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THE CLASSIFICATION OF NATURALLY  
REDUCTIVE HOMOGENEOUS SPACES IN  
DIMENSION 7 AND 8

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# Introduction and Summary

Riemannian homogeneous spaces have been an interesting research subject over the last century, one of the first milestones being the classification of Riemannian symmetric spaces in [Car26]. In this work Cartan classified all Riemannian manifolds which have parallel curvature tensor, i.e.  $\nabla^g R^g = 0$ , where  $\nabla^g$  is the Levi-Civita connection of the Riemannian metric  $g$ . It turned out that these spaces are remarkable homogeneous manifolds. In fact Cartan reduced the classification to the classification of simple real Lie algebras. The way in which Lie theory is linked to the geometry of symmetric spaces in [Car26] is later generalized in [Nom54] to reductive homogeneous spaces. In this paper Nomizu investigates invariant connections on homogeneous spaces, and he established the correspondence between connections  $\nabla$  with parallel torsion and curvature and group-theoretical data. This is now known as the Nomizu construction. Symmetric spaces are exactly the spaces for which the torsion of the canonical connection (of the second kind) vanishes. Later it is proved in [AS58] that a complete simply connected Riemannian manifold is a homogeneous Riemannian manifold if and only if there exists a metric connection  $\nabla$  on it such that  $\nabla T = 0$  and  $\nabla R = 0$ , where  $T$  is the torsion of  $\nabla$  and  $R$  is the curvature of  $\nabla$ . Such a connection is called an Ambrose-Singer connection.

A particularly interesting class of Riemannian homogeneous spaces is the class of naturally reductive spaces. They can be seen as natural generalizations of symmetric spaces. They also form the simplest examples of Riemannian manifolds with a metric connection with skew torsion. Such a connection has the same geodesics as the Levi-Civita connection. Over the past years there has been an increasing interest in such connections because they arise in several fields in theoretical and mathematical physics like supersymmetric heterotic string theory or heterotic supergravity, see for example [FI02] and references therein. In this paper Friedrich and Ivanov also prove that many non-integrable geometries admit a unique connection with skew torsion which is adapted to the geometric structure, see also [AFH13] for a general theorem. Before, we mentioned that naturally reductive homogeneous spaces are amongst the simplest class of Riemannian homogeneous spaces. This is also nicely illustrated in

[Reg10] and [OR12]. Herein Olmos and Reggiani give an easy way to compute the full isometry algebra of a compact naturally reductive space. Another very interesting result by these authors states that an irreducible Riemannian homogeneous space admits at most one naturally reductive structure except when the Riemannian homogeneous space is, up to covering, a compact Lie group with bi-invariant metric, or its symmetric dual or a round sphere, see [OR12, OR13].

The rich class of naturally reductive structures together with their simple algebraic and geometric properties make them a very useful source of examples. For instance the structure group of naturally reductive homogeneous spaces is always contained in the holonomy group of the naturally reductive connection. Usually this is a relatively small subgroup of  $SO(n)$ . Therefore, naturally reductive spaces allow many interesting  $G$ -structures. If the principal holonomy bundle is contained in some  $G$ -reduction of the  $SO(n)$ -frame bundle, the  $G$ -structure is preserved by the naturally reductive connection. Of particular interest are  $G_2$ -structures in dimension 7 and  $Spin(7)$ -structures in dimension 8. The naturally reductive connection preserves such a  $G$ -structure precisely when there exists a parallel spinor for the naturally reductive connection. Most of these naturally reductive spaces with a compatible  $G_2$ -structure are already known, see [Fri07], [FKMS97].

Another area where naturally reductive spaces have been used is in the study of homogeneous Einstein metrics. D'Atri and Ziller investigated naturally reductive Einstein metrics on compact Lie groups and classified these in [DZ79]. This then led to the study of non-naturally reductive Einstein metrics, see for example [Mor96], [ASS15] and [AMS12]. Some famous naturally reductive spaces, which are also Einstein, are of course the compact isotropy irreducible spaces, see [Wol68, Wol84], [Krä75], [Man61a, Man61b, Man66].

A larger class of homogeneous spaces containing the naturally reductive spaces is the class of the geodesic orbit spaces, see [KV91]. Geodesic orbit spaces are classified up to dimension 5 and non-naturally reductive spaces up to dimension 6 in [KV91]. Geodesic orbit spaces still form an active and interesting research area.

The algebraic description of naturally reductive spaces allows one to classify them. The existing classifications of naturally reductive spaces are in dimension 3 in [TV83], in dimension 4 in [KV83], in dimension 5 in [KV85] and in dimension 6 in [AFF15].

In this thesis we deal with the classification problem of naturally reductive homogeneous spaces. A complete list of all naturally reductive homogeneous spaces is far out of reach. In this thesis a new construction of naturally reductive homogeneous spaces is presented. Moreover, we prove that this construction exhausts all naturally reductive homogeneous spaces in all dimensions. This allows us to describe the most general form of all of these spaces. It also gives us a new approach to classifying nat-

urally reductive homogeneous spaces. Previous classification results relied on normal forms of either  $R$  or  $T$ . This approach breaks down in higher dimensions. The approach we follow here doesn't use any normal forms and works in all dimensions. Using this we will classify all naturally reductive homogeneous spaces in dimensions 7 and 8. Our approach does rely on the classification of semisimple real Lie algebras and of finite-dimensional representations of semisimple Lie algebras.

In Chapter 1 we discuss the basics of naturally reductive spaces and we introduce some terminology. Since many of our considerations are for infinitesimal models it is convenient to have the concept of a homogeneous fiber bundle on an infinitesimal level, see Definition 1.2.2. We call this an *infinitesimal fiber bundle*. Let  $(V, g)$  be a finite dimensional vector space with a metric  $g$ . We define a 3-form  $T$  on  $(V, g)$  to be *reducible* if there exists a non-trivial orthogonal decomposition  $V = V_1 \oplus V_2$  with respect to  $g$  such that  $T \in \Lambda^3 V_1 \oplus \Lambda^3 V_2$ . Let  $(M, g, \nabla)$  be a simply connected Riemannian manifold with a metric connection  $\nabla$  which has parallel and skew torsion  $T$ . We prove that  $M$  is isometric to a product and  $\nabla$  is the product connection, i.e.

$$(M, g, \nabla) = (M_1, g_1, \nabla_1) \times (M_2, g_2, \nabla_2),$$

if and only if  $T_x$  is reducible for some  $x \in M$ . This result is very useful to decide whether a naturally reductive homogeneous space is reducible. For naturally reductive spaces this was already known, see [Tsu96].

In Chapter 2 we define a new construction of naturally reductive spaces, the majority of this chapter will appear in [Sto17]. This construction produces many new naturally reductive spaces. For most of these spaces the transvection algebra is not a reductive Lie algebra, see Remark 2.2.2. In particular the naturally reductive structure is not induced from a normal homogeneous structure. Two examples of families of such spaces are the naturally reductive structures on 2-step nilpotent Lie groups by Gordon in [Gor85] and the naturally reductive structures on the tangent space of a compact Lie group in [AF16]. The construction presented here produces both of the above mentioned families of naturally reductive spaces and many more. We start with the following pieces of data. We take a naturally reductive space  $M$  together with a Lie algebra  $\mathfrak{k}$  with an  $\text{ad}(\mathfrak{k})$ -invariant metric  $B$  on  $\mathfrak{k}$ . The algebra  $\mathfrak{k}$  is a certain subalgebra of derivations of the transvection algebra of  $M$ . From these data we can construct a new naturally reductive space which, if it is regular, is a homogeneous fiber bundle over  $M$ . The construction also works when  $M$  is not globally homogeneous. We will call the constructed space the  $(\mathfrak{k}, B)$ -extension of  $M$ . If the naturally reductive space we started with is the symmetric space  $\mathbb{R}^n$  and  $\mathfrak{k} \subset \mathfrak{so}(n)$  is a subalgebra together with any  $\text{ad}(\mathfrak{k})$ -invariant metric, then we obtain exactly the 2-step nilpotent Lie groups with a naturally reductive structure from

[Gor85], see Section 2.2.2. Let  $G$  be a simple compact Lie group with bi-invariant metric and its flat naturally reductive structure. Then the transvection algebra of this space is equal to  $\mathfrak{g} := \text{Lie}(G)$ . For the algebra  $\mathfrak{k}$  of derivations we pick  $\mathfrak{k} = \mathfrak{g}$  with an  $\text{ad}(\mathfrak{k})$ -invariant metric on  $\mathfrak{k}$ . In this case our construction produces the naturally reductive spaces from [AF16]. For this base space  $G$  we can pick any semisimple subalgebra  $\mathfrak{k} \subset \mathfrak{g}$  and our construction will produce an irreducible naturally reductive structure on  $G \times \mathbb{R}^{\dim(\mathfrak{k})}$ , see Section 2.2.1.

In Chapter 3 we distinguish two types of naturally reductive spaces:

Type I: The transvection algebra is semisimple.

Type II: The transvection algebra is not semisimple.

With the help of a classical theorem by Kostant [Kos56] and the classification of semisimple real Lie algebras it is relatively easy to describe all naturally reductive spaces of type I. This is done in Section 3.1. Moreover in small dimensions it is possible to classify all of these spaces. This is done in dimension 7 and 8, see Chapter 4. In the lower dimensions 6, 5, 4 and 3 it becomes considerably easier to classify all naturally reductive spaces by our approach. We can also apply our classification approach in higher dimensions. However, it becomes increasingly more difficult, mainly because there will be more cases to consider. In our classification of 7- and 8-dimensional spaces of type I we only list the compact spaces, because every non-compact naturally reductive space of type I corresponds to a compact space in an easy way. This correspondence is induced from the duality of symmetric spaces, see Remark 3.1.6. We do mention for every space if there exist corresponding non-compact spaces. For the spaces of type II we use that every non-semisimple Lie algebra contains a non-trivial abelian ideal. This allows us to prove that these spaces are infinitesimal fiber bundles over other naturally reductive spaces. We then derive a formula for the infinitesimal model in terms of the infinitesimal model of the base space and a certain Lie algebra representation, see Proposition 3.2.9. The infinitesimal model of the type II space is then a certain  $(\mathfrak{k}, B)$ -extension of the base space. One of our main results is Theorem 3.3.6. This theorem says that we obtain every infinitesimal model of a naturally reductive space of type II by applying the construction we defined in Chapter 2 to a (locally) naturally reductive base space of the form  $M \times \mathbb{R}^n$ , where  $M$  is of type I. This proves that every naturally reductive space is of the form described in Section 2.2.3. In other words Theorem 3.3.6 proves that every simply connected and complete naturally reductive space can be presented as:

$$(G \times Nil \times \mathbb{R}^n)/(H \times \mathbb{R}^k),$$

where  $G$  is semisimple, and  $H \subset G$  is some subgroup, and  $Nil$  is a simply connected 2-step nilpotent Lie group just as in Section 2.2.2, and  $\mathbb{R}^k \subset G \times Nil$  an abelian subgroup. For this realization of the homogeneous space we explicitly describe the naturally reductive structure, see Section 2.2.3.

At the end of Chapter 3 we answer the question when two naturally reductive spaces of type II are isomorphic. We also give an easy criterion when a naturally reductive space of type II can be written as a product. Our classification approach makes it easy to argue that none of the naturally reductive spaces we list are isomorphic. In the previous classification results this problem is not addressed. The contents of Chapter 2 and Chapter 3 are not just abstract results, but also applicable in practice. This is demonstrated in Chapter 4 where we classify all irreducible naturally reductive spaces in dimension 7 and 8. This is another one of our main results, see Theorem 4.1.9 and Theorem 4.2.6. It should be noted that for type II spaces we can also apply the correspondence of type I spaces described above to the type I factor  $M$  of the base space. The construction is in some sense compatible with this correspondence of type I spaces. Hence we only list the type II spaces for which  $M$  is compact and mention if there exist corresponding non-compact spaces to  $M$ .

It is my hope that these results will find further interesting applications in the study of naturally reductive spaces and other neighbouring fields.



# Einführung und Zusammenfassung

Riemannsche homogene Räume sind seit mehr als 100 Jahren ein interessantes Forschungsobjekt. Einer der ersten Meilensteine ist die Klassifizierung der Riemannschen symmetrischen Räume [Car26]. In jener Arbeit klassifizierte Cartan alle Riemannschen Mannigfaltigkeiten, deren Krümmungstensor parallel ist, d.h.  $\nabla^g R^g = 0$ , wobei  $\nabla^g$  der Levi-Civita Zusammenhang ist. Es stellte sich heraus, dass diese Räume bemerkenswerte homogene Mannigfaltigkeiten sind. Tatsächlich reduzierte Cartan die Klassifizierung auf die Klassifizierung von einfachen reellen Lie-Algebren.

Die Art und Weise, wie die Lie-Theorie mit der Geometrie der symmetrischen Räume in [Car26] verknüpft ist, wird später in [Nom54] auf reduktive homogene Räume verallgemeinert. In jener Arbeit untersuchte Nomizu invariante Zusammenhänge auf homogenen Räumen und etablierte die Entsprechung zwischen Zusammenhängen  $\nabla$  mit paralleler Torsion und Krümmung und gruppentheoretischen Daten. Dies ist heute bekannt als die Nomizu Konstruktion. Symmetrische Räume sind genau die Räume, für die die Torsion des kanonischen Zusammenhangs (der zweiten Art) verschwindet. Später wird bewiesen, dass eine vollständige, einfach zusammenhängende Riemannsche Mannigfaltigkeit genau dann ein Riemannscher homogener Raum ist, wenn sie einen metrischen Zusammenhang  $\nabla$  zulässt, sodass  $\nabla T = 0$  und  $\nabla R = 0$ , wobei  $T$  die Torsion von  $\nabla$  ist und  $R$  die Krümmung. Ein solcher Zusammenhang heißt Ambrose-Singer Zusammenhang.

Eine besonders interessante Klasse von Riemannschen homogenen Räumen ist die Klasse der natürlich reduktiven Räume. Sie können als natürliche Verallgemeinerungen von symmetrischen Räumen betrachtet werden. Sie bilden auch die einfachsten Beispiele für Riemannsche Mannigfaltigkeiten mit einem metrischen Zusammenhang mit schief-symmetrischer Torsion. Ein solcher Zusammenhang hat die gleichen Geodäten wie der Levi-Civita Zusammenhang. In den letzten Jahren hat es ein zunehmendes Interesse an derartigen Zusammenhängen gegeben, weil sie auf mehreren Gebieten der theoretischen und mathematischen Physik wie der supersymmetrischen heterotischen Stringtheorie oder der heterotischen Supergravitation auftauchen, siehe z.B. [FI02] und die dort genannten Referenzen. Dort beweisen

Friedrich und Ivanov, dass viele nicht integrierbare Geometrien einen eindeutigen an die geometrische Struktur angepassten Zusammenhang mit schief-symmetrischer Torsion zulassen. Siehe [AFH13] für einen allgemeineren Satz. Vorher haben wir erwähnt, dass natürlich reductive homogene Räume zu den einfachsten Riemannschen homogenen Räumen gehören. Dies wird auch in [Reg10] und [OR12] schön illustriert. Olmos und Reggiani geben hier eine einfache Methode zur Berechnung der vollständigen Isometriealgebra eines kompakten natürlich reductiven Raumes. Ein weiteres sehr interessantes Ergebnis dieser Autoren ist, dass ein irreduzibler Riemannscher homogener Raum höchstens eine natürlich reductive Struktur zulässt, außer wenn er, bis auf Überlagerung, eine kompakte Lie-Gruppe mit bi-invarianter Metrik oder deren symmetrischer Dualraum oder eine runde Sphäre ist, siehe [OR12, OR13].

Die große Klasse von natürlich reductiven Strukturen zusammen mit ihren einfachen algebraischen und geometrischen Eigenschaften machen sie zu einer sehr nützlichen Quelle von Beispielen. Die Strukturgruppe der natürlich reductiven homogenen Räume ist zum Beispiel immer in der Holonomiegruppe des natürlich reductiven Zusammenhangs enthalten. Normalerweise ist dies eine relativ kleine Untergruppe von  $SO(n)$ . Daher erlauben natürlich reductive Räume viele interessante  $G$ -Strukturen. Wenn das Holonomiebündel in einer  $G$ -Reduktion des  $SO(n)$ -Rahmenbündels enthalten ist, wird die  $G$ -Struktur durch den natürlich reductiven Zusammenhang erhalten. Von besonderem Interesse sind  $G_2$ -Strukturen in Dimension 7 und  $Spin(7)$ -Strukturen in Dimension 8. Der natürlich reductive Zusammenhang erhält eine solche  $G$ -Struktur genau dann, wenn es einen parallelen Spinor für den natürlich reductiven Zusammenhang gibt. Die meisten dieser natürlich reductiven Räume mit kompatiblen  $G_2$ -Strukturen sind bereits bekannt, siehe [Fri07] und [FKMS97].

Ein weiterer Bereich, in dem natürlich reductive Räume verwendet wurden, ist die Untersuchung homogener Einstein-Metriken. D'Atri und Ziller untersuchten natürlich reductive Einstein-Metriken auf kompakten Lie-Gruppen und klassifizierten diese in [DZ79]. Dies führte dann zur Untersuchung von nicht-natürlich reductiven Einstein-Metriken, siehe z.B. [Mor96], [ACS11] und [AMS12]. Einige berühmte natürlich reductive Räume, die auch Einstein sind, sind die kompakten Isotropieirreduzibelen Räume, siehe [Wol68, Wol84], [Krä75], [Man61a, Man61b, Man66].

Eine größere Klasse homogener Räume, die die natürlich reductiven Räume enthält, ist die Klasse der geodätischen Orbiträume, siehe [KV91]. Geodätische Orbiträume werden bis zur Dimension 5 und nicht-natürlich reductive Räume bis zur Dimension 6 klassifiziert in [KV91]. Geodätische Orbiträume bilden nach wie vor ein aktives und interessantes Forschungsgebiet.

Die algebraische Beschreibung natürlich reductiver Räume erlaubt es, sie zu klassifizieren. Die vorhandenen Klassifikationen von natürlich reductiven Räumen sind

in Dimension 3 in [TV83], in Dimension 4 in [KV83], in Dimension 5 in [KV85] und in Dimension 6 in [AFF15] zu finden.

In dieser Arbeit beschäftigen wir uns mit dem Klassifikationsproblem von natürlich reductiven homogenen Räumen. Von einer vollständigen Auflistung aller natürlich reductiven homogenen Räume sind wir weit entfernt. In dieser Arbeit wird eine neue Konstruktion von natürlich reductiven homogenen Räumen vorgestellt. Außerdem beweisen wir, dass diese Konstruktion alle natürlich reductiven homogenen Räume in allen Dimensionen abdeckt. Dies erlaubt uns, die allgemeinste Form all dieser Räume zu beschreiben. Es gibt uns auch einen neuen Ansatz zur Klassifikation natürlich reductiver homogener Räume. Bisherige Klassifikationsergebnisse beruhten auf Normalenformen von entweder  $R$  oder  $T$ . Dieser Ansatz versagt in höheren Dimensionen. Der Ansatz, dem wir hier folgen, verwendet keine Normalformen, sondern beruht auf der zuvor genannten Konstruktion und der Klassifikation von halbeinfachen reellen Lie-Algebren und ihren endlichdimensionalen Darstellungen. Dieser Ansatz funktioniert in allen Dimensionen und liefert uns eine vollständige Klassifikation von natürlich reductiven homogenen Räumen in den Dimensionen 7 und 8.

In Kapitel 1 diskutieren wir die Grundlagen der Theorie der natürlich reductiven Räume und führen einige Begriffe ein. Da viele unserer Betrachtungen für infinitesimale Modelle gelten, ist es nützlich, den Begriff eines homogenen Faserbündels auf einer infinitesimalen Ebene zu verwenden, siehe Definition 1.2.2. Wir nennen dies ein *infinitesimales Faserbündel*. Wir definieren eine 3-Form  $T$  auf  $(V, g)$  als *reduzibel*, wenn es eine nicht-triviale orthogonale Zerlegung  $V = V_1 \oplus V_2$  bezüglich  $g$  gibt, sodass  $T \in \Lambda^3 V_1 \oplus \Lambda^3 V_2$ . Sei  $(M, g, \nabla)$  eine einfach zusammenhängende Riemannsche Mannigfaltigkeit mit einem metrischen Zusammenhang  $\nabla$  mit paralleler und schiefsymmetrischer Torsion  $T$ . Wir beweisen, dass  $M$  genau dann isometrisch zu einem Produkt ist und  $\nabla$  der Produktzusammenhang ist, d.h.

$$(M, g, \nabla) = (M_1, g_1, \nabla_1) \times (M_2, g_2, \nabla_2),$$

wenn  $T_x$  für ein beliebiges  $x \in M$  reduzibel ist. Dieses Ergebnis ist sehr nützlich, um zu entscheiden ob ein natürlich reductiver homogener Raum reduzibel ist. Für natürlich reductive Räume war das Ergebnis schon bekannt, siehe [Tsu96].

In Kapitel 2 geben wir eine neue Konstruktion von natürlich reductiven Räumen. Die Großteil dieses Kapitels wird in [Sto17] erscheinen. Diese Konstruktion erzeugt viele neue, natürlich reductive Räume. Für die meisten dieser Räume ist die Transvektionsalgebra keine reductive Lie-Algebra, siehe 2.2.2. Insbesondere wird die natürlich reductive Struktur nicht von einer normalen homogenen Struktur induziert. Zwei Beispiele für Familien solcher Räume sind die natürlich reductiven Strukturen auf

bestimmten 2-stufigen nilpotenten Lie-Gruppen in [Gor85] und die natürlich reduktiven Strukturen auf dem Tangentialraum einer kompakten Lie-Gruppe in [AF16]. Die in dieser Dissertation vorgestellte Konstruktion erzeugt neben den oben erwähnten Familien noch viele weitere Beispiele natürlich reduktiver Räume. Wir beginnen mit den folgenden Daten. Wir nehmen einen natürlich reduktiven Raum  $M$  zusammen mit einer Lie-Algebra  $\mathfrak{k}$  mit einer  $\text{ad}(\mathfrak{k})$ -invarianten Metrik  $B$  auf  $\mathfrak{k}$ . Die Algebra  $\mathfrak{k}$  ist eine gewisse Unteralgebra von Derivationen der Transvektionsalgebra von  $M$ . Aus diesen Daten können wir einen neuen, natürlich reduktiven Raum konstruieren, der, wenn er regulär ist, ein homogenes Faserbündel über  $M$  ist. Die Konstruktion funktioniert auch, wenn  $M$  nicht global homogen ist. Wir nennen den konstruierten Raum die  $(\mathfrak{k}, B)$ -Erweiterung von  $M$ . Wenn der natürlich reduktive Raum, mit dem wir angefangen haben, der symmetrische Raum  $\mathbb{R}^n$  ist und  $\mathfrak{k} \subset \mathfrak{so}(n)$  eine Unter- algebra mit einer  $\text{ad}(\mathfrak{k})$ -invarianten Metrik ist, so erhalten wir genau die 2-stufigen nilpotenten Lie-Gruppen mit einer natürlich reduktiven Struktur aus [Gor85], siehe Abschnitt 2.2.2. Sei  $G$  eine einfache kompakte Lie-Gruppe mit bi-invarianter Metrik und ihrer flachen natürlich reduktiven Struktur. Dann ist die Transvektionsalgebra dieses Raumes gleich  $\mathfrak{g} := \text{Lie}(G)$ . Für die Algebra  $\mathfrak{k}$  wählen wir  $\mathfrak{k} = \mathfrak{g}$  mit einer  $\text{ad}(\mathfrak{k})$ - invarianten Metrik auf  $\mathfrak{k}$ . In diesem Fall erzeugt unsere Konstruktion die natürlich reduktiven Räume aus [AF16]. Für diesen Basisraum  $G$  können wir jede halbeinfache Unteralgebra  $\mathfrak{k} \subset \mathfrak{g}$  auswählen und unsere Konstruktion ergibt eine irreduzible natürlich reduktive Struktur auf  $G \times \mathbb{R}^{\dim(\mathfrak{k})}$ , siehe Abschnitt 2.2.1.

In Kapitel 3 unterscheiden wir zwei Arten natürlich reduktiver Räume:

Typ I: Die Transvektionsalgebra ist halbeinfach.

Typ II: Die Transvektionsalgebra ist nicht halbeinfach.

Mit Hilfe eines klassischen Theorems von Kostant [Kos56] und der Klassifikation von halbeinfachen reellen Lie-Algebren ist es relativ einfach, alle natürlich reduktiven Räume des Typs I zu beschreiben. Dies geschieht in Abschnitt 3.1. Darüber hinaus ist es in kleinen Dimensionen möglich, alle diese Räume zu klassifizieren. In Kapitel 4 klassifizieren wir alle natürlich reduktiven Räume von Typ I in Dimension 7 und 8. In den niederen Dimensionen 6, 5, 4 und 3 wird es wesentlich einfacher, alle natürlich reduktiven Räume nach unserem Ansatz zu klassifizieren. Wir können unseren Klassifizierungsansatz auch in höheren Dimensionen anwenden. Allerdings wird es immer schwieriger, vor allem weil es mehr Fälle zu berücksichtigen gibt. In unserer Klassifikation von 7- und 8-dimensionalen Räumen des Typs I haben wir nur die kompakten Räume aufgelistet, weil jeder nicht-kompakte natürlich reduktive Raum des Typs I auf einfache Weise zu einem kompakten Raum korrespondiert. Diese Korrespondenz

wird von der Dualität der symmetrischen Räume induziert, siehe Bemerkung 3.1.6. Wir erwähnen für jeden Raum, wenn es entsprechende nicht-kompakte Räume gibt. Für die Räume des Typs II verwenden wir, dass jede nicht halbeinfache Lie-Algebra ein nicht-triviales abelsches Ideal enthält. Dies erlaubt uns zu beweisen, dass diese Räume infinitesimale Faserbündel über anderen natürlich reduktiven Räumen sind. Wir geben eine Formel für das infinitesimale Modell in Bezug auf das infinitesimale Modell des Basisraums und eine gewisse Lie-Algebra-Darstellung, siehe Satz 3.2.9. Das infinitesimale Modell des Typ-II-Raums ist dann eine gewisse  $(\mathfrak{k}, B)$ -Erweiterung des Basisraums. Eines unserer Hauptergebnisse ist Theorem 3.3.6. Dieser Satz sagt, dass wir jedes infinitesimale Modell eines natürlich reduktiven Raumes des Typs II durch Anwendung der Konstruktion, die wir in Kapitel 2 auf einem (lokal) natürlich reduktiven Basisraum der Form  $M \times \mathbb{R}^n$  besprochen haben, erhalten, wobei  $M$  vom Typ I ist. Dies beweist, dass jeder natürlich reduktive Raum die in Abschnitt 2.2.3 beschriebene Form hat. Mit anderen Worten: Abschnitt 2.2.3 zusammen mit Theorem 3.3.6 beweist, dass alle einfach zusammenhängenden und vollständigen natürlich reduktiven Räume dargestellt werden können als:

$$(G \times Nil \times \mathbb{R}^n)/(H \times \mathbb{R}^k),$$

wobei  $G$  halbeinfach,  $H \subset G$  eine Untergruppe,  $Nil$  eine einfach zusammenhängende 2-Schritt-Nilpotent-Lie-Gruppe und  $\mathbb{R}^k \subset G \times Nil$  eine abelsche Untergruppe ist. Für diese Realisierung des homogenen Raumes beschreiben wir explizit die natürlich reduktive Struktur, siehe Abschnitt 2.2.3. Am Ende von Kapitel 3 beantworten wir die Frage, wann zwei natürlich reduktive Räume des Typs II isomorph sind. Wir geben auch ein einfaches Kriterium an, das bestimmt, wann ein natürlich reduktiver Raum des Typs II als Produkt geschrieben werden kann. Die Ergebnisse von Kapitel 2 und Kapitel 3 sind auch in der Praxis anwendbar. Dies wird in Kapitel 4 illustriert, wo wir alle natürlich reduktiven Räume der dimension 7 und 8 klassifizieren. Die Resultate sind zusammengefasst in Theorem 4.1.9 und Theorem 4.2.6.

Es ist anzumerken, dass für Typ-II-Räume auch die Korrespondenz der oben beschriebenen Typ-I-Räume auf den Typ-I-Faktor  $M$  des Basisraums angewendet werden kann. Die Konstruktion ist in gewissem Sinne kompatibel mit dieser Korrespondenz von Typ-I-Räume. Daher haben wir nur die Typ-II-Räume aufgelistet, für die  $M$  kompakt ist und erwähnen, wenn es zu  $M$  korrespondierende, nicht-kompakte Räume gibt.

Es ist meine Hoffnung, dass diese Ergebnisse weitere interessante Anwendungen bei der Untersuchung natürlich reduktiver Räume und anderen Nachbarfeldern finden.



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# Chapter 1

## Preliminaries

In this chapter we first discuss some basics of naturally reductive homogeneous spaces. At the end of Section 1.1 we prove a formula for the curvature tensors which will be useful in the sequel, see Lemma 1.1.19. As is common for naturally reductive homogeneous spaces one mostly works with the infinitesimal model. For this reason it is useful to have a notion of a fiber bundle for infinitesimal models. This is discussed in Section 1.2. In Section 1.3 we give a criterion when a Riemannian manifold with a metric connection, which has totally skew symmetric parallel torsion, is locally a product. This result tells us in particular when a simply connected naturally reductive space is a product of two naturally reductive spaces. Having this result will be of great use in our classification in Chapter 4.

### 1.1 Basics of naturally reductive spaces

Let  $(M = G/H, g)$  be a Riemannian homogeneous manifold. Let  $\mathfrak{g}$  and  $\mathfrak{h}$  be the Lie algebras of  $G$  and  $H$ , respectively. Let

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

be some reductive decomposition, i.e.  $\text{Ad}(H)\mathfrak{m} \subset \mathfrak{m}$ . The reductive decomposition induces a left invariant connection on the principal  $H$ -bundle  $G \rightarrow G/H$  called the canonical connection of the complement  $\mathfrak{m}$ . Its horizontal distribution is defined by

$$T_g G \supset \mathcal{H}_g = dL_g(\mathfrak{m}),$$

where  $L_g : G \rightarrow G$  is the left multiplication by  $g \in G$ . The tangent bundle of  $M$  is the associated bundle  $TM \cong G \times_{\text{Ad}(H)} \mathfrak{m}$ . For  $x \in \mathfrak{g}$  let  $\bar{x}$  denote the induced Killing

vector field:

$$\bar{x}(p) := \left. \frac{d}{dt} \right|_{t=0} e^{tx} \cdot p \in T_p M.$$

We will denote the chosen origin of our homogeneous space by  $o$ . Note that  $\mathfrak{m}$  is canonically identified with the tangent space at the origin by

$$x \mapsto \bar{x}(o) \in T_o M. \quad (1.1.1)$$

The covariant derivative on  $TM$  associated to the canonical connection, denoted  $\nabla$ , has parallel torsion and curvature:  $\nabla T = \nabla R = 0$ . The following theorem of Ambrose and Singer [AS58] gives a characterization of metric connections with parallel torsion and curvature on a complete simply connected Riemannian manifold, see also [Kos60].

**Theorem 1.1.2** (Ambrose-Singer). *A complete simply connected Riemannian manifold  $(M, g)$  is a homogeneous Riemannian manifold if and only if there exists a metric connection  $\nabla$  with torsion  $T$  and curvature  $R$  such that*

$$\nabla T = 0 \quad \text{and} \quad \nabla R = 0. \quad (1.1.3)$$

*Remark 1.1.4.* A Riemannian manifold is locally homogeneous if its pseudogroup of local isometries acts transitively on it. It should be noted that there exist locally homogeneous Riemannian manifolds which are not locally isometric to a globally homogeneous space, see [Kow90]. Of course such spaces have to be non-complete.

A metric connection satisfying (1.1.3) is called an *Ambrose-Singer connection*. The torsion  $T$  and curvature  $R$  of an Ambrose-Singer connection evaluated at a point  $p \in M$  are linear maps

$$T_p : \Lambda^2 T_p M \rightarrow T_p M, \quad R_p : \Lambda^2 T_p M \rightarrow \mathfrak{so}(T_p M), \quad (1.1.5)$$

which satisfy

$$R_p(x, y) \cdot T_p = R_p(x, y) \cdot R_p = 0 \quad (1.1.6)$$

$$\mathfrak{S}^{x,y,z} R_p(x, y)z - T_p(T_p(x, y), z) = 0 \quad (1.1.7)$$

$$\mathfrak{S}^{x,y,z} R_p(T_p(x, y), z) = 0, \quad (1.1.8)$$

where  $\mathfrak{S}^{x,y,z}$  denotes the cyclic sum over  $x, y$  and  $z$  and  $\cdot$  denotes the natural action of  $\mathfrak{so}(T_p M)$  on tensors. The first equation encodes that  $T$  and  $R$  are parallel objects for  $\nabla$  and under this condition the first and second Bianchi identity become equations (1.1.7) and (1.1.8), respectively. A pair of tensors  $(T, R)$ , as in (1.1.5), on a vector

space  $\mathfrak{m}$  with a metric  $g$  satisfying (1.1.6), (1.1.7) and (1.1.8) is called an *infinitesimal model* on  $(\mathfrak{m}, g)$ . From the infinitesimal model  $(T, R)$  of a homogeneous space one can construct a homogeneous space with infinitesimal model  $(T, R)$ . This construction is known as the *Nomizu construction*, see [Nom54]. This construction is now briefly discussed.

Let

$$\mathfrak{h} := \{h \in \mathfrak{so}(\mathfrak{m}) : h \cdot T = 0, h \cdot R = 0\}.$$

The Nomizu construction associates to every infinitesimal model a Lie algebra

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \tag{1.1.9}$$

by defining the following Lie bracket for all  $h, k \in \mathfrak{h}$  and  $x, y \in \mathfrak{m}$ :

$$[h + x, k + y] := [h, k]_{\mathfrak{so}(\mathfrak{m})} - R(x, y) + h(y) - k(x) - T(x, y), \tag{1.1.10}$$

where  $[-, -]_{\mathfrak{so}(\mathfrak{m})}$  denotes the Lie bracket in  $\mathfrak{so}(\mathfrak{m})$ . The bracket from (1.1.10) satisfies the Jacobi identity if and only if  $R$  and  $T$  satisfy the equations (1.1.6), (1.1.7) and (1.1.8). We will call  $\mathfrak{g}$  the *symmetry algebra* of the infinitesimal model  $(T, R)$ . Let  $G$  be the simply connected Lie group with Lie algebra  $\mathfrak{g}$  and let  $H$  be the connected subgroup with Lie algebra  $\mathfrak{h}$ . The infinitesimal model is called *regular* if  $H$  is a closed subgroup of  $G$ . If this is the case, then clearly the canonical connection on  $G/H$  has the infinitesimal model  $(T, R)$  we started with. In [Tri92, Thm. 5.2] it is proved that every infinitesimal model coming from a globally homogeneous Riemannian manifold is regular.

Two infinitesimal models  $(T, R)$  and  $(T', R')$  on  $(\mathfrak{m}, g)$  and  $(\mathfrak{m}', g')$ , respectively, are called *isomorphic* if there exists a linear isometry  $M : \mathfrak{m} \rightarrow \mathfrak{m}'$  such that

$$M \cdot T = T' \quad \text{and} \quad M \cdot R = R',$$

where  $\cdot$  denotes the induced linear map on tensors. Note for all  $x, y \in \mathfrak{m}$  that  $M \cdot R(x, y) = (M \cdot R)(M^{-1}x, M^{-1}y) = R'(M^{-1}x, M^{-1}y)$ . This implies that  $M$  induces a linear isomorphism from  $\text{im}(R)$  to  $\text{im}(R')$ . Let  $\hat{M}$  be the linear isomorphism given by

$$\hat{M} : \text{im}(R) \oplus \mathfrak{m} \rightarrow \text{im}(R') \oplus \mathfrak{m}', \quad h + x \mapsto M \cdot h + M(x).$$

**Lemma 1.1.11.** *Let  $(T, R)$  and  $(T', R')$  be two infinitesimal models on  $(\mathfrak{m}, g)$  and  $(\mathfrak{m}', g')$ , respectively. Let  $M : \mathfrak{m} \rightarrow \mathfrak{m}'$  be a linear isometry. The following are equivalent*

$$i) \quad M \cdot T = T' \quad \text{and} \quad M \cdot R = R',$$

ii)  $\hat{M} : \text{im}(R) \oplus \mathfrak{m} \rightarrow \text{im}(R') \oplus \mathfrak{m}'$  is a Lie algebra isomorphism.

The proof is straight forward and can be found in [TV83]. We will call a Riemannian manifold  $(M, g)$  *naturally reductive* if there exists a transitive group action of a group of isometries  $G$  with isotropy group  $H$  and a reductive decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  such that the canonical connection of  $\mathfrak{m}$  has skew torsion. The torsion of the canonical connection is given by

$$T(x, y)_o = -[x, y]_{\mathfrak{m}},$$

where  $[x, y]_{\mathfrak{m}}$  is the  $\mathfrak{m}$ -component of  $[x, y]$ . Therefore, the naturally reductive condition on the Lie algebra  $\mathfrak{g}$  is given by

$$g([x, y]_{\mathfrak{m}}, z) = -g(y, [x, z]_{\mathfrak{m}}), \quad \forall x, y, z \in \mathfrak{m},$$

where the metric on  $\mathfrak{m}$ , which we also denote by  $g$ , comes from the linear isomorphism (1.1.1). From now on every homogeneous space will be naturally reductive. We use the metric to make the identification  $\Lambda^2 \mathfrak{m} \cong \mathfrak{so}(\mathfrak{m})$ . For naturally reductive spaces the curvature tensor  $R : \Lambda^2 \mathfrak{m} \rightarrow \Lambda^2 \mathfrak{m}$  is a symmetric map with respect to the Killing form of  $\mathfrak{so}(\mathfrak{m})$  and equation (1.1.8) holds automatically, see [AFF15]. Throughout this paper we will identify  $\mathfrak{m}$  with its dual  $\mathfrak{m}^*$  using the metric  $g$ . In this way we see  $T$  as an element in  $\Lambda^3 \mathfrak{m}$  and  $R$  as an element in  $\Lambda^2 \mathfrak{m} \odot \Lambda^2 \mathfrak{m}$ , where  $\odot$  denotes the symmetric tensor product.

**Definition 1.1.12.** Let  $(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, g)$  be a Lie algebra together with a subalgebra  $\mathfrak{h} \subset \mathfrak{g}$ , a complement  $\mathfrak{m}$  of  $\mathfrak{h}$  and a metric  $g$  on  $\mathfrak{m}$ . Suppose  $\text{ad}(\mathfrak{h})\mathfrak{m} \subset \mathfrak{m}$  and for all  $x, y, z \in \mathfrak{m}$  that

$$g([x, y]_{\mathfrak{m}}, z) = -g(y, [x, z]_{\mathfrak{m}}).$$

Then we call  $(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, g)$  a *naturally reductive decomposition* with  $\mathfrak{h}$  the *isotropy algebra*. We will mostly refer to just  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  as a naturally reductive decomposition and let the metric be implicit. The infinitesimal model of the naturally reductive decomposition is defined by

$$T(x, y) := -[x, y]_{\mathfrak{m}}, \quad \forall x, y \in \mathfrak{m}, \quad (1.1.13)$$

$$R(x, y) := -\text{ad}([x, y]_{\mathfrak{h}}) \in \mathfrak{so}(\mathfrak{m}), \quad \forall x, y \in \mathfrak{m}, \quad (1.1.14)$$

where  $[x, y]_{\mathfrak{h}}$  is the  $\mathfrak{h}$ -component of  $[x, y]$ . We call the decomposition an *effective* naturally reductive decomposition if the restricted adjoint map  $\text{ad} : \mathfrak{h} \rightarrow \mathfrak{so}(\mathfrak{m})$  is injective. We will say that  $\mathfrak{g}$  is the *transvection algebra* of the naturally reductive decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  if the decomposition is effective and  $\text{im}(R) = \text{ad}(\mathfrak{h}) \subset \mathfrak{so}(\mathfrak{m})$ . Note that (1.1.6) implies that  $\text{im}(R) \subset \mathfrak{so}(\mathfrak{m})$  is a subalgebra.

As mentioned before the fact that the pair  $(T, R)$  defines an infinitesimal model on  $(\mathfrak{m}, g)$  can easily be derived from the skew-symmetry and Jacobi identity of the Lie bracket and the fact that the decomposition is reductive. Let  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  be a naturally reductive decomposition. Let  $G$  be a Lie group with  $\text{Lie}(G) = \mathfrak{g}$  and let  $H \subset G$  be the connected subgroup with  $\text{Lie}(H) = \mathfrak{h}$ . Then  $\text{Ad}(H)\mathfrak{m} \subset \mathfrak{m}$ . Hence, if  $H \subset G$  is closed, then a naturally reductive decomposition is in particular a reductive decomposition for the homogeneous space  $G/H$ .

In many cases the following result helps to determine if an infinitesimal model is regular. The proof of the lemma also gives us a formula of the Nomizu map of all the naturally connections in Section 2.2.

**Lemma 1.1.15.** *Let  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  be a naturally reductive decomposition and let  $(T, R)$  be the infinitesimal model defined by (1.1.13) and (1.1.14). Suppose that  $\mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{m}'$  is a subalgebra of  $\mathfrak{g}$ , with  $\mathfrak{h}' := \mathfrak{g}' \cap \mathfrak{h}$  and  $\mathfrak{m}'$  a complement of  $\mathfrak{h}'$  with  $\text{ad}(\mathfrak{h}')\mathfrak{m}' \subset \mathfrak{m}'$ . Furthermore, we suppose that  $\pi_{\mathfrak{m}}(\mathfrak{m}') = \mathfrak{m}$ , where  $\pi_{\mathfrak{m}}$  is the projection in  $\mathfrak{g}$  onto  $\mathfrak{m}$  along  $\mathfrak{h}$ . Let  $G'$  be the simply connected Lie group with  $\text{Lie}(G') = \mathfrak{g}'$  and let  $H'$  be the connected subgroup with  $\text{Lie}(H') = \mathfrak{h}'$ . If  $H' \subset G'$  is closed, then the infinitesimal model  $(T, R)$  is regular.*

*Proof.* Let  $\phi : \mathfrak{m} \rightarrow \mathfrak{h}$  be such that

$$A : \mathfrak{m} \rightarrow \mathfrak{m}'; \quad x \mapsto x + \phi(x) \in \mathfrak{m}'$$

is a linear isomorphism. We define the metric on  $\mathfrak{m}'$  such that  $A$  becomes an isometry. If  $h' \in \mathfrak{h}'$ , then

$$[h', x + \phi(x)] = [h', x] + [h', \phi(x)] \in \mathfrak{m}'.$$

This implies that  $\phi([h', x]) = [h', \phi(x)]$ . In other words  $\phi$  is  $\mathfrak{h}'$ -equivariant. Since  $H'$  is connected it follows that  $\phi(h' \cdot x) = \text{Ad}(h')\phi(x)$  for all  $h' \in H'$ . Note that for all  $h' \in H'$  we have

$$A \cdot \text{Ad}(h')_{\mathfrak{m}} = \text{Ad}(h')_{\mathfrak{m}'} \cdot A,$$

where  $\text{Ad}(h')_{\mathfrak{m}} \in SO(\mathfrak{m})$  and  $\text{Ad}(h')_{\mathfrak{m}'} \in SO(\mathfrak{m}')$  denote the restricted adjoint representations. We define a  $G'$ -invariant connection on  $G'/H'$  by the  $\text{Ad}(H')$ -equivariant Nomizu map  $\Lambda_{\mathfrak{m}'} : \mathfrak{m}' \rightarrow \mathfrak{so}(\mathfrak{m}')$  defined by

$$\Lambda_{\mathfrak{m}'}(x + \phi(x)) = A \cdot \text{ad}(\phi(x)) \cdot A^{-1},$$

where  $\text{ad}(\phi(x)) \in \mathfrak{so}(\mathfrak{m})$  denotes the restricted adjoint action. We extend it by  $\Lambda(h' + x') = \text{ad}(h') + \Lambda_{\mathfrak{m}'}(x')$  for all  $h' \in \mathfrak{h}'$  and  $x' \in \mathfrak{m}'$ . The induced  $G'$ -invariant

connection is denoted by  $\nabla^\Lambda$ . Let  $x' := A(x)$  for all  $x \in \mathfrak{m}$ . Let  $o$  be the identity coset of  $G'/H'$ . By [KN63] the curvature of  $\nabla^\Lambda$  is

$$\begin{aligned} R^\Lambda(x', y')_o &= [\Lambda(x'), \Lambda(y')] - \Lambda([x', y']) \\ &= A[\text{ad}(\phi(x)), \text{ad}(\phi(y))]A^{-1} - A \cdot \text{ad}([x', y']_{\mathfrak{h}})A^{-1} \\ &= A([\text{ad}(\phi(x)), \text{ad}(\phi(y))] - \text{ad}([x, y]_{\mathfrak{h}}) - \text{ad}([\phi(x), \phi(y)]))A^{-1} \\ &= -A \cdot \text{ad}([x, y]_{\mathfrak{h}}) \cdot A^{-1} = A \cdot R(x, y) \cdot A^{-1}. \end{aligned}$$

The torsion of  $\nabla^\Lambda$  is given by

$$\begin{aligned} T^\Lambda(x', y')_o &= \Lambda(x')y' - \Lambda(y')x' - [x', y']_{\mathfrak{m}'} \\ &= A \cdot \text{ad}(\phi(x)) \cdot A^{-1}y' - A \cdot \text{ad}(\phi(y)) \cdot A^{-1}x' - A[x', y']_{\mathfrak{m}} \\ &= A[\phi(x), y] - A[\phi(y), x] - A[x', y']_{\mathfrak{m}} \\ &= A[\phi(x), y] - A[\phi(y), x] - A([x, y]_{\mathfrak{m}} + [\phi(x), y] - [\phi(y), x]) \\ &= -A[x, y]_{\mathfrak{m}} = A \cdot T(x, y). \end{aligned}$$

If  $F$  is an  $\text{ad}(\mathfrak{h})$ -invariant tensor on  $\mathfrak{m}$ , then  $F' := A \cdot F$  is an  $\text{ad}(\mathfrak{h}')$ -invariant tensor on  $\mathfrak{m}'$  and thus defines a  $G'$ -invariant tensor field on  $G'/H'$ . This tensor field is parallel with respect to  $\nabla^\Lambda$ , because for all  $x' \in \mathfrak{m}'$  we have

$$(\nabla_{x'}^\Lambda F')_o = \mathcal{L}_{\overline{x'}} F' + \Lambda(x')F' = \Lambda(x')F' = A \cdot \text{ad}(\phi(x)) \cdot A^{-1}F' = A \cdot \text{ad}(\phi(x)) \cdot F = 0,$$

where  $\mathcal{L}_{\overline{x'}}$  is the Lie derivative with respect to  $\overline{x'}$ . We conclude that

$$\nabla^\Lambda R^\Lambda = 0 \quad \text{and} \quad \nabla^\Lambda T^\Lambda = 0.$$

In other words  $\nabla^\Lambda$  is a naturally reductive connection on  $G'/H'$ . By [Tri92, Thm. 5.2] the infinitesimal model  $(T^\Lambda, R^\Lambda)$  is regular. Since  $A$  is an isomorphism between  $(T, R)$  and  $(T^\Lambda, R^\Lambda)$  we conclude that  $(T, R)$  is a regular infinitesimal model.  $\square$

The following result is due to [Kos56], see also [DZ79].

**Theorem 1.1.16** (Kostant). *Let  $(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, g)$  be an effective naturally reductive decomposition. Then  $\mathfrak{k} := [\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}} \oplus \mathfrak{m}$  is an ideal in  $\mathfrak{g}$  and there exists a unique  $\text{ad}(\mathfrak{k})$ -invariant non-degenerate symmetric bilinear form  $\overline{g}$  on  $\mathfrak{k}$  such that  $\overline{g}|_{\mathfrak{m} \times \mathfrak{m}} = g$  and  $[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}} \perp \mathfrak{m}$ . Conversely, any  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear form on  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  with  $\mathfrak{m} = \mathfrak{h}^\perp$  and  $\overline{g}|_{\mathfrak{m} \times \mathfrak{m}}$  positive definite gives a naturally reductive decomposition.*

We can write  $R$  as  $R = S \circ P$ , where  $P : \Lambda^2 \mathfrak{m} \rightarrow \text{im}(R)$  is the orthogonal projection with respect to the Killing form of  $\mathfrak{so}(\mathfrak{m})$  and  $S : \text{im}(R) \rightarrow \text{im}(R)$  is an  $\text{im}(R)$ -equivariant symmetric isomorphism. Let

$$B_{\Lambda^2}(h, k) := -\frac{1}{2} \text{tr}(hk), \quad \forall h, k \in \mathfrak{so}(\mathfrak{m}). \quad (1.1.17)$$

This is a multiple of the Killing form and satisfies  $B_{\Lambda^2}(h, x \wedge y) = g(h(x), y)$ , where  $g$  is the metric on  $\mathfrak{m}$  for which the Lie algebra  $\mathfrak{so}(\mathfrak{m})$  is defined. Note that  $S$  is symmetric with respect to  $B_{\Lambda^2}$ .

**Lemma 1.1.18.** *Let  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  be a naturally reductive decomposition with  $\mathfrak{g}$  as transvection algebra. We define for  $h, h' \in \mathfrak{h}$  and  $x, y \in \mathfrak{m}$  a symmetric bilinear form by:*

$$\begin{aligned} \tilde{g}(h, h') &:= -B_{\Lambda^2}(S^{-1} \text{ad}(h), \text{ad}(h')), \\ \tilde{g}(h, x) &:= 0, \\ \tilde{g}(x, y) &:= g(x, y). \end{aligned}$$

*This is a symmetric non-degenerate  $\text{ad}(\mathfrak{g})$ -invariant bilinear form.*

*Proof.* By assumption  $\text{ad} : \mathfrak{h} \rightarrow \text{ad}(\mathfrak{h}) = \text{im}(R)$  is an isomorphism. This allow us to slightly abuse the notation and write  $h$  for  $\text{ad}(h) \in \mathfrak{so}(\mathfrak{m})$ . Clearly  $\tilde{g}$  is symmetric and non-degenerate. If  $h, h', h'' \in \mathfrak{h}$ , then

$$\begin{aligned} \tilde{g}([h, h'], h'') &= -B_{\Lambda^2}(S^{-1}[h, h'], h'') = -B_{\Lambda^2}([h, S^{-1}h'], h'') \\ &= B_{\Lambda^2}(S^{-1}h', [h, h'']) = -\tilde{g}(h', [h, h'']). \end{aligned}$$

For  $x, y, z \in \mathfrak{m}$  we get

$$\tilde{g}([x, y], z) = g([x, y], z) = g(x, [y, z]) = \tilde{g}(x, [y, z])$$

and

$$\begin{aligned} \tilde{g}(h, [x, y]) &= -\tilde{g}(h, R(x \wedge y)) = B_{\Lambda^2}(S^{-1}h, R(x \wedge y)) \\ &= B_{\Lambda^2}(h, P(x \wedge y)) = B_{\Lambda^2}(h, x \wedge y) = g([h, x], y) = \tilde{g}([h, x], y). \end{aligned}$$

The last case to consider is

$$\tilde{g}([h, h'], x) = 0 = \tilde{g}(h, [h', x]).$$

This shows that  $\tilde{g}$  is  $\text{ad}(\mathfrak{g})$ -invariant. □

Note that the above lemma is just a description for the unique  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear form from Kostant's theorem, Theorem 1.1.16. This allows us to write the curvature in a simple way as the following lemma demonstrates.

**Lemma 1.1.19.** *Let  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  be a naturally reductive decomposition with  $\mathfrak{g}$  its transvection algebra. Let  $\bar{g}$  be the unique  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear form on  $\mathfrak{g}$ . Let  $h_1, \dots, h_l$  be a pseudo-orthonormal basis of  $\mathfrak{h}$ , i.e.  $\bar{g}(h_i, h_j) = \epsilon_i \delta_{ij}$ , where  $\epsilon_i$  is either 1 or  $-1$ . The curvature defined by (1.1.14) is given by*

$$R = - \sum_{i=1}^l \epsilon_i \text{ad}(h_i) \odot \text{ad}(h_i).$$

*Proof.* Note that the formula for  $R$  corresponds to the inverse metric tensor of  $\bar{g}|_{\mathfrak{h} \times \mathfrak{h}}$ . In particular  $\sum_{i=1}^l \epsilon_i \text{ad}(h_i) \odot \text{ad}(h_i)$  is independent of the choice of pseudo-orthonormal basis  $h_1, \dots, h_l$  and it suffices to prove the formula for one pseudo-orthonormal basis. Just as before we identify  $\mathfrak{h}$  with  $\text{ad}(\mathfrak{h})$ . Let  $\tilde{h}_1, \dots, \tilde{h}_l$  be an orthonormal basis with respect to  $B_{\Lambda^2}$  and such that  $S$  is diagonal with respect to this basis of  $\mathfrak{h}$ . Then  $S(\tilde{h}_i) = -\epsilon_i \lambda_i^2 \tilde{h}_i$  for  $i = 1, \dots, l$  and  $\epsilon_i$  is either 1 or  $-1$ . For  $h_i := \lambda_i \tilde{h}_i$  we get

$$\bar{g}(h_i, h_j) = -B_{\Lambda^2}(S^{-1}h_i, h_j) = \epsilon_i \lambda_i^{-2} B_{\Lambda^2}(\lambda_i \tilde{h}_i, \lambda_j \tilde{h}_j) = \epsilon_i \delta_{ij}.$$

Computing  $R(x, y)$  yields

$$\begin{aligned} R(x, y) &= R(x \wedge y) = R(P(x \wedge y)) = R\left(\sum_{i=1}^l B_{\Lambda^2}(x \wedge y, \text{ad}(\tilde{h}_i)) \text{ad}(\tilde{h}_i)\right) \\ &= - \sum_{i=1}^l B_{\Lambda^2}(x \wedge y, \text{ad}(\tilde{h}_i)) \epsilon_i \lambda_i^2 \text{ad}(\tilde{h}_i) = - \sum_{i=1}^l \epsilon_i B_{\Lambda^2}(x \wedge y, \text{ad}(h_i)) \text{ad}(h_i), \end{aligned}$$

for all  $x, y \in \mathfrak{m}$ . Thus, we conclude that  $R = - \sum_{i=1}^l \epsilon_i \text{ad}(h_i) \odot \text{ad}(h_i)$ .  $\square$

## 1.2 Infinitesimal fiber bundles

Suppose that  $(M = G/H, g)$  is a naturally reductive space with respect to the canonical connection of  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ . Let

$$\pi : M \rightarrow N$$

be a homogeneous fiber bundle, i.e.  $G$  also acts on  $N$  and  $\pi$  is a  $G$ -equivariant bundle map. Then the group  $G$  acts transitively on  $N$  and  $N = G/B$  with

$$B := \{g \in G : g \cdot \pi(e) = \pi(e)\},$$

where  $e \in G$  is the identity element. Let us also assume that  $N$  is simply connected and  $G$  is connected. The long homotopy exact sequence for the fiber bundle  $G \rightarrow G/B$  tells us that  $B$  is connected. Let  $\mathfrak{b}$  be the Lie algebra of  $B$ . Then  $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{m}^+$ , where  $\mathfrak{m}^+ \subset \mathfrak{m}$  is the projection of  $\mathfrak{b}$  onto  $\mathfrak{m}$ . Let  $\mathfrak{m}^-$  be the orthogonal complement of  $\mathfrak{m}^+$  in  $\mathfrak{m}$ . It is easy to see that  $\text{Ad}(B)\mathfrak{m}^- \subset \mathfrak{m}^-$  and that the restricted metric  $g|_{\mathfrak{m}^- \times \mathfrak{m}^-}$  is  $B$ -invariant. The canonical connection of  $\mathfrak{b} \oplus \mathfrak{m}^-$  together with the restricted metric  $g|_{\mathfrak{m}^- \times \mathfrak{m}^-}$  define a naturally reductive connection on  $G/B$ .

Next we will investigate the above situation on the level of Lie algebras. For many considerations this is good enough. For this reason we discuss the following lemma and definition.

**Lemma 1.2.1.** *Let  $(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, g)$  be an effective naturally reductive decomposition. Furthermore, suppose  $\mathfrak{m} = \mathfrak{m}^+ \oplus \mathfrak{m}^-$  is an orthogonal decomposition of  $\mathfrak{h}$ -modules. Then the following hold:*

- i)  $[\mathfrak{m}^+, \mathfrak{m}^-] \subset \mathfrak{m}$ ,*
- ii)  $[\mathfrak{m}^+, \mathfrak{m}^-] \subset \mathfrak{m}^-$  if and only if  $[\mathfrak{m}^+, \mathfrak{m}^+]_{\mathfrak{m}} \subset \mathfrak{m}^+$ .*

*If we assume that  $[\mathfrak{m}^+, \mathfrak{m}^-] \subset \mathfrak{m}^-$ , then*

- iii)  $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{m}^+$  is a subalgebra of  $\mathfrak{g}$ ,*
- iv)  $(\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{m}^-, g|_{\mathfrak{m}^- \times \mathfrak{m}^-})$  is a naturally reductive decomposition.*

*Proof.* *i)* Since  $\mathfrak{m}^+$  and  $\mathfrak{m}^-$  are  $\mathfrak{h}$ -invariant we conclude

$$g(R(u, v)x^+, x^-) = 0, \quad \forall x^\pm \in \mathfrak{m}^\pm, \forall u, v \in \mathfrak{m}.$$

Combining this with the fact that  $R : \Lambda^2 \mathfrak{m} \rightarrow \Lambda^2 \mathfrak{m}$  is symmetric with respect to the Killing form on  $\mathfrak{so}(\mathfrak{m}) \cong \Lambda^2 \mathfrak{m}$  it follows that  $R(x^+, x^-) = 0$  for all  $x^\pm \in \mathfrak{m}^\pm$ . The tensor  $R$  is defined by  $R(x^+, x^-) = -\text{ad}([x^+, x^-]_{\mathfrak{h}})$ . Since we assume our decomposition to be effective  $\text{ad}([x^+, x^-]_{\mathfrak{h}}) = 0$  implies that  $[x^+, x^-]_{\mathfrak{h}} = 0$ . Hence  $[\mathfrak{m}^+, \mathfrak{m}^-] \subset \mathfrak{m}$ .

*ii)* Suppose that  $[\mathfrak{m}^+, \mathfrak{m}^-] \subset \mathfrak{m}^-$ . If  $x_1^+, x_2^+ \in \mathfrak{m}^+$  and  $x^- \in \mathfrak{m}^-$ , then

$$0 = g([x_1^+, x^-], x_2^+) = -g(x^-, [x_1^+, x_2^+]).$$

This implies  $[x_1^+, x_2^+]_{\mathfrak{m}} \in \mathfrak{m}^+$ . The converse follows from the same equation and *i*).

*iii*) From *ii*) we can easily conclude that  $\mathfrak{b}$  is a subalgebra of  $\mathfrak{g}$ .

*iv*) For the decomposition  $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{m}^-$  we clearly have  $[\mathfrak{b}, \mathfrak{m}^-] \subset \mathfrak{m}^-$  and the decomposition is naturally reductive with respect to the metric  $g|_{\mathfrak{m}^- \times \mathfrak{m}^-}$ .  $\square$

**Definition 1.2.2.** Let  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  be a naturally reductive decomposition. Suppose that  $[\mathfrak{m}^+, \mathfrak{m}^-] \subset \mathfrak{m}^-$ , with the notation from Lemma 1.2.1. In this case we will call  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  the *decomposition of the total space* of the *infinitesimal fiber bundle* and the naturally reductive decomposition  $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{m}^-$  with isotropy algebra  $\mathfrak{b}$  the *decomposition of the base space*. Furthermore, we will call  $\mathfrak{m}^+$  the *fiber direction*.

There is no reason for the connected subgroup  $B \subset G$  with  $\text{Lie}(B) = \mathfrak{b}$  to be closed. However, the decomposition  $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{m}^-$  still defines a naturally reductive decomposition and therefore a locally naturally reductive space. This is the reason why we consider infinitesimal fiber bundles.

### 1.3 Reducibility of naturally reductive spaces

In this section we prove that a metric connection with parallel skew torsion is locally a product if and only if the torsion is locally a product. This will in particular apply to every naturally reductive space. For naturally reductive spaces this result is essentially already known, see [Tsu96].

**Lemma 1.3.1.** *Let  $(M, g, \nabla)$  be a complete simply connected Riemannian manifold with metric connection  $\nabla$ . Suppose that the tangent space splits into  $\nabla$  parallel distributions  $TM = V_1 \oplus V_2$ . Let  $A$  be the connection form such that  $\nabla = \nabla^g + A$ , where  $\nabla^g$  is the Levi-Civita connection. If the connection form splits:  $A = A_1 + A_2$ , with*

$$A_i \in \Gamma(V_i^* \otimes \mathfrak{so}(V_i)),$$

*then the manifold is a product*

$$(M, g) = (M_1, g_1) \times (M_2, g_2).$$

*If  $A$  is parallel, i.e.  $\nabla A = 0$ , then this implies that  $\nabla^{g_i} + A_i$  define connections on  $M_i$  and  $\nabla$  is the product connection of these two connections.*

*Proof.* Let  $X \in \Gamma(TM)$  and  $Y \in \Gamma(V_i)$  for  $i = 1, 2$ . Then

$$\Gamma(V_i) \ni \nabla_X Y = \nabla_X^g Y + A(X)Y.$$

We also know that  $A(X)Y \in \Gamma(V_i)$ . Together these imply that

$$\nabla_X^g Y \in \Gamma(V_i), \quad \text{if } Y \in \Gamma(V_i).$$

Hence  $V_1$  and  $V_2$  are parallel with respect to  $\nabla^g$ . Now De Rham's theorem implies that

$$(M, g) = (M_1, g_1) \times (M_2, g_2).$$

Suppose that  $\nabla A = 0$ . Then  $\nabla A_i = 0$  for  $i = 1, 2$ . Let  $Y_2 \in \Gamma(V_2)$ . Then

$$0 = \nabla_{Y_2} A_1 = \nabla_{Y_2}^g A_1 + A_2(Y_2) \cdot A_1 = \nabla_{Y_2}^g A_1,$$

because  $A_1(Y_2) = 0$  and  $A_2(Y_2)$  acts trivially on all tensors in  $V_1$ . Let  $x_1, \dots, x_k$  be local coordinates of  $M_1$  and let  $x_{k+1}, \dots, x_m$  be local coordinates of  $M_2$ . Let  $e_1 = \frac{d}{dx_1}, \dots, e_m = \frac{d}{dx_m}$  be the corresponding local frame. Then

$$0 = \nabla_{Y_2}^g A_1 = d_{Y_2} A_1 + \omega(Y_2) \cdot A_1 = d_{Y_2} A_1,$$

where we used that  $\nabla_{e_i}^g e_j = \omega(e_i) \cdot e_j = 0$  if  $j \leq k$  and  $k+1 \leq i$ . Hence  $A_1$  is independent of  $x_{k+1}, \dots, x_m$  and thus  $A_1$  restricts to a well defined endomorphism valued 1-form on  $M_1$ . Similarly  $A_2$  restricts to a well defined endomorphism valued 1-form on  $M_2$ . The connection  $\nabla$  clearly is the product connection of the two restricted connections.  $\square$

Let  $(V, g)$  be some vector space with a metric  $g$ . Let  $T \in \Lambda^3 V$  be a 3-form. We consider  $T$  as a linear map

$$T : V \rightarrow \Lambda^2 V; \quad T : x \mapsto x \lrcorner T.$$

We define the kernel of  $T$  as the kernel of this linear map.

**Lemma 1.3.2.** *Let  $(V, g)$  be some vector space with a metric  $g$ . Let  $T \in \Lambda^3 V$  be a 3-form. Let  $h \in \mathfrak{so}(V)$  with  $h \cdot T = 0$ . Suppose that either*

- i)  $T$  has no kernel and  $T = T_1 + T_2 \in \Lambda^3 V_1 \oplus \Lambda^3 V_2$ , with  $V_1 = (V_2)^\perp$  or,*
- ii)  $T$  has a kernel and we set  $V_2 = \ker(T)$  and  $V_1 = (V_2)^\perp$ , so  $T = T_1 + T_2 \in \Lambda^3 V_1 \oplus \Lambda^3 V_2$  with  $T_2 = 0$ .*

*Then for both cases  $h$  leaves  $V_1$  and  $V_2$  invariant. In other words*

$$\{h \in \mathfrak{so}(V) : h \cdot T = 0\} \cong \{h_1 \in \mathfrak{so}(V_1) : h_1 \cdot T_1 = 0\} \oplus \{h_2 \in \mathfrak{so}(V_2) : h_2 \cdot T_2 = 0\}.$$

*Proof.* We view  $h$  as a skew-symmetric endomorphism of  $V$  and we write  $h$  as

$$h = \left( \begin{array}{c|c} A & -B^T \\ \hline B & C \end{array} \right),$$

where  $A \in \mathfrak{so}(V_1)$ ,  $B \in \text{Lin}(V_1, V_2)$ ,  $C \in \mathfrak{so}(V_2)$ . Since the torsion is invariant under  $h$  we get

$$0 = h \cdot T = A \cdot T_1 + B \cdot T_1 - B^T \cdot T_2 + C \cdot T_2.$$

If any two of these summands are non-zero, then they are linearly independent, since

$$\begin{aligned} A \cdot T_1 &\in \Lambda^3 V_1, \\ B \cdot T_1 &\in \Lambda^2 V_1 \otimes V_2, \\ -B^T T_2 &\in V_1 \otimes \Lambda^2 V_2, \\ C \cdot T_2 &\in \Lambda^3 V_2. \end{aligned}$$

Hence all terms vanish. We get

$$0 = B \cdot T_1 = (B - B^T) \cdot T_1 = \sum_i B(e_i) \wedge (e_i \lrcorner T_1),$$

where the sum is over an orthonormal basis of  $V_1$  and  $(B - B^T)$  is considered as a block matrix in  $\mathfrak{so}(V)$ . For the last equality we used Lemma 2.1.6. The 2-forms  $e_i \lrcorner T_1$  are all linearly independent, because  $T_1$  has no kernel for both case *i*) and case *ii*). Since  $B(e_i) \in V_2$  and  $e_i \lrcorner T_1 \in \Lambda^2 V_1$  we obtain the equation  $B(e_i) \wedge (e_i \lrcorner T_1) = 0$  for all  $i$ . This implies  $B(e_i) = 0$  for all  $e_i$ . We conclude that  $B = 0$  and thus  $h$  leaves  $V_1$  and  $V_2$  invariant.  $\square$

**Definition 1.3.3.** Let  $(V, g)$  be some vector space with a metric  $g$ . A 3-form  $T \in \Lambda^3 V$  is called *reducible* if it can be written as  $T = T_1 + T_2$  with  $T_i \in \Lambda^3 V_i$  for some non-zero  $V_1 \subset V$  and  $V_2 \subset V$  such that  $V_1 \perp V_2$ . Otherwise  $T$  is called *irreducible*.

**Definition 1.3.4.** A naturally reductive decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  is *reducible* if its torsion, defined by (1.1.13), is given by  $T = T_1 + T_2 \in \Lambda^3 \mathfrak{m}_1 \oplus \Lambda^3 \mathfrak{m}_2$ , for some non-trivial orthogonal decomposition  $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ . Otherwise the decomposition is *irreducible*.

**Theorem 1.3.5.** *Let  $(M, g, \nabla)$  be a complete simply connected manifold with a metric connection  $\nabla$  with non-zero parallel skew torsion  $T$ . Then the following are equivalent*

i)  $M$  is isometric to a product and  $\nabla$  is the product connection:

$$(M, g, \nabla) \cong (M_1, g_1, \nabla_1) \times (M_2, g_2, \nabla_2),$$

where  $\nabla_1$  and  $\nabla_2$  are connections on  $M_1$  and  $M_2$ , respectively. Both  $\nabla_1$  and  $\nabla_2$  have parallel skew torsion.

ii) The torsion at some point  $x \in M$  is reducible, i.e.  $T(x) = T_1(x) + T_2(x) \in \Lambda^3 V_1(x) \oplus \Lambda^3 V_2(x)$ , for certain orthogonal subspaces  $V_1(x), V_2(x) \subset T_x M$  and  $T_i(x) \in \Lambda^3 V_i(x)$ .

*Proof.* It is clear that i) implies ii).

Suppose that ii) holds. From Lemma 1.3.2 we know for any loop  $\gamma$  based at  $x$  that  $P_\gamma(V_i) = V_i$ , where  $P_\gamma$  denotes the parallel transport of  $\nabla$  along  $\gamma$ . Hence we can define two distributions  $V_1$  and  $V_2$  by  $V_i(y) = P_\gamma(V_i(x))$ , where  $\gamma$  is any curve from  $x$  to  $y$ . Now we are in the situation of Lemma 1.3.1 and thus we obtain i).  $\square$

*Remark 1.3.6.* Note that if in Theorem 1.3.5 we don't assume that the connection is the product connection, then the statement is false. For example on  $\mathbb{R}^n$  there exist naturally reductive connections which are not the Levi-Civita connection, see Remark 2.2.6.

Theorem 1.3.5 applies in particular to naturally reductive spaces. In [Tsu96] it is proved that if a simply connected naturally reductive space is irreducible as Riemannian manifold, then the torsion is irreducible.

We will give a slightly different characterisation of reducibility which will be useful later on.

**Lemma 1.3.7.** *Let  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  be a naturally reductive decomposition with  $\mathfrak{g}$  its transvection algebra. Let  $\bar{g}$  be the unique  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear form from Kostant's theorem, see Theorem 1.1.16. The reductive decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  is reducible if and only if there exist two orthogonal ideals  $\mathfrak{g}_1 \subset \mathfrak{g}$  and  $\mathfrak{g}_2 \subset \mathfrak{g}$  with respect to  $\bar{g}$  such that  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ ,  $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$  with  $\mathfrak{h}_i \subset \mathfrak{g}_i$ , and  $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$  with  $\mathfrak{m}_i \subset \mathfrak{g}_i$  and  $\mathfrak{m}_i \neq \{0\}$  for  $i = 1, 2$ .*

*Proof.* Assume two such ideals exist. Then clearly  $T \in \Lambda^3 \mathfrak{m}_1 \oplus \Lambda^3 \mathfrak{m}_2$ , where  $T$  is defined by (1.1.13), and the decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  is reducible, see Definition 1.3.4.

Conversely suppose that  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  is the transvection algebra of a reducible naturally reductive decomposition, i.e.  $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$  with  $\mathfrak{m}_1 \neq \{0\}$ ,  $\mathfrak{m}_2 \neq \{0\}$ ,  $\mathfrak{m}_1 \perp \mathfrak{m}_2$ , and  $[\mathfrak{m}_1, \mathfrak{m}_2] = \{0\}$ . Then

$$\mathfrak{g} = [\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}} \oplus \mathfrak{m} = ([\mathfrak{m}_1, \mathfrak{m}_1]_{\mathfrak{h}} \oplus \mathfrak{m}_1) + ([\mathfrak{m}_2, \mathfrak{m}_2]_{\mathfrak{h}} \oplus \mathfrak{m}_2) = (\mathfrak{h}_1 \oplus \mathfrak{m}_1) + (\mathfrak{h}_2 \oplus \mathfrak{m}_2),$$

where  $\mathfrak{h}_i := [\mathfrak{m}_i, \mathfrak{m}_i]_{\mathfrak{h}}$ . Let  $m, m' \in \mathfrak{m}_1$  and  $n \in \mathfrak{m}_2$ . Then we have

$$[[m, m']_{\mathfrak{h}}, n] = [[m, m'], n] = [[m, n], m'] + [m, [m', n]] = 0 + 0 = 0.$$

Since elements of the form  $[m, m']_{\mathfrak{h}}$  span  $\mathfrak{h}_1$  it follows that  $[\mathfrak{h}_1, \mathfrak{m}_2] = \{0\}$ . In the same way we get  $[\mathfrak{h}_2, \mathfrak{m}_1] = \{0\}$ . This also implies that  $[\mathfrak{h}_1 \cap \mathfrak{h}_2, \mathfrak{m}] = \{0\}$  and because the reductive decomposition is effective we get  $\mathfrak{h}_1 \cap \mathfrak{h}_2 = \{0\}$ . From Lemma 1.1.18 we see that  $\mathfrak{h}_1 \perp \mathfrak{h}_2$  with respect to  $\bar{g}$ . We conclude that  $\mathfrak{g} = (\mathfrak{h}_1 \oplus \mathfrak{m}_1) \oplus (\mathfrak{h}_2 \oplus \mathfrak{m}_2)$  is the direct sum of two ideals in the way required.  $\square$

## Chapter 2

# A new construction of naturally reductive spaces

In this chapter we will describe a new construction of naturally reductive spaces. This construction produces many new examples of naturally reductive spaces. We will see in Chapter 3 that practically all of these spaces do not have a semisimple transvection algebra and that they are not normal homogeneous with its canonical naturally reductive structure. Such a construction of naturally reductive spaces also appear in [Gor85] and [AF16]. The construction presented here is a generalisation of these constructions. Our construction starts with the following pieces of data. We take a naturally reductive space  $M$  together with a Lie algebra  $\mathfrak{k}$  with an  $\text{ad}(\mathfrak{k})$ -invariant metric on  $\mathfrak{k}$ . The algebra  $\mathfrak{k}$  is a certain subalgebra of derivations of the transvection algebra of  $M$ . From this data we can construct a new naturally reductive space which is an infinitesimal homogeneous fiber bundle over  $M$  as in Definition 1.2.2. If the naturally reductive space we start with is the symmetric space  $\mathbb{R}^n$  and  $\mathfrak{k} \subset \mathfrak{so}(n)$  is a subalgebra together with any  $\text{ad}(\mathfrak{k})$ -invariant metric, then we obtain exactly the 2-step nilpotent Lie groups with a naturally reductive structure from [Gor85]. If we start with a compact simple Lie group  $G$  and choose  $\mathfrak{k} = \text{Lie}(G) \cong \text{Der}(\text{Lie}(G))$ , we obtain the spaces from [AF16]. However, we can start with any base space and a suitable subalgebra  $\mathfrak{k}$  and obtain many new examples of naturally reductive spaces which are not normally homogeneous with respect to their canonical connection. In fact in Chapter 3 we prove that every naturally reductive space can be obtained by our construction. The majority of this chapter will appear in [Sto17].

## 2.1 The construction

Let  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  be a naturally reductive decomposition with infinitesimal model  $(T_0, R_0)$ . Furthermore, suppose that  $\mathfrak{g}$  is the transvection algebra of  $(T_0, R_0)$ . We define the following Lie algebra of derivations

$$\mathfrak{s}(\mathfrak{g}) := \{f \in \text{Der}(\mathfrak{g}) : f(\mathfrak{h}) = \{0\}, f(\mathfrak{m}) \subset \mathfrak{m}, f|_{\mathfrak{m}} \in \mathfrak{so}(\mathfrak{m})\}.$$

We will sometimes simply write  $\mathfrak{s}$  instead of  $\mathfrak{s}(\mathfrak{g})$ . It will always be clear from the context what the reductive decomposition of  $\mathfrak{g}$  is. We make one exception for the definition of  $\mathfrak{s}(\mathfrak{g})$ , namely we set

$$\mathfrak{s}(\{0\}) := \mathfrak{so}(\infty).$$

Here one should think of  $\{0\}$  as the transvection algebra of a point space.

In the following we show that if  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  is a naturally reductive decomposition with  $\mathfrak{g}$  its transvection algebra and  $\mathfrak{g} \neq \{0\}$ , then  $\mathfrak{s}(\mathfrak{g})$  can be identified with all  $\mathfrak{h}$ -equivariant module endomorphisms of  $\mathfrak{m}$  which act trivially on  $T_0$ .

**Lemma 2.1.1.** *Let  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  be a naturally reductive decomposition with  $\mathfrak{g} \neq \{0\}$  its transvection algebra. Let  $(T_0, R_0)$  be the infinitesimal model of the decomposition. Let*

$$\mathfrak{so}_{\mathfrak{h}}(\mathfrak{m}) = \{k \in \mathfrak{so}(\mathfrak{m}) : [k, \text{ad}(h)]_{\mathfrak{so}(\mathfrak{m})} = 0, \forall h \in \mathfrak{h}\}.$$

Then

$$\mathfrak{s}(\mathfrak{g}) \cong \{h \in \mathfrak{so}_{\mathfrak{h}}(\mathfrak{m}) : h \cdot T_0 = 0\}.$$

*Proof.* For all  $k \in \mathfrak{s}(\mathfrak{g})$ ,  $h \in \mathfrak{h}$  and  $m \in \mathfrak{m}$  we have

$$k([h, m]) = [k(h), m] + [h, k(m)] = [h, k(m)].$$

In other words  $\varphi(k) \in \mathfrak{so}_{\mathfrak{h}}(\mathfrak{m})$ . Furthermore, for all  $m_1, m_2 \in \mathfrak{m}$  we have

$$\begin{aligned} k(T_0(m_1, m_2)) &= -k([m_1, m_2]_{\mathfrak{m}}) = -k([m_1, m_2]) \\ &= -[k(m_1), m_2]_{\mathfrak{m}} - [m_1, k(m_2)]_{\mathfrak{m}} = T_0(k(m_1), m_2) + T_0(m_1, k(m_2)). \end{aligned}$$

We conclude that  $\varphi(k) \cdot T_0 = 0$ .

To find a map in the other direction we let  $k \in \mathfrak{so}_{\mathfrak{h}}(\mathfrak{m})$  with  $k \cdot T_0 = 0$ . We define

$$\hat{k} : \mathfrak{g} \rightarrow \mathfrak{g}; \quad \hat{k}(h + m) := k(m)$$

and we show that  $\hat{k} \in \mathfrak{s}(\mathfrak{g})$ . For all  $h, h' \in \mathfrak{h}$  and  $m \in \mathfrak{m}$  we have

$$\begin{aligned}\hat{k}([h, h' + m]) &= \hat{k}([h, h' + m]_{\mathfrak{m}}) = \hat{k}([h, m]) = [h, \hat{k}(m)] \\ &= [\hat{k}(h), h' + m] + [h, \hat{k}(h' + m)],\end{aligned}$$

where in the before last equality we used  $k \in \mathfrak{so}_{\mathfrak{h}}(\mathfrak{m})$ . It remains to show that for all  $m_1, m_2 \in \mathfrak{m}$  we have

$$\hat{k}([m_1, m_2]) = [\hat{k}(m_1), m_2] + [m_1, \hat{k}(m_2)].$$

From  $k \cdot T_0 = 0$  we immediately get

$$\hat{k}([m_1, m_2]) = \hat{k}([m_1, m_2]_{\mathfrak{m}}) = [\hat{k}(m_1), m_2]_{\mathfrak{m}} + [m_1, \hat{k}(m_2)]_{\mathfrak{m}}.$$

Furthermore, we have

$$\begin{aligned}\text{ad}([\hat{k}(m_1), m_2]_{\mathfrak{h}} + [m_1, \hat{k}(m_2)]_{\mathfrak{h}}) &= -R_0(\hat{k}(m_1), m_2) - R_0(m_1, \hat{k}(m_2)) \\ &= -R_0(\hat{k}(m_1) \wedge m_2 + m_1 \wedge \hat{k}(m_2)) \\ &= -R_0(k \cdot (m_1 \wedge m_2)).\end{aligned}$$

The right-hand-side vanishes precisely when  $k \cdot (m_1 \wedge m_2) \in \text{ad}(\mathfrak{h})^{\perp}$ , where  $\text{ad}(\mathfrak{h})^{\perp}$  is the orthogonal complement of  $\text{ad}(\mathfrak{h})$  in  $\mathfrak{so}(\mathfrak{m})$  with respect to the Killing form  $B_{\mathfrak{so}(\mathfrak{m})}$  of  $\mathfrak{so}(\mathfrak{m})$ . Note that Lemma 2.1.6 gives us  $k \cdot (m_1 \wedge m_2) = [k, m_1 \wedge m_2]_{\mathfrak{so}(\mathfrak{m})}$ . For all  $h \in \mathfrak{h}$  we have

$$B_{\mathfrak{so}(\mathfrak{m})}(\text{ad}(h), [k, m_1 \wedge m_2]_{\mathfrak{so}(\mathfrak{m})}) = B_{\mathfrak{so}(\mathfrak{m})}([\text{ad}(h), k], m_1 \wedge m_2) = 0.$$

This implies that  $R_0(k \cdot (m_1 \wedge m_2)) = 0$  and thus also  $[\hat{k}(m_1), m_2]_{\mathfrak{h}} + [m_1, \hat{k}(m_2)]_{\mathfrak{h}} = 0$ . From this we now obtain

$$\begin{aligned}\hat{k}([m_1, m_2]) &= [\hat{k}(m_1), m_2]_{\mathfrak{m}} + [m_1, \hat{k}(m_2)]_{\mathfrak{m}} \\ &= [\hat{k}(m_1), m_2] + [m_1, \hat{k}(m_2)].\end{aligned}$$

Consequently,  $\hat{k}$  defines a derivation of  $\mathfrak{g}$  and  $\hat{k} \in \mathfrak{s}(\mathfrak{g})$ . It is clear that the above two maps are inverse to each other. We conclude that

$$\mathfrak{s}(\mathfrak{g}) \cong \{h \in \mathfrak{so}_{\mathfrak{h}}(\mathfrak{m}) : h \cdot T_0 = 0\}.$$

□

Let  $\mathfrak{k} \subset \mathfrak{s}$  be a subalgebra and let  $\varphi : \mathfrak{k} \rightarrow \mathfrak{so}(\mathfrak{m})$  be the natural faithful Lie algebra representation. Because of this faithful representation we know that  $\mathfrak{k}$  is a compact Lie algebra and thus  $\mathfrak{k}$  admits positive definite  $\text{ad}(\mathfrak{k})$ -invariant metrics. Let  $B$  be some  $\text{ad}(\mathfrak{k})$ -invariant metric on  $\mathfrak{k}$ . Later on we will have two copies of the Lie algebra  $\mathfrak{k}$ . To keep notation consistent with the sequel we let  $\mathfrak{n} = \mathfrak{k}$  be another copy of  $\mathfrak{k}$ , even though at this moment this notation has no real use.

**Definition 2.1.2.** Let  $(T_0, R_0)$  be an infinitesimal model of a naturally reductive space on  $(\mathfrak{m}, \mathfrak{g}_0)$ . Let  $g := B \oplus g_0$  be a metric on  $\mathfrak{n} \oplus \mathfrak{m}$  with  $B$  any  $\text{ad}(\mathfrak{k})$ -invariant metric on  $\mathfrak{n}$ . Let  $k_1, \dots, k_l$  be an orthonormal basis of  $\mathfrak{k}$  and denote by  $n_1, \dots, n_l$  the corresponding basis of  $\mathfrak{n}$ . We define a pair of tensors  $(T, R)$ ,  $T \in \Lambda^3(\mathfrak{n} \oplus \mathfrak{m})$  and  $R \in \Lambda^2(\mathfrak{n} \oplus \mathfrak{m}) \odot \Lambda^2(\mathfrak{n} \oplus \mathfrak{m})$  by

$$T := T_0 + \sum_{i=1}^l \varphi(k_i) \wedge n_i + 2T_{\mathfrak{n}}, \quad (2.1.3)$$

where

$$T_{\mathfrak{n}}(x, y, z) := B([x, y], z), \quad \text{for } x, y, z \in \mathfrak{n},$$

and  $[-, -]$  is the Lie bracket of  $\mathfrak{n} = \mathfrak{k}$ , and  $\varphi(k_i) \in \mathfrak{so}(\mathfrak{m}) \cong \Lambda^2 \mathfrak{m}$  is identified with a 2-form on  $\mathfrak{m}$ . We define a representation of  $\mathfrak{k}$  by

$$\psi := \text{ad} \oplus \varphi : \mathfrak{k} \rightarrow \mathfrak{so}(\mathfrak{n} \oplus \mathfrak{m}),$$

where  $\text{ad} : \mathfrak{k} \rightarrow \mathfrak{so}(\mathfrak{k}) = \mathfrak{so}(\mathfrak{n})$  is just the adjoint representation. The curvature tensor  $R$  is defined as

$$R := R_0 + \sum_{i=1}^l \psi(k_i) \odot \psi(k_i). \quad (2.1.4)$$

We denote the last summand by

$$R_{\psi} := \sum_{i=1}^l \psi(k_i) \odot \psi(k_i).$$

We call the pair  $(T, R)$  the  $(\mathfrak{k}, B)$ -*extension* of  $(T_0, R_0)$ .

We will prove that  $(T, R)$  defines an infinitesimal model of a naturally reductive space on  $(\mathfrak{n} \oplus \mathfrak{m}, g)$ . For this we have to show that  $T$  and  $R$  are invariant under  $\text{im}(R)$  and that the first Bianchi identity (1.1.7) is satisfied. To prove this we first consider the following algebraic lemma.

**Definition 2.1.5.** Let  $(V, g)$  be a finite dimensional vector space with a metric  $g$ . Let  $\alpha \in \Lambda^p V$  and  $\beta \in \Lambda^q V$ . We define a  $(p + q - 2)$ -form by

$$\alpha \bar{\wedge} \beta = \sum_{i=1}^n (e_i \lrcorner \alpha) \wedge (e_i \lrcorner \beta),$$

where  $e_1, \dots, e_n$  is an orthonormal basis of  $V$ .

Note that the operation  $\alpha \bar{\wedge} \beta$  is independent of the basis. One easily checks the following:

**Lemma 2.1.6.** *Let  $(V, g)$  be a finite dimensional vector space with a metric  $g$ . If  $\alpha \in \Lambda^2 V \cong \mathfrak{so}(V)$  and  $\beta \in \Lambda^q V$ , then  $\alpha \bar{\wedge} \beta = \pi^{\wedge q}(\alpha)\beta \equiv \alpha \cdot \beta$ , where  $\pi$  is the vector representation of  $\mathfrak{so}(V)$  and  $\pi^{\wedge q}$  is the induced tensor representation on  $\Lambda^q V$ . Furthermore if  $\alpha, \beta \in \Lambda^2 V$ , then  $\alpha \cdot \beta = [\alpha, \beta]_{\mathfrak{so}(V)}$ .*

**Proposition 2.1.7.** *Let  $T_0, T, R_0$  and  $R$  be as in Definition 2.1.2. Then the tensors  $T_0, R_0, T$  and  $R$  are  $(\text{im}(R_0) + \psi(\mathfrak{k}))$ -invariant. In particular these tensors are  $\text{im}(R)$ -invariant.*

*Proof.* First note that

$$\text{im}(R) \subset \text{im}(R_0) + \psi(\mathfrak{k}) \subset \mathfrak{so}(\mathfrak{n} \oplus \mathfrak{m}).$$

For every  $k \in \mathfrak{k}$  and  $x, y, z \in \mathfrak{m}$  we have

$$\begin{aligned} (\psi(k) \cdot T_0)(x, y, z) &= -T_0(\psi(k)x, y, z) - T_0(x, \psi(k)y, z) - T_0(x, y, \psi(k)z) \\ &= g([\psi(k)x, y]_{\mathfrak{m}}, z) + g([x, \psi(k)y]_{\mathfrak{m}}, z) + g([x, y]_{\mathfrak{m}}, \psi(k)z) \\ &= g([\psi(k)x, y]_{\mathfrak{m}}, z) + g([x, \psi(k)y]_{\mathfrak{m}}, z) - g(\psi(k)[x, y]_{\mathfrak{m}}, z) \\ &= 0. \end{aligned}$$

Hence  $\psi(\mathfrak{k})$  leaves  $T_0$  invariant. The invariance of  $T_{\mathfrak{n}}$  under  $\psi(\mathfrak{k})$  is just the Jacobi identity of  $\mathfrak{n} = \mathfrak{k}$ . To see that the second term in (2.1.3) is invariant under  $\mathfrak{k}$  we do the following computation. If  $k \in \mathfrak{k}$ , then

$$\psi(k) \cdot \left( \sum_{i=1}^l \varphi(k_i) \wedge n_i \right) = \left( \sum_{i=1}^l [\varphi(k), \varphi(k_i)]_{\mathfrak{so}(\mathfrak{m})} \wedge n_i + \varphi(k_i) \wedge \text{ad}(k)(n_i) \right).$$

For the second term of this we have

$$\begin{aligned}
\sum_{i=1}^l \varphi(k_i) \wedge \text{ad}(k)(n_i) &= \sum_{i,j=1}^l \varphi(k_i) \wedge B([k, n_i], n_j) n_j \\
&= \sum_{i,j=1}^l -\varphi(B([k, n_j], n_i) k_i) \wedge n_j \\
&= \sum_{i,j=1}^l -\varphi(B([k, k_j], k_i) k_i) \wedge n_j \\
&= \sum_{j=1}^l -\varphi([k, k_j]) \wedge n_j = \sum_{j=1}^l -[\varphi(k), \varphi(k_j)]_{\mathfrak{so}(\mathfrak{m})} \wedge n_j. \quad (2.1.8)
\end{aligned}$$

Plugging this result back into the first line we see that

$$\psi(k) \cdot \left( \sum_{i=1}^l \varphi(k_i) \wedge n_i \right) = 0.$$

We conclude that  $T$  is invariant under  $\psi(\mathfrak{k})$ . We have  $\text{im}(R_0) \subset \mathfrak{so}(\mathfrak{m}) \subset \mathfrak{so}(\mathfrak{n} \oplus \mathfrak{m})$ , so  $\text{im}(R_0)$  acts trivially on  $\mathfrak{n}$ . This immediately shows that  $\text{im}(R_0)$  leaves  $T_{\mathfrak{n}}$  invariant. From Lemma 2.1.1 we know that  $\varphi(k)$  and  $\psi(k)$  commute with every element of  $\text{ad}(\mathfrak{h})$ . Applying Lemma 2.1.6 it follows that  $\text{im}(R_0)$  also leaves the second summand of (2.1.3) invariant. This concludes that  $T_0$  and  $T$  are invariant under  $\text{im}(R_0) + \psi(\mathfrak{k})$ .

The tensor  $R_0$  is invariant under  $\psi(\mathfrak{k})$ , since  $\psi(\mathfrak{k})$  commutes with  $\text{im}(R_0)$ . The same argument also implies that  $\sum_{i=1}^l \psi(k_i) \odot \psi(k_i)$  is invariant under  $\text{im}(R_0)$ . Lastly, by a similar computation as (2.1.8) one can see that the tensor  $\sum_{i=1}^l \psi(k_i) \odot \psi(k_i)$  is invariant under  $\psi(\mathfrak{k})$ . We conclude that  $R_0$  and  $R$  are  $(\text{im}(R_0) + \psi(\mathfrak{k}))$ -invariant.  $\square$

*Remark 2.1.9.* The first Bianchi identity is equivalent to (cf. [AFF15])

$$R^{\Lambda^4} = 2\sigma_T := \sum_{i=1}^m (e_i \lrcorner T) \wedge (e_i \lrcorner T), \quad (2.1.10)$$

where  $R^{\Lambda^4}$  denotes the 4-form component of the curvature tensor  $R$ . In other words  $R^{\Lambda^4} = b(R)$ , where  $b$  is the Bianchi map:

$$b(R)(x, y, z, v) = \frac{1}{3}(R(x, y, z, v) + R(y, z, x, v) + R(z, x, y, v)).$$

**Proposition 2.1.11.** *The pair of tensors  $(T, R)$  from Definition 2.1.2 satisfies the first Bianchi identity (1.1.7).*

*Proof.* Let  $e_1, \dots, e_m$  be an orthonormal basis of  $\mathfrak{m}$ . We will compute  $\sigma_T$  from (2.1.10):

$$\sigma_T = \frac{1}{2} \left( \sum_{p=1}^m (e_p \lrcorner T) \wedge (e_p \lrcorner T) + \sum_{i=1}^l (n_i \lrcorner T) \wedge (n_i \lrcorner T) \right).$$

For  $(e_p \lrcorner T) \wedge (e_p \lrcorner T) \equiv (e_p \lrcorner T)^{\wedge 2}$  we have

$$\begin{aligned} (e_p \lrcorner T)^{\wedge 2} &= (e_p \lrcorner T_0)^{\wedge 2} + 2 \sum_{i=1}^l (e_p \lrcorner T_0) \wedge \varphi(k_i)(e_p) \wedge n_i \\ &\quad + \sum_{i,j=1}^l \varphi(k_i)(e_p) \wedge n_i \wedge \varphi(k_j)(e_p) \wedge n_j \\ &= (e_p \lrcorner T_0)^{\wedge 2} + 2 \sum_{i=1}^l (e_p \lrcorner T_0) \wedge \varphi(k_i)(e_p) \wedge n_i \\ &\quad - \sum_{i,j=1}^l \varphi(k_i)(e_p) \wedge \varphi(k_j)(e_p) \wedge n_i \wedge n_j. \end{aligned}$$

Now we sum these three summands over  $p$ . For the first summand this gives

$$2\sigma_{T_0} = \sum_{p=1}^m (e_p \lrcorner T_0) \wedge (e_p \lrcorner T_0).$$

For the second summand we obtain

$$\begin{aligned} 2 \sum_{p=1}^m \sum_{i=1}^l (e_p \lrcorner T_0) \wedge \varphi(k_i)(e_p) \wedge n_i &= 2 \sum_{i=1}^l (\varphi(k_i) \bar{\lrcorner} T_0) \wedge n_i \\ &= 2 \sum_{i=1}^l (\varphi(k_i) \cdot T_0) \wedge n_i = 0, \end{aligned}$$

where we used Lemma 2.1.6 in the before last equality and the last equality follows from the fact that  $\varphi(k_i)$  acts trivially on  $T_0$  by Proposition 2.1.7. For the third

summand we use Lemma 2.1.6 again and obtain

$$\begin{aligned} -\sum_{p=1}^m \sum_{i,j=1}^l \varphi(k_i)(e_p) \wedge \varphi(k_j)(e_p) \wedge n_i \wedge n_j &= -\sum_{i,j=1}^l [\varphi(k_i), \varphi(k_j)] \wedge n_i \wedge n_j \\ &= -2 \sum_{i=1}^l \varphi(k_i) \wedge \text{ad}(k_i), \end{aligned}$$

where  $\text{ad}(k_i) \in \mathfrak{so}(\mathfrak{k}) \cong \Lambda^2 \mathfrak{n}$  and the last equality follows from:

$$\begin{aligned} \left( \sum_{i=1}^l \varphi(k_i) \wedge \text{ad}(k_i) \right) (n_a, n_b) &= \sum_{i=1}^l \varphi(B([k_i, n_a], n_b)k_i) \\ &= \sum_{i=1}^l \varphi(B(k_i, [k_a, k_b])k_i) \\ &= \varphi([k_a, k_b]) = [\varphi(k_a), \varphi(k_b)] \\ &= \frac{1}{2} \left( \sum_{i,j=1}^l [\varphi(k_i), \varphi(k_j)] \wedge n_i \wedge n_j \right) (n_a, n_b), \end{aligned}$$

for all  $1 \leq a, b \leq l$ . Computing the last term for  $\sigma_T$  yields

$$\begin{aligned} \sum_{i=1}^l (n_i \lrcorner T) \wedge (n_i \lrcorner T) &= \sum_{i=1}^l (\varphi(k_i) \wedge \varphi(k_i) + 4\varphi(k_i) \wedge (n_i \lrcorner T_{\mathfrak{n}})) + 2\sigma_{2T_{\mathfrak{n}}} \\ &= \sum_{i=1}^l (\varphi(k_i) \wedge \varphi(k_i) + 4\varphi(k_i) \wedge (n_i \lrcorner T_{\mathfrak{n}})) \\ &= \sum_{i=1}^l (\varphi(k_i) \wedge \varphi(k_i) + 4\varphi(k_i) \wedge \text{ad}(k_i)). \end{aligned}$$

Here we used the Jacobi identity for  $\mathfrak{n}$  in the form  $\sigma_{2T_{\mathfrak{n}}} = 0$ :

$$\begin{aligned} \sigma_{2T_{\mathfrak{n}}}(x, y, z) &= 2 \sum_{i=1}^l ((n_i \lrcorner T_{\mathfrak{n}}) \wedge (n_i \lrcorner T_{\mathfrak{n}}))(x, y, z) = 2 \left( \sum_{i=1}^l \text{ad}(k_i) \wedge \text{ad}(k_i) \right) (x, y, z) \\ &= 2\mathfrak{G}^{x,y,z} \sum_{i=1}^l B([k_i, x], y)[k_i, z] = 2\mathfrak{G}^{x,y,z} [[x, y], z] = 0. \end{aligned} \quad (2.1.12)$$

Summing all the terms we obtain

$$\sigma_T = \sigma_{T_0} + \frac{1}{2} \sum_{i=1}^l (2\varphi(k_i) \wedge \text{ad}(k_i) + \varphi(k_i) \wedge \varphi(k_i)).$$

Computing  $R^{\Lambda^4}$  is a bit easier. We have

$$\begin{aligned} R^{\Lambda^4} &= R_0^{\Lambda^4} + \sum_{i=1}^l (\varphi(k_i) + \text{ad}(k_i)) \wedge (\varphi(k_i) + \text{ad}(k_i)) \\ &= R_0^{\Lambda^4} + \sum_{i=1}^l \varphi(k_i) \wedge \varphi(k_i) + \text{ad}(k_i) \wedge \text{ad}(k_i) + 2\varphi(k_i) \wedge \text{ad}(k_i) \\ &= R_0^{\Lambda^4} + \sum_{i=1}^l \varphi(k_i) \wedge \varphi(k_i) + 2\varphi(k_i) \wedge \text{ad}(k_i). \end{aligned}$$

Here we used that  $\sum_{i=1}^l \text{ad}(k_i) \wedge \text{ad}(k_i) = 0$  by (2.1.12). We conclude that this torsion and curvature satisfy the first Bianchi identity.  $\square$

Combining Propositions 2.1.7 and Proposition 2.1.11 we obtain the following result.

**Theorem 2.1.13.** *Let  $(T_0, R_0)$  be an infinitesimal model of a naturally reductive space on  $(\mathfrak{m}, g_0)$ . Any  $(\mathfrak{k}, B)$ -extension  $(T, R)$  defines an infinitesimal model of a naturally reductive space on  $(\mathfrak{n} \oplus \mathfrak{m}, g = B \oplus g_0)$ .*

Note that there can be a multitude of different Lie algebras  $\mathfrak{k} \subset \mathfrak{s}$  for a given infinitesimal model  $(T_0, R_0)$ , see Example 2.2.23. Also any  $\text{ad}(\mathfrak{k})$ -invariant metric  $B$  on  $\mathfrak{k}$  gives us a  $(\mathfrak{k}, B)$ -extension. When we parametrize the space of all  $\text{ad}(\mathfrak{k})$ -invariant metrics on  $\mathfrak{k}$ , we get a parameter family of naturally reductive structures. In other words the newly constructed naturally reductive structures always come in parameter families. Now we will apply the Nomizu construction to the  $(\mathfrak{k}, B)$ -extensions.

**Definition 2.1.14.** Let  $(T_0, R_0)$  be an infinitesimal model of a naturally reductive space on  $(\mathfrak{m}, g_0)$ . Let  $\mathfrak{h} := \text{im}(R_0)$  and let  $(T, R)$  be a  $(\mathfrak{k}, B)$ -extension of  $(T_0, R_0)$ . We define the following vector space

$$\mathfrak{g}(\mathfrak{k}) := \mathfrak{h} \oplus \mathfrak{k} \oplus \mathfrak{n} \oplus \mathfrak{m},$$

together with  $g := B \oplus g_0$  as metric on  $\mathfrak{n} \oplus \mathfrak{m}$ . We define a skew-symmetric bilinear map,  $[-, -] : \mathfrak{g}(\mathfrak{k}) \oplus \mathfrak{g}(\mathfrak{k}) \rightarrow \mathfrak{g}(\mathfrak{k})$  similar to (1.1.10) by

$$\begin{aligned} [h + k, n + m] &= \psi(k)(n + m) + h(m), \quad \forall h \in \mathfrak{h}, \forall k \in \mathfrak{k}, \forall n \in \mathfrak{n}, \forall m \in \mathfrak{m}, \\ [h_1 + k_1, h_2 + k_2] &= [h_1, h_2] + [k_1, k_2], \quad \forall h_1, h_2 \in \mathfrak{h}, \forall k_1, k_2 \in \mathfrak{k}, \\ [x, y] &= -R_0(x, y) - R_{\mathfrak{k}}(x, y) - T(x, y) \quad \forall x, y \in \mathfrak{n} \oplus \mathfrak{m}, \end{aligned}$$

where

$$R_{\mathfrak{k}}(x, y) := \sum_{i=1}^l \psi(k_i)(x, y)k_i \in \mathfrak{k}.$$

Note that if the representation

$$\text{id} + \psi : \mathfrak{h} \oplus \mathfrak{k} \rightarrow \mathfrak{so}(\mathfrak{n} \oplus \mathfrak{m})$$

is faithful, we can just apply the original Nomizu construction to see that  $[-, -]$  defines a Lie bracket, because the image of  $\text{id} + \psi$  is contained in  $\{h \in \mathfrak{so}(\mathfrak{n} \oplus \mathfrak{m}) : h \cdot T = 0, h \cdot R = 0\}$  by Proposition 2.1.7. Since this is not necessarily the case we have to consider this slightly altered definition above.

**Lemma 2.1.15.** *The vector space  $\mathfrak{g}(\mathfrak{k})$  with the bilinear map defined in Definition 2.1.14 is a Lie algebra.*

*Proof.* We have to show that the Jacobi identity is satisfied. Because the expression  $[[x, y], z]$  is multilinear in  $x, y$  and  $z$  it suffices to prove the Jacobi identity for the following three cases:

$$\begin{aligned} (i) \quad \mathfrak{S}^{h_1, h_2, x}[[h_1, h_2], x] &= 0 & \forall h_1, h_2 \in \mathfrak{h} \oplus \mathfrak{k}, \forall x \in \mathfrak{g}(\mathfrak{k}), \\ (ii) \quad \mathfrak{S}^{h, m_1, m_2}[h, [m_1, m_2]] &= 0 & \forall h \in \mathfrak{h} \oplus \mathfrak{k}, \forall m_1, m_2 \in \mathfrak{n} \oplus \mathfrak{m}, \\ (iii) \quad \mathfrak{S}^{m_1, m_2, m_3}[[m_1, m_2], m_3] &= 0 & \forall m_1, m_2, m_3 \in \mathfrak{n} \oplus \mathfrak{m}. \end{aligned}$$

Remember from Lemma 2.1.1 that  $[\mathfrak{h}, \psi(\mathfrak{k})]_{\mathfrak{so}(\mathfrak{n} \oplus \mathfrak{m})} = \{0\}$ , where  $[-, -]_{\mathfrak{so}(\mathfrak{n} \oplus \mathfrak{m})}$  denotes the Lie bracket in  $\mathfrak{so}(\mathfrak{n} \oplus \mathfrak{m})$ . This tells us that

$$\text{id} + \psi : \mathfrak{h} \oplus \mathfrak{k} \rightarrow \mathfrak{so}(\mathfrak{n} \oplus \mathfrak{m})$$

is a Lie algebra representation. It is easy to see that case (i) for  $x \in \mathfrak{n} \oplus \mathfrak{m}$  is equivalent to the fact that  $\text{id} + \psi$  is a representation. The same applies when  $x \in \mathfrak{h} \oplus \mathfrak{k}$  then (i) is satisfied because the Lie bracket is the adjoint representation of  $\mathfrak{h} \oplus \mathfrak{k}$ .

For case (ii) we use Proposition 2.1.7 where we showed that  $T$ ,  $R_0$  and  $R_\psi$  are  $(\mathfrak{h} + \psi(\mathfrak{k}))$ -invariant. Clearly the invariance of  $R_\psi$  under  $\psi(\mathfrak{k})$  is equivalent to the invariance of  $R_\mathfrak{k}$  under  $\mathfrak{k}$ . We get

$$\begin{aligned}
[h, [m_1, m_2]] &= -[h, R_0(m_1, m_2) + R_\mathfrak{k}(m_1, m_2) + T(m_1, m_2)] \\
&= -(\text{id} + \psi)(h) \cdot (R_0(m_1, m_2) + R_\mathfrak{k}(m_1, m_2) + T(m_1, m_2)) \\
&= -R_0([h, m_1], m_2) - R_\mathfrak{k}([h, m_1], m_2) - R_0(m_1, [h, m_2]) \\
&\quad - R_\mathfrak{k}(m_1, [h, m_2]) - T([h, m_1], m_2) - T(m_1, [h, m_2]) \\
&= [[h, m_1], m_2]_{\mathfrak{h} \oplus \mathfrak{k}} + [m_1, [h, m_2]]_{\mathfrak{h} \oplus \mathfrak{k}} \\
&\quad + [[h, m_1], m_2]_{\mathfrak{n} \oplus \mathfrak{m}} + [m_1, [h, m_2]]_{\mathfrak{n} \oplus \mathfrak{m}} \\
&= [[h, m_1], m_2] + [m_1, [h, m_2]].
\end{aligned}$$

For case (iii) we have

$$\begin{aligned}
\mathfrak{S}^{m_1, m_2, m_3} [[m_1, m_2], m_3]_{\mathfrak{n} \oplus \mathfrak{m}} &= \mathfrak{S}^{m_1, m_2, m_3} [[m_1, m_2]_{\mathfrak{h} \oplus \mathfrak{k}}, m_3] + T(T(m_1, m_2), m_3) \\
&= \mathfrak{S}^{m_1, m_2, m_3} - R(m_1, m_2)m_3 + T(T(m_1, m_2), m_3) \\
&= 0.
\end{aligned}$$

The last equality is the first Bianchi identity for  $(T, R)$ . Lastly we still have to check that  $\mathfrak{S}^{m_1, m_2, m_3} [[m_1, m_2], m_3]_{\mathfrak{h} \oplus \mathfrak{k}} = 0$ . We have

$$\begin{aligned}
[[m_1, m_2], m_3]_{\mathfrak{h} \oplus \mathfrak{k}} &= [[m_1, m_2]_{\mathfrak{n} \oplus \mathfrak{m}}, m_3]_{\mathfrak{h} \oplus \mathfrak{k}} \\
&= -R_0(T(m_1, m_2), m_3) - R_\mathfrak{k}(T(m_1, m_2), m_3). \tag{2.1.16}
\end{aligned}$$

Consider the first component  $R_0(T(m_1, m_2), m_3)$ . It is easily seen that if at least two elements of  $m_1, m_2, m_3$  are in  $\mathfrak{n}$ , then  $R_0(T(m_1, m_2), m_3) = 0$ . If all elements of  $m_1, m_2, m_3$  are in  $\mathfrak{m}$ , then

$$\mathfrak{S}^{m_1, m_2, m_3} R_0(T(m_1, m_2), m_3) = \mathfrak{S}^{m_1, m_2, m_3} R_0(T_0(m_1, m_2), m_3) = 0$$

by the second Bianchi identity for  $(T_0, R_0)$ . The last case to consider is when exactly one of the elements of  $m_1, m_2, m_3$  is contained in  $\mathfrak{n}$ . Let  $m_1 \in \mathfrak{n}$ ,  $m_2, m_3 \in \mathfrak{m}$  and let  $k_1 \in \mathfrak{k}$  be the element corresponding to  $m_1$ . Then we have

$$\begin{aligned}
\mathfrak{S}^{m_1, m_2, m_3} R_0(T(m_1, m_2), m_3) &= R_0(\varphi(k_1)m_2, m_3) - R_0(\varphi(k_1)m_3, m_2) \\
&= R_0(\varphi(k_1)m_2, m_3) + R_0(m_2, \varphi(k_1)m_3) \\
&= -\varphi(k_1) \cdot R_0(m_2, m_3) = 0,
\end{aligned}$$

where the last two equalities follow because  $\varphi(k_1) \cdot R_0 = 0$  and  $\varphi(k_1)$  commutes with  $\text{im}(R_0) = \mathfrak{h}$ . For the second component  $R_{\mathfrak{k}}(T(m_1, m_2), m_3)$  of (2.1.16) we consider

$$\mathfrak{S}^{m_1, m_2, m_3}(\text{id} + \psi)([[m_1, m_2], m_3]_{\mathfrak{h} \oplus \mathfrak{k}}) = \mathfrak{S}^{m_1, m_2, m_3} R(T(m_1, m_2), m_3) = 0,$$

where the last equality is the second Bianchi identity for the pair  $(T, R)$ . We just saw that

$$\mathfrak{S}^{m_1, m_2, m_3} R_0(T(m_1, m_2), m_3) = 0$$

and thus we conclude that also  $\mathfrak{S}^{m_1, m_2, m_3} \psi(R_{\mathfrak{k}}(T(m_1, m_2), m_3)) = 0$ . Since  $\ker(\psi) = \{0\}$  we conclude that  $\mathfrak{S}^{m_1, m_2, m_3} R_{\mathfrak{k}}(T(m_1, m_2), m_3) = 0$ . In total we obtain  $\mathfrak{g}(\mathfrak{k})$  is Lie algebra.  $\square$

*Remark 2.1.17.* The constructed Lie algebra  $\mathfrak{g}(\mathfrak{k})$  is known as the double extension of  $\mathfrak{g}$  by  $\mathfrak{k}$ . This construction is used in [MR85] to describe the set of all Lie algebras which possess an invariant non-degenerate symmetric bilinear form.

Note that  $\mathfrak{g}(\mathfrak{k}) = \mathfrak{h} \oplus \mathfrak{k} \oplus \mathfrak{n} \oplus \mathfrak{m}$  defines a naturally reductive decomposition as in Definition 1.1.12, with  $\mathfrak{h} \oplus \mathfrak{k}$  as isotropy algebra. Whenever we construct a naturally reductive decomposition  $\mathfrak{g}(\mathfrak{k}) = \mathfrak{h} \oplus \mathfrak{k} \oplus \mathfrak{n} \oplus \mathfrak{m}$  from  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  we call  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  the *base space* and  $\mathfrak{g}(\mathfrak{k}) = \mathfrak{h} \oplus \mathfrak{k} \oplus \mathfrak{n} \oplus \mathfrak{m}$  the *total space*. In the next section we briefly discuss that if both the base space and the total space are regular, then the total space is a homogeneous fiber bundle over the base space and the fiber direction is  $\mathfrak{n}$ . Note that it is also possible to start from an infinitesimal model of a locally homogeneous space of which we can obtain a globally homogeneous  $(\mathfrak{k}, B)$ -extension, see Example 2.2.15.

## 2.2 Further investigation of the extensions

In this section we will investigate when the  $(\mathfrak{k}, B)$ -extensions  $(T, R)$  are regular. In the Subsections 2.2.1, 2.2.2 and 2.2.3 we will give explicit isometric transitive group actions for the  $(\mathfrak{k}, B)$ -extensions with particular base spaces. We describe the naturally reductive structure with respect to this group action. From now on  $G(\mathfrak{k})$  will denote the simply connected Lie group with Lie algebra  $\mathfrak{g}(\mathfrak{k})$  and  $H(\mathfrak{k})$  will be the connected subgroup with subalgebra  $\mathfrak{h}(\mathfrak{k})$ .

First we want to point out that the diagonal

$$\Delta \mathfrak{k} \subset \mathfrak{k} \oplus \mathfrak{k} \cong \mathfrak{k} \oplus \mathfrak{n} \subset \mathfrak{g}(\mathfrak{k})$$

is a non-trivial abelian ideal of  $\mathfrak{g}(\mathfrak{k})$ . In particular  $\mathfrak{g}(\mathfrak{k})$  is never semisimple. We will denote  $\Delta \mathfrak{k} \subset \mathfrak{k} \oplus \mathfrak{n}$  by  $\mathfrak{a}$ .

**Lemma 2.2.1.** *Let  $\mathfrak{g}(\mathfrak{k})$  be the Lie algebra from Definition 2.1.14. Then the following hold:*

- i)  $\mathfrak{a}$  commutes with  $\mathfrak{h} \oplus \mathfrak{m}$ ,*
- ii) the linear subspace  $\mathfrak{a} \subset \mathfrak{g}(\mathfrak{k})$  is an abelian ideal in  $\mathfrak{g}(\mathfrak{k})$ ,*
- iii)  $\mathfrak{l} := \mathfrak{h} \oplus \mathfrak{a} \oplus \mathfrak{m}$  is an ideal in  $\mathfrak{g}(\mathfrak{k})$ ,*
- iv)  $\mathfrak{g}(\mathfrak{k}) \cong \mathfrak{k} \ltimes (\mathfrak{h} \oplus \mathfrak{a} \oplus \mathfrak{m})$ .*

*Proof.* *i)* Let  $k + n \in \mathfrak{a}$  and  $m \in \mathfrak{m}$ . Then

$$\begin{aligned} [k + n, m] &= \varphi(k)m - T(n, m) - R_0(n, m) - R_{\mathfrak{k}}(n, m) \\ &= \varphi(k)m - T(n, m) = \varphi(k)m - \varphi(k)m = 0. \end{aligned}$$

We used that  $R_0(n, m) = 0$  and  $R_{\mathfrak{k}}(n, m) = 0$  for  $n \in \mathfrak{n}$  and  $m \in \mathfrak{m}$ . This follows directly from the symmetries of  $R_0$  and  $R_{\mathfrak{k}}$  and the  $(\mathfrak{h} \oplus \mathfrak{k})$ -invariance of the direct sum  $\mathfrak{n} \oplus \mathfrak{m}$ . By Definition 2.1.14 we have  $[\mathfrak{h}, \mathfrak{k} \oplus \mathfrak{n}] = \{0\}$ . In particular  $h$  commutes with  $\mathfrak{a}$ .

*ii)* Let  $n + k \in \mathfrak{a}$  and  $n' + k' \in \mathfrak{a}$ . Let  $[-, -]_{\mathfrak{n}}$  denote the Lie bracket in  $\mathfrak{n}$ . Then

$$\begin{aligned} [n', n + k] &= [n', n] + [n', k] = -2[n', n]_{\mathfrak{n}} - \sum_{i=1}^l B([k_i, n'], n)k_i + [n', k] \\ &= -[n', n]_{\mathfrak{n}} - \sum_{i=1}^l B([k_i, n'], n)k_i = -[n', n]_{\mathfrak{n}} - \sum_{i=1}^l B(k_i, [k', k])k_i \\ &= -[n', n]_{\mathfrak{n}} - [k', k] \in \mathfrak{a}. \end{aligned}$$

Furthermore, we have  $[k', n + k] = [n', n]_{\mathfrak{n}} + [k', k] \in \mathfrak{a}$ . In particular we see that  $[n' + k', n + k] = 0$ , thus  $\mathfrak{a}$  is abelian. We showed that  $[\mathfrak{k} \oplus \mathfrak{n}, \mathfrak{a}] \subset \mathfrak{a}$  and together with *i)* this implies that  $\mathfrak{a}$  is an ideal in  $\mathfrak{g}(\mathfrak{k})$ .

*iii)* We already know that  $[\mathfrak{h} \oplus \mathfrak{k}, \mathfrak{l}] \subset \mathfrak{l}$ . For  $n \in \mathfrak{n}$  and  $m \in \mathfrak{m}$  we have  $[n, m] = [k, m] \in \mathfrak{m}$  by *i)*, where  $k \in \mathfrak{k}$  is such that  $n + k \in \mathfrak{a}$ . This gives us that  $[\mathfrak{h} \oplus \mathfrak{k} \oplus \mathfrak{n}, \mathfrak{l}] \subset \mathfrak{l}$ . The only remaining thing to check is that if  $m_1, m_2 \in \mathfrak{m}$ , then  $[m_1, m_2] \in \mathfrak{h} \oplus \mathfrak{a} \oplus \mathfrak{m}$ . This is equivalent to  $[m_1, m_2]_{\mathfrak{k} \oplus \mathfrak{n}} \in \mathfrak{a}$ . A short computation yields

$$\begin{aligned} [m_1, m_2]_{\mathfrak{k} \oplus \mathfrak{n}} &= - \sum_{i=1}^l \varphi(k_i)(m_1, m_2)n_i + \psi(k_i)(m_1, m_2)k_i \\ &= - \sum_{i=1}^l \varphi(k_i)(m_1, m_2)(n_i + k_i) \in \mathfrak{a}. \end{aligned}$$

iv) This follows from iii) and the fact that  $\mathfrak{k} \subset \mathfrak{g}(\mathfrak{k})$  is a subalgebra.  $\square$

*Remark 2.2.2.* In many cases  $\mathfrak{g}(\mathfrak{k})$  is the transvection algebra of the  $(\mathfrak{k}, B)$ -extension  $(T, R)$  and by Lemma 2.2.1 this is not a reductive Lie algebra. In particular the naturally reductive structure is not induced from a normal homogeneous structure. It was pointed out to us by Nikonorov that the vectors in  $\mathfrak{a}$  constitute Killing vectors of constant length on  $G(\mathfrak{k})/H(\mathfrak{k})$ . This is proven in [Nik13].

**Lemma 2.2.3.** *Let  $(T, R)$  be some  $(\mathfrak{k}, B)$ -extension. The subgroup  $H(\mathfrak{k}) \subset G(\mathfrak{k})$  is closed if and only if the infinitesimal model  $(T, R)$  is regular.*

*Proof.* Suppose that  $H(\mathfrak{k})$  is closed. Note that  $\text{Ad}(H(\mathfrak{k}))(\mathfrak{n} \oplus \mathfrak{m}) \subset \mathfrak{n} \oplus \mathfrak{m}$ , because  $H(\mathfrak{k})$  is connected. The definition of  $\mathfrak{g}(\mathfrak{k})$  implies that the canonical connection of the reductive decomposition  $\mathfrak{h} \oplus \mathfrak{k} \oplus \mathfrak{n} \oplus \mathfrak{m}$ , where  $\mathfrak{h} \oplus \mathfrak{k}$  is the isotropy algebra, has infinitesimal model  $(T, R)$ . By [Tri92, Thm. 5.2] the infinitesimal model  $(T, R)$  is regular.

Let  $\bar{\mathfrak{g}}$  be the symmetry algebra of  $(T, R)$  as in (1.1.9) with  $\bar{\mathfrak{h}}$  as isotropy algebra. Let  $\bar{G}$  be the simply connected Lie group with Lie algebra  $\bar{\mathfrak{g}}$  and let  $\bar{H}$  be the connected subgroup with Lie subalgebra  $\bar{\mathfrak{h}}$ . The subgroup  $\bar{H}$  is closed in  $\bar{G}$  if and only if the infinitesimal model  $(T, R)$  is regular. Since  $\text{ad}(\mathfrak{h} \oplus \mathfrak{k}) \subset \bar{\mathfrak{h}}$  we get a Lie algebra homomorphism

$$q : \mathfrak{g}(\mathfrak{k}) \longrightarrow \text{ad}(\mathfrak{h} \oplus \mathfrak{k}) \oplus \mathfrak{n} \oplus \mathfrak{m} \subset \bar{\mathfrak{g}}; \quad h + k + n + m \mapsto \text{ad}(h + k) + n + m, \quad (2.2.4)$$

where the Lie bracket on  $\text{ad}(\mathfrak{h} \oplus \mathfrak{k}) \oplus \mathfrak{n} \oplus \mathfrak{m}$  is induced from (1.1.10). Since  $G(\mathfrak{k})$  is simply connected this induces a group homomorphism  $G(\mathfrak{k}) \rightarrow \bar{G}$ . If the infinitesimal model  $(T, R)$  is regular, then  $G(\mathfrak{k})$  acts transitively on  $\bar{G}/\bar{H}$  and the isotropy group  $D \subset G(\mathfrak{k})$  is closed in  $G(\mathfrak{k})$ . Note that  $\text{Lie}(D) = \mathfrak{h} \oplus \mathfrak{k}$ . Hence the connected component of the identity of  $D$ , which is equal to  $H(\mathfrak{k})$ , is closed in  $G(\mathfrak{k})$ . We conclude that  $H(\mathfrak{k})$  is closed in  $G(\mathfrak{k})$  if and only if the infinitesimal model  $(T, R)$  is regular.  $\square$

For now we assume that both the infinitesimal model  $(T_0, R_0)$  as well as the  $(\mathfrak{k}, B)$ -extension  $(T, R)$  are regular. Furthermore, we assume that the base space  $G/H$  is simply connected with  $\mathfrak{g} := \text{Lie}(G)$  which is the symmetry algebra of  $(T_0, R_0)$ . From Lemma 2.2.1 we know that  $\mathfrak{e} := \mathfrak{h} \oplus \mathfrak{k} \oplus \mathfrak{n}$  is a subalgebra of  $\mathfrak{g}(\mathfrak{k})$  and that  $\mathfrak{e} \oplus \mathfrak{m}$  is a reductive decomposition. Note that  $\text{ad} : \mathfrak{e} \rightarrow \mathfrak{so}(\mathfrak{m})$  maps into  $\{h \in \mathfrak{so}(\mathfrak{m}) : h \cdot T_0 = 0, h \cdot R_0 = 0\}$  by Proposition 2.1.7. This gives us a Lie algebra homomorphism

$$\mathfrak{e} \oplus \mathfrak{m} \rightarrow \text{ad}(\mathfrak{e}) \oplus \mathfrak{m} \subset \mathfrak{g}; \quad e + m \mapsto \text{ad}(e) + m.$$

Since  $G(\mathfrak{k})$  is simply connected we obtain an induced Lie group homomorphism  $G(\mathfrak{k}) \rightarrow G$ . Hence the group  $G(\mathfrak{k})$  acts on  $G/H$  by isometries. Let  $E \subset G(\mathfrak{k})$  be the stabilizer group of the origin of  $G/H$ . Then  $E$  a closed subgroup of  $G(\mathfrak{k})$ . We readily see that  $\text{Lie}(E) = \mathfrak{e}$ . This means we have a homogeneous fiber bundle which is a Riemannian submersion:

$$E/H(\mathfrak{k}) \rightarrow G(\mathfrak{k})/H(\mathfrak{k}) \rightarrow G(\mathfrak{k})/E \cong G/H.$$

The fibers  $E/H(\mathfrak{k})$  are connected by the long homotopy exact sequence and are described by the reductive decomposition

$$\mathfrak{h} \oplus \mathfrak{k} \oplus \mathfrak{n},$$

where  $\mathfrak{h} \oplus \mathfrak{k}$  is the isotropy algebra. This is clearly a naturally reductive decomposition. Let  $A$  be the connected subgroup of  $G(\mathfrak{k})$  with Lie algebra  $\mathfrak{a}$ . Note that  $A \subset E$  acts isometric and transitive on the fibers. Since  $A$  is an abelian Lie group the universal cover of  $E/H(\mathfrak{k})$  is isometric to the Euclidean space  $\mathbb{R}^l$ . The torsion  $T_f$  and curvature  $R_f$  of the naturally reductive connection on the fiber  $\mathbb{R}^l$  are given by

$$T_f = 2T_{\mathfrak{n}} \quad \text{and} \quad R_f = \sum_{i=1}^l \text{ad}(k_i) \odot \text{ad}(k_i), \quad (2.2.5)$$

where  $T_{\mathfrak{n}}$  is as in Definition 2.1.2.

*Remark 2.2.6.* Note that the infinitesimal model  $(T_f, R_f)$  on  $(\mathfrak{n}, B)$  is the  $(\mathfrak{k}, B)$ -extension of a point space. For this reason we defined  $\mathfrak{s}(\{0\}) = \mathfrak{so}(\infty)$ . By Theorem 2.1.13 this defines a naturally reductive structure on  $(\mathbb{R}^l, g_{eucl})$ , where  $l = \dim(\mathfrak{n})$ . In particular the infinitesimal model  $(T_f, R_f)$  is always regular by Lemma 1.1.15. The naturally reductive structure is irreducible precisely when  $\mathfrak{k}$  is a compact simple Lie algebra, see Lemma 3.3.19. In that case we have a 1-parameter family of naturally reductive structures parametrized by  $B = -\lambda B_{\mathfrak{k}}$ , where  $B_{\mathfrak{k}}$  denotes the Killing form of  $\mathfrak{k}$  and  $\lambda > 0$ .

## 2.2.1 Extensions with $\mathfrak{g}$ semisimple

In this subsection we will assume that  $\mathfrak{g}$  is a semisimple Lie algebra. Let  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  be a naturally reductive decomposition with  $\mathfrak{g}$  its transvection algebra and with infinitesimal model  $(T_0, R_0)$ , see Definition 1.1.12. Let  $\bar{g}$  be the  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear form on  $\mathfrak{g}$  from Kostant's theorem, see Theorem 1.1.16. Note that the non-degenerate symmetric bilinear form  $\bar{g}$  can have signature.

Since  $\mathfrak{g}$  is semisimple all derivations of  $\mathfrak{g}$  are inner derivations. We consider an inner derivation  $\text{ad}(h + m) \in \mathfrak{s}(\mathfrak{g})$ . For all  $h' \in \mathfrak{h}$  we have

$$0 = [h + m, h'] = [h, h'] + [m, h'].$$

This implies that  $h \in \mathfrak{z}$  and  $m \in \mathfrak{p}$ , where  $\mathfrak{z}$  is the center of  $\mathfrak{h}$  and  $\mathfrak{p}$  are all vectors on which  $\mathfrak{h}$  acts as zero:

$$\mathfrak{p} := \{m \in \mathfrak{m} : [h, m] = 0, \forall h \in \mathfrak{h}\}.$$

It is easy to see that  $\text{ad}(\mathfrak{z} \oplus \mathfrak{p}) \subset \mathfrak{s}(\mathfrak{g})$ . We conclude that  $\mathfrak{s}(\mathfrak{g})$  is given by

$$\mathfrak{s}(\mathfrak{g}) \cong \mathfrak{z} \oplus \mathfrak{p}.$$

*Remark 2.2.7.* The dimension of  $\mathfrak{p}$  appears in [ORT14] as the index of symmetry of the naturally reductive space  $G/H$ .

Let  $\mathfrak{k} = \mathfrak{k}_{\mathfrak{z}} \oplus \mathfrak{k}_{\mathfrak{p}} \subset \mathfrak{s}$  be a subalgebra with  $\mathfrak{k}_{\mathfrak{z}} \subset \mathfrak{z}$  and  $\mathfrak{k}_{\mathfrak{p}} \subset \mathfrak{p}$ . Let  $B$  be an  $\text{ad}(\mathfrak{k})$ -invariant metric on  $\mathfrak{k}$  for which  $\mathfrak{k}_{\mathfrak{z}} \perp \mathfrak{k}_{\mathfrak{p}}$ . In Proposition 3.3.14 we prove that we only have to consider subalgebras  $\mathfrak{k}$  with metrics of this form. We will denote the corresponding subalgebra of  $\mathfrak{k}$  in  $\mathfrak{g}$  by  $\mathfrak{b}$ . We denote the corresponding decompositions of  $\mathfrak{b}$ ,  $\mathfrak{n}$  and  $\mathfrak{a}$  by  $\mathfrak{b} = \mathfrak{b}_{\mathfrak{z}} \oplus \mathfrak{b}_{\mathfrak{p}}$ ,  $\mathfrak{n} = \mathfrak{n}_{\mathfrak{z}} \oplus \mathfrak{n}_{\mathfrak{p}}$  and  $\mathfrak{a} = \mathfrak{a}_{\mathfrak{z}} \oplus \mathfrak{a}_{\mathfrak{p}}$ , respectively. Let  $b_1, \dots, b_l$  be an orthonormal basis of  $\mathfrak{b}$  with respect to  $B$ . We denote the corresponding basis of  $\mathfrak{n}$  by  $n_1, \dots, n_l$  and of  $\mathfrak{k}$  by  $k_1, \dots, k_l$ . We define the following linear map

$$a : \mathfrak{g} \rightarrow \mathfrak{g}(\mathfrak{k}); \quad a(x) := \sum_{i=1}^l \bar{g}(x, b_i)(n_i + k_i).$$

With the help of the following lemma we will see that  $\mathfrak{g}$  is a subalgebra of  $\mathfrak{g}(\mathfrak{k})$ . It will always be clear from the context whether an element  $x \in \mathfrak{m}$  or  $x \in \mathfrak{h}$  belongs to  $\mathfrak{g}$  or  $\mathfrak{g}(\mathfrak{k})$ .

**Lemma 2.2.8.** *The linear map  $f : \mathfrak{g} \rightarrow \mathfrak{g}(\mathfrak{k})$  defined by*

$$f(x) = x - a(x)$$

*is an injective Lie algebra homomorphism.*

*Proof.* In the following we will denote the Lie bracket on  $\mathfrak{g}(\mathfrak{k})$  by  $[-, -]$  and the Lie bracket on  $\mathfrak{g}$  by  $[-, -]_{\mathfrak{g}}$ . Let  $h \in \mathfrak{h} \subset \mathfrak{g}$  and  $x \in \mathfrak{g}$ . Using that  $a(x) \in \mathfrak{a}$  for all  $x \in \mathfrak{g}$  it follows from Lemma 2.2.1 that

$$[f(h), f(x)] = [h - a(h), x - a(x)] = [h, x] = [h, x]_{\mathfrak{g}} = f([h, x]_{\mathfrak{g}}),$$

where the last equality follows because

$$\bar{g}([h, x], z + p) = -\bar{g}(x, [h, z + p]) = 0, \quad \forall z \in \mathfrak{z}, \forall p \in \mathfrak{p}.$$

If  $m_1, m_2 \in \mathfrak{m}$ , then

$$\begin{aligned} [f(m_1), f(m_2)] &= [m_1 - a(m_1), m_2 - a(m_2)] = [m_1, m_2] \\ &= -T(m_1, m_2) - R_0(m_1, m_2) - R_{\mathfrak{k}}(m_1, m_2) \\ &= -T_0(m_1, m_2) - R_0(m_1, m_2) - \sum_{i=1}^l \varphi(k_i)(m_1, m_2)(n_i + k_i) \\ &= [m_1, m_2]_{\mathfrak{g}} - \sum_{i=1}^l \bar{g}([b_i, m_1]_{\mathfrak{g}}, m_2)(n_i + k_i) \\ &= [m_1, m_2]_{\mathfrak{g}} - \sum_{i=1}^l \bar{g}([m_1, m_2]_{\mathfrak{g}}, b_i)(n_i + k_i) \\ &= f([m_1, m_2]_{\mathfrak{g}}). \end{aligned}$$

We conclude that  $f$  is an injective Lie algebra homomorphism.  $\square$

**Notation 2.2.9.** We will denote a direct sum of Lie algebras by  $\oplus_{L.a.}$  to emphasize that it is not just a direct sum of vector spaces.

Note that the ideal  $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{a} \oplus \mathfrak{m}$  from Lemma 2.2.1 *iii*) is equal to  $f(\mathfrak{g}) \oplus_{L.a.} \mathfrak{a}$ . The projection of

$$f(\mathfrak{g}) \oplus_{L.a.} \mathfrak{a}_{\mathfrak{p}} \subset \mathfrak{g}(\mathfrak{k})$$

along  $\mathfrak{h} \oplus \mathfrak{k}$  onto  $\mathfrak{n} \oplus \mathfrak{m}$  is surjective. We define an injective Lie algebra homomorphism

$$\phi := f \oplus i : \mathfrak{g} \oplus_{L.a.} \mathfrak{a}_{\mathfrak{p}} \rightarrow \mathfrak{g}(\mathfrak{k}),$$

where  $i : \mathfrak{a}_{\mathfrak{p}} \rightarrow \mathfrak{g}(\mathfrak{k})$  is the inclusion. Let  $l_{\mathfrak{p}} = \dim(\mathfrak{a}_{\mathfrak{p}})$  and  $\Phi : G \times \mathbb{R}^{l_{\mathfrak{p}}} \rightarrow G(\mathfrak{k})$  be the induced Lie group homomorphism on the simply connected Lie group  $G \times \mathbb{R}^{l_{\mathfrak{p}}}$ . Suppose that  $H(\mathfrak{k}) \subset G(\mathfrak{k})$  is closed, i.e.  $(T, R)$  is regular. Then  $G \times \mathbb{R}^{l_{\mathfrak{p}}}$  acts transitively on  $G(\mathfrak{k})/H(\mathfrak{k})$  through the map  $\Phi$ .

Let  $b_1^{\mathfrak{z}}, \dots, b_{l_{\mathfrak{z}}}^{\mathfrak{z}}$  be an orthonormal basis of  $\mathfrak{b}_{\mathfrak{z}}$  with respect to  $B$ . We have  $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}] \oplus \mathfrak{z}$  and  $[\mathfrak{h}, \mathfrak{h}] \perp \mathfrak{z}$  with respect to  $\bar{g}$ . Therefore, there exist elements  $h_1, \dots, h_{l_{\mathfrak{z}}}$  of  $\mathfrak{z}$  such that  $\bar{g}(h_i, b_j^{\mathfrak{z}}) = \delta_{ij}$ . Let  $\mathfrak{h}_0 := \ker(a) \cap \mathfrak{h}$  and  $\mathfrak{h}_1 := \text{span}\{h_1, \dots, h_{l_{\mathfrak{z}}}\}$ . Note that  $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$  and both  $\mathfrak{h}_0$  and  $\mathfrak{h}_1$  are ideals in  $\mathfrak{h}$ . The isotropy algebra for the action of  $G \times \mathbb{R}^{l_{\mathfrak{p}}}$  on  $G(\mathfrak{k})/H(\mathfrak{k})$  is

$$\phi^{-1}(\mathfrak{h} \oplus \mathfrak{k}) = \mathfrak{h}_0. \tag{2.2.10}$$

Let  $H_0$  be the connected subgroup of  $G$  with Lie algebra  $\mathfrak{h}_0$ . If the infinitesimal model  $(T, R)$  is regular, then  $H_0$  is a closed subgroup. Conversely, if  $H_0$  is closed, then the infinitesimal model  $(T, R)$  is regular by Lemma 1.1.15. The homogeneous space  $G(\mathfrak{k})/H(\mathfrak{k})$  can be presented by

$$G/H_0 \times \mathbb{R}^{l_p}. \quad (2.2.11)$$

Next we will describe the naturally reductive structure directly on the reductive decomposition associated to (2.2.11):

$$\mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{m} \oplus \mathfrak{a}_p \quad \text{and} \quad \mathfrak{h}_0 \text{ is the isotropy algebra.} \quad (2.2.12)$$

Let  $n_1^{\mathfrak{s}}, \dots, n_{l_3}^{\mathfrak{s}}$  be an orthonormal basis of  $\mathfrak{n}_3$  and let  $n_1^{\mathfrak{p}}, \dots, n_{l_p}^{\mathfrak{p}}$  be an orthonormal basis of  $\mathfrak{n}_p$ . Let  $m_1, \dots, m_n$  be an orthonormal basis of  $\mathfrak{m}$ . We know the formula for the torsion and curvature in this basis. Thus, all we have to do is find a basis  $f_1^{\mathfrak{s}}, \dots, f_{l_3}^{\mathfrak{s}}, f_1^{\mathfrak{p}}, \dots, f_{l_p}^{\mathfrak{p}}, e_1, \dots, e_n$  of

$$\mathfrak{h}_1 \oplus \mathfrak{m} \oplus \mathfrak{a}_p,$$

such that for the induced Killing vector fields the following hold:

$$\overline{f_i^{\mathfrak{s}}}(o) = \overline{n_i^{\mathfrak{s}}}(o), \quad \overline{f_i^{\mathfrak{p}}}(o) = \overline{n_i^{\mathfrak{p}}}(o) \quad \text{and} \quad \overline{e_j}(o) = \overline{m_j}(o), \quad (2.2.13)$$

where  $o$  is the chosen origin. Such a basis will give an orthonormal basis of the tangent space at the origin and thus describe the left invariant metric. Also in this basis the formula for the torsion and curvature of the naturally reductive connection are as in Definition 2.1.2. We define the following vectors:

$$\begin{aligned} f_i^{\mathfrak{p}} &:= n_i^{\mathfrak{p}} + k_i^{\mathfrak{p}} \in \mathfrak{a}_p, & \text{for } i = 1, \dots, l_p, \\ f_i^{\mathfrak{s}} &:= -h_i, & \text{for } i = 1, \dots, l_3, \\ e_i &:= m_i + a(m_i), & \text{for } i = 1, \dots, n. \end{aligned}$$

This basis satisfies (2.2.13). To illustrate this we consider

$$\overline{f_i^{\mathfrak{s}}}(o) = \overline{\phi(-h_i)}(o) = \overline{-h_i + n_i^{\mathfrak{s}} + k_i^{\mathfrak{s}}}(o) = \overline{n_i^{\mathfrak{s}}}(o).$$

Note that the two factors of  $G/H_0 \times \mathbb{R}^{l_p}$  are in general not orthogonal with respect to the naturally reductive metric. Alternatively, we can also describe the naturally reductive connection by a Nomizu map. The proof of Lemma 1.1.15 gives the explicit formula of this map for the reductive decomposition (2.2.12). Next we give two examples of the above discussion.

**Example 2.2.14.** In this example we will construct a 3-parameter family of naturally reductive structures on  $SU(2) \times SU(2) \times \mathbb{R}^3$ . The transvection algebra of the base space is given by

$$\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) = \mathfrak{m}.$$

Let  $\bar{g} := \frac{-1}{8\lambda_1^2} B_{\mathfrak{su}(2)} \oplus \frac{-1}{8\lambda_2^2} B_{\mathfrak{su}(2)}$  be the  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear form on  $\mathfrak{g}$ , where  $B_{\mathfrak{su}(2)}$  is the Killing form of  $\mathfrak{su}(2)$  and  $\lambda_1, \lambda_2 > 0$ . We pick the following basis of  $\mathfrak{su}(2)$ :

$$x_1 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad x_2 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad x_3 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

This basis satisfies:

$$[x_1, x_2] = -2x_3, \quad [x_2, x_3] = -2x_1, \quad [x_3, x_1] = -2x_2.$$

The following is an orthonormal basis of  $\mathfrak{g}$  with respect to  $\bar{g}$ :

$$m_i := \mu_1(x_i, x_i), \quad m_{i+3} := \mu_2(\lambda_1^2 x_i, -\lambda_2^2 x_i),$$

where  $\mu_1 := \frac{1}{\sqrt{\lambda_1^{-2} + \lambda_2^{-2}}}$ ,  $\mu_2 := \frac{1}{\sqrt{\lambda_1^2 + \lambda_2^2}}$  and  $i = 1, 2, 3$ . The infinitesimal model of the base space is  $(T_0, 0)$  with

$$T_0 = 2\mu_1(m_{123} + m_{156} + m_{264} + m_{345}) + 2(\lambda_1^2 - \lambda_2^2)\mu_2 m_{456}.$$

The Lie algebra  $\mathfrak{s}$  is given by  $\mathfrak{s} = \text{Der}(\mathfrak{g}) \cong \mathfrak{g}$ . Let  $\mathfrak{k}$  be the following subalgebra:

$$\mathfrak{k} \cong \mathfrak{b} = \text{span}\{m_1, m_2, m_3\}.$$

This corresponds to the diagonal subalgebra in  $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ . Let  $k_1, k_2, k_3$  be an orthonormal basis of  $\mathfrak{k}$  with respect to some  $\text{ad}(\mathfrak{k})$ -invariant metric  $B$ . Then there is some  $c > 0$  such that

$$\begin{aligned} \varphi(k_1) &= c \cdot \text{ad}(m_1) = -c\mu_1(m_{23} + m_{56}), \\ \varphi(k_2) &= c \cdot \text{ad}(m_2) = -c\mu_1(m_{31} + m_{64}), \\ \varphi(k_3) &= c \cdot \text{ad}(m_3) = -c\mu_1(m_{12} + m_{45}). \end{aligned}$$

The Lie algebra  $\mathfrak{g}(\mathfrak{k}) = \mathfrak{k} \oplus \mathfrak{n} \oplus \mathfrak{m}$  is isomorphic to  $\mathfrak{g}(\mathfrak{k}) \cong \mathfrak{k} \ltimes (\mathfrak{m} \oplus \mathfrak{a})$  by Lemma 2.2.1. Let  $f : \mathfrak{g} \rightarrow \mathfrak{m} \oplus \mathfrak{a}$  be the Lie algebra homomorphism from Lemma 2.2.8. This gives us that  $\mathfrak{g}(\mathfrak{k}) \cong \mathfrak{k} \ltimes (f(\mathfrak{g}) \oplus_{L.a.} \mathfrak{a})$ . By the discussion above,  $(T, R)$  is always regular and the connected Lie subgroup of  $f(\mathfrak{g}) \oplus_{L.a.} \mathfrak{a}$  acts transitively on our space.

Consequently, the naturally reductive space  $G(\mathfrak{k})/H(\mathfrak{k})$  can as a homogeneous space be presented as  $SU(2) \times SU(2) \times \mathbb{R}^3$ , where the Lie algebra of  $\mathbb{R}^3$  is

$$\mathfrak{a} = \text{span}\{f_1 := n_1 + k_1, f_2 := n_2 + k_2, f_3 := n_3 + k_3\} \subset \mathfrak{g}(\mathfrak{k}).$$

For  $i = 1, 2, 3$  we have

$$f(m_i) = m_i - \sum_{j=1}^3 \bar{g}(m_i, b_j)(n_j + k_j) = m_i - cf_i, \quad f(m_{i+3}) = m_{i+3}.$$

For  $i = 1, 2, 3$  let

$$e_i := m_i + cf_i, \quad e_{i+3} := m_{i+3}.$$

Then  $e_1, \dots, e_6, f_1, f_2, f_3$  spans  $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{a}$  and defines an orthonormal basis for the naturally reductive metric. The torsion is given by

$$T = T_0 + f_1 \wedge \varphi(k_1) + f_2 \wedge \varphi(k_2) + f_3 \wedge \varphi(k_3) - 4c\mu_1 f_{123}$$

with

$$T_0 = 2\mu_1 (e_{123} + e_{156} + e_{264} + e_{345}) + 2(\lambda_1^2 - \lambda_2^2)\mu_2 e_{456}.$$

The curvature is given by

$$R = \sum_{i=1}^3 \psi(k_i) \odot \psi(k_i),$$

where

$$\psi(k_i) = \varphi(k_i) - \frac{1}{2} \sum_{j,k=1}^3 2c\mu_1 \varepsilon_{ijk} f_{jk}, \quad i = 1, 2, 3,$$

and  $\varepsilon_{ijk}$  is the Levi-Civita symbol. It is important to note that, even though the homogeneous space is a product, the constructed naturally reductive structures can not be written as products, see Lemma 3.3.19. By permuting the  $\mathfrak{su}(2)$ -factors we assume that  $\lambda_1 \geq \lambda_2$ . Under this extra condition all of the naturally reductive structures are non-isomorphic, see Proposition 3.3.16. Hence we constructed a 3-parameter family of naturally reductive structures on  $SU(2) \times SU(2) \times \mathbb{R}^3$ , parametrized by  $\lambda_1, \lambda_2$  and  $c$ .

Now we give a low dimensional example were the model for the base space is not regular but we still construct a regular extension of it.

**Example 2.2.15.** Let  $x_1, x_2, x_3$  be as in Example 2.2.14. Let  $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$  and

$$\mathfrak{h} := \text{span}\{(x_1, \alpha x_2)\},$$

where  $\alpha$  is an irrational number. Let  $\bar{g} := \frac{-1}{8\lambda_1^2} B_{\mathfrak{su}(2)} \oplus \frac{-1}{8\lambda_2^2} B_{\mathfrak{su}(2)}$ . Kostant's theorem tells us that  $\mathfrak{h} \oplus \mathfrak{m}$  is a naturally reductive decomposition and this defines an infinitesimal model  $(T_0, R_0)$ . Note that  $\mathfrak{g}$  is the transvection algebra and that the connected subgroup  $H$  of  $G$  with Lie subalgebra  $\mathfrak{h}$  is not closed. By [Kow90] this infinitesimal model is not regular. If  $\mathfrak{k} := \mathfrak{h}$ , then (2.2.10) together with Lemma 1.1.15 tell us that for any metric  $B$  the  $(\mathfrak{k}, B)$ -extension is regular and defines a naturally reductive structure on  $SU(2) \times SU(2)$ .

## 2.2.2 Extensions with $\mathfrak{g} = \mathbb{R}^n$

If we take the symmetric space  $\mathbb{R}^n$  as base space for a  $(\mathfrak{k}, B)$ -extension, we obtain naturally reductive structures on 2-step nilpotent Lie groups. These structures were first described in [Gor85]. As naturally reductive decomposition of the base space we have

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{m} = \mathbb{R}^n,$$

where  $\mathbb{R}^n$  is an abelian Lie algebra, thus  $(T_0, R_0) = (0, 0)$ . The algebra  $\mathfrak{s}$  is given by

$$\mathfrak{s} = \mathfrak{so}(n).$$

Let  $\mathfrak{k} \subset \mathfrak{s}$  be a subalgebra. The torsion and curvature of the  $(\mathfrak{k}, B)$ -extension are given by

$$T = \sum_{i=1}^l \varphi(k_i) \wedge n_i + 2T_{\mathfrak{n}} \quad \text{and} \quad R = \sum_{i=1}^l \psi(k_i) \odot \psi(k_i).$$

By Lemma 2.2.1 the Lie algebra

$$\mathfrak{g}(\mathfrak{k}) = \mathbb{R}^n(\mathfrak{k}) = \mathfrak{k} \oplus \mathfrak{n} \oplus \mathbb{R}^n$$

has the following ideal

$$\mathfrak{l} := \mathfrak{a} \oplus \mathbb{R}^n.$$

For  $m_1, m_2 \in \mathbb{R}^n$  we have  $[m_1, m_2] \in \mathfrak{a}$ , because  $T_0 = 0$  By Lemma 2.2.1  $\mathfrak{a}$  commutes with  $\mathbb{R}^n$ . Hence  $\mathfrak{l}$  is a 2-step nilpotent Lie algebra. This gives us a naturally reductive structure on the 2-step nilpotent Lie group  $L$ , where  $L$  is the simply connected Lie group with  $\text{Lie}(L) = \mathfrak{l}$ . In particular the infinitesimal model  $(T, R)$  is always regular by Lemma 1.1.15. For this case it is easy to describe an orthonormal basis of the

tangent space at the origin. Let  $n_1, \dots, n_l$  be an orthonormal basis of  $\mathfrak{n}$  and let  $m_1, \dots, m_n$  be an orthonormal basis of  $\mathbb{R}^n$ . Then  $f_i := n_i + k_i \in \mathfrak{a}$  for  $i = 1, \dots, l$  and  $m_1, \dots, m_n \in \mathbb{R}^n$  form a basis of  $\mathfrak{l}$  such that the induced Killing vector fields span an orthonormal basis of the tangent space at the origin. In this basis the formula for the torsion and curvature are given by the formulas above. The Nomizu map of the naturally reductive connection takes a particularly simple form for these spaces. It is given by  $\Lambda(m_j) = 0$  and  $\Lambda(f_i) = \psi(k_i)$ , where we made the natural identification  $\mathfrak{so}(\mathfrak{n} \oplus \mathbb{R}^n) \cong \mathfrak{so}(\mathfrak{a} \oplus \mathbb{R}^n)$ . To illustrate how this works we give a concrete example.

**Example 2.2.16.** We start with  $\mathbb{R}^4$  as base space and let  $\mathfrak{k} = \mathfrak{su}(2) \subset \mathfrak{so}(4)$  be the subalgebra corresponding to the standard representation of  $\mathfrak{su}(2)$  on  $\mathbb{C}^2 \cong \mathbb{R}^4$ . The  $\text{ad}(\mathfrak{k})$ -invariant metric  $B$  on  $\mathfrak{k}$  has to be a negative multiple of the Killing form  $B := -\frac{1}{8\lambda^2} B_{\mathfrak{su}(2)}$ . In a natural basis we get

$$\begin{aligned}\varphi(k_1) &= \lambda(e_{12} - e_{34}), \\ \varphi(k_2) &= \lambda(e_{13} + e_{24}), \\ \varphi(k_3) &= \lambda(e_{14} - e_{23}),\end{aligned}$$

where  $k_1, k_2, k_3$  is an orthonormal basis with respect to  $B$ . The formula for the torsion is

$$T = \lambda(e_{12} - e_{34}) \wedge f_1 + \lambda(e_{13} + e_{24}) \wedge f_2 + \lambda(e_{14} - e_{23}) \wedge f_3 - 4\lambda f_{123},$$

and the curvature is

$$R = \lambda^2 \left( (-2f_{23} + e_{12} - e_{34})^{\odot 2} + (-2f_{31} + e_{13} + e_{24})^{\odot 2} + (-2f_{12} + e_{14} - e_{23})^{\odot 2} \right).$$

By the discussion above this defines a naturally reductive structure on a 2-step nilpotent Lie group, which is known as the 7-dimensional quaternionic Heisenberg group. We will denote it by  $QH^7$ . On this space we have a 1-parameter family of naturally reductive structures. The higher dimensional quaternionic Heisenberg groups  $QH^{4n+3}$  are in the same way obtained from the representation

$$\bigoplus^n \varphi : \mathfrak{su}(2) \rightarrow \text{End}(\mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \dots \oplus \mathbb{C}^2).$$

More about the geometry of the quaternionic Heisenberg group can be read in [\[AFS15\]](#).

For a Lie subalgebra  $\mathfrak{k} \subset \mathfrak{so}(n)$  defined by a representation  $\varphi : \mathfrak{k} \rightarrow \mathfrak{so}(n)$  we will denote the 2-step nilpotent Lie algebra  $\mathfrak{l}$  obtained above by  $\mathfrak{nil}(\mathfrak{k})$  or by  $\mathfrak{nil}(\varphi)$  and the corresponding simply connected Lie group by  $Nil(\mathfrak{k})$  or  $Nil(\varphi)$ .

### 2.2.3 Extensions with $\mathfrak{g}$ a reductive Lie algebra

In this subsection the base space is a product of the base spaces considered in the last two subsections:

$$G/H \times \mathbb{R}^k.$$

As before, we suppose that  $\mathfrak{g}$  is semisimple and that  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  is a naturally reductive decomposition with  $\mathfrak{g}$  its transvection algebra. Let  $\bar{g}$  be the  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear form on  $\mathfrak{g}$  from Kostant's theorem, see Theorem 1.1.16. We want to describe the naturally reductive spaces which are constructed from derivations of  $\mathfrak{g}' := \mathfrak{g} \oplus_{L.a.} \mathbb{R}^k$ . Any such derivation preserves the direct sum:

$$\mathfrak{s}(\mathfrak{g}') = \mathfrak{s}(\mathfrak{g}) \oplus_{L.a.} \mathfrak{so}(k).$$

Let  $\mathfrak{k} \subset \mathfrak{s}(\mathfrak{g}')$  be a subalgebra with an  $\text{ad}(\mathfrak{k})$ -invariant metric  $B$ . Let  $\varphi_1 : \mathfrak{k} \rightarrow \mathfrak{so}(\mathfrak{m})$  and  $\varphi_2 : \mathfrak{k} \rightarrow \mathfrak{so}(\mathbb{R}^k)$  be the Lie algebra representations defining  $\mathfrak{k}$ . Let  $\mathfrak{k}_1 := \ker(\varphi_2)$ ,  $\mathfrak{k}_3 := \ker(\varphi_1)$  and let  $\mathfrak{k}_2$  be the orthogonal complement of  $\mathfrak{k}_1 \oplus \mathfrak{k}_3$  in  $\mathfrak{k}$  with respect to  $B$ . The projection  $\pi : \mathfrak{s}(\mathfrak{g}') \rightarrow \mathfrak{s}(\mathfrak{g})$  along  $\mathfrak{so}(k)$  restricted to  $\mathfrak{k}_1 \oplus \mathfrak{k}_2$  is injective. We use  $\pi$  to identify  $\mathfrak{k}_1 \oplus \mathfrak{k}_2$  with a subalgebra  $\mathfrak{b}_1 \oplus \mathfrak{b}_2$  of  $\mathfrak{z} \oplus \mathfrak{p} \cong \mathfrak{s}(\mathfrak{g})$ , where  $\pi(\mathfrak{k}_i) = \mathfrak{b}_i$  for  $i = 1, 2$ . As in Subsection 2.2.1 we suppose that  $\mathfrak{b}_1 = \mathfrak{b}_{1,\mathfrak{z}} \oplus \mathfrak{b}_{1,\mathfrak{p}}$  and  $\mathfrak{b}_{1,\mathfrak{z}} \perp \mathfrak{b}_{1,\mathfrak{p}}$  with respect to  $B$ , where  $\mathfrak{b}_{1,\mathfrak{z}} \subset \mathfrak{z}$  and  $\mathfrak{b}_{1,\mathfrak{p}} \subset \mathfrak{p}$ . Proposition 3.3.14 implies that we do not obtain more space with out this restriction. In total we now have the following decomposition into ideals

$$\mathfrak{k} = \mathfrak{k}_{1,\mathfrak{z}} \oplus \mathfrak{k}_{1,\mathfrak{p}} \oplus \mathfrak{k}_2 \oplus \mathfrak{k}_3,$$

with  $\mathfrak{k}_1 = \mathfrak{k}_{1,\mathfrak{z}} \oplus \mathfrak{k}_{1,\mathfrak{p}}$ . We define

$$a : \mathfrak{g} \rightarrow \mathfrak{g}'(\mathfrak{k}); \quad a(x) := \sum_{i=1}^l \bar{g}(x, b_i)(n_i + k_i),$$

where  $b_1, \dots, b_l$  is an orthonormal basis of  $\mathfrak{b}_1 \oplus \mathfrak{b}_2$  with respect to  $B$  and  $n_i, k_i$  are the corresponding bases of  $\mathfrak{n}_1 \oplus \mathfrak{n}_2$  and  $\mathfrak{k}_1 \oplus \mathfrak{k}_2$ . We get analogue to Lemma 2.2.8 the following result.

**Lemma 2.2.17.** *The linear map  $f : \mathfrak{g} \rightarrow \mathfrak{g}'(\mathfrak{k})$  defined by*

$$f(x) = x - a(x)$$

*is an injective Lie algebra homomorphism.*

Let the following be orthonormal bases with respect to  $B$ :

$$\text{span}\{b_1^{1,\mathfrak{z}}, \dots, b_{l_{1,\mathfrak{z}}}^{1,\mathfrak{z}}\} = \mathfrak{b}_{1,\mathfrak{z}}, \quad \text{span}\{b_1^{1,\mathfrak{p}}, \dots, b_{l_{1,\mathfrak{p}}}^{1,\mathfrak{p}}\} = \mathfrak{b}_{1,\mathfrak{p}}.$$

Furthermore, let  $b_1^{1,\mathfrak{z}}, \dots, b_{l_{1,\mathfrak{z}}}^{1,\mathfrak{z}}, b_1^{2,\mathfrak{z}}, \dots, b_{l_{2,\mathfrak{z}}}^{2,\mathfrak{z}}$  be an orthonormal basis of

$$(\mathfrak{p} \cap (\mathfrak{b}_1 \oplus \mathfrak{b}_2))^\perp$$

with respect to  $B$ . We define  $\mathfrak{b}_{2,\mathfrak{z}} := \text{span}\{b_1^{2,\mathfrak{z}}, \dots, b_{l_{2,\mathfrak{z}}}^{2,\mathfrak{z}}\} \subset \mathfrak{b}_2$ . Let  $n_i^\bullet, k_i^\bullet, a_i^\bullet$  be the corresponding bases of  $\mathfrak{n}_\bullet, \mathfrak{k}_\bullet, \mathfrak{a}_\bullet$  with  $\bullet$  replaced by one of the above superscripts.

Note that the vectors  $b_1^{1,\mathfrak{z}}, \dots, b_{l_{1,\mathfrak{z}}}^{1,\mathfrak{z}}, b_1^{2,\mathfrak{z}}, \dots, b_{l_{2,\mathfrak{z}}}^{2,\mathfrak{z}}$  are linearly independent. We chose the  $\mathfrak{b}_i^{2,\mathfrak{z}}$  such that  $\pi_{\mathfrak{h}}(b_1^{1,\mathfrak{z}}), \dots, \pi_{\mathfrak{h}}(b_{l_{1,\mathfrak{z}}}^{1,\mathfrak{z}}), \pi_{\mathfrak{h}}(b_1^{2,\mathfrak{z}}), \dots, \pi_{\mathfrak{h}}(b_{l_{2,\mathfrak{z}}}^{2,\mathfrak{z}})$  is a basis of  $\pi_{\mathfrak{h}}(\mathfrak{b}_1 \oplus \mathfrak{b}_2)$ , where  $\pi_{\mathfrak{h}} : \mathfrak{g} \rightarrow \mathfrak{h}$  is the projection along  $\mathfrak{m}$ . Just as before, let  $h_1^{i,\mathfrak{z}}, \dots, h_{l_{i,\mathfrak{z}}}^{i,\mathfrak{z}}$  be elements of  $\mathfrak{z}$  such that  $\bar{g}(h_n^{i,\mathfrak{z}}, b_m^{j,\mathfrak{z}}) = \delta_{nm} \delta_{ij}$  for  $i = 1, 2$ . Let  $\mathfrak{h}_{i,\mathfrak{z}} := \text{span}\{h_1^{i,\mathfrak{z}}, \dots, h_{l_{i,\mathfrak{z}}}^{i,\mathfrak{z}}\}$ . Lastly set  $\mathfrak{h}_0 := \ker(a) \cap \mathfrak{h}$ . Then we have a decomposition of  $\mathfrak{h}$  into ideals:

$$\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_{1,\mathfrak{z}} \oplus \mathfrak{h}_{2,\mathfrak{z}}.$$

**Lemma 2.2.18.** *The algebras  $\text{nil}(\mathfrak{k}_2 \oplus \mathfrak{k}_3) = \mathfrak{a}_2 \oplus \mathfrak{a}_3 \oplus \mathbb{R}^k$  and  $\mathfrak{a}_{1,\mathfrak{p}}$  are subalgebras of  $\mathfrak{g}'(\mathfrak{k})$ , they commute with each other and with  $f(\mathfrak{g})$ . They both have zero intersection with each other and with  $f(\mathfrak{g})$ , i.e.:*

$$f(\mathfrak{g}) \oplus_{L.a.} \text{nil}(\mathfrak{k}_2 \oplus \mathfrak{k}_3) \oplus_{L.a.} \mathfrak{a}_{1,\mathfrak{p}} \subset \mathfrak{g}'(\mathfrak{k}).$$

Moreover, the projection of  $f(\mathfrak{g}) \oplus \text{nil}(\mathfrak{k}_2 \oplus \mathfrak{k}_3) \oplus \mathfrak{a}_{1,\mathfrak{p}}$  along  $\mathfrak{h} \oplus \mathfrak{k}$  onto  $\mathfrak{n} \oplus \mathfrak{m} \oplus \mathbb{R}^k \subset \mathfrak{g}'(\mathfrak{k})$  is surjective.

*Proof.* By Lemma 2.2.17  $f(\mathfrak{g})$  is a subalgebra. By Lemma 2.2.1 we know that  $\mathfrak{a}_{1,\mathfrak{p}}, \mathfrak{a}_2$  and  $\mathfrak{a}_3$  are subalgebras of  $\mathfrak{g}'(\mathfrak{k})$ . We have  $[\mathbb{R}^k, \mathbb{R}^k] \subset \mathfrak{a}_2 \oplus \mathfrak{a}_3$  and thus  $\mathfrak{a}_2 \oplus \mathfrak{a}_3 \oplus \mathbb{R}^k$  is a subalgebra. The pairwise intersections and commutators of these three subalgebras clearly vanish. The projection of  $\text{nil}(\mathfrak{k}_2 \oplus \mathfrak{k}_3)$  onto  $\mathfrak{n} \oplus \mathfrak{m} \oplus \mathbb{R}^k$  is equal to  $\mathfrak{n}_2 \oplus \mathfrak{n}_3 \oplus \mathbb{R}^k$ . The projection of  $f(\mathfrak{g}) \oplus \mathfrak{a}_{1,\mathfrak{p}}$  onto  $\mathfrak{n} \oplus \mathfrak{m} \oplus \mathbb{R}^k$  is equal to  $\mathfrak{n}_1 \oplus \mathfrak{m}$ .  $\square$

If  $H'(\mathfrak{k}) \subset G'(\mathfrak{k})$  is closed, then the connected subgroup of  $G'(\mathfrak{k})$  with Lie subalgebra  $f(\mathfrak{g}) \oplus_{L.a.} \text{nil}(\mathfrak{k}_2 \oplus \mathfrak{k}_3) \oplus_{L.a.} \mathfrak{a}_{1,\mathfrak{p}}$  acts transitively on the homogeneous space  $G'(\mathfrak{k})/H'(\mathfrak{k})$  by the above lemma. As before let

$$\phi := f \oplus i : \mathfrak{g} \oplus \text{nil}(\mathfrak{k}_2 \oplus \mathfrak{k}_3) \oplus \mathfrak{a}_{1,\mathfrak{p}} \rightarrow \mathfrak{g}'(\mathfrak{k}),$$

where  $i : \text{nil}(\mathfrak{k}_2 \oplus \mathfrak{k}_3) \oplus \mathfrak{a}_{1,\mathfrak{p}} \rightarrow \mathfrak{g}'(\mathfrak{k})$  is the inclusion. Let  $G \times \text{Nil}(\mathfrak{k}_2 \oplus \mathfrak{k}_3) \times A_{1,\mathfrak{p}}$  be the simply connected Lie group with Lie algebra  $\mathfrak{g} \oplus \text{nil}(\mathfrak{k}_2 \oplus \mathfrak{k}_3) \oplus \mathfrak{a}_{1,\mathfrak{p}}$ . Let

$$\Phi : G \times \text{Nil}(\mathfrak{k}_2 \oplus \mathfrak{k}_3) \times A_{1,\mathfrak{p}} \rightarrow G'(\mathfrak{k})$$

be the Lie group homomorphism with its derivative at the identity given by  $\phi$ . The isotropy algebra of  $G \times Nil(\mathfrak{k}_2 \oplus \mathfrak{k}_3) \times A_{1,p}$  is

$$\phi^{-1}(\mathfrak{h} \oplus \mathfrak{k}) = \mathfrak{h}_0 \oplus \phi^{-1}(\mathfrak{b}_{2,3}).$$

We denote  $\phi^{-1}(\mathfrak{b}_{2,3})$  by

$$\tilde{\mathfrak{b}}_{2,3} := \phi^{-1}(\mathfrak{b}_{2,3}) = \text{span}\{h_i^{2,3} + n_i^{2,3} + k_i^{2,3} : i = 1, \dots, l_{2,3}\}. \quad (2.2.19)$$

Let  $H_0$  be the connected subgroup of  $G$  with Lie subalgebra  $\mathfrak{h}_0$  and let  $B_{2,3}$  be the connected subgroup of  $G \times Nil(\mathfrak{k}_2 \oplus \mathfrak{k}_3)$  with Lie subalgebra  $\mathfrak{b}_{2,3}$ . Note that  $B_{2,3}$  is always a closed subgroup and is isomorphic to  $\mathbb{R}^{l_{2,3}}$ . It follows that

$$G'(\mathfrak{k})/H'(\mathfrak{k}) \cong (G \times Nil(\mathfrak{k}_2 \oplus \mathfrak{k}_3) \times A_{1,p})/(H_0 \times B_{2,3}).$$

The  $(\mathfrak{k}, B)$ -extension  $(T, R)$  is regular precisely when  $H_0$  is closed in  $G$  by Lemma 1.1.15. Now we describe the naturally reductive structure on the following reductive decomposition associated to the action of  $G \times Nil(\mathfrak{k}_2 \oplus \mathfrak{k}_3) \times A_{1,p}$ :

$$\mathfrak{h}_0 \oplus \tilde{\mathfrak{b}}_{2,3} \oplus \mathfrak{b}_{1,3} \oplus \mathfrak{m} \oplus \mathfrak{a}_2 \oplus \mathfrak{a}_3 \oplus \mathbb{R}^k \oplus \mathfrak{a}_{1,p} \subset \mathfrak{g}'(\mathfrak{k}),$$

where  $\mathfrak{h}_0 \oplus \tilde{\mathfrak{b}}_{2,3}$  is the isotropy algebra. Let  $n_1^3, \dots, n_{l_3}^3$  be an orthonormal basis of  $\mathfrak{n}_3$ . Let  $n_1^{2,3}, \dots, n_{l_{2,3}}^{2,3}, n_1^{2,p}, \dots, n_{l_{2,p}}^{2,p}$  be an orthonormal basis of  $\mathfrak{n}_2$ , with  $n_i^{2,3}$  as before. Let  $m_1, \dots, m_n$  be an orthonormal basis of  $\mathfrak{m}$ . Lastly let  $m_{n+1}, \dots, m_{n+k}$  be an orthonormal basis of  $\mathbb{R}^k$ . We know the formula for the metric, torsion and curvature in this basis. Thus all we have to do is to find a basis

$$f_1^{i,3}, \dots, f_{l_{i,3}}^{i,3}, f_1^{i,p}, \dots, f_{l_{i,p}}^{i,p}, f_1^3, \dots, f_{l_3}^3, e_1, \dots, e_{n+k}$$

of

$$\mathfrak{b}_{1,3} \oplus \mathfrak{m} \oplus \mathfrak{a}_2 \oplus \mathfrak{a}_3 \oplus \mathbb{R}^k \oplus \mathfrak{a}_{1,p},$$

such that for the induced Killing vector fields the following holds:

$$\overline{f_j^{i,3}}(o) = \overline{n_j^{i,3}}(o), \quad \overline{f_j^{i,p}}(o) = \overline{n_j^{i,p}}(o), \quad \overline{f_j^3}(o) = \overline{n_j^3}(o), \quad \overline{e_j}(o) = \overline{m_j}(o), \quad (2.2.20)$$

where  $o$  is the chosen origin. This gives us a correspondence

$$\mathfrak{b}_{1,3} \oplus \mathfrak{a}_{1,p} \oplus \mathfrak{a}_2 \oplus \mathfrak{a}_3 \oplus \mathfrak{m} \oplus \mathbb{R}^k \cong \mathfrak{n} \oplus \mathfrak{m} \oplus \mathbb{R}^k,$$

such that the metric, torsion and curvature tensor correspond to how we defined them on  $\mathfrak{n} \oplus \mathfrak{m} \oplus \mathbb{R}^n$  in Definition 2.1.2. We define the following vectors:

$$\begin{aligned}
f_i^{1,\mathfrak{h}} &:= -h_i^{1,\mathfrak{h}}, & \text{for } i = 1, \dots, l_{1,\mathfrak{h}}, \\
f_i^{2,z} &:= n_i^{2,z} + k_i^{2,z} \in \mathfrak{a}_{2,z}, & \text{for } i = 1, \dots, l_{2,\mathfrak{h}}, \\
f_i^{j,\mathfrak{p}} &:= n_i^{j,\mathfrak{p}} + k_i^{j,\mathfrak{p}} \in \mathfrak{a}_{j,\mathfrak{p}}, & \text{for } i = 1, \dots, l_{j,\mathfrak{p}} \text{ and } j = 1, 2, \\
f_i^3 &:= n_i^3 + k_i^3 \in \mathfrak{a}_3, & \text{for } i = 1, \dots, l_3, \\
e_i &:= m_i + a(m_i), & \text{for } i = 1, \dots, n, \\
e_i &:= m_i \in \mathbb{R}^k, & \text{for } i = n + 1, \dots, n + k.
\end{aligned}$$

Just as in Subsection 2.2.1 this basis satisfies (2.2.20). Alternatively, the naturally reductive connection can also be described by the Nomizu map, which can be read off from the proof of Lemma 1.1.15.

*Remark 2.2.21.* Note that  $\mathcal{Z}(\mathfrak{k}_1)$  doesn't have to be perpendicular to  $\mathcal{Z}(\mathfrak{k}_3)$ .

A simple example of one of these spaces is in [KV85]. Where they describe the spaces  $(SU(2) \times H^3)/\mathbb{R}$ , where  $H^3$  is a simply connected 3-dimensional Heisenberg group. In the following example we show how we can obtain these spaces from our construction.

**Example 2.2.22.** As base space we take  $S^2 \times \mathbb{R}^2$ . The first factor is the symmetric space  $S^2 = SU(2)/S^1$ . Let  $h, e_1, e_2$  be an orthonormal basis of  $\mathfrak{su}(2)$  with respect to  $\frac{-1}{8\lambda^2} B_{\mathfrak{su}(2)}$ . The transvection algebra of the base space is given by

$$\mathfrak{g}' = \mathfrak{su}(2) \oplus \mathbb{R}^2 = \mathfrak{h} \oplus \mathfrak{m} \oplus_{L.a.} \mathbb{R}^2,$$

where  $\mathfrak{h} := \text{span}\{h\}$  and  $\mathfrak{m} := \text{span}\{e_1, e_2\}$ . Let  $e_3, e_4$  be an orthonormal basis of  $\mathbb{R}^2$ . The  $\text{ad}(\mathfrak{g}')$ -invariant non-degenerate symmetric bilinear form on  $\mathfrak{g}'$  is  $\bar{g} = \frac{-1}{8\lambda^2} B_{\mathfrak{su}(2)} \oplus B_{eucl}$ . The infinitesimal model is given by  $(T_0, R_0) = (0, -\text{ad}(h) \odot \text{ad}(h))$ . The Lie algebra  $\mathfrak{s}$  is given by  $\mathfrak{s} = \text{span}\{h\} \oplus \mathfrak{so}(2)$ . We pick an element  $k = (c_1 h, c_2 e_{34}) \in \mathfrak{s}$ , with  $c_1 \neq 0, c_2 \neq 0$ . Let  $\mathfrak{k}$  be the 1-dimensional Lie algebra spanned by  $k$  and with a metric such that  $k$  has norm 1. Then

$$\varphi(k) = c_1 \text{ad}(h) + c_2 e_{34}.$$

In this case  $\mathfrak{k} = \mathfrak{k}_{2,3}$ . By Lemma 2.2.18 the connected Lie subgroup  $SU(2) \times H^3$  of  $G'(\mathfrak{k})$  with  $\text{Lie}(SU(2) \times H^3) = \mathfrak{su}(2) \oplus \mathfrak{a} \oplus \mathbb{R}^2$  acts transitively and by isometries, where  $H^3$  is the simply connected 3-dimensional Heisenberg group. Let  $e_3, e_4, f$  be a basis of  $\mathfrak{h}^3 = \mathfrak{a} \oplus \mathbb{R}^2$ , with  $f := n + k$ . The only non-vanishing Lie bracket in  $\mathfrak{h}^3$  is

$$[e_3, e_4] = -c_2 f.$$

The isotropy algebra for the action of  $SU(2) \times H^3$  is

$$\phi^{-1}(\mathfrak{h} \oplus \mathfrak{k}) = \tilde{\mathfrak{b}}_{2,3} = \text{span}\{h + c_1 f\}.$$

As reductive complement of  $\tilde{\mathfrak{b}}_{2,3}$  in  $\mathfrak{su}(2) \oplus \mathfrak{h}^3$  we have

$$\mathfrak{a} \oplus \mathfrak{m} \oplus \mathbb{R}^2 = \text{span}\{f, e_1, e_2, e_3, e_4\}.$$

From the description above  $\bar{f}(o), \bar{e}_1(o), \bar{e}_2(o), \bar{e}_3(o), \bar{e}_4(o)$  is an orthonormal basis at the origin and the torsion and curvature in this basis are given by

$$T = f \wedge (c_1 \text{ad}(h) + c_2 e_{34}), \quad \text{and} \quad R = -\text{ad}(h) \odot \text{ad}(h) + \varphi(k) \odot \varphi(k).$$

The isotropy group is the connected Lie subgroup with Lie subalgebra  $\tilde{\mathfrak{b}}_{2,3}$ . It is isomorphic to  $\mathbb{R}$  and it is closed for all  $c_1, c_2 \in \mathbb{R} \setminus \{0\}$ . So we obtain a 3-parameter family of naturally reductive structures on

$$(SU(2) \times H^3)/\mathbb{R}.$$

The parameters are  $\lambda, c_1, c_2$ .

We give an example which is a bit more interesting and which illustrates better how the  $(\mathfrak{k}, B)$ -extensions can look like.

**Example 2.2.23.** As base space we take the product of a particular Aloff-Wallach space  $SU(3)/S^1$  and  $\mathbb{R}^4$ . We consider the following basis of  $\mathfrak{su}(3)$ :

$$\begin{aligned} h &= \begin{pmatrix} \frac{-i\lambda}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{-i\lambda}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{2i\lambda}{\sqrt{3}} \end{pmatrix}, \quad m_1 = \begin{pmatrix} i\lambda & 0 & 0 \\ 0 & -i\lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad m_2 = \begin{pmatrix} 0 & -\lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ m_3 &= \begin{pmatrix} 0 & i\lambda & 0 \\ i\lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad m_4 = \begin{pmatrix} 0 & 0 & -\lambda \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{pmatrix}, \quad m_5 = \begin{pmatrix} 0 & 0 & i\lambda \\ 0 & 0 & 0 \\ i\lambda & 0 & 0 \end{pmatrix}, \\ m_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\lambda \\ 0 & \lambda & 0 \end{pmatrix}, \quad m_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i\lambda \\ 0 & i\lambda & 0 \end{pmatrix}. \end{aligned}$$

This is an orthonormal basis of  $\mathfrak{su}(3)$  with respect to  $\frac{-1}{12\lambda^2} B_{\mathfrak{su}(3)}$ , where  $B_{\mathfrak{su}(3)}$  denotes the Killing form of  $\mathfrak{su}(3)$ . We consider the subgroup  $S^1 \subset SU(3)$  with  $\mathfrak{h} := \text{Lie}(S^1) =$

$\text{span}\{h\}$ . We define  $\mathfrak{m} := \text{span}\{m_1, \dots, m_7\}$ . The transvection algebra of the base space is

$$\mathfrak{g}' = \mathfrak{su}(3) \oplus \mathbb{R}^4 = \mathfrak{h} \oplus \mathfrak{m} \oplus_{L.a.} \mathbb{R}^4.$$

The  $\text{ad}(\mathfrak{g}')$ -invariant non-degenerate symmetric bilinear form on  $\mathfrak{g}'$  is  $\bar{g} = \frac{-1}{12\lambda^2} B_{\mathfrak{su}(3)} \oplus B_{\text{eucl}}$ . The torsion is given by

$$T_0 = 2\lambda m_{123} + \lambda m_1 \wedge (m_{45} - m_{67}) - \lambda m_2 \wedge (m_{46} + m_{57}) + \lambda m_3 \wedge (m_{47} - m_{56}).$$

The curvature is

$$R_0 = -\text{ad}(h) \odot \text{ad}(h) = -(\sqrt{3}\lambda(m_{45} + m_{67})) \odot (\sqrt{3}\lambda(m_{45} + m_{67})).$$

The  $S^1$ -invariant vectors in  $\mathfrak{m}$  are spanned by  $m_1, m_2$  and  $m_3$ . Let  $m_8, m_9, m_{10}, m_{11}$  be an orthonormal basis of  $\mathbb{R}^4$ . The Lie algebra  $\mathfrak{s}$  is

$$\text{span}\{h, m_1, m_2, m_3\} \oplus \mathfrak{so}(4) \cong \mathfrak{u}(2) \oplus \mathfrak{so}(4).$$

We define the subalgebra  $\mathfrak{k} \subset \mathfrak{s}$  by

$$\begin{aligned} \varphi(k_1) &:= c_1(\text{ad}(m_1) + \lambda(m_{8,9} - m_{10,11})), \\ \varphi(k_2) &:= c_1(\text{ad}(m_2) + \lambda(m_{8,10} + m_{9,11})), \\ \varphi(k_3) &:= c_1(\text{ad}(m_3) + \lambda(m_{8,11} - m_{9,10})), \\ \varphi(k_4) &:= c_2 \text{ad}(h) + c_3(m_{8,9} + m_{10,11}), \end{aligned}$$

where  $k_1, k_2, k_3, k_4$  is an orthonormal basis of  $(\mathfrak{k}, B)$ . We consider the case that  $c_3 = 0$ . Then  $\mathfrak{k}_1 = \mathfrak{k}_{1,3} = \text{span}\{k_4\}$  and  $\mathfrak{k}_2 = \text{span}\{k_1, k_2, k_3\}$ . From the discussion above, we know that the  $(\mathfrak{k}, B)$ -extension  $(T, R)$  is always regular. The connected subgroup of  $G'(\mathfrak{k})$  with Lie subalgebra  $f(\mathfrak{g}) \oplus_{L.a.} \mathfrak{nil}(\mathfrak{k}_2)$  acts transitively on  $G'(\mathfrak{k})/H'(\mathfrak{k})$  and the isotropy algebra of this group action is trivial. The simply connected Lie group with Lie algebra  $\mathfrak{g} \oplus \mathfrak{nil}(\mathfrak{k}_2)$  is  $SU(3) \times QH^7$ . We describe the naturally reductive structure directly on  $SU(3) \times QH^7$ . As basis for the Lie algebra of  $QH^7$  we take

$$m_8, m_9, m_{10}, m_{11}, f_1, f_2, f_3,$$

where  $m_i$  corresponds to  $e_{i-7}$  from Example 2.2.16. An orthonormal basis at the origin is defined by

$$\{f_1, f_2, f_3, f_4 = \frac{-1}{c_2} h, e_1 := m_1 + c_1 f_1, e_2 := m_2 + c_1 f_2, e_3 := m_3 + c_1 f_3, e_j = m_j\},$$

for  $j = 4, \dots, 11$ . The torsion in this basis is

$$\begin{aligned}
T &= 2\lambda e_{123} + \lambda e_1 \wedge (e_{45} - e_{67}) - \lambda e_2 \wedge (e_{46} + e_{57}) + \lambda e_3 \wedge (e_{47} - e_{56}) \\
&\quad + c_1 \lambda f_1 \wedge (-2e_{23} - e_{45} + e_{67} + e_{8,9} - e_{10,11}) \\
&\quad + c_1 \lambda f_2 \wedge (-2e_{31} + e_{46} + e_{57} + e_{8,10} + e_{9,11}) \\
&\quad + c_1 \lambda f_3 \wedge (-2e_{12} - e_{47} + e_{56} + e_{8,11} - e_{9,10}) \\
&\quad + c_2 \lambda f_4 \wedge (\sqrt{3}(m_{45} + m_{67})) - 4c_1 \lambda f_{123}.
\end{aligned}$$

The curvature is

$$\begin{aligned}
R &= c_1^2 \lambda^2 (-2e_{23} - e_{45} + e_{67} + e_{8,9} - e_{10,11} - 2f_{23})^{\odot 2} + \\
&\quad c_1^2 \lambda^2 (-2e_{31} + e_{46} + e_{57} + e_{8,10} + e_{9,11} - 2f_{31})^{\odot 2} + \\
&\quad c_1^2 \lambda^2 (-2e_{12} - e_{47} + e_{56} + e_{8,11} - e_{9,10} - 2f_{12})^{\odot 2} + 3c_2^2 \lambda^2 (e_{45} + e_{67})^{\odot 2}.
\end{aligned}$$

The base space of this example illustrates nicely that there are multiple different choices for the Lie algebra  $\mathfrak{k}$ . For simplicity we decided to pick  $c_3 = 0$ . When  $c_3 \neq 0$  we simply obtain a different naturally reductive space. Another option is to pick  $c_2 = 0$ ,  $\varphi(k_5) := c_3(m_{8,10} - m_{9,11})$  and  $\varphi(k_6) := c_3(m_{8,11} + m_{9,10})$ . In this case  $\mathfrak{k} = \mathfrak{k}_2 \oplus \mathfrak{k}_3$ , with  $\mathfrak{k}_2 = \text{span}\{k_1, k_2, k_3\} \cong \mathfrak{su}(2)$ ,  $\mathfrak{k}_3 = \text{span}\{k_4, k_5, k_6\} \cong \mathfrak{su}(2)$ . We readily see from our previous discussion that the  $(\mathfrak{k}, B)$ -extensions define naturally reductive structures on

$$SU(3)/S^1 \times Nil(\mathfrak{so}(4)),$$

where  $Nil(\mathfrak{so}(4))$  is a 10-dimensional 2-step nilpotent Lie group described in Section 2.2.2. The naturally reductive structures in this example are not reducible by Lemma 3.3.19.



# Chapter 3

## Two types of naturally reductive spaces

In this chapter we define two types of naturally reductive spaces.

Let  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  be a naturally reductive decomposition with  $\mathfrak{g}$  its transvection algebra. We define two types of naturally reductive spaces:

Type I: The transvection algebra is semisimple.

Type II: The transvection algebra is not semisimple.

First we discuss some basic results for spaces of type I. Most of this chapter is about describing the spaces of type II. If a Lie algebra is not semisimple, then it contains a non-trivial abelian ideal. This fact will allow us to show that every naturally reductive space of type II is an infinitesimal fiber bundle over another naturally reductive space, see Definition 1.2.2. We can derive a formula for the infinitesimal model of the total space in terms of the infinitesimal model of the base space and a certain Lie algebra representation. This brings us precisely to the situation presented in Chapter 2 where these spaces have explicitly been described. The main result of this chapter is Theorem 3.3.6. This says that all naturally reductive spaces of type II are obtained as a  $(\mathfrak{k}, B)$ -extensions with the base space of the same form as in Section 2.2.3. At the end of this chapter we give a criterion when two spaces of type II are isomorphic in proposition 3.3.16. We also give a criterion when a naturally reductive decomposition of type II is irreducible, see Lemma 3.3.19.

### 3.1 Naturally reductive spaces of type I

Throughout this section we assume that every decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  is a naturally reductive decomposition with  $\mathfrak{g}$  its transvection algebra, which is semisimple. The main tool for spaces of type I is Kostant's theorem, Theorem 1.1.16. With this result and the other small results in this section we will classify all naturally reductive spaces of type I of dimension 7 and 8 in Chapter 4. A naturally reductive decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  with  $\mathfrak{g}$  its transvection algebra is always effective. Therefore, we can apply Kostant's theorem. The classification of naturally reductive spaces of type I relies on the classification of semisimple Lie algebras. Moreover, for a semisimple Lie algebra  $\mathfrak{g}$  we know all the  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear forms on  $\mathfrak{g}$ , because the restriction of such a bilinear form to a simple factor of  $\mathfrak{g}$  has to be a multiple of the Killing form of this simple Lie algebra by Schur's lemma.

*Remark 3.1.1.* Suppose that the  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear form  $\bar{g}$  on  $\mathfrak{g}$  has signature  $(n, m)$ , with  $n$  positive eigenvalues and  $m$  negative. Kostant's theorem tells us that  $\bar{g}$  is non-degenerate on  $\mathfrak{h}$  and  $\mathfrak{m}$ . The signature is an invariant of the bilinear form. Thus,  $\bar{g}|_{\mathfrak{m} \times \mathfrak{m}}$  is positive definite if and only if  $\bar{g}|_{\mathfrak{h}}$  has  $m$  negative eigenvalues. To illustrate this we give a short example. Consider  $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$  with  $\mathfrak{h} \subset \mathfrak{g}$  given by

$$\mathfrak{h} = \{(h, h, h) : h \in \mathfrak{su}(2)\}.$$

Let  $\bar{g} = -r_1 B_{\mathfrak{su}(2)} \oplus -r_2 B_{\mathfrak{su}(2)} \oplus -r_3 B_{\mathfrak{su}(2)}$  be the  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear form on  $\mathfrak{g}$ . The restricted non-degenerate symmetric bilinear form  $\bar{g}|_{\mathfrak{h} \times \mathfrak{h}}$  is given by

$$\bar{g}|_{\mathfrak{h} \times \mathfrak{h}} = -(r_1 + r_2 + r_3) B_{\mathfrak{su}(2)}.$$

If  $r_1 + r_2 + r_3 > 0$ , then  $\bar{g}|_{\mathfrak{h} \times \mathfrak{h}}$  is positive definite and thus the non-degenerate symmetric bilinear form needs to have signature  $(9, 0)$ , i.e.  $r_1 > 0$ ,  $r_2 > 0$ ,  $r_3 > 0$ . If  $r_1 + r_2 + r_3 < 0$ , then the signature on  $\bar{g}$  has to be  $(6, 3)$  and thus up to permuting the  $r_i$  we have  $r_2 > 0$ ,  $r_3 > 0$  and  $r_1 < -r_2 - r_3$ . These are the only two possibilities for the signature of  $\bar{g}$ .

**Lemma 3.1.2.** *Let  $\mathfrak{g}$  be a compact simple Lie algebra together with a negative multiple of its Killing form as  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear form. Any proper subalgebra  $\mathfrak{h} \subset \mathfrak{g}$  gives a reductive decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ , with  $\mathfrak{m} = (\mathfrak{h})^\perp$ . This is either an irreducible naturally reductive decomposition with non-zero torsion or the decomposition of an irreducible symmetric space.*

*Proof.* If the torsion is zero, then  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  is a decomposition of an irreducible symmetric space. Suppose that the torsion  $T$  defined by (1.1.13) is non-zero and

$T \in \Lambda^3 \mathfrak{m}_1 \oplus \Lambda^3 \mathfrak{m}_2$  for some orthogonal decomposition  $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ . By Lemma 1.3.2  $\mathfrak{h} \oplus \mathfrak{m}_1$  defines a non-zero ideal of  $\mathfrak{g}$ . Hence it has to be equal to  $\mathfrak{g}$ , which means  $\mathfrak{m}_1 = \mathfrak{m}$ . We conclude that  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  is irreducible.  $\square$

The next result gives a criterion when  $\mathfrak{g}$  is the transvection algebra of a reductive decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ , with  $\mathfrak{g}$  semisimple.

**Lemma 3.1.3.** *Let  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ , be a naturally reductive decomposition with  $\mathfrak{g}$  semisimple. Let  $(T, R)$  be the infinitesimal model defined by (1.1.13) and (1.1.14). The following hold:*

*i) If  $[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}} = \mathfrak{h}$ , then  $\mathfrak{g}$  is the transvection algebra of  $(T, R)$ .*

*ii) If  $\mathfrak{g}$  is simple, then  $[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}} = \mathfrak{h}$  and by i) the transvection algebra is equal to  $\mathfrak{g}$ .*

*Proof.* *i)* Let  $\text{ad}|_{\mathfrak{h}} : \mathfrak{h} \rightarrow \mathfrak{so}(\mathfrak{m})$  denote the restricted adjoint representation. Let  $\mathfrak{l} := \ker(\text{ad}|_{\mathfrak{h}})$ . Then  $\mathfrak{l} \subset \mathfrak{h}$  is an ideal in  $\mathfrak{g}$ . This ideal is either semisimple or  $\{0\}$ . Let  $\mathfrak{l}^{\perp} = \{g \in \mathfrak{g} : [g, l] = 0, \forall l \in \mathfrak{l}\}$  be the complementary ideal. Then  $\mathfrak{m} \subset \mathfrak{l}^{\perp}$  and  $[\mathfrak{m}, \mathfrak{m}] \subset [\mathfrak{l}^{\perp}, \mathfrak{l}^{\perp}] = \mathfrak{l}^{\perp}$ . This implies

$$[\mathfrak{h}, \mathfrak{l}] = [[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}}, \mathfrak{l}] = [[\mathfrak{m}, \mathfrak{m}], \mathfrak{l}] = [[\mathfrak{m}, \mathfrak{l}], \mathfrak{m}] + [\mathfrak{m}, [\mathfrak{m}, \mathfrak{l}]] = \{0\}$$

and thus  $\mathfrak{l} \subset \mathfrak{h} \subset \mathfrak{l}^{\perp}$ . We conclude that  $\mathfrak{l} = \{0\}$  and  $\text{ad}|_{\mathfrak{h}}$  is injective. In particular  $\mathfrak{g}$  is the transvection algebra.

*ii)* Let  $\mathfrak{k}$  be the subalgebra  $\mathfrak{k} := [\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}} \oplus \mathfrak{m}$ . By Kostant's theorem, Theorem 1.1.16,  $\mathfrak{k}$  is a non-zero ideal in  $\mathfrak{g}$  and thus  $\mathfrak{k} = \mathfrak{g}$ . This gives us  $[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}} = \mathfrak{h}$  and thus by *i)* the transvection algebra of  $(T, R)$  is  $\mathfrak{g}$ .  $\square$

The case that  $\mathfrak{g}$  is simple and non-compact is very different from the compact case as the following lemma shows.

**Lemma 3.1.4.** *Let  $\mathfrak{g}$  be a non-compact simple Lie algebra and  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  a naturally reductive decomposition. Then  $(\mathfrak{g}, \mathfrak{h})$  is a symmetric pair.*

*Proof.* By [Wol11, Thm. 12.1.4] we know that any subalgebra  $\mathfrak{h}$  of a reductive Lie algebra  $\mathfrak{g}$  is reductive in  $\mathfrak{g}$  if and only if there is a Cartan involution of  $\mathfrak{g}$  which stabilizes  $\mathfrak{h}$ , i.e.  $\sigma(\mathfrak{h}) = \mathfrak{h}$ . Let  $\sigma$  be a Cartan involution which stabilizes  $\mathfrak{h}$  and let  $\mathfrak{h} = \mathfrak{h}^+ \oplus \mathfrak{h}^-$ , with

$$\mathfrak{h}^{\pm} = \{h \in \mathfrak{h} : \sigma(h) = \pm h\}.$$

The metric on  $\mathfrak{m}$  is induced from a multiple of the Killing form and  $\mathfrak{m} = \mathfrak{h}^\perp$ . The Killing form is invariant under all automorphisms. This implies that  $\sigma$  preserves  $\mathfrak{m}$  as well. Hence we also have  $\mathfrak{m} = \mathfrak{m}^+ \oplus \mathfrak{m}^-$  with

$$\mathfrak{m}^\pm = \{m \in \mathfrak{m} : \sigma(m) = \pm m\}.$$

Let  $\mathfrak{g}^- = \mathfrak{h}^- \oplus \mathfrak{m}^-$  and  $\mathfrak{g}^+ = \mathfrak{h}^+ \oplus \mathfrak{m}^+$ . Since  $\sigma$  is a Lie algebra automorphism we immediately get

$$[\mathfrak{g}^+, \mathfrak{g}^+] \subset \mathfrak{g}^+, \quad [\mathfrak{g}^-, \mathfrak{g}^+] \subset \mathfrak{g}^-, \quad \text{and} \quad [\mathfrak{g}^-, \mathfrak{g}^-] \subset \mathfrak{g}^+.$$

The Killing form is positive definite on  $\mathfrak{m}^-$  and negative definite on  $\mathfrak{m}^+$ . This implies that either  $\mathfrak{m}^+ = \{0\}$  or  $\mathfrak{m}^- = \{0\}$ . Suppose that  $\mathfrak{m}^- = \{0\}$ . Then we have

$$[\mathfrak{h}^-, \mathfrak{m}] = [\mathfrak{h}^-, \mathfrak{m}^+] \subset \mathfrak{g}^- \cap \mathfrak{m} = \{0\}.$$

This implies that  $\mathfrak{h}^- \subset \ker(\text{ad}|_{\mathfrak{h}})$  and by Lemma 3.1.3 this implies that  $\mathfrak{h}^- = \{0\}$ . In this case we have  $\mathfrak{g}^- = \{0\}$  and this contradicts the non-compactness of  $\mathfrak{g}$ . Suppose that  $\mathfrak{m}^+ = \{0\}$ . Then we have

$$[\mathfrak{m}, \mathfrak{m}] = [\mathfrak{m}^-, \mathfrak{m}^-] \subset \mathfrak{g}^+ = \mathfrak{h}^+ \subset \mathfrak{h}.$$

This means  $(\mathfrak{g}, \mathfrak{h})$  is a symmetric pair. □

We sometimes choose to adopt the following notation:

**Notation 3.1.5.** Kostant's theorem, Theorem 1.1.16, allows us to describe every naturally reductive infinitesimal model by a triple

$$(\mathfrak{g}, \mathfrak{h}, \bar{g}),$$

where  $\mathfrak{h} \subset \mathfrak{g}$  is a subalgebra and  $\bar{g}$  an  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear form on  $\mathfrak{g}$  whose restriction to  $\mathfrak{m} := \mathfrak{h}^\perp$  is positive definite.

*Remark 3.1.6.* From Lemma 3.1.4 we can exactly describe how to obtain all non-compact naturally reductive decompositions of Type I from the compact ones in the following way. Suppose that  $(\mathfrak{g}, \mathfrak{h}, \bar{g})$  is a naturally reductive decomposition of Type I and that  $\mathfrak{g}$  is its transvection algebra. Let

$$\mathfrak{g} = \mathfrak{g}_1 \oplus_{L.a.} \mathfrak{g}_2$$

be a direct sum of ideals with  $\mathfrak{g}_1$  non-compact and simple and suppose for now that  $\mathfrak{g}_2$  is compact. Furthermore, the  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear

form on  $\mathfrak{g}_1 \oplus \mathfrak{g}_2$  from Kostant's theorem is given by  $\alpha B_{\mathfrak{g}_1} \oplus B_2$ , with  $B_{\mathfrak{g}_1}$  the Killing form of  $\mathfrak{g}_1$ ,  $B_2$  some  $\text{ad}(\mathfrak{g}_2)$ -invariant non-degenerate symmetric bilinear form on  $\mathfrak{g}_2$ , and  $\alpha > 0$ . Let

$$i = i_1 \oplus i_2 : \mathfrak{h} \rightarrow \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

denote the inclusion of the isotropy algebra. Note that  $\mathfrak{n} := i_1(\mathfrak{h})^\perp \subset \mathfrak{g}_1$  is contained in  $\mathfrak{m} = \mathfrak{h}^\perp$ . Thus,  $\bar{g}|_{\mathfrak{n} \times \mathfrak{n}} = \alpha B_1|_{\mathfrak{n} \times \mathfrak{n}}$  is positive definite. This means  $(g_1, i_1(\mathfrak{h}), \alpha B_1)$  defines a naturally reductive decomposition. By Lemma 3.1.4  $(\mathfrak{g}_1, i_1(\mathfrak{h})) \equiv (\mathfrak{g}_1, \mathfrak{k})$  is a non-compact symmetric pair, where  $\mathfrak{k} \subset \mathfrak{g}_1$  is the +1 eigenspace of a Cartan involution. We denote the map  $i_1$  with restricted codomain by  $\varphi : \mathfrak{h} \rightarrow \mathfrak{k}$  and the inclusion of  $\mathfrak{k}$  in  $\mathfrak{g}_1$  by  $j : \mathfrak{k} \rightarrow \mathfrak{g}_1$ . We have  $i_1 = j \circ \varphi$ . Let  $(\mathfrak{g}_1^*, \mathfrak{k})$  be the dual symmetric pair of  $(\mathfrak{g}_1, \mathfrak{k})$  and  $j^* : \mathfrak{k} \rightarrow \mathfrak{g}_1^*$  the natural inclusion. As  $\text{ad}(\mathfrak{g}_1^* \oplus \mathfrak{g}_2)$ -invariant non-degenerate symmetric bilinear form on  $\mathfrak{g}_1^* \oplus \mathfrak{g}_2$  we take  $\bar{g}^* := -\alpha B_{\mathfrak{g}_1^*} \oplus B_2$ , where  $B_{\mathfrak{g}_1^*}$  denotes the Killing form on  $\mathfrak{g}_1^*$ . Let

$$\mathfrak{h}^* := ((j^* \circ \varphi) \oplus i_2)(\mathfrak{h}) \subset \mathfrak{g}_1^* \oplus \mathfrak{g}_2.$$

Kostant's theorem tells us that with respect to the non-degenerate symmetric bilinear form  $g^*$  the triple

$$(\mathfrak{g}^* := \mathfrak{g}_1^* \oplus \mathfrak{g}_2, \mathfrak{h}^*, \bar{g}^*)$$

defines a naturally reductive decomposition. We will call this the *compact dual* of  $(\mathfrak{g}, \mathfrak{h}, \bar{g})$ . If there is more than one non-compact simple factor in  $G$ , then we simply apply the above procedure for every factor.

Of course we can also reverse this process. Let  $\mathfrak{g}_1$  be compact semisimple and suppose that  $(\mathfrak{g}_1, i_1(\mathfrak{h})) = (\mathfrak{g}_1, \mathfrak{k})$  is an irreducible compact symmetric pair. Let  $(\mathfrak{g}_1^*, \mathfrak{k})$  be the dual non-compact symmetric pair. Then just as above we obtain a naturally reductive decomposition

$$(\mathfrak{g}^* := \mathfrak{g}_1^* \oplus \mathfrak{g}_2, \mathfrak{h}^*, \bar{g}^*),$$

if  $\bar{g}|_{\mathfrak{h}^* \times \mathfrak{h}^*}$  is non-degenerate. Example 3.1.8 below nicely illustrates when  $\bar{g}|_{\mathfrak{h}^* \times \mathfrak{h}^*}$  is non-degenerate. Note that by Lemma 3.1.4 there is exactly one compact dual for every non-compact naturally reductive decomposition of type I. However, there can be many different non-compact naturally reductive decompositions having the same compact dual.

From Lemma 1.3.7 and the above remark we see immediately that a non-compact naturally reductive space of type I is irreducible if and only if its compact dual is irreducible. Dual naturally reductive spaces are algebraically very similar. Also it is quite easy to obtain all non-compact duals from a compact naturally reductive decomposition. To save ourself some space we will only list the compact spaces of type I in our classification.

*Remark 3.1.7.* We should point out that we are not defining a complete duality for naturally reductive spaces, because we only define dual spaces for a very small class of naturally reductive spaces.

We will illustrate how this works by an example:

**Example 3.1.8.** We consider the following basis of  $\mathfrak{su}(3)$ :

$$\begin{aligned} x_1 &= \begin{pmatrix} i\lambda_1 & 0 & 0 \\ 0 & -i\lambda_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 & -\lambda_1 & 0 \\ \lambda_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 0 & i\lambda_1 & 0 \\ i\lambda_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ x_4 &= \begin{pmatrix} \frac{-i\lambda_1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{-i\lambda_1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{2i\lambda_1}{\sqrt{3}} \end{pmatrix}, \quad x_5 = \begin{pmatrix} 0 & 0 & -\lambda_1 \\ 0 & 0 & 0 \\ \lambda_1 & 0 & 0 \end{pmatrix}, \quad x_6 = \begin{pmatrix} 0 & 0 & i\lambda_1 \\ 0 & 0 & 0 \\ i\lambda_1 & 0 & 0 \end{pmatrix}, \\ x_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\lambda_1 \\ 0 & \lambda_1 & 0 \end{pmatrix}, \quad x_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i\lambda_1 \\ 0 & i\lambda_1 & 0 \end{pmatrix}. \end{aligned}$$

This is an orthonormal basis of  $\mathfrak{su}(3)$  with respect to  $\frac{-1}{12\lambda_1^2}B_{\mathfrak{su}(3)}$  and  $\lambda_1 > 0$ . Furthermore, let

$$y_1 = \begin{pmatrix} i\lambda_2 & 0 \\ 0 & -i\lambda_2 \end{pmatrix}, \quad y_2 = \begin{pmatrix} 0 & -\lambda_2 \\ \lambda_2 & 0 \end{pmatrix}, \quad y_3 = \begin{pmatrix} 0 & i\lambda_2 \\ i\lambda_2 & 0 \end{pmatrix},$$

for some  $\lambda_2 > 0$ . This is an orthonormal basis of  $\mathfrak{su}(2)$  with respect to  $\frac{-1}{8\lambda_2^2}B_{\mathfrak{su}(2)}$ . We will now define a triple  $(\mathfrak{su}(3) \oplus \mathfrak{su}(2), \mathfrak{su}(2), \bar{g})$  which describes an irreducible naturally reductive decomposition. Let  $\bar{g} := \frac{-1}{12\lambda_1^2}B_{\mathfrak{su}(3)} \oplus \frac{-1}{8\lambda_2^2}B_{\mathfrak{su}(2)}$ . For  $i = 1, 2, 3$  let

$$h_i := v_1(\lambda_1^{-1}x_i, \lambda_2^{-1}y_i)$$

where  $v_1 = (\lambda_1^{-2} + \lambda_2^{-2})^{-\frac{1}{2}}$ . Note that  $\mathfrak{su}(2)_\Delta := \text{span}\{h_1, h_2, h_3\}$  is a subalgebra of  $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$  isomorphic to  $\mathfrak{su}(2)$ . We define  $h_4 := (x_4, 0)$ . For  $i = 1, 2, 3, 4$  let

$$e_i := (x_{4+i}, 0)$$

and for  $i = 1, 2, 3$  let

$$e_{4+i} := v_2(\lambda_1 x_i, -\lambda_2 y_i),$$

where  $v_2 = (\lambda_1^2 + \lambda_2^2)^{-\frac{1}{2}}$ . Let  $\mathfrak{h} := \text{span}\{h_1, h_2, h_3, h_4\}$  and  $\mathfrak{m} := \text{span}\{e_1, \dots, e_7\}$ . Then  $\mathfrak{m} \perp \mathfrak{h}$  with respect to  $\bar{g}$ . Thus, the triple

$$(\mathfrak{g} = \mathfrak{su}(3) \oplus \mathfrak{su}(2), \mathfrak{h}, \bar{g})$$

defines a naturally reductive decomposition. Let  $\pi_1 : \mathfrak{su}(3) \oplus \mathfrak{su}(2) \rightarrow \mathfrak{su}(3)$  be the projection on the first factor. Note that  $(\mathfrak{su}(3), \pi_1(\mathfrak{h}))$  is a symmetric pair. Hence by Remark 3.1.6 we can define a dual space. We will now explicitly describe the dual space. We consider the following basis of  $\mathfrak{su}(2, 1)$ :

$$\begin{aligned} x_1 &= \begin{pmatrix} i\lambda_1 & 0 & 0 \\ 0 & -i\lambda_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 & -\lambda_1 & 0 \\ \lambda_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 0 & i\lambda_1 & 0 \\ i\lambda_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ x_4 &= \begin{pmatrix} \frac{-i\lambda_1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{-i\lambda_1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{2i\lambda_1}{\sqrt{3}} \end{pmatrix}, \quad x_5 = \begin{pmatrix} 0 & 0 & -i\lambda_1 \\ 0 & 0 & 0 \\ i\lambda_1 & 0 & 0 \end{pmatrix}, \quad x_6 = \begin{pmatrix} 0 & 0 & -\lambda_1 \\ 0 & 0 & 0 \\ -\lambda_1 & 0 & 0 \end{pmatrix}, \\ x_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i\lambda_1 \\ 0 & i\lambda_1 & 0 \end{pmatrix}, \quad x_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\lambda_1 \\ 0 & -\lambda_1 & 0 \end{pmatrix}. \end{aligned}$$

This is an pseudo-orthonormal basis of  $\mathfrak{su}(2, 1)$  with respect to  $\frac{1}{12\lambda_1^2}B_{\mathfrak{su}(2,1)}$ . We let  $y_1, y_2, y_3$  be as before. We require that  $\lambda_1 \neq \lambda_2$ . For  $i = 1, 2, 3$  let

$$h_i := u_1(\lambda_1^{-1}x_i, \lambda_2^{-1}y_i),$$

where  $u_1 = (-\lambda_1^{-2} + \lambda_2^{-2})^{-\frac{1}{2}}$ . Note that  $\mathfrak{su}(2)_\Delta = \text{span}\{h_1, h_2, h_3\}$  is a subalgebra of  $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$  which is isomorphic to  $\mathfrak{su}(2)$ . We define  $h_4 := (x_4, 0)$ . For  $i = 1, 2, 3, 4$  let

$$e_i := (x_{4+i}, 0).$$

For  $i = 1, 2, 3$  let

$$e_{4+i} := u_2(\lambda_1 x_i, -\lambda_2 y_i),$$

where  $u_2 = (-\lambda_1^2 + \lambda_2^2)^{-\frac{1}{2}}$ . Let  $\mathfrak{h}^* := \text{span}\{h_1, h_2, h_3, h_4\}$  and  $\mathfrak{m} := \text{span}\{e_1, \dots, e_7\}$ . Let  $\bar{g}^* = \frac{1}{12\lambda_1^2}B_{\mathfrak{su}(2,1)} \oplus \frac{-1}{8\lambda_2^2}B_{\mathfrak{su}(2)}$ . Then  $\mathfrak{m} \perp \mathfrak{h}^*$  with respect to  $\bar{g}$ . Thus, this defines a dual naturally reductive decomposition

$$(\mathfrak{g} = \mathfrak{su}(2, 1) \oplus \mathfrak{su}(2), \mathfrak{h}^*, \bar{g}^*).$$

For our classification of naturally reductive spaces of type I in Chapter 4 we still require a tool to prove all the spaces we list are non-isomorphic. The following lemma gives a criterion when two subalgebras  $\mathfrak{h}_1, \mathfrak{h}_2$  of  $\mathfrak{g} = \mathfrak{so}(n)$  or  $\mathfrak{g} = \mathfrak{su}(n)$  are conjugate by an element of  $\text{Aut}(\mathfrak{g})$ . Note that this implies that the naturally reductive structures defined by the triples  $(\mathfrak{g}, \mathfrak{h}_1, \bar{g})$  and  $(\mathfrak{g}, \mathfrak{h}_2, \bar{g})$  define isomorphic naturally reductive structures.

**Lemma 3.1.9.** *Let  $\mathfrak{g} = \mathfrak{so}(n)$  or  $\mathfrak{g} = \mathfrak{su}(n)$ . Let  $\pi : \mathfrak{g} \rightarrow \text{End}(K^n)$  be the vector representation, with  $K = \mathbb{R}^n, \mathbb{C}^n$ . Let  $f_i : \mathfrak{h} \rightarrow \mathfrak{g}$  be an injective Lie algebra homomorphism for  $i = 1, 2$ . We denote the image of  $f_i$  by  $\mathfrak{h}_i := f_i(\mathfrak{h})$ . If the representations  $\pi \circ f_1$  and  $\pi \circ f_2$  are equivalent, then the subalgebras  $\mathfrak{h}_1$  and  $\mathfrak{h}_2$  are conjugate by an automorphism of  $\mathfrak{g}$ .*

*Proof.* Let  $K^n = V_{1,i} \oplus V_{2,i} \oplus \dots \oplus V_{m,i}$  be an orthogonal decomposition of irreducible submodules for the representation  $\pi \circ f_i$  and with respect to the  $\mathfrak{g}$ -invariant inner product  $\langle -, - \rangle$ . By assumption there exists an intertwining linear map  $A : K^n \rightarrow K^n$ , i.e.  $A \circ \pi(f_1(h)) = \pi(f_2(h)) \circ A$  for all  $h \in \mathfrak{h}$ . We know that  $A_j := A|_{V_{j,1}} : V_{j,1} \rightarrow V_{\sigma(j),2}$  is an isomorphism of irreducible  $\mathfrak{h}$ -modules for all  $j = 1, \dots, m$  and a certain permutation  $\sigma$ . For all  $x, y \in V_{1,j}$  let  $(x, y) := \langle A_j x, A_j y \rangle$  define a second inner product on  $V_{1,j}$ . The inner product  $(-, -)$  is  $\pi(f_1(\mathfrak{h}))$ -invariant, because  $A$  is an intertwining map. Therefore, we obtain by Schur's lemma  $(x, y) = c_i \langle x, y \rangle$ , with  $c_i > 0$ . The linear map  $\hat{A} := (c_1^{-1/2} A_1 | c_2^{-1/2} A_2 | \dots | c_m^{-1/2} A_m)$  is still an intertwining map for the actions and satisfies  $\langle x, y \rangle = \langle Ax, Ay \rangle$  for all  $x, y \in K^n$ . Thus,  $A \in O(n)$  if  $K = \mathbb{R}$  and  $A \in U(n)$  if  $K = \mathbb{C}$ . In both cases we see that  $\tau : \mathfrak{g} \mapsto AgA^{-1}$  defines an automorphism of  $\mathfrak{g}$  such that  $\tau(\mathfrak{h}_1) = \mathfrak{h}_2$ .  $\square$

## 3.2 Naturally reductive spaces of type II

In this section we will investigate the structure of type II spaces on the Lie algebra level. In Proposition 3.2.9 a formula for the torsion and curvature of these spaces is given in terms of the torsion and curvature of a base space and a Lie algebra representation. The main result will be Theorem 3.3.6. This says that these spaces are infinitesimal fiber bundles over other naturally reductive spaces in the sense of Definition 1.2.2 and that they can be reconstructed as a  $(\mathfrak{k}, B)$ -extension. This fact, together with some other technical details at the end of this chapter, will allow us to classify all naturally reductive spaces of type II. We will do this in dimension 7 and 8 in Chapter 4.

**Lemma 3.2.1.** *Let  $(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, g)$  be an effective naturally reductive decomposition. Let  $\mathfrak{a} \subset \mathfrak{g}$  be an abelian ideal. Let  $\mathfrak{m}_\mathfrak{a} := \mathfrak{a} \cap \mathfrak{m}$  and let  $\mathfrak{m}_0$  be the orthogonal complement of  $\mathfrak{m}_\mathfrak{a}$  in  $\mathfrak{m}$ . Let  $\mathfrak{a}' := (\pi_\mathfrak{m}|_\mathfrak{a})^{-1}(\mathfrak{m}_0)$ , where  $\pi_\mathfrak{m}$  is the projection in  $\mathfrak{g}$  onto  $\mathfrak{m}$  along  $\mathfrak{h}$ . Then the following hold:*

- i)  $[\mathfrak{m}_\mathfrak{a}, \mathfrak{m}] = \{0\}$ ,
- ii)  $\mathfrak{g}' := \mathfrak{h} \oplus \mathfrak{m}_0$  is a subalgebra of  $\mathfrak{g}$  and a naturally reductive decomposition,

iii)  $\mathfrak{a}'$  is an abelian ideal of  $\mathfrak{g}'$  and  $\mathfrak{g}$  and satisfies  $\mathfrak{a}' \cap \mathfrak{m}_0 = \mathfrak{a}' \cap \mathfrak{h} = \{0\}$ .

*Proof.* i) Since the decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  is reductive and  $\mathfrak{a}$  is an ideal we have

$$[\mathfrak{h}, \mathfrak{m}_\mathfrak{a}] \subset \mathfrak{a} \cap \mathfrak{m} = \mathfrak{m}_\mathfrak{a}.$$

Hence  $\mathfrak{m}_\mathfrak{a}$  and its orthogonal complement  $\mathfrak{m}_0$  are  $\mathfrak{h}$ -invariant. Since  $\mathfrak{a}$  is abelian we have

$$[\mathfrak{m}_\mathfrak{a}, \mathfrak{m}_\mathfrak{a}] = \{0\}.$$

Let  $m \in \mathfrak{m}_\mathfrak{a}$  and  $n \in \mathfrak{m}_0$ . Then we can apply Lemma 1.2.1.i) to see that  $[m, n] \subset \mathfrak{m}$ . Combining this with the fact that  $\mathfrak{a}$  is an ideal gives us

$$k := [m, n] \in \mathfrak{a} \cap \mathfrak{m} = \mathfrak{m}_\mathfrak{a}.$$

We obtain

$$g(k, k) = g([m, n], k) = -g(n, [m, k]) = 0$$

and thus  $k = 0$ . We conclude  $[\mathfrak{m}, \mathfrak{m}_\mathfrak{a}] = \{0\}$ .

ii) We already know that  $[\mathfrak{h}, \mathfrak{m}_0] \subset \mathfrak{m}_0$ . We just saw that  $[\mathfrak{m}_0, \mathfrak{m}_\mathfrak{a}] = \{0\} \subset \mathfrak{m}_\mathfrak{a}$ . Lemma 1.2.1.ii) now implies  $[\mathfrak{m}_0, \mathfrak{m}_0]_\mathfrak{m} \subset \mathfrak{m}_0$ . Consequently,  $\mathfrak{g}' = \mathfrak{h} \oplus \mathfrak{m}_0$  is a subalgebra and defines a naturally reductive decomposition with respect to the metric  $g|_{\mathfrak{m}_0 \times \mathfrak{m}_0}$ .

iii) We know that  $\mathfrak{a}' \subset \mathfrak{g}'$  and by ii)  $\mathfrak{g}' \subset \mathfrak{g}$  is a subalgebra. Hence  $[\mathfrak{g}', \mathfrak{a}'] \subset \mathfrak{g}' \cap \mathfrak{a} = \mathfrak{a}'$ . This means  $\mathfrak{a}'$  is an abelian ideal in  $\mathfrak{g}'$ . Clearly  $\mathfrak{a}'$  is still an abelian ideal in  $\mathfrak{g}$ . Note that  $\mathfrak{a}' \cap \mathfrak{m}_0 \subset \mathfrak{m}_\mathfrak{a} \cap \mathfrak{m}_\mathfrak{a}^\perp = \{0\}$ . Suppose that  $h \in \mathfrak{h} \cap \mathfrak{a}$ . Then for every  $n \in \mathfrak{m}_0$  we have

$$[h, n] \in \mathfrak{m}_0 \cap \mathfrak{a} = \{0\}.$$

If  $m \in \mathfrak{m}_\mathfrak{a}$ , then  $[h, m] = 0$  holds because both  $h$  and  $m$  are in  $\mathfrak{a}$  and  $\mathfrak{a}$  is abelian. By assumption the map  $\text{ad} : \mathfrak{h} \rightarrow \mathfrak{so}(\mathfrak{m})$  has trivial kernel. Since  $\mathfrak{h} \cap \mathfrak{a}$  is contained in the kernel we conclude that  $\mathfrak{h} \cap \mathfrak{a} = \{0\}$ . In particular  $\mathfrak{h} \cap \mathfrak{a}' = \{0\}$ .  $\square$

Next, we prove that an irreducible naturally reductive decomposition satisfies  $\mathfrak{m}_\mathfrak{a} = \{0\}$ .

**Lemma 3.2.2.** *Suppose that  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  is an irreducible effective naturally reductive decomposition with infinitesimal model  $(T, R)$  and  $T \neq 0$ . Let  $\mathfrak{a} \subset \mathfrak{g}$  be an abelian ideal. Then  $\mathfrak{a} \cap \mathfrak{m} = \mathfrak{a} \cap \mathfrak{h} = \{0\}$ .*

*Proof.* From Lemma 3.2.1 we know that  $\mathfrak{h} \cap \mathfrak{a} = \{0\}$  and

$$T \in \Lambda^3 \mathfrak{m}_\mathfrak{a} \oplus \Lambda^3 \mathfrak{m}_\mathfrak{a}^\perp.$$

By assumption  $T$  is irreducible and non-zero. This implies that  $\mathfrak{a} \cap \mathfrak{m} = \mathfrak{m}_\mathfrak{a} = \{0\}$ .  $\square$

**Definition 3.2.3.** Let  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  be an effective naturally reductive decomposition. Let  $\mathfrak{a} = \mathfrak{a}' \oplus \mathfrak{m}_a$  be as in Lemma 3.2.1. Let  $\mathfrak{m}^+ := \pi_{\mathfrak{m}}(\mathfrak{a}') \subset \mathfrak{m}_0$ , where  $\pi_{\mathfrak{m}} : \mathfrak{g} \rightarrow \mathfrak{m}$  is the projection along  $\mathfrak{h}$ . Let  $\mathfrak{m}^-$  be the orthogonal complement of  $\mathfrak{m}^+$  inside  $\mathfrak{m}_0$ . Furthermore, let  $\mathfrak{h}^+ := \pi_{\mathfrak{h}}(\mathfrak{a}')$ , where  $\pi_{\mathfrak{h}} : \mathfrak{g} \rightarrow \mathfrak{h}$  is the projection along  $\mathfrak{m}$ . Note that  $\mathfrak{h}^+$  is an ideal in  $\mathfrak{h}$  because  $\pi_{\mathfrak{h}}$  is  $\mathfrak{h}$ -equivariant and  $\mathfrak{a}$  is an ideal. Let  $\mathfrak{h}^-$  be a complementary ideal in  $\mathfrak{h}$ , which exists because  $\mathfrak{h}$  is a reductive Lie algebra. It will be irrelevant which complement we pick. This gives us the following decomposition:

$$\mathfrak{g} = \mathfrak{h}^+ \oplus \mathfrak{h}^- \oplus \mathfrak{m}^+ \oplus \mathfrak{m}^- \oplus \mathfrak{m}_a.$$

We call this the *fiber decomposition with respect to  $\mathfrak{a}$* .

**Lemma 3.2.4.** *Let the notation be as in Definition 3.2.3. Then*

- i) The decomposition  $\mathfrak{m} = \mathfrak{m}^+ \oplus \mathfrak{m}^- \oplus \mathfrak{m}_a$  is  $\mathfrak{h}$ -invariant,*
- ii)  $[\mathfrak{m}^+, \mathfrak{m}^+]_{\mathfrak{m}} \subset \mathfrak{m}^+$ ,*
- iii)  $[\mathfrak{h}^-, \mathfrak{m}^+] = \{0\}$  and  $[\mathfrak{h}^-, \mathfrak{h}^+] = \{0\}$ ,*
- iv)  $[\mathfrak{a}, \mathfrak{m}^- \oplus \mathfrak{m}_a] = \{0\}$ .*

*Proof.* *i)* From Lemma 3.2.1 we know that  $\mathfrak{m}_a$  and  $\mathfrak{m}_0$  are  $\mathfrak{h}$ -invariant. Let  $m \in \mathfrak{m}^+$  and pick  $h \in \mathfrak{h}^+$  such that  $h + m \in \mathfrak{a}'$ . Then by Lemma 3.2.1.iii) for every  $k \in \mathfrak{h}$  we have

$$\mathfrak{a}' \ni [k, h + m] = \underbrace{[k, h]}_{\in \mathfrak{h}} + \underbrace{[k, m]}_{\in \mathfrak{m}_0}.$$

Hence  $[k, m] \in \mathfrak{m}^+$  and thus  $\mathfrak{m}^+$  is  $\mathfrak{h}$ -invariant. The orthogonal complement  $\mathfrak{m}^-$  in  $\mathfrak{m}_0$  is automatically also  $\mathfrak{h}$ -invariant.

*ii)* Let  $m' \in \mathfrak{m}^+$  and pick  $h' \in \mathfrak{h}^+$  such that  $h' + m' \in \mathfrak{a}$ . Then we have

$$0 = [h + m, h' + m'] = \underbrace{[h, h']}_{\in \mathfrak{h}} + \underbrace{[h, m']}_{\in \mathfrak{m}^+} - \underbrace{[h', m]}_{\in \mathfrak{m}^+} + [m, m'].$$

This implies that  $[m, m']_{\mathfrak{m}} \in \mathfrak{m}^+$ .

*iii)* Because  $\mathfrak{h}^-$  and  $\mathfrak{h}^+$  are both ideals in  $\mathfrak{h}$  we get  $[\mathfrak{h}^-, \mathfrak{h}^+] = \{0\}$ . Let  $h^- \in \mathfrak{h}^-$ . Then

$$\mathfrak{a}' \ni [h + m, h^-] = [h, h^-] + [m, h^-] = [m, h^-] \in \mathfrak{m}^+.$$

Combining this with  $\mathfrak{a}' \cap \mathfrak{m}^+ = \{0\}$  we obtain  $[\mathfrak{h}^-, \mathfrak{m}^+] = \{0\}$ .

iv) Let  $m^- \in \mathfrak{m}^-$ . Then

$$\mathfrak{a} \ni [h + m, m^-] = \underbrace{[h, m^-]}_{\mathfrak{m}^-} + \underbrace{[m, m^-]}_{\mathfrak{m}^-}$$

implies that  $[\mathfrak{a}, \mathfrak{m}^-] \subset \mathfrak{a} \cap \mathfrak{m}^- = \{0\}$ . Since  $\mathfrak{a}$  is abelian it follows that  $[\mathfrak{a}, \mathfrak{m}_\alpha] = \{0\}$ .  $\square$

In the following we assume that we have an abelian ideal  $\mathfrak{a} \subset \mathfrak{g}$  with  $\mathfrak{a} \cap \mathfrak{m} = \mathfrak{a} \cap \mathfrak{h} = \{0\}$ . We let

$$\rho : \mathfrak{h}^+ \rightarrow \mathfrak{m}^+$$

be the linear map defined by the graph  $\mathfrak{a} \subset \mathfrak{h}^+ \oplus \mathfrak{m}^+$ . Let  $k \in \mathfrak{h}^+$  and  $h + m \in \mathfrak{a}$ . Then

$$\mathfrak{a} \ni [k, h + m] = [k, h] + [k, m].$$

This implies that  $\rho([k, h]) = [k, m] = [k, \rho(h)]$ , i.e. the linear map  $\rho : \mathfrak{h}^+ \rightarrow \mathfrak{m}^+$  is an isomorphism of  $\mathfrak{h}^+$ -modules. Let  $h + m, h' + m' \in \mathfrak{a}$ . Then we have

$$0 = [h + m, h' + m'] = [h, h'] + [h, m'] + [m, h'] + [m, m'],$$

or equivalently

$$\begin{aligned} [h, h'] &= -[m, m']_{\mathfrak{h}}, \\ [m, m']_{\mathfrak{m}} &= [h', m] + [m', h] = -2\rho([h, h']). \end{aligned} \tag{3.2.5}$$

*Remark 3.2.6.* Rewriting (3.2.5) we get  $[m, m']_{\mathfrak{m}} = -2[h, m']$ . This implies that if  $\mathfrak{v} \subset \mathfrak{m}^+$  is  $\mathfrak{h}^+$ -invariant, i.e. a submodule, then also  $[\mathfrak{v}, \mathfrak{v}]_{\mathfrak{m}} \subset \mathfrak{v}$ .

Let  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  be an effective naturally reductive decomposition of type II. If we combine Lemma 3.2.4 with Lemma 1.2.1, then we obtain an infinitesimal fiber bundle, in the sense of Definition 1.2.2, for every abelian ideal  $\mathfrak{a} \subset \mathfrak{g}$ .

**Definition 3.2.7.** Let  $(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, g)$  be an effective naturally reductive decomposition of type II with infinitesimal model  $(T, R)$ . Let  $\mathfrak{a} \subset \mathfrak{g}$  be an abelian ideal and let  $\mathfrak{g} = \mathfrak{h}^+ \oplus \mathfrak{h}^- \oplus \mathfrak{m}^+ \oplus \mathfrak{m}^- \oplus \mathfrak{m}_\alpha$  be the fiber decomposition with respect to  $\mathfrak{a}$ , see Definition 3.2.3. Let  $\mathfrak{e} := \mathfrak{h} \oplus \mathfrak{m}^+ \oplus \mathfrak{m}_\alpha$ . The base space associated to  $\mathfrak{a}$  is given by the naturally reductive decomposition

$$(\mathfrak{e} \oplus \mathfrak{m}^-, g|_{\mathfrak{m}^- \times \mathfrak{m}^-}),$$

where  $\mathfrak{e}$  is the isotropy algebra. We will denote the infinitesimal model of the base space by  $(T_0, R_0)$ . This is defined by (1.1.13) and (1.1.14).

**Notation 3.2.8.** Let  $B = \rho^*g|_{\mathfrak{m}^+ \times \mathfrak{m}^+}$  be the pullback metric on  $\mathfrak{h}^+$ . This metric is  $\text{ad}(\mathfrak{h}^+)$ -invariant. We define a 3-form  $T_{\mathfrak{h}^+}$  on  $\mathfrak{h}^+$  by  $T(h_1, h_2, h_3) := B([h_1, h_2], h_3)$ . We define  $T_{\mathfrak{m}^+} := \rho(T_{\mathfrak{h}^+})$ , where  $\rho$  is the natural extension  $\rho : \Lambda^3 \mathfrak{h}^+ \rightarrow \Lambda^3 \mathfrak{m}^+$ . Let

$$\varphi : \mathfrak{h}^+ \rightarrow \mathfrak{so}(\mathfrak{m}^-)$$

and

$$\psi : \mathfrak{h}^+ \rightarrow \mathfrak{so}(\mathfrak{m}^+ \oplus \mathfrak{m}^-)$$

denote the restricted adjoint representations in  $\mathfrak{g}$ .

As the notation suggests, the representations  $\varphi$  and  $\psi$  correspond to the representations from Definition 2.1.2. Note that  $T_0$  is invariant under  $\varphi(\mathfrak{h}^+)$ . From the discussion above we now derive a formula for the torsion and curvature of a naturally reductive space of type II in terms of  $(T_0, R_0)$  and the representations  $\varphi$  and  $\psi$ .

**Proposition 3.2.9.** *Let  $\mathfrak{g} = \mathfrak{h}^+ \oplus \mathfrak{h}^- \oplus \mathfrak{m}^+ \oplus \mathfrak{m}^-$  be an irreducible effective naturally reductive decomposition of type II associated with an abelian ideal  $\mathfrak{a} \subset \mathfrak{g}$ . Its torsion and curvature are given by*

$$T = T_0 + \sum_{i=1}^l \varphi(h_i) \wedge m_i + 2T_{\mathfrak{m}^+},$$

and

$$R = R_0 + \sum_{i=1}^l \psi(h_i) \odot \psi(h_i),$$

respectively, where  $m_1, \dots, m_l$  is an orthonormal basis of  $\mathfrak{m}^+$ , and  $h_i := \rho^{-1}(m_i)$ .

*Proof.* We know by Lemma 3.2.4 that  $[\mathfrak{m}^+, \mathfrak{m}^+]_{\mathfrak{m}} \subset \mathfrak{m}^+$ . Thus Lemma 1.2.1 implies that  $[\mathfrak{m}^+, \mathfrak{m}^-] \subset \mathfrak{m}^-$ . These two inclusions tell us that

$$T \in \Lambda^3 \mathfrak{m}^+ \oplus \Lambda^2 \mathfrak{m}^- \otimes \mathfrak{m}^+ \oplus \Lambda^3 \mathfrak{m}^-.$$

The component in  $\Lambda^3 \mathfrak{m}^-$  is exactly  $T_0$  by the definition of  $T_0$ . Let  $h + m \in \mathfrak{a}$ . Then by Lemma 3.2.4. *iv)* we have

$$0 = [h + m, n] = [h, n] + [m, n] = \varphi(h)(n) + [m, n],$$

for every  $n \in \mathfrak{m}^-$ . This means that  $T(m, n) = -[m, n] = \varphi(h)n$ . This proves that the summand in  $\Lambda^2 \mathfrak{m}^- \otimes \mathfrak{m}^+$  is given by  $\sum_{i=1}^l \varphi(h_i) \wedge m_i$ . From (3.2.5) we know

that  $[m, m'] = -2\rho([h, h'])$ . This shows that the summand in  $\Lambda^3\mathfrak{m}^+$  is given by  $2\rho(T_{\mathfrak{h}^+}) = 2T_{\mathfrak{m}^+}$ .

The curvature of the base space is by definition given by

$$R_0(x, y) = -\text{ad}([x, y]_{\mathfrak{e}}) \in \mathfrak{so}(\mathfrak{m}^-), \quad \forall x, y \in \mathfrak{m}^-.$$

Let  $x, y, u, v \in \mathfrak{m}^-$ . Then

$$\begin{aligned} R(x, y, u, v) &= g(R(x, y)u, v) = -g([x, y]_{\mathfrak{h}}, u, v) \\ &= -g([x, y]_{\mathfrak{e}} - [x, y]_{\mathfrak{m}^+}, u, v) \\ &= R_0(x, y, u, v) + g([x, y]_{\mathfrak{m}^+}, u, v) \\ &= R_0(x, y, u, v) + \sum_{i=1}^l -g(g(\psi(h_i)x, y)[m_i, u], v) \\ &= R_0(x, y, u, v) + \sum_{i=1}^l g(\psi(h_i)x, y)g(\psi(h_i)u, v) \\ &= (R_0 + \sum_{i=1}^l \psi(h_i) \odot \psi(h_i))(x, y, u, v). \end{aligned}$$

Let  $x, y \in \mathfrak{m}^+$  and  $u, v \in \mathfrak{m}$ . From (3.2.5) it follows that

$$[x, y]_{\mathfrak{m}} = \sum_{i=1}^l g([x, y], m_i)m_i = \sum_{i=1}^l g([m_i, x], y)m_i = -2 \sum_{i=1}^l g([h_i, x], y)m_i,$$

and  $[x, y]_{\mathfrak{h}} = \frac{1}{2}\rho^{-1}([x, y]_{\mathfrak{m}})$ . Combining these gives

$$[x, y]_{\mathfrak{h}} = - \sum_{i=1}^l g([h_i, x], y)h_i.$$

Consequently,

$$\begin{aligned} R(x, y, u, v) &= -g([x, y]_{\mathfrak{h}}u, v) \\ &= \sum_{i=1}^l (\psi(h_i) \odot \psi(h_i))(x, y, u, v). \end{aligned}$$

From the symmetries of the curvature tensor  $R$  we conclude that

$$R = R_0 + \sum_{i=1}^l \psi(h_i) \odot \psi(h_i).$$

□

### 3.3 General form of any naturally reductive space

The main result of this section is Theorem 3.3.6. This proves that every naturally reductive space of type II can be constructed as a  $(\mathfrak{k}, B)$ -extension. Moreover, Theorem 3.3.6 together with Proposition 3.3.14 implies that every naturally reductive space of type II is of the form discussed in Section 2.2.3. In other words every naturally reductive homogeneous space is described by the explicit formulas in Section 2.2.3. In the following lemma we will prove that every effective naturally reductive decomposition admits a maximal abelian ideal. This result will be very useful for the main theorem of this chapter.

**Lemma 3.3.1.** *Let  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  be an effective naturally reductive decomposition. The sum over all abelian ideals inside  $\mathfrak{g}$  is again an abelian ideal in  $\mathfrak{g}$ . In other words there always exists a maximal abelian ideal. Every derivation of  $\mathfrak{g}$  preserves the maximal abelian ideal.*

*Proof.* Let  $\mathfrak{a} := \sum_i \mathfrak{a}_i$  be the sum of all abelian ideals  $\mathfrak{a}_i$  in  $\mathfrak{g}$ . Then  $\mathfrak{a} \subset \mathfrak{g}$  is an ideal. We have to show for all  $x, y \in \mathfrak{a}$  that  $[x, y] = \sum_{i,j} [x_i, y_j] = 0$ , where  $x = \sum_i x_i$ ,  $y = \sum_i y_i$ , and  $x_i, y_i \in \mathfrak{a}_i$ . In other words the sum of two abelian ideals  $\mathfrak{a}_i$  and  $\mathfrak{a}_j$  is an abelian ideal in  $\mathfrak{g}$ . It is clear that  $\mathfrak{a}_i + \mathfrak{a}_j$  is an ideal and that  $[\mathfrak{a}_i, \mathfrak{a}_j] \subset \mathfrak{a}_i \cap \mathfrak{a}_j$ . This means that if  $\mathfrak{a}_{ij} := \mathfrak{a}_i \cap \mathfrak{a}_j$  is equal to  $\{0\}$ , then  $\mathfrak{a}_i + \mathfrak{a}_j$  is also abelian.

Let  $\mathfrak{g} = \mathfrak{h}_i^+ \oplus \mathfrak{h}_i^- \oplus \mathfrak{m}_i^+ \oplus \mathfrak{m}_i^- \oplus \mathfrak{m}_{\mathfrak{a}_i}$  be the fiber decomposition of  $\mathfrak{g}$  with respect to  $\mathfrak{a}_i$ , see Definition 3.2.3. The intersection  $\mathfrak{a}_{ij} = \mathfrak{a}_i \cap \mathfrak{a}_j$  is an abelian ideal of  $\mathfrak{g}$  and  $\mathfrak{a}_{ij} \subset \mathfrak{a}_i$ . Let  $\mathfrak{m}_{ij}^+$  be the projection of  $\mathfrak{a}_{ij}$  onto  $\mathfrak{m}$ . Just as in Lemma 3.2.4, it follows that  $\mathfrak{m}_{ij}^+ \subset \mathfrak{m}_i^+ \oplus \mathfrak{m}_{\mathfrak{a}_i}$  is  $\mathfrak{h}$ -invariant. Let  $\mathfrak{v}_i$  be the orthogonal complement of  $\mathfrak{m}_{ij}^+$  in  $\mathfrak{m}_i^+ \oplus \mathfrak{m}_{\mathfrak{a}_i}$ . Then  $\mathfrak{v}_i$  is also  $\mathfrak{h}$ -invariant. Remark 3.2.6 implies  $[\mathfrak{v}_i, \mathfrak{v}_i]_{\mathfrak{m}} \subset \mathfrak{v}_i$  and  $[\mathfrak{m}_{ij}^+, \mathfrak{m}_{ij}^+]_{\mathfrak{m}} \subset \mathfrak{m}_{ij}^+$ . Therefore, Lemma 1.2.1.ii) implies that  $[\mathfrak{v}_i, \mathfrak{m}_{ij}^+] \subset \mathfrak{v}_i \cap \mathfrak{m}_{ij}^+ = \{0\}$ . Let  $\mathfrak{o}_i := (\pi_{\mathfrak{m}|\mathfrak{a}_i})^{-1}(\mathfrak{v}_i)$ , where  $\pi_{\mathfrak{m}} : \mathfrak{g} \rightarrow \mathfrak{m}$  is the projection along  $\mathfrak{h}$ . Then  $\mathfrak{o}_i \subset \mathfrak{a}_i$  and thus Lemma 3.2.4 implies that  $[\mathfrak{o}_i, \mathfrak{m}_i^- \oplus \mathfrak{m}_{\mathfrak{a}_i}] = \{0\}$ . Since  $\mathfrak{v}_i$  is  $\mathfrak{h}$ -invariant and  $\pi_{\mathfrak{m}}$  is  $\mathfrak{h}$ -equivariant we see that also  $\mathfrak{o}_i$  is  $\mathfrak{h}$ -invariant, i.e.  $[\mathfrak{h}, \mathfrak{o}_i] \subset \mathfrak{o}_i$ . Finally, we have  $[\mathfrak{o}_i, \mathfrak{a}_i] = \{0\}$ . In total this tells us that  $[\mathfrak{g}, \mathfrak{o}_i] = [\mathfrak{h} \oplus \mathfrak{a}_i \oplus \mathfrak{m}_i^- \oplus \mathfrak{m}_{\mathfrak{a}_i}, \mathfrak{o}_i] \subset \mathfrak{o}_i$  and thus that  $\mathfrak{o}_i$  is an ideal in  $\mathfrak{g}$ . Moreover  $\mathfrak{o}_i$  is abelian because  $\mathfrak{o}_i \subset \mathfrak{a}_i$ . We have  $\mathfrak{a}_i = \mathfrak{o}_i \oplus \mathfrak{a}_{ij}$ . By construction we have  $\mathfrak{o}_i \cap \mathfrak{a}_j = \{0\}$ . This implies that  $\mathfrak{o}_i \oplus \mathfrak{a}_j$  is again an abelian ideal. Since  $\mathfrak{a}_{ij} \subset \mathfrak{a}_j$  we obtain

$$\mathfrak{a}_i + \mathfrak{a}_j = (\mathfrak{o}_i \oplus \mathfrak{a}_{ij}) + \mathfrak{a}_j = \mathfrak{o}_i \oplus \mathfrak{a}_j.$$

We conclude that  $\mathfrak{a}_i + \mathfrak{a}_j$  is an abelian ideal and thus also  $\mathfrak{a} = \sum_i \mathfrak{a}_i$  is an abelian ideal. Moreover,  $\mathfrak{a}$  is maximal in the sense that it contains all other abelian ideals.

The maximal abelian ideal of  $\mathfrak{g}$  is the sum over all abelian ideals. The image of an abelian ideal under an automorphism is an abelian ideal. Therefore, we see that any automorphism preserves the maximal abelian ideal. This implies that also all derivations preserve the maximal abelian ideal.  $\square$

**Lemma 3.3.2.** *Let  $(\mathfrak{g} = \mathfrak{h}^+ \oplus \mathfrak{h}^- \oplus \mathfrak{m}^+ \oplus \mathfrak{m}^-, g)$  be an effective naturally reductive decomposition for some abelian ideal  $\mathfrak{a} \subset \mathfrak{g}$  with  $\mathfrak{a} \cap \mathfrak{m} = \{0\}$ . Let  $\mathfrak{l} := \ker(\varphi)$  and  $\mathfrak{l}^\perp$  the orthogonal complement in  $\mathfrak{h}^+$  with respect to  $\rho^*g$ . Then we have the following decomposition of ideals*

$$\mathfrak{g} = (\mathfrak{l} \oplus \rho(\mathfrak{l})) \oplus_{L.a.} (\mathfrak{l}^\perp \oplus \mathfrak{h}^- \oplus \rho(\mathfrak{l}^\perp) \oplus \mathfrak{m}^-).$$

The restricted representation  $\alpha = \text{ad}|_{\mathfrak{l}^\perp \oplus \mathfrak{h}^-} : \mathfrak{l}^\perp \oplus \mathfrak{h}^- \rightarrow \mathfrak{so}(\mathfrak{m}^-)$  is faithful.

*Proof.* Let  $\mathfrak{m}_\mathfrak{l}^+ := \rho(\mathfrak{l})$  and let  $\mathfrak{m}_{\mathfrak{l}^\perp}^+ := \rho(\mathfrak{l}^\perp)$  be the orthogonal complement in  $\mathfrak{m}^+$ . Since  $\mathfrak{l}$  is an ideal we obtain  $\mathfrak{m}_\mathfrak{l}^+ \subset \mathfrak{m}^+$  is an  $\mathfrak{h}^+$ -invariant subspace and so is  $\mathfrak{m}_{\mathfrak{l}^\perp}^+$ . Combining this with Remark 3.2.6 we see that  $\mathfrak{l} \oplus \mathfrak{m}_\mathfrak{l}^+$  commutes with  $\mathfrak{l}^\perp \oplus \mathfrak{m}_{\mathfrak{l}^\perp}^+$ . Let  $n \in \mathfrak{m}^-$  and  $h + m \in \mathfrak{a}$  with  $h \in \mathfrak{l}$ ,  $m \in \mathfrak{m}_\mathfrak{l}^+$ . Then by Lemma 3.2.4 we have

$$0 = [h + m, n] = [h, n] + [m, n] = [m, n].$$

Hence  $\mathfrak{m}_\mathfrak{l}^+$  also commutes with  $\mathfrak{m}^-$  and thus it commutes with its orthogonal complement in  $\mathfrak{m}$ . From Lemma 3.2.4.iii) it follows that  $\mathfrak{l} \oplus \mathfrak{m}_\mathfrak{l}^+$  commutes with  $\mathfrak{h}^-$ . Since  $\mathfrak{l} \oplus \mathfrak{m}_\mathfrak{l}^+$  is a subalgebra we obtain it is an ideal and it commutes with  $\mathfrak{l}^\perp \oplus \mathfrak{h}^- \oplus \mathfrak{m}_{\mathfrak{l}^\perp}^+ \oplus \mathfrak{m}^-$ . From Proposition 3.2.9 we can immediately see that  $\mathfrak{l}^\perp \oplus \mathfrak{h}^- \oplus \mathfrak{m}_{\mathfrak{l}^\perp}^+ \oplus \mathfrak{m}^-$  is a subalgebra and thus also an ideal.

Suppose that  $h \in \ker(\alpha)$ . For all  $m \in \mathfrak{m}^+$  and  $n \in \mathfrak{m}^-$  we have  $[m, n] \in \mathfrak{m}^-$  by Lemma 3.2.4.ii) and Lemma 1.2.1.ii). Thus

$$0 = [h, [m, n]] = [[h, m], n] + [m, [h, n]] = [[h, m], n].$$

We conclude that  $[h, m] \in \mathfrak{m}^+$  commutes with  $\mathfrak{m}^-$ . This implies  $\rho^{-1}([h, m]) \in \mathfrak{l}$ . On the other hand  $\rho^{-1}([h, m]) = [h, \rho^{-1}(m)] \in \mathfrak{l}^\perp$ , because  $h \in \mathfrak{l}^\perp \oplus \mathfrak{h}^-$  and  $\rho^{-1}(m) \in \mathfrak{h}^+$ . We obtain  $\rho^{-1}([h, m]) \in \mathfrak{l} \cap \mathfrak{l}^\perp = \{0\}$ . Thus  $[h, m] = 0$  for all  $m \in \mathfrak{m}^+$ . In total we have  $[h, \mathfrak{m}] = \{0\}$ . This implies  $h = 0$ , because we assumed the reductive decomposition to be effective. We conclude  $\ker(\alpha) = \{0\}$ .  $\square$

By Lemma 1.3.7 the above Lemma 3.3.2 implies that for an irreducible naturally reductive decomposition of type II and any abelian ideal there are two possible cases:  $\ker(\varphi) = \{0\}$  or  $\mathfrak{m}^- = \{0\}$ . The case  $\mathfrak{m}^- = \{0\}$  corresponds to the extensions discussed in Remark 2.2.6.

**Lemma 3.3.3.** *Let  $\mathfrak{g} = \mathfrak{h}^+ \oplus \mathfrak{h}^- \oplus \mathfrak{m}^+ \oplus \mathfrak{m}^-$  be an effective irreducible naturally reductive decomposition of type II associated with an abelian ideal  $\mathfrak{a} \subset \mathfrak{g}$ . Let  $\mathfrak{h}_0 := \pi_{\mathfrak{h}}([\mathfrak{m}^-, \mathfrak{m}^-])$ , where this time  $\pi_{\mathfrak{h}}$  is the projection onto  $\mathfrak{h}$  along  $\mathfrak{a} \oplus \mathfrak{m}^-$  in  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{a} \oplus \mathfrak{m}^-$ . Let  $\mathfrak{h}_0^\perp \subset \mathfrak{h}$  be a complementary ideal of  $\mathfrak{h}_0$  in  $\mathfrak{h}$ . Then  $\mathfrak{a} \oplus \mathfrak{h}_0 \oplus \mathfrak{m}^-$  is a subalgebra of  $\mathfrak{g}$  and*

$$\mathfrak{g} \cong \mathfrak{h}_0^\perp \times (\mathfrak{a} \oplus \mathfrak{h}_0 \oplus \mathfrak{m}^-).$$

Moreover,  $\mathfrak{a}$  is contained in the center of  $\mathfrak{a} \oplus \mathfrak{h}_0 \oplus \mathfrak{m}^-$ . If we define a Lie algebra structure on  $\mathfrak{g}^- := \mathfrak{h}_0 \oplus \mathfrak{m}^-$  by the quotient  $\mathfrak{h}_0 \oplus \mathfrak{m}^- \cong (\mathfrak{a} \oplus \mathfrak{h}_0 \oplus \mathfrak{m}^-)/\mathfrak{a}$ , then  $\mathfrak{g}^- = \mathfrak{h}_0 \oplus \mathfrak{m}^-$  is a naturally reductive decomposition of the base space, with  $\mathfrak{g}^-$  its transvection algebra.

*Proof.* To see that  $\mathfrak{a} \oplus \mathfrak{h}_0 \oplus \mathfrak{m}^-$  is a subalgebra of  $\mathfrak{g}$  we first note that  $[\mathfrak{h}_0, \mathfrak{m}^-] \subset \mathfrak{m}^-$  and  $[\mathfrak{a}, \mathfrak{m}^-] = \{0\}$ , see Lemma 3.2.4.iv). Therefore, the inclusions which we still need to check are:

$$[\mathfrak{m}^-, \mathfrak{m}^-] \subset \mathfrak{a} \oplus \mathfrak{h}_0 \oplus \mathfrak{m}^- \quad \text{and} \quad [\mathfrak{a}, \mathfrak{h}_0] \subset \mathfrak{a} \oplus \mathfrak{h}_0 \oplus \mathfrak{m}^-.$$

Clearly we have  $[\mathfrak{m}^-, \mathfrak{m}^-] \subset \mathfrak{a} \oplus \mathfrak{h}_0 \oplus \mathfrak{m}^-$ . We know that  $[\mathfrak{a}, \mathfrak{m}^-] = \{0\}$  and thus

$$[\mathfrak{a}, \mathfrak{h}_0] = [\mathfrak{a}, \pi_{\mathfrak{h}}([\mathfrak{m}^-, \mathfrak{m}^-])] = [\mathfrak{a}, [\mathfrak{m}^-, \mathfrak{m}^-]] = [[\mathfrak{a}, \mathfrak{m}^-], \mathfrak{m}^-] + [\mathfrak{m}^-, [\mathfrak{a}, \mathfrak{m}^-]] = \{0\}.$$

Thus,  $\mathfrak{a} \oplus \mathfrak{h}_0 \oplus \mathfrak{m}^-$  is a subalgebra and  $\mathfrak{a}$  is contained in its center. By definition of  $\mathfrak{h}_0^\perp$  we have  $[\mathfrak{h}_0^\perp, \mathfrak{h}_0] = \{0\}$ . Furthermore, we know  $[\mathfrak{h}_0^\perp, \mathfrak{a} \oplus \mathfrak{m}^-] \subset \mathfrak{a} \oplus \mathfrak{m}^-$ . We conclude that  $\mathfrak{g} \cong \mathfrak{h}_0^\perp \times (\mathfrak{a} \oplus \mathfrak{h}_0 \oplus \mathfrak{m}^-)$ . We have shown  $(\mathfrak{a} \oplus \mathfrak{h}_0) \oplus \mathfrak{m}^-$  is a naturally reductive decomposition of the base space. We also know that  $[\mathfrak{a}, \mathfrak{m}^-] = \{0\}$ . Therefore, the quotient  $\mathfrak{h}_0 \oplus \mathfrak{m}^-$  still defines a naturally reductive decomposition of the base space. Moreover, this decomposition is effective by Lemma 3.3.2 both for the case  $\mathfrak{m}^- = \{0\}$  and for the case  $\ker(\varphi) = \{0\}$ . By definition we have  $[\mathfrak{m}^-, \mathfrak{m}^-]_{\mathfrak{h}_0} = \mathfrak{h}_0$  and thus we conclude that  $\mathfrak{g}^-$  is the transvection algebra of the base space.  $\square$

**Lemma 3.3.4.** *Let  $\mathfrak{g} = \mathfrak{h}^+ \oplus \mathfrak{h}^- \oplus \mathfrak{m}^+ \oplus \mathfrak{m}^-$  be an irreducible naturally reductive decomposition of type II with  $\mathfrak{g}$  its transvection algebra and with  $\ker(\varphi) = \{0\}$ . Let  $\mathfrak{g}^-$  be the Lie algebra from Lemma 3.3.3. Then  $\mathfrak{h}^+$  can be identified with a subalgebra of  $\mathfrak{s}(\mathfrak{g}^-)$ . Moreover, the maximal abelian ideal of  $\mathfrak{g}^-$  is preserved by  $\mathfrak{h}^+$ .*

*Proof.* By Lemma 3.3.3 we know that  $[\mathfrak{a}, \mathfrak{h}_0] = \{0\}$  and this implies that  $[\mathfrak{h}^+, \mathfrak{h}_0] = \{0\}$ . Thus, we obtain

$$\varphi(\mathfrak{h}^+) \subset \{h \in \mathfrak{so}_{\mathfrak{h}_0}(\mathfrak{m}^-) : h \cdot T_0 = 0\}.$$

Since  $\mathfrak{g}^-$  is the transvection algebra of  $\mathfrak{h}_0 \oplus \mathfrak{m}^-$  it follows by Lemma 2.1.1 that  $\mathfrak{h}^+$  is identified with a subalgebra of  $\mathfrak{s}(\mathfrak{g}^-)$ . By Lemma 3.3.1 all derivations of  $\mathfrak{g}^-$  preserve the maximal abelian ideal, so in particular  $\mathfrak{h}^+$  preserves it.  $\square$

Let the notation be as in Lemma 3.3.3 and let

$$p : \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{a} \cong \mathfrak{h}_0^\perp \ltimes \mathfrak{g}^- \quad (3.3.5)$$

be the quotient map. Now we come to the main result of this chapter.

**Theorem 3.3.6.** *Let  $(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, g)$  be an irreducible naturally reductive decomposition of type II with  $\mathfrak{g}$  its transvection algebra. Let*

$$\mathfrak{g} = \mathfrak{h}^+ \oplus \mathfrak{h}^- \oplus \mathfrak{m}^+ \oplus \mathfrak{m}^-$$

*be the fiber decomposition with respect to the maximal abelian ideal  $\mathfrak{a}$ . Then the base space associated to  $\mathfrak{a}$  is isomorphic to the following space*

$$(\mathfrak{g}^- = (\mathfrak{h}_0 \oplus \mathfrak{m}_0) \oplus_{L.a.} \mathbb{R}^n, g|_{\mathfrak{m}^- \times \mathfrak{m}^-}),$$

*where  $\mathfrak{h}_0 \oplus \mathfrak{m}_0$  is a naturally reductive decomposition of a space of type I or  $\{0\}$ . Moreover,  $(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, g)$  is isomorphic to the  $(\varphi(\mathfrak{h}^+), \rho^*g|_{\mathfrak{m}^+ \times \mathfrak{m}^+})$ -extension of  $\mathfrak{g}^- = \mathfrak{h}_0 \oplus \mathfrak{m}^-$ .*

*Proof.* By assumption our naturally reductive decomposition is irreducible. Therefore, either  $\mathfrak{l} := \ker(\varphi) = \mathfrak{h}^+$  and  $\mathfrak{m}^- = \{0\}$  or  $\mathfrak{l} = \{0\}$  holds by Lemma 3.3.2. In case  $\mathfrak{l} = \mathfrak{h}^+$  we have  $\mathfrak{g}^- = \{0\}$  and thus the base space is of the required form.

Now we consider the case  $\mathfrak{l} = \ker(\varphi) = \{0\}$ . Let  $\mathfrak{g}^- = \mathfrak{h}_0 \oplus \mathfrak{m}^-$  be the effective naturally decomposition of the base space described in Lemma 3.3.3. Let  $\mathfrak{b}$  be the maximal abelian ideal in  $\mathfrak{g}^-$ , which exists by Lemma 3.3.1. From Lemma 3.3.4 we know that  $\mathfrak{b}$  is also an abelian ideal of

$$\mathfrak{h}_0^\perp \ltimes \mathfrak{g}^- \cong \mathfrak{g}/\mathfrak{a},$$

where  $\mathfrak{h}_0^\perp$  is defined in Lemma 3.3.3. By Lemma 3.2.1 we can decompose  $\mathfrak{b}$  as

$$\mathfrak{b} = \mathfrak{b}' \oplus \mathfrak{m}_\mathfrak{b}^-,$$

where  $\mathfrak{b}'$  satisfies  $\mathfrak{b}' \cap \mathfrak{m}^- = \mathfrak{b}' \cap \mathfrak{h}_0 = \{0\}$ . By Lemma 3.3.4 we know that  $\mathfrak{h}^+ \subset \mathfrak{s}(\mathfrak{g}^-)$  preserves  $\mathfrak{b}$  and  $\mathfrak{m}^-$ . In particular this tells us that  $\mathfrak{m}_\mathfrak{b}^-$  is  $\mathfrak{h}^+$ -invariant and thus also the orthogonal complement of  $\mathfrak{m}_\mathfrak{b}^-$  in  $\mathfrak{m}^-$  is  $\mathfrak{h}^+$ -invariant. This in turn implies that  $\mathfrak{b}'$  is  $\mathfrak{h}^+$ -invariant. In Lemma 3.3.4 we saw that  $[\mathfrak{h}^+, \mathfrak{h}_0] = \{0\}$ . We can write every  $b \in \mathfrak{b}'$  as  $b = h + m^-$  with  $h \in \mathfrak{h}_0$  and  $m^- \in \mathfrak{m}^-$ . If  $d \in \mathfrak{h}^+$ , then

$$\mathfrak{b}' \ni d(b) = d(h) + d(m^-) = d(m^-) \in \mathfrak{m}^-. \quad (3.3.7)$$

Since  $\mathfrak{b}' \cap \mathfrak{m}^- = \{0\}$  we obtain  $d(\mathfrak{b}') = \{0\}$ . Let  $\bar{\mathfrak{a}} := p^{-1}(\mathfrak{b}')$ , where  $p$  is the map from (3.3.5). Then  $\bar{\mathfrak{a}}$  is an ideal in  $\mathfrak{g}$  and  $\mathfrak{a} \subset \bar{\mathfrak{a}}$ . Note that  $p([\bar{\mathfrak{a}}, \bar{\mathfrak{a}}]) = [\mathfrak{b}', \mathfrak{b}'] = \{0\}$  and thus  $[\bar{\mathfrak{a}}, \bar{\mathfrak{a}}] \subset \mathfrak{a}$ . Let  $\pi_{\mathfrak{m}} : \mathfrak{g} \rightarrow \mathfrak{m}$  be the projection onto  $\mathfrak{m}$  along  $\mathfrak{h}$ . Then  $\pi_{\mathfrak{m}}|_{\bar{\mathfrak{a}}}$  is injective. This implies that for  $x_1, x_2 \in \bar{\mathfrak{a}}$  we have

$$[x_1, x_2] = 0 \Leftrightarrow [x_1, x_2]_{\mathfrak{m}} = [x_1, x_2]_{\mathfrak{m}^+} = 0.$$

We know that  $x_i = a_i + h_{0,i} + m_i^-$ , with  $h_{0,i} + m_i^- \in \mathfrak{b}'$  and  $a_i \in \mathfrak{a}$  for  $i = 1, 2$ . Let  $m_1, \dots, m_l$  be an orthonormal basis of  $\mathfrak{m}^+$  and  $h_j = \rho^{-1}(m_j)$ . Then

$$[x_1, x_2]_{\mathfrak{m}^+} = [a_1 + h_{0,1} + m_1^-, a_2 + h_{0,2} + m_2^-]_{\mathfrak{m}^+} = [m_1^-, m_2^-]_{\mathfrak{m}^+} = \sum_{j=1}^l g([h_j, m_1^-], m_2^-) m_j,$$

where in the second equality we use  $[h_{0,1}, h_{0,2}]_{\mathfrak{m}} = 0$ ,  $[h_{0,i}, m_j^-] \in \mathfrak{m}^-$  and that  $\mathfrak{a}$  commutes with  $\mathfrak{h}_0 \oplus \mathfrak{m}^-$ . All the summands vanish by (3.3.7). We conclude  $[x_1, x_2] = 0$  and thus  $\bar{\mathfrak{a}}$  is an abelian ideal. The maximality of  $\mathfrak{a}$  implies  $\bar{\mathfrak{a}} = \mathfrak{a}$ . Hence  $\mathfrak{b}' = \{0\}$ . We have

$$\mathfrak{g}^- = \mathfrak{h}_0 \oplus \mathfrak{m}_0 \oplus \mathfrak{m}_{\mathfrak{b}}^-,$$

where  $\mathfrak{m}_0 := (\mathfrak{m}_{\mathfrak{b}}^-)^\perp \subset \mathfrak{m}^-$ . We know from Lemma 3.2.1.i) that  $[\mathfrak{m}_0, \mathfrak{m}_{\mathfrak{b}}^-] = \{0\}$ . In Lemma 3.3.3 we saw that  $\mathfrak{g}^-$  is the transvection algebra of  $\mathfrak{g}^- = \mathfrak{h}_0 \oplus \mathfrak{m}^-$ , i.e.  $\mathfrak{h}_0 = [\mathfrak{m}^-, \mathfrak{m}^-]_{\mathfrak{h}_0}$ . Thus, we have

$$\begin{aligned} [\mathfrak{h}_0, \mathfrak{m}_{\mathfrak{b}}^-] &= [[\mathfrak{m}^-, \mathfrak{m}^-]_{\mathfrak{h}_0}, \mathfrak{m}_{\mathfrak{b}}^-] = [[\mathfrak{m}^-, \mathfrak{m}^-], \mathfrak{m}_{\mathfrak{b}}^-] \\ &= [[\mathfrak{m}^-, \mathfrak{m}_{\mathfrak{b}}^-], \mathfrak{m}^-] + [\mathfrak{m}^-, [\mathfrak{m}_{\mathfrak{b}}^-, \mathfrak{m}^-]] = \{0\}. \end{aligned}$$

Hence  $\mathfrak{m}_{\mathfrak{b}}^-$  is in the center of  $\mathfrak{g}^-$ . By Lemma 3.2.1.ii) we know that  $\mathfrak{h}_0 \oplus \mathfrak{m}_0$  is a subalgebra of  $\mathfrak{g}^-$ . We conclude that

$$\mathfrak{g}^- = (\mathfrak{h}_0 \oplus \mathfrak{m}_0) \oplus_{L.a.} \mathfrak{m}_{\mathfrak{b}}^-.$$

The subalgebra  $\mathfrak{h}_0 \oplus \mathfrak{m}_0$  has no non-trivial abelian ideals, since  $\mathfrak{b}$  is the maximal abelian ideal of  $\mathfrak{g}^-$ . In other words  $\mathfrak{h}_0 \oplus \mathfrak{m}_0$  is semisimple or equal to  $\{0\}$ . The infinitesimal model of the  $(\varphi(\mathfrak{h}^+), \rho^*g|_{\mathfrak{m}^+ \times \mathfrak{m}^+})$ -extension is identified with the infinitesimal model of  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  through the isometry  $\rho \oplus \text{id} : \mathfrak{h}^+ \oplus \mathfrak{m}^- \rightarrow \mathfrak{m}^+ \oplus \mathfrak{m}^-$ . It follows directly from Proposition 3.2.9 and the equations (2.1.3) and (2.1.4) that  $\rho \oplus \text{id}$  is an isomorphism of the infinitesimal models. We conclude that  $(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, g)$  is isomorphic to the  $(\varphi(\mathfrak{h}^+), \rho^*g|_{\mathfrak{m}^+ \times \mathfrak{m}^+})$ -extension of  $(\mathfrak{g}^- = \mathfrak{h}_0 \oplus \mathfrak{m}^-, g|_{\mathfrak{m}^- \times \mathfrak{m}^-})$ .  $\square$

**Definition 3.3.8.** Let the notation be as in Theorem 3.3.6. We call the base space associated with the maximal abelian ideal the *canonical base space* of the space of type II. Furthermore, we will call  $\mathfrak{m}^+$  the *canonical fiber direction*. We call the naturally reductive decomposition

$$\mathfrak{h}_0 \oplus \mathfrak{m}_0$$

the *type I part of the canonical base space* and  $\mathbb{R}^n$  the *Euclidian part of the canonical base space*.

*Remark 3.3.9.* In [MR85] the authors proved that the class of Lie algebras which admit an invariant non-degenerate symmetric bilinear form on it is the smallest class which contains the simple and abelian Lie algebras and which is stable under direct sums and double extensions. Theorem 3.3.6 is similar in the sense that every irreducible infinitesimal model is obtained as a  $(\mathfrak{k}, B)$ -extension of an naturally reductive infinitesimal model which has a reductive transvection algebra. The biggest difference is that we do not obtain any new spaces by repeated  $(\mathfrak{k}, B)$ -extensions.

We now know that we can construct every naturally space of type II as a  $(\mathfrak{k}, B)$ -extension. For a classification we also want to make sure that all the naturally reductive spaces we list are non-isomorphic. For this it is important that if we construct a  $(\mathfrak{k}, B)$ -extension of  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ , then the canonical base space of the  $(\mathfrak{k}, B)$ -extension is equal to  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ . This is equivalent to the canonical fiber being equal to  $\mathfrak{n}$ .

We now adopt the notation from Chapter 2. Let  $(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, g)$  be a naturally reductive decomposition of the form

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}_0 \oplus_{L.a.} \mathbb{R}^n, \quad (3.3.10)$$

with  $\mathfrak{g}$  its transvection algebra and  $\mathfrak{h} \oplus \mathfrak{m}_0$  a semisimple Lie algebra. Let

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k \oplus \mathbb{R}^n,$$

where  $\mathfrak{g}_1, \dots, \mathfrak{g}_k$  are simple ideals of  $\mathfrak{g}$ . Furthermore, let  $\mathfrak{k} \subset \mathfrak{s}(\mathfrak{g})$  and  $(T, R)$  be the infinitesimal model of the  $(\mathfrak{k}, B)$ -extension. The transvection algebra of  $(T, R)$  is given by

$$\mathfrak{f} := \text{im}(R) \oplus \mathfrak{n} \oplus \mathfrak{m},$$

with the Lie bracket defined by (1.1.10). Let  $\mathfrak{d} \subset \mathfrak{f}$  be the maximal abelian ideal. We will give a criterion when  $\pi_{\mathfrak{n} \oplus \mathfrak{m}}(\mathfrak{d}) = \mathfrak{n}$ , i.e. when the base space  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  is equal to the canonical base space of the  $(\mathfrak{k}, B)$ -extension. We introduce the following notation

$$\mathfrak{w} := \ker(R|_{\text{ad}(\mathfrak{h} \oplus \mathfrak{k})}) \subset \mathfrak{so}(\mathfrak{n} \oplus \mathfrak{m}).$$

*Remark 3.3.11.* Remember that  $R : \text{ad}(\mathfrak{h} \oplus \mathfrak{k}) \rightarrow \text{ad}(\mathfrak{h} \oplus \mathfrak{k})$  is symmetric with respect to  $B_{\Lambda^2}$ , see (1.1.17). This means we have an orthogonal direct sum

$$\text{ad}(\mathfrak{h} \oplus \mathfrak{k}) = \mathfrak{w} \oplus \text{im}(R).$$

**Lemma 3.3.12.** *Let  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  be a naturally reductive decomposition with  $\mathfrak{g}$  its transvection algebra as in (3.3.10) and let  $\mathfrak{k} \subset \mathfrak{s}(\mathfrak{g})$ . Let  $(T, R)$  be the infinitesimal model of the  $(\mathfrak{k}, B)$ -extension. Let  $\mathfrak{h}^{ss}$  and  $\mathfrak{k}^{ss}$  be the semisimple commutator ideals of  $\mathfrak{h}$  and  $\mathfrak{k}$ , respectively. Then*

$$\text{ad}(\mathfrak{h}^{ss}) \oplus \text{ad}(\mathfrak{k}^{ss}) = \text{ad}(\mathfrak{h}^{ss} \oplus \mathfrak{k}^{ss}) \subset \text{im}(R) \quad \text{and} \quad \mathfrak{w} \subset \text{ad}(\mathcal{Z}(\mathfrak{h} \oplus \mathfrak{k})).$$

Moreover, if  $\mathfrak{k}_1 = \{0\}$ , then  $\mathfrak{w} = \{0\}$ .

*Proof.* Note that  $\mathfrak{h} = \mathfrak{h}^{ss} \oplus_{L.a.} \mathcal{Z}(\mathfrak{h})$ . If  $h_1, h_2 \in \mathfrak{h}^{ss}$  and  $k \in \mathfrak{k}$ , then

$$B_{\Lambda^2}(\text{ad}([h_1, h_2]), \psi(k)) = B_{\Lambda^2}(\text{ad}(h_1), \text{ad}([h_2, k])) = 0.$$

The Lie algebra  $\mathfrak{h}^{ss}$  is semisimple, so  $[\mathfrak{h}^{ss}, \mathfrak{h}^{ss}] = \mathfrak{h}^{ss}$ . Therefore, for all  $h \in \mathfrak{h}^{ss}$  and  $k \in \mathfrak{k}$  we obtain  $B_{\Lambda^2}(\text{ad}(h), \psi(k)) = 0$ . This implies that

$$R_\psi(\text{ad}(h)) = \sum_{i=1}^l B_{\Lambda^2}(\text{ad}(h), \psi(k_i))\psi(k_i) = 0$$

for all  $h \in \mathfrak{h}^{ss}$  and thus  $R(\text{ad}(h)) = R_0(\text{ad}(h)) \neq 0$ . Furthermore, for every  $z \in \mathcal{Z}(\mathfrak{h})$  and  $h_1, h_2 \in \mathfrak{h}^{ss}$  we have

$$B_{\Lambda^2}(R_0([\text{ad}(h_1), \text{ad}(h_2)]), \text{ad}(z)) = B_{\Lambda^2}(R_0(\text{ad}(h_1)), [\text{ad}(h_2), \text{ad}(z)]) = 0.$$

By assumption we have that  $R_0(\text{ad}(\mathfrak{h})) = \text{ad}(\mathfrak{h})$ . Thus, we use Lemma 1.1.19 to conclude that  $R(\text{ad}(\mathfrak{h}^{ss})) = R_0(\text{ad}(\mathfrak{h}^{ss})) = \text{ad}(\mathfrak{h}^{ss})$ . Similarly we prove that  $R(\psi(\mathfrak{k}^{ss})) = \psi(\mathfrak{k}^{ss})$ . Consequently,  $\mathfrak{w} \subset \text{ad}(\mathcal{Z}(\mathfrak{h} \oplus \mathfrak{k}))$ , because  $\mathfrak{w} \perp \text{im}(R)$  and  $\text{ad}(\mathcal{Z}(\mathfrak{h} \oplus \mathfrak{k})) \perp \text{ad}(\mathfrak{h}^{ss} \oplus \mathfrak{k}^{ss})$ .

If  $\mathfrak{k}_1 = \{0\}$ , then  $R_0(\text{ad}(\mathfrak{h})) \cap R_\psi(\psi(\mathfrak{k})) = \{0\}$ . Therefore,  $\mathfrak{w} = \{0\}$ , because  $R_0 : \text{ad}(\mathfrak{h}) \rightarrow \text{ad}(\mathfrak{h})$  and  $R_\psi : \psi(\mathfrak{k}) \rightarrow \psi(\mathfrak{k})$  are both injective.  $\square$

Let  $p : \mathfrak{g}(\mathfrak{k}) \rightarrow \mathfrak{g}(\mathfrak{k})/\mathfrak{a}$  be the quotient Lie algebra homomorphism and  $q$  the Lie algebra homomorphisms from (2.2.4). Note that Proposition 2.1.7 implies  $\mathfrak{f}$  is a ideal of  $\text{ad}(\mathfrak{h} \oplus \mathfrak{k}) \oplus \mathfrak{n} \oplus \mathfrak{m}$ . We summarize this in the following diagram:

$$\begin{array}{ccc} \mathfrak{g}(\mathfrak{k}) & & \\ \downarrow q & \searrow p & \\ \mathfrak{f} \hookrightarrow \text{ad}(\mathfrak{h} \oplus \mathfrak{k}) \oplus \mathfrak{n} \oplus \mathfrak{m} & & \mathfrak{g}(\mathfrak{k})/\mathfrak{a} \cong \mathfrak{k} \times \mathfrak{g} . \end{array} \quad (3.3.13)$$

The following result gives us criteria when the canonical base space of a  $(\mathfrak{k}, B)$ -extension of  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  is again equal to  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ . We will use this result often in our classification of spaces of type II in Chapter 4. In the following we will denote the diagonal in  $\mathfrak{k}^{ss} \oplus \mathfrak{n}^{ss}$  by  $\mathfrak{a}^{ss}$ .

**Proposition 3.3.14.** *The following are equivalent*

$$\pi_{\mathfrak{n} \oplus \mathfrak{m}}(\mathfrak{d}) = \mathfrak{n} \iff \begin{cases} (i) & \pi_{\mathfrak{m}}(\mathcal{Z}(\mathfrak{b}_1)) = \{0\} \text{ and,} \\ (ii) & \mathfrak{w} = \{0\}, \end{cases}$$

where  $\pi_{\mathfrak{n} \oplus \mathfrak{m}}$  and  $\pi_{\mathfrak{m}}$  denote the projections in  $\mathfrak{f}$  onto  $\mathfrak{n} \oplus \mathfrak{m}$  and  $\mathfrak{m}$ , respectively.

*Proof.* Suppose that  $\pi_{\mathfrak{n}}(\mathfrak{d}) = \pi_{\mathfrak{n} \oplus \mathfrak{m}}(\mathfrak{d}) \cap \mathfrak{n} \subsetneq \mathfrak{n}$ . Let  $n \in \pi_{\mathfrak{n}}(\mathfrak{d})^\perp \cap \mathfrak{n}$  and  $n \neq 0$ . Let  $k \in \mathfrak{k}$  be the element corresponding to  $n$ . From Lemma 3.3.12 we know that  $\psi(\mathfrak{k}^{ss}) \subset \text{im}(R)$ , thus  $q(\mathfrak{a}^{ss}) \subset \mathfrak{f}$ . Note that  $\mathfrak{a}^{ss} \subset \mathfrak{g}(\mathfrak{k})$  is an abelian ideal. Thus, the subalgebra  $q(\mathfrak{a}^{ss})$  is also an abelian ideal in  $\text{ad}(\mathfrak{h} \oplus \mathfrak{k}) \oplus \mathfrak{n} \oplus \mathfrak{m}$ , because  $q$  is a surjective Lie algebra homomorphism. Therefore, by Lemma 3.3.1,  $q(\mathfrak{a}^{ss}) \subset \mathfrak{d}$ . Since  $q(\mathfrak{a}^{ss}) \subset \mathfrak{d}$  we obtain  $\mathfrak{n}^{ss} \subset \pi_{\mathfrak{n} \oplus \mathfrak{m}}(\mathfrak{d})$  and thus  $k \in \mathcal{Z}(\mathfrak{k})$ . Suppose that  $\psi(k) \in \text{im}(R)$ . It is easy to see that  $k + n \in \mathcal{Z}(\mathfrak{g}(\mathfrak{k}))$ . The homomorphism  $q$  is surjective and thus  $q(k + n) = \psi(k) + n \in \mathcal{Z}(\mathfrak{f})$  and

$$\text{span}\{\psi(k) + n\} \oplus \mathfrak{d} \subset \mathfrak{f}$$

is an abelian ideal. This contradicts the maximality of  $\mathfrak{d}$ . We conclude that  $\psi(k) \notin \text{im}(R)$  and thus that  $\mathfrak{w} \neq \{0\}$ . We have shown that (ii) doesn't hold. Now we can assume that  $\mathfrak{n} \subset \pi_{\mathfrak{n} \oplus \mathfrak{m}}(\mathfrak{d})$ . Suppose  $\text{ad}(h' + k') + m \in \mathfrak{d}$ , with  $m \in \mathfrak{m}$  and  $m \neq 0$ . We will use the diagram (3.3.13) to transfer the abelian ideal  $\mathfrak{d} \subset \mathfrak{f}$  to  $\mathfrak{k} \rtimes \mathfrak{g}$  and conclude that  $\pi_{\mathfrak{m}}(\mathcal{Z}(\mathfrak{b}_1)) \neq \{0\}$ . By Lemma 3.3.4 we know that  $\mathfrak{d}$  is also preserved by all derivations of  $\mathfrak{f}$ . As pointed out above,  $\mathfrak{f} \subset \text{ad}(\mathfrak{h} \oplus \mathfrak{k}) \oplus \mathfrak{n} \oplus \mathfrak{m}$  is an ideal. It follows that  $\mathfrak{d}$  is also an abelian ideal in  $\text{ad}(\mathfrak{h} \oplus \mathfrak{k}) \oplus \mathfrak{n} \oplus \mathfrak{m}$ . Note that  $\ker(q) \subset \mathcal{Z}(\mathfrak{h} \oplus \mathfrak{k})$  and  $\ker(q)$  commutes with  $\mathfrak{n} \oplus \mathfrak{m}$ , thus  $\ker(q) \subset \mathcal{Z}(\mathfrak{g}(\mathfrak{k}))$ . The subspace  $q^{-1}(\mathfrak{d})$  is a 2-step nilpotent ideal in  $\mathfrak{g}(\mathfrak{k})$  with  $\ker(q)$  as its center. Therefore, the subalgebra  $\tilde{\mathfrak{d}} := p(q^{-1}(\mathfrak{d}))$  is a 2-step nilpotent ideal in  $\mathfrak{k} \rtimes \mathfrak{g}$ . The reductive decomposition  $\text{ad}(\mathfrak{h} \oplus \mathfrak{k}) \oplus \mathfrak{n} \oplus \mathfrak{m}$  is effective. Thus, we know that  $q(\mathfrak{a}) + \mathfrak{d}$  is an abelian ideal in  $\text{ad}(\mathfrak{h} \oplus \mathfrak{k}) \oplus \mathfrak{n} \oplus \mathfrak{m}$ , see Lemma 3.3.1. If  $\text{ad}(x) + n \in \mathfrak{d}$  with  $n \in \mathfrak{n}$  and  $x \in \mathfrak{h} \oplus \mathfrak{k}$ , then  $\text{ad}(x - k) = \text{ad}(x) + n - (\text{ad}(k) + n) \in q(\mathfrak{a}) + \mathfrak{d}$ , where  $k + n \in \mathfrak{a}$ . From Lemma 3.2.1. iii) we obtain  $\text{ad}(x) = \text{ad}(k)$  and thus  $q(\mathfrak{a}) \subset \mathfrak{d}$ . In particular, for every  $k + n \in \mathfrak{a}$  we have

$$0 = [\text{ad}(h' + k') + m, \text{ad}(k) + n] = [\text{ad}(k'), \text{ad}(k) + n],$$

where we used Lemma 3.2.4. This implies that  $k' \in \mathcal{Z}(\mathfrak{k})$ . Let

$$\tilde{d} := p(h' + k' + m) = k' + h' + m = k' + g_1 + \cdots + g_k + x \in \tilde{\mathfrak{d}},$$

where  $g_i \in \mathfrak{g}_i$  for  $i = 1, \dots, k$  and  $x \in \mathbb{R}^n$ . Consider

$$[\tilde{d}, \mathfrak{g}_i] \in \tilde{\mathfrak{d}} \cap \mathfrak{g}_i,$$

for  $i = 1, \dots, k$ . If  $[\tilde{d}, \mathfrak{g}_i] \neq \{0\}$ , then this implies that  $\mathfrak{g}_i \subset \tilde{\mathfrak{d}}$ , because  $\mathfrak{g}_i$  is simple and  $\tilde{\mathfrak{d}}$  is an ideal. This is not possible because  $\tilde{\mathfrak{d}}$  is 2-step nilpotent and  $\mathfrak{g}_i$  is simple. We conclude that  $[\tilde{d}, \mathfrak{g}_i] = \{0\}$ . Suppose that  $y \in \mathbb{R}^n$  and  $[k', y] = z \neq 0$ . Then  $[\tilde{d}, y] = z \in \tilde{\mathfrak{d}} \cap \mathbb{R}^n$ . Moreover,  $w := [k', z] \in \tilde{\mathfrak{d}} \cap \mathbb{R}^n$  and  $g(w, y) = g([k', z], y) = g([k', [k', y]], y) = -g(z, z) \neq 0$ . In particular  $w \neq 0$ . We already saw that  $q(\mathfrak{a}) \subset \mathfrak{d}$ . Therefore,  $p^{-1}(\tilde{\mathfrak{d}}) = q^{-1}(\mathfrak{d})$  and  $q(p^{-1}(\tilde{\mathfrak{d}})) = \mathfrak{d}$ . It follows that  $z, w \in \mathfrak{d} \subset \mathfrak{f}$ . If we take the Lie bracket of  $z$  and  $w$  in  $\mathfrak{f}$ , we obtain

$$[z, w] = \sum_{i=1}^l g([k_i, z], w) \text{ad}(n_i + k_i) = \sum_{i=1}^l g([k_i, z], [k', z]) \text{ad}(n_i + k_i) \neq 0,$$

where  $k_1, \dots, k_l$  is an orthonormal basis of  $\mathfrak{k}$  with respect to  $B$  and  $n_i$  is the corresponding basis of  $\mathfrak{n}$ . This contradicts the fact that  $\mathfrak{d}$  is abelian. We conclude that  $[k', y] = 0$  for all  $y \in \mathbb{R}^n$ . In other words  $k' \in \mathfrak{k}_1$ . Remember that  $k' \in \text{Der}(\mathfrak{g})$  and we showed  $k' \in \text{Der}(\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k) \subset \text{Der}(\mathfrak{g})$  and  $k' = -\text{ad}(g_1 + \cdots + g_k) = -\text{ad}(h' + m)$ . Hence we see that  $0 \neq k' \in \mathcal{Z}(\mathfrak{k}_1)$  and  $\pi_{\mathfrak{m}}(h' + m) = m \neq 0$ . Remember that we defined  $\mathfrak{b}_1 \subset \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$  by the equation  $\text{ad}(\mathfrak{b}_1) = \mathfrak{k}_1$ . This proves that (i) doesn't hold.

For the converse, suppose  $\pi_{\mathfrak{n} \oplus \mathfrak{m}}(\mathfrak{d}) = \mathfrak{n}$ . Let  $n \in \mathfrak{n}$  and  $\omega \in \text{im}(R)$  such that  $\omega + n \in \mathfrak{d}$ . Let  $k \in \mathfrak{k}$  be the corresponding element of  $n$ . Note that Lemma 3.2.4.iv) and Remark 3.2.6 imply that  $\omega = \psi(k)$ . Thus,  $\psi(\mathfrak{k}) \subset \text{im}(R)$  and if  $\omega' \in \mathfrak{w}$ , then  $B_{\Lambda^2}(\omega', \psi(k)) = 0$  for all  $k \in \mathfrak{k}$ . It follows that

$$0 = R(\omega') = R_0(\omega') + R_\psi(\omega') = R_0(\omega') + \sum_{i=1}^l B_{\Lambda^2}(\omega', \psi(k_i)) \psi(k_i) = R_0(\omega'),$$

where  $k_1, \dots, k_l$  is an orthonormal basis of  $\mathfrak{k}$ . This implies  $\omega' \perp \text{ad}(\mathfrak{h})$ , because  $\text{im}(R_0) = \text{ad}(\mathfrak{h})$  and  $R_0$  is symmetric with respect to  $B_{\Lambda^2}$ . We have  $\omega' \perp \text{ad}(\mathfrak{h} \oplus \mathfrak{k})$  and thus  $\omega' = 0$ . We conclude that  $\mathfrak{w} = \{0\}$ .

Finally, we still need to show that if  $\mathfrak{w} = 0$  and  $\pi_{\mathfrak{m}}(\mathcal{Z}(\mathfrak{b}_1)) \neq 0$ , then  $\pi_{\mathfrak{n} \oplus \mathfrak{m}}(\mathfrak{d}) \neq \mathfrak{n}$ . Let  $b = h + m \in \mathcal{Z}(\mathfrak{b}_1) \subset \mathfrak{h} \oplus \mathfrak{m}$  with  $m \neq 0$ . Let  $n \in \mathfrak{n}$  and  $k \in \mathfrak{k}$  be the corresponding elements of  $b$ . Since  $\mathfrak{w} = 0$  we know that  $\psi(k) \in \text{im}(R)$  and  $\text{ad}(h) \in \text{im}(R)$ . We easily see that  $-\psi(k) + \text{ad}(h) + m \in \mathcal{Z}(\mathfrak{f})$  and thus in particular that  $-\psi(k) + \text{ad}(h) + m \in \mathfrak{d}$ . We have  $0 \neq m \in \pi_{\mathfrak{n} \oplus \mathfrak{m}}(\mathfrak{d})$  and  $m \notin \mathfrak{n}$  and thus  $\pi_{\mathfrak{n} \oplus \mathfrak{m}}(\mathfrak{d}) \neq \mathfrak{n}$ .  $\square$

*Remark 3.3.15.* From the above lemma we see that if  $\pi_{\mathfrak{n} \oplus \mathfrak{m}}(\mathfrak{d}) = \mathfrak{n}$ , then  $\text{im}(R) = \text{ad}(\mathfrak{h} \oplus \mathfrak{k})$  and  $\text{ad}(\mathcal{Z}(\mathfrak{k}_1)) \subset \text{ad}(\mathfrak{h})$ . More precisely, we obtain

$$\text{im}(R) = \text{ad}(\mathfrak{h} \oplus \mathfrak{k}) = \text{ad}(\mathfrak{h}) \oplus \psi(\mathfrak{k}_1^{ss} \oplus \mathfrak{k}_2 \oplus \mathfrak{k}_3).$$

Next we give a criterion when two  $(\mathfrak{k}, B)$ -extensions are isomorphic.

**Proposition 3.3.16.** *For  $i = 1, 2$  let  $\mathfrak{g}_i = \mathfrak{h}_i \oplus \mathfrak{m}_i$  be naturally reductive decompositions with  $\mathfrak{g}_i$  their transvection algebras and with  $\mathfrak{g}_i$  of the form*

$$\mathfrak{g}_i = \mathfrak{h}_i \oplus \mathfrak{m}_{0,i} \oplus_{L.a.} \mathbb{R}^{n_i},$$

where  $\mathfrak{h}_i \oplus \mathfrak{m}_{0,i}$  is semisimple or  $\{0\}$ . Let  $(T_i, R_i)$  be the infinitesimal model of  $\mathfrak{g}_i = \mathfrak{h}_i \oplus \mathfrak{m}_i$  for  $i = 1, 2$ . Furthermore, let  $\mathfrak{f}_i = \mathfrak{r}_i \oplus \mathfrak{n}_i \oplus \mathfrak{m}_i$  be the transvection algebra of the  $(\mathfrak{k}_i, B_i)$ -extension of  $(T_i, R_i)$ , where  $\mathfrak{r}_i$  is the isotropy algebra. Suppose  $\mathfrak{g}_i = \mathfrak{h}_i \oplus \mathfrak{m}_i$  is the canonical base space of the  $(\mathfrak{k}_i, B_i)$ -extension for  $i = 1, 2$ . Then the  $(\mathfrak{k}_1, B_1)$ -extension and the  $(\mathfrak{k}_2, B_2)$ -extension are isomorphic if and only if there is a Lie algebra isomorphism

$$\tau : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2,$$

such that  $\tau(\mathfrak{h}_1) = \mathfrak{h}_2$ ,  $\tau|_{\mathfrak{m}_1} : \mathfrak{m}_1 \rightarrow \mathfrak{m}_2$  is an isometry and  $\tau_* : \mathfrak{k}_1 \rightarrow \mathfrak{k}_2$  is an isometry, where  $\tau_* : \text{Der}(\mathfrak{g}_1) \rightarrow \text{Der}(\mathfrak{g}_2)$  is the induced map on derivations.

*Proof.* From Lemma 1.1.11 we obtain a Lie algebra isomorphism

$$\sigma : \mathfrak{f}_1 \rightarrow \mathfrak{f}_2,$$

such that  $\sigma(\mathfrak{r}_1) = \mathfrak{r}_2$  and  $\sigma$  preserves the unique bilinear form from Kostant's theorem, see Theorem 1.1.16. The maximal abelian ideal  $\mathfrak{a}_1$  of  $\mathfrak{f}_1$  is bijectively mapped to the maximal abelian ideal  $\mathfrak{a}_2$  of  $\mathfrak{f}_2$  by  $\sigma$ . This implies that  $\sigma(\mathfrak{n}_1) = \mathfrak{n}_2$  and thus we obtain  $\sigma(\mathfrak{m}_1) = \mathfrak{m}_2$ , because  $\sigma|_{\mathfrak{n}_1 \oplus \mathfrak{m}_1} : \mathfrak{n}_1 \oplus \mathfrak{m}_1 \rightarrow \mathfrak{n}_2 \oplus \mathfrak{m}_2$  is an isometry. For all  $x, y \in \mathfrak{m}_1$  we obtain

$$\sigma(T_1(x, y)) = -\sigma([x, y]_{\mathfrak{m}_1}) = -[\sigma(x), \sigma(y)]_{\mathfrak{m}_2} = T_2(\sigma(x), \sigma(y))$$

and

$$\sigma(R_1(x, y)) = -\sigma(\text{ad}([x, y]_{\mathfrak{r}_1 \oplus \mathfrak{n}_1})) = -\text{ad}([\sigma(x), \sigma(y)]_{\mathfrak{r}_2 \oplus \mathfrak{n}_2}) = R_2(\sigma(x), \sigma(y)),$$

where  $\sigma$  also denotes the linear map  $\Lambda^2 \mathfrak{m}_1 \rightarrow \Lambda^2 \mathfrak{m}_2$  induced by  $\sigma|_{\mathfrak{m}_1} : \mathfrak{m}_1 \rightarrow \mathfrak{m}_2$ . By Lemma 1.1.11 the isometry  $\sigma|_{\mathfrak{m}_1} : \mathfrak{m}_1 \rightarrow \mathfrak{m}_2$  induces a Lie algebra isomorphism

$\tau : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ , which satisfies  $\tau(\mathfrak{h}_1) = \mathfrak{h}_2$  and  $\tau|_{\mathfrak{m}_1} = \sigma|_{\mathfrak{m}_1}$  is an isometry. Recall from Lemma 2.1.1 that

$$\mathfrak{s}(\mathfrak{g}_i) \cong \{x \in \mathfrak{so}_{\mathfrak{h}_i}(\mathfrak{m}_i) : h \cdot T_i = 0\}.$$

Under this identification  $\tau_* : \mathfrak{s}(\mathfrak{g}_1) \rightarrow \mathfrak{s}(\mathfrak{g}_2)$  is given by  $\tau_*(x) = \sigma|_{\mathfrak{m}_1} \circ x \circ (\sigma|_{\mathfrak{m}_1})^{-1}$ . Let  $k_1 \in \mathfrak{k}_1$  and let  $n_1 \in \mathfrak{n}_1$  be the element corresponding to  $k_1$ . For every  $m_2 \in \mathfrak{m}_2$  we have

$$(\sigma|_{\mathfrak{m}_1} \circ k_1 \circ (\sigma|_{\mathfrak{m}_1})^{-1})(m_2) = \sigma|_{\mathfrak{m}_1}([n_1, (\sigma|_{\mathfrak{m}_1})^{-1}(m_2)]) = [\sigma(n_1), m_2].$$

Remember that by definition  $(\mathfrak{k}_i, B_i) = (\mathfrak{n}_i, B_i)$ . Therefore,  $\tau_*|_{\mathfrak{k}_1} : \mathfrak{k}_1 \rightarrow \mathfrak{k}_2$  is given by the isometry  $\sigma|_{\mathfrak{n}_1} : \mathfrak{n}_1 \rightarrow \mathfrak{n}_2$ .

Conversely, if we are given a Lie algebra isomorphism  $\tau$  as above, then clearly the two extensions are isomorphic.  $\square$

*Remark 3.3.17.* This proposition also implies that the canonical base space is unique for every space of type II. It can be quite non-trivial whether two infinitesimal models  $(T_1, R_2)$  and  $(T_2, R_2)$  on  $(\mathfrak{m}, g)$  are equivalent. We can view the canonical base space as an invariant of the infinitesimal model. If we look in the first column of Table 4.6, we see that there are some naturally reductive structures on the same homogeneous space which have a different canonical base space. Therefore, they are non-isomorphic. For a base space  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}_0 \oplus_{L.a.} \mathbb{R}^n$  it is also quite tractable to decide when two algebras  $\mathfrak{k}_1, \mathfrak{k}_2 \subset \mathfrak{s}(\mathfrak{g})$  are conjugate to each other. This is one of the facts which guarantees us that all the spaces in our classification in Chapter 4 are non-isomorphic.

*Remark 3.3.18.* The duality of spaces of type I extends to spaces of type II. From Theorem 3.3.6 we know that every naturally reductive space of type II is a  $(\mathfrak{k}, B)$ -extension of a reductive decomposition

$$(\mathfrak{g} = \mathfrak{h}_0 \oplus \mathfrak{m}_0 \oplus \mathbb{R}^n, \mathfrak{h}_0, g),$$

where  $\mathfrak{k} \subset \mathfrak{s}(\mathfrak{g})$  and  $B$  is some  $\text{ad}(\mathfrak{k})$ -invariant metric on  $\mathfrak{k}$ . For now we adopt the following notation  $\mathfrak{g}_0 := \mathfrak{h}_0 \oplus \mathfrak{m}_0$ . Suppose that  $(\mathfrak{g}_0, \mathfrak{h}_0, g|_{\mathfrak{m}_0 \times \mathfrak{m}_0})$  has a dual space:

$$(\mathfrak{g}_0^*, \mathfrak{h}_0^*, g|_{\mathfrak{m}_0 \times \mathfrak{m}_0}^*),$$

see Remark 3.1.6. Let  $\mathfrak{g}^* := \mathfrak{g}_0^* \oplus \mathbb{R}^n$  and let  $g^*$  be the product metric of  $g|_{\mathfrak{m}_0 \times \mathfrak{m}_0}^*$  on  $\mathfrak{m}_0$  and the Euclidean metric on  $\mathbb{R}^n$ . We have a natural Lie algebra isomorphism  $\tau : \mathfrak{s}(\mathfrak{g}) \rightarrow \mathfrak{s}(\mathfrak{g}^*)$ . Let  $\mathfrak{k}^* := \tau(\mathfrak{k})$  and let  $B^* := \tau_* B$ . We define a dual naturally reductive space of type II by the  $(\mathfrak{k}^*, B^*)$ -extension of

$$(\mathfrak{g}^*, \mathfrak{h}_0^*, g^*).$$

For the classification we are only interested in irreducible naturally reductive decompositions. Hence we would like to have a criterion when a  $(\mathfrak{k}, B)$ -extension of a naturally reductive decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \oplus_{L.a.} \mathbb{R}^n$  is irreducible. The following lemma will give us such a criterion.

**Lemma 3.3.19.** *Let  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \oplus_{L.a.} \mathbb{R}^n$  be an effective naturally reductive decomposition, with  $\mathfrak{g}$  its transvection algebra and  $\mathfrak{h} \oplus \mathfrak{m}$  of type I. Furthermore, let  $\mathfrak{k} \subset \mathfrak{s}(\mathfrak{g})$  and let  $B$  be some  $\text{ad}(\mathfrak{k})$ -invariant inner product on  $\mathfrak{k}$ . Consider the following decomposition*

$$\mathfrak{g} = (\mathfrak{h}_1 \oplus \mathfrak{m}_1) \oplus_{L.a.} \cdots \oplus_{L.a.} (\mathfrak{h}_p \oplus \mathfrak{m}_p) \oplus_{L.a.} \mathfrak{m}_{p+1} \oplus_{L.a.} \cdots \oplus_{L.a.} \mathfrak{m}_{p+q}, \quad (3.3.20)$$

where  $\mathfrak{h}_i \oplus \mathfrak{m}_i$  is an irreducible naturally reductive decomposition with  $\mathfrak{h}_i \subset \mathfrak{h}$  and  $\mathfrak{m}_i \subset \mathfrak{m}$  for  $i = 1, \dots, p$  and  $\mathfrak{m}_{p+j} \subset \mathbb{R}^n$  is an irreducible  $\mathfrak{k}$ -module for  $j = 1, \dots, q$ . We choose the  $\mathfrak{m}_1, \dots, \mathfrak{m}_{p+q}$  mutually orthogonal. Suppose that  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \oplus_{L.a.} \mathbb{R}^n$  is the canonical base space of the  $(\mathfrak{k}, B)$ -extension. The  $(\mathfrak{k}, B)$ -extension is reducible if and only if there exists a non-trivial partition:

$$\{\mathfrak{m}_1, \dots, \mathfrak{m}_p, \mathfrak{m}_{p+1}, \dots, \mathfrak{m}_{p+q}\} = W' \cup W'', \quad W' \cap W'' = \emptyset,$$

and an orthogonal decomposition of ideals  $\mathfrak{k} = \mathfrak{k}' \oplus \mathfrak{k}''$  with respect to  $B$  such that  $\mathfrak{k}'$  acts trivially on all elements of  $W''$  and  $\mathfrak{k}''$  acts trivially on all elements of  $W'$ .

*Proof.* If such a partition exists, then it is clear from the formula of the  $(\mathfrak{k}, B)$ -extension and Theorem 1.3.5 that the  $(\mathfrak{k}, B)$ -extension is reducible.

For the converse, we suppose that the  $(\mathfrak{k}, B)$ -extension is reducible. Let  $\mathfrak{v} := \{v \in \mathfrak{m} \oplus \mathbb{R}^n : \varphi(k)v = 0, \forall k \in \mathfrak{k}\}$ . Suppose that  $\mathfrak{m}_i \subset \mathfrak{v}$  for some  $i = 1, \dots, p+q$ . Then we can define a partition by  $W' := \{\mathfrak{m}_i\}$ ,  $W'' := \{\mathfrak{m}_1, \dots, \hat{\mathfrak{m}}_i, \dots, \mathfrak{m}_{p+q}\}$  and define  $\mathfrak{k}' := \{0\}$  and  $\mathfrak{k}'' := \mathfrak{k}$ . From now on we assume that no  $\mathfrak{m}_i$  contained in  $\mathfrak{v}$ . Let  $\mathfrak{f}$  be the transvection algebra of the  $(\mathfrak{k}, B)$ -extension  $(T, R)$ . If the  $(\mathfrak{k}, B)$ -extension is reducible, then by Lemma 1.3.7 there exist two orthogonal ideals  $\mathfrak{f}_1 \subset \mathfrak{f}$  and  $\mathfrak{f}_2 \subset \mathfrak{f}$  with respect to the unique bilinear form from Kostant's theorem, such that  $\mathfrak{f} = \mathfrak{f}_1 \oplus \mathfrak{f}_2$  and  $\text{im}(R) = \mathfrak{r}_1 \oplus \mathfrak{r}_2$  with  $\mathfrak{r}_i \subset \mathfrak{f}_i$ . Let  $\mathfrak{a} \subset \mathfrak{f}$  be the maximal abelian ideal. Let  $\pi_i : \mathfrak{f} \rightarrow \mathfrak{f}_i$  be the projections for  $i = 1, 2$ . Now  $\pi_i(\mathfrak{a}) \subset \mathfrak{f}_i$  is an abelian ideal in  $\mathfrak{f}_i$ . Hence also  $\pi_1(\mathfrak{a}) \oplus \pi_2(\mathfrak{a})$  is an abelian ideal of  $\mathfrak{f}$ . Since  $\mathfrak{a} \subset \pi_1(\mathfrak{a}) \oplus \pi_2(\mathfrak{a})$  and  $\mathfrak{a}$  is maximal we obtain  $\mathfrak{a} = \pi_1(\mathfrak{a}) \oplus \pi_2(\mathfrak{a})$ . Hence  $\mathfrak{n} = \mathfrak{n}' \oplus \mathfrak{n}''$  with  $\mathfrak{n}' \subset \mathfrak{f}_1$  and  $\mathfrak{n}'' \subset \mathfrak{f}_2$ . In particular this implies that  $\mathfrak{n}' \perp \mathfrak{n}''$ . Let  $\mathfrak{k} = \mathfrak{k}' \oplus \mathfrak{k}''$  be the corresponding orthogonal decomposition of  $\mathfrak{k}$ . We will now show for all  $\mathfrak{m}_i$  that either  $\mathfrak{m}_i \subset \mathfrak{f}_1$  or  $\mathfrak{m}_i \subset \mathfrak{f}_2$ . Since there is no  $\mathfrak{m}_i$  contained in  $\mathfrak{v}$  we have

$$\mathbb{R}^n = [\mathfrak{k}, \mathbb{R}^n] = [\mathfrak{k}', \mathbb{R}^n] + [\mathfrak{k}'', \mathbb{R}^n].$$

Note that  $[\mathfrak{k}', \mathbb{R}^n] \subset \mathfrak{f}_1$  and  $[\mathfrak{k}'', \mathbb{R}^n] \subset \mathfrak{f}_2$ , hence  $\mathbb{R}^n = [\mathfrak{k}', \mathbb{R}^n] \oplus [\mathfrak{k}'', \mathbb{R}^n]$ . This implies that  $\mathfrak{m}_{p+j}$  is contained in either  $\mathfrak{f}_1$  or  $\mathfrak{f}_2$  for all  $j = 1, \dots, q$ . We consider the case that  $\mathfrak{h}_i \oplus \mathfrak{m}_i$  is not a reductive decomposition of an irreducible symmetric space. Note that  $[\mathfrak{k}, \mathfrak{m}_i] \neq \{0\}$ , because we assumed that  $\mathfrak{m}_i$  is not contained in  $\mathfrak{v}$ . Suppose that  $v \in [\mathfrak{k}', \mathfrak{m}_i]$  for some  $v \neq 0$  and  $1 \leq i \leq p$ . Then  $v \in \mathfrak{f}_1 \cap \mathfrak{m}_i$ . Define  $V_0 := \{v\}$  and  $V_j := \text{span}\{V_{j-1}, [V_{j-1}, \mathfrak{m}_i]_{\mathfrak{m}_i}\}$  for  $j \geq 1$ . By assumption  $\mathfrak{h}_i \oplus \mathfrak{m}_i$  is an irreducible decomposition. It is easy to see that this implies there exists a  $p \in \mathbb{N}$  for which  $V_p = \mathfrak{m}_i$ . Since  $\mathfrak{f}_1$  is an ideal we conclude that  $\mathfrak{m}_i \subset \mathfrak{f}_1$ . Similarly with  $\mathfrak{k}'$  replaced by  $\mathfrak{k}''$  and  $\mathfrak{f}_1$  replaced by  $\mathfrak{f}_2$ . If  $\mathfrak{h}_i \oplus \mathfrak{m}_i$  defines an irreducible symmetric space, then  $\mathfrak{s}(\mathfrak{h}_i \oplus \mathfrak{m}_i) = \mathcal{Z}(\mathfrak{h}_i)$ . If  $\mathcal{Z}(\mathfrak{h}_i) = \{0\}$ , then  $\mathfrak{m}_i \subset \mathfrak{v}$  and this we assumed not to be the case. The irreducible symmetric spaces for which  $\mathcal{Z}(\mathfrak{h}_i) \neq 0$  are exactly the irreducible hermitian symmetric spaces and  $\mathcal{Z}(\mathfrak{h}_i)$  is then 1-dimensional. If  $z \in \mathcal{Z}(\mathfrak{h}_i) \setminus \{0\}$ , then  $\text{ad}(z)$  is a multiple of the almost complex structure on  $\mathfrak{m}_i$ , see [Hel01, Ch. VIII]. Thus,  $[z, \mathfrak{m}_i] = \mathfrak{m}_i$  holds. By assumption  $\varphi(\mathfrak{k})$  doesn't act trivially on  $\mathfrak{m}_i$ , so there either is some  $k' \in \mathfrak{k}'$  which acts on  $\mathfrak{m}_i$  by the derivation  $\text{ad}(z)$  or otherwise there is some  $k'' \in \mathfrak{k}''$  which acts on  $\mathfrak{m}_i$  by the derivation  $\text{ad}(z)$ . In this first case we have  $\mathfrak{m}_i = [z, \mathfrak{m}_i] = [k', \mathfrak{m}_i] \subset \mathfrak{f}_1$ . In the second case we have  $\mathfrak{m}_i = [k'', \mathfrak{m}_i] \subset \mathfrak{f}_2$ . This shows that either  $\mathfrak{m}_i$  is contained in  $\mathfrak{f}_1$  or that  $\mathfrak{m}_i$  is contained in  $\mathfrak{f}_2$ . We can define a partition by  $\mathfrak{m}_i \in W'$  if  $\mathfrak{m}_i \subset \mathfrak{f}_1$  and  $\mathfrak{m}_i \in W''$  if  $\mathfrak{m}_i \subset \mathfrak{f}_2$ . Then  $\mathfrak{k}'$  acts trivially on all elements of  $W''$  and  $\mathfrak{k}''$  acts trivially on all elements of  $W'$ .  $\square$

*Remark 3.3.21.* If we consider a reducible space of type II as  $(\mathfrak{k}_1 \oplus \mathfrak{k}_2, B)$ -extension of

$$(\mathfrak{g}^- = \mathfrak{g}_1^- \oplus \mathfrak{g}_2^-, \mathfrak{h}_0, g^-)$$

as above, then any dual space is a  $(\mathfrak{k}_1^* \oplus \mathfrak{k}_2^*, B^*)$ -extension of

$$((\mathfrak{g}_1^- \oplus \mathfrak{g}_2^-)^*, (\mathfrak{h}_0)^*, (g^-)^*)$$

and is thus also reducible by Lemma 3.3.19. It is possible to have a  $(\mathfrak{k}, B)$ -extension of  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \oplus_{L.a.} \mathbb{R}^n$  such that the canonical base space of the extension is  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \oplus_{L.a.} \mathbb{R}^n$ , but that this is not the case for the dual space. Clearly condition (i) in Proposition 3.3.14 holds for a  $(\mathfrak{k}, B)$ -extension if and only if it holds for its dual space. However, condition (ii) does not. So this is still something that needs to be checked in every case.

# Chapter 4

## Classification of naturally reductive spaces

In this chapter we will classify every irreducible naturally reductive decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  with  $\mathfrak{g}$  its transvection algebra and  $\dim(\mathfrak{m})$  equal to 7 or 8. Equivalently, we classify all the irreducible infinitesimal models of locally naturally reductive spaces in dimensions 7 and 8. We also point out which infinitesimal models come from globally homogeneous spaces. These results are summarized in Theorem 4.1.9 and Theorem 4.2.6. We will do this by applying the results from ???. For the spaces of type I we will only list the compact ones and in case a non-compact dual space exists we will mention this, see Remark 3.1.6. For the spaces of type II we will only list the ones for which the type I part of the canonical base space is compact and we will mention if the type I part has a non-compact dual space. From Remark 3.3.18 we can easily see how all dual spaces are obtained.

### 4.1 Classification of type I spaces in dimension 7 and 8

In this section we will list every irreducible naturally reductive decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  with  $\mathfrak{g}$  its transvection algebra of type I and  $\dim(\mathfrak{m}) = 7$  or  $\dim(\mathfrak{m}) = 8$ . We also point out which of these give globally homogeneous spaces.

This can be done in a certain dimension  $k$  by listing all semisimple Lie algebras  $\mathfrak{g}$  with all  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear forms  $\bar{g}$  on  $\mathfrak{g}$  and all subalgebras  $\mathfrak{h}$  such that:

1.  $\dim(\mathfrak{g}/\mathfrak{h}) = k$ ,

2.  $\bar{g}|_{\mathfrak{m} \times \mathfrak{m}}$  is positive definite, where  $\mathfrak{m} = \mathfrak{h}^\perp$ ,
3. the torsion  $T$  from (1.1.13) is irreducible,
4.  $[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}} = \mathfrak{h}$ .

We will refer to these as conditions 1 to 4, as we will use them regularly. Note that condition 4 implies that  $\mathfrak{g}$  is the transvection algebra, see Lemma 3.1.3. This produces all irreducible naturally reductive decompositions  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  with  $\mathfrak{g}$  its transvection algebra of type I. To obtain all regular ones we only have to investigate when  $H$  is closed in  $G$ , where  $G$  is the simply connected Lie group with Lie algebra  $\mathfrak{g}$  and  $H$  is the connected subgroup with Lie subalgebra  $\mathfrak{h}$ , see [Kow90, Tri92].

We start by giving an upper bound for the dimension of  $\mathfrak{h}$ . Of course  $\mathfrak{h}$  is always a subalgebra of  $\mathfrak{so}(k)$  and thus  $\dim(\mathfrak{h}) \leq \frac{1}{2}k(k-1)$ . However, since  $\mathfrak{h}$  is the stabilizer of an irreducible 3-form  $T \in \Lambda^3(\mathbb{R}^k)$  we can improve this estimate. In Table 4.1 we list stabilizers of irreducible 3-forms in dimension 3 to 8 which are of the largest dimension possible.

$k$	3	4	5	6	7	8
$\mathfrak{h}$	$\mathfrak{so}(3)$	n/a	$\mathfrak{u}(2)$	$\mathfrak{su}(3)$	$\mathfrak{g}_2$	$\mathfrak{su}(3)$
$d_k := \dim(\mathfrak{h})$	3	n/a	4	8	14	8

Table 4.1: Stabilizers of some irreducible 3-form of the largest dimension possible.

Let us briefly illustrate how one can obtain this table by explaining it in dimension 8. The largest dimensional stabilizer will be a proper subalgebra of  $\mathfrak{so}(8)$  of dimension bigger than or equal to 8, since the adjoint representation of  $\mathfrak{su}(3)$  stabilizes the irreducible 3-form defined by  $T(x, y, z) := B_{\mathfrak{su}(3)}([x, y], z)$ . Note that any stabilizer is a reductive Lie algebra and its commutator ideal is equal to one of the following semisimple Lie subalgebras of  $\mathfrak{so}(8)$ :

$$\mathfrak{su}(2), \mathfrak{su}(2)^2, \mathfrak{su}(3), \mathfrak{su}(2)^3, \mathfrak{sp}(2), \mathfrak{so}(4) \oplus \mathfrak{so}(4), \mathfrak{sp}(2) \oplus \mathfrak{sp}(1), \mathfrak{g}_2, \mathfrak{su}(4), \mathfrak{so}(7).$$

The only Lie algebras  $\mathfrak{h}$  with semisimple part  $\mathfrak{su}(2)$  and  $\text{rank}(\mathfrak{h}) \leq \text{rank}(\mathfrak{so}(8)) = 4$  are  $\mathfrak{h} = \mathfrak{su}(2) \oplus \mathbb{R}^i$  for  $i = 1, 2, 3$ . These are of dimension less than or equal to 6. Hence to find the stabilizer with the largest dimension, we can forget about these cases. For the other Lie algebras we can do this by listing the complexifications of all 8-dimensional real representations and check if there exists an irreducible invariant 3-form. The next step is to check if the representation can be extended to a larger Lie algebra and see if the 3-form is still stabilized by this larger Lie algebra. To see

if the representation can be extended we just need to compute the endomorphism ring of the representation.

The following lemma will exclude many Lie subalgebras  $\mathfrak{h} \subset \mathfrak{so}(k)$  from having an invariant irreducible 3-form.

**Lemma 4.1.1.** *Suppose that  $\mathfrak{so}(l) \subset \mathfrak{h} \subset \mathfrak{so}(k)$ , where the inclusion  $\mathfrak{so}(l) \subset \mathfrak{so}(k)$  is the standard block embedding and  $l \geq 3$ . Then there is no  $\mathfrak{h}$ -invariant irreducible 3-form  $T \in \Lambda^3 \mathbb{R}^k$ .*

*Proof.* We show that there is no irreducible 3-form invariant under  $\mathfrak{so}(l)$  and this implies that there is no invariant irreducible 3-form under the  $\mathfrak{h}$ -action. As an  $\mathfrak{so}(l)$  module  $\mathbb{R}^k$  splits into two orthogonal submodules:  $\mathbb{R}^k = \mathbb{R}^l \oplus \mathbb{R}^{k-l}$ . This implies that

$$T \in \Lambda^3 \mathbb{R}^l \oplus \Lambda^2 \mathbb{R}^l \otimes \mathbb{R}^{k-l} \oplus \mathbb{R}^l \otimes \Lambda^2 \mathbb{R}^{k-l} \oplus \Lambda^3 \mathbb{R}^{k-l},$$

and all direct sums are preserved by  $\mathfrak{so}(l)$ . Let  $T_2$  denote the component of  $T$  in  $\Lambda^2 \mathbb{R}^l \otimes \mathbb{R}^{k-l}$ . We can identify  $T_2$  with an  $\mathfrak{so}(l)$ -equivariant map

$$T_2 : \Lambda^2 \mathbb{R}^l \rightarrow \mathbb{R}^{k-l}.$$

Since  $\mathfrak{so}(l)$  acts trivially on  $\mathbb{R}^{k-l}$  and has no fixed 2-forms, while  $\Lambda^2 \mathbb{R}^l \cong \mathfrak{so}(l)$  is the adjoint representation. We conclude by Schur's lemma that  $T_2 = 0$ . By a similar argument the component of  $T$  in  $\mathbb{R}^l \otimes \Lambda^2 \mathbb{R}^{k-l}$  vanishes. We conclude

$$T \in \Lambda^3 \mathbb{R}^l \oplus \Lambda^3 \mathbb{R}^{k-l}$$

and thus  $T$  is reducible. □

Note that  $\mathfrak{su}(2)^2$  is a subalgebra of the following Lie algebras

$$\mathfrak{su}(2)^3, \mathfrak{sp}(2), \mathfrak{so}(4) \oplus \mathfrak{so}(4), \mathfrak{sp}(2) \oplus \mathfrak{sp}(1), \mathfrak{g}_2, \mathfrak{su}(4), \mathfrak{so}(7).$$

Therefore, if there is no representation of  $\mathfrak{su}(2)^2$  that stabilizes an irreducible 3-form, then there is also no representation of any of these Lie algebras which stabilizes an irreducible 3-form. In the following we will denote a highest weight representations of a semisimple Lie algebra  $\mathfrak{g}$  with highest weight  $n_1 \lambda_1 + \dots + n_p \lambda_p$  as  $R(n_1, \dots, n_p)$ , where  $\lambda_1, \dots, \lambda_p$  are the fundamental weights of  $\mathfrak{g}$  in the Bourbaki labelling. All complexifications of 8-dimension faithful real representations of  $\mathfrak{su}(2)^2$  are:

$$\begin{aligned} &R(1, 0) \oplus R(0, 1), \quad R(1, 0) \oplus R(0, 2) \oplus R(0, 0), \quad R(1, 1) \oplus 4R(0, 0), \\ &R(1, 1) \oplus R(0, 1), \quad R(1, 1) \oplus R(0, 2) \oplus R(0, 0), \quad R(1, 1) \oplus R(1, 1), \\ &R(4, 0) \oplus R(0, 2), \quad R(2, 0) \oplus R(0, 2) \oplus 2R(0, 0). \end{aligned}$$

For the representations  $R(1, 0) \oplus R(0, 2) \oplus R(0, 0)$ ,  $R(1, 1) \oplus R(0, 2) \oplus R(0, 0)$ ,  $R(2, 0) \oplus R(0, 2) \oplus 2R(0, 0)$ ,  $R(1, 1) \oplus 4R(0, 0)$  and  $R(4, 0) \oplus R(0, 2)$  we can apply Lemma 4.1.1 to see that there is no invariant irreducible 3-form. For the other three representations it follows that there are no irreducible invariant 3-forms by a similar argument as that in Lemma 4.1.1. We conclude that the stabilizer of some irreducible 3-form of the largest dimension possible has  $\mathfrak{su}(3)$  as its commutator ideal. The representation  $R(1, 1)$  is the complexified adjoint representation of  $\mathfrak{su}(3)$  and it is of real type. Hence the endomorphism ring is trivial and  $\mathfrak{su}(3)$  is the stabilizer of an irreducible 3-form with the largest dimension. We also see from the table that the stabilizer of an irreducible 3-form of the second largest dimension has  $\mathfrak{su}(2)$  as its semisimple part.

Lets consider the algebra  $\mathfrak{su}(2) \oplus \mathbb{R}^3 \cong \mathfrak{u}(2) \oplus \mathbb{R}^2$ . There is only one faithful Lie algebra representation of this algebra on  $\mathbb{R}^8$ , namely:  $\mathbb{R}^8 = \mathbb{R}^4 \oplus \mathbb{R}^2 \oplus \mathbb{R}^2$ , where  $\mathbb{R}^4 = \mathbb{C}^2$  is the vector representation of  $\mathfrak{u}(2)$  and both  $\mathbb{R}^2$ -summands are an irreducible  $\mathbb{R}$ -representation. We see that there is no irreducible invariant 3-form for this representation by a similar argument as in Lemma 4.1.1. We conclude the biggest dimension of a stabilizer of an irreducible 3-form of dimension less than 8 has dimension less than or equal to 5. So for the case  $k = 8$  we only have to list all semisimple Lie algebras  $\mathfrak{g}$  with  $\dim(\mathfrak{g}) \leq 13$  and add those of dimension 16. We see that for  $k = 7$  there is a stabilizer with a relatively large dimension and there is only one naturally reductive decomposition which has  $\mathfrak{g}_2$  as isotropy algebra, namely the decomposition of  $Spin(7)/G_2$ .

In Table 4.2 we listed all semisimple Lie algebras with their dimension between 8 and 14 together with all of their 7-dimensional faithful representations. In the third column we indicated if the representation admits an invariant irreducible 3-form.

$\mathfrak{h}$	$R_{\mathbb{C}}$	inv. irred. 3-form
$\mathfrak{su}(3)$	$R(1, 0) \oplus R(0, 0)$	✓
$\mathfrak{su}(2)^3$	$R(1, 1, 0) \oplus R(0, 0, 1)$	×
$\mathfrak{so}(5)$	$R(1, 0) \oplus 2R(0, 0)$	×
$\mathfrak{su}(3) \oplus \mathfrak{su}(2)$	$\emptyset$	n/a
$\mathfrak{su}(2)^4$	$\emptyset$	n/a
$\mathfrak{so}(5) \oplus \mathfrak{su}(2)$	$\emptyset$	n/a
$\mathfrak{g}_2$	$R(1, 0)$	✓
$\mathfrak{su}(3) \oplus \mathfrak{su}(2)^2$	$\emptyset$	n/a

Table 4.2: 7-dimensional representations with irreducible 3-forms.

Lemma 4.1.1 implies that there doesn't exist an invariant irreducible 3-form for the representations of  $\mathfrak{su}(2)^3$  and  $\mathfrak{so}(5)$ . The endomorphism ring of the  $\mathfrak{su}(3)$ -representation  $R(1, 0) \oplus R(0, 0)$  is 1-dimensional. We see that the stabilizer of an irreducible 3-form in dimension 7 with the second largest dimension is  $\mathfrak{u}(3)$ . For a particular choice of basis in  $\mathbb{R}^7$  the  $\mathfrak{u}(3)$ -invariant torsion forms are spanned by

$$e_7 \wedge (e_{12} + e_{34} + 2e_{56}).$$

So for the case  $k = 7$  we only have to list all semisimple Lie algebras  $\mathfrak{g}$  with  $\dim(\mathfrak{g}) \leq 16$  and add to these the pair  $(\mathfrak{so}(7), \mathfrak{g}_2)$ .

To classify all irreducible naturally reductive decompositions  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  with  $\mathfrak{g}$  its transvection algebra in dimension  $k$  we will list all semisimple Lie algebras  $\mathfrak{g}$  of the allowed dimensions, together with all reductive algebras  $\mathfrak{h}$  which satisfy  $\dim(\mathfrak{h}) = \dim(\mathfrak{g}) - k$  and  $\text{rank}(\mathfrak{h}) \leq \min\{\text{rank}(\mathfrak{g}), \text{rank}(\mathfrak{so}(k))\}$ . Once we have the list of all such pairs  $(\mathfrak{g}, \mathfrak{h})$  the only thing still to be done is to find all possible injective Lie algebra homomorphisms  $\mathfrak{h} \rightarrow \mathfrak{g}$  up to conjugation by an automorphism of  $\mathfrak{g}$ , such that condition 3 and 4 are satisfied. We can use Lemma 3.1.9 to list all conjugacy classes of subalgebras of  $\mathfrak{so}(n)$  and  $\mathfrak{su}(n)$  in small dimensions.

#### 4.1.1 Classification of type I in dimension 7

In the second column of Table 4.3 we list all compact semisimple Lie algebras  $\mathfrak{g}$  of dimension  $7 \leq k \leq 16$ . In the third column we list all Lie algebras  $\mathfrak{h}$  of dimension  $\dim(\mathfrak{g}) - 7$  with

$$\text{rank}(\mathfrak{h}) \leq \min\{\text{rank}(\mathfrak{g}), \text{rank}(\mathfrak{so}(7))\} \leq 3.$$

The following result will exclude many cases from satisfying condition 3.

**Lemma 4.1.2.** *Let  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k$ , with  $\mathfrak{g}_i$  simple for  $i = 1, \dots, k$ . Let  $\mathfrak{h} \subset \mathfrak{g}$  be a subalgebra with a naturally reductive decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ , where  $\mathfrak{m} = \mathfrak{h}^\perp$  with respect to some  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear form. If  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  is irreducible, then*

$$\text{rank}(\mathfrak{g}) \geq \text{rank}(\mathfrak{h}) + k - 1.$$

*Proof.* For  $k = 1$  the statement is true. Suppose that it is true for a certain  $k \in \mathbb{N}$ . Let  $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k \oplus \mathfrak{g}_{k+1}$  and let us denote  $\mathfrak{g}' = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ . Let  $\pi_1 : \mathfrak{g} \rightarrow \mathfrak{g}'$  and  $\pi_2 : \mathfrak{g} \rightarrow \mathfrak{g}_{k+1}$  be the projections. Let  $\mathfrak{h}_1 := \ker(\pi_2)$ ,  $\mathfrak{h}_3 := \ker(\pi_1)$  and  $\mathfrak{h}_2 \subset \mathfrak{h}$  a complementary ideal of  $\mathfrak{h}_1 \oplus \mathfrak{h}_3$ , which exists because  $\mathfrak{h}$  is a reductive Lie algebra. Note that  $\text{rank}(\mathfrak{h}_2) \geq 1$ , because otherwise the decomposition is reducible by Lemma 1.3.7. By our induction hypothesis we have

$$\text{rank}(\mathfrak{g}') \geq \text{rank}(\mathfrak{h}_1 \oplus \mathfrak{h}_2) + k - 1.$$

Furthermore, we have

$$\text{rank}(\mathfrak{g}_{k+1}) \geq \text{rank}(\mathfrak{h}_2) + \text{rank}(\mathfrak{h}_3).$$

Combining these yields

$$\text{rank}(\mathfrak{g}) \geq \text{rank}(\mathfrak{h}_1) + \text{rank}(\mathfrak{h}_2) + k - 1 + \text{rank}(\mathfrak{h}_2) + \text{rank}(\mathfrak{h}_3) \geq \text{rank}(\mathfrak{h}) + k.$$

□

$\dim(\mathfrak{g})$	$\mathfrak{g}$	$\mathfrak{h}$
8	$\mathfrak{su}(3)$	$\mathbb{R}$
9	$\mathfrak{su}(2)^3$	$\mathbb{R}^2$
10	$\mathfrak{so}(5)$	$\mathfrak{su}(2)$
11	$\mathfrak{su}(3) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(2) \oplus \mathbb{R}$
12	$\mathfrak{su}(2)^4$	$\mathfrak{su}(2) \oplus \mathbb{R}^2$
13	$\mathfrak{so}(5) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$
14	$\mathfrak{su}(3) \oplus \mathfrak{su}(2)^2$	$\mathfrak{su}(2)^2 \oplus \mathbb{R}$
14	$\mathfrak{g}_2$	n/a
15	$\mathfrak{su}(2)^5$	$\mathfrak{su}(3)$
15	$\mathfrak{su}(4)$	$\mathfrak{su}(3)$
16	$\mathfrak{so}(5) \oplus \mathfrak{su}(2)^2$	$\mathfrak{su}(3) \oplus \mathbb{R}, \mathfrak{su}(2)^3$
16	$\mathfrak{su}(3) \oplus \mathfrak{su}(3)$	$\mathfrak{su}(3) \oplus \mathbb{R}, \mathfrak{su}(2)^3$

Table 4.3: Candidates for 7-dimensional spaces of type I.

Now that we have all candidates for the pairs  $(\mathfrak{g}, \mathfrak{h})$ , it remains to find all possible conjugacy classes of injective Lie algebra homomorphisms  $\mathfrak{h} \rightarrow \mathfrak{g}$  such that condition 3 and 4 from the beginning of this section are satisfied. The pairs  $(\mathfrak{g}, \mathfrak{h})$  which are excluded by Lemma 4.1.2 are:

$$\begin{aligned} & (\mathfrak{su}(2)^4, \mathfrak{su}(2) \oplus \mathbb{R}^2), \quad (\mathfrak{su}(3) \oplus \mathfrak{su}(2)^2, \mathfrak{su}(2)^2 \oplus \mathbb{R}), \quad (\mathfrak{su}(2)^5, \mathfrak{su}(3)) \\ & (\mathfrak{so}(5) \oplus \mathfrak{su}(2)^2, \mathfrak{su}(3) \oplus \mathbb{R}), \quad (\mathfrak{so}(5) \oplus \mathfrak{su}(2)^2, \mathfrak{su}(2)^3). \end{aligned}$$

For the pairs  $(\mathfrak{su}(3) \oplus \mathfrak{su}(3), \mathfrak{su}(2)^3)$  there doesn't exist an injective Lie algebra homomorphism from  $\mathfrak{h}$  to  $\mathfrak{g}$ . It is easily seen that no injective Lie algebra homomorphism  $\mathfrak{su}(3) \oplus \mathbb{R} \rightarrow \mathfrak{su}(3) \oplus \mathfrak{su}(3)$  satisfies condition 3 or 4. The remaining pairs are

$$\begin{aligned} & (\mathfrak{su}(3), \mathbb{R}), \quad (\mathfrak{su}(2)^3, \mathbb{R}^2), \quad (\mathfrak{so}(5), \mathfrak{su}(2)), \quad (\mathfrak{su}(3) \oplus \mathfrak{su}(2), \mathfrak{su}(2) \oplus \mathbb{R}) \\ & (\mathfrak{so}(5) \oplus \mathfrak{su}(2), \mathfrak{su}(2) \oplus \mathfrak{su}(2)), \quad (\mathfrak{su}(4), \mathfrak{su}(3)), \quad (\mathfrak{so}(7), \mathfrak{g}_2). \end{aligned}$$

Now we briefly discuss all these pairs.

**Case  $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{su}(3), \mathbb{R})$ :** Every subalgebra  $\mathbb{R} \subset \mathfrak{su}(3)$  is conjugate to one spanned by

$$r(a, b) := \begin{pmatrix} ia & 0 & 0 \\ 0 & ib & 0 \\ 0 & 0 & -i(a+b) \end{pmatrix}, \quad (4.1.3)$$

with  $a, b \in \mathbb{R}$  and not both equal to zero. By Lemma 1.1.11 two pairs  $(a, b)$  and  $(c, d)$  will give an isomorphic infinitesimal model exactly when their subalgebras are conjugate by an element  $A \in \text{Aut}(\mathfrak{su}(3))$ . If  $A$  is an inner automorphism, then  $A(r(a, b))$  has the same eigenvalues as  $r(a, b)$ . Therefore  $A$  is a signed permutation matrix in  $SU(3)$ . An outer automorphism  $\tau : \mathfrak{su}(3) \rightarrow \mathfrak{su}(3)$  is given by taking the negative transpose in  $\mathfrak{su}(3)$ . We have  $\tau(r(a, b)) = r(-a, -b)$ . The outer automorphism group of  $\mathfrak{su}(3)$  is  $\mathbb{Z}_2$ . We can now see that all pairs  $(x, y)$  for which  $\text{span}\{r(x, y)\}$  is conjugate to  $\text{span}\{r(a, b)\}$  by an automorphism of  $\mathfrak{su}(3)$  are:

$$\pm(a, b), \pm(a, -a-b), \pm(b, a), \pm(b, -a-b), \pm(-a-b, a), \pm(-a-b, b). \quad (4.1.4)$$

The connected subgroup with this Lie algebra is closed precisely when its Lie algebra is spanned by  $r(k_1, k_2)$  with  $k_1, k_2 \in \mathbb{Z}$  and not both equal to zero. These homogeneous spaces are known as Aloff-Wallach spaces:  $SU(3)/S^1_{k_1, k_2}$ , where  $S^1_{k_1, k_2}$  is the image of

$$S^1 \rightarrow SU(3); \theta \mapsto \begin{pmatrix} e^{i\theta k_1} & 0 & 0 \\ 0 & e^{i\theta k_2} & 0 \\ 0 & 0 & e^{-i\theta(k_1+k_2)} \end{pmatrix}.$$

The  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear form  $\bar{g}$  on  $\mathfrak{g}$  is induced from the Killing form of  $\mathfrak{su}(3)$ , hence for every subalgebra  $\text{span}\{r(a, b)\}$  there is a 1-parameter family of naturally reductive metrics.

**Case  $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{su}(2)^3, \mathbb{R}^2)$ :** Let  $x_1, x_2, x_3$  be the following basis of  $\mathfrak{su}(2)$ :

$$x_1 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad x_2 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad x_3 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \quad (4.1.5)$$

The  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear form on  $\mathfrak{su}(2)^3$  is given by

$$\bar{g} = \frac{-1}{8\lambda_1^2} B_{\mathfrak{su}(2)} \oplus \frac{-1}{8\lambda_2^2} B_{\mathfrak{su}(2)} \oplus \frac{-1}{8\lambda_3^2} B_{\mathfrak{su}(2)}.$$

Every subalgebra  $\mathbb{R}^2 \subset \mathfrak{su}(2)^3$  is conjugate to one given by

$$\mathfrak{h} := \text{span}\{(a'_1 x_1, a'_2 x_1, a'_3 x_1), (b'_1 x_1, b'_2 x_1, b'_3 x_1)\}.$$

By permuting the  $\mathfrak{su}(2)$ -factors we choose a basis of  $\mathfrak{h}$  of the form:

$$h_1 := (a_1\lambda_1x_1, a_2\lambda_2x_1, a_3\lambda_3x_1), \quad h_2 := (b_1\lambda_1x_1, b_2\lambda_2x_1, 0),$$

with

$$a_3 > 0, \quad b_2 > 0, \quad b_2 \geq b_1, \quad \text{and if } b_1 = b_2 \text{ then } a_2 \geq a_1. \quad (4.1.6)$$

Under these conditions every conjugacy class of  $\mathbb{R}^2 \subset \mathfrak{su}(2)^3$  is exactly represented once. From Lemma 1.3.7 we see that the naturally reductive decomposition is reducible if and only if one of the following holds:  $a_1 = a_2 = 0$ ,  $a_1 = b_1 = 0$  or  $a_2 = b_1 = 0$ . The connected subgroup  $H$  of  $SU(2)^3$  with  $\text{Lie}(H) = \mathfrak{h}$  is a closed subgroup precisely when:

$$\frac{b_2}{b_1} \in \mathbb{Q} \quad \text{and} \quad \frac{a_1}{a_3} - \frac{a_2}{a_3} \cdot \frac{b_1}{b_2} \in \mathbb{Q}. \quad (4.1.7)$$

If  $H$  is closed then it is isomorphic to  $S^1 \times S^1$  and

$$\mathfrak{h} = \text{span}\{(k_1x_1, k_2x_1, k_3x_1), (l_1x_1, l_2x_1, l_3x_1)\},$$

for certain numbers  $k_1, k_2, k_3, l_1, l_2, l_3 \in \mathbb{Z}$ . All these spaces have to be normal homogeneous by Remark 3.1.1. We obtain a 3-parameter family of naturally reductive structures on  $SU(2)^3 / (S_{k_1, k_2, k_3}^1 \times S_{l_1, l_2, l_3}^1)$ , where the parameters are  $\lambda_1, \lambda_2, \lambda_3 > 0$  and  $\text{Lie}(S_{k_1, k_2, k_3}^1 \times S_{l_1, l_2, l_3}^1) = \mathfrak{h}$ . Note that  $(\mathfrak{su}(2), \mathbb{R})$  is a symmetric pair with  $(\mathfrak{sl}(2, \mathbb{R}), \mathbb{R})$  its dual symmetric pair. We obtain the dual spaces by replacing one or two  $\mathfrak{su}(2)$ -factors by  $\mathfrak{sl}(2, \mathbb{R})$ .

**Case  $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{so}(5), \mathfrak{su}(2))$ :** For this pair there are three inequivalent faithful 5-dimensional real representations of  $\mathfrak{su}(2)$ . They are given by  $\mathbb{R}^3 \oplus \mathbb{R} \oplus \mathbb{R}$ ,  $\mathbb{R}^4 \oplus \mathbb{R}$ ,  $\mathbb{R}^5$ , where each summand is irreducible. This gives us the following simply connected spaces:

$$SO(5)/SO(3)_{\text{ir}}, \quad SO(5)/SO(3)_{\text{st}}, \quad Sp(2)/Sp(1)_{\text{st}},$$

where  $SO(3)_{\text{ir}}$  denotes the subgroup given by the 5-dimensional irreducible representation of  $SO(3)$ , and  $SO(3)_{\text{st}}$  is the standard  $SO(3)$  subgroup of  $SO(5)$ , and  $Sp(1)_{\text{st}} \subset Sp(2)$  is the standard  $Sp(1)$  subgroup. The first space corresponds to the representation  $\mathbb{R}^5$ , the second space to  $\mathbb{R}^3 \oplus \mathbb{R} \oplus \mathbb{R}$  and the last space to  $\mathbb{R}^4 \oplus \mathbb{R}$ . In particular all the possible infinitesimal models for the pair  $(\mathfrak{so}(5), \mathfrak{su}(2))$  are regular. The metric is induced from the Killing form on  $\mathfrak{so}(5)$  and thus for each case we get a 1-parameter family of naturally reductive metrics. We can easily see that these three naturally reductive spaces are not isomorphic, because they have pairwise different isotropy representations and the isotropy representations are the same as the holonomy representations of the canonical connections.

**Case  $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{su}(3) \oplus \mathfrak{su}(2), \mathfrak{su}(2) \oplus \mathbb{R})$ :** Let  $f : \mathfrak{h} \rightarrow \mathfrak{g}$  be an injective Lie algebra homomorphism. If  $f(\mathfrak{su}(2)) \subset \mathfrak{su}(2)$ , then  $f(\mathbb{R}) \subset \mathfrak{su}(3)$ , since  $f(\mathfrak{su}(2))$  and  $f(\mathbb{R})$  commute. Now condition 3 and 4 from the beginning of this section are not satisfied. There are up to conjugation only two injective Lie algebra homomorphism from  $\mathfrak{su}(2)$  to  $\mathfrak{su}(3)$  associated to the irreducible representations on  $\mathbb{C}^2$  and  $\mathbb{C}^3$ . The irreducible representation  $\mathbb{C}^3$  defines the irreducible symmetric pair  $(\mathfrak{su}(3), \mathfrak{so}(3))$ . This implies that  $f(\mathbb{R}) \subset \mathfrak{su}(2)$  and thus results in a reducible space, see Lemma 1.3.7. In other words Condition 3 is not satisfied. Hence the inclusion of  $\mathfrak{su}(2)$  in  $\mathfrak{su}(3)$  can only be the standard inclusion. We obtain the following possibilities:

$$\mathfrak{su}(2)_{\text{st}} \oplus \mathbb{R}_{a,b} \subset \mathfrak{su}(3) \oplus \mathfrak{su}(2) \quad \text{and} \quad \mathfrak{su}(2)_{\Delta} \oplus \mathbb{R} \subset \mathfrak{su}(3) \oplus \mathfrak{su}(2).$$

In the first inclusion  $\mathfrak{su}(2)_{\text{st}}$  is  $i_{\text{st}}(\mathfrak{su}(2))$  with  $i_{\text{st}}$  the standard inclusion of  $\mathfrak{su}(2)$  in  $\mathfrak{su}(3)$ , and  $\mathbb{R}_{a,b}$  is the subalgebra spanned by

$$\left( \begin{pmatrix} ia & 0 & 0 \\ 0 & ia & 0 \\ 0 & 0 & -2ia \end{pmatrix}, \begin{pmatrix} ib & 0 \\ 0 & -ib \end{pmatrix} \right). \quad (4.1.8)$$

By Lemma 1.3.7 this naturally reductive decomposition is irreducible if and only if  $a$  and  $b$  are non-zero. In this case the connected subgroup of  $SU(3) \times SU(2)$  with Lie algebra  $\mathfrak{su}(2)_{\text{st}} \oplus \mathbb{R}_{a,b}$  is closed exactly when  $\frac{a}{b} \in \mathbb{Q}$ . Hence the infinitesimal model is regular if and only if  $\frac{a}{b} \in \mathbb{Q}$ . In this case  $\mathbb{R}_{a,b} = \mathbb{R}_{k_1, k_2}$  for certain  $k_1, k_2 \in \mathbb{Z}$ . The  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear form is given by  $\bar{g} = \frac{-\lambda_1}{12} B_{\mathfrak{su}(3)} \oplus \frac{-\lambda_2}{8} B_{\mathfrak{su}(2)}$ . For this case  $\bar{g}$  has to be positive definite, i.e.  $\lambda_1, \lambda_2 > 0$ . We obtain a 2-parameter family of naturally reductive structures on  $(SU(3) \times SU(2)) / (SU(2)_{\text{st}} \times S^1_{k_1, k_2})$ , where  $\text{Lie}(S^1_{k_1, k_2}) = \mathbb{R}_{k_1, k_2}$ .

For the second inclusion  $\mathfrak{su}(2)_{\Delta} := (i_{\text{st}} \oplus \text{id})(\mathfrak{su}(2))$  and  $\mathbb{R}$  is spanned by:

$$\left( \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right).$$

The corresponding naturally reductive decomposition is irreducible and regular. The  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear form is given by  $\bar{g} = \frac{-\lambda_1}{12} B_{\mathfrak{su}(3)} \oplus \frac{-\lambda_2}{8} B_{\mathfrak{su}(2)}$ . In this case the space can be normal homogeneous or not. The normal homogeneous metrics correspond to  $\lambda_1, \lambda_2 > 0$ . For the non-normal homogeneous case we have  $\lambda_1 > 0$ ,  $\lambda_2 < 0$  and  $\lambda_1 + \lambda_2 < 0$ , see Remark 3.1.1.

Note that for both cases  $(\mathfrak{su}(3), f(\mathfrak{su}(2) \oplus \mathbb{R}))$  is a symmetric pair. Therefore, by Remark 3.1.6 we see that both spaces have non-compact duals. For a non-compact

dual the  $\text{ad}(\mathfrak{g}^*)$ -invariant non-degenerate symmetric bilinear form is given by  $\bar{g}^* = \frac{\lambda_1}{12} B_{\mathfrak{su}(2,1)} \oplus \frac{-\lambda_2}{8} B_{\mathfrak{su}(2)}$ . Note that for the first space  $\bar{g}|_{\mathfrak{h} \times \mathfrak{h}}^*$  has to be negative definite and this is precisely when  $-3a^2\lambda_1 + b^2\lambda_2 < 0$ . For the second space  $\bar{g}^*|_{\mathfrak{m} \times \mathfrak{m}}$  is positive definite if and only if  $\lambda_1, \lambda_2 > 0$  and  $-\lambda_1 + \lambda_2 < 0$ .

For the first space also  $(\mathfrak{su}(2), i_2(\mathfrak{h}))$  is a symmetric pair. If we replace this pair with its symmetric dual we obtain a naturally reductive structure on

$$(SU(3) \times SL(2, \mathbb{R})) / (SU(2) \times S_{k_1, k_2}^1).$$

The  $\text{ad}(\mathfrak{g}^*)$ -invariant non-degenerate symmetric bilinear form is  $\bar{g}^* = \frac{-\lambda_1}{12} B_{\mathfrak{su}(3)} \oplus \frac{\lambda_2}{8} B_{\mathfrak{sl}(2, \mathbb{R})}$ . We have  $\bar{g}^*|_{\mathfrak{m} \times \mathfrak{m}}$  is positive definite if and only if  $\lambda_1, \lambda_2 > 0$  and  $3a^2\lambda_1 - b^2\lambda_2 < 0$ . Suppose we replace both factors by their non-compact dual. The invariant symmetric bilinear form is  $\frac{\lambda_1}{12} B_{\mathfrak{su}(2,1)} \oplus \frac{\lambda_2}{8} B_{\mathfrak{sl}(2, \mathbb{R})}$  with  $\lambda_1, \lambda_2 > 0$ , but this has signature  $(6, 5)$  and thus  $\bar{g}|_{\mathfrak{m} \times \mathfrak{m}}$  is never positive definite.

**Case  $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{so}(5) \oplus \mathfrak{su}(2), \mathfrak{su}(2) \oplus \mathfrak{su}(2))$ :** In order for condition 3 to be satisfied we see that both  $\mathfrak{su}(2)$  factors of  $\mathfrak{h}$  need to have a non-zero image in  $\mathfrak{so}(5)$ . There is only one 5-dimensional orthogonal faithful representation of  $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \cong \mathfrak{so}(4)$  and this corresponds to the standard inclusion of  $\mathfrak{so}(4)$  in  $\mathfrak{so}(5)$ . We will denote the image of the  $\mathfrak{su}(2)$ -summand which has non-zero image in both  $\mathfrak{so}(5)$  and  $\mathfrak{su}(2)$  by  $\mathfrak{su}(2)_\Delta$ . The associated infinitesimal model is always regular and this gives us the following naturally reductive space:

$$(Spin(5) \times SU(2)) / (SU(2)_\Delta \times SU(2)).$$

On this homogeneous space we have a 2-parameter family of  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear forms:  $\bar{g} := \frac{-\lambda_1}{6} B_{\mathfrak{so}(5)} \oplus \frac{-\lambda_2}{8} B_{\mathfrak{su}(2)}$ . The normal homogeneous spaces correspond to the parameter  $\lambda_1, \lambda_2 > 0$ . The non-normal homogeneous spaces correspond to  $\lambda_1 > 0, \lambda_2 < 0$  and  $2\lambda_1 + \lambda_2 < 0$ . The inequality ensures that  $\bar{g}|_{\mathfrak{su}(2)_\Delta \times \mathfrak{su}(2)}$  is negative definite and thus  $\bar{g}|_{\mathfrak{m} \times \mathfrak{m}}$  is positive definite as explained in Remark 3.1.1, where  $\mathfrak{m}$  is the orthogonal complement of  $\mathfrak{su}(2)_\Delta \oplus \mathfrak{su}(2)$  in  $\mathfrak{spin}(5) \oplus \mathfrak{su}(2)$  with respect to  $\bar{g}$ .

Note that  $(\mathfrak{so}(5), f(\mathfrak{su}(2) \oplus \mathfrak{su}(2)))$  is a symmetric pair. From Remark 3.1.6 we see that there exists a non-compact dual. For the non-compact dual the  $\text{ad}(\mathfrak{g}^*)$ -invariant non-degenerate symmetric bilinear form is given by  $\bar{g}^* = \frac{\lambda_1}{6} B_{\mathfrak{so}(4,1)} \oplus \frac{-\lambda_2}{8} B_{\mathfrak{su}(2)}$ . The parameters  $\lambda_1$  and  $\lambda_2$  have to satisfy  $\lambda_1, \lambda_2 > 0$  and  $-2\lambda_1 + \lambda_2 < 0$  for the metric  $g|_{\mathfrak{m} \times \mathfrak{m}}^*$  to be positive definite.

**Case  $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{su}(4), \mathfrak{su}(3))$ :** There are two non-equivalent faithful representations of  $\mathfrak{su}(3)$  on  $\mathbb{C}^4$ . They correspond to the reducible representations  $\mathbb{C}^3 \oplus \mathbb{C} = R(1, 0) \oplus R(0, 0)$  and  $\overline{\mathbb{C}^3} \oplus \mathbb{C} = R(0, 1) \oplus R(0, 0)$ . The two subalgebras defined by these representations are conjugate by an outer automorphism of  $\mathfrak{su}(4)$ . Therefore, there is only one injective Lie algebra homomorphism  $\mathfrak{su}(3) \rightarrow \mathfrak{su}(4)$  up to conjugation and this is the standard inclusion. This gives us the 7-dimensional Berger sphere as a naturally reductive space

$$SU(4)/SU(3).$$

The infinitesimal model for  $(\mathfrak{su}(4), \mathfrak{su}(3))$  is always regular and we get a 1-parameter family of metrics.

**Case  $Spin(7)/G_2$ :** The space  $Spin(7)/G_2$  is isotropy irreducible and is isometric to  $S^7$  with a round metric. The infinitesimal model for  $(\mathfrak{so}(7), \mathfrak{g}_2)$  is always regular and we get a 1-parameter family of metrics.

## 4.1.2 Classification of type I in dimension 8

In the second column of Table 4.4 we list all candidates of compact semisimple Lie algebras  $\mathfrak{g}$  of dimension  $8 \leq k \leq 16$ . We have already shown that  $\mathfrak{g}$  can have dimension less than or equal to 13 or the dimension of  $\mathfrak{g}$  is 16. In the third column of Table 4.4 we list all Lie algebras of dimension  $\dim(\mathfrak{g}) - 8$  which satisfy  $\text{rank}(\mathfrak{h}) \leq \min(\text{rank}(\mathfrak{g}), \text{rank}(\mathfrak{so}(8))) \leq 4$ .

$\dim(\mathfrak{g})$	$\mathfrak{g}$	$\mathfrak{h}$
8	$\mathfrak{su}(3)$	$\{0\}$
9	$\mathfrak{su}(2)^3$	$\mathbb{R}$
10	$\mathfrak{so}(5)$	$\mathbb{R}^2$
11	$\mathfrak{su}(3) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(2), \mathbb{R}^3$
12	$\mathfrak{su}(2)^4$	$\mathfrak{su}(2) \oplus \mathbb{R}, \mathbb{R}^4$
13	$\mathfrak{so}(5) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(2) \oplus \mathbb{R}^2$
16	$\mathfrak{so}(5) \oplus \mathfrak{su}(2)^2$	$\mathfrak{su}(3), \mathfrak{su}(2)^2 \oplus \mathbb{R}^2$
16	$\mathfrak{su}(3)^2$	$\mathfrak{su}(3), \mathfrak{su}(2)^2 \oplus \mathbb{R}^2$

Table 4.4: Candidates for 8-dimensional spaces of type I.

The pairs  $(\mathfrak{g}, \mathfrak{h})$  which are excluded by Lemma 4.1.2 are:

$$\begin{aligned} & (\mathfrak{su}(3) \oplus \mathfrak{su}(2), \mathbb{R}^3), \quad (\mathfrak{su}(2)^4, \mathfrak{su}(2) \oplus \mathbb{R}), \quad (\mathfrak{su}(2)^4, \mathbb{R}^4), \\ & (\mathfrak{so}(5) \oplus \mathfrak{su}(2), \mathfrak{su}(2) \oplus \mathbb{R}^2), \quad (\mathfrak{so}(5) \oplus \mathfrak{su}(2)^2, \mathfrak{su}(2)^2 \oplus \mathbb{R}^2) \\ & (\mathfrak{su}(3)^2, \mathfrak{su}(2)^2 \oplus \mathbb{R}^2). \end{aligned}$$

For the pair  $(\mathfrak{so}(5) \oplus \mathfrak{su}(2)^2, \mathfrak{su}(3))$  there does not exist an injective Lie algebra homomorphism from  $\mathfrak{su}(3)$  to  $\mathfrak{so}(5) \oplus \mathfrak{su}(2)^2$ . The remaining cases are:

$$(\mathfrak{su}(3), \{0\}), \quad (\mathfrak{su}(2)^3, \mathbb{R}), \quad (\mathfrak{so}(5), \mathbb{R}^2), \quad (\mathfrak{su}(3) \oplus \mathfrak{su}(2), \mathfrak{su}(2)), \quad (\mathfrak{su}(3)^2, \mathfrak{su}(3)).$$

We will now discuss these.

**Case  $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{su}(3), \{0\})$ :** The pair  $(\mathfrak{su}(3), \{0\})$  is always regular. The simply connected naturally reductive space for this case is  $SU(3)$  with as metric any negative multiple of the Killing form. In other words we have a 1-parameter family of naturally reductive structures.

**Case  $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{su}(2)^3, \mathbb{R})$ :** Let  $x_1, x_2, x_3$  be as in (4.1.5). Every subalgebra  $\mathbb{R} \subset \mathfrak{su}(2)^3$  is conjugate to one given by

$$\mathbb{R}_{a_1, a_2, a_3} = \text{span}\{(a_1 x_1, a_2 x_1, a_3 x_1)\}.$$

We can permute the  $\mathfrak{su}(2)$ -factors such that  $a_3 \geq a_2 \geq a_1$  and  $a_3 > 0$ . Under these conditions all of these are not conjugate to each other. The  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear form on  $\mathfrak{su}(2)^3$  is given by

$$\bar{g} = \frac{-1}{8\lambda_1^2} B_{\mathfrak{su}(2)} \oplus \frac{-1}{8\lambda_2^2} B_{\mathfrak{su}(2)} \oplus \frac{-1}{8\lambda_3^2} B_{\mathfrak{su}(2)}.$$

All these spaces have to be normal homogeneous. From Lemma 1.3.7 we see that the naturally reductive decomposition is irreducible if and only if all  $a_1, a_2, a_3$  are non-zero. Clearly the connected subgroup of  $SU(2)^3$  with this Lie algebra is a closed subgroup if and only if  $\frac{a_1}{a_2}, \frac{a_2}{a_3}, \frac{a_3}{a_1} \in \mathbb{Q}$ . If it is closed, then there are integers  $k_1, k_2, k_3 \in \mathbb{Z}$  such that  $\mathbb{R}_{a_1, a_2, a_3} = \text{Lie}(S_{k_1, k_2, k_3}^1)$ , where  $S_{k_1, k_2, k_3}^1$  is the image of the map

$$S^1 \rightarrow SU(2)^3; \theta \mapsto \left( \left( \begin{array}{cc} e^{i\theta k_1} & 0 \\ 0 & e^{-i\theta k_1} \end{array} \right), \left( \begin{array}{cc} e^{i\theta k_2} & 0 \\ 0 & e^{-i\theta k_2} \end{array} \right), \left( \begin{array}{cc} e^{i\theta k_3} & 0 \\ 0 & e^{-i\theta k_3} \end{array} \right) \right).$$

We obtain a 3-parameter family of naturally reductive structures on  $SU(2)^3/S_{k_1, k_2, k_3}^1$ . Note that  $(\mathfrak{su}(2), \mathbb{R})$  is a symmetric pair with  $(\mathfrak{sl}(2, \mathbb{R}), \mathbb{R})$  its dual symmetric pair. We obtain the dual spaces by replacing any  $\mathfrak{su}(2)$ -factor by  $\mathfrak{sl}(2, \mathbb{R})$ .

**Case  $(\mathfrak{so}(5), \mathbb{R}^2)$ :** The subalgebra  $\mathbb{R}^2 \subset \mathfrak{so}(5)$  has to be the maximal torus. In particular these spaces are always regular. The simply connected naturally reductive space for this case is  $SO(5)/(SO(2) \times SO(2))$ , where  $SO(2) \times SO(2)$  is embedded block diagonally. The metric is induced from any negative multiple of the Killing form of  $\mathfrak{so}(5)$ . In other words we have a 1-parameter family of naturally reductive structures.

**Case  $(\mathfrak{su}(3) \oplus \mathfrak{su}(2), \mathfrak{su}(2))$ :** Up to conjugation there are two injective Lie algebra homomorphisms  $\mathfrak{su}(2) \rightarrow \mathfrak{su}(3) \oplus \mathfrak{su}(2)$  such that condition 3 and 4 from the beginning of this section are satisfied. For the inclusion in the second factor there is only the identity. For the inclusion in  $\mathfrak{su}(3)$  there are two choices, namely the standard inclusion, denoted by  $i_{\text{st}}$  and the other given by the 3-dimensional irreducible representation of  $\mathfrak{su}(2)$ , denoted by  $i_{\text{ir}}$ . For both inclusions the infinitesimal model is regular. The simply connected homogeneous spaces are:

$$(SU(3) \times SU(2))/(i_{\text{st}} \times \text{id})(SU(2)) \quad \text{and} \quad (SU(3) \times SU(2))/(i_{\text{ir}} \times \text{id})(SU(2)),$$

where we denote the corresponding group homomorphism of  $i_{\text{st}}$  and  $i_{\text{ir}}$  also by  $i_{\text{st}}$  and  $i_{\text{ir}}$ , respectively. There is a 2-parameter family of  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear forms:  $\bar{g} = \frac{-\lambda_1}{12} B_{\mathfrak{su}(3)} \oplus \frac{-\lambda_2}{8} B_{\mathfrak{su}(2)}$ . The normal homogeneous spaces correspond to  $\lambda_1, \lambda_2 > 0$ . For the non-normal homogeneous spaces we have  $\lambda_1 > 0$  and  $\lambda_2 < 0$ . Furthermore, we require that the condition  $\lambda_1 + \lambda_2 < 0$  holds for the first space and  $4\lambda_1 + \lambda_2 < 0$  for the second space.

For the space  $(SU(3) \times SU(2))/(i_{\text{ir}} \times \text{id})(SU(2))$  there is a non-compact dual space

$$(SL(3, \mathbb{R}) \times SU(2))/(i_{\text{ir}} \times \text{id})(SU(2)).$$

The  $\text{ad}(\mathfrak{g}^*)$ -invariant non-degenerate symmetric bilinear forms are

$$\bar{g}^* = \frac{\lambda_1}{12} B_{SL(3, \mathbb{R})} \oplus \frac{-\lambda_2}{8} B_{\mathfrak{su}(2)}.$$

In order to get a positive definite metric on our space require that  $\lambda_1, \lambda_2 > 0$  and  $-4\lambda_1 + \lambda_2 < 0$ .

**Case  $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{su}(3)^2, \mathfrak{su}(3))$ :** There are two possible conjugacy classes of the subalgebra  $\mathfrak{su}(3)$ , namely  $\mathfrak{su}(3) \times \{0\}$  and the diagonal  $\mathfrak{su}(3)_\Delta$ . The first case clearly doesn't satisfy condition 4. The  $\text{ad}(\mathfrak{g})$ -invariant metrics are given by  $\bar{g} = -\lambda_1 B_{\mathfrak{su}(3)} \oplus -\lambda_2 B_{\mathfrak{su}(3)}$ , with  $\lambda_1 \neq 0$  and  $\lambda_2 \neq 0$ . By permuting the two  $\mathfrak{su}(3)$ -factors we can assume that  $\lambda_1 > \lambda_2$ . The normal homogeneous spaces correspond

to  $\lambda_1, \lambda_2 > 0$ . Note that for  $\lambda_1 = \lambda_2$  and  $\lambda_1 > 0$  we obtain a symmetric space. For the non-normal homogeneous spaces we require that the signature of  $\bar{g}$  is  $(8, 8)$  and that  $\bar{g}|_{\mathfrak{h} \times \mathfrak{h}}$  is negative definite, see Remark 3.1.1. This is the case if and only if  $\lambda_1 + \lambda_2 < 0$  and  $\lambda_1 > 0 > \lambda_2$ . All the naturally reductive structures are regular and irreducible. For every case the homogeneous space is diffeomorphic to  $(SU(3) \times SU(3))/SU(3)_\Delta \cong SU(3)$ .

This concludes the classification of all 7- and 8-dimensional naturally reductive spaces of type I. We summarize the discussion from Section 4.1.1 and Section 4.1.2 as the following result.

**Theorem 4.1.9.** *Every 7- and 8-dimensional compact globally homogeneous naturally reductive space of type I is presented in Table 4.5. In the first column  $\text{Lie}(G)$  is the transvection algebra of the naturally reductive space. The second column indicates if there exist non-compact dual spaces. The third column indicates the number of parameters of naturally reductive structures.*

$G/H$	dual space	# param.
$SU(3)/S^1_{k_1, k_2}$	×	1
$SU(2)^3/(S^1_{k_1, k_2, k_3} \times S^1_{l_1, l_2, l_3})$	✓	3
$SO(5)/SO(3)_{\text{ir}}$	×	1
$SO(5)/SO(3)_{\text{st}}$	×	1
$Sp(2)/Sp(1)_{\text{st}}$	×	1
$(SU(3) \times SU(2))/(SU(2) \times S^1_{a, b})$	✓	2
$(SU(3) \times SU(2))/(SU(2)_\Delta \times S^1)$	✓	2
$(SO(5) \times SU(2))/(SU(2)_\Delta \times SU(2))$	✓	2
$SU(4)/SU(3)$	×	1
$Spin(7)/G_2$	×	1
$SU(3)$	×	1
$SU(2)^3/S^1_{k_1, k_2, k_3}$	✓	3
$SO(5)/(SO(2) \times SO(2))$	×	1
$(SU(3) \times SU(2))/SU(2)_{\text{st} \times \text{id}}$	×	2
$(SU(3) \times SU(2))/SU(2)_{\text{ir} \times \text{id}}$	✓	2
$(SU(3) \times SU(3))/SU(3)_\Delta$	×	2

Table 4.5: 7- and 8-dimensional naturally reductive spaces of type I.

## 4.2 Classification of type II spaces in dimension 7 and 8

By Theorem 3.3.6 we can construct every infinitesimal model of a locally naturally reductive space of type II as a  $(\mathfrak{k}, B)$ -extension of a naturally reductive decomposition of the form

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}_0 \oplus_{L.a.} \mathbb{R}^n,$$

where  $\mathfrak{h} \oplus \mathfrak{m}_0$  is semisimple and  $\mathfrak{g}$  is the transvection algebra of this naturally reductive decomposition. In this section we will construct all 7 and 8 dimensional irreducible  $(\mathfrak{k}, B)$ -extensions of all naturally reductive decomposition of the above form.

For the spaces of type II we only list the ones for which the type I part of the base space is compact. We saw in Remark 3.3.18 that we can easily obtain all naturally reductive spaces of type II from this list by taking all duals. For every case we will mention if there exist dual spaces. We will systematically run through all possibilities for the canonical base space. Note that to classify the naturally reductive spaces of type II in some dimension  $k$  we need the classification of all naturally reductive spaces of type I up to dimension  $k - 1$ .

### 4.2.1 Classification of type II in dimension 7

We will first argue, by systematically excluding all other possibilities, that all possible canonical base spaces of irreducible naturally reductive decomposition of type II with a compact type I part are the following:

$$\begin{aligned} &\mathbb{R}^6, \mathbb{R}^4, S^2 \times \mathbb{R}^4, SU(2) \times \mathbb{R}^2, S^2 \times S^2 \times \mathbb{R}^2, \mathbb{C}P^2 \times \mathbb{R}^2, Sp(2)/(SU(2) \times S^1), \\ &SU(3)/(S^1 \times S^1), SO(5)/(SO(3) \times SO(2)), SU(4)/S(U(1) \times U(3)), \mathbb{C}P^2 \times S^2, \\ &S^2 \times S^2 \times S^2. \end{aligned}$$

The Euclidian factor can't be  $\mathbb{R}^5$ , because then the Lie algebra  $\mathfrak{k} \subset \mathfrak{so}(5)$  is two dimensional and its linear action on  $\mathbb{R}^5$  has a vector on which it acts trivially. From Lemma 3.3.19 we see that such a  $(\mathfrak{k}, B)$ -extension results in a reducible naturally reductive space.

Suppose that the Euclidian factor is  $\mathbb{R}^3$ , the Lie algebra  $\mathfrak{k} \subset \mathfrak{so}(3)$  has to be equal to  $\mathfrak{so}(3)$  in order not to have a vector on which it acts trivially. This means that the type I part of the base space has to be 1-dimensional, which is not possible.

Suppose that the Euclidian factor is  $\mathbb{R}^2$ . If the dimension of the type I part is two. Then  $\dim(\mathfrak{s}(\mathfrak{g})) \leq 2$  and thus we can't construct an irreducible  $(\mathfrak{k}, B)$ -extension of dimension 7. If the dimension of the type I part is three, then  $\dim(\mathfrak{k})$  has to be equal to 2 and thus  $\mathfrak{k}$  is abelian. The type I part is either  $SU(2)$  or

the symmetric space  $(SU(2) \times SU(2))/SU(2)_\Delta$ . For the symmetric space we have  $\dim(\mathfrak{s}(\mathfrak{g})) = \dim(\mathfrak{so}(2)) = 1$  and thus we can't make an irreducible 7-dimensional space out of it by Lemma 3.3.19. If the dimension of the type I part is four, then the type I part has to be a hermitian symmetric space by [KV83]. There are only two compact homogeneous spaces which allow a hermitian symmetric structure, these are  $S^2 \times S^2$  and  $\mathbb{C}P^2$ .

For all other 7-dimensional naturally reductive spaces of type II the base space has only a type I part. We check that every 7-dimensional  $(\mathfrak{k}, B)$ -extension of any naturally reductive space of type I of dimension less than or equal to 4 is reducible. This leaves us with the 5- and 6-dimensional cases. The only compact spaces of type I in dimension 5 with  $\dim(\mathfrak{s}) \geq 2$  are  $S^2 \times SU(2)$  and  $(SU(2) \times SU(2))/S^1$ . However, we see for any  $\mathfrak{k} \subset \mathfrak{s}$  that condition (i) of Proposition 3.3.14 is not satisfied in both cases. For the 6-dimensional spaces of type I we have the nearly Kähler spaces  $G_2/SU(3)$  and  $(SU(2) \times SU(2) \times SU(2))/SU(2)$ , for both of these  $\mathfrak{s} = \{0\}$  holds. Hence we can discard them. Similarly  $((SU(2) \times SU(2))/SU(2)_\Delta) \times ((SU(2) \times SU(2))/SU(2)_\Delta)$  satisfies  $\mathfrak{s} = \{0\}$  and can be discarded. The spaces  $SU(2) \times ((SU(2) \times SU(2))/SU(2)_\Delta)$  and  $SU(2) \times SU(2)$  don't satisfy condition (i) from Proposition 3.3.14 for any  $\mathfrak{k} \subset \mathfrak{s}$  and thus they can be discarded as well. All other 6-dimensional naturally reductive spaces of type I are possible.

For all of these base spaces we will give all possible  $(\mathfrak{k}, B)$ -extensions. How to obtain a globally homogeneous naturally reductive space from this is described in Chapter 2. We will mention in which cases dual naturally reductive spaces exist in the sense of Remark 3.3.18. Whenever  $\mathfrak{k}_1 \neq \{0\}$  we need to check condition (ii) from Proposition 3.3.14, see Lemma 3.3.12. We use Proposition 3.3.16 to conclude that all the spaces we list are non-isomorphic.

**The canonical base space is  $\mathbb{R}^6$ :** The Lie algebra  $\mathfrak{k}$  is 1-dimensional. Let  $k$  be a unit vector in  $\mathfrak{k}$ . Then there is an orthonormal basis  $e_1, \dots, e_6$  of  $\mathbb{R}^6$  such that

$$\varphi(k) = c_1 e_{12} + c_2 e_{34} + c_3 e_{56},$$

for  $c_1, c_2, c_3 \in \mathbb{R}$ . It is clear from Lemma 3.3.19 that the spaces are irreducible precisely when  $c_1, c_2, c_3 \in \mathbb{R} \setminus \{0\}$ . Therefore, from now on we suppose that  $c_1, c_2, c_3 \in \mathbb{R} \setminus \{0\}$ . The  $(\mathfrak{k}, B)$ -extensions describe naturally reductive structures on the 7-dimensional Heisenberg group, as explained in Section 2.2.2. We get a 3-parameter family of naturally reductive structures on the 7-dimensional Heisenberg group. We can ensure that  $c_1 \geq c_2 \geq c_3$  and  $c_1 > 0$  by choosing a different basis of  $\mathbb{R}^6$ . When we do this, all the described naturally reductive structures are non-isomorphic.

**The canonical base space is  $\mathbb{R}^4$ :** The Lie algebra  $\mathfrak{k}$  has to be  $\mathfrak{su}(2)$  and the resulting space is the quaternionic Heisenberg group as explained explicitly in Example 2.2.16. We get a 1-parameter family of naturally reductive structures on the quaternionic Heisenberg group

**The canonical base space is  $S^2 \times \mathbb{R}^4$ :** Let  $h, e_1, e_2$  be an orthonormal basis of  $\mathfrak{su}(2)$  with respect to  $\frac{-1}{8\lambda_1^2}B_{\mathfrak{su}(2)}$ . The transvection algebra of the base space is given by

$$\mathfrak{g} = \mathfrak{su}(2) \oplus_{L.a.} \mathbb{R}^4 = \mathfrak{h} \oplus \mathfrak{m} \oplus_{L.a.} \mathbb{R}^4,$$

where  $\mathfrak{h} := \text{span}\{h\}$  and  $\mathfrak{m} := \text{span}\{e_1, e_2\}$ . The  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear form on  $\mathfrak{g}$  is given by  $\bar{g} = \frac{-1}{8\lambda_1^2}B_{\mathfrak{su}(2)} \oplus B_{eucl}$ . We have  $\mathfrak{s}(\mathfrak{g}) = \text{span}\{h\} \oplus \mathfrak{so}(4)$ . Let  $k \in \mathfrak{k}$  be a unit vector. Then there is an orthonormal basis of  $\mathbb{R}^4$  such that

$$\varphi(k) = c_1 \text{ad}(h) + c_2 e_{34} + c_3 e_{56},$$

with  $c_1, c_2, c_3 \in \mathbb{R}$ . All these spaces are irreducible precisely when  $c_1, c_2, c_3 \in \mathbb{R} \setminus \{0\}$  by Lemma 3.3.19. Therefore, from now on we suppose that  $c_1, c_2, c_3 \in \mathbb{R} \setminus \{0\}$ . We have  $\mathfrak{k} = \mathfrak{k}_2$  and from Section 2.2.3 we know that the  $(\mathfrak{k}, B)$ -extension defines a naturally reductive structure on  $S^2 \times H^5$ , where  $H^5$  denotes the 5-dimensional Heisenberg group. On this homogeneous space we obtain a 4-parameter family of naturally reductive structures, with  $c_1, c_2, c_3$  and  $\lambda_1 > 0$  as parameters. By choosing a different basis of  $\mathbb{R}^4$  and changing the sign of  $k$  we can arrange that  $c_2 \geq c_3$  and  $c_1 > 0$ . When we do this, none of the naturally reductive structures are isomorphic. Note that we can replace the type I part  $S^2 = SU(2)/S^1$  by its non-compact dual symmetric space:  $SL(2, \mathbb{R})/S^1$ .

**The canonical base space is  $SU(2) \times \mathbb{R}^2$ :** Let  $e_1, e_2, e_3$  be an orthonormal basis of  $\mathfrak{su}(2)$  with respect to  $-\frac{1}{8\lambda^2}B_{\mathfrak{su}(2)}$ . The transvection algebra of the base space is given by

$$\mathfrak{g} = \mathfrak{su}(2) \oplus_{L.a.} \mathbb{R}^2 = \mathfrak{m} \oplus_{L.a.} \mathbb{R}^2,$$

where  $\mathfrak{m} := \text{span}\{e_1, e_2, e_3\}$ . The  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear form on  $\mathfrak{g}$  is given by  $\bar{g} = -\frac{1}{8\lambda^2}B_{\mathfrak{su}(2)} \oplus B_{eucl}$ . We have  $\mathfrak{s}(\mathfrak{g}) = \mathfrak{su}(2) \oplus \mathfrak{so}(2)$  and  $\mathfrak{k} \subset \mathfrak{s}(\mathfrak{g})$  is a 2-dimensional subalgebra. In particular  $\mathfrak{k}$  is abelian. Let  $B$  be some  $\text{ad}(\mathfrak{k})$ -invariant metric on  $\mathfrak{k}$ . We can choose an orthonormal basis  $k_1, k_2$  of  $\mathfrak{k}$  such that

$$\varphi(k_1) = c_1 \text{ad}(e_1) + c_2 e_{45} \quad \text{and} \quad \varphi(k_2) = d_1 \text{ad}(e_1),$$

where  $c_2 > 0$  and  $d_1 > 0$ . In particular we have  $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_3$ , where  $\mathfrak{k}_1$  and  $\mathfrak{k}_3$  are both 1-dimensional. From Lemma 3.3.19 we see that the  $(\mathfrak{k}, B)$ -extension is reducible

precisely when  $c_1 = 0$ . From Section 2.2.3 we know that this gives us a 4-parameter family of naturally reductive structures on  $SU(2) \times H^3 \times \mathbb{R}$ , with  $c_1, c_2, d_1$  and  $\lambda > 0$  as parameters, and  $H^3$  denotes the 3-dimensional Heisenberg group. None of these structures are isomorphic.

**The canonical base space is  $S^2 \times S^2 \times \mathbb{R}^2$ :** Let  $h_1, e_1, e_2$  be an orthonormal basis of  $\mathfrak{su}(2)$  with respect to  $\frac{-1}{8\lambda_1^2}B_{\mathfrak{su}(2)}$ . Let  $h_2, e_3, e_4$  be an orthonormal basis of  $\mathfrak{su}(2)$  with respect to  $\frac{-1}{8\lambda_2^2}B_{\mathfrak{su}(2)}$ . The transvection algebra of the base space is given by

$$\mathfrak{g} = \mathfrak{su}(2) \oplus_{L.a.} \mathfrak{su}(2) \oplus_{L.a.} \mathbb{R}^2 = \mathfrak{h} \oplus \mathfrak{m} \oplus_{L.a.} \mathbb{R}^2,$$

where  $\mathfrak{h} := \text{span}\{h_1, h_2\}$  and  $\mathfrak{m} := \text{span}\{e_1, e_2, e_3, e_4\}$ . The  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear form on  $\mathfrak{g}$  is given by  $\bar{g} = \frac{-1}{8\lambda_1^2}B_{\mathfrak{su}(2)} \oplus \frac{-1}{8\lambda_2^2}B_{\mathfrak{su}(2)} \oplus B_{\text{eucl}}$ . Let  $e_5, e_6$  be an orthonormal basis of  $\mathbb{R}^2$ . We have  $\mathfrak{s}(\mathfrak{g}) = \mathfrak{h} \oplus \mathfrak{so}(2)$ . Let  $k$  be a unit vector in  $\mathfrak{k}$ . Then

$$\varphi(k) = c_1 \text{ad}(h_1) + c_2 \text{ad}(h_2) + c_3 e_{56},$$

for  $c_1, c_2, c_3 \in \mathbb{R}$ . It is clear from Lemma 3.3.19 that the  $(\mathfrak{k}, B)$ -extension is irreducible if and only if  $c_1, c_2, c_3 \in \mathbb{R} \setminus \{0\}$ . From now on we suppose that  $c_1, c_2, c_3 \in \mathbb{R} \setminus \{0\}$ . From Section 2.2.3 we know that the  $(\mathfrak{k}, B)$ -extension defines a naturally reductive structure on  $S^2 \times S^2 \times H^3$ . On this space we get a 5-parameter family of naturally reductive structures, with  $c_1, c_2, c_3$  and  $\lambda_1, \lambda_2 > 0$  as parameters. By permuting the two  $S^2$  factors we can always assume that  $c_1 \geq c_2$  and by changing the sign of  $k$  we can assume that  $c_1 > 0$ . When we do this, all the described naturally reductive structures are non-isomorphic. Note that we can replace one or both of the type I factors  $S^2 = SU(2)/S^1$  by its non-compact dual symmetric space:  $SL(2, \mathbb{R})/S^1$ .

**The canonical base space is  $\mathbb{C}P^2 \times \mathbb{R}^2$ :** As reductive decomposition for  $\mathbb{C}P^2$  we take

$$(\mathfrak{su}(3), \mathfrak{u}(2), -\frac{1}{12\lambda^2}B_{\mathfrak{su}(3)}),$$

where  $B_{\mathfrak{su}(3)}$  is the Killing form of  $\mathfrak{su}(3)$ . We pick the following basis for  $\mathfrak{h} := \mathfrak{u}(2)$ :

$$\begin{aligned} h_1 &= \begin{pmatrix} \lambda i & 0 & 0 \\ 0 & -\lambda i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & -\lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h_3 = \begin{pmatrix} 0 & i\lambda & 0 \\ i\lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ h_4 &= \begin{pmatrix} \frac{-i\lambda}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{-i\lambda}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{2i\lambda}{\sqrt{3}} \end{pmatrix}. \end{aligned} \tag{4.2.1}$$

For the orthogonal complement  $\mathfrak{m}$  of  $\mathfrak{h}$  we choose the following orthonormal basis:

$$e_1 = \begin{pmatrix} 0 & 0 & -\lambda \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & i\lambda \\ 0 & 0 & 0 \\ i\lambda & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\lambda \\ 0 & \lambda & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i\lambda \\ 0 & i\lambda & 0 \end{pmatrix}. \quad (4.2.2)$$

The transvection algebra of  $\mathbb{C}P^2 \times \mathbb{R}^2$  is

$$\mathfrak{g} = \mathfrak{su}(3) \oplus_{L.a.} \mathbb{R}^2 = \mathfrak{u}(2) \oplus \mathfrak{m} \oplus_{L.a.} \mathbb{R}^2,$$

with  $\bar{g} = -\frac{1}{12\lambda^2} B_{\mathfrak{su}(3)} \oplus B_{eucl}$  as  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear form on  $\mathfrak{g}$ . Let  $e_5, e_6$  be an orthonormal basis for  $\mathbb{R}^2$ . We have  $\mathfrak{s}(\mathfrak{g}) = \mathcal{Z}(\mathfrak{u}(2)) \oplus \mathfrak{so}(2)$  and  $\mathcal{Z}(\mathfrak{u}(2)) = \text{span}\{h_4\}$ . Let  $k$  be a unit vector in  $\mathfrak{k}$ . Then

$$\varphi(k) = c_1 \text{ad}(h_4) + c_2 e_{56} = c_1 \sqrt{3} \lambda (e_{12} + e_{34}) + c_2 e_{56},$$

for  $c_1, c_2 \in \mathbb{R}$ . It is clear from Lemma 3.3.19 that the  $(\mathfrak{k}, B)$ -extension is irreducible if and only if  $c_1, c_2 \in \mathbb{R} \setminus \{0\}$ , which we will assume from now on. By changing the sign of  $k$  we can assume that  $c_1 > 0$ . From Chapter 2 we know that the  $(\mathfrak{k}, B)$ -extension defines a naturally reductive structure on  $\mathbb{C}P^2 \times H^3$ . On this space we have a 3-parameter family of naturally reductive structures, with  $c_1, c_2$  and  $\lambda > 0$  as parameters. All of these are non-isomorphic. Note that we can replace  $\mathbb{C}P^2$  by its non-compact dual symmetric space.

**The canonical base space is  $Sp(2)/(SU(2) \times S^1)$ :** We consider  $Sp(2) \subset GL(2, \mathbb{H})$ . We denote by  $i, j, k$  the imaginary quaternions, i.e.  $i^2 = j^2 = k^2 = ijk = -1$  and we pick the following basis for  $\mathfrak{u}(2)$ :

$$h_1 = \lambda \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \quad h_2 = \lambda \begin{pmatrix} j & 0 \\ 0 & 0 \end{pmatrix}, \quad h_3 = \lambda \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}, \quad h_4 = \lambda \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}.$$

For the orthogonal complement of  $\mathfrak{u}(2)$  we choose the bases:

$$e_1 = \frac{\lambda}{\sqrt{2}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \frac{\lambda}{\sqrt{2}} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad e_3 = \frac{\lambda}{\sqrt{2}} \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, \\ e_4 = \frac{\lambda}{\sqrt{2}} \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}, \quad e_5 = \lambda \begin{pmatrix} 0 & 0 \\ 0 & j \end{pmatrix}, \quad e_6 = \lambda \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}.$$

The basis  $\{e_1, \dots, e_6, h_1, \dots, h_4\}$  is an orthonormal basis with respect to the metric  $\bar{g} = \frac{-1}{6\lambda^2} B_{\mathfrak{sp}(2)}$ , where  $B_{\mathfrak{sp}(2)}$  is the Killing form of  $\mathfrak{sp}(2)$ . The Lie algebra  $\mathfrak{s}(\mathfrak{g})$  is given

by  $\mathcal{Z}(\mathfrak{u}(2)) = \text{span}\{h_4\}$ . Let  $k \in \mathfrak{k}$  be a unit vector. We have  $\varphi(k) = c \cdot \text{ad}(h_4)$  with  $c \in \mathbb{R} \setminus \{0\}$ . All these  $(\mathfrak{k}, B)$ -extensions are irreducible. From Lemma 1.1.19 we know that the curvature of the  $(\mathfrak{k}, B)$ -extension is given by

$$\begin{aligned} R &= - \sum_{i=1}^4 \text{ad}(h_i) \odot \text{ad}(h_i) + \varphi(k) \odot \varphi(k) \\ &= - \sum_{i=1}^3 \text{ad}(h_i) \odot \text{ad}(h_i) + (-1 + c^2) \text{ad}(h_4) \odot \text{ad}(h_4). \end{aligned}$$

Hence we see that  $R|_{\text{ad}(\mathfrak{h} \oplus \mathfrak{k})}$  has trivial kernel precisely when  $c \neq \pm 1$ . If  $c \neq \pm 1$ , then the canonical base space equals the base space  $Sp(2)/(SU(2) \times S^1)$ , see Proposition 3.3.14. From Chapter 2 we know that the  $(\mathfrak{k}, B)$ -extension defines a naturally reductive structure on  $Sp(2)/Sp(1)_{\text{st}}$ , where  $Sp(1)_{\text{st}}$  denotes the image of the standard embedding of  $Sp(1)$  in  $Sp(2)$ . On this space we have a 2-parameter family of naturally reductive structures, with  $\lambda > 0$  and  $c$  as parameters. If we assume that  $c > 0$ , then all of these structures are non-isomorphic.

**The canonical base space is  $SU(3)/(S^1 \times S^1)$ :** The Lie algebra of  $S^1 \times S^1$  is spanned by  $h_1$  and  $h_4$  from (4.2.1). Let  $\mathfrak{h} := \text{span}\{h_1, h_4\}$ . Let  $e_1, \dots, e_4$  be as in (4.2.2) and let  $e_5 := h_2$  and  $e_6 := h_3$  be as in (4.2.1). Then this is an orthonormal basis for the orthogonal complement of  $\mathfrak{h}$  with respect to  $\bar{g} = \frac{-1}{12\lambda^2} B_{\text{su}(3)}$ . Let  $k \in \mathfrak{k}$  be a unit vector with respect to  $B$ . We have

$$\varphi(k) = c_1 \text{ad}(h_1) + c_2 \text{ad}(h_4), \quad (4.2.3)$$

with  $c_1, c_2 \in \mathbb{R}$  and not both equal to zero. These  $(\mathfrak{k}, B)$ -extension are irreducible by Lemma 3.3.19. From Lemma 1.1.19 we know that the curvature is given by

$$R = -\text{ad}(h_1) \odot \text{ad}(h_1) - \text{ad}(h_4) \odot \text{ad}(h_4) + \varphi(k) \odot \varphi(k).$$

Let  $\omega_1, \omega_2 \in \text{ad}(\mathfrak{h})$  be such that  $B_{\Lambda^2}(\omega_i, \text{ad}(h_1)) = \delta_{i1}$  and  $B_{\Lambda^2}(\omega_i, \text{ad}(h_4)) = \delta_{i2}$ . Then

$$\begin{aligned} R(\omega_1) &= (-1 + c_1^2) \text{ad}(h_1) + c_1 c_2 \text{ad}(h_4), \\ R(\omega_2) &= c_1 c_2 \text{ad}(h_1) + (-1 + c_2^2) \text{ad}(h_4). \end{aligned}$$

We see that  $R|_{\text{ad}(\mathfrak{h} \oplus \mathfrak{k})}$  has rank 2 precisely when  $c_1^2 + c_2^2 \neq 1$ . In this case the base space equals the canonical base space by Proposition 3.3.14. The  $(\mathfrak{k}, B)$ -extensions

are not always regular in this case. As explained in Section 2.2.1, we require that the connected Lie subgroup with Lie subalgebra  $\mathfrak{h}_0$  as in (2.2.10) is a closed subgroup. In this case  $\mathfrak{h}_0 = \mathfrak{k}^\perp \subset \mathfrak{h}$ , where the orthogonal complement is taken with respect to  $B_{\mathfrak{su}(3)}$ . This subgroup is spanned by  $c_2 h_1 - c_1 h_4$  and is closed precisely when  $\frac{c_1 + \sqrt{3}c_2}{c_1 - \sqrt{3}c_2} \in \mathbb{Q}$ . From Chapter 2 we know that the  $(\mathfrak{k}, B)$ -extension defines a naturally reductive structure on  $SU(3)/S_{x,y}^1$ , where  $\text{Lie}(S_{x,y}^1)$  is spanned by  $r(x, y)$  from (4.1.3) and  $(x, y) = (c_1 + \sqrt{3}c_2, c_1 - \sqrt{3}c_2)$ . On this manifold we get a 2-parameter family of naturally reductive structures, with  $\lambda > 0$  and  $c_1^2 + c_2^2$  as parameters. Note that every pair  $(\mathfrak{k}, B)$  can be identified by a pair of numbers  $(c_1, c_2)$  by (4.2.3). Suppose that two pairs  $(c_1, c_2)$  and  $(d_1, d_2)$  give equivalent naturally reductive spaces. Then there is a Lie algebra isomorphism  $M : \mathfrak{su}(3) \rightarrow \mathfrak{su}(3)$  which preserves  $\mathfrak{h}$  and with  $M(c_1 h_1 + c_2 h_4) = d_1 h_1 + d_2 h_4$ , see Proposition 3.3.16. It follows that the  $(\mathfrak{k}, B)$ -extensions corresponding to  $(c_1, c_2)$  and  $(d_1, d_2)$  are isomorphic if and only if  $(c_1, c_2)$  is related to  $(d_1, d_2)$  by (4.1.4).

**The canonical base space is  $SO(5)/(SO(3) \times SO(2))$ :** The base space is an irreducible hermitian symmetric space. Let  $E_{ij} \in \mathfrak{so}(5)$  be the matrix with the  $ij$ th entry equal to -1, the  $ji$ th entry equal to 1 and all other entries equal to zero. We pick the following basis:

$$\begin{aligned} h_1 &= \lambda E_{12}, & e_1 &= \lambda E_{14}, & e_5 &= \lambda E_{34}, \\ h_2 &= \lambda E_{23}, & e_2 &= \lambda E_{15}, & e_6 &= \lambda E_{35}, \\ h_3 &= \lambda E_{31}, & e_3 &= \lambda E_{24}, \\ h_4 &= \lambda E_{45}, & e_4 &= \lambda E_{25}. \end{aligned}$$

This is an orthonormal basis with respect to  $\bar{g} = \frac{-1}{6\lambda^2} B_{\mathfrak{so}(5)}$ . Let

$$\mathfrak{h} := \text{span}\{h_1, h_2, h_3, h_4\} \quad \text{and} \quad \mathfrak{m} := \text{span}\{e_1, \dots, e_6\}.$$

This gives the reductive decomposition  $\mathfrak{so}(5) = \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  of a hermitian symmetric space. We have  $\mathfrak{s}(\mathfrak{g}) = \text{span}\{h_4\}$ . Let  $k$  be a unit vector in  $\mathfrak{k}$ . Then  $\varphi(k) = c \cdot \text{ad}(h)$ . We need to check when condition (ii) from Proposition 3.3.14 is satisfied. The curvature is given by

$$\begin{aligned} R &= - \sum_{i=1}^4 \text{ad}(h_i) \odot \text{ad}(h_i) + \varphi(k) \odot \varphi(k) \\ &= - \sum_{i=1}^3 \text{ad}(h_i) \odot \text{ad}(h_i) + (-1 + c^2) \text{ad}(h_4) \odot \text{ad}(h_4). \end{aligned}$$

We see that  $\ker(R|_{\text{ad}(\mathfrak{h} \oplus \mathfrak{k})}) = 0$  precisely when  $c \neq \pm 1$ . In this case the base space is the canonical base space. The  $(\mathfrak{k}, B)$ -extension is always irreducible by Lemma 3.3.19. From Section 2.2.1 we see that the  $(\mathfrak{k}, B)$ -extension is always regular and defines a naturally reductive structure on  $SO(5)/SO(3)_{\text{st}}$ . On this space we obtain a 2-parameter family of naturally reductive structures, with  $\lambda > 0$  and  $c$  as parameters. All of these naturally reductive structures are non-isomorphic if we choose  $c > 0$ . Note that we can replace  $SO(5)/(SO(3) \times SO(2))$  by its dual symmetric space:  $SO(3, 2)/(SO(3) \times SO(2))$ .

**The canonical base space is  $SU(4)/S(U(1) \times U(3))$ :** Because the base space is an irreducible hermitian symmetric space this case is completely analogous to the previous case:  $SO(5)/(SO(3) \times SO(2))$ . All in all we obtain a 2-parameter family of naturally reductive structures on  $SU(4)/SU(3)$  with the standard embedding of  $SU(3)$ . Note that we can replace  $SU(4)/S(U(1) \times U(3))$  by its dual symmetric space:  $SU(1, 3)/S(U(1) \times U(3))$ .

**The canonical base space is  $\mathbb{C}P^2 \times S^2$ :** As reductive decomposition for  $\mathbb{C}P^2 = SU(3)/S(U(2) \times U(1))$  we take the one from (4.2.1) and (4.2.2) with respect to  $\frac{-1}{12\lambda_1^2}B_{\mathfrak{su}(3)}$ . Let  $h_5, e_5, e_6$  be an orthonormal basis of  $\mathfrak{su}(2)$  with respect to  $\frac{-1}{8\lambda_2^2}B_{\mathfrak{su}(2)}$ . The transvection algebra of the base space is given by

$$\mathfrak{g} = \mathfrak{su}(3) \oplus \mathfrak{su}(2) = \mathfrak{h} \oplus \mathfrak{m},$$

where  $\mathfrak{h} := \text{span}\{h_1, \dots, h_5\}$  and  $\mathfrak{m} := \text{span}\{e_1, \dots, e_6\}$ . The  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear form is  $\bar{g} = \frac{-1}{12\lambda_1^2}B_{\mathfrak{su}(3)} \oplus \frac{-1}{8\lambda_2^2}B_{\mathfrak{su}(2)}$ . The algebra  $\mathfrak{k} \subset \mathfrak{s}(\mathfrak{g}) = \text{span}\{h_4, h_5\}$  is 1-dimensional. Let  $k \in \mathfrak{k}$  be a unit vector. Then

$$\varphi(k) = c_1 \text{ad}(h_4) + c_2 \text{ad}(h_5).$$

The curvature is given by

$$R = - \sum_{i=1}^5 \text{ad}(h_i) \odot \text{ad}(h_i) + \varphi(k) \odot \varphi(k).$$

From Lemma 3.3.12 we have  $\ker(R|_{\text{ad}(\mathfrak{h} \oplus \mathfrak{k})}) = \ker(R|_{\text{ad}(\mathcal{Z}(\mathfrak{h} \oplus \mathfrak{k}))}) = \ker(R|_{\text{ad}(\mathcal{Z}(\mathfrak{h}))})$ . We need to check when  $R|_{\text{ad}(\mathcal{Z}(\mathfrak{h}))}$  has trivial kernel. Note that the center of  $\mathfrak{h}$  is given by  $\text{span}\{h_4, h_5\}$ . Let  $\omega_1, \omega_2$  be such that  $B_{\Lambda^2}(\omega_1, h_j) = \delta_{4j}$  and  $B_{\Lambda^2}(\omega_2, h_j) = \delta_{5j}$  for  $j = 1, \dots, 5$ . Then

$$\begin{aligned} R(\omega_1) &= (-1 + c_1^2) \text{ad}(h_4) + c_1 c_2 \text{ad}(h_5), \\ R(\omega_2) &= c_1 c_2 \text{ad}(h_4) + (-1 + c_2^2) \text{ad}(h_5). \end{aligned}$$

We see that  $R|_{\text{ad}(\mathcal{Z}(\mathfrak{h}))}$  has rank 2 precisely when  $c_1^2 + c_2^2 \neq 1$ . In other words the base space is equal to the canonical base space if and only if  $c_1^2 + c_2^2 \neq 1$ . By Lemma 3.3.19 the  $(\mathfrak{k}, B)$ -extension is reducible precisely when either  $c_1 = 0$  or  $c_2 = 0$ . Suppose that the  $(\mathfrak{k}, B)$ -extension is irreducible. By changing the sign of  $k$  we can assume that  $c_1 > 0$ . Under this condition none of the described  $(\mathfrak{k}, B)$ -extensions are isomorphic. The  $(\mathfrak{k}, B)$ -extension is regular if and only if the connected subgroup  $H_0$  with Lie subalgebra  $\mathfrak{h}_0 = \mathfrak{k}^\perp \subset \mathfrak{h}$  is closed in  $SU(3) \times SU(2)$ . We have  $\mathfrak{h}_0 = \text{span}\{c_2 h_4 - c_1 h_5\}$ . We see that  $H_0$  is closed precisely when  $\frac{c_2 \lambda_1}{\sqrt{3} c_1 \lambda_2} \in \mathbb{Q}$ . The  $(\mathfrak{k}, B)$ -extension describes a naturally reductive structure on  $(SU(3) \times SU(2))/(SU(2) \times S_{x,y}^1)$ , where  $SU(2)$  is the standard subgroup of  $SU(3)$  and  $S_{x,y}^1$  is the subgroup with Lie subalgebra given by (4.1.8) with  $(x, y) = (\frac{-c_2 \lambda_1}{\sqrt{3}}, -c_1 \lambda_2)$ . To obtain all of these naturally reductive structures on the fixed homogeneous space  $(SU(3) \times SU(2))/(SU(2) \times S_{x,y}^1)$  we start by defining  $\mathfrak{h}_0 := \text{Lie}(S_{x,y}^1)$  and  $\mathfrak{k} := \mathfrak{h}_0^\perp \subset \mathfrak{h}$  with respect to  $\bar{g}$ . We have a 1-parameter family of  $\text{ad}(\mathfrak{k})$ -invariant metrics on  $\mathfrak{k}$ . Together with the parameters  $\lambda_1, \lambda_2$  this gives us a 3-parameter family of naturally reductive structures on  $(SU(3) \times SU(2))/(SU(2) \times S_{x,y}^1)$ . Note that we can replace  $SU(3)/S(U(2) \times U(1))$  by its symmetric dual  $SU(2, 1)/S(U(2) \times U(1))$ .

**The canonical base space is  $S^2 \times S^2 \times S^2$ :** Let  $h_1, e_1, e_2$  be an orthonormal basis of  $\mathfrak{su}(2)$  with respect to  $\frac{-1}{8\lambda_1^2} B_{\mathfrak{su}(2)}$ . Let  $h_2, e_3, e_4$  be an orthonormal basis of  $\mathfrak{su}(2)$  with respect to  $\frac{-1}{8\lambda_2^2} B_{\mathfrak{su}(2)}$ . Let  $h_3, e_5, e_6$  be an orthonormal basis of  $\mathfrak{su}(2)$  with respect to  $\frac{-1}{8\lambda_3^2} B_{\mathfrak{su}(2)}$ . The transvection algebra of the base space is given by

$$\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) = \mathfrak{h} \oplus \mathfrak{m},$$

where  $\mathfrak{h} := \text{span}\{h_1, h_2, h_3\}$  and  $\mathfrak{m} := \text{span}\{e_1, \dots, e_6\}$ . The  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear form on  $\mathfrak{g}$  is  $\bar{g} := \frac{-1}{8\lambda_1^2} B_{\mathfrak{su}(2)} \oplus \frac{-1}{8\lambda_2^2} B_{\mathfrak{su}(2)} \oplus \frac{-1}{8\lambda_3^2} B_{\mathfrak{su}(2)}$ . The algebra  $\mathfrak{s}(\mathfrak{g})$  is equal to  $\mathfrak{h}$ . Let  $k \in \mathfrak{k}$  be a unit vector. Then

$$\varphi(k) = c_1 \text{ad}(h_1) + c_2 \text{ad}(h_2) + c_3 \text{ad}(h_3).$$

By Lemma 3.3.19 the  $(\mathfrak{k}, B)$ -extension is irreducible precisely when  $c_1, c_2, c_3 \in \mathbb{R} \setminus \{0\}$ . From now on we assume that the  $(\mathfrak{k}, B)$ -extension is irreducible. The curvature of the  $(\mathfrak{k}, B)$ -extension is given by

$$R = - \sum_{i=1}^3 \text{ad}(h_i) \odot \text{ad}(h_i) + \varphi(k) \odot \varphi(k).$$

We need to know when  $R|_{\text{ad}(\mathfrak{h} \oplus \mathfrak{k})}$  has trivial kernel. Let  $\omega_1, \omega_2, \omega_3$  be the dual basis of  $h_1, h_2, h_3$  in  $\mathfrak{h}$  with respect to  $B_{\Lambda^2}$ . Then

$$\begin{aligned} R(\omega_1) &= (-1 + c_1^2)\text{ad}(h_1) + c_1c_2\text{ad}(h_2) + c_1c_3\text{ad}(h_3), \\ R(\omega_2) &= c_1c_2\text{ad}(h_1) + (-1 + c_2^2)\text{ad}(h_2) + c_2c_3\text{ad}(h_3), \\ R(\omega_3) &= c_1c_3\text{ad}(h_1) + c_2c_3\text{ad}(h_2) + (-1 + c_3^2)\text{ad}(h_3). \end{aligned}$$

These are linearly independent precisely when

$$\det \begin{pmatrix} -1 + c_1^2 & c_1c_2 & c_1c_3 \\ c_1c_2 & -1 + c_2^2 & c_2c_3 \\ c_1c_3 & c_2c_3 & -1 + c_3^2 \end{pmatrix} \neq 0.$$

The determinant of this matrix is given by  $c_1^2 + c_2^2 + c_3^2 - 1$ . Hence we see that  $R|_{\text{ad}(\mathfrak{h} \oplus \mathfrak{k})}$  has no kernel precisely when  $c_1^2 + c_2^2 + c_3^2 \neq 1$ . If this is the case, then the base space is equal to the canonical base space by Proposition 3.3.14. By permuting the  $S^2$  factors we can assume that  $c_1 \geq c_2 \geq c_3$  and by changing the sign of  $k$  we can assume that  $c_1 > 0$ . Under these assumptions all the  $(\mathfrak{k}, B)$ -extensions are non-isomorphic. Let  $\mathfrak{h}_0 := \mathfrak{k}^\perp \subset \mathfrak{h}$ . Then  $\mathfrak{h}_0$  is spanned by  $c_2h_1 - c_1h_2$  and  $c_1c_3h_1 + c_2c_3h_2 - (c_1^2 + c_2^2)h_3$ . The  $(\mathfrak{k}, B)$ -extension is regular precisely when the connected subgroup  $H_0$  with  $\text{Lie}(H_0) = \mathfrak{h}_0$  is a closed subgroup of  $SU(2)^3$ , see (4.1.7). In this case the  $(\mathfrak{k}, B)$ -extension defines a naturally reductive structure on  $SU(2)^3 / (S_{k_1, k_2, k_3}^1 \times S_{l_1, l_2, l_3}^1)$  for certain number  $k_1, k_2, k_3, l_1, l_2, l_3 \in \mathbb{Z}$ . To obtain all of these naturally reductive structures on the fixed homogeneous space  $SU(2)^3 / (S_{k_1, k_2, k_3}^1 \times S_{l_1, l_2, l_3}^1)$  we start by defining  $\mathfrak{h}_0 := \text{Lie}(S_{k_1, k_2, k_3}^1 \times S_{l_1, l_2, l_3}^1)$  and  $\mathfrak{k} := \mathfrak{h}_0^\perp \subset \mathfrak{h}$  with respect to  $\bar{g}$ . We have a 1-parameter family of  $\text{ad}(\mathfrak{k})$ -invariant metrics on  $\mathfrak{k}$ . Together with the parameters  $\lambda_1, \lambda_2, \lambda_3$  this gives us a 4-parameter family of naturally reductive structures on  $SU(2)^3 / (S_{k_1, k_2, k_3}^1 \times S_{l_1, l_2, l_3}^1)$ . Note that we can replace any number of  $S^2$ -factors by their symmetric dual  $SL(2, \mathbb{R})/S^1$ .

## 4.2.2 Classification of type II in dimension 8

We will first argue, by systematically excluding all other possibilities, that all possible canonical base spaces of irreducible naturally reductive decomposition of type II with a compact type I part are the following:

$$\begin{aligned} &\mathbb{R}^6, \mathbb{R}^5, \mathbb{R}^4, S^2 \times \mathbb{R}^4, SU(2) \times \mathbb{R}^4, \mathbb{C}P^2 \times \mathbb{R}^2, S^2 \times S^2 \times \mathbb{R}^2, \\ &(SU(2) \times SU(2))/S_{k_1, k_2}^1 \times \mathbb{R}^2, SU(3)/SU(2) \times \mathbb{R}^2, SU(2) \times S^2 \times \mathbb{R}^2, SU(3)/S_{k_1, k_2}^1, \\ &(SU(3) \times SU(2))/(SU(2) \times S_{k_1, k_2}^1), (SU(3) \times SU(2))/(SU(2)_\Delta \times S^1), \\ &SU(2)^3 / (S_{k_1, k_2, k_3}^1 \times S_{l_1, l_2, l_3}^1), S^2 \times (SU(2) \times SU(2))/S_{k_1, k_2}^1, SU(3)/(S^1 \times S^1), \\ &S^2 \times S^2 \times S^2, S^2 \times \mathbb{C}P^2, \{*\}, \end{aligned}$$

where  $\{*\}$  denotes a point space. Even though we write all above base spaces as globally homogeneous spaces they can also be locally homogeneous spaces. This is discussed below when this occurs.

The Euclidean factor can't be  $\mathbb{R}^7$ , because then  $\dim(\mathfrak{k}) = 1$  and the linear action of  $\mathfrak{k}$  on  $\mathbb{R}^7$  has a vector on which it acts trivially and by Lemma 3.3.19 any such  $(\mathfrak{k}, B)$ -extension is reducible.

If the Euclidean factor is  $\mathbb{R}^6$ , then the type I part needs to have dimension zero and  $\dim(\mathfrak{k}) = 2$ .

If the Euclidean factor is  $\mathbb{R}^5$  and the type I part is 2-dimensional, then  $\dim(\mathfrak{k}) = 1$ . Just as for  $\mathbb{R}^7$  we see that the linear action of  $\mathfrak{k}$  on  $\mathbb{R}^5$  has a vector on which it acts trivially and by Lemma 3.3.19 any such  $(\mathfrak{k}, B)$ -extension is reducible. Thus, also for  $\mathbb{R}^5$  the type I part has to be zero dimensional.

Suppose that the Euclidean factor is  $\mathbb{R}^4$ . The type I part can be 2- or 3-dimensional. If it is 2-dimensional, then it is  $S^2$ . If it is 3-dimensional, then it either is the symmetric space  $(SU(2) \times SU(2))/SU(2)$  or the Lie group  $SU(2)$ . In the first case we see by Lemma 3.3.19 that any  $(\mathfrak{k}, B)$ -extension is reducible.

If the Euclidean factor is  $\mathbb{R}^3$ , then  $\mathfrak{k}$  has to contain  $\mathfrak{so}(3)$  in order for the linear representation of  $\mathfrak{k}$  on  $\mathbb{R}^3$  not to have a vector on which it acts trivially. We see that if the type I part is 0-dimensional, then we can't construct an irreducible 8-dimensional  $(\mathfrak{k}, B)$ -extension. The only other possibility is that the type I part is 2-dimensional. In this case we immediately see by Lemma 3.3.19 that any such  $(\mathfrak{so}(3), B)$ -extension is reducible.

Suppose that the Euclidean factor is  $\mathbb{R}^2$ . The type I part can either be 3-, 4- or 5-dimensional. Suppose that the type I part is 5-dimensional. We will call the factors  $\mathfrak{h}_i \oplus \mathfrak{m}_i$  in (3.3.20) irreducible factors. Let  $\mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{m}'$  be an irreducible factor of the transvection algebra of the type I part. For the  $(\mathfrak{k}, B)$ -extension to be irreducible we see from Lemma 3.3.19 that we require  $\mathfrak{s}(\mathfrak{g}') \neq \{0\}$ . We see that there are three possibilities:  $(SU(2) \times SU(2))/S_{k_1, k_2}^1$ ,  $SU(3)/SU(2)$  and  $SU(2) \times S^2$ . Suppose that the type I part is 4-dimensional. If it is irreducible, then it can only be  $\mathbb{C}P^2$ . If it is reducible, then it can only be  $S^2 \times S^2$ . Suppose that the type I part is 3-dimensional. For the symmetric space  $(SU(2) \times SU(2))/SU(2)$  we have  $\mathfrak{s}(\mathfrak{su}(2) \oplus \mathfrak{su}(2)) = \{0\}$  and thus from Lemma 3.3.19 we see that any  $(\mathfrak{k}, B)$ -extension is reducible. The other possibility is that the type I part is equal to  $SU(2)$  and  $\mathfrak{s}(\mathfrak{g}) = \mathfrak{su}(2) \oplus \mathfrak{so}(2)$ . The Lie algebra  $\mathfrak{k} \subset \mathfrak{s}(\mathfrak{g}) = \mathfrak{su}(2) \oplus \mathfrak{so}(2)$  is a 3-dimensional subalgebra. Hence  $\mathfrak{k} = \mathfrak{su}(2) \subset \mathfrak{s}(\mathfrak{g})$  and thus  $\mathfrak{k}$  acts trivially on  $\mathbb{R}^2$ . Therefore, by Lemma 3.3.19, any such  $(\mathfrak{k}, B)$ -extension is reducible. If the type I part is 2-dimensional, then  $\dim(\mathfrak{s}(\mathfrak{g})) \leq 2$  and thus we can't make an irreducible 8-dimensional  $(\mathfrak{k}, B)$ -extension from this.

Thus, only base spaces with no Euclidean part remain. For now we make the extra assumption that  $\mathfrak{k}$  is abelian. As before, we should list all spaces for which every irreducible factor  $\mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{m}'$  of the transvection algebra satisfies  $\mathfrak{s}(\mathfrak{g}') \neq \{0\}$ . Note that in this case  $\mathfrak{k} = \mathfrak{k}_1$  and thus if  $\mathfrak{k}$  is abelian, then by Proposition 3.3.14 condition (i) we require that  $\pi_{\mathfrak{m}}(\mathfrak{k}) = \pi_{\mathfrak{m}}(\mathcal{Z}(\mathfrak{h}_1)) = \{0\}$ . We need this condition in order for the canonical base space to be the base space we start with. Note that  $\pi_{\mathfrak{m}}(\mathfrak{k}) = \{0\}$  if and only if  $\mathfrak{k} \subset \mathcal{Z}(\mathfrak{h})$ . Hence in this case we require that  $\mathcal{Z}(\mathfrak{h}') \neq \{0\}$  for every irreducible factor  $\mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{m}'$  of the transvection algebra.

Now we discuss the case for which the base space has an irreducible 3-dimensional factor. There are only two compact irreducible 3-dimensional naturally reductive spaces of type I:  $SU(2)$  and the symmetric space  $(SU(2) \times SU(2))/SU(2)$ . For the symmetric space  $\mathfrak{s} = \{0\}$  and thus this factor is not allowed for an irreducible  $(\mathfrak{k}, B)$ -extension. If we have  $SU(2)$  as a 3-dimensional factor, then  $\mathfrak{k}$  has to be at least 3-dimensional. Otherwise the space will be either reducible or condition (i) from Proposition 3.3.14 is not satisfied. The only possibility for a base space is  $SU(2) \times S^2$ , but just as for the case  $SU(2) \times \mathbb{R}^2$  any 8-dimensional  $(\mathfrak{k}, B)$ -extension of this space is reducible. We conclude that if there is no Euclidean factor, then the type I part can't contain a 3-dimensional factor.

If the base space is 7-dimensional, then  $\dim(\mathfrak{k}) = 1$  and thus  $\mathfrak{k}$  is abelian. Hence, by the discussion above we require that every irreducible factor  $\mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{m}'$  of the transvection algebra satisfies  $\mathcal{Z}(\mathfrak{h}') \neq \{0\}$ . We noted above that there can't be a 3-dimensional factor, hence the 7-dimensional space either is irreducible or it is a product of a 5-dimensional irreducible space and a 2-dimensional space. We see that all possible spaces are:  $SU(3)/S_{k_1, k_2}^1$ ,  $(SU(3) \times SU(2))/(SU(2) \times S_{k_1, k_2}^1)$ ,  $(SU(3) \times SU(2))/(SU(2)_{\Delta} \times S^1)$ ,  $SU(2)^3/(S_{k_1, k_2, k_3}^1 \times S_{l_1, l_2, l_3}^1)$  and  $(SU(2) \times SU(2))/S_{k_1, k_2}^1 \times S^2$ .

For a 6-dimensional base space  $\mathfrak{k}$  is abelian and thus by the discussion above we need that  $\dim(\mathcal{Z}(\mathfrak{h}')) \geq 1$  for every irreducible factor  $\mathfrak{h}' \oplus \mathfrak{m}'$  of the transvection algebra. We can easily check that all possibilities are:  $SU(3)/(S^1 \times S^1)$ ,  $\mathbb{C}P^2 \times S^2$  and  $S^2 \times S^2 \times S^2$ .

We check that every 5-dimensional irreducible naturally reductive space of type I satisfies  $\dim(\mathcal{Z}(\mathfrak{h})) \leq 1$  and thus we can't make an 8-dimensional irreducible  $(\mathfrak{k}, B)$ -extension from this. Every reducible 5-dimensional space contains a 3-dimensional factor and thus can be discarded by the above discussion. Similarly for every 4-dimensional space of type I we have  $\dim(\mathfrak{s}) \leq 2$  and thus we can not make an irreducible 8-dimensional  $(\mathfrak{k}, B)$ -extension of this.

The Lie algebra  $\mathfrak{su}(3)$  has dimension 8 and is a compact simple Lie algebra. Therefore, we also a point space is a possible base space, see Remark 2.2.6.

For all of these base spaces we will give all possible  $(\mathfrak{k}, B)$ -extensions. How to

obtain a globally homogeneous naturally reductive space from this is described in Chapter 2. From Lemma 3.3.12 we see that condition (ii) from Proposition 3.3.14 only needs to be checked if  $\mathfrak{k}_1 \neq 0$ . We will mention in which cases dual naturally reductive spaces exist in the sense of Remark 3.3.18. Whenever  $\mathfrak{k}_1 \neq \{0\}$  we need to check condition (ii) from Proposition 3.3.14, see Lemma 3.3.12. We use Proposition 3.3.16 to conclude that all the spaces we list are non-isomorphic.

**The canonical base space is  $\mathbb{R}^6$ :** The Lie algebra  $\mathfrak{k}$  is 2-dimensional and abelian. Let  $k_1, k_2$  be an orthonormal basis of  $\mathfrak{k}$ . Then

$$\varphi(k_1) := c_1 e_{12} + c_2 e_{34} + c_3 e_{56}, \quad \varphi(k_2) := d_1 e_{12} + d_2 e_{34} + d_3 e_{56},$$

for  $c_1, c_2, c_3, d_1, d_2, d_3 \in \mathbb{R}$ . By picking a different basis of  $\mathbb{R}^6$  and  $\mathfrak{k}$  we can bring this in the form:

$$\varphi(k_1) := c_1 e_{12} + c_2 e_{34} + c_3 e_{56}, \quad \varphi(k_2) := d_1 e_{12} + d_2 e_{34},$$

with

$$c_3 > 0, \quad d_2 > 0, \quad d_2 \geq d_1, \quad \text{and if } d_1 = d_2 \text{ then } c_2 \geq c_1.$$

From Lemma 3.3.19 we see that the  $(\mathfrak{k}, B)$ -extension is reducible if and only if one of the following holds:  $c_1 = d_1 = 0$ ,  $c_2 = d_1 = 0$  or  $c_1 = c_2 = 0$ . From Section 2.2.2 we see that we obtain a 5-parameter family of naturally reductive structures on an 8-dimensional 2-step nilpotent Lie group, with  $c_1, c_2, c_3, d_1, d_2$  as parameters. None of these naturally reductive structures are isomorphic under the above conditions.

**The canonical base space is  $\mathbb{R}^5$ :** The Lie algebra  $\mathfrak{k}$  has to be 3-dimensional and in order to have a 5-dimensional representation without vectors on which  $\mathfrak{k}$  acts trivially. The only possibility is  $\mathfrak{k} = \mathfrak{su}(2)$  and the representation of  $\mathfrak{k}$  is the 5-dimensional irreducible representation of  $\mathfrak{su}(2)$ . Let  $k_1, k_2, k_3$  be an orthonormal basis of  $\mathfrak{su}(2)$  with respect to  $B = -\frac{1}{2\lambda^2} B_{\mathfrak{su}(2)}$ . We choose a basis such that

$$\begin{aligned} \varphi(k_1) &= \lambda(\sqrt{3}e_{13} - e_{24} - e_{35}), \\ \varphi(k_2) &= \lambda(-\sqrt{3}e_{12} + e_{34} - e_{25}), \\ \varphi(k_3) &= \lambda(e_{23} + 2e_{45}). \end{aligned}$$

The  $(\mathfrak{k}, B)$ -extension defines a naturally reductive structure on an 8-dimensional 2-step nilpotent Lie group, as described in Section 2.2.2. On this homogeneous space we obtain a 1-parameter family of naturally reductive structures, with the  $\lambda > 0$  as parameter.

**The canonical base space is  $\mathbb{R}^4$ :** The Lie algebra  $\mathfrak{k} \subset \mathfrak{so}(4)$  has to be 4-dimensional. This implies that  $\mathfrak{k} \cong \mathfrak{u}(2) \cong \mathfrak{su}(2) \oplus \mathbb{R}$ . The  $\text{ad}(\mathfrak{k})$ -invariant metric  $B$  on  $\mathfrak{k}$  is given by  $B = -\frac{1}{8\lambda^2}B_{\mathfrak{su}(2)} \oplus B_{\mathbb{R}}$ . Let  $k_1, k_2, k_3, k_4$  be an orthonormal basis with respect to  $B$ . For some basis of  $\mathbb{R}^4$  we have:

$$\begin{aligned}\varphi(k_1) &= \lambda(e_{13} + e_{24}), & \varphi(k_3) &= \lambda(-e_{14} + e_{23}), \\ \varphi(k_2) &= \lambda(-e_{12} + e_{34}), & \varphi(k_4) &= \mu(e_{12} + e_{34}),\end{aligned}$$

with  $\mu, \lambda \in \mathbb{R} \setminus \{0\}$ . The  $(\mathfrak{k}, B)$ -extension defines a naturally reductive structure on an 8-dimensional 2-step nilpotent Lie group, as described in Section 2.2.2. On this space we have a 2-parameter family of naturally reductive structures, with  $\lambda, \mu > 0$  as parameters.

**The canonical base space is  $S^2 \times \mathbb{R}^4$ :** Let  $h, e_1, e_2$  be an orthonormal basis of  $\mathfrak{su}(2)$  with respect to  $\frac{-1}{8\lambda^2}B_{\mathfrak{su}(2)}$ . The Lie algebra  $\mathfrak{k}$  is 2-dimensional. We choose a orthonormal bases  $k_1, k_2$  of  $\mathfrak{k}$  and  $e_3, e_4, e_5, e_6$  of  $\mathbb{R}^4$  such that

$$\varphi(k_1) := c_1 \text{ad}(h) + c_2 e_{34} + c_3 e_{56}, \quad \varphi(k_2) := d_1 e_{34} + d_2 e_{56},$$

for  $c_1, c_2, c_3, d_1, d_2 \in \mathbb{R}$  and  $d_2 > 0$ . Suppose that  $(c_2, c_3)$  and  $(d_1, d_2)$  are linearly independent. We have  $\mathfrak{k} = \mathfrak{k}_{2,3} \oplus \mathfrak{k}_3$ , where both  $\mathfrak{k}_{2,3}$  and  $\mathfrak{k}_3$  are 1-dimensional. The  $(\mathfrak{k}, B)$ -extension is reducible precisely when one of the following holds:  $c_2 = c_3 = 0$ ,  $c_2 = d_1 = 0$  or  $c_3 = d_1 = 0$ . Suppose the  $(\mathfrak{k}, B)$ -extension is irreducible. By choosing a different basis we can always assume that  $c_1 > 0$ ,  $c_2 \geq c_3$  and if  $c_2 = c_3$ , then  $d_1 \geq d_2$ . Under these extra assumptions all naturally reductive structures are non-isomorphic. By Section 2.2.3 the  $(\mathfrak{k}, B)$ -extension defines a naturally reductive structure on  $(SU(2) \times N^6)/\mathbb{R}$ , where  $N^6$  is a 6-dimensional 2-step nilpotent Lie group as described in Section 2.2.2 and  $\text{Lie}(\mathbb{R})$  is described by (2.2.19). On this space we have a 6-parameter family of naturally reductive structures, with  $\lambda > 0$  and  $c_1, c_2, c_3, d_1, d_2$  as parameters.

Suppose that  $(d_1, d_2) = \mu(c_2, c_3)$  for some  $\mu \neq 0$ . We have  $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_3$ . The  $(\mathfrak{k}, B)$ -extension is irreducible precisely when  $c_1, c_2, c_3 \in \mathbb{R} \setminus \{0\}$ . If we choose a basis such that  $c_1 > 0$  and  $c_2 \geq c_3$ , then all structures are non-isomorphic. By Section 2.2.3 the  $(\mathfrak{k}, B)$ -extensions define a 5-parameter family of naturally reductive structures on  $SU(2) \times H^5$ , with  $c_1, c_2, c_3$  and  $\lambda, \mu > 0$  as parameters. For both cases we can replace  $S^2 = SU(2)/S^1$  by its non-compact dual symmetric space  $SL(2, \mathbb{R})/S^1$ .

**The canonical base space is  $SU(2) \times \mathbb{R}^4$ :** Let  $e_1, e_2, e_3$  be an orthonormal basis of  $\mathfrak{su}(2)$  with respect to  $-\frac{1}{8\lambda^2}B_{\mathfrak{su}(2)}$ . The Lie algebra  $\mathfrak{k}$  is 1-dimensional. Let  $k \in \mathfrak{k}$  be

a unit vector. Up to a conjugation in  $SU(2) \times \mathbb{R}^4$  and a choice of orthonormal basis  $e_4, \dots, e_7$  of  $\mathbb{R}^4$  we have

$$\varphi(k) = c_1 \text{ad}(e_1) + c_2 e_{45} + c_3 e_{67} = -2c_1 \lambda e_{23} + c_2 e_{45} + c_3 e_{67},$$

with  $c_1, c_2, c_3 \in \mathbb{R}$ . From Lemma 3.3.19 we see that this  $(\mathfrak{k}, B)$ -extension is irreducible precisely when  $c_1, c_2, c_3 \in \mathbb{R} \setminus \{0\}$ , which we assume from now on. The  $(\mathfrak{k}, B)$ -extension defines a naturally reductive structure on  $SU(2) \times H^5$  as is described in Section 2.2.3. On this homogeneous space we obtain a 4-parameter family of naturally reductive structures, with  $\lambda > 0$  and  $c_1, c_2, c_3$  as parameters. By choosing a different basis and changing the sign of  $k$  we can always assume that  $c_2 \geq c_3$  and  $c_1 > 0$ . Under these extra assumptions all naturally reductive structures are non-isomorphic.

**The canonical base space is  $\mathbb{C}P^2 \times \mathbb{R}^2$  :** Let  $h_1, h_2, h_3, h_4$  and  $e_1, e_2, e_3, e_4$  be as in (4.2.1) and (4.2.2), respectively. Also let  $\bar{g} = -\frac{1}{12\lambda^2} B_{\text{su}(3)} \oplus B_{\text{eucl}}$  and let  $e_5, e_6$  be an orthonormal basis of  $\mathbb{R}^2$ . We have  $\mathfrak{s}(\bar{\mathfrak{g}}) = \text{span}\{h_4\} \oplus \mathfrak{so}(2)$ . We choose an orthonormal basis  $k_1, k_2$  of  $\mathfrak{k}$  such that

$$\varphi(k_1) = c_1 \text{ad}(h_4) + c_2 e_{56}, \quad \varphi(k_2) = d_1 \text{ad}(h_4),$$

with  $c_2 \neq 0$  and  $d_1 \neq 0$ . Hence  $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ . We need to check when condition (ii) of Proposition 3.3.14 is satisfied. The curvature of the  $(\mathfrak{k}, B)$ -extensions is given by

$$R = - \sum_{i=1}^4 \text{ad}(h_i) \odot \text{ad}(h_i) + \varphi(k_1) \odot \varphi(k_1) + \varphi(k_2) \odot \varphi(k_2).$$

For now we denote  $e_{56}$  by  $\text{ad}(h_5)$ . Let  $\omega_1, \omega_2 \in \text{ad}(\mathfrak{h}) + \psi(\mathfrak{k})$  be such that  $B_{\Lambda^2}(\omega_1, h_j) = \delta_{j4}$  and  $B_{\Lambda^2}(\omega_2, h_j) = \delta_{j5}$  for  $j = 1, \dots, 5$ . From Lemma 3.3.12 we know that  $h_1, h_2, h_3 \in \text{im}(R)$  and thus  $\ker(R|_{\text{ad}(\mathfrak{h} \oplus \mathfrak{k})}) \subset \text{span}\{\omega_1, \omega_2\}$ . We have

$$\begin{aligned} R(\omega_1) &= (-1 + c_1^2 + d_1^2) \text{ad}(h_1) + c_1 c_2 e_{56}, \\ R(\omega_2) &= c_1 c_2 \text{ad}(h_1) + c_2^2 e_{56}. \end{aligned}$$

These are linearly independent if and only if  $(-1 + d_1^2)c_2^2 \neq 0$ . Since  $c_2 \neq 0$  we obtain that  $\ker(R|_{\text{ad}(\mathfrak{h} \oplus \mathfrak{k})}) = \{0\}$  if and only if  $d_1^2 \neq 1$ . From Lemma 3.3.19 we see that the  $(\mathfrak{k}, B)$ -extension is reducible precisely when  $c_1 = 0$ . From Section 2.2.3 we see that the  $(\mathfrak{k}, B)$ -extension is always regular and it defines a naturally reductive structure on  $SU(3)/SU(2) \times H^3$ . On this homogeneous space we obtain a 4-parameter family of naturally reductive structures, with  $\lambda > 0$  and  $c_1, c_2, d_1$  as parameters. If we assume that  $c_2 > 0$  and  $d_2 > 0$  by changing the signs of  $k_1$  and  $k_2$ , then all these structures are non-isomorphic. Note that we can replace  $\mathbb{C}P^2 = SU(3)/S(U(1) \times U(2))$  by its non-compact symmetric dual  $SU(1, 2)/S(U(1) \times U(2))$ .

**The canonical base space is  $S^2 \times S^2 \times \mathbb{R}^2$  :** Let  $h_1, e_1, e_2$  be an orthonormal basis of  $\mathfrak{su}(2)$  with respect to  $\frac{-1}{8\lambda_1^2}B_{\mathfrak{su}(2)}$ . Let  $h_2, e_3, e_4$  be an orthonormal basis of  $\mathfrak{su}(2)$  with respect to  $\frac{-1}{8\lambda_2^2}B_{\mathfrak{su}(2)}$ . The transvection algebra of the base space is given by

$$\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus_{L.a.} \mathbb{R}^2 = \mathfrak{h} \oplus \mathfrak{m} \oplus_{L.a.} \mathbb{R}^2,$$

where  $\mathfrak{h} := \text{span}\{h_1, h_2\}$  and  $\mathfrak{m} := \text{span}\{e_1, e_2, e_3, e_4\}$ . The  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear form on  $\mathfrak{g}$  is  $\bar{g} = \frac{-1}{8\lambda_1^2}B_{\mathfrak{su}(2)} \oplus \frac{-1}{8\lambda_2^2}B_{\mathfrak{su}(2)} \oplus B_{eucl}$ . We have  $\mathfrak{s}(\mathfrak{g}) = \text{span}\{h_1, h_2\} \oplus \mathfrak{so}(2)$ . We can pick an orthonormal basis  $k_1, k_2$  of  $\mathfrak{k}$  such that

$$\varphi(k_1) = c_1\text{ad}(h_1) + c_2\text{ad}(h_2) + c_3e_{56}, \quad \varphi(k_2) = d_1\text{ad}(h_1) + d_2\text{ad}(h_2).$$

From Lemma 3.3.19 we see that the  $(\mathfrak{k}, B)$ -extension is reducible if and only if one of the following holds:  $c_3 = 0, d_1 = c_2 = 0, d_2 = c_1 = 0, d_1 = c_1 = 0, d_2 = c_2 = 0$  or  $c_1 = c_2 = 0$ . Therefore, we assume that  $c_3 > 0$  from now on. We need to check if condition (ii) of Proposition 3.3.14 is satisfied. The curvature of the  $(\mathfrak{k}, B)$ -extension is given by

$$R = -\text{ad}(h_1) \odot \text{ad}(h_1) - \text{ad}(h_2) \odot \text{ad}(h_2) + \varphi(k_1) \odot \varphi(k_1) + \varphi(k_2) \odot \varphi(k_2).$$

We denote  $e_{56}$  by  $\text{ad}(h_3)$  for now. Let  $\omega_1, \omega_2, \omega_3$  be such that  $B_{\Lambda^2}(\omega_i, \text{ad}(h_j)) = \delta_{ij}$ . Then

$$\begin{aligned} R(\omega_1) &= (-1 + c_1^2 + d_1^2)\text{ad}(h_1) + (c_1c_2 + d_1d_2)\text{ad}(h_2) + c_1c_3e_{56}, \\ R(\omega_2) &= (c_1c_2 + d_1d_2)\text{ad}(h_1) + (-1 + c_2^2 + d_2^2)\text{ad}(h_2) + c_2c_3e_{56}, \\ R(\omega_3) &= c_1c_3\text{ad}(h_1) + c_2c_3\text{ad}(h_2) + c_3^2e_{56}. \end{aligned}$$

These are linearly independent precisely when

$$\det \begin{pmatrix} -1 + c_1^2 + d_1^2 & c_1c_2 + d_1d_2 & c_1c_3 \\ c_1c_2 + d_1d_2 & -1 + c_2^2 + d_2^2 & c_2c_3 \\ c_1c_3 & c_2c_3 & c_3^2 \end{pmatrix} \neq 0.$$

This is equivalent to  $-c_3^2(d_1^2 + d_2^2 - 1) \neq 0$ . Thus, by Proposition 3.3.14, the canonical base space is equal to the base space precisely when  $d_1^2 + d_2^2 \neq 1$ , because  $c_3 \neq 0$ . By permuting the  $S^2$  factors and changing the sign of  $k_2$  we can always arrange that  $d_1 > 0, c_1 \geq c_2$  and if  $c_1 = c_2$ , that  $d_1 \geq d_2$ . Under these assumptions all the  $(\mathfrak{k}, B)$ -extensions are non-isomorphic. Suppose that  $(c_1, c_2)$  and  $(d_1, d_2)$  are linearly independent. Then  $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_{2,3}$ . From Section 2.2.3 we obtain a 6-parameter family of naturally reductive structures on  $(SU(2) \times SU(2) \times H^3)/\mathbb{R}_\alpha$ , where the

image of  $\text{Lie}(\mathbb{R}_\alpha)$  in  $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$  is spanned by  $h$  from (4.2.4) with  $\alpha = \frac{-d_1\lambda_2}{d_2\lambda_1}$ . The  $(\mathfrak{k}, B)$ -extension is always regular. To obtain all of these naturally reductive structures on the homogeneous space  $(SU(2) \times SU(2) \times H^3)/\mathbb{R}_\alpha$  for some fixed  $\alpha$  we start by defining  $\mathfrak{h}_0 := \text{span}\{h\}$  and  $\mathfrak{k}_1 := \mathfrak{h}_0^\perp \subset \mathfrak{h}$  with respect to  $\bar{g}$ . Next we pick  $\mathfrak{k}_2 = \text{span}\{c_1h_1 + c_2h_2 + c_3e_{56}\}$ , with  $c_1h_1 + c_2h_2$  linear independent from  $h$  and  $\mathfrak{k}_1 \perp \mathfrak{k}_2$  with respect to the metric on  $\mathfrak{k}$ . This gives us a 6-parameter family of naturally reductive structures on  $(SU(2) \times SU(2) \times H^3)/\mathbb{R}_\alpha$ , with the parameters  $\lambda_1, \lambda_2 > 0$ , and  $c_1, c_2, c_3$ , and one parameter from the metric on  $\mathfrak{k}_1$ .

Suppose that  $(d_1, d_2) = \mu(c_1, c_2)$  for some  $\mu \neq 0$ . Then  $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_3$ . The  $(\mathfrak{k}, B)$ -extension is regular if and only if  $\alpha = \frac{-c_1\lambda_2}{c_2\lambda_1} \in \mathbb{Q}$ . If the  $(\mathfrak{k}, B)$ -extension is regular, then it defines a naturally reductive structure on  $(SU(2) \times SU(2))/S_{k_1, k_2}^1 \times H^3$ , see Section 2.2.3. We have  $\text{Lie}(S_{k_1, k_2}^1) = \mathfrak{h}_0 = \mathfrak{k}_1^\perp = \text{span}\{c_2h_1 - c_1h_2\}$  for certain numbers  $k_1, k_2 \in \mathbb{Z}$ . In the same way as in the case above we obtain a 5-parameter family of naturally reductive structures on  $(SU(2) \times SU(2))/S_{k_1, k_2}^1 \times H^3$ .

For both cases we can replace one or both of the  $S^2$  factors by its non-compact symmetric dual  $SL(2, \mathbb{R})/S^1$ .

**The canonical base space is  $(SU(2) \times SU(2))/S^1 \times \mathbb{R}^2$  :** The Lie algebra  $\mathfrak{k}$  is 1-dimensional. Let  $k \in \mathfrak{k}$  be a unit vector. To keep notation short we consider  $\mathfrak{su}(2) \cong \mathfrak{sp}(1) \subset \mathfrak{gl}(1, \mathbb{H})$ . The non-degenerate symmetric bilinear form on  $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$  is given by  $-\frac{1}{8\lambda_1^2}B_{\mathfrak{sp}(1)} \oplus -\frac{1}{8\lambda_2^2}B_{\mathfrak{sp}(1)}$ , where  $B_{\mathfrak{sp}(1)}$  denotes the Killing form of  $\mathfrak{sp}(1)$ . Let

$$\begin{aligned} e_1 &:= (\lambda_1 j, 0), & e_3 &:= (0, \lambda_2 \beta j), & e_5 &:= (\alpha^2 \lambda_1^2 + \lambda_2^2)^{-1/2} (\lambda_1^2 \alpha i, -\lambda_2^2 i), \\ e_2 &:= (\lambda_1 k, 0), & e_4 &:= (0, \lambda_2 \beta k), & h &:= \frac{\lambda_1 \lambda_2}{\sqrt{\alpha^2 \lambda_1^2 + \lambda_2^2}} (i, \alpha i), \end{aligned} \quad (4.2.4)$$

where  $e_1, \dots, e_5$  is an orthonormal basis of  $\mathfrak{m} := \mathfrak{h}^\perp$  with respect to the metric above and  $\alpha \in \mathbb{R} \setminus \{0\}$ . For  $k$  we have

$$\varphi(k) = c_1 \text{ad}(h) + c_2 \text{ad}(e_5) + c_3 e_{67},$$

where  $e_6, e_7$  is an orthonormal basis of  $\mathbb{R}^2$ . The  $(\mathfrak{k}, B)$ -extension is reducible precisely when  $c_3 = 0$  or  $c_1 = c_2 = 0$ . If  $c_1 \neq 0$ , then  $\mathfrak{k} = \mathfrak{k}_{2,3}$  and the  $(\mathfrak{k}, B)$ -extension defines a naturally reductive structure on

$$(SU(2) \times SU(2) \times H^3)/\mathbb{R}_\alpha,$$

where  $\text{Lie}(\mathbb{R}_\alpha)$  is described by (2.2.19) and its image in  $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$  is spanned by  $h$ . By changing the sign of  $k$  we can assume that  $c_3 > 0$  and under this extra

assumption all the naturally reductive structures are non-isomorphic. This  $(\mathfrak{k}, B)$ -extension is regular for all values of  $\alpha$  even though the base space is only regular when  $\alpha \in \mathbb{Q}$ . For every  $\alpha \in \mathbb{R}$  we obtain in this way a 5-parameter family of naturally reductive structures, with  $\lambda_1, \lambda_2 > 0$  and  $c_1, c_2, c_3$  as parameters.

If  $c_1 = 0$ , then  $\mathfrak{k} = \mathfrak{k}_2$  and  $\mathfrak{k}_{2,3} = \{0\}$ . If  $\alpha \in \mathbb{Q}$ , the  $(\mathfrak{k}, B)$ -extension defines a naturally reductive structure on

$$(SU(2) \times SU(2))/S_\alpha^1 \times H^3.$$

On this homogeneous space we obtain a 4-parameter family of naturally reductive structures, with  $\lambda_1, \lambda_2 > 0$  and  $c_2, c_3$  as parameters.

For both spaces we can replace one  $S^2$  factor by its symmetric dual  $SL(2, \mathbb{R})/S^1$ .

**The canonical base space is  $SU(3)/SU(2) \times \mathbb{R}^2$  :** Let  $h_1, h_2, h_3$  be as in (4.2.1),  $e_1, e_2, e_3, e_4$  as in (4.2.2) and let  $e_5 := h_4$  from (4.2.1). Let the non-degenerate symmetric bilinear form on  $\mathfrak{su}(3)$  be  $\bar{g} := -\frac{1}{12\lambda^2}B_{\mathfrak{su}(3)}$ . Let  $k \in \mathfrak{k}$  be a unit vector. Then

$$\varphi(k) = c_1 \text{ad}(e_5) + c_2 e_{67}.$$

By Lemma 3.3.19 we see that the  $(\mathfrak{k}, B)$ -extension is irreducible precisely when  $c_1, c_2 \in \mathbb{R} \setminus \{0\}$ . The  $(\mathfrak{k}, B)$ -extension is always regular in this case. If the  $(\mathfrak{k}, B)$ -extension is irreducible, we have  $\mathfrak{k} = \mathfrak{k}_2$  and it defines a naturally reductive structure on  $SU(3)/SU(2) \times H^3$ . On this homogeneous space we have a 3-parameter family of naturally reductive structures, with  $\lambda > 0$  and  $c_1, c_2$  as parameters. If we assume that  $c_1 > 0$  by changing the sign of  $k$ , then all of these are non-isomorphic.

**The canonical base space is  $SU(2) \times S^2 \times \mathbb{R}^2$  :** Let  $e_1, e_2, e_3$  be an orthonormal basis of  $\mathfrak{su}(2)$  with respect to  $-\frac{1}{8\lambda_1^2}B_{\mathfrak{su}(2)}$ . Let  $h, e_4, e_5$  be an orthonormal basis of  $\mathfrak{su}(2)$  with respect to  $-\frac{1}{8\lambda_2^2}B_{\mathfrak{su}(2)}$ . The transvection algebra of the base space is given by

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \oplus_{L.a.} \mathbb{R}^2,$$

with  $\mathfrak{h} := \text{span}\{h\}$  and  $\mathfrak{m} := \text{span}\{e_1, \dots, e_5\}$ . The  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear form on  $\mathfrak{g}$  is given by  $\bar{g} := -\frac{1}{8\lambda_1^2}B_{\mathfrak{su}(2)} \oplus -\frac{1}{8\lambda_2^2}B_{\mathfrak{su}(2)} \oplus B_{\text{eucl}}$ . Let  $e_6, e_7$  be an orthonormal basis of  $\mathbb{R}^2$ . Let  $k \in \mathfrak{k}$  be a unit vector. Up to an automorphism of the first  $\mathfrak{su}(3)$  summand we have

$$\varphi(k) = c_1 \text{ad}(e_1) + c_2 \text{ad}(h) + c_3 e_{67}.$$

From Lemma 3.3.19 we see that the  $(\mathfrak{k}, B)$ -extension is irreducible precisely when  $c_1, c_2, c_3 \in \mathbb{R} \setminus \{0\}$ , which we will assume from now on. By Section 2.2.3 the  $(\mathfrak{k}, B)$ -extension defines a naturally reductive structure on  $SU(3) \times S^2 \times H^3$ . On this space we get a 5-parameter family of naturally reductive structures, with  $\lambda_1, \lambda_2 > 0$  and  $c_1, c_2, c_3$  as parameters. If we change the sign of  $k$  such that  $c_1 > 0$ , then all of these structures are non-isomorphic. Note that we can replace  $S^2 = SU(2)/S^1$  by its non-compact symmetric dual  $SL(2, \mathbb{R})/S^1$ .

**The canonical base space is  $SU(3)/S^1$ :** Remember that from condition (i) of Proposition 3.3.14 we require that  $\mathfrak{k} \subset \mathcal{Z}(\mathfrak{h})$ . Let  $\mathfrak{h}_\theta := \text{span}\{r(\cos(\theta), \sin(\theta))\}$ , where  $r(a, b)$  is defined in (4.1.3) and  $\theta \in [0, 2\pi)$ . Let  $h \in \mathfrak{h}_\theta$  with  $\bar{g}(h, h) = 1$ , where  $\bar{g} = \frac{-1}{12\lambda^2} B_{\mathfrak{su}(3)}$ . Let  $\mathfrak{m} = \mathfrak{h}_\theta^\perp$  with respect to  $\bar{g}$ . The transvection algebra of the base space is given by

$$\mathfrak{su}(3) = \mathfrak{g} = \mathfrak{h}_\theta \oplus \mathfrak{m}.$$

Let  $k \in \mathfrak{k}$  be a unit vector. Then

$$\varphi(k) = c \cdot \text{ad}(h),$$

for some  $c \in \mathbb{R} \setminus \{0\}$ . The curvature is given by

$$R = -\text{ad}(h) \odot \text{ad}(h) + \varphi(k) \odot \varphi(k).$$

Condition (ii) of Proposition 3.3.14 is satisfied precisely when  $c \neq \pm 1$ . When this is satisfied we see from Section 2.2.1 that the  $(\mathfrak{k}, B)$ -extension is always regular and describes a naturally reductive structure on  $SU(3)$ . All of these structures are irreducible by Lemma 3.3.19. We obtain a 3-parameter family of metrics on  $SU(3)$ , with  $\lambda > 0$ ,  $c > 0$  and  $\theta \in [0, 2\pi)$  as parameters. Two of these naturally reductive structures  $(\lambda, c, \theta)$  and  $(\lambda', c', \theta')$  are isomorphic precisely when  $\lambda = \lambda'$ ,  $c = c'$  and  $r(\cos(\theta), \sin(\theta))$  is related to  $r(\cos(\theta'), \sin(\theta'))$  as in (4.1.4).

**The canonical base space is  $SU(2)^3/(S^1 \times S^1)$ :** We take the description of this space from Section 4.1.1. Let  $k \in \mathfrak{k}$  be a unit vector. Then

$$\varphi(k) = c_1 \text{ad}(h_1) + c_2 \text{ad}(h_2).$$

The  $(\mathfrak{k}, B)$ -extension is always irreducible by Lemma 3.3.19. We need to check that the curvature of the  $(\mathfrak{k}, B)$ -extension satisfies condition (ii) from Proposition 3.3.14, i.e. when  $\ker(R|_{\text{ad}(\mathfrak{h} \oplus \mathfrak{k})}) = \{0\}$  holds. The curvature is given by

$$R = -\text{ad}(h_1) \odot \text{ad}(h_1) - \text{ad}(h_2) \odot \text{ad}(h_2) + \varphi(k) \odot \varphi(k).$$

Let  $\omega_1, \omega_2$  be such that  $B_{\Lambda^2}(\omega_i, \text{ad}(h_j)) = \delta_{ij}$ . Then

$$\begin{aligned} R(\omega_1) &= (-1 + c_1^2)\text{ad}(h_1) + c_1c_2\text{ad}(h_2), \\ R(\omega_2) &= c_1c_2\text{ad}(h_1) + (-1 + c_2^2)\text{ad}(h_2). \end{aligned}$$

We see that  $\ker(R|_{\text{ad}(\mathfrak{h} \oplus \mathfrak{k})}) = \{0\}$  precisely when  $c_1^2 + c_2^2 \neq 1$ . The isotropy algebra  $\mathfrak{h}_0$  is in this case given by  $\mathfrak{h}_0 := \mathfrak{k}^\perp \subset \mathfrak{h}$  and is spanned by  $c_2h_1 - c_1h_2$ . The  $(\mathfrak{k}, B)$ -extension is regular precisely when the connected subgroup with Lie subalgebra  $\mathfrak{h}_0$  is closed. If the  $(\mathfrak{k}, B)$ -extension is regular, then it defines a naturally reductive structure on  $SU(2)^3/S_{k_1, k_2, k_3}^1$ . This is discussed for the type I space  $(SU(2)^3, \mathbb{R})$  in Section 4.1.2. Two tuples  $(c_1, c_2, \lambda_1, \lambda_2, \lambda_3)$  and  $(c'_1, c'_2, \lambda'_1, \lambda'_2, \lambda'_3)$  yield isomorphic spaces if and only if  $\lambda_i = \lambda'_i$  and  $(c_1, c_2) = \pm(c'_1, c'_2)$ . This follows directly from Proposition 3.3.16 and because any automorphism of  $\mathfrak{su}(2)^3$  that preserves  $\mathfrak{h} = \text{span}\{h_1, h_2\} \subset \mathfrak{su}(2)^3$  restricts to the identity on  $\mathfrak{h}$ . To obtain all of these naturally reductive structures on the fixed homogeneous space  $SU(2)^3/S_{k_1, k_2, k_3}^1$  we start by defining  $\mathfrak{h}_0 := \text{Lie}(S_{k_1, k_2, k_3}^1)$  and  $\mathfrak{k} := \mathfrak{h}_0^\perp \subset \mathfrak{h}$  with respect to  $\bar{g}$ . We have a 1-parameter family of  $\text{ad}(\mathfrak{k})$ -invariant metrics on  $\mathfrak{k}$ . Together with the parameters  $\lambda_1, \lambda_2, \lambda_3$  this gives us a 4-parameter family of naturally reductive structures on  $SU(2)^3/S_{k_1, k_2, k_3}^1$ . Note that we can replace one or two of the  $SU(2)$  factor by  $SL(2, \mathbb{R})$  as described in Remark 3.3.18.

**The canonical base space is  $(SU(2) \times SU(2))/S^1 \times S^2$  :** Let  $e_1, \dots, e_5$  and  $h_1 := h$  be as in (4.2.4). Let  $h_2, e_6, e_7$  be an orthonormal basis of  $\mathfrak{su}(2)$  with respect to  $-\frac{1}{8\lambda_3^2}B_{\mathfrak{su}(2)}$ . The transvection algebra of the base space is given by

$$\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) = \mathfrak{h} \oplus \mathfrak{m},$$

where  $\mathfrak{h} := \text{span}\{h_1, h_2\}$  and  $\mathfrak{m} := \text{span}\{e_1, \dots, e_7\}$ . The  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear form on  $\mathfrak{g}$  is  $\bar{g} = \frac{-1}{8\lambda_1^2}B_{\mathfrak{su}(2)} \oplus \frac{-1}{8\lambda_2^2}B_{\mathfrak{su}(2)} \oplus \frac{-1}{8\lambda_3^2}B_{\mathfrak{su}(2)}$ . Let  $k \in \mathfrak{k}$  be a unit vector. Note that by condition (i) of Proposition 3.3.14 we only have to consider the case  $\mathfrak{k} \subset \mathcal{Z}(\mathfrak{h})$ . Hence we get

$$\varphi(k) = c_1\text{ad}(h_1) + c_2\text{ad}(h_2).$$

From Lemma 3.3.19 we see that the  $(\mathfrak{k}, B)$ -extension is irreducible precisely when  $c_1, c_2 \in \mathbb{R} \setminus \{0\}$ . Suppose that  $c_1, c_2 \in \mathbb{R} \setminus \{0\}$ . If we change the sign of  $k$  such that  $c_1 > 0$ , then all of these  $(\mathfrak{k}, B)$ -extensions are non-isomorphic, because all automorphisms of  $\mathfrak{g}$  which preserves  $\mathfrak{h}$  restrict to the identity on  $\mathfrak{h}$ . We now check when condition (ii) of Proposition 3.3.14 is satisfied. We have  $\text{ad}(\mathfrak{h} \oplus \mathfrak{k}) = \text{ad}(\mathfrak{h})$  and

$$R = -\text{ad}(h_1) \odot \text{ad}(h_1) - \text{ad}(h_2) \odot \text{ad}(h_2) + \varphi(k) \odot \varphi(k).$$

Let  $\omega_1, \omega_2$  be such that  $B_{\Lambda^2}(\omega_i, \text{ad}(h_j)) = \delta_{ij}$ . Then

$$\begin{aligned} R(\omega_1) &= (-1 + c_1^2)\text{ad}(h_1) + c_1c_2\text{ad}(h_2), \\ R(\omega_2) &= c_1c_2\text{ad}(h_1) + (-1 + c_2^2)\text{ad}(h_2). \end{aligned}$$

We see that  $\ker(R|_{\text{ad}(\mathfrak{h} \oplus \mathfrak{k})}) = \{0\}$  precisely when  $c_1^2 + c_2^2 \neq 1$ . We should still check which of these  $(\mathfrak{k}, B)$ -extensions are regular. From Section 2.2.1 we know that it is regular precisely when the connected Lie subgroup with Lie algebra  $\mathfrak{h}_0$  as in (2.2.10) is a closed subgroup. In this case  $\mathfrak{h}_0 = \mathfrak{k}^\perp \subset \mathfrak{h}$ , where the orthogonal complement is taken with respect to  $\bar{g}$ . We have  $\mathfrak{h}_0 = \text{span}\{c_2h_1 - c_1h_2\}$ . For the connected Lie subgroup  $H_0$  with  $\text{Lie}(H_0) = \mathfrak{h}_0$  to be closed in  $SU(2)^3$  we require that the parameter  $\alpha$  from (4.2.4) is in  $\mathbb{Q}$  and also  $c_1\lambda_3\sqrt{\alpha^2\lambda_1^2 + \lambda_2^2}(c_2\lambda_1\lambda_2)^{-1} \in \mathbb{Q}$ . In this case there are integers  $k_1, k_2, k_3 \in \mathbb{Z}$  such that  $H_0 = S_{k_1, k_2, k_3}^1$ . To obtain all of these naturally reductive structures on the fixed homogeneous space  $SU(2)^3/S_{k_1, k_2, k_3}^1$  we start by defining  $\mathfrak{h}_0 := \text{Lie}(S_{k_1, k_2, k_3}^1)$  and  $\mathfrak{k} := \mathfrak{h}_0^\perp \subset \mathfrak{h}$  with respect to  $\bar{g}$ . We have a 1-parameter family of  $\text{ad}(\mathfrak{k})$ -invariant metrics on  $\mathfrak{k}$ . Together with the parameters  $\lambda_1, \lambda_2, \lambda_3$  this gives us a 4-parameter family of naturally reductive structures on  $SU(2)^3/S_{k_1, k_2, k_3}^1$ . To obtain all of the dual spaces we can replace one of the  $SU(2)$  factors in  $(SU(2) \times SU(2))/S_{k_1, k_2}^1$  by  $SL(2, \mathbb{R})$  and we can replace  $S^2$  by its non-compact symmetric dual  $SL(2, \mathbb{R})/S^1$ .

**The canonical base space is  $(SU(3) \times SU(2))/(SU(2)_\Delta \times S^1)$  or  $(SU(3) \times SU(2))/(SU(2) \times S_{k_1, k_2}^1)$ :** Let  $\bar{g} := \frac{-1}{12\lambda_1^2}B_{\mathfrak{su}(3)} \oplus \frac{-1}{8\lambda_2^2}B_{\mathfrak{su}(2)}$  be some  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear form on  $\mathfrak{g}$ . Remember that by condition (i) of Proposition 3.3.14 we require that  $\mathfrak{k} \subset \mathcal{Z}(\mathfrak{h})$ . We consider the first space  $(SU(3) \times SU(2))/(SU(2)_\Delta \times S^1)$ . We have  $\text{Lie}(S^1) = \text{span}\{h_4\}$  with  $h_4$  as in (4.2.1) and with  $\lambda$  replaced by  $\lambda_1$ . Hence for a unit vector  $k \in \mathfrak{k}$  we have

$$\varphi(k) = c \cdot \text{ad}(h_4),$$

for some  $c \in \mathbb{R} \setminus \{0\}$ . The curvature is given by

$$R = - \sum_{i=1}^3 \text{ad}(h_i) \odot \text{ad}(h_i) - \text{ad}(h_4) \odot \text{ad}(h_4) + \varphi(k) \odot \varphi(k),$$

where  $h_1, h_2, h_3, h_4$  is an orthonormal basis of  $\mathfrak{h}$ . We see that the curvature has rank 4 precisely when  $c \neq \pm 1$ . In this case the base space is equal to the canonical base space by Proposition 3.3.14. The  $(\mathfrak{k}, B)$ -extension is always regular. We obtain a 3-parameter family of irreducible naturally reductive structures on  $(SU(3) \times$

$SU(2))/SU(2)_\Delta$ , with  $\lambda_1, \lambda_2$  and  $c \neq 0$  as parameters. If we choose  $k \in \mathfrak{k}$  such that  $c > 0$ , then none of these spaces are isomorphic. There is a dual naturally reductive space for the base space  $(SU(2, 1) \times SU(2))/(SU(2)_\Delta \times S^1)$ . For the dual  $(\mathfrak{k}, B)$ -extension condition (ii) of Proposition 3.3.14 is automatically satisfied.

The  $(\mathfrak{k}, B)$ -extensions of the base space  $(SU(3) \times SU(2))/(SU(2) \times S^1_{k_1, k_2})$  are analogous. We should point out that in this case the  $(\mathfrak{k}, B)$ -extensions are all regular, whether the subgroup with Lie algebra  $\mathbb{R}_{a,b}$  from (4.1.8) is closed or not. In Section 4.1.1 we saw that there are two dual spaces for the base space.

**The canonical base space is  $SU(3)/(S^1 \times S^1)$  :** We pick the following orthonormal basis with respect to  $\bar{g} = \frac{-1}{12\lambda^2} B_{\mathfrak{su}(3)}$  of  $\mathfrak{h} := \text{Lie}(S^1 \times S^1)$ :

$$h_1 := \begin{pmatrix} i\lambda & 0 & 0 \\ 0 & -i\lambda & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad h_2 := \begin{pmatrix} \frac{-i\lambda}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{-i\lambda}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{2i\lambda}{\sqrt{3}} \end{pmatrix}.$$

In this case we have  $\mathfrak{k} = \text{ad}(\mathfrak{h})$ . The only freedom is in the choice of a metric  $B$  on  $\mathfrak{k}$ . We define a quadratic form on  $\mathcal{Z}(\mathfrak{u}(3))$  by

$$\begin{pmatrix} ia & 0 & 0 \\ 0 & ib & 0 \\ 0 & 0 & ic \end{pmatrix} \mapsto x_1 a^2 + x_2 b^2 + x_3 c^2.$$

Restricting this to  $\mathfrak{h}$  gives us in the basis  $h_1, h_2$  the following symmetric bilinear form:

$$B_{x_1, x_2, x_3} := \begin{pmatrix} x_1 + x_2 & \frac{1}{\sqrt{3}}(-x_1 + x_2) \\ \frac{1}{\sqrt{3}}(-x_1 + x_2) & \frac{1}{3}(x_1 + x_2 + 4x_3) \end{pmatrix}.$$

This is positive definite if and only if its trace and determinant are positive, i.e.

$$\frac{3}{4} \text{tr}(B_{x_1, x_2, x_3}) = x_1 + x_2 + x_3 > 0 \quad \text{and} \quad \frac{3}{4} \det(B_{x_1, x_2, x_3}) = x_1 x_2 + x_2 x_3 + x_1 x_3 > 0.$$

This parametrizes exactly all metric tensors on  $\mathfrak{h}$ . From Proposition 3.3.16 we know that two of these metrics induce an isomorphic naturally reductive structure precisely when they are conjugate by an automorphism of  $\mathfrak{su}(3)$  which preserves  $\mathfrak{h}$ , i.e. an element of the normalizer  $N_{\mathfrak{su}(3)}(\mathfrak{h})$  of  $\mathfrak{h}$  in  $\mathfrak{su}(3)$ . Two metrics are conjugate by an element of  $N_{\mathfrak{su}(3)}(\mathfrak{h})$  if and only if they are conjugate by an element of the Weyl group of  $\mathfrak{su}(3)$ . The Weyl group of  $\mathfrak{su}(3)$  is isomorphic to  $S_3$  and the action of the Weyl group on  $\mathfrak{h}$  is given by conjugation with permutation matrices. Therefore, the Weyl

group acts on these metrics  $B_{x_1, x_2, x_3}$  by simply permuting the parameters  $x_1, x_2, x_3$ . We see that under the conditions

$$x_3 \geq x_2 \geq x_1$$

every  $S_3$ -orbit of these metrics is parametrized exactly ones. We still need to know when condition (ii) from Proposition 3.3.14 is satisfied. The curvature of the  $(\mathfrak{k}, B)$ -extension is given by

$$R = -\text{ad}(h_1) \odot \text{ad}(h_1) - \text{ad}(h_2) \odot \text{ad}(h_2) + \sum_{i,j=1}^2 (B^{-1})_{ij} \text{ad}(h_i) \odot \text{ad}(h_j).$$

In the basis  $h_1, h_2$  this becomes

$$R = 6\lambda^2 \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + 6\lambda^2 \det(B)^{-1} \begin{pmatrix} \frac{1}{3}(x_1 + x_2 + 4x_3) & \frac{1}{\sqrt{3}}(x_1 - x_2) \\ \frac{1}{\sqrt{3}}(x_1 - x_2) & x_1 + x_2 \end{pmatrix}.$$

This has full rank if and only if

$$x_1x_2 + x_2x_3 + x_1x_3 - x_1 - x_2 - x_3 + \frac{3}{4} \neq 0. \quad (4.2.5)$$

Under this condition the canonical base space is equal to  $SU(3)/(S^1 \times S^1)$  by Proposition 3.3.14. The  $(\mathfrak{k}, B)$ -extension is always regular and irreducible. Under the above conditions we obtain a 4-parameter family of naturally reductive structures on  $SU(3)$ , with  $\lambda > 0$  and  $x_1, x_2, x_3$  as parameters. None of these structures are isomorphic under the condition  $x_3 \geq x_2 \geq x_1$ . For a fixed  $\lambda > 0$  all the isomorphism classes of naturally reductive structures are depicted yellow in Figure 4.1. The blue cone depicts all parameters  $(x_1, x_2, x_3)$  such that  $x_1 + x_2 + x_3 \geq 0$  and  $x_1x_2 + x_2x_3 + x_1x_3 = 0$ . The red cone consists of the parameter values for which (4.2.5) is not satisfied. In the point  $(x_1, x_2, x_3) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$  the rank of the curvature is zero and this is the naturally reductive structure of type I described in Section 4.1.2. For the other points on the red cone the curvature has rank one and these correspond to naturally reductive structures of type II with canonical base space equal to  $SU(3)/S^1$ . Note that the yellow part has three connected components.

**The canonical base space is  $\mathbb{C}P^2 \times S^2$  :** As reductive decomposition for  $\mathbb{C}P^2 = SU(3)/S(U(1) \times U(2))$  we take the one from (4.2.1) and (4.2.2) with respect to

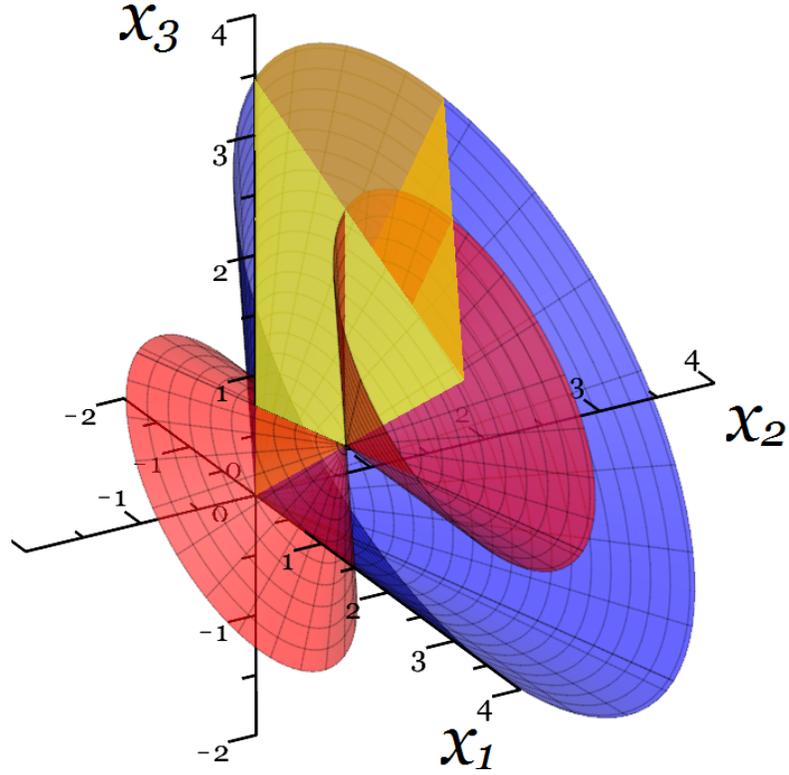


Figure 4.1: Parameter space of type II structures on  $SU(3)$ .

$\frac{-1}{12\lambda_1^2}B_{\mathfrak{su}(3)}$ . Furthermore, let  $h_5, e_5, e_6$  be an orthonormal basis of  $\mathfrak{su}(2)$  with respect to  $\frac{-1}{8\lambda_2^2}B_{\mathfrak{su}(2)}$ . The transvection algebra of the base space is given by

$$\mathfrak{g} = \mathfrak{su}(3) \oplus \mathfrak{su}(2) = \mathfrak{h} \oplus \mathfrak{m},$$

where  $\mathfrak{h} := \text{span}\{h_1, \dots, h_5\}$  and  $\mathfrak{m} := \text{span}\{e_1, \dots, e_6\}$  and the  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear form is  $\bar{g} = \frac{-1}{12\lambda_1^2}B_{\mathfrak{su}(3)} \oplus \frac{-1}{8\lambda_2^2}B_{\mathfrak{su}(2)}$ . The algebra  $\mathfrak{k} \subset \mathfrak{s}(\mathfrak{g}) = \text{span}\{h_4, h_5\}$  is 2-dimensional. We pick an orthonormal basis of  $\mathfrak{k}$  such that

$$\varphi(k_1) = c_1\text{ad}(h_4) + c_2\text{ad}(h_5), \quad \varphi(k_2) = d_1\text{ad}(h_4),$$

where  $c_2 > 0$  and  $d_1 > 0$ . The  $(\mathfrak{k}, B)$ -extension is reducible precisely when  $c_1 = 0$ , see Lemma 3.3.19. From now on let  $c_1 \neq 0$ . We need to check when  $R|_{\text{ad}(\mathfrak{h} \oplus \mathfrak{k})}$  has trivial kernel or equivalently when  $R|_{\text{ad}(\mathfrak{z}(\mathfrak{h} \oplus \mathfrak{k}))}$  has trivial kernel, see Lemma 3.3.12. We

have  $\text{ad}(\mathcal{Z}(\mathfrak{h} \oplus \mathfrak{k})) = \text{span}\{\text{ad}(h_4), \text{ad}(h_5)\}$ . Let  $\omega_1, \omega_2$  be such that  $B_{\Lambda^2}(\omega_1, h_j) = \delta_{4j}$  and  $B_{\Lambda^2}(\omega_2, h_j) = \delta_{5j}$  for  $j = 1, \dots, 5$ . Then

$$\begin{aligned} R(\omega_1) &= (-1 + c_1^2 + d_1^2)\text{ad}(h_4) + c_1c_2\text{ad}(h_5), \\ R(\omega_2) &= c_1c_2\text{ad}(h_4) + (-1 + c_2^2)\text{ad}(h_5). \end{aligned}$$

Condition (ii) from Proposition 3.3.14 is satisfied precisely when

$$1 - c_1^2 - c_2^2 - d_1^2 + d_1^2c_2^2 = \det \begin{pmatrix} -1 + c_1^2 + d_1^2 & c_1c_2 \\ c_1c_2 & -1 + c_2^2 \end{pmatrix} \neq 0.$$

Thus, condition (ii) is satisfied precisely when  $c_1^2 + c_2^2 + d_1^2 - c_2^2d_1^2 \neq 1$ . The base space is then equal to the canonical base space. From Section 2.2.1 we know that the  $(\mathfrak{k}, B)$ -extension is always regular and it defines a naturally reductive structure on  $(SU(3)/SU(2)) \times SU(2)$ , where  $SU(2) \subset SU(3)$  is the standard embedding. On this space we obtain a 5-parameter family of naturally reductive structures, with  $\lambda_1, \lambda_2 > 0$  and  $c_1, c_2 > 0, d_1 > 0$  as parameters. Every automorphism of  $\mathfrak{g}$  which preserves  $\mathfrak{s}(\mathfrak{g}) = \text{span}\{h_4, h_5\}$  acts trivially on  $\mathfrak{s}(\mathfrak{g})$ , thus all of these structures are non-isomorphic. We can replace both  $\mathbb{C}P^2 = SU(3)/S(U(1) \times U(2))$  and  $S^2 = SU(2)/S^1$  by their non-compact dual symmetric spaces:  $SU(1, 2)/S(U(1) \times U(2))$  and  $SL(2, \mathbb{R})/S^1$ .

**The canonical base space is  $S^2 \times S^2 \times S^2$  :** Let  $h_1, e_1, e_2$  be an orthonormal basis of  $\mathfrak{su}(2)$  with respect to  $\frac{-1}{8\lambda_1^2}B_{\mathfrak{su}(2)}$ . Let  $h_2, e_3, e_4$  be an orthonormal basis of  $\mathfrak{su}(2)$  with respect to  $\frac{-1}{8\lambda_2^2}B_{\mathfrak{su}(2)}$ . Let  $h_3, e_5, e_6$  be an orthonormal basis of  $\mathfrak{su}(2)$  with respect to  $\frac{-1}{8\lambda_3^2}B_{\mathfrak{su}(2)}$ . The transvection algebra of the base space is given by

$$\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) = \mathfrak{h} \oplus \mathfrak{m},$$

where  $\mathfrak{h} := \text{span}\{h_1, h_2, h_3\}$  and  $\mathfrak{m} := \text{span}\{e_1, \dots, e_6\}$ . The  $\text{ad}(\mathfrak{g})$ -invariant non-degenerate symmetric bilinear form on  $\mathfrak{g}$  is  $\bar{g} := \frac{-1}{8\lambda_1^2}B_{\mathfrak{su}(2)} \oplus \frac{-1}{8\lambda_2^2}B_{\mathfrak{su}(2)} \oplus \frac{-1}{8\lambda_3^2}B_{\mathfrak{su}(2)}$ . The algebra  $\mathfrak{s}(\mathfrak{g})$  is equal to  $\mathfrak{h}$ . We pick an orthonormal basis of  $\mathfrak{k}$  such that

$$\varphi(k_1) = c_1\text{ad}(h_1) + c_2\text{ad}(h_2) + c_3\text{ad}(h_3), \quad \varphi(k_2) = d_1\text{ad}(h_1) + d_2\text{ad}(h_2),$$

with  $c_3 > 0$ . By Lemma 3.3.19 the  $(\mathfrak{k}, B)$ -extension is reducible precisely when one of the following holds:  $c_1 = d_2 = 0, c_2 = d_1 = 0, d_1 = c_1 = 0, d_2 = c_2 = 0$  or  $c_1 = c_2 = 0$ . The curvature of the  $(\mathfrak{k}, B)$ -extension is given by

$$R = - \sum_{i=1}^3 \text{ad}(h_i) \odot \text{ad}(h_i) + \varphi(k_1) \odot \varphi(k_1) + \varphi(k_2) \odot \varphi(k_2).$$

We need to know when  $R|_{\text{ad}(\mathfrak{h} \oplus \mathfrak{k})}$  has trivial kernel. Let  $\omega_1, \omega_2, \omega_3$  be the dual basis of  $\text{ad}(h_1), \text{ad}(h_2), \text{ad}(h_3)$  in  $\text{ad}(\mathfrak{h})$  with respect to  $B_{\Lambda^2}$ . Then

$$\begin{aligned} R(\omega_1) &= (-1 + c_1^2 + d_1^2)\text{ad}(h_1) + (c_1c_2 + d_1d_2)\text{ad}(h_2) + c_1c_3\text{ad}(h_3), \\ R(\omega_2) &= (c_1c_2 + d_1d_2)\text{ad}(h_1) + (-1 + c_2^2 + d_2^2)\text{ad}(h_2) + c_2c_3\text{ad}(h_3), \\ R(\omega_3) &= c_1c_3\text{ad}(h_1) + c_2c_3\text{ad}(h_2) + (-1 + c_3^2)\text{ad}(h_3). \end{aligned}$$

Hence  $R$  has rank 3 precisely when

$$\det \begin{pmatrix} -1 + c_1^2 + d_1^2 & c_1c_2 + d_1d_2 & c_1c_3 \\ c_1c_2 + d_1d_2 & -1 + c_2^2 + d_2^2 & c_2c_3 \\ c_1c_3 & c_2c_3 & -1 + c_3^2 \end{pmatrix} \neq 0,$$

or equivalently if  $c_1^2 + c_2^2 + c_3^2(1 - d_1^2 - d_2^2) - (c_1d_2 - c_2d_1)^2 + d_1^2 + d_2^2 \neq 1$ . If this is the case, then the base space is equal to the canonical base space by Proposition 3.3.14. Let  $\mathfrak{h}_0 := \mathfrak{k}^\perp \subset \mathfrak{h}$ . The  $(\mathfrak{k}, B)$ -extension is regular precisely when the connected subgroup  $H_0$  with  $\text{Lie}(H_0) = \mathfrak{h}_0 = \text{span}\{c_3d_2h_1 - c_3d_1h_2 - (c_1d_2 - c_2d_1)h_3\}$  is a closed subgroup of  $SU(2)^3$ . This is the case exactly when  $\frac{d_2\lambda_1}{d_1\lambda_2} \in \mathbb{Q}$  and  $\frac{c_3d_2\lambda_1}{(c_1d_2 + c_2d_1)\lambda_3} \in \mathbb{Q}$ . Then the  $(\mathfrak{k}, B)$ -extension defines a naturally reductive structure on  $SU(2)^3/S^1_{k_1, k_2, k_3}$ , for certain integers  $k_1, k_2, k_3 \in \mathbb{Z}$ . By permuting the factors and changing the sign of  $\mathfrak{k}_2$  we can always assume that  $c_1 \leq c_2 \leq c_3$ ,  $0 < d_2$  and if  $c_1 = c_2$ , then  $d_1 \leq d_2$ . If we do this, then none of the  $(\mathfrak{k}, B)$ -extensions are isomorphic. To obtain all of these naturally reductive structures on the fixed homogeneous space  $SU(2)^3/S^1_{k_1, k_2, k_3}$  we start by defining  $\mathfrak{h}_0 := \text{Lie}(S^1_{k_1, k_2, k_3})$  and  $\mathfrak{k} := \mathfrak{h}_0^\perp \subset \mathfrak{h}$  with respect to  $\bar{g}$ . We have a 3-parameter family of  $\text{ad}(\mathfrak{k})$ -invariant metrics on  $\mathfrak{k}$ . Together with the parameters  $\lambda_1, \lambda_2, \lambda_3$  this gives us a 6-parameter family of naturally reductive structures on  $SU(2)^3/S^1_{k_1, k_2, k_3}$ . Note that we can replace any of the  $S^2$  factors by its non-compact symmetric dual  $SL(2, \mathbb{R})/S^1$ .

**The canonical base space is  $\{*\}$  :** We write  $\mathfrak{g} = \{0\}$  for the 0-dimensional Lie algebra. Let  $\mathfrak{k} = \mathfrak{su}(3)$  and let  $B = \frac{-1}{\lambda^2} B_{\mathfrak{su}(3)}$ . Let  $x_1, \dots, x_8$  be an orthonormal basis of  $\mathfrak{su}(3)$  with respect to  $B$ . The torsion and curvature are given by

$$T(x, y, z) = 2B([x, y], z) \quad \text{and} \quad R = \sum_{i=1}^8 \text{ad}(x_i) \odot \text{ad}(x_i),$$

just as in (2.2.5). The Lie algebra  $\mathfrak{g}(\mathfrak{k})$  is given by

$$\mathfrak{g}(\mathfrak{k}) = \mathfrak{k} \oplus \mathfrak{n} = \mathfrak{k} \times \mathfrak{a} \cong \mathfrak{su}(3) \times \mathbb{R}^8.$$

The infinitesimal model is always irreducible and regular. The homogeneous space is just  $\mathbb{R}^8$  with the euclidean metric and we obtain a 1-parameter family of naturally reductive structures, with  $\lambda > 0$  as parameter.

This concludes the classification of all 7- and 8-dimensional naturally reductive spaces of type II. We summarize the discussion from Section 4.2.1 and Section 4.2.2 in the following result.

**Theorem 4.2.6.** *Every 7- and 8-dimensional naturally reductive space of type II for which the type I part of the canonical base space is compact is presented in Table 4.6. In the fourth column is the number of parameters of naturally reductive structures of type II on the homogeneous space  $G/H$ . The canonical base space of the naturally reductive structure is in the second column. The third column indicates if dual spaces exist.*

$G/H$	canonical base space	dual	# param.
$H^7$	$\mathbb{R}^6$	×	3
$QH^7$	$\mathbb{R}^4$	×	1
$S^2 \times H^5$	$S^2 \times \mathbb{R}^4$	✓	4
$SU(2) \times H^3 \times \mathbb{R}$	$SU(2) \times \mathbb{R}^2$	×	4
$S^2 \times S^2 \times H^3$	$S^2 \times S^2 \times \mathbb{R}^2$	✓	5
$\mathbb{C}P^2 \times H^3$	$\mathbb{C}P^2 \times \mathbb{R}^2$	✓	3
$Sp(2)/Sp(1)_{\text{st}}$	$Sp(2)/(SU(2) \times S^1)$	×	2
$SU(3)/S^1_{k_1, k_2}$	$SU(3)/(S^1 \times S^1)$	×	2
$SO(5)/SO(3)_{\text{st}}$	$SO(5)/(SO(3) \times SO(2))$	✓	2
$SU(4)/SU(3)$	$SU(4)/S(U(1) \times U(3))$	✓	2
$(SU(3) \times SU(2))/(SU(2) \times S^1_{k_1, k_2})$	$\mathbb{C}P^2 \times S^2$	✓	3
$SU(2)^3/(S^1_{k_1, k_2, k_3} \times S^2_{l_1, l_2, l_3})$	$S^2 \times S^2 \times S^2$	✓	4
$Nil(\mathbb{R}^2 \rightarrow \mathfrak{so}(6))$	$\mathbb{R}^6$	×	5
$Nil(\varphi_{\text{ir}} : \mathfrak{so}(3) \rightarrow \mathfrak{so}(5))$	$\mathbb{R}^5$	×	1
$Nil(\mathfrak{u}(2) \rightarrow \mathfrak{so}(4))$	$\mathbb{R}^4$	×	2
$(SU(2) \times N^6)/\mathbb{R}$	$S^2 \times \mathbb{R}^4$	✓	6
$SU(2) \times H^5$	$S^2 \times \mathbb{R}^4$	✓	5
$SU(2) \times H^5$	$SU(2) \times \mathbb{R}^4$	×	4
$SU(3)/SU(2)_{\text{st}} \times H^3$	$\mathbb{C}P^2 \times \mathbb{R}^2$	✓	4
$(SU(2) \times SU(2) \times H^3)/\mathbb{R}_\alpha$	$S^2 \times S^2 \times \mathbb{R}^2$	✓	6
$(SU(2) \times SU(2))/S^1_{k_1, k_2} \times H^3$	$S^2 \times S^2 \times \mathbb{R}^2$	✓	5
$(SU(2) \times SU(2) \times H^3)/\mathbb{R}_\alpha$	$(SU(2) \times SU(2))/S^1_\alpha \times \mathbb{R}^2$	✓	5
$(SU(2) \times SU(2))/S^1_\alpha \times H^3$	$(SU(2) \times SU(2))/S^1_\alpha \times \mathbb{R}^2$	✓	4
$SU(3)/SU(2)_{\text{st}} \times H^3$	$SU(3)/SU(2)_{\text{st}} \times \mathbb{R}^2$	×	3
$SU(2) \times S^2 \times H^3$	$SU(2) \times S^2 \times \mathbb{R}^2$	✓	5
$SU(3)$	$SU(3)/S^1_{k_1, k_2}$	×	3
$SU(2)^3/S^1_{n_1, n_2, n_3}$	$SU(2)^3/(S^1_{k_1, k_2, k_3} \times S^1_{l_1, l_2, l_3})$	✓	4
$SU(2)^3/S^1_{l_1, l_2, l_3}$	$(SU(2) \times SU(2))/S^1_{k_1, k_2} \times S^2$	✓	4
$(SU(3) \times SU(2))/SU(2)_{\text{st}} \times \text{id}$	$(SU(3) \times SU(2))/(SU(2)_{\text{st}} \times \text{id} \times S^1)$	✓	3
$(SU(3) \times SU(2))/SU(2)_{\text{st}}$	$(SU(3) \times SU(2))/(SU(2)_{\text{st}} \times S^1_{k_1, k_2})$	✓	3
$SU(3)$	$SU(3)/(S^1 \times S^1)$	×	4
$SU(3)/SU(2)_{\text{st}} \times SU(2)$	$\mathbb{C}P^2 \times S^2$	✓	5
$SU(2)^3/S^1_{k_1, k_2, k_3}$	$S^2 \times S^2 \times S^2$	✓	6
$\mathbb{R}^8$	$\{*\}$	×	1

Table 4.6: 7- and 8-dimensional type II spaces.

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