{Tr}\left[(-1)^F e^{-\beta H_{harm}}\right] = \mathrm{Tr}\left[e^{-\beta H_{harm,+}}\right] - \mathrm{Tr}\left[e^{-\beta H_{harm,-}}\right] = 0. \tag{6.27}$$

At the inverse temperature $\beta = 1$ the frequency becomes zero. This means that all states suddenly go over to the vacuum with zero energy, but $\mathrm{Tr}\left[(-1)^F e^{-\beta H_{harm}}\right] = 0$ holds independently of β. From the supersymmetric point of view all bosonic and fermionic contributions cancel each other, even, if at $\beta = 1$ all states go over to the vacuum.

However, if we consider the isolated bosonic ensemble, the partition function is

$$Z_B(\beta) = \mathrm{Tr}\left[(-1)^F e^{-\beta H_{harm,+}}\right] = \sum_{n=0}^{\infty} e^{-\beta E_n} = \sum_{n=0}^{\infty} \beta^n = \frac{1}{1-\beta}, \tag{6.28}$$

where we have used the convergence of the geometric series for $\beta < 1$. Moreover, the isolated fermionic ensemble has the partition function

$$Z_F(\beta) = \mathrm{Tr}\left[(-1)^F e^{-\beta H_{harm,-}}\right] = -\mathrm{Tr}\left[e^{-\beta H_{harm,+}}\right] = \frac{1}{\beta - 1}, \tag{6.29}$$

where we explicitly take account to supersymmetry, i.e. that we take care of the right sign, which arises through the operator $(-1)^F$.

If we consider the bosonic and fermionic ensemble isolated, both systems become instable at $\beta = 1$, since the vacuum is then infinitely degenerated. Since this leads to a divergent partition function, both, the bosonic and the fermionic system reach its Hagedorn temperature at $\beta = 1$.

From the physics point of view, it is clear that both systems must have different microscopic descriptions before and after the Hagedorn temperature is reached. The explanation in condensed matter physics is that at the Hagedorn temperature a phase transition occurs, where the density of states growths exponentially [12, 27, 36–40, 61]. In the case of hadronic matter, which can be described by the statistical bootstrap model [27, 36, 61], one has a transition into quark matter [36–40].

Now, we take focus on the isolated fermionic ensemble (6.29). As we already mentioned, a Hagedorn transition occurs at $\beta = 1$, where all fermions go over to the vacuum with zero energy. As a consequence, all fermions must occupy the same state, what is not allowed due to the Pauli exclusion principle. In oder to obtain a physical description of the fermionic ensemble (6.29) for $\beta > 1$ we have to find a mechanism in accordance with quantum statistics. Since only bosons are able to occupy a single quantum state multiple, a good candidate for the searched mechanism would be a transition of fermions into bosons, thus Eq. (6.29) can be continued into a boson-like description.

For so far, it is completely unclear how such a transition can be realized for the isolated system (6.29). However, in the context of the primon gas, there are investigations [8] on the transformation between bosonic and fermionic ensembles by considering parafermionic states. Our approach will be motivated from the BCS theory [9, 10, 21].

Now, we consider an interacting system described by the Hamiltonian

$$H = H_{harm,-} + H' + H_{int}, \tag{6.30}$$

where $H_{harm,-}$ is the Hamiltonian of the fermionic harmonic oscillator gas (6.29), H' is a restriction of H_{prim} to a "smaller" supersymmetric subspace. The Hamiltonian H_{int} describes a weak-interaction between the fermions of the harmonic oscillator gas and the primon gas fermions and has no contributions to H in the regime $0 < \beta < 1$. Therefore, we consider an effective interaction H_{int}^{eff} given by

$$H_{int}^{eff} = \theta(\beta - 1)H_{int} \tag{6.31}$$

with the Heaviside step function

$$\theta(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } x \geq 0. \end{cases} \tag{6.32}$$

Since, we want to consider a suitable subsystem of the supersymmetric primon gas, we have to require

$$\mathrm{Tr}\left[(-1)^F e^{-\beta H'}\right] = 1. \tag{6.33}$$

The full supersymmetric primon gas is to "big" for our considerations, since in Spector's approach [70], every energy level $E_n = \ln n$ has as much supersymmetric realizations $|n, d\rangle = |\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3, \alpha_4, \beta_4, \ldots, \alpha_k, \beta_k, \ldots\rangle$ as the number of square-free divisors of n allows to have. Rather, due to the Pauli exclusion principle, it is reasonable to consider a system, where an energy level can only be occupied from a single fermion. Therefore, we consider a subsystem, which consists of states with only one supersymmetric realization.

To achieve this, we only take states of the form $|n, d^*\rangle$ into account, where d^* has to be the smallest prime factor of n. For example, let $p_1 = 2$ be the smallest prime factor of $n = \prod_k p_k^{\alpha_k}$. Then, we have the bosonic realization $|\alpha_1, \beta_1 = 0, \alpha_2, \beta_2, \alpha_3, \beta_3, \alpha_4, \beta_4, \ldots, \alpha_k, \beta_k, \ldots\rangle$ and the fermionic realization $|\alpha_1 - 1, \beta_1 = 1, \alpha_2, \beta_2, \alpha_3, \beta_3, \alpha_4, \beta_4, \ldots, \alpha_k, \beta_k, \ldots\rangle$. As a consequence, we obtain an unique supersymmetric representation of the state $|n\rangle$. Moreover, we denote by $|n, +\rangle$ the bosonic states and by $|n, -\rangle$ the fermionic states. Furthermore, we denote by $\mathcal{H}'$ the Hilbert space of this states, which is a subspace of $\mathcal{H}_{prime}$. Obviously, we have

$$\mathrm{Tr}\left[(-1)^F e^{-\beta H'}\right] = \mathrm{Tr}\left[e^{-\beta H'_+}\right] + \mathrm{Tr}\left[e^{-\beta H'_-}\right] = 1. \tag{6.34}$$

Following the standard procedure of supersymmetric quantum mechanics, we introduce a separation $\mathcal{H}'$ into $\mathcal{H}' = \mathcal{H}'_+ \oplus H'_-$, where $\mathcal{H}'_+$ ($\mathcal{H}'_-$) contains all bosonic (fermionic) states. Moreover, for energies greater than zero, all bosonic and fermionic contributions cancel each other. Thus, only the vacuum $|1\rangle$ survives as the only "visible" state.

By $\mathcal{H}_{harm,-}$ we denote the Hilbert space of the fermionic harmonic oscillator states. Since we assumed that for $0 < \beta < 1$ the interaction part of (6.30) can be neglected ($H_{int} = 0$), we have isolated systems, i.e that we can separate the Hamiltonian into

$$H|\psi\rangle = \begin{cases} H_{harm,-}|\psi\rangle & \text{for } |\psi\rangle \in \mathcal{H}_{harm,-}, \\ H'|\psi\rangle & \text{for } |\psi\rangle \in \mathcal{H}' \end{cases} \tag{6.35}$$

and further

$$H_{harm,-}|\psi\rangle = 0 \tag{6.36}$$

for every $|\psi\rangle \in \mathcal{H}'$ as well as

$$H'|\psi\rangle = 0 \tag{6.37}$$

for every $|\psi\rangle \in \mathcal{H}_{harm,-}$. In this situation the Hilbert space can be written as $\mathcal{H}_{harm,-} \oplus \mathcal{H}'$. The partition function becomes

$$Z(\beta) = \mathrm{Tr}\left[(-1)^F e^{-\beta H}\right] = \mathrm{Tr}\left[e^{-\beta H_{harm,-}}\right] - \mathrm{Tr}\left[e^{-\beta H'}\right] \tag{6.38}$$

by explicitly taking account of supersymmetry. Therefore, we obtain

$$Z(\beta) = \frac{1}{\beta - 1} + 1. \tag{6.39}$$

By using Eq. (6.34) we can express the partition function in terms of

$$\begin{aligned} \mathrm{Tr}\left[(-1)^F e^{-\beta H}\right] &= \mathrm{Tr}\left[(-1)^F e^{-\beta H_{harm,-}}\right] + \mathrm{Tr}\left[e^{-\beta H'_+}\right] - \mathrm{Tr}\left[e^{-\beta H'_-}\right] \\ &= \frac{1}{\beta - 1} + \sum_{n=1}^{\infty} \frac{1}{n^\beta} - \sum_{n=1}^{\infty} \frac{1}{(n+1)^\beta} \qquad (6.40) \\ &= \frac{1}{\beta - 1} + \zeta'_B(\beta) + \zeta'_F(\beta). \qquad (6.41) \end{aligned}$$

Here, $\zeta'_B(\beta)$ describes the bosonic part of the ensemble given by H'. Furthermore, $\zeta'_F(\beta)$ is the fermionic part with partition function

$$\zeta'_F(\beta) = -\sum_{n=1}^{\infty} \frac{1}{(n+1)^\beta}. \tag{6.42}$$

The rearranged expression (6.40) corresponds to the consideration of $\mathcal{H}$ as

$$\mathcal{H} = \mathcal{H}_{harm,-} \oplus \mathcal{H}'_+ \oplus \mathcal{H}'_-. \tag{6.43}$$

In Ref. [1], it is showed that one easily can derivate

$$\frac{1}{\beta-1}-\sum_{n=1}^{\infty}\frac{1}{(n+1)^{\beta}}=\sum_{n=1}^{\infty}\int_0^1 dt\frac{t\beta}{(n+t)^{\beta+1}}, \tag{6.44}$$

which leads directly, by rearranging the terms in Eq. (6.38), to an expression of the partition functions in terms of

$$Z(\beta)=\zeta(\beta)+\sum_{n=1}^{\infty}\int_0^1 dt\frac{t\beta}{(n+t)^{\beta+1}}. \tag{6.45}$$

Since both terms in Eq. (6.45) are defined for every $\beta \geq 1$, i.e. for temperatures smaller than the Hagedorn temperature, the ensemble (6.45) is a continuation of the initial ensemble (6.38) after the Hagedorn transition.

Since we assume that for $\beta > 1$ the interaction part H_{int} of the Hamiltonian (6.30) takes effect on the system, the fermions taking account to the term

$$\sum_{n=1}^{\infty}\int_0^1 dt\frac{t\beta}{(n+t)^{\beta+1}}. \tag{6.46}$$

have to be eigenstates of $H_{harm,-}+H'_{-}+H_{int}$, now with $H_{int}\neq 0$. Since at the Hagedorn temperature all fermions of the harmonic oscillator gas go over to the vacuum, the ensemble containing these fermions, has to become boson-like. Therefore, we interpret the interaction between the fermions of the harmonic oscillator gas and the fermions of the supersymmetric primon gas, induced by H_{int}, as a coupling of fermions to boson-like pairs. Such a coupling process between fermions is a well-known phenomenon in condensed matter physics. The BCS theory [9, 10, 21] explains the transition into the superconducting phase as formation of Cooper pairs. This boson-like states are pairs of electrons coupled via a weak attractive interaction. The concept of Cooper pairs has nowadays applications to other phenomena in condensed matter physics [63].

Moreover, at $\beta = 1$, the number of degrees of freedom growths exponentially, because, due to the coupling of the fermions, for every harmonic oscillator state one bosonic primon is created. Thus, a "doubling" of states occurs at the Hagedorn temperature.

For a better understanding of the coupling process, we have to analyze the eigenstates of $H_{harm,-}+H'_{-}+H_{int}$. Since we consider pairs of fermions, we introduce the notation

$$H_{pair}=H_{harm,-}+H'_{-}+H_{int} \tag{6.47}$$

Then, the full Hilbert space is $\mathcal{H}'_{+}\oplus\mathcal{H}_{pair}$, where $\mathcal{H}_{pair}$ is the corresponding Hilbert space to H_{pair}.

In order to understand the coupling of the fermions and the eigenstates of H_{pair}, we consider the ensemble (6.45) in the absence of the bosonic primon gas by formally setting $\zeta \equiv 0$. Then, the partition function becomes

$$Z(\beta) = \sum_{n=1}^{\infty} \int_0^1 dt \frac{t\beta}{(n+t)^{\beta+1}}. \tag{6.48}$$

The mean value theorem for integrals, which states that there exists a $\gamma \in (0,1)$ with

$$\int_0^1 dt \frac{t\beta}{(n+t)^{\beta+1}} = \frac{\gamma\beta}{(n+\gamma)^{\beta+1}} \tag{6.49}$$

implies

$$Z(\beta) = \sum_{n=1}^{\infty} \frac{\gamma\beta}{(n+\gamma)^{\beta+1}}. \tag{6.50}$$

Since generally the partition function of a canonical ensemble has the form

$$Z(\beta) = \sum_{n=1}^{\infty} e^{\beta E_n}, \tag{6.51}$$

we are able to give an explicit expression for E_n in terms of

$$e^{-\beta E_n} = \frac{\beta\gamma}{(n+\gamma)^{\beta+1}}. \tag{6.52}$$

Thus, we obtain

$$E_n = \frac{(\beta+1)\ln\gamma \ln n}{\beta} - \frac{\ln\beta \ln\gamma}{\beta}, \tag{6.53}$$

which are the energy eigenvalues of H_{pair}.

First, the energy of the new vacuum is

$$E_1 = -\frac{\ln\beta \ln\gamma}{\beta}, \tag{6.54}$$

where we have set $n = 1$.

The new vacuum energy is positive, since $\beta > 1$ and $\gamma < 1$. Furthermore, as one would expect, the vacuum energy becomes zero at the Hagedorn temperature.

Since $\omega(\beta)$ tends to zero for low temperatures, we would expect, that in the low temperature limit the contributions from the primon gas fermions are dominant. Indeed, in the limit of low temperatures, we obtain

$$\lim_{\beta\to\infty} E_n = \ln\gamma \ln n, \tag{6.55}$$

which are exactly the energies of the fermionic primon gas states shifted by the factor $\ln\gamma$.

At the Hagedorn temperature we have

$$E_n = 2 \ln \gamma \ln n. \tag{6.56}$$

Since $\ln \gamma$ is negative for $0 < \gamma < 1$, this amount of energy is needed to break a pair of fermions, which occupied the n-th energy level. Thus, this amount of energy can be considered as the pure binding energy between the fermions. Moreover, at $\beta = 1$, the vacuum with zero energy is an ordinary state in the discrete spectrum of H_{pair}, thus the coupling between the fermions, which built a pair occupying the energy level $n = 1$, will be automatically broken at the Hagedorn temperature.

Summarizing, the phase transition at the Hagedorn temperature can be interpreted as follows: in the phase with $0 < \beta < 1$ the initial ensemble can be considered as a non-interacting system of fermionic harmonic oscillators and supersymmetric primons, where bosonic and fermionic contributions cancel each other up to the single vacuum, independently of β. At the Hagedorn temperature, where all fermions of the harmonic oscillator gas go over to the vacuum with zero energy, there must be a transition, where fermions are transformed into bosons, due to the Pauli exclusion principle. Moreover, the infinitely degenerated vacuum leads to a diverging partition function, thus the system becomes instable. As a consequence of a coupling between the fermions of the harmonic oscillator gas and the fermions of the supersymmetric primon gas, via an attractive interaction, which occurs for $\beta \geq 1$, the fermions built boson-like pairs. One can compare this to the formation of Cooper pairs in the BCS theory. Then, the system, in the phase with $\beta > 1$, can be described as a non-interacting ensemble of bosonic primons and boson-like pairs of fermions.

In a more realistic situation, for temperatures in the range $0 < \beta < 1$, the states of the fermionic harmonic oscillator gas are filled up to a finite energy, the Fermi energy E_F. Here, we have to mention that in reality, for temperatures $T > 0$, the Fermi energy is only an approximately sharp level, since some thermal fluctuations around the Fermi energy possibly occur. Since we consider only a finite number of fermions, divergences cannot occur in a strict sense. Moreover, the boson-like pairs of fermions tend to occupy the lowest energy level. Therefore, in a realistic scenario, the coupled fermions would all occupy the state with energy $E_1 = -\frac{\ln \beta \ln \gamma}{\beta}$, due to Eq. (6.54). Then, at $\beta = 1$, the thermal energy is high enough to break the pairs of fermions and, consequently, one has a transition into the fermionic harmonic oscillator gas.

6.3 A Link to the Riemann Hypothesis

In the previous section we worked out an explicit physical explanation of an analytic continuation of a partition function, which describes a system, closely related to the primon gas. In particular, the analytic continuation is necessary in order to understand a phase transition at the Hagedorn temperature, which separates two phase, one in the regime $0 < \beta < 1$ and one for $\beta > 1$. Now, we apply some results of our physical model to the Riemann hypothesis.

First, we consider the Hurwitz zeta function

$$\zeta(\beta+1,\gamma) = \sum_{n=0}^{\infty} \frac{1}{(n+\gamma)^{\beta+1}} \tag{6.57}$$

and yield for Eq. (6.51) a more technical expression

$$Z(\beta) = \beta\gamma\zeta(\beta+1,\gamma) - \frac{\beta}{\gamma^{\beta}}. \tag{6.58}$$

Furthermore, an important object in thermodynamics is the free energy $F(\beta)$, which is defined as

$$F(\beta) = -\frac{1}{\beta}\ln Z(\beta). \tag{6.59}$$

With Eq. (6.58), i.e., in the absence of the bosonic primon gas, we have an explicit expression for the free energy given by

$$F(\beta) = -\frac{\ln(\beta\gamma)}{\beta}\ln\left(\zeta(\beta+1,\gamma) - \frac{1}{\gamma^{\beta+1}}\right). \tag{6.60}$$

In the previous considerations we assumed the absence of the bosonic primon gas in order to analyze the boson-like pairs of fermions. Now, we take focus on the assumption $\zeta \equiv 0$.

Obviously, $\zeta \equiv 0$ is realized on the set of zeros of ζ in the critical strip. Moreover, for $\beta F(\beta) = -\ln Z(\beta)$ not too large, we can approximate the partition function, defined by Eq. (6.59), in terms of

$$Z(\beta) = 1 - \beta F(\beta). \tag{6.61}$$

Then, we have

$$1 - \beta F(\beta) = \frac{1}{\beta - 1} + 1. \tag{6.62}$$

Since the partition function of a canonical ensemble cannot vanish for real-valued temperatures, Eq. (6.62) can only be fulfilled for a complex-valued β with a non-vanishing imaginary part.

At temperatures, where $F \neq 0$, Eq. (6.62) is equivalent to

$$\beta^2 - \beta + \frac{1}{F} = 0. \tag{6.63}$$

As solutions of this quadratic equation we get

$$\beta = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{F}}. \tag{6.64}$$

Since β has to be complex-valued with a non-vanishing imaginary part, it must have real part one half, assuming that the free energy F is still real-valued.

In other words, the absence of the bosonic primon gas can only be realized for complex-valued temperatures, where $\Re(\beta) = \frac{1}{2}$. Moreover, the complex-valued inverse temperatures has to be symmetric with respect to the imaginary axis. The fact that all zeros lie on a line in the complex plane with real part $\frac{1}{2}$ might correlate with a possible phase transition of the bosonic primon gas at $\beta = \frac{1}{2}$. Possibly, the vanishing of the bosonic primons is a consequence of such a phase transition.

Therefore, our approach shows a physical evidence for the truth of the Riemann hypothesis, although this approach is not rigorous. Rather it becomes clear that one has to understand the behavior of the free energy in the critical strip. The assumption that the free energy is still real-valued at the zeros of the Riemann zeta function makes sense from the physics point of view, since it describes a realizable physical system. Certainly, this is hard to prove.

Chapter 7

Summary and Outlook

In this dissertation I have derived a supersymmetric analogue of the conformal geometric quantum mechanics, which I have introduced in my Master's thesis [77]. This unifying approach yields fundamental connections between conformal geometric quantum mechanics, the spectral geometry of Schrödinger operators and topology. In particular, I have shown that the famous Yang-Yau estimate for the first eigenvalue of the Laplacian [80] can be derived in a physics-based way. Furthermore, the supersymmetry approach shows how the eigenvalue estimate can be extended to Schrödinger operators. In this way I provide an estimate for the energy gap between the vacuum and the first excited state valid for arbitrary (non-periodic) potentials. As I illustrate for the Coulomb potential and the harmonic oscillator the estimates provide a good approximation of the true value. Furthermore, as in mathematics, the geometrical properties of eigenvalue estimates for Laplacians and Schrödinger Operators are still intensively studied (see, for example, Refs. [16–19, 24, 25, 57, 79, 81]). The developed approach might provide the basis for novel mathematical insights.

Moreover, I introduced concepts of supersymmetry to spin chain models by describing the 1D nearest neighbor Ising model by supercharges. By doing so, it was shown that supersymmetry naturally occurs in spin chain models. Next, motivated from the framework of Knauf's number-theoretical spin chain [53], I rigorously derived an expression for n-point correlation functions in terms of transformed Boltzmann weights, which I introduced as Witten indices defined on the dual configuration spaces. The results were transfered to subspaces and by using methods from finite Fourier analysis I worked out an explicit expression of the vacuum expectation value for the density matrix in terms of n-point correlation functions. Moreover, I analyzed the result for the special case of supersymmetric theories and showed that phase transitions cannot occur in supersymmetric spin chains. Furthermore, I showed the independence of the result from the choice of a basis. In other words, the results are invariant under unitary transformations.

Based on Spector's theory of the supersymmetric primon gas [70] I analyzed the behavior of a canonical ensemble, which is closely related to the primon gas. To do so, I introduced a coupling process of fermions to boson-like pairs, comparable with the formation of Cooper pairs in the BCS theory [9, 10, 21]. In this way, I obtained a physical explanation of the Hagedorn transition of the canonical ensemble. Based on this I worked out a novel link to the Riemann hypothesis, which provides another hint about its validity. However, this physical approach yields novel insights into the problem of the Hagedorn transition using supersymmetry and concepts inspired from condensed matter physics. Furthermore, our application to the distribution of the zeros of the Riemann zeta function, which is so far a phenomenological one, shows an interesting evidence for the truth of the Riemann hypothesis. One can expect that a deeper understanding of our approach, by using methods of condensed matter physics, might give new insights into a rigorous treatment of the Riemann hypothesis.

Chapter 8

Appendix

8.1 Riemannian Geometry

We briefly introduce some concepts of Riemannian geometry that are used in Chapter 2, Chapter 3 and Chapter 4 of this dissertation. This section on the foundations of Riemannian geometry is identical with Chapter 2 of my Master's thesis [77]. For better understanding we often use both the local and the global representation of some basic concepts. A more detailed presentation can be found in various books which are convenient for both mathematicians and physicists. In this section, for the most part, we follow Refs. [58, 62].

8.1.1 Tensor Fields

Let M be a differentiable manifold of dimension $\dim(M) = n$. Furthermore, let $p \in M$ be a point. We denote the tangent space in p at M with T_pM. We define then a covariant tensor of rank s as map

$$V_p : \underbrace{(T_pM) \times \ldots \times (T_pM)}_{s \text{ times}} \longrightarrow \mathbb{R}, \tag{8.1}$$

that is linear in each argument. For a given basis

$$\partial_1, \ldots, \partial_n \tag{8.2}$$

of T_pM and its dual basis

$$dx^1, \ldots, dx^n \tag{8.3}$$

of the cotangent space $(T_pM)^*$ we get

$$\Big(dx^{j_1} \otimes \ldots \otimes dx^{j_s}\Big)_{j_1,\ldots,j_s=1,\ldots,n} \tag{8.4}$$

as a basis of the space of all covariant tensors of rank s in p. Thus, an arbitrary tensor is given by

$$V_p = \sum V_{j_1\ldots j_s} dx^{j_1} \otimes \ldots \otimes dx^{j_s} = V_{j_1\ldots j_s} dx^{j_1} \otimes \ldots \otimes dx^{j_s}, \tag{8.5}$$

where in the last identity we used summation convention, i.e. repeated upper and lower indices have to be summed over. Upon using

$$\Big(dx^{j_1} \otimes \ldots \otimes dx^{j_s}\Big)(\partial_{l_1}, \ldots, \partial_{l_s}) = \delta^{j_1}_{l_1} \cdot \ldots \cdot \delta^{j_s}_{l_s}, \tag{8.6}$$

we obtain

$$V_{j_1\ldots j_s} = V_p(\partial_{j_1}, \ldots, \partial_{j_s}), \tag{8.7}$$

for the coefficients. Thus, in the Ricci calculus a covariant tensor of rank s is expressed by $V_{j_1\ldots j_s}$. Moreover, a differentiable covariant tensor field of rank s is a map $p \mapsto V_p$ where the coefficient functions are differentiable. A map

$$V_p : \underbrace{(T_pM)^* \times \ldots \times (T_pM)^*}_{r \text{ times}} \times \underbrace{(T_pM) \times \ldots \times (T_pM)}_{s \text{ times}} \longrightarrow \mathbb{R}, \tag{8.8}$$

which is linear in each argument is a tensor of rank $r + s$ where r is the rank of contravariance and s is the rank of covariance. A basis of the space of all these tensors is given by

$$\Big(\partial_{i_1} \otimes \ldots \partial_{i_r} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s}\Big)_{i_1,\ldots,i_r,j_1,\ldots,j_s=1,\ldots,n}. \tag{8.9}$$

Therefore,

$$V_p = V^{i_1\ldots i_r}_{j_1\ldots j_s} \partial_{i_1} \otimes \ldots \partial_{i_r} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s}. \tag{8.10}$$

Furthermore, the identity

$$(\partial_{i_1} \otimes \ldots \otimes dx^{j_s})(dx^{k_1}, \ldots, dx^{k_r}, \partial_{l_1}, \ldots, \partial_{l_s}) = \delta^{k_1}_{i_1} \cdot \ldots \cdot \delta^{k_r}_{i_r} \cdot \delta^{j_1}_{l_1} \cdot \ldots \cdot \delta^{j_s}_{l_s}, \tag{8.11}$$

implies

$$V^{i_1\ldots i_r}_{j_1\ldots j_s} = V_p(dx^{i_1}, \ldots, dx^{i_r}, \partial_{j_1}, \ldots, \partial_{j_s}). \tag{8.12}$$

In the Ricci calculus we denote $V^{i_1\dots i_r}_{j_1\dots j_s}$ for a tensor of rank $r+s$. Moreover, a differentiable tensor field of rank $r+s$ is a map $p \mapsto V_p$ where the coefficient functions $V^{i_1\dots i_r}_{j_1\dots j_s}$ are differentiable. Since $(T_pM)^* = \mathrm{Hom}(T_pM, \mathbb{R})$ we can grasp a tensor of rank $r+s$ in p as an element of

$$\underbrace{(T_pM) \otimes \ldots \otimes (T_pM)}_{s \text{ times}} \otimes \underbrace{(T_pM)^* \otimes \ldots \otimes (T_pM)^*}_{r \text{ times}} \equiv \mathcal{T}_p^{(r,s)}, \tag{8.13}$$

and a tensor field as map $p \mapsto V_p \in \mathcal{T}_p^{(r,s)}$. Furthermore, a tensor (field) of rank zero is called scalar (field) and a covariant respectively contravariant tensor (field) is called vector (field) respectively covector (field).

8.1.2 Riemannian Metrics

Now, we can define the Riemannian metric g on M in a formal way as differentiable tensor field $p \mapsto g_p \in \mathcal{T}_p^{(0,2)}$ for any $p \in M$ with the properties

1. $g_p(X,Y) = g_p(Y,X) \quad$ for all $X, Y \in T_pM$,

2. $g_p(X,X) > 0 \quad$ for all $X \neq 0$.

Furthermore, the metric tensor is given by the coefficients

$$g_{ij} = g_p(\partial_i, \partial_j). \tag{8.14}$$

The pair (M, g) is called a Riemannian manifold. A metric defines an inner product g_p on T_pM at every point p. In this way it induces a measure for lengths and angles on T_pM respectively M. More precisely, for two vectors $X = X^i\partial_i$ and $Y = Y^j\partial_j$ the inner product $\langle X, Y\rangle$ is given by

$$\langle X, Y\rangle = g_p(X,Y) = g_{ij}X^iY^j = X_jY^j. \tag{8.15}$$

Therefore, g is an isomorphism between T_pM and $(T_pM)^*$ through

$$g(\,\cdot\,, X) : T_pM \to (T_pM)^*, \tag{8.16}$$

implying

$$g_{ij}X^j = X_i, \tag{8.17}$$

and

$$g^{ij}X_j = X^i. \tag{8.18}$$

Here, g^{ij} denotes the inverse g^{-1} of the metric tensor. As the last argumentations show the metric tensor and its inverse raise and lower, respectively, the indices of the components of tensors.

On a Riemannian manifold (M, g) the volume element for integration is given by

$$dM = \sqrt{\det g} \cdot dx^i \dots dx^n. \tag{8.19}$$

8.1.3 Isometries and Conformal Maps

For a differentiable map $F : M \to \widetilde{M}$ between two manifolds M and $\widetilde{M}$ and $F(p) = q$ for fixed points $p \in M$ and $q \in \widetilde{M}$ the differential of F is defined by

$$DF|_p : T_pM \longrightarrow T_q\widetilde{M}, \tag{8.20}$$

with $(DF|_p(X))(f) \equiv X(f \circ F)$ for each differentiable function $f : \widetilde{M} \to \mathbb{R}$ and each $X \in T_pM$. Now, let (M, g) and $(\widetilde{M}, \widetilde{g})$ be two Riemannian manifolds. A differentiable map $F : M \longrightarrow \widetilde{M}$ is called isometry if

$$\widetilde{g}_{F(p)}(DF|_p(X), DF|_p(Y)) = g_p(X, Y), \tag{8.21}$$

holds for each $p \in M$ and $X, Y \in T_pM$. Moreover, F is called conformal map if a function $\lambda : M \to \mathbb{R}^+$ exists with

$$\widetilde{g}_{F(p)}(DF|_p(X), DF|_p(Y)) = \lambda(p)g_p(X, Y), \tag{8.22}$$

for all $p \in M$ and $X, Y \in T_pM$.

Levi-Civita Connection

The Levi-Civita connection or covariant derivative ∇ of a Riemannian manifold (M, g) is a map

$$(X, Y) \mapsto \nabla_X Y, \tag{8.23}$$

which maps two differentiable vector fields X, Y onto the differentiable vector field $\nabla_X Y$ in such a way that

1. $\nabla_{fX_1+gX_2}Y = f \cdot \nabla_{X_1}Y + g \cdot \nabla_{X_2}Y,$

2. $\nabla_X(Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2,$

3. $\nabla_X(fY) = (X(f)) \cdot Y + f \cdot \nabla_X Y$,

4. $X\left(g(Y,Z)\right) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$,

5. $(\nabla_X Y)(f) - (\nabla_Y X)(f) - [X,Y](f) = 0$.

Here, $X, Y, Z, X_1, X_2, Y_1, Y_2$ are differentiable vector fields, f and g are differentiable functions and $[X,Y](f) \equiv X(Y(f)) - Y(X(f))$ is the Lie bracket. For $X = X^i\partial_i$ and $Y = Y^j\partial_j$ we obtain for the expression in local coordinates by using the above properties of the covariant derivative

$$\begin{aligned}\nabla_X Y &= \nabla_{X^i\partial_i}(Y^j\partial_j) \\ &= X^i(\nabla_{\partial_i}(Y^j\partial_j)) = X^i(Y^j\nabla_{\partial_i}\partial_j + (\partial_i Y^j)\partial_i) \\ &= (X^i\partial_i Y^k + \Gamma^k_{ij}X^iY^j)\partial_k. \qquad (8.24)\end{aligned}$$

Here, Γ^k_{ij} are the Christoffel symbols (of the second kind) with $\nabla_i\partial_j = \Gamma^k_{ij}\partial_k$. Furthermore, the last of the above properties implies $\Gamma^k_{ij} = \Gamma^k_{ji}$. Moreover, it can be shown that the covariant derivative is unique and

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}). \qquad (8.25)$$

Furthermore, we can write the identity (8.24) in terms of

$$\begin{aligned}\nabla_X Y &= (X^i\partial_i Y^k + \Gamma^k_{ij}X^iY^j)\partial_k \\ &= X^i(\partial_i Y^k + \Gamma^k_{ij}Y^j)\partial_k \\ &= X^i(\nabla_i Y^k)\partial_k. \qquad (8.26)\end{aligned}$$

Therefore, the components of the covariant derivative ∇_i of a vector field Y^k in the Ricci calculus are given by

$$\nabla_i Y^k = \partial_i Y^k + \Gamma^k_{ij}Y^j. \qquad (8.27)$$

In general, the covariant derivative can be extended to tensor fields of rank $r+s$

$$\begin{aligned}\nabla_k V^{i_1\ldots i_r}_{j_1\ldots j_s} = \partial_k V^{i_1\ldots i_r}_{j_1\ldots j_s} &- \Gamma^j_{kj_1}V^{i_1\ldots i_r}_{jj_2\ldots j_s} - \ldots - \Gamma^j_{kj_s}V^{i_1\ldots i_r}_{j_1\ldots j_{s-1}j} \\ &+ \Gamma^{i_1}_{ki}V^{ii_2\ldots i_r}_{j_1\ldots j_s} + \ldots + \Gamma^{i_r}_{ki}V^{i_1\ldots i_{r-1}i}_{j_1\ldots j_s}. \qquad (8.28)\end{aligned}$$

8.1.4 The Riemannian Curvature Tensor

For a Riemannian manifold (M, g) the covariant derivative induces an intrinsic measure of curvature in terms of the so called Riemannian curvature tensor

$$R(X,Y)Z = \Big(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\Big)\, Z. \tag{8.29}$$

Then, $R(X,Y)Z$ is a measure of the non-commutativity of covariant derivatives. To express this quantity in the Ricci calculus we have to determine the components of the curvature tensor in the local basis of the tangent spaces. In doing so, we obtain

$$R(\partial_k, \partial_j)\partial_i = R^l_{ikj}\partial_l, \tag{8.30}$$

with

$$R^l_{ikj} = \partial_k \Gamma^l_{ij} - \partial_j \Gamma^l_{ik} + \Gamma^r_{ij}\Gamma^l_{rk} - \Gamma^r_{ik}\Gamma^l_{rj}. \tag{8.31}$$

A Riemannian manifold (M, g) is called flat if and only if the coefficient functions of the curvature Tensor $R^l_{ikj}(p) = 0$ for all $p \in M$. Moreover, we can see the Riemannian curvature tensor as an operator which acts on vector fields. This operator is denoted by

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}. \tag{8.32}$$

In local coordinates

$$R(\partial_i, \partial_j) = \nabla_i \nabla_j - \nabla_j \nabla_i, \tag{8.33}$$

where we have used that the term $\nabla_{[X,Y]}$ vanishes since $[\partial_i, \partial_j] = 0$. Moreover, by contraction of the Riemannian curvature tensor we get the Ricci tensor

$$R_{ij} = R^k_{ikj}, \tag{8.34}$$

and the scalar curvature

$$R = g^{ij} R_{ij}. \tag{8.35}$$

For a Riemannian manifold (M, g) with $\dim(M) = 2$ the quantity

$$K \equiv \frac{g_{2l} R^l_{121}}{g_{11} g_{22} - g_{12}^2} = \frac{R_{2121}}{\det(g)}, \tag{8.36}$$

is called Gaussian curvature.

Now, we represent the metric tensor as matrix

$$g = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \tag{8.37}$$

where E, F, G are any differentiable functions on M. For orthogonal parameters ($F = 0$) we get

$$K = -\frac{1}{2\sqrt{EG}} \left(\partial_1 \left(\frac{\partial_1 E}{\sqrt{EG}} \right) + \partial_2 \left(\frac{\partial_2 G}{\sqrt{EG}} \right) \right). \tag{8.38}$$

For $E = G = \lambda$ we obtain

$$K = -\frac{1}{2\lambda} \left(\partial_1 \left(\frac{\partial_1 \lambda}{\lambda} \right) + \partial_2 \left(\frac{\partial_2 \lambda}{\lambda} \right) \right) = -\frac{1}{2\lambda} \Delta(\ln \lambda). \tag{8.39}$$

This is the so-called isothermal parametrization with $E = G = \lambda$ that defines a special conformal class on M.

8.1.5 The Laplace-Beltrami Operator on Riemannian Manifolds

Let T be a linear operator on a Hilbert space H. The spectrum $\sigma(T)$ of T is defined by

$$\sigma(T) \equiv \{\lambda \in \mathbb{C} |\ T - \lambda \mathbb{I} \text{ has no bounded inverse}\}, \tag{8.40}$$

with the identity $\mathbb{I}$ on H.

In this situation, we can consider that $T - \lambda\mathbb{I}$ is not injective, which implies $\ker(T - \lambda\mathbb{I}) \neq \{0\}$, so that a non-zero solution $\psi \in H$ of the eigenvalue equation $T\psi = \lambda\psi$ exists. Then the point spectrum $\sigma_p(T)$ is

$$\sigma_p(T) \equiv \{\lambda \in \mathbb{C} |\ \ker(T - \lambda\mathbb{I}) \neq \{0\}\}. \tag{8.41}$$

Another case is that $T\psi = \lambda\psi$ has an *almost* non-zero solution $\psi \in H$, i.e. the solution ψ is the limit of a Weyl sequence for T and $\lambda \in \sigma(T)$. A sequence $(\psi_n)_{n\in\mathbb{N}}$ in H is called a Weyl sequence if

1. $\|\psi_n\| = 1$ for all $n \in \mathbb{N}$,
2. $\|(T - \lambda\mathbb{I})\psi_n\| \to 0$ as $n \to \infty$,
3. $\langle \phi, \psi_n \rangle \to 0$ for all $\phi \in H$ and $n \to \infty$.

Here $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ are the inner product and the norm of H. So we can define the continuous spectrum $\sigma_c(T)$ as

$$\sigma_c(T) \equiv \{\lambda \in \mathbb{C} |\ \exists \text{ Weyl sequence for } T \text{ and } \lambda\}. \tag{8.42}$$

An important theorem by Weyl states that for a self-adjoint operator T

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T). \tag{8.43}$$

Now let (M, g) be a closed, connected and orientable Riemannian manifold with dimension $\dim(M) = n$. Then, the Hilbert space of square-integrable functions on M is defined as

$$L^2(M) \equiv \{\psi : M \to \mathbb{C} |\ \int_M |\psi|^2\, dM < \infty\}. \tag{8.44}$$

The Sobolev space is defined as

$$H^2(M) \equiv \{\psi \in L^2(M) |\ \nabla^\alpha \psi \in L^2(M)\}, \tag{8.45}$$

where ∇^α are the mixed covariant derivatives of order $|\alpha| = 2$ with the multi index $\alpha = (\alpha_1, \ldots, \alpha_n)$ and the order $|\alpha| \equiv \sum_{i=1}^{N} \alpha_i$.

Now we can define the Laplace-Beltrami operator on (M, g) as differential operator

$$\Delta_g : H^2(M) \to L^2(M), \tag{8.46}$$

by

$$\Delta_g \equiv \mathrm{div}(\mathrm{grad}). \tag{8.47}$$

In local coordinates

$$\Delta_g = \frac{1}{\sqrt{\det g}}\ \partial_i \left(\sqrt{\det g}\ g^{ij} \partial_j\right). \tag{8.48}$$

We can express the Laplace-Beltrami operator also in terms of the covariant derivative by

$$\begin{aligned} \nabla_i(\nabla^i f) &= \nabla_i(\partial^i f) \\ &= \nabla_i(g^{ij}\partial_j f) \\ &= \partial_i(g^{ij}\partial_j f) + \Gamma^i_{ij} g^{jk} \partial_k f \\ &= \frac{1}{\sqrt{\det g}}\ \partial_i(\sqrt{\det g}\ g^{ij}\partial_j) f \\ &= \Delta_g f, \end{aligned} \tag{8.49}$$

for any differentiable function f on M. The Christoffel symbols are given by

$$\Gamma^i_{ij} = \partial_j \ln(\sqrt{\det g}). \tag{8.50}$$

Green's identity

$$\int_M \varphi \Delta_g \psi \; dM = -\int_M \langle \mathrm{grad}\varphi, \mathrm{grad}\psi \rangle \; dM, \tag{8.51}$$

implies that Δ is self-adjoint. For $\varphi = \psi$ with $-\Delta\psi = \lambda\psi$ we obtain

$$\lambda = \frac{\int_M \|\mathrm{grad}\psi\|^2 \; dM}{\int_M |\psi|^2 \; dM} = \frac{\|\|\mathrm{grad}\psi\|\|_{L_2}}{\|\psi\|_{L_2}} \geq 0, \tag{8.52}$$

so that the point spectrum $\sigma_p(-\Delta_g)$ is non-negative. Since M is compact, we can prove easily that $-\Delta_g$ has only a point spectrum.

Theorem 1. *Let M be a compact and orientable manifold, then*

$$\sigma(-\Delta_g) = \sigma_p(-\Delta_g). \tag{8.53}$$

Furthermore, it is well-known that the point spectrum is discrete, i.e. only has isolated eigenvalues.

Gauss-Bonnet Theorem

For a two-dimensional compact Riemannian manifold the Gaussian curvature (8.36) establishes a relation between the geometry of M and its topology through the Gauss-Bonnet theorem.

Theorem 2. *Let (M, g) be a closed and orientable two-dimensional Riemannian manifold with Gaussian curvature K. Then,*

$$\int_M K dM = 2\pi\chi(M), \tag{8.54}$$

where $\chi(M)$ is the Euler characteristic of M.

It is worth remarking that the total curvature $\int_M K dM$ is independent of the metric g on M. The Euler characteristic of a manifold is a topological invariant, what means that it don't change under homeomorphisms. By means of the Euler characteristic all two-dimensional compact manifolds without boundary can be classified by

Theorem 3. *Two connected and closed two-dimensional manifold are homeomorphic if and only if they have the same Euler characteristic and are both either orientable or non-orientable.*

From the point of view of algebraic topology the Euler characteristic of a two-dimensional manifold M can be calculated by triangulating M. A triangulation is a triple $(M, \mathcal{K}, h)$ where $\mathcal{K}$ is a geometric simplicial complex and h a homeomorphism $h : \mathcal{K} \to M$. Then, the Euler characteristic of M is

$$\chi(M) = V - E + F, \tag{8.55}$$

where V is the number of vertices, E the number of edges and F the number of faces in the given $\mathcal{K}$. Another way to determine the Euler characteristic is through the genus γ of M where the genus is the number of holes of M. Therefore, the sphere has genus zero, the torus has genus one and in general the n-fold torus has genus n. The Euler characteristic is given by

$$\chi = 2 - 2\gamma, \tag{8.56}$$

for orientable manifolds and

$$\chi = 2 - \gamma, \tag{8.57}$$

for non-orientable manifolds. Typical examples for orientable compact two-dimensional manifolds without boundary are the sphere and the n-fold torus. A projective plane and the Klein bottle are examples for non-orientable compact manifolds without boundary with genus one and two.

8.1.6 Complex Line Bundles

We consider smooth manifolds L and M and a surjective function $\pi : L \to M$. Then, a map $s : M \to L$ is called section if $\pi \circ s = \mathrm{id}_M$ where $\mathrm{id}_M : M \to M$ is the identity map. The set of all smooth sections is denoted by $\Gamma^\infty(L, M)$.

Definition 1. *A complex line bundle is a triple* (L, M, π) *with the properties:*

1. *The fibres* $\pi^{-1}(p)$ *are complex one-dimensional vector spaces for each* $p \in M$*.*
2. *An open covering* $\{U_i\}_{i \in I}$ *of* M *and diffeomorphisms* $\phi_i : \mathbb{C} \times U_i \to \pi^{-1}(U_i)$ *exists such that* $\pi \circ \phi_i = id|_{U_i}$*.*

A line bundle is labeled by the shortened form $\pi : L \to M$. A covariant derivative ∇ on a line bundle is a map which assign a differentiable vector field X and a section s to another section $\nabla_X s$ such that

1. $\nabla_X(s_1 + s_2) = \nabla_X s_1 + \nabla_X s_2$,

2. $\nabla_{fX_1+gX_2} s = f \cdot \nabla_{X_1} s + g \cdot \nabla_{X_2} s$,

3. $\nabla_X(fs) = X(f)s + f \cdot \nabla_X s$,

for any differentiable vector field X_1, X_2 and any differentiable function f and g. Then, the curvature tensor Ω is given by

$$\Omega(X,Y)s = \left(\nabla_X \nabla_Y - \nabla_Y \nabla Y - \nabla_{[X,Y]}\right) s. \tag{8.58}$$

By analogy with the Levi-Cevita connection we consider the curvature tensor as operator on the sections and therefore we get in the Ricci calculus

$$\Omega_{ij} s = (\nabla_i \nabla_j - \nabla_j \nabla_i) s. \tag{8.59}$$

8.2 Differential Forms and the De Rham Cohomology

In this section we introduce some basics of differential forms and de Rham Cohomology that are used in Chapter 4. Here, we follow Refs. [48, 62, 66].

8.2.1 Differential Forms

Starting with a Riemannian manifold M of dimension n and $\omega_1, \omega_2 \in \Lambda^1 = (T_pM)^*$, where p is a fixed point of M, one defines a exterior product with

$$\omega_1 \wedge \omega_2 = -\omega_2 \wedge \omega_1 \tag{8.60}$$

and in particular

$$\omega \wedge \omega = 0. \tag{8.61}$$

This product leads to a new space $\omega_1 \wedge \omega_2 \in \Lambda^2 = (T_pM)^* \wedge (T_pM)^*$. Moreover, one can define successively the space of k-forms

$$\Lambda^k = \underbrace{(T_pM)^* \wedge \cdots \wedge (T_pM)^*}_{k \text{ times}}. \tag{8.62}$$

With the basis $dx^1, \ldots, dx^n$ of $(T_pM)^*$ one gets for an arbitrary k-form in local coordinates

$$\omega(p) = \omega_{i_1,\ldots,i_k}(p) dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \tag{8.63}$$

where $\omega_{i_1,\ldots,i_k}(p)$ is a smooth function on M.

By goinig over to the cotangent bundle

$$(TM)^* = \bigcup_{p \in M} (T_pM)^* \tag{8.64}$$

a differential form in $p \in M$ can be extended to M. The space of k-forms, which are defined on entire M, are denoted by Ω^k.

Furthermore, one has the exterior derivative

$$d : \Omega^k \to \Omega^{k+1} \tag{8.65}$$

given by

$$d(\omega_{i_1,\ldots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = \frac{\partial \omega_{i_1,\ldots,i_k}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}. \tag{8.66}$$

A simple calculation shows

$$d^2 = 0. \tag{8.67}$$

An inner product for k-forms $\omega_1, \omega_2 \in \Omega^k$ is defined by

$$\langle \omega_1, \omega_2 \rangle = \int_M g(\omega_1, \omega_2) dM, \tag{8.68}$$

where $g(\omega_1, \omega_2)$ is the inner product in respect of the metric g, defined by its local representation in the basis of $(T_pM)^*$

$$g^{ij} = g_p(dx^i, dx^j) = (g_{ij})^{-1}. \tag{8.69}$$

Furthermore, the adjoint exterior derivative δ is given by

$$\langle \delta\omega_1, \omega_2 \rangle = \langle \omega_1, d\omega_2 \rangle \tag{8.70}$$

and is a map

$$\delta : \Omega^k \to \Omega^{k-1}. \tag{8.71}$$

with $\delta^2 = 0$. Moreover, the Hodge Laplacian $\Delta : \Omega^k \to \Omega^k$ is defined by

$$\Delta = d\delta + \delta d. \tag{8.72}$$

8.2.2 The De Rham Cohomology

Now, let M be compact and orientable. One call a k-form closed if

$$d\omega = 0 \tag{8.73}$$

and exact if there exists a $(k-1)$-form θ with

$$\omega = d\theta. \tag{8.74}$$

As a direct consequence of $d^2 = 0$ one obtains

$$d\omega = 0 \tag{8.75}$$

for any exact k-form ω. Thus, any exact k-form is also closed. But, on the other hand, one can ask, what are the closed k-forms, which are not exact. Therefore, one introduce the k-th de Rham cohomology group

$$H^k_{dR} = \{\omega \in \Omega^k \mid d\omega = 0\}/\{d\theta \mid \theta \in \Omega^{k-1}\}. \tag{8.76}$$

If two compact and orientable manifolds M_1 and M_2 are homeomorphic, then the de Rham cohomology groups of M_1 and M_2 are isomorphic. The dimension of H^k_{dR} is called the Betti number b_k

$$b_k = \dim H^k_{dR} \tag{8.77}$$

and is a topological invariant.

Moreover, the connection between the Betti numbers and the Euler characteristic is given by

$$\chi(M) = \sum_k (-1)^k b_k. \tag{8.78}$$

Furthermore, the Hodge theorem states that $\ker(\Delta^k)$ and H^k_{dR} are isomorphic, where Δ^k denotes the Hodge Laplacian on k-forms. Therefore, it follows that

$$\chi(M) = \sum_k (-1)^k b_k = \sum_k (-1)^k \dim H^k_{dR} = \sum_k (-1)^k \dim \ker(\Delta^k). \tag{8.79}$$

8.3 Finite Fourier Analysis

In this section we introduce some basics of finite Fourier analysis, which are used in Chapter 5. Here, we follow Ref. [68].

8.3.1 Dual Groups

For a finite abelian group G a character of G is defined as a homomorphism

$$\chi : G \to \mathbb{C}^{\times} \tag{8.80}$$

with $\mathbb{C}^{\times} = \mathbb{C} \setminus \{0\}$ and $|\chi(g)| = 1$ for every $g \in G$. The multiplication between characters χ_1 and χ_2 is defined as

$$(\chi_1 \cdot \chi_2)(g) = \chi_1(g) \cdot \chi_2(g) \tag{8.81}$$

for every $g \in G$. Together with $\chi \equiv 1$, as the neutral element, the set of all characters build an abelian group, the dual group $\widehat{G}$.

For example, if we consider $G = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, then for $\bar{1} \in \mathbb{Z}_2$ one has $\bar{1}^2 = \bar{0}$, thus $\bar{1}$ is a generator and for every $\chi_2 \in \widehat{\mathbb{Z}}_2$ we have

$$\chi_2(\bar{1})^2 = \chi_2(\bar{1}^2) = \chi_2(\bar{0}) = 1, \tag{8.82}$$

where in the second step we have used, that χ_2 is homomorphic. Obviously, $\widehat{\mathbb{Z}}_2$ is generated by $\chi_{2,1}(\tilde{\sigma}) = e^{i\pi\tilde{\sigma}}$ with $\tilde{\sigma} \in \{\bar{0}, \bar{1}\}$, i.e. $\widehat{\mathbb{Z}}_2 = \{\chi_{2,0}, \chi_{2,1}\}$ with $\chi_{2,0} = \chi_{2,1}^2 = 1$

8.3.2 Fourier Transforms

The Fourier transform $\hat{f}$ of a function $f : G \to \mathbb{C}$ is defined by

$$\hat{f}(\chi) = \sum_{g \in G} \overline{\chi}(g) f(g). \tag{8.83}$$

Furthermore, one has

$$\hat{\hat{f}}(g) = |G| f(-g), \tag{8.84}$$

where $|\cdot|$ denotes the cardinality of a group or a set.

An important result is the Poisson summation formula, which states that

$$\frac{1}{|A|} \sum_{\tilde{\boldsymbol{\sigma}} \in A} f(\tilde{\boldsymbol{\sigma}}) = \frac{1}{|G|} \sum_{\chi \in \widehat{G}^A} \hat{f}(\chi), \tag{8.85}$$

for any function f, where $\widehat{G}^A$ is the annihilator for a subgroup $A \subseteq G$ defined by

$$\widehat{G}^A = \{\chi \in \widehat{G} \mid \chi|_A = 1\}, \tag{8.86}$$

where $\chi|_A$ denote the restriction of χ to A.

8.4 Knauf's Number-Theoretical Spin Chain

In this section, we summarize the basic framework of Knauf's Number-Theoretical Spin Chain [53]. Some results concerning the application of character theory were already mentioned in Sect. 5.2 of Chapter 5. Moreover, we briefly introduce some key results of the further investigations in Refs. [54–56], which based on the framework of Ref. [53].

In Ref. [53] Knauf considers $G_N = \mathbb{Z}_2^N$ as configuration space of a spin chain of length N. The canonical energy function is given by $H_N = E_0 \ln h_N$ with the coefficients h_N, which are inductively defined by [53]

$$h_{N+1}(\tilde{\boldsymbol{\sigma}}, 0) = h_N(\tilde{\boldsymbol{\sigma}}) \tag{8.87}$$

and

$$h_{N+1}(\tilde{\boldsymbol{\sigma}}, 1) = h_N(\tilde{\boldsymbol{\sigma}}) + h_N(1 - \tilde{\boldsymbol{\sigma}}) \tag{8.88}$$

for $\tilde{\boldsymbol{\sigma}} = |\tilde{\sigma}_1, \ldots, \tilde{\sigma}_N\rangle \in G_N$ with $h_0(0) = 1$ and $1 - \tilde{\boldsymbol{\sigma}} = |1 - \tilde{\sigma}_1, \ldots, 1 - \tilde{\sigma}_N\rangle$.

For example, for $k = 4$ one gets $h_4(0000) = 1$, $h_4(1000) = 2$,$h_4(0100) = 3$, $h_4(0010) = 4$, $h_4(0001) = h_3(000) + h_3(111) = 1 + 4 = 5$, $h_4(1100) = 3$, $h_4(1010) = 5$, $h_4(1001) = h_3(100) + h_3(011) = 2 + 5 = 7$, $h_4(0110) = 5$, $h_4(0101) = h_3(010) + h_3(101) = 3 + 5 = 8$, $h_4(0011) = h_3(001) + h_3(110) = 4 + 3 = 7$, $h_4(1110) = 4$, $h_4(0111) = h_3(011) + h_3(100) = 5+2 = 7$, $h_4(1101) = h_3(101)+h_3(010) = 5+3 = 8$, $h_4(1101) = h_3(110)+h_3(001) = 3+4 = 7$, $h_4(1111) = h_3(111) + h_3(000) = 4 + 1 = 5$.

The order of degeneracy for a state with energy $\ln n$ is given by [53]

$$\varphi_N(n) = |\{\tilde{\boldsymbol{\sigma}} \in G \mid h_N(\tilde{\boldsymbol{\sigma}}) = n\}|. \tag{8.89}$$

As a directly consequence one obtains

$$\varphi_{N+1}(n) = \varphi_N(n) + |\{\tilde{\boldsymbol{\sigma}} \in G_{N+1} \mid h_N(\tilde{\boldsymbol{\sigma}}) + h_N(1 - \tilde{\boldsymbol{\sigma}}) = n\}|. \tag{8.90}$$

The partition function

$$Z_N(\beta) = \sum_{\tilde{\boldsymbol{\sigma}} \in G} e^{-\beta H_N(\tilde{\boldsymbol{\sigma}})} \tag{8.91}$$

then becomes [53]

$$Z_N(s) = \sum_{n=1}^{\infty} \varphi_N(n) \cdot n^{-s} \tag{8.92}$$

with the inverse temperature $\beta = \frac{s}{E_0}$. Here, β is a complex-valued extension with $\Re(s) > 2 + \epsilon$ for an arbitrary $\epsilon > 0$, where Knauf is setting $E_0 = 1$. The complex extension of the inverse temperature and the fugacity are well-known approaches by describing phase transitions of canonical an grand-canonical ensembles in terms of the zeros of partition functions [33–35, 60, 78].

Moreover, in Ref. [53], it is showed that $Z_N(s)$ converges for $N \to \infty$ uniformly to

$$Z(s) = \sum_{n=1}^{\infty} \varphi(n) n^{-s} \tag{8.93}$$

in the half plane $\{s \in \mathbb{C} \mid \Re(s) > 2 + \epsilon\}$ with Eulers's totient function

$$\varphi(n) = |\{i \in \{1, \ldots, n\} \mid \gcd(i, n) = 1\}|, \tag{8.94}$$

which returns the number of positive integers not exceeding n, which are relatively prime to n.

Moreover, it is a fundamental statement of number-theory [6] that

$$Z(s) = \sum_{n=1}^{\infty} \varphi(n) n^{-s} = \frac{\zeta(s-1)}{\zeta(s)}, \tag{8.95}$$

thus Knauf's spin chain reaches its Hagedorn temperature at $s = 2$.

Furthermore, one can prove that [53]

$$\varphi_N \leq \varphi_{N+1} \leq \varphi \tag{8.96}$$

and

$$\varphi_N(n) = \varphi(n) \tag{8.97}$$

for $1 \leq n \leq N + 1$. One can show further [53] that for $n > N + 1$ one has $\varphi_N(n) < \varphi(n)$ and

$$\varphi_N(n) = 0 \tag{8.98}$$

for $n > F(N + 2)$ with the N-th Fibonacci number, defined recursively by

$$F(N+2) = F(N) + F(N+1), \quad F(1) = F(2) = 1. \tag{8.99}$$

In Ref. [55], the existence of the thermodynamic limit of the free energy is proven and, interestingly, it is shown in Ref. [54] that a first-order phase transition occurs at a critical β_{cr} with

$$\beta_{cr} \geq \frac{\ln 2}{\ln 3/2}. \tag{8.100}$$

Knauf makes further, very noteworthy, investigations on the grand-canonical description and on the spin-spin interaction [53, 54]. Moreover, he relates his spin chain and the Riemann hypothesis with Markov chains [56].

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