Deformed Fomin-Kirillov Algebras and Applications

Dissertation

zur Erlangung des Doktorgrades der
Naturwissenschaften (Dr. rer. nat)
vorgelegt dem
Fachbereich Mathematik und Informatik der
Philipps-Universität Marburg von

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aus Gießen

Marburg, Dezember 2016

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Einreichungstermin: 25. Oktober 2016
Prüfungstermin: 21. Dezember 2016
Hochschulkennziffer: 1180

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Introduction

The Fomin-Kirillov algebra $\mathcal{E}_n$ is a quadratic algebra generated by the edges of the complete graph on n vertices which was first introduced in [**FK98**] in order to study Schubert calculus. Later it was observed ([**MS00**], see also [**GV16**] for history) that for $n = 3, 4, 5$ the Fomin-Kirillov algebra $\mathcal{E}_n$ is a Nichols algebra of nonabelian group type. This is conjectured to be true for $n > 5$ aswell.

To this day very elementary properties like the vector space dimension of $\mathcal{E}_6$ remain unknown.

Recently, [**BLM13**] extended the scope to "hyperbolic" subalgebras $\mathcal{E}_S$ generated by the edges of subgraphs of the complete graph. While those algebras have nice properties in their own right (for example, well behaved Hilbert series), they also very much widen the range of examples one can use to try out new methods for the study of Fomin-Kirillov algebras.

The goal of this work is to give new methods to study Fomin-Kirillov algebras, in particular to calculate their dimensions as vector spaces or decide finite dimensionality.

Our main tool are *deformed* Fomin-Kirillov algebras, denoted by $\mathcal{D}_S$. The second chapter introduces those algebras and connects them to regular Fomin-Kirillov algebras by realizing the deformed Fomin-Kirillov algebra $\mathcal{D}_S$ as a subalgebra of the linear endomorphisms of the regular Fomin-Kirillov algebra $\mathcal{E}_S$. In this context also note that the Fomin-Kirillov algebra is known to be *not* Koszul [**Roo99**].

Note that more general deformations of Fomin-Kirillov algebras were already considered in [**FK98**], however with a very different purpose and without investigating the relation between the deformations and the regular algebras.

In the third and fourth chapter we use the deformation and the results from the second chapter to introduce new methods. Note that chapter three and four are largely independent from each other.

In chapter three we study subalgebras of deformed Fomin-Kirillov algebras that correspond to certain subgroups of the symmetric group. Those subalgebras of $\mathcal{D}_S$ are relatively well behaved - they are related to

Fomin-Kirillov algebras $\mathcal{E}_{S'}$ belonging to graphs on fewer vertices than S. This allows us to rederive most of the known dimensions of algebras $\mathcal{E}_S$ by hand, which with different methods is mostly only feasible using Groebner basis methods, see also [**BLM13**]. Most importantly, we are able to calculate the dimension of the full Fomin-Kirillov algebra $\mathcal{E}_5$ for the first time without using computer calculations. Our approach for this is similar to the one given in [**FP00**]. We also apply our method to $\mathcal{E}_6$, and while our results are still only very much partial we believe they look somewhat promising and warrant further investigation.

In chapter four we attach groups to Fomin-Kirillov algebras. For Nichols algebras these groups were already considered in [**Loc13**]. It turns out that finite dimensionality of a Fomin-Kirillov algebra is related to finiteness of the attached group.

Our original motivation for investigating these groups was to prove our conjecture that the deformed Fomin-Kirillov algebras are always semisimple by realizing them as quotients of group algebras. However, much to our surprise, the attached groups turned to be quite interesting themselves. In particular, the alternating group, some sporadic simple groups, and an exceptional group of Lie type occured. In the way those groups occur they naturally fit into the class of groups classified in [**FLZ01**]. Based on this observation we propose a tentative strategy to tackle infinite dimensionality of $\mathcal{E}_6$.

Most of the examples in the fourth chapter are computational and based on the algorithms described in [**BHLGO15**]. We consider giving a theoretical explanation of the occurence of the mentioned groups to be a tantalizing question.

CHAPTER 1

Fomin-Kirillov Algebras

We recall some of the known statements about Fomin-Kirillov algebras that will be used later on. Note that treatment in this section is in no way meant to be exhaustive.

1. Fomin-Kirillov Algebras

Here and throughout let $\Bbbk$ be a field, n a natural number, and $T = \{(i,j) \mid 1 \leq i,j \leq n, i \neq j\}$.

We denote by $\mathcal{E}_n = \mathcal{E}_n(\Bbbk)$ the associative unital $\Bbbk$-algebra given by generators $x_{ij} = x_{(i,j)}$, where $(i,j) \in T$, and relations

$$\begin{aligned} x_{ij} &= -x_{ji} && \text{for all } (i,j) \in T, \\ x_{ij}^2 &= 0 && \text{for all } (i,j) \in T, \\ x_{ij}x_{jk} + x_{jk}x_{ki} + x_{ki}x_{ij} &= 0 && \text{if } \#\{i,j,k\} = 3, \\ x_{ij}x_{kl} - x_{kl}x_{ij} &= 0 && \text{if } \#\{i,j,k,l\} = 4. \end{aligned}$$

The algebras $\mathcal{E}_n$ are known as *Fomin-Kirillov algebras*. They have been introduced by Fomin and Kirillov in [**FK98**]. We recall some of the basic facts from [**FK98**].

The algebra $\mathcal{E}_n$ is $\mathbb{N}_0$-graded such that $\deg x_{ij} = 1$ for any $(i,j) \in T$. We write $\mathcal{E}_n(l)$ for the homogeneous component of $\mathcal{E}_n$ of degree $l \in \mathbb{N}_0$ and $\mathcal{E}_n^l = \bigoplus_{j=0}^{l} \mathcal{E}_n(j)$ for the elements up to degree $l \in \mathbb{N}_0$.

There is an algebra isomorphism $\tau : \mathcal{E}_n \to \mathcal{E}_n^{op}$ from the Fomin-Kirillov algebra to its opposite algebra that is the identity on generators.

Moreover, $\mathcal{E}_n$ has a unique grading by the symmetric group $\mathbb{S}_n$ such that x_{ij} is homogeneous of degree $(i\,j)$ for any $(i,j) \in T$. The homogeneous components of degree $\sigma \in \mathbb{S}_n$ with respect to this grading are denoted by $(\mathcal{E}_n)_\sigma$.

Furthermore, there is a unique action of the symmetric group $\mathbb{S}_n$ on $\mathcal{E}_n$ such that $\sigma(xy) = (\sigma x)(\sigma y)$ for all $\sigma \in \mathbb{S}_n$ and $x, y \in \mathcal{E}_n$ given by $\sigma x_{ij} = x_{\sigma(i)\sigma(j)}$ for $\sigma \in \mathbb{S}_n$ and $(i,j) \in T$. In particular, $\mathcal{E}_n$ is a $\Bbbk\mathbb{S}_n$-module algebra.

Note that we have $\sigma(\mathcal{E}_n)_\tau \subseteq (\mathcal{E}_n)_{\sigma\tau\sigma^{-1}}$ for $\sigma, \tau \in \mathbb{S}_n$. In other words, $\mathcal{E}_n$ is a Yetter-Drinfeld module over $\Bbbk\mathbb{S}_n$. It was observed in [**MS00**] that one can turn $\mathcal{E}_n$ into a Hopf-algebra in the braided category of Yetter-Drinfeld modules over $\Bbbk\mathbb{S}_n$ (a *braided* Hopf-algebra). We will find it more convenient to work with a slightly different version of this structure as described in Proposition 1.1. Note that we abuse notation and identify elements $t \in T$ with the corresponding transposition in $\mathbb{S}_n$.

Proposition 1.1. [**FP00**] *The smash product algebra $\mathcal{E}_n \# \Bbbk\mathbb{S}_n$ is a Hopf algebra where $\varepsilon(x_t) = 0$, $\varepsilon(t) = 1$,*

$$\Delta(x_t) = x_t \otimes 1 + t \otimes x_t,$$
$$\Delta(t) = t \otimes t,$$

and $S(x_t) = -t^{-1}x_t$, $S(t) = t^{-1}$ for any $t \in T$. This Hopf algebra is known as the bosonization or Radford biproduct of $\mathcal{E}_n$ and $\Bbbk\mathbb{S}_n$.

We will need some derivations acting on $\mathcal{E}_n$.

Lemma 1.2. [**FK98**] [**BLM13**]

(1) *Let $t = (i, j) \in T$. There is a unique $\Bbbk$-linear skew-derivation ∂_t of $\mathcal{E}_n$ such that*

$$\partial_t(xy) = \partial_t(x)y + (tx)\,\partial_t(y), \quad \partial_t(x_s) = \begin{cases} 1 \text{ if } (i,j) = (k,l), \\ -1 \text{ if } (i,j) = (l,k), \\ 0 \text{ otherwise.} \end{cases}$$

for all $x, y \in \mathcal{E}_n$ and $(k, l) = s \in T$.

(2) *For any $t \in T$ there is a unique $\Bbbk$-linear map ∂_t^* of $\mathcal{E}_n$ such that*

$$\partial_s^*(xy) = x\partial_s^*(y) + \partial_{\sigma(k)\sigma(l)}^*(x)y, \quad \partial_s^*(x_{s'}) = \partial_s(x_{s'})$$

for all $x \in \mathcal{E}_n$, $y \in (\mathcal{E}_n)_\sigma$, and $(k, l) = s, s' \in T$.

The operators ∂_t and ∂_t^* satisfy the defining relations of $\mathcal{E}_n$ and give a left (respectively right) action of $\mathcal{E}_n$ on itself.

Remark 1.3. [**BLM13**] ∂_t maps $\mathbb{S}_n$-homogeneous elements of $\mathcal{E}_n$ of degree σ to $\mathbb{S}_n$-homogeneous elements of degree $t\sigma$. Similarly, ∂_t^* maps $\mathbb{S}_n$-homogeneous elements of $\mathcal{E}_n$ of degree σ to $\mathbb{S}_n$-homogeneous elements of degree σt.

In relation to the structure of $\mathcal{E}_n$ as a braided Hopf-algebra the following is conjectured.

Conjecture 1.4. [**FK98**] *$\mathcal{E}_n$ is a Nichols algebra.*

This conjecture is known to be true for $n = 3, 4$ by [**MS00**] and for $n = 5$ by computer calculations (see [**GV16**] and the references there).

The conjecture can be restated equivalently in the following way: For each $x \in \mathcal{E}_n$ the property $\partial_t(x) = 0$ for all $t \in T$ already implies $x = 0$. In Subsection 5.4 we will operate under the assumption that the conjecture is true and use this statement.

In [**BLM13**] for a subset $S \subseteq T$ the subalgebra of $\mathcal{E}_n$ generated by $x_s, s \in S$, is called *Fomin-Kirillov algebra* $\mathcal{E}_S$. Note that since $\mathcal{E}_n$ is naturally a subalgebra of $\mathcal{E}_{n+1}$ (this was observed in [**MS00**], [**FP00**], [**BLM13**] each with different methods) this definition only depends on the set S and not on the ambient set T. Furthermore, it is convenient to think of S as an (undirected) graph in the obvious way. One observes that if S and S' are isomorphic as undirected graphs we also have $\mathcal{E}_S \cong \mathcal{E}_{S'}$ as algebras by acting with a suitable permutation. Moreover, if S and S' are graphs on disjoint vertex sets, we have $\mathcal{E}_S \otimes \mathcal{E}_{S'} \cong \mathcal{E}_{S \cup S'}$ as algebras.

We will not give an overview of the properties of the algebras $\mathcal{E}_S$. Let us only collect the statements we shall use in the following.

Remark 1.5. [**BLM13**] For each $S \subseteq T$ and $s \in S$ we have $\partial_s^*(\mathcal{E}_S) \subseteq \mathcal{E}_S$. Note that the same does not hold for ∂_s since $\mathcal{E}_S$ is generally not closed under the action of $\mathbb{S}_n$.

Theorem 1.6. [**BLM13**] *Let $S \subseteq T$ and $S_1, S_2 \subseteq S$ complementary subgraphs such that any two vertices in the same connected component of S_2 have the same neighbors in S_1. Then the multiplication map $\mathcal{E}_{S_1} \otimes \mathcal{E}_{S_2} \to \mathcal{E}_S$ is an isomorphism of vector spaces.*

The following consequence of Theorem 1.6 will be used frequently.

Remark 1.7. Let $S \subset T$ and assume that S contains a star graph S_n on n vertices. Denote by S' the complementary subgraph of S_n in S. Then Theorem 1.6 implies that $\dim_{\Bbbk} \mathcal{E}_S = \dim_{\Bbbk} \mathcal{E}_{S'} \dim_{\Bbbk} \mathcal{E}_{S_n}$.

CHAPTER 2

Deformed Fomin-Kirillov Algebras

In this chapter we consider a deformation of the Fomin-Kirillov algebra given by generators and relations. We will realize this algebra as a subalgebra of the linear endomorphisms of the original Fomin-Kirillov algebra and use this to investigate the relation between Fomin-Kirillov algebras and their deformations.

In the second part of this chapter we look at relations in the deformed Fomin-Kirillov algebras, this is an adaption of the relations that are known in the original Fomin-Kirillov algebra, see [**FK98**], [**BLM13**], and [**Kir97**].

2. Deformed Fomin-Kirillov Algebras

Recall $T = \{(i,j) \,|\, 1 \leq i,j \leq n, i \neq j\}$. Let $\mathcal{D}_n = \mathcal{D}_n(\Bbbk)$ denote the associative unital $\Bbbk$-algebra given by generators $y_{ij} = y_{(i,j)}$, where $(i,j) \in T$, and relations

$$\begin{aligned} y_{ij} &= -y_{ji} && \text{for all } (i,j) \in T, \\ y_{ij}^2 &= 1 && \text{for all } (i,j) \in T, \\ y_{ij}y_{jk} + y_{jk}y_{ki} + y_{ki}y_{ij} &= 0 && \text{if } \#\{i,j,k\} = 3, \\ y_{ij}y_{kl} - y_{kl}y_{ij} &= 0 && \text{if } \#\{i,j,k,l\} = 4. \end{aligned}$$

We call the algebra $\mathcal{D}_n$ the *deformed Fomin-Kirillov algebra.*

As the original Fomin-Kirillov algebra $\mathcal{E}_n$ the deformed Fomin-Kirillov algebra $\mathcal{D}_n$ has a unique grading by the symmetric group $\mathbb{S}_n$ such that y_{ij} is homogeneous of degree $(i\,j)$ for any $(i,j) \in T$. Similar to before the homogeneous components of degree $\sigma \in \mathbb{S}_n$ with respect to this grading are denoted by $(\mathcal{D}_n)_\sigma$. Also as in the non deformed case, there is a unique $\mathbb{S}_n$-action on $\mathcal{D}_n$ that satisfies $\sigma(xy) = \sigma(x)\sigma(y)$ for $\sigma \in \mathbb{S}_n$ given by $\sigma y_{ij} = y_{\sigma(i)\sigma(j)}$ for $\sigma \in \mathbb{S}_n$.

As for the Fomin-Kirillov algebra we denote for a subset $S \subseteq T$ by $\mathcal{D}_S$ the subalgebra of $\mathcal{D}_n$ generated by $y_s, s \in S$. Notice that none of the proofs mentioned in Section 1 for $\mathcal{E}_n$ being a subalgebra of $\mathcal{E}_{n+1}$ carry over to the deformed case. However, it will follow immediately from the results of the current section that in fact $\mathcal{D}_n$ can be regarded as a subalgebra of $\mathcal{D}_{n+1}$ in the natural way. The remaining remarks regarding the definition of $\mathcal{E}_S$ from Section 1 then carry over verbatim to $\mathcal{D}_S$.

We introduce an algebra filtration on $\mathcal{D}_S$. For any $k \in \mathbb{N}_0$, let $\mathcal{D}_S^k$ be the linear subspace of $\mathcal{D}_S$ spanned by $\{y_{s_1} \cdots y_{s_l} \,|\, 0 \leq l \leq k, s_1, \ldots, s_l \in S\}$. It is then clear that $(\mathcal{D}_S^k)_{k \in \mathbb{N}_0}$ is an algebra filtration of $\mathcal{D}_S$.

Remark 2.1. (1) More general deformations of Fomin-Kirillov algebras were already considered in [**FK98**], however for a very different purpose and without investigating the relation between the deformations and the original algebras.

(2) We only deformed the algebra structure of $\mathcal{E}_n$. The deformed Fomin-Kirillov algebra does *not* have a Hopf-algebra structure anymore. In fact it was recently observed in [**AKM15**, Thm 7.2] that $\mathcal{E}_n$ is rigid as a braided bialgebra.

Our goal is to identify the deformed Fomin-Kirillov algebra with a subalgebra of the endomorphisms of the original Fomin-Kirillov algebra.

For any $t = (i, j) \in T$ we define $\hat{y}_t = \hat{y}_{ij} \in \mathrm{End}_{\Bbbk}(\mathcal{E}_n)$ by

$$\hat{y}_t(x) = x x_t + \partial_t^*(x)$$

for $x \in \mathcal{E}_n$. It is clear that $\hat{y}_{ij} = -\hat{y}_{ji}$ for any $(i, j) \in T$. We denote the subalgebra of $\mathrm{End}_{\Bbbk}(\mathcal{E}_n)$ generated by $\hat{y}_t$, $t \in T$, by $\mathcal{Y}_n$. For any subset $S \subseteq T$ the algebra generated by $\hat{y}_s$, $s \in S$, is denoted by $\mathcal{Y}_S$. The algebra $\mathcal{Y}_S$ has a filtration $(\mathcal{Y}_S^k)_{k \in \mathbb{N}_0}$ defined analogously to the filtration of $\mathcal{D}_S$.

Note that in later sections we will stop distinguishing between the elements $y_t \in \mathcal{D}_n$ and $\hat{y}_t \in \mathcal{Y}_n$, this will be justified by the results of this section.

By definition we have $\mathcal{Y}_S \subseteq \mathrm{End}_K(\mathcal{E}_n)$. For subalgebras $\mathcal{E}_S$ we have the following.

Proposition 2.2. *Let $S \subseteq T$. Then $\hat{y}_s(\mathcal{E}_S) \subseteq \mathcal{E}_S$ for all $s \in S$. In particular, $\mathcal{E}_S$ is a left $\mathcal{Y}_S$-module.*

Proof. Follows immediately from Remark 1.5 and the definition of $\hat{y}_s$. □

It is more natural to think of $\hat{y}_s$ as acting on the right, in particular in light of the upcoming Remark 2.3. However, since the results of this section will imply that $\mathcal{Y}_S$ is isomorphic to its opposite algebra $\mathcal{Y}_S^{op}$ this is only a minor issue and we will continue acting on the left.

REMARK 2.3. If an element $x \in \mathcal{E}_n$ is $\mathbb{S}_n$-homogeneous of degree $\sigma \in \mathbb{S}_n$, then $\hat{y}_t(x)$ is homogeneous of degree σt for $t \in T$ by Remark 1.3. The following Lemma 2.4 shows that $\hat{y}_t$ is invertible. Combined with Proposition 2.2 this implies that all the non zero $\mathbb{S}_n$-homogeneous components of $\mathcal{E}_S$ are isomorphic as vector spaces ($\mathcal{E}_S$ is *balanced*), provided that the graph S is connected. Operators that are essentially the same as our $\hat{y}_t$ were used in [**Loc13**] to show an analogous statement for a certain class of Nichols algebras of non-abelian group type. Similar operators also appeared in [**MS00**].

The following lemma shows that $\mathcal{Y}_n$ is isomorphic to a quotient of $\mathcal{D}_n$.

LEMMA 2.4. *The relations*

$$\begin{aligned} \hat{y}_t^2 &= 1 && \text{for all } t \in T, \\ \hat{y}_{ij}\hat{y}_{jk} + \hat{y}_{jk}\hat{y}_{ki} + \hat{y}_{ki}\hat{y}_{ij} &= 0 && \text{if } \#\{i,j,k\} = 3, \\ \hat{y}_{ij}\hat{y}_{kl} - \hat{y}_{kl}\hat{y}_{ij} &= 0 && \text{if } \#\{i,j,k,l\} = 4 \end{aligned}$$

hold in $\mathrm{End}_{\Bbbk}(\mathcal{E}_n)$, *where* $1 = \mathrm{id}_{\mathcal{E}_n}$. *In particular,* $\hat{y}_t \in \mathrm{Aut}_{\Bbbk}(\mathcal{E}_n)$ *for all* $t \in T$.

PROOF. Let $x \in \mathcal{E}_n$ and $t \in T$. Compute

$$\begin{aligned} \hat{y}_t^2(x) &= \hat{y}_t(xx_t + \partial_t^*(x)) \\ &= xx_tx_t + \partial_t^*(xx_t) + \partial_t^*(x)x_t + \partial_t^*\partial_t^*(x) \\ &= \partial_t^*(xx_t) + \partial_t^*(x)x_t, \end{aligned}$$

because $x_t^2 = 0$ in $\mathcal{E}_n$ and since ∂^* gives a right action of $\mathcal{E}_n$ on itself by Lemma 1.2. Writing $t = (ij)$ we have

$$\begin{aligned} \hat{y}_{ij}^2(x) &= x\partial_{ij}^*(x_{ij}) + \partial_{i^{(ij)}j^{(ij)}}^*(x)x_{ij} + \partial_{ij}^*(x)x_{ij} \\ &= x - \partial_{ij}^*(x)x_{ij} + \partial_{ij}^*(x)x_{ij} \\ &= x, \end{aligned}$$

using Lemma 1.2(2).

For the second relation let $i,j,k \in \{1,\dots,n\}$ with $\#\{i,j,k\} = 3$. Then

$$\begin{aligned} \hat{y}_{ij}\hat{y}_{jk}(x) &= \hat{y}_{ij}(xx_{jk} + \partial_{jk}^*(x)) \\ &= xx_{jk}x_{ij} + \partial_{jk}^*(x)x_{ij} + \partial_{ij}^*(xx_{jk}) + \partial_{ij}^*\partial_{jk}^*(x) \\ &= xx_{jk}x_{ij} + \partial_{jk}^*(x)x_{ij} + x\partial_{ij}^*(x_{jk}) + \partial_{i^{(jk)}j^{(jk)}}^*(x)x_{jk} + \partial_{ij}^*\partial_{jk}^*(x) \\ &= xx_{jk}x_{ij} + \partial_{jk}^*(x)x_{ij} + \partial_{ik}^*(x)x_{jk} + \partial_{ij}^*\partial_{jk}^*(x), \end{aligned}$$

which implies

$$\begin{aligned}
&(\hat{y}_{ij}\hat{y}_{jk}+\hat{y}_{jk}\hat{y}_{ki}+\hat{y}_{ki}\hat{y}_{ij})(x)\\
&= xx_{jk}x_{ij}+\partial^*_{jk}(x)x_{ij}+\partial^*_{ik}(x)x_{jk}+\partial^*_{ij}\partial^*_{jk}(x)\\
&\qquad + xx_{ki}x_{jk}+\partial^*_{ki}(x)x_{jk}+\partial^*_{ji}(x)x_{ki}+\partial^*_{jk}\partial^*_{ki}(x)\\
&\qquad + xx_{ij}x_{ki}+\partial^*_{ij}(x)x_{ki}+\partial^*_{kj}(x)x_{ij}+\partial^*_{ki}\partial^*_{ij}(x)\\
&= x(x_{jk}x_{ij}+x_{ki}x_{jk}+x_{ij}x_{ki})\\
&\qquad + (\partial^*_{ij}\partial^*_{jk}+\partial^*_{jk}\partial^*_{ki}+\partial^*_{ki}\partial^*_{ij})(x)\\
&\qquad + (\partial^*_{jk}+\partial^*_{kj})(x)x_{ij}+(\partial^*_{ik}+\partial^*_{ki})(x)x_{jk}\\
&\qquad + (\partial^*_{ji}+\partial^*_{ij})(x)x_{ki}\\
&= 0,
\end{aligned}$$

using again the defining relations of $\mathcal{E}_n$ and that ∂^* is a right action of $\mathcal{E}_n$ on itself.

Finally, let $i,j,k,l \in \{1,\dots,n\}$ with $\#\{i,j,k,l\}=4$. Then clearly

$$\begin{aligned}
(\hat{y}_{ij}\hat{y}_{kl})(x) &= \hat{y}_{ij}(xx_{kl}+\partial^*_{kl}(x))\\
&= xx_{kl}x_{ij}+\partial^*_{kl}(x)x_{ij}+\partial^*_{ij}(xx_{kl})+\partial^*_{ij}\partial^*_{kl}(x)\\
&= xx_{kl}x_{ij}+\partial^*_{kl}(x)x_{ij}+x\partial^*_{ij}(x_{kl})+\partial^*_{i^{(kl)}j^{(kl)}}(x)x_{kl}+\partial^*_{ij}\partial^*_{kl}(x)\\
&= xx_{ij}x_{kl}+\partial^*_{kl}(x)x_{ij}+\partial^*_{ij}(x)x_{kl}+\partial^*_{kl}\partial^*_{ij}(x)\\
&= (\hat{y}_{kl}\hat{y}_{ij})(x),
\end{aligned}$$

similar to the first two relations. □

As a consequence of 2.4 the associated graded algebra of $\mathcal{Y}_n$ is a quotient of $\mathcal{E}_n$.

Corollary 2.5. *There is a surjective algebra homomorphism $\psi : \mathcal{E}_n \to \operatorname{gr}\mathcal{Y}_n$ mapping x_t to $\hat{y}_t+\mathcal{Y}_n^0$ for $t \in T$.*

The existence of this algebra homomorphism allows us to prove the following statement concerning the relation of $\mathcal{Y}_S$ and $\mathcal{E}_S$. For this we consider $\mathcal{Y}_S$ as a left module over itself in the natural way and recall that $\mathcal{E}_S$ has a $\mathcal{Y}_S$-structure as noted in Proposition 2.2.

Proposition 2.6. *Let $S \subseteq T$. The $\mathcal{Y}_S$-module homomorphism*

$$\phi : \mathcal{Y}_S \to \mathcal{E}_S, y \mapsto y(1)$$

is bijective.

Proof. It is clear that ϕ is a well defined module homomorphism.

For surjectivity we show that for $k \geq 0$ each element in $\mathcal{E}_S(k)$ is in the image of ϕ. This is clear for $k=0$. We proceed by induction on k. For $k \geq 1$ it suffices to show that every monomial of degree k is in the image of ϕ. Let $s_1, \ldots, s_k \in S$ and $x = x_{s_1} \cdots x_{s_k} \in \mathcal{E}_S$. Write $y = \hat{y}_{s_k} \cdots \hat{y}_{s_1} \in \mathcal{Y}_S$ and observe

$$x' = \phi(y) - x = y(1) - x \in \mathcal{E}_S^{k-1}.$$

By the induction hypothesis there is an $y' \in \mathcal{Y}_S$ with $\phi(y') = x'$. We deduce $\phi(y - y') = x$. This implies surjectivity.

For injectivity we consider the extended map $\phi : \mathcal{Y}_n \to \mathcal{E}_n, y \mapsto y(1)$ and show by induction on k that $\phi_{|\mathcal{Y}_n^k}$ is injective. The case $k = 0$ is trivial since $\mathcal{Y}_n^0 = \Bbbk\mathrm{id}$. Let $k \geq 1$. Let $y \in \mathcal{Y}_n^k$ such that $\phi(y) = 0$. Then there is a subset $J \subset \cup_{l=0}^k T^l$ and $\lambda_j \in \Bbbk$ for any $j = (j_1, \ldots, j_l) \in J$ such that writing $y_j = \hat{y}_{j_1} \cdots \hat{y}_{j_l} \in \mathcal{Y}_n$ for $j \in J$ we have $y = \sum_{j \in J} \lambda_j y_j$. Write $x_j = x_{j_1} \cdots x_{j_l} \in \mathcal{E}_n$ for $j \in J$. Recall the isomorphism $\tau : \mathcal{E}_n \to \mathcal{E}_n^{op}$ from Section 1. By definition we have

$$0 = \phi(y) = y(1) = \sum_{j \in J, |j| = k} \lambda_j \tau(x_j) + x''$$

for some $x'' \in \mathcal{E}_n^{k-1}$. We obtain $\sum_{j \in J, |j|=k} \lambda_j x_j = 0$. Using the notation from Corollary 2.5 this yields

$$0 = \psi\left(\sum_{j \in J, |j| = k} \lambda_j x_j\right) = \sum_{j \in J, |j| = k} \lambda_j y_j + \mathcal{Y}_n^{k-1},$$

in other words $y \in \mathcal{Y}_n^{k-1}$. By the induction hypothesis we obtain $y = 0$. This implies injectivity and completes the proof. □

The next technical lemma is an easy consequence. We use the notation from Proposition 2.6.

Lemma 2.7. *Let $S \subseteq T$ and $k \geq 0$. Then the following hold.*

(1) $\phi(\mathcal{Y}_S^k) = \mathcal{E}_S^k$
(2) $\mathcal{Y}_n^k \cap \mathcal{Y}_S = \mathcal{Y}_S^k$

Proof. (1) The inclusion $\phi(\mathcal{Y}_S^k) \subseteq \mathcal{E}_S^k$ is obvious. The other inclusion has implicitly been shown in the proof of the surjectivity in Proposition 2.6.

(2) Since the map $\phi : \mathcal{Y}_n \to \mathcal{E}_n$ from Proposition 2.6 is bijective, the claim is equivalent to

$$\phi(\mathcal{Y}_n^k) \cap \phi(\mathcal{Y}_S) = \phi(\mathcal{Y}_S^k),$$

which by (1) is equivalent to

$$\mathcal{E}_n^k \cap \mathcal{E}_S = \mathcal{E}_S^k.$$

This however is clearly true since $\mathcal{E}_n$ is graded. □

Proposition 2.6 yields a couple of corollaries which reveal the relation between the algebra structures of $\mathcal{Y}_S, \mathcal{D}_S$, and $\mathcal{E}_S$.

Corollary 2.8. *Let $S \subseteq T$. The associated graded algebra of $\mathcal{Y}_S$ is isomorphic to $\mathcal{E}_S$ as a graded algebra.*

Proof. By Corollary 2.5, there is a surjective graded algebra map $\psi : \mathcal{E}_n \to \mathrm{gr}\ \mathcal{Y}_n$, $x_t \mapsto \hat{y}_t + \mathcal{Y}_n^0$ for any $t \in T$. From the preceding Lemma we obtain

$$(\mathcal{Y}_S^k + \mathcal{Y}_n^{k-1})/\mathcal{Y}_n^{k-1} = \mathcal{Y}_S^k/\mathcal{Y}_S^k \cap \mathcal{Y}_n^{k-1} = \mathcal{Y}_S^k/\mathcal{Y}_S^{k-1}$$

for any $k \geq 1$. This means that we can identify the subalgebra $\psi(\mathcal{E}_S)$ of $\mathrm{gr}\,\mathcal{Y}_n$ with $\mathrm{gr}\,\mathcal{Y}_S$ and therefore ψ induces a surjective graded algebra map $\psi_S : \mathcal{E}_S \to \mathrm{gr}\ \mathcal{Y}_S$.

Proposition 2.6 implies that $\dim_{\Bbbk} \mathcal{E}_S^k = \dim_{\Bbbk} \mathcal{Y}_S^k$ for any $k \geq 0$ (and this number is finite). Therefore the map ψ_S is an isomorphism. $\square$

Corollary 2.9. *Let $S \subseteq T$. Then*

$$\mathcal{Y}_S \cong \mathcal{D}_S$$

as filtered algebras.

Proof. Due to 2.4 we have a surjective homomorphism of filtered algebras $\mathcal{D}_n \to \mathcal{Y}_n$ by mapping generators to generators. This together with 2.8 and the defining relations of $\mathcal{D}_n$ implies that we have surjective homomorphisms of graded algebras

$$\mathcal{E}_n \to \mathrm{gr}\,\mathcal{D}_n \to \mathrm{gr}\,\mathcal{Y}_n \to \mathcal{E}_n,$$

and the composition of these homomorphisms is the identity. In particular this implies $\mathrm{gr}\,\mathcal{D}_n \cong \mathrm{gr}\,\mathcal{Y}_n$ as graded algebras, which in turn implies $\mathcal{D}_n \cong \mathcal{Y}_n$ since we have a surjective algebra homomorphism $\mathcal{D}_n \to \mathcal{Y}_n$ which respects the filtration. $\square$

Remark 2.10. We want to put this result into context. From Corollary 2.8 and Corollary 2.9 it follows that the associated graded algebra of $\mathcal{D}_n$ is isomorphic to $\mathcal{E}_n$. In the literature (see for example [**SS15**], [**BG96**]) this situation would be described as $\mathcal{D}_n$ being a *PBW deformation of* $\mathcal{E}_n$ (with respect to the defining relations we have given). Note that for quadratic algebras that are *koszul* [**BG96**] gives a criterion for this to be the case. However, [**Roo99**] proved that $\mathcal{E}_n$ is in fact not koszul.

At the end of this section we want to state a conjecture that is a motivating factor for considering the deformed Fomin-Kirillov algebras. This conjecture was also the original motivation for our investigation of the groups considered in Chapter 4.

CONJECTURE 2.11. *Assume* $\operatorname{char}\Bbbk = 0$. *For each* $S \subseteq T$ *the deformed Fomin-Kirillov algebra* $\mathcal{D}_S$ *is a semisimple algebra.*

REMARK 2.12. Instead of realizing the algebra $\mathcal{D}_S$ using the right derivations ∂_s^* we could also use the left derivations ∂_s from Lemma 1.2. Since the situations are very similar but *not* completely analogous, we want to discuss this approach briefly. The introduced notation will be used exclusively in this remark.

For any $t = (i, j) \in T$ let $y_t^* = y_{ij}^* \in \operatorname{End}_\Bbbk(\mathcal{E}_n)$ with

$$y_t^*(x) = x_t x + \partial_t(x)$$

for all $x \in \mathcal{E}_n$. For $S \subseteq T$ we denote the subalgebra of $\operatorname{End}_\Bbbk(\mathcal{E}_n)$ generated by y_s^*, $s \in S$, by $\mathcal{Y}_S^*$. We observe that $\mathcal{E}_S$ is *not* a $\mathcal{Y}_S^*$-module in the natural way, because $\mathcal{E}_S$ is not invariant under the derivations $\partial_s, s \in S$. However, all statements of this section bar Proposition 2.6 are still true if we replace $\mathcal{Y}_S$ by $\mathcal{Y}_S^*$. Since the now failing Proposition 2.6 is crucial in our proofs of Corollary 2.8 and Corollary 2.9, we want to give a short argument why those statements still hold.

It is easy to check that Lemma 2.4 and therefore also Corollary 2.5 still hold. This implies

$$\dim_\Bbbk (\mathcal{Y}_n^*)^k \leq \dim_\Bbbk \mathcal{E}_n^k \tag{2.1}$$

for all $k \in \mathbb{N}_0$. To show a variant of Corollary 2.8 for $\mathcal{Y}_n^*$, namely $\operatorname{gr}\mathcal{Y}_n^* \cong \mathcal{E}_n$, it then suffices to show that equality holds in (2.1). We show by induction on k that for every linear independent set of monomials in $\mathcal{E}_n^k$ the "corresponding" monomials in $\mathcal{Y}_n^*$ are linearly independent (by the corresponding monomial of $x_{t_1} \cdots x_{t_k}$ we mean the element $y_{t_1}^* \cdots y_{t_k}^*$ where $k \geq 0, t_1, \dots, t_k \in T$). Since $\mathcal{E}_n^k$ has a monomial basis, this implies the desired equality. For $k = 0$ the statement is trivial. Let $k \geq 1$. Let $J \subset \cup_{l=0}^k T^l$ be a subset such that $(x_j)_{j \in J}$ is linearly independent where $x_j = x_{j_1} \cdots x_{j_l} \in \mathcal{E}_n$ for any $j = (j_1, \dots, j_l) \in J$. Let $y_j^* = y_{j_1}^* \cdots y_{j_l}^* \in \mathcal{Y}_n^*$ for $j \in J$. We show that $(y_j^*)_{j \in J}$ is linearly independent. Let $\lambda_j \in \Bbbk$ for $j \in J$ such that $\sum_{j \in J} \lambda_j y_j^* = 0$. Then

$$\sum_{j \in J} \lambda_j x_j = \sum_{j \in J} \lambda_j x_j - \sum_{j \in J} \lambda_j y_j^*(1) \in \mathcal{E}_n^{k-1}.$$

In particular, $\lambda_j = 0$ for all $j \in J$ with $|j| = k$. Then $\lambda_j = 0$ for all $j \in J$ by induction. This proves equality in (2.1).

From $\operatorname{gr}\mathcal{Y}_n^* \cong \mathcal{E}_n$ it follows in the same way as above that $\mathcal{Y}_n^* \cong \mathcal{D}_n$. We can then refer to the results above to deduce from that the statements concerning $\mathcal{Y}_S^*$ for any $S \subset T$.

It is of course also possible to prove these statements for $\mathcal{Y}_S^*$ from scratch without involving the right derivations, to the best of our knowledge this however requires a far more intricate argument to replace Proposition 2.6.

3. Some Relations in the Deformed Fomin-Kirillov Algebra

In this subsection we look at some very well known relations in $\mathcal{E}_S$ that are also relations in $\mathcal{D}_S$. It is not clear a priori that those relations need not be deformed, because of this we give complete proofs. The proofs however end up being identical or similar to the well known proofs in the graded case that can be found for example in [**FK98**], [**BLM13**], and [**Kir97**].

Also note that on 6 vertices there appear to be graphs $S \subseteq T$ such that some of the defining relations of $\mathcal{E}_S$ in fact do need to be deformed in order to produce relations in $\mathcal{D}_S$, at least if we assume that $\mathcal{E}_6$ is the same as the corresponding Nichols algebra.

Statement (1) of the following lemma is very well known.

Lemma 3.1. *Let $n \geq 3$ and $i, j, k \in \{1, \ldots, n\}$ with $\#\{i, j, k\} = 3$. Then*

(1) $y_{ij}y_{jk}y_{ij} = y_{jk}y_{ij}y_{jk}, \qquad y_{ji}y_{jk}y_{ji} = -y_{jk}y_{ji}y_{jk}.$
(2) $y_{ij}y_{jk}y_{ik} = y_{ik}y_{jk}y_{ij}.$

Proof. (1) We use the relations in Lemma 2.4.

$$\begin{aligned} y_{ij}y_{jk}y_{ij} &= (-y_{jk}y_{ki} - y_{ki}y_{ij})y_{ij} \\ &= -y_{jk}(-y_{ij}y_{jk} - y_{jk}y_{ki}) - y_{ki} \\ &= y_{jk}y_{ij}y_{jk}. \end{aligned}$$

The second equation in (1) follows directly from the first.

(2)

$$\begin{aligned} y_{ij}y_{jk}y_{ik} &= (-y_{jk}y_{ki} - y_{ki}y_{ij})y_{ik} \\ &= y_{jk} - y_{ik}y_{ji}y_{ik} \\ &= y_{jk} - y_{ik}(-y_{ik}y_{kj} - y_{kj}y_{ji}) \\ &= y_{ik}y_{jk}y_{ij}. \end{aligned}$$

□

The first relation in Lemma 3.1 is often referred to as *braid relation*. We will do so in following. The braid relation is a special case of the so called *cyclic relations*, which we look at in Lemma 3.2.

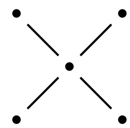


FIGURE 1. The star on 5 vertices.

3.1. Relations in the Star. In this section we consider relations for the star. We begin with the cyclic relations.

The graded analogue of the following Lemma 3.2 was proved in [**FK98**, Lem. 7.2].

LEMMA 3.2. *Let* $n \geq 3$. *Then*

$$\sum_{i=2}^{n} y_{1i}y_{1i+1}\cdots y_{1n}y_{12}y_{13}\cdots y_{1i} = 0.$$

PROOF. For $m \geq 3$ and $k \geq 2$ let

$$y(m,k) = \sum_{i=k}^{m} y_{1i}y_{1i+1}\cdots y_{1m}y_{1k}y_{1k+1}\cdots y_{1i}.$$

We show $y(n,l) = 0$ by induction on $n-l$. The case $n-l=1$ has been treated in Lemma 3.1(1). For the induction step we may assume $l=2$ by using the action of $\mathbb{S}_n$ on $\mathcal{D}_n$. In $y(n,2)$ use Lemma 2.4 to replace $y_{1n}y_{12}$ by $y_{12}y_{2n} - y_{2n}y_{1n}$ in all possible summands excluding the last term $y_{1n}y_{12}y_{13}\cdots y_{1n}$. Observe that the factor y_{2n} commutes with y_{1i} where $3 \leq i \leq n-1$. We obtain

$$\begin{aligned} y(n,2) = &\sum_{i=2}^{n-1} y_{1i}y_{1i+1}\cdots y_{1n-1}y_{12}y_{13}\cdots y_{1i}\cdot y_{2n} \\ &\quad - y_{2n}\sum_{i=3}^{n-1} y_{1i}y_{1i+1}\cdots y_{1n}y_{13}\cdots y_{1i} \\ &\quad - (y_{12}y_{2n})y_{13}\cdots y_{1n} + y_{1n}y_{12}y_{13}\cdots y_{1n} \\ = &\, y(n-1,2)y_{2n} - y_{2n}\sum_{i=3}^{n-1} y_{1i}y_{1i+1}\cdots y_{1n}y_{13}\cdots y_{1i} \\ &\quad - y_{1n}y_{12}y_{13}\cdots y_{1n} - y_{2n}y_{1n}y_{13}\cdots y_{1n} \\ &\quad + y_{1n}y_{12}y_{13}\cdots y_{1n} \\ = &\, y(n-1,2)y_{2n} - y_{2n}y(n,3) \\ = &\, 0, \end{aligned}$$

by induction. In the third to last equation we again used $y_{12}y_{2n} = y_{1n}y_{12} + y_{2n}y_{1n}$. The proof is complete. □

REMARK 3.3. For $n = 4$ the relation in Lemma 3.2 becomes

$$y_{12}y_{13}y_{14}y_{12} + y_{13}y_{14}y_{12}y_{13} + y_{14}y_{12}y_{13}y_{14} = 0,$$

which is also often referred to as *claw relation*. We shall do so in the following.

Next we consider a relation in the star on 5 vertices.

The graded analogue of the following Lemma 3.4 was proved in [**Kir97**, Lem. 8.5].

LEMMA 3.4. *Assume* $\#\{i, j, k, l, n\} = 5$. *Then*

$$\begin{aligned} &y_{ni}y_{nk}y_{nl}y_{nk}y_{nj}y_{ni} + y_{ni}y_{nj}y_{ni}y_{nk}y_{nl}y_{nk} + y_{nj}y_{ni}y_{nk}y_{nl}y_{nk}y_{nj} \\ &\qquad + y_{nk}y_{nl}y_{ni}y_{nj}y_{ni}y_{nk} + y_{nl}y_{ni}y_{nj}y_{ni}y_{nk}y_{nl} \\ &= y_{ni}y_{nj}y_{nk}y_{nl}y_{nk}y_{ni} + y_{nj}y_{nk}y_{nl}y_{nk}y_{ni}y_{nj} + y_{nk}y_{ni}y_{nj}y_{ni}y_{nl}y_{nk} \\ &\quad + y_{nk}y_{nl}y_{nk}y_{ni}y_{nj}y_{ni} + y_{nl}y_{nk}y_{ni}y_{nj}y_{ni}y_{nl}. \end{aligned}$$

PROOF. Use Lemma 3.2 for $n = 5$ and obtain

$$\begin{aligned} &y_{ik}\big(y_{ni}y_{nl}y_{nk}y_{nj}y_{ni} + y_{nl}y_{nk}y_{nj}y_{ni}y_{nl} + y_{nk}y_{nj}y_{ni}y_{nl}y_{nk} + y_{nj}y_{ni}y_{nl}y_{nk}y_{nj}\big) \\ &\quad - \big(y_{ni}y_{nj}y_{nk}y_{nl}y_{ni} + y_{nj}y_{nk}y_{nl}y_{ni}y_{nj} + y_{nk}y_{nl}y_{ni}y_{nj}y_{nk} \\ &\qquad + y_{nl}y_{ni}y_{nj}y_{nk}y_{nl}\big)y_{ik} = 0. \end{aligned}$$

Using the commutation relations and the quadratic relations $y_{ki}y_{in} + y_{in}y_{nk} + y_{nk}y_{ki} = y_{ni}y_{ik} + y_{ik}y_{kn} + y_{kn}y_{ni} = 0$ yields

$$\begin{aligned} 0 = {} & y_{ni}y_{nk}y_{nl}y_{nk}y_{nj}y_{ni} - y_{nk}y_{ki}y_{nl}y_{nk}y_{nj}y_{ni} \\ & - y_{nl}y_{nk}y_{ni}y_{nj}y_{ni}y_{nl} + y_{nl}y_{ni}y_{ik}y_{nj}y_{ni}y_{nl} \\ & - y_{nk}y_{ni}y_{nj}y_{ni}y_{nl}y_{nk} + y_{ni}y_{ik}y_{nj}y_{ni}y_{nl}y_{nk} \\ & + y_{nj}y_{ni}y_{nk}y_{nl}y_{nk}y_{nj} - y_{nj}y_{nk}y_{ki}y_{nl}y_{nk}y_{nj} \\ & + y_{ni}y_{nj}y_{nk}y_{nl}y_{ik}y_{kn} - y_{ni}y_{nj}y_{nk}y_{nl}y_{nk}y_{ni} \\ & + y_{nj}y_{nk}y_{nl}y_{ik}y_{kn}y_{nj} - y_{nj}y_{nk}y_{nl}y_{nk}y_{ni}y_{nj} \\ & - y_{nk}y_{nl}y_{ni}y_{nj}y_{ki}y_{in} + y_{nk}y_{nl}y_{ni}y_{nj}y_{ni}y_{nk} \\ & - y_{nl}y_{ni}y_{nj}y_{ki}y_{in}y_{nl} + y_{nl}y_{ni}y_{nj}y_{ni}y_{nk}y_{nl}. \end{aligned}$$

We sort and get

$$
\begin{aligned}
0 = {} & y_{ni}y_{nk}y_{nl}y_{nk}y_{nj}y_{ni} - y_{ni}y_{nj}y_{nk}y_{nl}y_{nk}y_{ni} \\
& + (y_{ni}y_{nj}y_{nk}y_{ki}y_{nl}y_{nk} + y_{ni}y_{nj}y_{ki}y_{in}y_{nl}y_{nk}) \\
& + y_{nj}y_{ni}y_{nk}y_{nl}y_{nk}y_{nj} - y_{nj}y_{nk}y_{nl}y_{nk}y_{ni}y_{nj} \\
& + (y_{nj}y_{nk}y_{ki}y_{nl}y_{nk}y_{nj} - y_{nj}y_{nk}y_{ki}y_{nl}y_{nk}y_{nj}) \\
& - y_{nk}y_{ni}y_{nj}y_{ni}y_{nl}y_{nk} + y_{nk}y_{nl}y_{ni}y_{nj}y_{ni}y_{nk} \\
& - (y_{nk}y_{nl}y_{ni}y_{ik}y_{nj}y_{ni} + y_{nk}y_{nl}y_{ik}y_{kn}y_{nj}y_{ni}) \\
& - y_{nl}y_{nk}y_{ni}y_{nj}y_{ni}y_{nl} + y_{nl}y_{ni}y_{nj}y_{ni}y_{nk}y_{nl} \\
& + (y_{nl}y_{ni}y_{ik}y_{nj}y_{ni}y_{nl} - y_{nl}y_{ni}y_{ik}y_{nj}y_{ni}y_{nl}),
\end{aligned}
$$

which implies the claim after we again use the quadratic relations mentioned above. □

For any algebra A we denote by $[\,,\,]$ the usual commutator given by $[a,b] = ab - ba$ for $a, b \in A$.

In the deformed Fomin-Kirillov algebra we can then restate the relation from Lemma 3.4 in a simpler way, see the following Lemma 3.5.

LEMMA 3.5. *Assume* $\#\{i,j,k,l,n\} = 5$. *Then*

$$
[y_{ni}y_{nk}y_{ni}, y_{nj}y_{nl}y_{nj}][y_{nk}y_{nj}y_{nk}, y_{ni}y_{nl}y_{ni}] = [y_{nk}y_{nl}y_{nk}, y_{ni}y_{nj}y_{ni}].
$$

PROOF. We compute

$$
\begin{aligned}
& [y_{ni}y_{nk}y_{ni}, y_{nj}y_{nl}y_{nj}][y_{nk}y_{nj}y_{nk}, y_{ni}y_{nl}y_{ni}] \\
& \quad = -y_{ni}y_{nk}y_{ni}y_{nj}y_{nl}y_{nj}y_{nj}y_{nk}y_{nj}y_{ni}y_{nl}y_{ni} \\
& \qquad - y_{ni}y_{nk}y_{ni}y_{nl}y_{nj}y_{nl}y_{nl}y_{ni}y_{nl}y_{nk}y_{nj}y_{nk} \\
& \qquad + y_{nj}y_{nl}y_{nj}y_{nk}y_{ni}y_{nk}y_{nk}y_{nj}y_{nk}y_{ni}y_{nl}y_{ni} \\
& \qquad + y_{nj}y_{nl}y_{nj}y_{ni}y_{nk}y_{ni}y_{ni}y_{nl}y_{ni}y_{nk}y_{nj}y_{nk} \\
& \quad = -y_{ni}y_{nk}y_{ni}(y_{nj}y_{nl}y_{nk}y_{nj})y_{ni}y_{nl}y_{ni} \\
& \qquad - y_{ni}y_{nk}y_{ni}(y_{nl}y_{nj}y_{ni}y_{nl})y_{nk}y_{nj}y_{nk} \\
& \qquad + y_{nj}y_{nl}y_{nj}(y_{nk}y_{ni}y_{nj}y_{nk})y_{ni}y_{nl}y_{ni} \\
& \qquad + y_{nj}y_{nl}y_{nj}(y_{ni}y_{nk}y_{nl}y_{ni})y_{nk}y_{nj}y_{nk}
\end{aligned}
$$

using the braid relations and the fact that the y's square to one. Apply the claw relations to the bracketed terms and obtain

$$\begin{aligned}
&[y_{ni}y_{nk}y_{ni}, y_{nj}y_{nl}y_{nj}][y_{nk}y_{nj}y_{nk}, y_{ni}y_{nl}y_{ni}] \\
&\quad = -y_{ni}y_{nk}y_{ni}y_{nl}y_{nk}y_{nj}y_{nl}y_{nl}y_{ni}y_{nl} \\
&\qquad - y_{nk}y_{ni}y_{nk}y_{nk}y_{nj}y_{nl}y_{nk}y_{ni}y_{nl}y_{ni} \\
&\qquad - y_{ni}y_{nk}y_{ni}y_{nj}y_{ni}y_{nl}y_{nj}y_{nj}y_{nk}y_{nj} \\
&\qquad + y_{ni}y_{nk}y_{ni}y_{ni}y_{nl}y_{nj}y_{ni}y_{nk}y_{nj}y_{nk} \\
&\qquad - y_{nj}y_{nl}y_{nj}y_{ni}y_{nj}y_{nk}y_{ni}y_{ni}y_{nl}y_{ni} \\
&\qquad - y_{nj}y_{nl}y_{nj}y_{nj}y_{nk}y_{ni}y_{nj}y_{ni}y_{nl}y_{ni} \\
&\qquad - y_{nj}y_{nl}y_{nj}y_{nk}y_{nl}y_{ni}y_{nk}y_{nk}y_{nj}y_{nk} \\
&\qquad + y_{nl}y_{nj}y_{nl}y_{nl}y_{ni}y_{nk}y_{nl}y_{nk}y_{nj}y_{nk} \\
&\quad = y_{ni}y_{nk}y_{ni}(-y_{nl}y_{nk}y_{nj}y_{ni}y_{nl} - y_{nj}y_{ni}y_{nl}y_{nk}y_{nj}) \\
&\qquad + (-y_{nk}y_{ni}y_{nj}y_{nl}y_{nk} - y_{nj}y_{nl}y_{nk}y_{ni}y_{nj})y_{ni}y_{nl}y_{ni} \\
&\qquad + (y_{ni}y_{nk}y_{nl}y_{nj}y_{ni} + y_{nl}y_{nj}y_{ni}y_{nk}y_{nl})y_{nk}y_{nj}y_{nk} \\
&\qquad + y_{nj}y_{nl}y_{nj}(-y_{ni}y_{nj}y_{nk}y_{nl}y_{ni} - y_{nk}y_{nl}y_{ni}y_{nj}y_{nk}),
\end{aligned}$$

where in the first step we again used the braid relations and in the second step the fact that the y's square to one. Now use the cyclic relation from Lemma 3.2 on the bracketed terms and obtain

$$\begin{aligned}
&[y_{ni}y_{nk}y_{ni}, y_{nj}y_{nl}y_{nj}][y_{nk}y_{nj}y_{nk}, y_{ni}y_{nl}y_{ni}] \\
&\quad = y_{ni}y_{nk}y_{ni}(y_{nk}y_{nj}y_{ni}y_{nl}y_{nk} + y_{ni}y_{nl}y_{nk}y_{nj}y_{ni}) \\
&\qquad + (y_{ni}y_{nj}y_{nl}y_{nk}y_{ni} + y_{nl}y_{nk}y_{ni}y_{nj}y_{nl})y_{ni}y_{nl}y_{ni} \\
&\qquad - (y_{nk}y_{nl}y_{nj}y_{ni}y_{nk} + y_{nj}y_{ni}y_{nk}y_{nl}y_{nj})y_{nk}y_{nj}y_{nk} \\
&\qquad + y_{nj}y_{nl}y_{nj}(y_{nj}y_{nk}y_{nl}y_{ni}y_{nj} + y_{nl}y_{ni}y_{nj}y_{nk}y_{nl}) \\
&\quad = -y_{nk}y_{ni}y_{nj}y_{ni}y_{nl}y_{nk} + y_{ni}y_{nk}y_{nl}y_{nk}y_{nj}y_{ni} \\
&\qquad - y_{ni}y_{nj}y_{nk}y_{nl}y_{nk}y_{ni} - y_{nl}y_{nk}y_{ni}y_{nj}y_{ni}y_{nl} \\
&\qquad + y_{nk}y_{nl}y_{ni}y_{nj}y_{ni}y_{nk} + y_{nj}y_{ni}y_{nk}y_{nl}y_{nk}y_{nj} \\
&\qquad - y_{nj}y_{nk}y_{nl}y_{nk}y_{ni}y_{nj} + y_{nl}y_{ni}y_{nj}y_{ni}y_{nk}y_{nl} \\
&\quad = [y_{nk}y_{nl}y_{nk}, y_{ni}y_{nj}y_{ni}].
\end{aligned}$$

which is the claim. In the last step we used Lemma 3.4, in the second to last step the braid relations and the fact that the y's square to one. □

REMARK 3.6. We will give a somewhat different direct proof of the formula in Lemma 3.5 in Lemma 6.13. However, we consider it to be of interest to show how the known formulation of the relation in Lemma 3.4 is connected to our version in Lemma 3.5 and Lemma 6.13.

3.2. Relations in the Circle. We look at some relations in $\mathcal{D}_S$ when S contains a circle.

The graded analogue of the following Lemma 3.7 was proved in [**BLM13**, Lem. 7.4].

LEMMA 3.7. *Let $n \geq 3$. Then*

$$\sum_{i=1}^{n} y_{i,i+1}y_{i+1,i+2}\cdots y_{n-1,n}y_{n,1}y_{12}\cdots y_{i-3,i-2}y_{i-2,i-1} = 0.$$

PROOF. We proceed by induction on n. For $n = 3$ the claim follows from Lemma 2.4. Assume now that $n > 3$. For all $m \geq 3$ let

$$y(m) = \sum_{i=1}^{m} y_{i,i+1}y_{i+1,i+2}\cdots y_{m-1,m}y_{m,1}y_{12}\cdots y_{i-3,i-2}y_{i-2,i-1}$$

where each summand has precisely $m-1$ factors y_t, $t \in T$. Replace $y_{n-1,n}y_{n1}$ by $y_{n1}y_{n-1,1} + y_{n-1,1}y_{n-1,n}$ in each summand of $y(n)$, where it appears. This is the case for the summands with $2 \leq i \leq n-1$. The factors y_{n1} commute with $y_{j,j+1}$ for all $2 \leq j \leq n-2$, and the factors $y_{n-1,n}$ commute with $y_{j,j+1}$ for all $1 \leq j \leq n-3$. Therefore

$$\begin{aligned}
y(n) &= y_{12}y_{23}\cdots y_{n-1,n} \\
&\quad + y_{n1}\sum_{i=2}^{n-1} y_{i,i+1}\cdots y_{n-2,n-1}y_{n-1,1}y_{12}\cdots y_{i-2,i-1} \\
&\quad + \sum_{i=2}^{n-1} y_{i,i+1}\cdots y_{n-2,n-1}y_{n-1,1}y_{12}\cdots y_{i-2,i-1}\cdot y_{n-1,n} \\
&\quad + y_{n1}y_{12}y_{23}\cdots y_{n-2,n-1} \\
&= y_{n1}y(n-1) + y(n-1)y_{n-1,n} \\
&= 0,
\end{aligned}$$

by induction. $\square$

We want to derive more relations in the circle on n vertices. For this let us introduce some convenient notation. First, we will allow the indices of the elements y_{ij} to range over all of $\mathbb{Z}$ and then consider them modulo n. Furthermore, in this section we will write y_j instead of $y_{j,j+1}$ for $j \in \mathbb{Z}$.

For example, we would state the previous Lemma 3.7 simply as

$$\sum_{i=1}^{n} y_i y_{i+1}\cdots y_{i+n-3}y_{i+n-2} = 0. \tag{3.1}$$

Note that we also have

$$\sum_{i=1}^{n} y_i y_{i-1} \cdots y_{i-(n-3)} y_{i-(n-2)} = 0 \tag{3.2}$$

by using the isomorphism between $\mathcal{D}_n$ and its opposite algebra.

To state the relations we want to prove let us introduce some notation. For $i \in \mathbb{Z}$ let

$$m_i = y_i y_{i+1} \cdots y_{i-2}.$$

For example, we have $m_1 = y_1 y_2 \cdots y_{n-1}$ and $m_n = y_n y_1 \cdots y_{n-2}$. Of course $m_i = m_j$ if $i \equiv j \mod n$.

We begin with a trivial lemma.

LEMMA 3.8. *Let $i, j \in \mathbb{Z}$. Then*

$$m_i y_j = \begin{cases} y_{j+1} m_i, & \text{if } j \not\equiv i-2, i-1 \mod n, \\ y_{j+1} m_{i+1}, & \text{if } j \equiv i-1 \mod n, \\ y_{j+1} m_{i-1}, & \text{if } j \equiv i-2 \mod n. \end{cases}$$

PROOF. This is an obvious consequence of the braid and commutation relations. □

For the remainder of the section the following observation will be useful. By definition and Lemma 3.1 (1) for each $1 \le i \le n$ the elements $y_i, y_{i+1}, \ldots, y_{i-2}$ satisfy all Coxeter relations of $\mathbb{S}_n$, in fact the group generated by those elements is isomorphic to $\mathbb{S}_n$.

We continue with a technical lemma.

LEMMA 3.9. *Let $2 \le d \le n-2$ and $2 \le k_1 \le k_2 \le \cdots \le k_d \le n-d+1$. Then there is $\tilde{w}_{k_1,\ldots,k_d} \in \Bbbk\langle y_{2-d}, y_{2-d+1}, \ldots, y_{n-d-1}\rangle$ such that*

$$m_{k_1} \cdots m_{k_d} = w_{k_1,\ldots,k_d} y_n y_{n-1} \cdots y_{n-d+1} \tilde{w}_{k_1,\ldots,k_d},$$

where $w_{k_1,\ldots,k_d} \in \Bbbk\langle y_2, \cdots, y_{n-1}\rangle$ is given as an element of $\mathbb{S}_{n-1}$ by the word $i_1 \ldots i_{n-d-1}\, k_1\, k_2+1 \ldots k_d+d-1$ where $i_1 < \cdots < i_{n-d-1} \in \{2, \ldots, n\} \setminus \{k_1, k_2+1, \ldots, k_d+d-1\}$ (the identity is represented as $23 \ldots n$).

PROOF. We begin with the case $k_1 = k_2 = \cdots = k_d = 2$. Considered as an element in $\mathbb{S}_n$ generated by $y_2, \ldots, y_n$ and represented as a word in the standard way we obviously have $m_2 = 2\,3 \ldots n\,1$, hence $m_2^d = d+1\, d+2 \ldots d+n$. Multiplying this equation from the right with $y_{n-d+1} y_{n-d+2} \cdots y_{n-1} y_n$ immediately implies the claim because y_j corresponds to the transposition of j and $j+1$. Also note that 1 is clearly mapped to 1 so we indeed have an element in $\Bbbk\langle y_2, \cdots, y_{n-1}\rangle$.

Next, assume that we are given $2 \le k_1 \le \cdots \le k_d \le n-d+1$ such that the statement is known for $m_{k_1} \cdots m_{k_d}$ and assume that there is some $1 \leq l \leq d$ with $k_l + 1 \leq k_{l+1}$ where we assign k_{d+1} to mean $n - d + 1$. We obtain using Lemma 3.8 repeatedly (we are always in the first case due to our restrictions on $k_1, \dots, k_d$) that

$$\begin{aligned}
& m_{k_1} \cdots m_{k_{l-1}} m_{k_l+1} m_{k_{l+1}} \cdots m_{k_d} \\
&= m_{k_1} \cdots m_{k_{l-1}} y_{k_l} m_{k_l} y_{k_l-1} m_{k_{l+1}} \cdots m_{k_d} \\
&= y_{k_l+l-1} m_{k_1} \cdots m_{k_{l-1}} m_{k_l} m_{k_{l+1}} \cdots m_{k_d} y_{k_l-1-d+l} \\
&= y_{k_l+l-1} w_{k_1,\dots,k_d} y_n y_{n-1} \cdots y_{n-d+1} (\tilde{w}_{k_1,\dots,k_d})^{-1} y_{k_l-1-d+l} \\
&= (y_{k_l+l-1} w_{k_1,\dots,k_d}) y_n y_{n-1} \cdots y_{n-d+1} (y_{k_l-1-d+l} \tilde{w}_{k_1,\dots,k_d})^{-1}.
\end{aligned}$$

We observe that by the restrictions above we have

$$2 \leq k_l + l - 1 \leq n - d + d - 1 \leq n - 1$$

and

$$2 - d \leq k_l - 1 - d + l \leq n - d - 1 - d + d = n - d - 1.$$

It only takes an easy calculation in $\mathbb{S}_n$ to see that

$$y_{k_l+l-1} w_{k_1,\dots,k_d} = w_{k_1,\dots,k_{l-1},k_l+1,k_{l+1},\dots,k_d},$$

and the claim follows. □

The next lemma is the analog of Lemma 3.9 for terms starting with m_1.

LEMMA 3.10. *Let $2 \leq d \leq n-2$ and $1 \leq k_2 \leq \cdots \leq k_d \leq n-d+1$. Then there is $\tilde{v}_{k_2,\dots,k_d} \in \Bbbk\langle y_{2-d}, y_{2-d+1}, \dots, y_{n-d-1}\rangle$ such that*

$$m_1 m_{k_2} \cdots m_{k_d} = v_{k_2,\dots,k_d} y_1 y_2 \cdots y_{n-d} \tilde{v}_{k_2,\dots,k_d},$$

where $v_{k_2,\dots,k_d} \in \Bbbk\langle y_2, \cdots, y_{n-1}\rangle$ is given as an element of $\mathbb{S}_{n-1}$ by the word $i_1 \dots i_{n-d}\, k_2 + 1\, k_3 + 2 \dots k_d + d - 1$ where $i_1 < \cdots < i_{n-d} \in \{2, \dots, n\} \setminus \{k_2 + 1, k_3 + 2, \dots, k_d + d - 1\}$ (the identity is represented as $23 \dots n$).

PROOF. We will skip this proof since it is analogous to the proof of Lemma 3.9. □

We can now formulate and prove our desired relations. The proof is very similar to [**BLM13**, Pro. 7.2].

PROPOSITION 3.11. *Let $1 \leq d \leq n - 1$. Then*

$$\sum_{1 \leq k_1 \leq k_2 \leq \cdots \leq k_d \leq n+1-d} m_{k_1} \cdots m_{k_d} = 0.$$

Proof. The cases $d = 1$ and $d = n-1$ are Equations (3.1) and (3.2), respectively. In this proof we will write $\mathbb{S}_{n-1}$ for the subgroup of $\mathbb{S}_n$ generated by $y_2, \dots, y_{n-1}$.

Now obviously,

$$\begin{aligned}
&\sum_{1\le k_1\le k_2\le\cdots\le k_d\le n+1-d} m_{k_1}\cdots m_{k_d} \\
&= \sum_{1\le k_2\le\cdots\le k_d\le n+1-d} m_1 m_{k_2}\cdots m_{k_d} \\
&\quad + \sum_{2\le k_1\le k_2\le\cdots\le k_d\le n+1-d} m_{k_1}\cdots m_{k_d}.
\end{aligned} \tag{3.3}$$

We write $x = y_{n-d+1,1}$ and compute using Equation (3.1) and Lemma 3.10 for the first term of (3.3)

$$\begin{aligned}
&\sum_{1\le k_2\le\cdots\le k_d\le n+1-d} m_1 m_{k_2}\cdots m_{k_d} \\
&= \sum_{2\le k_1\le k_2\le\cdots\le k_d\le n+1-d} v_{k_2,\dots,k_d} y_1 y_2\cdots y_{n-d}\tilde{v}_{k_2,\dots,k_d} \\
&= -\sum_{2\le k_1\le k_2\le\cdots\le k_d\le n+1-d}\ \sum_{s=2}^{n-d+1} v_{k_2,\dots,k_d} y_s y_{s+1}\cdots y_{n-d}\cdot x \\
&\qquad\qquad \cdot y_1 y_2\cdots y_{s-2}\tilde{v}_{k_2,\dots,k_d}.
\end{aligned} \tag{3.4}$$

Note that $v_{k_2,\dots,k_d}$ in (3.4) runs over all elements in $\mathbb{S}_{n-1}$ with at most one descent occuring from $n-d+1$ to $n-d+2$. This implies that $v_{k_2,\dots,k_d}$ runs over the minimal (i.e. of minimal length in the Coxeter generators) left coset representatives in $\mathbb{S}_{n-1}$ of the subgroup generated by $y_2, \dots, y_{n-d}, y_{n-d+2}, \dots, y_{n-1}$, which we can identify with $\mathbb{S}_{n-d}\times\mathbb{S}_{d-1}$. Moreover $y_s y_{s+1}\cdots y_{n-d}$ in (3.4) ranges over the minimal left coset representatives of the subgroup generated by $y_2, \dots, y_{n-d-1}$ in the group generated by $y_2, \dots, y_{n-d}$; put more suggestively: $y_s y_{s+1}\cdots y_{n-d}$ in (3.4) ranges over the minimal left coset representatives of $\mathbb{S}_{n-d-1}\times 1$ in $\mathbb{S}_{n-d}$. Put together, this implies that $v_{k_2,\dots,k_d} y_s y_{s+1}\cdots y_{n-d}$ in (3.4) ranges over the minimal left coset representatives of $\mathbb{S}_{n-d-1}\times 1\times\mathbb{S}_{d-1}$ in $\mathbb{S}_{n-1}$ with the identifications as above.

Similarly using Equation (3.2) and Lemma 3.9 we obtain for the second term of (3.3)

$$\sum_{2\le k_1\le k_2\le\cdots\le k_d\le n+1-d} m_{k_1}\cdots m_{k_d}$$

$$= \sum_{2 \le k_1 \le k_2 \le \cdots \le k_d \le n+1-d} w_{k_1,\dots,k_d} y_n y_{n-1} \cdots y_{n-d+1} \tilde{w}_{k_1,\dots,k_d}$$

$$= \sum_{2 \le k_1 \le k_2 \le \cdots \le k_d \le n+1-d} \sum_{s=1}^{d} w_{k_1,\dots,k_d} y_{n-s} y_{n-s-1} \cdots y_{n-d+1} \cdot x$$

$$\cdot y_n y_{n-1} \cdots y_{n-s+2} \tilde{w}_{k_1,\dots,k_d} \tag{3.5}$$

Similar to before note that $w_{k_1,\dots,k_d}$ in (3.5) runs over all elements in $\mathbb{S}_{n-1}$ with at most one descent occuring from $n-d$ to $n-d+1$. Again this implies that $w_{k_1,\dots,k_d}$ from (3.5) runs over the minimal left coset representatives in $\mathbb{S}_{n-1}$ of the subgroup generated by $y_2, \dots, y_{n-d-1}, y_{n-d+1}, \dots, y_{n-1}$, which we can identify with $\mathbb{S}_{n-d-1} \times \mathbb{S}_d$. Also similar to above the terms $y_{n-s} y_{n-s-1} \cdots y_{n-d+1}$ in (3.5) range over the minimal left coset representatives of the subgroup generated by $y_{n-d+2}, \dots, y_{n-1}$ in the group generated by $y_{n-d+1}, \dots, y_{n-1}$, or put differently $1 \times \mathbb{S}_{d-1}$ in $\mathbb{S}_d$. As above, put together this implies that $w_{k_1,\dots,k_d} y_{n-s} y_{n-s-1}$ in (3.5) ranges over the minimal left coset representatives of $\mathbb{S}_{n-d-1} \times 1 \times \mathbb{S}_{d-1}$ in $\mathbb{S}_{n-1}$.

This implies that the terms $v_{k_2,\dots,k_d} y_s y_{s+1} \cdots y_{n-d}$ in (3.4) and the terms $w_{k_1,\dots,k_d} y_{n-s} y_{n-s-1}$ in (3.5) range over the same elements.

Furthermore, observe that the terms $y_1 y_2 \cdots y_{s-2} \tilde{v}_{k_2,\dots,k_d}$ from (3.4) and $y_n y_{n-1} \cdots y_{n-s+2} \tilde{w}_{k_1,\dots,k_d}$ from (3.5) contain no y_{n-d}, this implies that they are equal if their $\mathbb{S}_n$-degrees are equal. Since all the occuring summands in (3.4) and (3.5) have the same $\mathbb{S}_n$-degree (namely $1{+}d\,2{+}d\dots n{+}d$, see the proof of Lemma 3.9) it follows from the considerations above that exactly the same summands occur in both (3.4) and (3.5), but with different signs. This implies the claim. $\square$

We want to rewrite our relations slightly for use later. For this we start with a trivial lemma. Recall that the indices of the $m_i, i \in \mathbb{Z}$, are understood to be modulo n.

LEMMA 3.12. *Let $i, j \in \mathbb{Z}$. Then $m_i m_j^{-1} = m_{j+1}^{-1} m_{i+1}$.*

PROOF. This is a trivial consequence of Lemma 3.8. Note that the potentially critical third case of Lemma 3.8 only happens if $i \equiv j \mod n$ (in which case the statement is just trivial). $\square$

Let us denote with $e_1, \dots, e_{n-1} \in \mathbb{k}[x_1, \dots, x_{n-1}]$ the elementary symmetric polynomials. Using these we can rewrite the relations from Proposition 3.11.

PROPOSITION 3.13. *For $1 \le l \le n-1$ we have*

$$e_l\left(m_2 m_1^{-1}, \dots, m_n m_1^{-1}\right) = (-1)^l.$$

Proof. We use induction on l. For $l = 1$ simply multiply the $d = 1$ relation from Proposition 3.11 from the right by m_1^{-1}.

Let $l > 1$. Multiply the $d = l$ relation from Proposition 3.11 from the right by $m_{n+1-l+1}^{-1} m_{n+1-l+2}^{-1} \cdots m_{n+1}^{-1}$, repeatedly use the preceding Lemma 3.12, and split up the sum to obtain

$$\begin{aligned} 0 &= \sum_{1 \le k_2 \le \cdots \le k_l \le n+1-l} m_1 m_1^{-1} m_{k_2+1} m_1^{-1} \cdots m_{k_l+l-1} m_1^{-1} \\ &\quad + \sum_{2 \le k_1 \le k_2 \le \cdots \le k_l \le n+1-l} m_{k_1} m_1^{-1} m_{k_2+1} m_1^{-1} \cdots m_{k_l+l-1} m_1^{-1} \\ &= e_{l-1}\left(m_2 m_1^{-1}, \ldots, m_n m_1^{-1}\right) + e_l\left(m_2 m_1^{-1}, \ldots, m_n m_1^{-1}\right). \end{aligned}$$

The claim now follows from the induction hypothesis. $\square$

CHAPTER 3

Subalgebras Related to Subgroups

In this chapter we investigate deformed Fomin-Kirillov Algebras by considering subalgebras corresponding to certain subgroups of $\mathbb{S}_n$.

We start with a few very general remarks in Section 4.

The most interesting case is if we consider $\mathbb{S}_{n-1}$ as a subgroup of $\mathbb{S}_n$ in the natural way. We go on to demonstrate this in Section 5 on examples by calculating upper bounds for the dimensions (as vector spaces) of $\mathcal{D}_S$ for many small graphs S. Our upper bounds in all cases coincide with the exact values given in [**BLM13**] for $\mathbb{k} = \mathbb{Q}$, which were obtained with different and mostly computational methods.

In Section 6 we apply the same method to the full deformed Fomin-Kirillov Algebra. In particular we give a way to calculate the dimension of $\mathcal{E}_5$ without using computers. To the best of our knowledge there currently is no other way to do this. We also consider the case $n = 6$, but only with partial results.

Finally, in Section 7 we consider the subalgebras corresponding to the trivial subgroup. We exhibit the examples D_n and the circle which turn out to have intriguing algebra structures.

4. Preliminaries

In this subsection we work in a much more general setting than needed in order not to obscure the very simple arguments.

4.1. Subalgebras Related to Subgroups. Let G be a group and A a G-graded $\mathbb{k}$-algebra. For a subgroup $H \subseteq G$ let

$$A_H = \bigoplus_{h \in H} A_h,$$

where A_h is the homogeneous component of A of degree h. Then A_H is clearly a subalgebra of A.

We first show that the algebra structure of A_H does only depend on the conjugacy class of H under suitable conditions on A.

Proposition 4.1. *Assume A is generated by invertible, homogeneous elements, the degrees of which generate G. Assume H, H' are conjugate subgroups of G. Then $A_H \cong A_{H'}$ as algebras.*

Proof. Let x be a homogeneous, invertible element of A of degree g. We show $xA_Hx^{-1} = A_{gHg^{-1}}$. Indeed, if $a \in A$ is homogeneous of degree $h \in H$, then xax^{-1} is homogeneous of degree ghg^{-1}. To show equality consider

$$A_H = x^{-1}xA_Hx^{-1}x \subseteq x^{-1}A_{gHg^{-1}}x \subseteq A_H,$$

hence $x^{-1}xA_Hx^{-1}x = x^{-1}A_{gHg^{-1}}x$. Since conjugation with x^{-1} is injective we obtain $xA_Hx^{-1} = A_{gHg^{-1}}$. Because A is generated by invertible, homogeneous elements the degrees of which generate G the claim follows. $\square$

The following propostion relates the dimension of A to the dimension of A_H.

Proposition 4.2. *Assume A is generated by invertible, homogeneous elements, the degrees of which generate G. Let $H \subseteq G$ be a subgroup and $n = \#(G/H)$. Then*

(1) *$A_g \cong A_{g'}$ as $\Bbbk$-vector spaces for all $g, g' \in G$.*
(2) *A has a basis of cardinality n as an A_H-left module.*
(3) *$\dim_\Bbbk A < \infty$ if and only if $\dim_\Bbbk A_H < \infty$. If $\dim_\Bbbk A < \infty$, then $\dim_\Bbbk A = n \dim_\Bbbk A_H$.*

Proof. (1) Let x be a homogeneous, invertible element of A. The map $A_g \to A_{\deg(x)g}, a \mapsto xa$ is an isomorphism of K-vector spaces. The claim follows since A is generated by such elements and the degrees of those generators generate G.

(2) Let $\{g_1, \dots, g_n\}$ be a set of right coset representatives of H in G. For each $j \in \{1, \dots, n\}$ choose a monomial u_j in the generators of A of degree g_j. Then $u_1, \dots, u_n$ form a A_H-basis of A. First it is clear that $u_j \neq 0$ for all j, since each u_j is invertible being a monomial in the invertible generators of A. Linear independence over A_H is clear since for each $a \in A_H$ we have $au_j \in \bigoplus_{g \in Hg_j} A_g$ and the g_j belong to different right cosets. To show that they also generate A as a A_H-left module, let $y \in A$ be homogeneous of degree $g \in Hg_j$. Then $yu_j^{-1} \in A_H$ and $y = (yu_j^{-1})u_j$.

(3) Follows from either (1) or (2). $\square$

We now specialize to our case of interest. For $G = \mathbb{S}_n$, $i \in \{1, \dots, n\}$, and $H_i = \{\sigma \in \mathbb{S}_n \mid \sigma(i) = i\}$ we write $\varphi_i(A)$ for A_{H_i}. In particular, if

$S \subseteq T$ we have

$$\varphi_i(\mathcal{D}_S) = (\mathcal{D}_S)_{H_i} = \bigoplus_{\sigma \in \mathbb{S}_n, \sigma(i)=i} (\mathcal{D}_S)_\sigma,$$

where $(\mathcal{D}_S)_\sigma$ is the homogeneous component of $\mathcal{D}_S$ of $\mathbb{S}_n$-degree σ.

The following corollaries are trivial applications of Proposition 4.1 and Proposition 4.2, respectively.

COROLLARY 4.3. *Let $S \subseteq T$ be a connected graph on n vertices and $1 \leq i, j \leq n$. Then $\varphi_i(\mathcal{D}_S) \cong \varphi_j(\mathcal{D}_S)$ as algebras.*

COROLLARY 4.4. *Let $S \subseteq T$ be a connected graph on n vertices and $1 \leq i \leq n$. Then*

(1) *$(\mathcal{D}_S)_\sigma \cong (\mathcal{D}_S)_\tau$ as $\Bbbk$-vector spaces for all $\sigma, \tau \in \mathbb{S}_n$.*
(2) *$\mathcal{D}_S$ has a basis of cardinality n as a $\varphi_i(\mathcal{D}_S)$-left module.*
(3) *$\mathcal{D}_S$ has a basis of cardinality $n!$ as a $(\mathcal{D}_S)_{()}$-left module.*
(4) *$\dim_\Bbbk \mathcal{D}_S < \infty$ if and only if $\dim_\Bbbk \varphi_i(\mathcal{D}_S) < \infty$. If $\dim_\Bbbk \mathcal{D}_S < \infty$, then $\dim_\Bbbk \mathcal{D}_S = n \dim_\Bbbk \varphi_i(\mathcal{D}_S)$.*
(5) *$\dim_\Bbbk \mathcal{D}_S < \infty$ if and only if $\dim_\Bbbk (\mathcal{D}_S)_{()} < \infty$. If $\dim_\Bbbk \mathcal{D}_S < \infty$, then $\dim_\Bbbk \mathcal{D}_S = n! \dim_\Bbbk (\mathcal{D}_S)_{()}$.*

4.2. Generators and Relations of Subalgebras. Given a $\Bbbk$-algebra B and a subalgebra A we give a method to find generators and relations of A under strong technical assumptions.

We begin with the following trivial but useful lemma.

LEMMA 4.5. *Let B be an algebra and $A \subseteq B$ a subalgebra such that B has a basis $1 = u_1, \ldots, u_n$ as a left A-module. Assume that we have $a_1, \ldots, a_m \in A$ such that $u_1, \ldots, u_n$ form a generating set of B as a left $\Bbbk\langle a_1, \ldots, a_m\rangle$-module. Then $A = \Bbbk\langle a_1, \ldots, a_m\rangle$.*

PROOF. The inclusion $\Bbbk\langle a_1, \ldots, a_m\rangle \subseteq A$ is trivial.

For the other inclusion let $a \in A$. Since $A \subseteq B$ we have $x_i \in \Bbbk\langle a_1, \ldots, a_m\rangle$ for $i \in \{1, \ldots, n\}$ such that

$$au_1 = a = \sum_{i=1}^n x_i u_i$$

due to our assumptions. Since $x_i \in A$ for all i it follows that $a = x_1$ because $u_1, \ldots, u_n$ form a basis of B as an A-module. This implies the claim. $\square$

We now discuss a method to convert relations in B to relations in A.

Let S be some finite set and $l \geq 1$. Let B be the $\Bbbk$-algebra given by generators $b_s, s \in S$, and relations $r_k(b_S)$ where $r_k \in \Bbbk\langle X_s | s \in S\rangle$

is a noncommutative polynomial for $1 \leq k \leq l$. Let $a_1, \dots, a_m \in B$ such that B has a basis $1 = u_1, \dots, u_n$ as a left A-module, where $A = \Bbbk\langle a_1, \dots, a_m\rangle$. We give a method to find defining relations of A.

For $i, j \in \{1, \dots, n\}$, $s \in S$ let $f_{ij}^s \in \Bbbk\langle X_1, \dots, X_m\rangle$ such that

$$u_j b_s = \sum_{i=1}^n f_{ij}^s(a_1, \dots, a_m) u_i.$$

Let $\tilde{A}$ and $\tilde{B}$ be the tensor algebras of the vector spaces with basis $\tilde{a}_1, \dots, \tilde{a}_m$ and $\tilde{b}_s, s \in S$, respectively.

Repeat the following procedure for $1 \leq k \leq l$.

Let M be the left $\tilde{A}$-module with basis $\tilde{u}_1, \dots, \tilde{u}_n$. We define a $\tilde{A} - \tilde{B}$-Bimodule structure on M by letting

$$\tilde{u}_j \tilde{b}_s = \sum_{i=1}^n f_{ij}^s(\tilde{a}_1, \dots, \tilde{a}_m) \tilde{u}_i$$

for $j \in \{1, \dots, n\}$ and $s \in S$. This is well defined by construction. For $j \in \{1, \dots, n\}$ there are $r_{ij}^k \in \Bbbk\langle x_1, \dots, x_m\rangle$ such that

$$\tilde{u}_j r_k(\tilde{b}_S) = \sum_{i=1}^n r_{ij}^k(\tilde{a}_1, \dots, \tilde{a}_m) \tilde{u}_i.$$

We now replace $\tilde{B}$ by $\tilde{B}/\langle r_k\rangle$ and $\tilde{A}$ by $\tilde{A}/\langle r_{ij}^k | 1 \leq i, j \leq n\rangle$ and repeat the procedure.

After having done this for $k = 1, \dots, l$ we obviously obtain an isomorphism $\phi : \tilde{B} \to B$ of $\Bbbk$-algebras by mapping $\tilde{b}_s \mapsto b_s$ for $s \in S$. Moreover, we also claim $\tilde{A} \cong A$ by mapping $\tilde{a}_p \mapsto a_p$ for $1 \leq p \leq m$. For this observe that $r_{ij}^k(a_1, \dots, a_m) = 0$ in A is clear since by construction we have

$$0 = u_j r_k(b_S) = \sum_{i=1}^n r_{ij}^k(a_1, \dots, a_m) u_i$$

and $u_1, \dots, u_n$ are linearly independent over A by assumption. Furthermore, if $r \in \Bbbk\langle X_1, \dots, X_m\rangle$ with $r(a_1, \dots, a_m) = 0$ in A then we obtain

$$0 = \phi^{-1}(r(a_1, \dots, a_m)) = r(\phi^{-1}(a_1), \dots, \phi^{-1}(a_m)),$$

which we can do since $A \subseteq B$ is a subalgebra. Since $u_1 = 1$ we have $\tilde{u}_1 \phi^{-1}(a_j) = \tilde{a}_j \tilde{u}_1$ for $1 \leq j \leq m$ by construction (express a_j as a polynomial in the $b_s, s \in S$). This implies

$$0 = \tilde{u}_1 r(\phi^{-1}(a_1), \dots, \phi^{-1}(a_m)) = r(\tilde{a}_1, \dots, \tilde{a}_m) u_1,$$

which in turn implies $r(\tilde{a}_1, \dots, \tilde{a}_m) = 0$ in $\tilde{A}$ since u_1 is part of a $\tilde{A}$-Basis of M.

Hence we know that our generators of $\tilde{A}$ satisfy exactly the same relations as the given generators of A. The claim follows.

REMARK 4.6. It is obvious how the content of this section applies to our case of interest. Since Lemma 4.5 is trivial, we will generally not mention using it.

Our method to convert relations is really only an improvement if the defining relations are difficult to deal with or if we want to make sure we have found *all* defining relations of the subalgebra. Since we do not have a method to find all defining relations in the bigger algebra $\mathcal{D}_S$ without involving Gröbner Basis in the first place, the second aspect is really only useful for $S = T$. For graphs on fewer than 6 vertices we will most of the time simply guess and prove relations of the subalgebra.

5. Examples for Fixing a Point

In this section we will demonstrate our first method for investigating $\mathcal{E}_S$ by giving upper bounds for the dimensions of $\mathcal{E}_S$ for some subgraphs $S \subseteq T$. Our strategy is to give a graph S' on fewer vertices than S and a finite dimensional algebra A such that we have a surjective linear map $\mathcal{E}_{S'} \otimes A \to \operatorname{gr} \varphi_i(\mathcal{D}_S)$. Except for some examples on 6 vertices we will do this by direct calculation.

Notice that most of the dimensions for graphs on at most 5 vertices can be easily derived knowing Theorem 1.6 and the dimension of $\mathcal{E}_n$ for $n \leq 5$, which we will compute directly in Section 6.

In [**BLM13**] all of the dimensions in this section, except the ones on 6 vertices, have already been calculated. In the cases except D_n this was done over $\mathbb{Q}$ by computer calculations.

When investigating $\mathcal{E}_S$ or $\mathcal{D}_S$ for some subgraph $S \subseteq T$ we are of course faced with the problem of not knowing a complete list of defining relations. Where it is important to make the distinction, we will write $\widehat{\mathcal{D}_S}$ for the algebra given by the generators of $\mathcal{D}_S$ with the quadratic relations and the relations discussed in Section 3. Of course any upper bound for the dimension of $\widehat{\mathcal{D}_S}$ is also an upper bound for the dimension of $\mathcal{E}_S$ and $\mathcal{D}_S$. This notation will only be used in this section.

5.1. $n \leq 3$**.** For $n = 2$ there is only the graph $S = \{(1,2)\}$ to consider. It is clear that $\varphi_1(\mathcal{D}_S) = \mathbb{k}$. In particular $\dim_{\mathbb{k}} \mathcal{D}_S = 2$.

For $n = 3$ we have the following.

EXAMPLE 5.1. Consider $S = \{(1,2), (2,3)\}$. It is clear that

$$\varphi_1(\mathcal{D}_S) = \mathbb{k}\langle y_{23} \rangle \cong \mathcal{D}_2.$$

Observe that as an easy consequence we already get $\dim_{\Bbbk} \mathcal{E}_3 = 12$ independently from Section 6, in fact keeping the notation S from Example 5.1 we have

$$\begin{aligned}\dim_{\Bbbk} \mathcal{E}_3 &= \dim_{\Bbbk} \mathcal{E}_2 \dim_{\Bbbk} \mathcal{E}_S \\ &= \dim_{\Bbbk} \mathcal{D}_2 \dim_{\Bbbk} \mathcal{D}_S \\ &= 3 \dim_{\Bbbk} \mathcal{D}_2 \dim_{\Bbbk} \varphi_1(\mathcal{D}_S) \\ &= 3(\dim_{\Bbbk} \mathcal{D}_2)^2 = 12,\end{aligned}$$

where we used Theorem 1.6, Corollary 2.8, Corollary 4.4, Example 5.1, and the remark on the dimension on $\mathcal{D}_2$ at the beginning of this section.

5.2. $n = 4$**.** The dimensions of the Fomin-Kirillov algebras belonging to graphs on at most 4 vertices can be easily derived from the dimension of $\mathcal{E}_4$, which we compute in Section 6, and the dimensions of the Fomin-Kirillov algebras to graphs on at most 3 vertices. Indeed, from $\dim_{\Bbbk} \mathcal{E}_3 = 12$ and $\dim_{\Bbbk} \mathcal{E}_4 = 576$ it follows from Remark 1.7 that the Fomin-Kirillov algebra to the star graph on 4 vertices has dimension 48. Again by Remark 1.7 we then know the dimension of the Fomin-Kirillov algebras for all graphs with at most 4 vertices that contain a star on 4 vertices. Hence the only missing example is the circle on 4 vertices. But, by removing two suitable edges from the complete graph on 4 vertices we can see using Theorem 1.6 that the Fomin-Kirillov algebra to the circle on 4 vertices has dimension 144.

While we already know the dimensions of the examples on 4 vertices, the specifics of the following treatment will be used again in the later parts of Section 5.

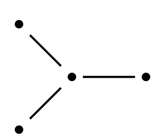


FIGURE 1. The graph from Example 5.2.

The following Example 5.2 is the first in the series D_n which we will consider later in Section 5.5.

EXAMPLE 5.2. Consider $S = \{(1,2),(1,3),(1,4)\}$. We will write

$$t_{23} = y_{12}y_{13}y_{12}, t_{42} = y_{14}y_{12}y_{14} \text{ and } t_{34} = y_{13}y_{14}y_{13}.$$

By Corollary 4.4 the set $\{y_{12}, y_{13}, y_{14}\}$ forms a basis of $\mathcal{D}_S$ as a left $\varphi_1(\mathcal{D}_S)$-module. It is easy to check that this set also generates $\mathcal{D}_S$ as a left $\Bbbk\langle t_{23}, t_{34}, t_{42}\rangle$-module, for example by showing that for $2 \le j \le 4$

and $s \in S$ the element $y_{1j}y_s$ is in the left $\Bbbk\langle t_{23}, t_{34}, t_{42}\rangle$-submodule of $\mathcal{D}_S$ generated by $\{y_{12}, y_{13}, y_{14}\}$.

Due to freeness this implies

$$\varphi_1(\mathcal{D}_S) = \Bbbk\langle t_{23}, t_{34}, t_{42}\rangle,$$

as we have expanded on in Lemma 4.5. We observe $t_{ij}^2 = (y_{1i}y_{1j}y_{1i})^2 = 1$ for $(i, j) \in \{(2, 3), (3, 4), (4, 2)\}$ and

$$\begin{aligned} t_{23}t_{42} &= (y_{12}y_{13}y_{12})(y_{14}y_{12}y_{14}) \\ &= -y_{12}y_{13}y_{12}y_{12}y_{14}y_{12} \\ &= -y_{12}y_{13}y_{14}y_{12} \\ &= y_{13}y_{14}y_{12}y_{13} + y_{14}y_{12}y_{13}y_{14} \\ &= y_{13}y_{14}y_{13}y_{13}y_{12}y_{13} + y_{14}y_{12}y_{14}y_{14}y_{13}y_{14} \\ &= -(y_{13}y_{14}y_{13})(y_{12}y_{13}y_{12}) - (y_{14}y_{12}y_{14})(y_{13}y_{14}y_{13}) \\ &= -t_{34}t_{23} - t_{42}t_{34} \end{aligned} \tag{5.1}$$

by the quadratic, braid, and claw relations. In the same way we get $t_{42}t_{23} + t_{23}t_{34} + t_{34}t_{42} = 0$. The generators of $\varphi_1(\mathcal{D}_S)$ fulfill the defining relations of $\mathcal{D}_3$, hence $\varphi_1(\mathcal{D}_S)$ is a quotient of $\mathcal{D}_3$. Using this in the second step we obtain

$$\dim_\Bbbk \mathcal{D}_S = 4 \dim_\Bbbk \varphi_1(\mathcal{D}_S) \leq 4 \dim_\Bbbk \mathcal{D}_3 \leq 48,$$

where in the first step we use Corollary 4.4(4) and in the last step the remark after Example 5.1.

Note that going on we will most of the time skip easy calculations involving only quadratic, braid, and claw relations as (5.1) in Example 5.2.

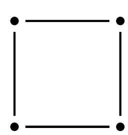


FIGURE 2. The graph from Example 5.3.

EXAMPLE 5.3. Consider $S = \{(1, 2), (2, 3), (3, 4), (4, 1)\}$. Let $a_{(234)} = -y_{12}y_{23}y_{34}y_{41}$ and

$$t_{23} = y_{23}, t_{34} = y_{34}, t_{42} = y_{41}y_{12}y_{41}.$$

Similar to the last example we can check

$$\varphi_1(\mathcal{D}_S) = \Bbbk\langle t_{23}, t_{34}, t_{42}, (a_{(234)})^{\pm 1}\rangle.$$

We have

$$\begin{aligned} a_{(234)} &= -y_{12}y_{23}y_{34}y_{41} \\ &= (y_{23}y_{34}y_{41} + y_{34}y_{41}y_{12} + y_{41}y_{12}y_{23})y_{41} \\ &= y_{23}y_{34} + y_{34}(y_{41}y_{12}y_{41}) + (y_{41}y_{12}y_{41})y_{23} \\ &= t_{23}t_{34} + t_{34}t_{42} + t_{42}t_{23} \end{aligned}$$

using the well known relation from Lemma 3.7 for $n = 4$. In the same way we obtain $a_{(234)}^{-1} = t_{34}t_{23}+t_{23}t_{42}+t_{42}t_{34}$. Hence our generators fulfill the defining relations of $\mathcal{D}_3$ up to elements of $\Bbbk[a_{(234)}, a_{(234)}^{-1}]$. Moreover, it is easy to check that

$$a_{(234)}t_{ij} = t_{i^{(234)}j^{(234)}}a_{(234)} \text{ for } (i,j) \in \{(2,3),(3,4),(4,2)\}.$$

We want to approximate the order of $a_{(234)}$. For this first observe that the $d = 2$ version of Proposition 3.11 is

$$\begin{aligned} &y_{12}y_{34}y_{23}y_{41} + y_{34}y_{41}y_{23}y_{34} + y_{41}y_{12}y_{34}y_{41} \\ &\quad + y_{12}y_{23}y_{41}y_{12} + y_{23}y_{34}y_{12}y_{23} + y_{23}y_{41}y_{12}y_{34} = 0 \end{aligned} \tag{5.2}$$

Using this in the fifth step and Lemma 3.7 for $n = 4$ in the second step we obtain

(5.3)

$$\begin{aligned} a_{(234)}^2 &= y_{12}y_{23}y_{34}(y_{41}y_{12}y_{23})y_{34}y_{41} \\ &= -y_{12}y_{23}y_{34}(y_{12}y_{23}y_{34} + y_{23}y_{34}y_{41} + y_{34}y_{41}y_{12})y_{34}y_{41} \\ &= -y_{12}y_{23}y_{34}y_{12}y_{23}y_{41} - y_{12}y_{34}y_{23}y_{34}y_{41}y_{34} - y_{12}y_{23}y_{41}y_{12}y_{34}y_{41} \\ &= -y_{12}y_{23}(y_{12}y_{34}y_{23}y_{41} + y_{34}y_{41}y_{23}y_{34} + y_{41}y_{12}y_{34}y_{41}) \\ &= y_{12}y_{23}(y_{12}y_{23}y_{41}y_{12} + y_{23}y_{34}y_{12}y_{23} + y_{23}y_{41}y_{12}y_{34}) \\ &= y_{23}y_{12}y_{41}y_{12} + y_{34}y_{23} + y_{12}y_{41}y_{12}y_{34} \\ &= t_{23}t_{42} + t_{34}t_{23} + t_{42}t_{34} \\ &= a_{(234)}^{-1}. \end{aligned}$$

Hence $a_{(234)}^3 = 1$.

If we introduce a filtration on $\varphi_1(\mathcal{D}_S)$ by letting $\deg t_{23} = \deg t_{34} = \deg t_{42} = 1$ and $\deg a_{(234)} = \deg a_{(234)}^{-1} = 0$ this implies that we have a surjective linear map $\mathcal{E}_3 \otimes \Bbbk[a_{(234)}] \to \operatorname{gr} \varphi_1(\mathcal{D}_S)$. Since $a_{(234)}^3 = 1$ this implies $\dim_\Bbbk \mathcal{D}_S \le 144$.

We will also deal with the cirlce on n vertices again using a different method in Section 7.2.

5.3. $n = 5$**.** Again as in the last section knowing the dimension of $\mathcal{E}_5$ already tells us the dimension of the Fomin-Kirillov algebras for many graphs on 5 vertices by Remark 1.7 and Theorem 1.6. The exceptions which we cannot deal with like this are discussed in the current section and in Section 5.5. The Fomin-Kirillov algebra to the graph in Example 5.8 is discussed for illustrative purposes.

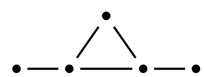


FIGURE 3. The graph from Example 5.4.

EXAMPLE 5.4. Consider $S = \{(1,2),(2,3),(3,4),(3,5),(5,2)\}$. Let $a_{(23)} = y_{35}y_{23}y_{25} = y_{25}y_{23}y_{35}$ and

$$\begin{aligned} &t_{12} = y_{12}, t_{23} = y_{23}, t_{34} = y_{34},\\ &t_{24} = y_{35}y_{23}y_{34}y_{23}y_{35}, t_{13} = y_{25}y_{23}y_{12}y_{23}y_{25}. \end{aligned}$$

It is easy to check

$$\varphi_5(\mathcal{D}_S) = \mathbb{k}\langle t_{12}, t_{24}, t_{34}, t_{13}, t_{23}, a_{23}\rangle.$$

Denote by S' the full graph on 4 vertices with the edge $(1,4)$ removed, and by S'' the graph S' with the edge $(2,3)$ removed (S'' is a circle). It is clear that $\dim_{\mathbb{k}} \widehat{\mathcal{D}_{S'}} \leq 2 \dim_{\mathbb{k}} \widehat{\mathcal{D}_{S''}} \leq 288$ by Example 5.3.

It is easy to check that $t_{12}, t_{23}, t_{34}, t_{24}, t_{13}$ satisfy the relations required in $\mathcal{D}_{S'}$. Notice that we do not need to check the special relations of the circle on 4 points due to the presence of the diagonal, see the proofs in Section 3.2. Furthermore we can check

$$a_{(23)}t_{ij} = t_{i^{(23)}j^{(23)}}a_{(23)} \text{ for } (i,j) \in \{(1,2),(2,4),(3,4),(1,3)\}.$$

For the remaining generator we have

$$\begin{aligned} t_{23}a_{(23)} &= y_{23}y_{35}y_{23}y_{25}\\ &= y_{35}y_{23}y_{35}y_{25}\\ &= y_{35}y_{25}y_{35}y_{23}\\ &= y_{35}(y_{35}y_{23} - y_{23}y_{25})y_{23}\\ &= -y_{35}y_{23}y_{25}y_{23} + 1\\ &= -a_{(23)}t_{23} + 1. \end{aligned}$$

It follows that we have a surjective linear map $\widehat{\mathcal{D}_{S'}} \otimes K[a_{(23)}] \to \varphi_1(\mathcal{D}_S)$; due to $a_{(23)}^2 = 1$ this implies $\dim_{\mathbb{k}} \mathcal{D}_S \leq 2880$.

We continue with

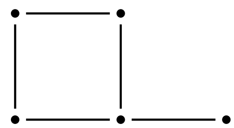


FIGURE 4. The graph from Example 5.5.

EXAMPLE 5.5. Consider $S = \{(1,2),(2,3),(3,4),(4,1),(3,5)\}$. Let $a_{(234)} = -y_{12}y_{23}y_{34}y_{41}$ and

$$t_{23} = y_{23}, t_{34} = y_{34}, t_{35} = y_{35}, t_{42} = y_{12}y_{41}y_{12},$$
$$t_{45} = y_{12}y_{23}y_{34}y_{35}y_{34}y_{23}y_{12}, t_{25} = a_{(234)}t_{45}a_{(234)}^{-1}.$$

We can check

$$\varphi_1(\mathcal{D}_S) = \Bbbk\langle t_{23}, t_{34}, t_{35}, t_{42}, t_{45}, t_{25}, (a_{(234)})^{\pm 1}\rangle.$$

The commutation relations

$$a_{(234)}t_{ij} = t_{i^{(234)}j^{(234)}}a_{(234)}$$
$$\text{for } (i,j) \in \{(2,3),(3,4),(3,5),(4,2),(4,5),(2,5)\},$$

are easy to check.

In Example 5.3 we have seen that

$$a_{(234)} = t_{23}t_{34} + t_{34}t_{42} + t_{42}t_{23} \text{ and } a_{(234)}^{-1} = t_{34}t_{23} + t_{23}t_{42} + t_{42}t_{34}.$$

The other defining relations of $\mathcal{D}_4$ are easy consequences of quadratic, braid, and claw relations and their proofs will be skipped.

Introducing a suitable filtration it follows that there is a surjective linear map $\mathcal{E}_4 \otimes \Bbbk[a_{(234)}] \to \operatorname{gr}\varphi_1(\mathcal{D}_S)$; due to $a_{(234)}^3 = 1$ this implies $\dim_{\Bbbk} \mathcal{D}_S \leq 8640$.

In the next example the situation is more difficult and we will give more details.

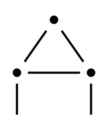


FIGURE 5. The graph from Example 5.6.

EXAMPLE 5.6. Consider $S = \{(1,2),(2,3),(3,4),(4,5),(5,1),(2,4)\}$. Write

$$t_{23} = y_{23}, t_{34} = y_{34}, t_{45} = y_{45}, t_{24} = y_{24},$$
$$t_{52} = y_{12}y_{51}y_{12}, t_{35} = a_{(245)}t_{34}a_{(245)}^{-1},$$

where

$$a_{(245)} = -y_{12}y_{24}y_{45}y_{51}, \text{ and } a_{(2345)} = -y_{12}y_{23}y_{34}y_{45}y_{51}.$$

We can check

$$\varphi_1(\mathcal{D}_S) = \Bbbk\langle t_{52}, t_{23}, t_{34}, t_{45}, t_{24}, t_{35}, (a_{(245)})^{\pm 1}, (a_{(2345)})^{\pm 1}\rangle.$$

From Example 5.3 we know

$$\begin{aligned} t_{52}t_{24} + t_{24}t_{45} + t_{45}t_{52} &= a_{(245)}, \\ t_{24}t_{52} + t_{45}t_{24} + t_{52}t_{45} &= a_{(245)}^{-1}, \end{aligned}$$

some commutation relations

$$a_{(245)}t_{ij} = t_{i^{(245)}j^{(245)}}a_{(245)} \text{ for } (i,j) \in \{(2,4),(4,5),(5,2),(3,4)\},$$

and $a_{(245)}^3 = 1$. Furthermore we can easily show

$$a_{(2435)}t_{ij} = t_{i^{(2435)}j^{(2435)}}a_{(2435)} \text{ for } (i,j) \in \{(2,3),(3,4),(4,5),(5,2)\},$$

using the braid relations. Before we turn to the remaining commutation relations we derive some relations between $a_{(245)}$ and $a_{(2345)}$. First we calculate

$$\begin{aligned} a_{(245)}a_{(2345)}^{-1} &= y_{12}y_{24}y_{45}y_{51}y_{51}y_{45}y_{34}y_{23}y_{12} \\ &= y_{12}(y_{24}y_{34}y_{23})y_{12} \\ &= y_{12}y_{23}y_{34}y_{24}y_{12} \\ &= y_{12}y_{23}y_{34}y_{45}y_{51}y_{51}y_{45}y_{24}y_{12} \\ &= a_{(2345)}a_{(245)}^{-1}, \end{aligned}$$

and in a similar way $a_{(245)}^{-1}a_{(2345)} = a_{(2345)}^{-1}a_{(245)}$.

The remaining commutation relations are a bit more involved. We first compute

$$\begin{aligned} a_{(245)}t_{23} &= -y_{12}y_{24}y_{45}y_{51}y_{23} \\ &= -y_{12}(y_{24}y_{23})y_{45}y_{51} \\ &= -y_{12}(y_{23}y_{34} - y_{34}y_{24})y_{45}y_{51} \\ &= a_{(2345)} - t_{34}a_{(245)}, \end{aligned}$$

and using this obtain

$$\begin{aligned} a_{(245)}t_{35} &= a_{(245)}^{-1}t_{34}a_{(245)}a_{(245)} \\ &= a_{(245)}^{-1}(a_{(2345)} - a_{(245)}t_{23})a_{(245)} \\ &= -t_{23}a_{(245)} + a_{(245)}^{-1}a_{(2345)}a_{(245)} \\ &= -t_{23}a_{(245)} + a_{(2345)}^{-1}a_{(245)}^{-1}. \end{aligned}$$

Going on we observe

$$\begin{aligned}
a_{(2345)}t_{24} &= a_{(2345)}a_{(245)}^{-1}t_{45}a_{(245)}\\
&= a_{(245)}a_{(2345)}^{-1}t_{45}a_{(245)}\\
&= a_{(245)}t_{34}a_{(2345)}^{-1}a_{(245)}\\
&= t_{35}a_{(245)}a_{(245)}^{-1}a_{(2345)}\\
&= t_{35}a_{(2345)},
\end{aligned}$$

and

$$\begin{aligned}
a_{(2345)}t_{35} &= a_{(2345)}a_{(245)}^{-1}a_{(245)}t_{35}\\
&= a_{(245)}a_{(2345)}^{-1}(-t_{23}a_{(245)} + a_{(2345)}^{-1}a_{(245)}^{-1})\\
&= -a_{(245)}a_{(2345)}^{-1}t_{23}a_{(245)} + a_{(245)}a_{(2345)}^{-2}a_{(245)}^{-1}\\
&= -a_{(245)}t_{52}a_{(2345)}^{-1}a_{(245)} + a_{(245)}a_{(2345)}^{-2}a_{(245)}^{-1}\\
&= -t_{24}a_{(2345)} + a_{(245)}a_{(2345)}^{-2}a_{(245)}^{-1}.
\end{aligned}$$

So we have commutation relations for $a_{(245)}$ and $a_{(2345)}$ with all generators t_{ij}. We discuss the relations between those generators required for $\mathcal{D}_4$. Begin with

$$\begin{aligned}
t_{34}t_{35} &= t_{34}a_{(245)}t_{34}a_{(245)}^{-1}\\
&= -a_{(245)}t_{23}t_{34}a_{(245)}^{-1} + a_{(2345)}t_{34}a_{(245)}^{-1}\\
&= -a_{(245)}(t_{34}t_{24} + t_{24}t_{23})a_{(245)}^{-1} + t_{45}a_{(2345)}a_{(245)}^{-1}\\
&= -t_{35}t_{45}a_{(245)}a_{(245)}^{-1} - t_{45}a_{(245)}t_{23}a_{(245)}^{-1} + t_{45}a_{(2345)}a_{(245)}^{-1}\\
&= -t_{35}t_{45} + t_{45}t_{34} - t_{45}a_{(2345)}a_{(245)}^{-1} + t_{45}a_{(2345)}a_{(245)}^{-1}\\
&= -t_{35}t_{45} + t_{45}t_{34},
\end{aligned}$$

by going to the opposite algebra this also yields $t_{35}t_{34}+t_{45}t_{35}-t_{34}t_{45} = 0$. Using this we compute

$$\begin{aligned}
t_{52}t_{35} &= t_{52}a_{(245)}t_{34}a_{(245)}^{-1}\\
&= a_{(245)}t_{45}t_{34}a_{(245)}^{-1}\\
&= a_{(245)}(t_{34}t_{35} + t_{35}t_{45})a_{(245)}^{-1}\\
&= t_{35}a_{(245)}t_{35}a_{(245)}^{-1} - t_{23}a_{(245)}t_{45}a_{(245)}^{-1} + a_{(2345)}^{-1}a_{(245)}^{-1}t_{45}a_{(245)}^{-1}\\
&= -t_{35}t_{23}a_{(245)}a_{(245)}^{-1} + t_{35}a_{(2345)}^{-1}a_{(245)}^{-2}\\
&\qquad - t_{23}t_{52}a_{(245)}a_{(245)}^{-1} + a_{(2345)}^{-1}t_{24}a_{(245)}^{-2}\\
&= -t_{35}t_{23} - t_{23}t_{52} + t_{35}a_{(2345)}^{-1}a_{(245)}^{-2} - t_{35}a_{(2345)}^{-1}a_{(245)}^{-2}
\end{aligned}$$

$$+ a^{-1}_{(2345)}a_{(245)}a^{-2}_{(2345)}a^{-1}_{(245)}a^{-1}_{(2345)}a^{-2}_{(245)}$$
$$= -t_{35}t_{23} - t_{23}t_{52} + a_{(245)}a^2_{(2345)},$$

which again immediately implies $t_{35}t_{52} + t_{23}t_{35} + t_{52}t_{23} = a^{-2}_{(2345)}a^{-1}_{(245)}$. Finally we compute

$$\begin{aligned} t_{24}t_{35} &= (a_{(245)}t_{52}a^{-1}_{(245)})(a_{(245)}t_{34}a^{-1}_{(245)}) \\ &= a_{(245)}t_{34}t_{52}a^{-1}_{(245)} \\ &= t_{35}t_{24} \end{aligned}$$

and are done (recall $t_{ij} = y_{ij}$ for $(i, j) \in \{(2, 3), (3, 4), (4, 5), (2, 4)\}$).

We need to approximate the order of $a_{(2345)}$. For this consider

$$\begin{aligned} 0 &= a_{(2345)}(t_{52}t_{24} + t_{24}t_{45} + t_{45}t_{52} - a_{(245)}) \\ &= t_{23}t_{35}a_{(2345)} + t_{35}t_{52}a_{(2345)} + t_{52}t_{23}a_{(2345)} - a_{(2345)}a_{(245)} \\ &= a^{-2}_{(2345)}a^{-1}_{(245)}a_{(2345)} - a_{(2345)}a_{(245)} \\ &= a^{-3}_{(2345)}a_{(245)} - a_{(2345)}a_{(245)}, \end{aligned}$$

and hence $a^4_{(2345)} = 1$.

Using an appropriate filtration we obtain a surjective linear map $\mathcal{E}_4 \otimes \Bbbk\langle a_{(245)}, a_{(2345)}\rangle \to \operatorname{gr}\varphi_1(\mathcal{D}_S)$. From the relations regarding $a_{(245)}$ and $a_{(2345)}$ from above we see that $\Bbbk\langle a_{(245)}, a_{(2345)}\rangle$ is a quotient of the group algebra of the symmetric group on 4 points. This implies $\dim_\Bbbk \mathcal{D}_S \leq 69120$.

Using Example 5.6 it is now easy to treat the circle on 5 vertices. Note that since the relations from Proposition 3.11 become somewhat difficult to deal with already for $n = 5$ we choose to argue by using extra edges.

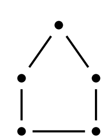


FIGURE 6. The graph from Example 5.6.

EXAMPLE 5.7. Let $S = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1)\}$. We will use the notation from the previous Example 5.6. We will refer to the circle on 4 vertices by S'. It is easy to check that

$$\varphi_1(\mathcal{D}_S) = \Bbbk\langle t_{52}, t_{23}, t_{34}, t_{45}, a^{\pm 1}_{(2345)}\rangle.$$

The commutation relations

$$a_{(2345)}t_{ij} = t_{i^{(2345)}j^{(2345)}}a_{(2345)} \text{ for } (i, j) \in \{(5, 2), (2, 3), (3, 4), (4, 5)\}$$

were already discussed in Example 5.6, as were most of the appropriate Coxeter relations among $t_{52}, t_{23}, t_{34}, t_{45}$ as in the 4-circle S'. Note that the braid relations involving t_{52} are not completely immediate from the previous example, since the quadratic relations involving t_{23} or t_{45} and t_{52} are only fulfilled up to summands in $\Bbbk\langle a_{(245)}, a_{(2345)}\rangle$. They are however easy to check, and we shall do so later in the proof of Proposition 5.13.

We need to discuss the special relations from Section 3.2. First we check

$$\begin{aligned}
&t_{52}t_{23}t_{34} + t_{23}t_{34}t_{45} + t_{34}t_{45}t_{52} + t_{45}t_{52}t_{23}\\
&= y_{12}y_{51}y_{12}y_{23} + y_{12}y_{12}y_{23}y_{34}y_{45} + y_{12}y_{34}y_{45}y_{51}y_{12} + y_{12}y_{45}y_{51}y_{12}y_{23}\\
&= y_{12}(y_{51}y_{12}y_{23} + y_{12}y_{23}y_{34}y_{45} + y_{34}y_{45}y_{51}y_{12} + y_{45}y_{51}y_{12}y_{23})\\
&= 0,
\end{aligned}$$

using the relation from Lemma 3.7 for $n = 5$. The last remaining relation we discuss in the form (5.2) as in Example 5.3. We want to show

$$\begin{aligned}
a^2_{(2345)} = {} & t_{45}t_{23}t_{34}t_{52} + t_{45}t_{34}t_{52}t_{45} + t_{52}t_{45}t_{23}t_{52}\\
&+ t_{23}t_{34}t_{52}t_{23} + t_{34}t_{45}t_{23}t_{34} + t_{34}t_{52}t_{45}t_{23}.
\end{aligned} \tag{5.4}$$

We imitate the proof of Proposition 3.11 and using relations from Example 5.6 replace $t_{23}t_{34}$ by $t_{34}t_{24} + t_{24}t_{23}$ and $t_{52}t_{45}$ by $-t_{24}t_{52} - t_{45}t_{24} + a^{-1}_{(245)}$ in the right hand side of (5.4) whereever possible. After cancelling the right hand side of (5.4) becomes

$$\begin{aligned}
&t_{45}t_{34}a^{-1}_{(245)} + a^{-1}_{(245)}t_{23}t_{52} + t_{34}a^{-1}_{(245)}t_{23}\\
&= t_{45}t_{34}a^{-1}_{(245)} - t_{35}a^{-1}_{(245)}t_{52} + a_{(245)}a_{(2345)}t_{52} - t_{34}t_{35}a^{-1}_{(245)} + t_{34}a_{(245)}a_{(2345)}\\
&= (t_{45}t_{34} - t_{35}t_{45} - t_{34}t_{35})a^{-1}_{(245)} + a_{(245)}t_{23}a_{(2345)} + t_{34}a_{(245)}a_{(2345)}\\
&= a_{(2345)}a_{(2345)} - t_{34}a_{(245)}a_{(2345)} + t_{34}a_{(245)}a_{(2345)}\\
&= a^2_{(2345)},
\end{aligned}$$

where we used the commutation relations from Example 5.6.

If we use that the Fomin-Kirillov algebra $\mathcal{E}_{S'}$ for the 4-circle S' has the relations we discussed in Section 3 plus the quadratic relations as defining relations, it follows that if we introduce an appropriate filtration we have a surjective linear map $\mathcal{E}_{S'} \otimes \Bbbk[a_{(2345)}] \to \operatorname{gr}\varphi_1(\mathcal{D}_S)$. In particular we would obtain $\dim_\Bbbk \mathcal{D}_S \leq 2880$ since $a^4_{(2345)} = 1$ by Example 5.6.

However, since we do not have a method (that does not involve computing Groebner basis) to show what the defining relations of $\mathcal{E}_{S'}$ are, we proceed differently and simply apply our method of fixing a point again to $\varphi_1(\mathcal{D}_S)$. Notice that the elements $t_{52}, t_{23}, t_{34}, t_{45} \in \varphi_1(\mathcal{D}_S)$ fulfill almost exactly the defining relations of $\mathcal{D}_{S'}$, the only difference being that relation (5.2) from Example 5.3 is deformed to (5.4). Hence we choose to imitate the procedure from Example 5.3. Let $b_{(345)} = -t_{23}t_{34}t_{45}t_{52}$ and

$$s_{34} = t_{34}, s_{45} = t_{45}, s_{53} = t_{52}t_{23}t_{52}, c_{(345)} = t_{23}a_{(2345)}.$$

Again, we have

$$\varphi_2(\varphi_1(\mathcal{D}_S)) = \Bbbk\langle s_{34}, s_{45}, s_{53}, c_{(345)}, b_{(345)}^{\pm 1}\rangle,$$

and it is shown as in Example 5.3 that s_{34}, s_{45}, s_{53} fulfill the defining relations of $\mathcal{D}_3$ upto elements in $\Bbbk[b_{(345)}, b_{(345)}^{-1}]$ and have appropriate commutation relations with $b_{(345)}^{\pm 1}$. Moreover, it is a trivial consequence of the braid relations, the quadratic relations, and the commutation relations for $a_{(2345)}$ that we have

$$c_{(345)}s_{ij} = s_{i^{(345)}j^{(345)}}c_{(345)} \text{ for } (i,j) \in \{(3,4),(4,5),(5,3)\}.$$

This implies that with an appropriate filtration there is a surjective linear map $\mathcal{E}_3 \otimes A \to \operatorname{gr}\varphi_2(\varphi_1(\mathcal{D}_S))$, where $A = \Bbbk\langle b_{(345)}^{\pm 1}, c_{(345)}\rangle$. It follows $\dim_\Bbbk \mathcal{D}_S \leq 5 \cdot 4 \cdot 12 \cdot \dim_\Bbbk A$.

We investigate relations among the generators of A. First, imitating the calculations in (5.3) from Example 5.3 with our deformed relation (5.4) instead of (5.2) we get

$$b_{(345)}^2 - b_{(345)}^{-1} = -t_{23}t_{34}a_{(2345)}^2 = -c_{(345)}^2. \tag{5.5}$$

And analogously we obtain

$$b_{(345)}^{-2} - b_{(345)} = -a_{(2345)}^2 t_{34}t_{23}.$$

Multiplying these two equations yields

$$b_{(345)}^3 + b_{(345)}^{-3} = 1 \tag{5.6}$$

due to $a_{(2345)}^4 = 1$.

Combining the relations (5.5) and (5.6) one can see with some elementary calculations that $c_{(345)}^4 = -b_{(345)}$ and $c_{(345)}^{12} + c_{(345)}^6 + 1 = 0$. This implies that A is a quotient of $\Bbbk[x]/\langle x^{12}+x^6+1\rangle$. In particular we have $\dim_\Bbbk A \leq 12$. Also we observe that the algebra A is semisimple if $\operatorname{char}\Bbbk \neq 2, 3$.

In Section 7.2 we will see that the upper bound we have given in Example 5.7 is in fact exact.

In all examples we treated so far we found some graph S' on fewer vertices than S and some finite group G such that we had a surjective linear map $\mathcal{E}_{S'} \otimes \Bbbk G \to \operatorname{gr} \varphi_n(\mathcal{D}_S)$. Sadly, this does not appear to work for all examples as we will see now.

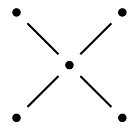


FIGURE 7. The graph from Example 5.8.

EXAMPLE 5.8. This example is essentially only part of the treatment in Section 6 where we will derive the results in a more conceptual way. We still present the more elementary ad hoc approach.

We consider the star on five vertices $S = \{(5,1),(5,2),(5,3),(5,4)\}$ and for $1 \le i,j \le 4$ with $i \ne j$ let $t_{ij} = y_{ni}y_{nj}y_{ni}$. It is easy to check

$$\varphi_5(\mathcal{D}_S) = \Bbbk\langle t_{12}, t_{13}, t_{14}, t_{23}, t_{24}, t_{34}\rangle.$$

We know from Example 5.2 that those generators satisfy all defining relations of the full graph on 4 vertices bar the commutation relations. We set then

$$t_{ij;kl} = [t_{ij}, t_{kl}]$$

for $1 \le i,j,k,l \le 4$ with $\#\{i,j,k,l\} = 4$. Note that Lemma 3.5 states $t_{ij;kl}t_{ik;jl} = t_{il;kj}$. We compute

$$\begin{aligned}
t_{ij;kl}t_{jk} &= [y_{5i}y_{5j}y_{5i}, y_{5k}y_{5l}y_{5k}]y_{5j}y_{5k}y_{5j} \\
&= -y_{5i}y_{5j}y_{5i}(y_{5k}y_{5l}y_{5j}y_{5k}) + y_{5k}y_{5l}y_{5k}(y_{5j}y_{5i}y_{5k}y_{5j}) \\
&= (y_{5i}y_{5j}y_{5i})y_{5l}y_{5j}y_{5k}y_{5l} - y_{5j}y_{5i}y_{5k}y_{5l}y_{5j} \\
&\qquad - y_{5k}y_{5l}y_{5k}y_{5i}y_{5k}y_{5j}y_{5i} - y_{5k}y_{5l}y_{5j}y_{5i}y_{5k} \\
&= -y_{5j}y_{5i}(y_{5j}y_{5l}y_{5j})y_{5k}y_{5l} - y_{5k}y_{5l}(y_{5k}y_{5i}y_{5k})y_{5j}y_{5i} \\
&\qquad + y_{5i}y_{5k}y_{5l}y_{5j}y_{5i} + y_{5l}y_{5j}y_{5i}y_{5k}y_{5l} \\
&= +y_{5j}y_{5i}y_{5l}y_{5j}y_{5l}y_{5k}y_{5l} + y_{5k}y_{5l}y_{5i}y_{5k}y_{5i}y_{5j}y_{5i} \\
&\qquad + y_{5i}y_{5k}y_{5l}y_{5i}y_{5i}y_{5j}y_{5i} + y_{5l}y_{5j}y_{5i}y_{5l}y_{5l}y_{5k}y_{5l} \\
&= -y_{5i}y_{5l}y_{5j}y_{5i}y_{5l}y_{5k}y_{5l} - y_{5l}y_{5i}y_{5k}y_{5l}y_{5i}y_{5j}y_{5i} \\
&= -y_{5i}y_{5l}y_{5i}y_{5i}y_{5j}y_{5i}y_{5l}y_{5k}y_{5l} - (y_{5l}y_{5i}y_{5l})y_{5l}y_{5k}y_{5l}y_{5i}y_{5j}y_{5i} \\
&= -y_{5i}y_{5l}y_{5i}y_{5i}y_{5j}y_{5i}y_{5l}y_{5k}y_{5l} + y_{5i}y_{5l}y_{5i}y_{5l}y_{5k}y_{5l}y_{5i}y_{5j}y_{5i} \\
&= y_{5i}y_{5l}y_{5i}[y_{5i}y_{5j}y_{5i}, y_{5k}y_{5l}y_{5k}] \\
&= t_{il}t_{ij;kl}.
\end{aligned}$$

using the cyclic relations from Lemma 3.2. We note that $t_{ij;kl}t_{ij} = -t_{ij}t_{ij;kl}$ is trivial. Put together and using a suitable filtration this implies the existence of a surjective linear map $\mathcal{E}_4 \otimes A \to \operatorname{gr} \varphi_5(\mathcal{D}_S)$

where A is the subalgebra of $\varphi_5(\mathcal{D}_S)$ generated by $t_{12;34}, t_{13;24}, t_{14;23}$. We will see later in Lemma 6.16 that $\dim_{\Bbbk} A \leq 5$. This implies $\dim_{\Bbbk} \mathcal{E}_S \leq 14400$.

Note that the algebra A in the previous Example 5.8 is not a group algebra since it is non commutative and at most 5 dimensional. If $\operatorname{char} \Bbbk = 0$ it is at least semisimple and we can check $A \cong \Bbbk \times \Bbbk^{2\times 2}$ as $\Bbbk$-algebras.

Furthermore, as already explained it follows easily from $\dim_{\Bbbk} \mathcal{E}_5 = 8294400$ and Remark 1.7 that our upper bound from Example 5.8 is in fact exact.

5.4. $n = 6$**. In this section we assume that** $\mathcal{E}_6$ **is the same as the associated Nichols-algebra, as is conjectured in** [**FK98**]. We use a computer aided approach with the method in Section 4.2 and a procedure similar to the one shown in the examples in Section 5.3 to give upper bounds for the dimensions of some algebras on 6 vertices. The strategy is again to give a graph S' on fewer vertices than S and a finite dimensional algebra A such that we have a surjective linear map $\mathcal{E}_{S'} \otimes A \to \operatorname{gr} \varphi_i(\mathcal{D}_S)$ using an appropriate filtration.

However, while the algebras A in the last sections were all rather well behaved (most were group algebras, the others at least semisimple if $\operatorname{char} \Bbbk = 0$) the situation here is much less satisfactory.

The complication that necessitates and limits the computer aided approach are complicated relations of high degree. We first search for new relations in the Nichols algebra and then deform them to relations in the deformed Fomin-Kirillov algebra using Proposition 2.6.

We will work with $\Bbbk = \mathbb{Q}$ and for Groebner basis computations use the package GBNP [**CK16**] implemented in GAP [**GAP16**].

Note that the easier examples on 6 vertices are dealt with on the go in Section 5.6.

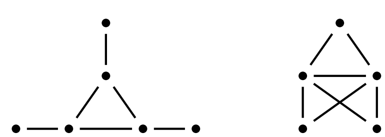


FIGURE 8. The graphs S and S' from Proposition 5.9.

PROPOSITION 5.9. *Let S be the graph on the left in Figure 8 and S' be the graph on the right in Figure 8. If we choose a suitable filtration on $\varphi_1(\mathcal{D}_S)(\mathbb{Q})$ there is a finite dimensional $\mathbb{Q}$-algebra A and a surjective linear map $\mathcal{E}_{S'}(\mathbb{Q}) \otimes A \to \operatorname{gr} \varphi_1(\mathcal{D}_S)(\mathbb{Q})$. The algebra A is*

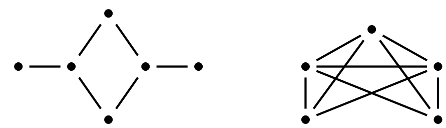


FIGURE 9. The graphs S and S' from Proposition 5.10.

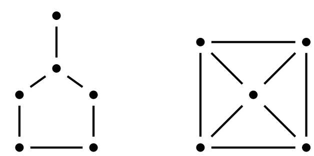


FIGURE 10. The graphs S and S' from Proposition 5.11.

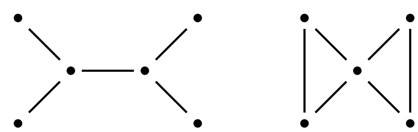


FIGURE 11. The graphs S and S' from Proposition 5.12.

not semisimple, not commutative, and of dimension 24*. In particular we obtain* $\dim_{\mathbb{Q}} \mathcal{E}_S(\mathbb{Q}) \leq 99532800$*.*

PROPOSITION 5.10. *Let S be the graph on the left in Figure 9 and S' be the graph on the right in Figure 9. If we choose a suitable filtration on $\varphi_1(\mathcal{D}_S)(\mathbb{Q})$ there is a finite dimensional $\mathbb{Q}$-algebra A and a surjective linear map $\mathcal{E}_{S'}(\mathbb{Q}) \otimes A \to \operatorname{gr} \varphi_1(\mathcal{D}_S)(\mathbb{Q})$. The algebra A is not semisimple, not commutative, and of dimension* 108*. In particular we obtain* $\dim_{\mathbb{Q}} \mathcal{E}_S(\mathbb{Q}) \leq 2687385600$*.*

PROPOSITION 5.11. *Let S be the graph on the left in Figure 10 and S' be the graph on the right in Figure 10. If we choose a suitable filtration on $\varphi_1(\mathcal{D}_S)(\mathbb{Q})$ there is a finite dimensional $\mathbb{Q}$-algebra A and a surjective linear map $\mathcal{E}_{S'}(\mathbb{Q}) \otimes A \to \operatorname{gr} \varphi_1(\mathcal{D}_S)(\mathbb{Q})$. The algebra A is not semisimple, not commutative, and of dimension* 8*. In particular we obtain* $\dim_{\mathbb{Q}} \mathcal{E}_S(\mathbb{Q}) \leq 99532800$*.*

PROPOSITION 5.12. *Let S be the graph on the left in Figure 11 and S' be the graph on the right in Figure 11. If we choose a suitable filtration on $\varphi_1(\mathcal{D}_S)(\mathbb{Q})$ there is a finite dimensional $\mathbb{Q}$-algebra A and a surjective linear map $\mathcal{E}_{S'}(\mathbb{Q}) \otimes A \to \operatorname{gr} \varphi_1(\mathcal{D}_S)(\mathbb{Q})$. The algebra A is semisimple, commutative, and of dimension* 6*. In particular we obtain* $\dim_{\mathbb{Q}} \mathcal{E}_S(\mathbb{Q}) \leq 2073600$*.*

5.5. D_n**.** Let $n \geq 4$ and

$$D_n = \{(1,2),(1,3),(1,4),(i,i+1) \,|\, 4 \leq i \leq n-1\}.$$

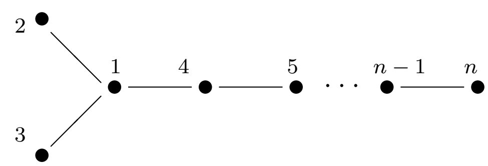


FIGURE 12. D_n

In this section we identify $\varphi_1(\mathcal{D}_{D_n})$ and by doing so give an upper bound for the dimension of $\mathcal{E}_{D_n}$. This coincides with the exact value given in [**BLM13**, Thm. 6.2] by different methods. We will also revisit this example in Section 7.1 and using a different method also show that our upper bound is in fact exact.

The case $n = 4$ was already treated in Example 5.2.

We will write D'_{n-1} for the graph obtained from D_{n-1} by adding the edge connecting 2 and 3. We have the following result.

PROPOSITION 5.13. *Let $n \geq 4$. Then $\varphi_1(\mathcal{D}_{D_n})$ has a set of generators which satisfy the quadratic, braid, and claw relations of $\mathcal{D}_{D'_{n-1}}$. In other words, $\varphi_1(\mathcal{D}_{D_n})$ is a quotient of $\widehat{\mathcal{D}_{D'_{n-1}}}$.*

PROOF. It is easy to check that $\varphi_1(\mathcal{D}_{D_n})$ is generated as an algebra by

$$t_{24} = y_{12}y_{14}y_{12}, t_{43} = y_{14}y_{13}y_{14}, t_{32} = y_{13}y_{12}y_{13},$$

and

$$t_{ii+1} = y_{ii+1} \text{ where } 4 \leq i \leq n-1.$$

We check that these generators satisfy the same defining relations as the generators of $\mathcal{D}_{D'_{n-1}}$. Commutation of t_{ii+1} where $4 \leq i \leq n-1$ with t_{23} is obvious, as is commutation of t_{ii+1} where $5 \leq i \leq n-1$ with t_{43}, t_{32}. The remaining quadratic relations we have seen in Example 5.2. Note that the quadratic relations in this case already imply the claw relations involving t_{24}, t_{43}, t_{45} as one can see in the proof of Lemma 3.2. What remains to be shown are the braid relations concerning t_{24} or t_{43}

and t_{45}. Consider

$$\begin{aligned} t_{45}t_{24}t_{45} &= y_{45}y_{12}y_{14}y_{12}y_{45} \\ &= y_{12}y_{45}y_{14}y_{45}y_{12} \\ &= y_{12}y_{14}y_{45}y_{14}y_{12} \\ &= y_{12}y_{14}(y_{12}y_{12})y_{45}y_{14}y_{12} \\ &= y_{12}y_{14}y_{12}y_{45}y_{12}y_{14}y_{12} \\ &= t_{24}t_{45}t_{24}. \end{aligned}$$

The relation $t_{43}t_{45}t_{43} = -t_{45}t_{43}t_{45}$ is obtained in a similar way. □

In fact, it is true that $\varphi_1(\mathcal{D}_{D_n}) \cong \mathcal{D}_{D'_{n-1}}$, which follows immediately from this statement if we use that the dimension of $\mathcal{E}_{D_n}$ was computed in [**BLM13**, Thm. 6.2]. We will later in Section 7.1 give an easy independent argument for this statement for $\Bbbk = \mathbb{C}$.

At this point we can only show the following estimate.

Corollary 5.14. *Let $n \geq 4$. Then* $\dim_{\Bbbk} \mathcal{D}_{D_n} \leq n!2^{n-3}$.

Proof. We already know the statement for $n = 4$ from Example 5.2.

Since we only used the braid and claw relations of $\mathcal{D}_{D_n}$ in the proof of Proposition 5.13, it follows

$$\begin{aligned} \dim_{\Bbbk} \mathcal{D}_{D_n} &= n \dim_{\Bbbk} \varphi_1(\mathcal{D}_{D_n}) \\ &\leq n \dim_{\Bbbk} \widehat{\mathcal{D}_{D'_{n-1}}} \\ &\leq 2n \dim_{\Bbbk} \widehat{\mathcal{D}_{D_{n-1}}} \\ &\leq n!2^{n-3} \end{aligned}$$

by induction. □

5.6. $E_n(n = 6, 7, 8)$**.** Let $n \geq 6$ and

$$E_n = \{(3, n), (i, i+1) \mid 1 \leq i \leq n-2\}.$$

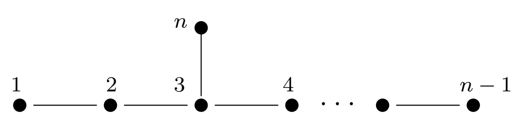


Figure 13. E_n

In this section we will investigate $\varphi_3(\mathcal{D}_{E_n})$, in particular we will give upper bounds on the dimension of $\mathcal{D}_{E_n}$ for $n = 6, 7, 8$. For the most part we will be able to do this by very easy direct computations, however for $n = 8$ we will require some minor Groebner basis calculations.

The dimensions were already obtained in [**BLM13**, Thm. 6.3] for $\Bbbk = \mathbb{Q}$ by computer calculation.

In passing we will also obtain the dimensions of $\mathcal{D}_S$ for some other graphs S.

We begin with a lemma. We introduce the notation

$$H_n = \{(3,n),(n,2),(i,i+1) \mid 1 \leq i \leq n-2\}.$$

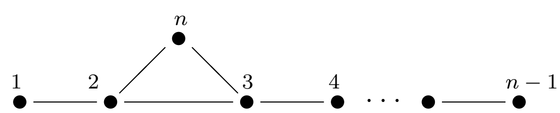


FIGURE 14. H_n

LEMMA 5.15. *$\varphi_3(\mathcal{D}_{E_n})$ has a set of generators which satisfy the quadratic, braid, and claw relations of $\mathcal{D}_{H_{n-1}}$. In other words, $\varphi_3(\mathcal{D}_{E_n})$ is a quotient of $\widehat{\mathcal{D}_{H_{n-1}}}$.*

PROOF. It is easy to check that $\varphi_3(\mathcal{D}_{E_n})$ is as an algebra generated by

$$t_{12} = y_{12}, t_{ii+1} = y_{ii+1} \text{ where } 4 \leq i \leq n-2,$$

and

$$t_{42} = y_{34}y_{32}y_{34}, t_{2n} = y_{32}y_{3n}y_{32}, t_{n4} = y_{3n}y_{34}y_{3n}.$$

By Example 5.2 and the proof of Proposition 5.13 we know that these generators satisfy the quadratic, braid, and claw relations of $\mathcal{D}_{H_{n-1}}$. $\square$

We obtained an upper bound for the dimension of $\mathcal{D}_{H_5}$ already in Example 5.4. Since we only used quadratic, braid, and claw relations in our investigation there, we can deduce

$$\dim_\Bbbk \mathcal{D}_{E_6} = 6 \dim_\Bbbk \varphi_3\left(\mathcal{D}_{E_6}\right) \leq 6 \dim_\Bbbk \widehat{\mathcal{D}_{H_5}} \leq 17280.$$

We expand on Example 5.4. We introduce the notation

$$K_{4,n} = \{(1,3),(2,4),(i,i+1) \mid 1 \leq i \leq n-1\} \setminus \{(2,3)\}.$$

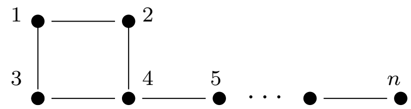


FIGURE 15. $K_{4,n}$

Again we have a lemma.

LEMMA 5.16. *There is a surjective linear map*

$$\widehat{\mathcal{D}_{K_{4,n-1}\cup\{(2,3)\}}} \otimes \Bbbk[a] \to \varphi_n(\mathcal{D}_{H_n}),$$

where $a^2 = 1$.

PROOF. It is easy to check that the algebra $\varphi_n(\mathcal{D}_{H_n})$ is generated by $a_{(23)} = y_{3n}y_{23}y_{2n} = y_{2n}y_{23}y_{3n}$ and

$$t_{24} = y_{3n}y_{23}y_{34}y_{23}y_{3n}, t_{13} = y_{2n}y_{23}y_{12}y_{23}y_{2n},$$
$$t_{ii+1} = y_{ii+1} \text{ for } 1 \leq i \leq n-2.$$

It is clear that $t_{ii+1}a_{(23)} = a_{(23)}t_{ii+1}$ for $i \geq 4$. We already discussed that these generators satisfy most of the relations of the graph $K_{4,n-1} \cup \{(2,3)\}$ in Example 5.4, the relations involving t_{ii+1} where $i \geq 4$ remain to be checked. It is obvious that t_{ii+1} commutes with t_{12}, t_{13}, and t_{23} for $i \geq 4$, as are the relations involving only t_{ii+1} where $i \geq 3$. The braid relation $t_{24}t_{45}t_{24} = t_{45}t_{24}t_{45}$ follows easily from the braid relation $y_{34}y_{45}y_{34} = y_{45}y_{34}y_{45}$ and the claw relation involving t_{24}, t_{45}, t_{34} already follows from the quadratic relations since there is one triangle involved (see the proof of Lemma 3.2). This implies the claim. □

In particular by Example 5.5 we have

$$\dim_\Bbbk \mathcal{D}_{H_6} \leq \dim_\Bbbk \widehat{\mathcal{D}_{H_6}} = 6 \dim_\Bbbk \varphi_6\left(\widehat{\mathcal{D}_{H_6}}\right) \leq 6{\cdot}2{\cdot}2{\cdot}\dim_\Bbbk \widehat{\mathcal{D}_{K_{4,5}}} \leq 207360.$$

We can now give an upper bound for the dimension of $\mathcal{D}_{E_7}$. We have

$$\dim_\Bbbk \mathcal{D}_{E_7} \leq 7 \dim_\Bbbk \widehat{\mathcal{D}_{H_6}} \leq 1451520.$$

Similar to before we can expand on Example 5.5. Again for use only in the remainder of the section we write

$$S_5' = \{(i,j), (5,6) \,|\, i,j \in \{2,3,4,5\}\}.$$

We need to know defining relations for $\mathcal{E}_{S_5'}$; it can be checked by computing Groebner basis and using the existence of the derivation ∂^* that the quadratic relations and the graded versions of the relations we discussed in Section 3 suffice to define $\mathcal{E}_{S_5'}$. Note furthermore that of the relations involving the generator $(5,6)$ only the Coxeter relations need to be checked, the others follow from the presence of the quadratic relations. Again, one can see this by either using Groebner basis methods or by checking the proofs in Section 3.

Of course by using Groebner basis calculations we limit ourselves to the case $\Bbbk = \mathbb{Q}$.

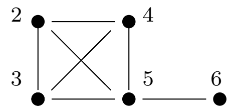


FIGURE 16. S_5'

LEMMA 5.17. *There is a filtration on the algebra $\varphi_1(\mathcal{D}_{K_{4,6}}(\mathbb{Q}))$ such that there is a surjective linear map*

$$\mathcal{E}_{S_5'}(\mathbb{Q}) \otimes \mathbb{Q}[a] \to \operatorname{gr} \varphi_1(\mathcal{D}_{K_{4,6}}(\mathbb{Q}))$$

where $a^3 = 1$.

PROOF. As in Example 5.5 it is easily verified that $\varphi_1(\mathcal{D}_{K_{4,6}}(\mathbb{Q}))$ is as an algebra generated by $a_{(243)} = -y_{12}y_{24}y_{43}y_{31}$ and

$$t_{24} = y_{24}, t_{34} = y_{34}, t_{45} = y_{45}, t_{56} = y_{56},$$
$$t_{32} = y_{12}y_{31}y_{12}, t_{35} = y_{12}y_{24}y_{43}y_{45}y_{43}y_{24}y_{12}, t_{25} = a_{(243)}t_{35}a_{(243)}^{-1}.$$

We know almost everything we need from Example 5.5, what remains to be shown are the relations involving t_{56}. This is easy to do using the braid and quadratic relations. $\square$

By Example 5.8 and Theorem 1.6 this implies

$$\begin{aligned}\dim_{\mathbb{Q}} \mathcal{D}_{K_{4,6}}(\mathbb{Q}) &\le \dim_{\mathbb{Q}} \widehat{\mathcal{D}_{K_{4,6}}}(\mathbb{Q}) \\ &\le 6 \cdot 3 \dim_{\mathbb{Q}} \mathcal{E}_{S_5'}(\mathbb{Q}) \\ &\le 18 \cdot 12 \cdot 14400 = 3110400,\end{aligned}$$

which in turn gives

$$\dim_{\mathbb{Q}} \mathcal{D}_{H_7}(\mathbb{Q}) \le \dim_{\mathbb{Q}} \widehat{\mathcal{D}_{H_7}}(\mathbb{Q}) \le 7 \cdot 2 \cdot 2 \dim_{\mathbb{Q}} \widehat{\mathcal{D}_{K_{4,6}}}(\mathbb{Q}) \le 87091200.$$

This finally allows us to give an upper bound for the dimension of $\mathcal{D}_{E_8}(\mathbb{Q})$ as

$$\dim_{\mathbb{Q}} \mathcal{D}_{E_8}(\mathbb{Q}) \le 8 \dim_{\mathbb{Q}} \widehat{\mathcal{D}_{H_7}}(\mathbb{Q}) \le 696729600.$$

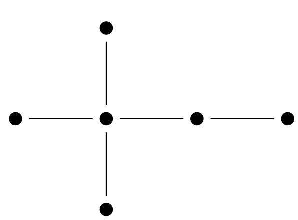


FIGURE 17. The cross.

REMARK 5.18. We briefly consider a problem for the computation of the dimension of $\mathcal{D}_{E_9}$. For this we should consider S_5' with the edge $(6,7)$ added. However, even the algebra to the subgraph in Figure 17 is not known to be finite dimensional yet and seems to require new relations: in the associated Nichols algebra we found additional relations of degree $14, 15$, and 16. This hints at additional relations appearing in $\mathcal{D}_{E_9}$ as is also stated in [**BLM13**].

6. The Complete Graph

In this section we extend the treatment of the examples in Section 5 to the full algebra $\mathcal{D}_n$. While the treatment in this section is inspired by the previous section, it is independent of the results given there.

After a little preparation we can finally calculate the dimension of $\mathcal{E}_5$ in Section 6.3.

In Section 6.4 we try to use our method for $n = 6$, however our results there are still partial.

6.1. The General Case.

Our approach is similar to the one given in [**FP00**].

For all $i, j \in \{1, \dots, n-1\}$ with $i \neq j$ let $t_{ij} = y_{ni}y_{nj}y_{ni} \in \mathcal{D}_n$. With the method from Section 4.2 we can easily see that $\varphi_n(\mathcal{D}_n)$ is as an algebra generated by y_{ij} and t_{ij} where $i, j \in \{1, \dots, n-1\}$, $i \neq j$ and has defining relations

$$
\begin{aligned}
y_{ij} + y_{ji} &= 0 && \text{if } i \neq j,\\
y_{ij}^2 &= 1 && \text{if } i \neq j,\\
y_{ij}y_{jk} + y_{jk}y_{ki} + y_{ki}y_{ij} &= 0 && \text{if } \#\{i,j,k\} = 3,\\
y_{ij}y_{kl} &= y_{kl}y_{ij} && \text{if } \#\{i,j,k,l\} = 4,\\
t_{ij} + t_{ji} &= 0 && \text{if } i \neq j,\\
t_{ij}^2 &= 1 && \text{if } i \neq j,\\
t_{ij}t_{jk} + t_{jk}t_{ki} + t_{ki}t_{ij} &= 0 && \text{if } \#\{i,j,k\} = 3,\\
y_{ij}t_{ij} &= 1 - t_{ij}y_{ij} && \text{if } i \neq j,\\
y_{ij}t_{ik} &= t_{jk}y_{ij} + t_{ik}t_{kj} && \text{if } \#\{i,j,k\} = 3,\\
y_{ij}t_{kl} &= t_{kl}y_{ij} && \text{if } \#\{i,j,k,l\} = 4.
\end{aligned}
$$

In the following we will identify the subalgebra of $\varphi_n(\mathcal{D}_n)$ generated by y_{ij} where $1 \leq i, j \leq n-1, i \neq j$ with $\mathcal{D}_{n-1}$ in the natural way. We refer by $\mathcal{T}_{n-1}$ to the subalgebra of $\varphi_n(\mathcal{D}_n)$ generated by t_{ij} where $1 \leq i, j \leq n-1, i \neq j$. We note that the action of $\mathbb{S}_n$ on $\mathcal{D}_n$ introduced in Section 2 induces an action of $\mathbb{S}_{n-1}$ on $\mathcal{T}_{n-1}$ and turns $\mathcal{T}_{n-1}$ into a $\Bbbk\mathbb{S}_{n-1}$-module algebra. It is easy to check that $\sigma t_{ij} = t_{\sigma(i)\sigma(j)}$ for $\sigma \in \mathbb{S}_{n-1}$.

Let $1 \le i, j \le n-1, i \neq j$. We define a linear map

$$d_{ij} : \mathcal{T}_{n-1} \to \varphi_n(\mathcal{D}_n), t \mapsto y_{ij}t - ((ij)t)y_{ij}.$$

This map is obviously a (ij)-skew derivation, i.e. we have

$$d_{ij}(tt') = d_{ij}(t)t' + ((ij)t)d_{ij}(t')$$

for all $t, t' \in \mathcal{T}_{n-1}$. From the defining relations of $\varphi_n(\mathcal{D}_n)$ we immediately obtain

$$\begin{aligned} &d_{ij}(t_{ij}) = 1 \in \mathcal{T}_{n-1}, \\ &d_{ij}(t_{ik}) = -d_{ij}(t_{ki}) = t_{ik}t_{kj} \in \mathcal{T}_{n-1} \text{ if } k \notin \{i,j\}, \\ &d_{ij}(t_{kl}) = 0 \in \mathcal{T}_{n-1} \text{ if } k, l \notin \{i,j\}. \end{aligned}$$

This combined with the property that d_{ij} is a (ij)-skew derivation implies $d_{ij}(\mathcal{T}_{n-1}) \subseteq \mathcal{T}_{n-1}$.

We will need some lemmas regarding these skew derivations. We postpone the proofs for now.

Lemma 6.1. *Let $1 \le i, j \le n-1$ and $i \neq j$. Let $\sigma \in \mathbb{S}_{n-1}$. Then $\sigma d_{ij}(t) = d_{\sigma(i)\sigma(j)}(\sigma t)$ for all $t \in \mathcal{T}_{n-1}$.*

Proof. Follows from Lemma 6.5. □

The following lemma shows that our skew derivations give an action of $\mathcal{E}_{n-1}$ on $\mathcal{T}_{n-1}$.

Lemma 6.2. *The following relations hold in $\mathrm{End}_{\Bbbk}(\mathcal{T}_{n-1})$. Let $1 \le i, j, k, l \le n-1$.*

(1) $d_{ij} + d_{ji} = 0$ *if* $i \neq j$.
(2) $d_{ij}^2 = 0$.
(3) $d_{ij}d_{jk} + d_{jk}d_{ki} + d_{ki}d_{ij} = 0$ *if* $\#\{i,j,k\} = 3$.
(4) $d_{ij}d_{kl} = d_{kl}d_{ij}$ *if* $\#\{i,j,k,l\} = 4$.

Proof. (1) is obvious.
(2),(3),(4) follow from the Lemmas 6.7, 6.6, and 6.8, respectively. □

Proposition 6.3. *The multiplication map $\mathcal{T}_{n-1} \otimes \mathcal{D}_{n-1} \to \varphi_n(\mathcal{D}_n)$ is an isomorphism of vector spaces.*

Proof. Let $p : \mathcal{T}_{n-1} \otimes \mathcal{D}_{n-1} \to \varphi_n(\mathcal{D}_n)$ be the multiplication map. It is clear from the defining relations of $\varphi_n(\mathcal{D}_n)$ that this map is surjective. We construct the inverse. We define a $\varphi_n(\mathcal{D}_n)$-module structure on $\mathcal{T}_{n-1} \otimes \mathcal{D}_{n-1}$ by

$$t_{ij} \cdot (t \otimes y) = t_{ij} t \otimes y,$$
$$y_{ij} \cdot (t \otimes y) = d_{ij}(t) \otimes y + ((ij)t) \otimes y_{ij} y,$$

for $1 \leq i, j, \leq n-1, i \neq j, t \in \mathcal{T}_{n-1}, y \in \mathcal{D}_{n-1}$. We can check that this satisfies the defining relations of $\varphi_n(\mathcal{D}_n)$. The relations involving only the t's are trivial, the relations involving only the y's follow by using Lemma 6.1 and the corresponding relations from Lemma 6.2 for the d's. The relations involving both t's and y's follow using that the d's are skew derivations and using the values of the d's on the generators of $\mathcal{T}_{n-1}$.

We can now use this structure to define a map

$$\pi : \varphi_n(\mathcal{D}_n) \to \mathcal{T}_{n-1} \otimes \mathcal{D}_{n-1}, a \mapsto a \cdot (1 \otimes 1).$$

It is clear that $\pi \circ p = \mathrm{id}$ by the definition of the module structure. Therefore p is injective aswell and hence an isomorphism. □

We want to know the defining relations of $\mathcal{T}_{n-1}$. For this we use the existence of the skew derivation d.

We rebuild the situation we are in and abuse notation. Let V be the $\Bbbk$-vector space with basis $\{t_{ij} \mid 1 \leq i < j \leq n-1\}$ and let $t_{ji} = -t_{ij}$ for $1 \leq i < j \leq n-1$. We define an action of $\mathbb{S}_{n-1}$ on $T(V)$: For $\sigma \in \mathbb{S}_{n-1}$ there is a unique algebra automorphism $\sigma \in \mathrm{Aut}(T(V))$ given by $\sigma(t_{ij}) = t_{\sigma(i)\sigma(j)}$ for $1 \leq i < j \leq n-1$.

Furthermore, $T(V)$ has a $\mathbb{S}_{n-1}$-grading by assigning $t_{ij} \in V$ the degree $(i\,j)$ for $1 \leq i < j \leq n-1$.

For $1 \leq i, j \leq n-1$ with $i \neq j$ let d_{ij} be the (ij)-skew derivation on $T(V)$ with

$$\begin{aligned} d_{ij}(t_{ij}) &= 1, & d_{ij}(t_{kl}) &= 0 & &(k, l \notin \{i, j\}, k \neq l), \\ d_{ij}(t_{ik}) &= t_{ik} t_{kj}, & d_{ij}(t_{kj}) &= t_{jk} t_{ki} & &(k \notin \{i, j\}). \end{aligned}$$

We then clearly have $d_{ij} = -d_{ji}$.

Let I_0 be the ideal of $T(V)$ generated by $t_{ij}^2 - 1$, $1 \leq i < j \leq n-1$. Let I_1 be the ideal of $T(V)$ generated by I_0 and the elements $t_{ij} t_{jk} + t_{jk} t_{ki} + t_{ki} t_{ij}$ where $1 \leq i, j, k \leq n-1, \#\{i, j, k\} = 3$.

Lemma 6.4. *Let $1 \leq i, j \leq n-1$ with $i \neq j$. Then $d_{ij}(I_0) \subseteq I_1$ holds.*

PROOF. Since d_{ij} is a (ij)-skew derivation and $(ij)(I_0) \subseteq I_0$, it suffices to show that d_{ij} maps the generators of I_0 to I_1. We have

$$\begin{aligned} d_{ij}(t_{ij}^2 - 1) =& d_{ij}(t_{ij})t_{ij} + ((ij)t_{ij})d_{ij}(t_{ij}) = t_{ij} + t_{ji} = 0, \\ d_{ij}(t_{ik}^2 - 1) =& d_{ij}(t_{ik})t_{ik} + ((ij)t_{ik})d_{ij}(t_{ik}) \\ =& t_{ik}t_{kj}t_{ik} - t_{kj}t_{ik}t_{kj} \in I_1, \\ d_{ij}(t_{jk}^2 - 1) =& -d_{ji}(t_{jk}^2 - 1) \in I_1, \\ d_{ij}(t_{kl}^2 - 1) =& d_{ij}(t_{kl})t_{kl} + ((ij)t_{kl})d_{ij}(t_{kl}) = 0 \end{aligned}$$

for all $k, l \notin \{i, j\}$ with $k \neq l$. See the proof of Lemma 3.1 for $t_{ik}t_{kj}t_{ik} - t_{kj}t_{ik}t_{kj} \in I_1$. $\square$

Let $d : V \to \mathrm{End}(T(V))$ denote the linear map given by $d(t_{ij}) = d_{ij}$ for $1 \leq i < j \leq n-1$. For $r \in \mathbb{N}$ with $r \geq 2$ let $I_r \subseteq T(V)$ be the ideal that is generated by I_{r-1} and $d(V)(I_{r-1})$. Let $I = \cup_{r \geq 0} I_r$.

For all $x, y \in T(V)$ and $r \in \mathbb{N}_0$ we write $x \equiv_r y$ if $x - y \in I_r$.

We want to find relations among the elements of $d(V)$. We start with a lemma.

LEMMA 6.5. *Let $1 \leq i, j \leq n-1$ and $i \neq j$. Let $\sigma \in \mathbb{S}_{n-1}$. Then $\sigma d_{ij}(t) = d_{\sigma(i)\sigma(j)}(\sigma t)$ for all $t \in T(V)$.*

PROOF. We start by evaluating both sides on the basis of V. Let $1 \leq k, l \leq n-1$ and $k, l \notin \{i, j\}$. Then

$$\begin{aligned} \sigma d_{ij}(t_{ij}) &= 1, \\ \sigma d_{ij}(t_{kl}) &= 0, \\ \sigma d_{ij}(t_{ik}) &= t_{\sigma(i)\sigma(k)} t_{\sigma(k)\sigma(j)}, \\ \sigma d_{ij}(t_{kj}) &= t_{\sigma(j)\sigma(k)} t_{\sigma(k)\sigma(i)}, \end{aligned}$$

and we get the same results for the evaluation of $d_{\sigma(i)\sigma(j)}\sigma$ on the basis of V. Moreover, we observe for t, t' in $T(V)$:

$$\sigma d_{ij}(tt') = (\sigma d_{ij})(t)\sigma(t') + ((\sigma(ij))t)(\sigma d_{ij})(t'),$$

and

$$d_{\sigma(i)\sigma(j)}(\sigma(tt')) = (d_{\sigma(i)\sigma(j)}\sigma)(t)\sigma(t') + ((\sigma(i)\sigma(j))\sigma)t(d_{\sigma(i)\sigma(j)}\sigma)(t').$$

Since $\sigma(ij) = (\sigma(i)\sigma(j))\sigma$ in $\mathbb{S}_{n-1}$ this implies that both σd_{ij} and $d_{\sigma(i)\sigma(j)}\sigma$ fulfill the same Leibniz rule. The claim follows. $\square$

Note that this together with $\mathbb{S}_{n-1} I_0 \subseteq I_0$ and $\mathbb{S}_{n-1} I_1 \subseteq I_1$ implies $\mathbb{S}_{n-1} I_r \subseteq I_r$ for all $r \geq 0$.

We derive some relations between the elements of $d(V)$.

LEMMA 6.6. *Let $1 \leq i,j,k \leq n-1$ with $\#\{i,j,k\} = 3$. Then $(d_{ij}d_{jk} + d_{jk}d_{ki} + d_{ki}d_{ij})(t) = 0$ for all $t \in T(V)$.*

PROOF. We evaluate $d_{ij}d_{jk} + d_{jk}d_{ki} + d_{ki}d_{ij}$ on a basis of V. It is clear that it evaluates to zero on $t_{l,m}$ where $l, m \notin \{i,j,k\}$. Due to symmetry we only need to consider the following cases. We start with

$$\begin{aligned} d_{ij}d_{jk}(t_{ij}) &= -d_{ij}d_{kj}(t_{ij}) \\ &= d_{ij}(t_{ji}t_{ik}) \\ &= t_{ik} - t_{ij}t_{ik}t_{kj}, \\ d_{ki}d_{ij}(t_{ij}) &= d_{ki}(1) = 0, \\ d_{jk}d_{ki}(t_{ij}) &= -d_{jk}d_{ki}(t_{ji}) \\ &= d_{jk}(t_{ji}t_{jk}) \\ &= t_{ji}t_{ik}t_{jk} + t_{ki}, \end{aligned}$$

and hence have $(d_{ij}d_{jk} + d_{jk}d_{ki} + d_{ki}d_{ij})(t_{ij}) = 0$. If $1 \leq l \leq n-1$, $l \notin \{i,j,k\}$ we compute

$$\begin{aligned} d_{ij}d_{jk}(t_{il}) &= 0, \\ d_{jk}d_{ki}(t_{il}) &= -d_{jk}d_{ik}(t_{il}) \\ &= -d_{jk}(t_{il}t_{lk}) \\ &= -t_{il}t_{kl}t_{lj}, \\ d_{ki}d_{ij}(t_{il}) &= -d_{ik}(t_{il}t_{lj}) \\ &= -t_{il}t_{lk}t_{lj}, \end{aligned}$$

and hence obtain $(d_{ij}d_{jk} + d_{jk}d_{ki} + d_{ki}d_{ij})(t_{il}) = 0$. Moreover we compute for $t, t' \in T(V)$ using Lemma 6.5

$$\begin{aligned} d_{ij}d_{jk}(tt') &= d_{ij}(d_{jk}(t)t' + ((jk)t)d_{jk}(t')) \\ &= d_{ij}d_{jk}(t)t' + ((ijk)t)d_{ij}d_{jk}(t') \\ &\quad + (d_{ik}((ij)t))d_{ij}(t') + d_{ij}((jk)t)d_{jk}(t'), \end{aligned}$$

which implies that $d_{ij}d_{jk} + d_{jk}d_{ki} + d_{ki}d_{ij}$ is a (ijk)-skew derivation. The claim now follows. $\square$

LEMMA 6.7. *Let $1 \leq i,j \leq n-1$ with $i \neq j$. Then $d_{ij}^2(t) \equiv_0 0$ for all $t \in T(V)$.*

PROOF. We evaluate d_{ij}^2 on the basis of V. It is clear that it evaluates to zero on t_{kl} where $k, l \notin \{i,j\}$. Let $1 \leq k \leq n-1$ with $k \notin \{i,j\}$.

We obtain

$$\begin{aligned}
d_{ij}^2(t_{ij}) &= d_{ij}(1) = 0,\\
d_{ij}^2(t_{ik}) &= d_{ij}(t_{ik}t_{kj})\\
&= t_{ik}t_{kj}^2 + t_{jk}^2 t_{ki}\\
&= t_{ik}(t_{kj}^2 - 1) + (t_{jk}^2 - 1)t_{ki}\\
&\equiv_0 0.
\end{aligned}$$

For $t, t' \in T(V)$ we have

$$\begin{aligned}
d_{ij}^2(tt') =& d_{ij}(d_{ij}(t)t' + ((ij)t)d_{ij}(t'))\\
=& d_{ij}^2(t)t' + ((ij)d_{ij}(t))d_{ij}(t') + d_{ij}((ij)t)d_{ij}(t') + ((ij)^2 t)d_{ij}^2(t')\\
=& d_{ij}^2(t)t' + td_{ij}^2(t'),
\end{aligned}$$

due to Lemma 6.5. Hence d_{ij}^2 is a skew derivation and the claim follows. □

LEMMA 6.8. *Let $1 \leq i, j, k, l \leq n-1$ with $\#\{i, j, k, l\} = 4$. Then $(d_{ij}d_{kl} - d_{kl}d_{ij})(t) \equiv_2 0$ for all $t \in T(V)$.*

PROOF. We evaluate $[d_{ij}, d_{kl}] = d_{ij}d_{kl} - d_{kl}d_{ij}$ on the basis of V. It is clear that it evaluates to zero on $t_{mm'}$ where $m, m' \notin \{i, j, k, l\}$. Due to symmetry we only need to consider the following cases. Let $1 \leq m \leq n-1$ with $m \notin \{i, j, k, l\}$. We compute

$$\begin{aligned}
[d_{ij}, d_{kl}](t_{ij}) =& d_{ij}(0) - d_{kl}(1) = 0,\\
[d_{ij}, d_{kl}](t_{ik}) =& -d_{ij}(t_{ki}t_{il}) - d_{kl}(t_{ik}t_{kj})\\
=& t_{ik}t_{kj}t_{il} - t_{kj}t_{il}t_{lj} + t_{ki}t_{il}t_{kj} - t_{il}t_{kj}t_{jl}\\
=& t_{ik}(t_{kj}t_{il} - t_{il}t_{kj}) + (t_{kj}t_{il} - t_{il}t_{kj})t_{jl},\\
[d_{ij}, d_{kl}](t_{im}) =& d_{ij}(0) - d_{kl}(t_{ij}t_{im} - t_{jm}t_{ij}) = 0.
\end{aligned}$$

We will check later (Lemma 6.11) that $t_{ik}(t_{kj}t_{il} - t_{il}t_{kj}) + (t_{kj}t_{il} - t_{il}t_{kj})t_{jl} \equiv_2 0$. Moreover, for $t, t' \in T(V)$ we have

$$\begin{aligned}
d_{ij}d_{kl}(tt') =& d_{ij}(d_{kl}(t)t' + ((kl)t)d_{kl}(t'))\\
=& d_{ij}d_{kl}(t)t' + d_{kl}((ij)t)d_{ij}(t')\\
&+ d_{ij}((kl)t)d_{kl}(t') + ((ij)(kl)t)d_{ij}d_{kl}(t'),
\end{aligned}$$

using Lemma 6.5. This implies $d_{ij}d_{kl} - d_{kl}d_{ij}$ is a $(ij)(kl)$-skew derivation. The claim follows. □

PROPOSITION 6.9. *The kernel of the algebra homomorphism*

$$\phi : T(V) \to \mathcal{T}_{n-1}, t_{ij} \mapsto t_{ij}$$

is I.

Proof. It is clear that I is contained in the kernel of ϕ. This means we can obtain a surjective linear map

$$p : T(V)/I \otimes \mathcal{D}_{n-1} \xrightarrow{\phi\otimes\mathrm{id}} \mathcal{T}_{n-1} \otimes \mathcal{D}_{n-1} \xrightarrow{\text{mult}} \varphi_n(\mathcal{D}_n).$$

We now proceed as in the proof of Proposition 6.3. Define a $\varphi_n(\mathcal{D}_n)$-module structure on $T(V)/I \otimes \mathcal{D}_{n-1}$ by

$$\begin{aligned} t_{ij} \cdot (t \otimes y) &= t_{ij}t \otimes y, \\ y_{ij} \cdot (t \otimes y) &= d_{ij}(t) \otimes y + ((ij)t) \otimes y_{ij}y, \end{aligned}$$

for $1 \leq i, j, \leq n-1, i \neq j, t \in T(V)/I, y \in \mathcal{D}_{n-1}$. Observe that on the left side we refer to the element $t_{ij} \in \mathcal{T}_{n-1}$ and on the right to the element $t_{ij} \in V$. By the same argument as in Proposition 6.3 we can see that this in fact gives a $\varphi_n(\mathcal{D}_n)$-module structure. Moreover, as in the proof of Proposition 6.3 we can use the existence of this structure to show that p is an isomorphism. It then follows that $\phi : T(V)/I \to \mathcal{T}_{n-1}$ is an isomorphism aswell. □

Remark 6.10. The Propositions 6.3 and 6.9 and their proofs are very similar to Theorem 1 in [**FP00**].

6.2. Description of I for $n \leq 5$. By Proposition 6.9 we know the defining relations of $\mathcal{T}_{n-1}$ in principle. In the following section we work towards a more concrete description. While our goal in this section is to achieve a description for $n \leq 5$, this is also the beginning of the considerations for the case $n = 6$ in Section 6.4.

We want to introduce some notation. In preparation of Section 6.4 we use graded commutators: For any group G and any G-graded $\Bbbk G$-module algebra A we define a bilinear map $A \times A \to A$ by letting

$$[\![a, b]\!] = ab - g(b)a$$

for $a \in A$ homogeneous of degree g and $b \in A$. Observe that if A is moreover a Yetter-Drinfeld module over $\Bbbk G$ we have

$$g\left([\![a, b]\!]\right) = [\![g(a), g(b)]\!]$$

for $g \in G$ and $a, b \in A$.

Returning to our setting we introduce the notation

$$t_{ij;kl} = [\![t_{ij}, t_{kl}]\!] = [t_{ij}, t_{kl}] \in T(V),$$

where $1 \leq i, j, k, l \leq n-1$ and $\#\{i, j, k, l\} = 4$. The degree of this element is $(i\,j)(k\,l)$. The formulas $t_{ij;kl} = -t_{kl;ij}$ and $t_{ij;kl} = -t_{ji;kl}$ are obvious. Moreover, we have

$$t_{ij;kl}t_{ij} \equiv_0 t_{ij}t_{kl}t_{ij} - t_{kl} \equiv_0 -t_{ij}t_{ij;kl}. \tag{6.1}$$

To obtain a description of the ideal I we start by applying the elements of $d(V)$ to the generators of I_1.

LEMMA 6.11. *Let $1 \leq i, j, k \leq n-1$ with $\#\{i, j, k\} = 3$.*

(1) $d_{ij}(t_{ji}t_{ik} + t_{ik}t_{kj} + t_{kj}t_{ji}) \equiv_1 0$
(2) *Let $1 \leq l \leq n-1$ with $l \notin \{i, j, k\}$. Then*

$$d_{il}(t_{ji}t_{ik} + t_{ik}t_{kj} + t_{kj}t_{ji}) \equiv_1 [\![t_{il;jk}, t_{ij}]\!].$$

PROOF. (1)We compute

$$\begin{aligned} d_{ij}(t_{ji}t_{ik} + t_{ik}t_{kj} + t_{kj}t_{ji}) &= -t_{ik} - t_{ji}d_{ij}(t_{ik}) + d_{ij}(t_{ik})t_{kj} + t_{jk}d_{ij}(t_{kj}) \\ &\quad - d_{ij}(t_{kj})t_{ij} - t_{ki} \\ &= -t_{ji}t_{ik}t_{kj} + t_{ik}t_{kj}t_{kj} + t_{jk}t_{jk}t_{ki} - t_{jk}t_{ki}t_{ij} \\ &\equiv_0 t_{ij}t_{ik}t_{kj} - t_{kj}t_{ik}t_{ij} + t_{ik} + t_{ki} \\ &\equiv_1 0, \end{aligned}$$

where the final equation follows from the proof of Lemma 3.1.

(2)We compute

$$\begin{aligned} d_{il}(t_{ji}t_{ik} + t_{ik}t_{kj} + t_{kj}t_{ji}) &= -d_{il}(t_{ij})t_{ik} + t_{jl}d_{il}(t_{ik}) \\ &\quad + d_{il}(t_{ik})t_{kj} - t_{kj}d_{il}(t_{ij}) \\ &= -(t_{ij}t_{jl})t_{ik} + t_{jl}(t_{ik}t_{kl}) \\ &\quad + (t_{ik}t_{kl})t_{kj} - t_{kj}(t_{ij}t_{jl}) \\ &\equiv_1 t_{jl}t_{li}t_{ik} + t_{li}t_{ij}t_{ik} - t_{jl}t_{kl}t_{li} - t_{jl}t_{li}t_{ik} \\ &\quad - t_{kl}t_{li}t_{kj} - t_{li}t_{ik}t_{kj} + t_{kj}t_{jl}t_{li} + t_{kj}t_{li}t_{ij} \\ &= t_{li}(-t_{ji}t_{ik} - t_{ik}t_{kj}) + (t_{jl}t_{lk} + t_{kj}t_{jl})t_{li} \\ &\quad - t_{kl}t_{li}t_{kj} + t_{kj}t_{li}t_{ij} \\ &\equiv_1 t_{li}t_{kj}t_{ji} - t_{lk}t_{kj}t_{li} - t_{kl}t_{li}t_{kj} + t_{kj}t_{li}t_{ij} \\ &= t_{jk}t_{il}t_{ij} - t_{il}t_{jk}t_{ij} - t_{lk}t_{jk}t_{il} + t_{lk}t_{il}t_{jk} \\ &= [\![t_{il;jk}, t_{ij}]\!], \end{aligned}$$

as was claimed. □

Let us consider the evaluation of some derivations on $t_{ij,kl}$ as preparation.

LEMMA 6.12. *Let $1 \leq i, j, k, l \leq n-1$ with $\#\{i, j, k, l\} = 4$. The following hold.*

(1) $d_{ij}(t_{ij;kl}) = 0$.
(2) $d_{ik}(t_{ij;kl}) \equiv_1 t_{jk;il}t_{ik} + t_{ik}t_{ij;kl} \equiv_2 t_{lj}t_{jk;il} + t_{ik}t_{ij;kl}$

PROOF. (1) Clear.

(2) We compute

$$\begin{aligned}
d_{ik}(t_{ij;kl}) &= d_{ik}(t_{ij})t_{kl} + t_{kj}d_{ik}(t_{kl}) - d_{ik}(t_{kl})t_{ij} - t_{il}d_{ik}(t_{ij}) \\
&= t_{ij}t_{jk}t_{kl} + t_{kj}t_{kl}t_{il} - t_{kl}t_{il}t_{ij} - t_{il}t_{ij}t_{jk} \\
&\equiv_1 -(t_{jk}t_{ki} + t_{ki}t_{ij})t_{kl} + t_{kj}t_{kl}t_{il} \\
&\qquad - (t_{li}t_{ik} + t_{ik}t_{kl})t_{ij} - t_{il}t_{ij}t_{jk} \\
&\equiv_1 -t_{jk}t_{li}t_{ik} - t_{ki}t_{ij}t_{kl} - t_{li}t_{jk}t_{ki} - t_{ik}t_{kl}t_{ij} \\
&= t_{jk;il}t_{ik} + t_{ik}t_{ij;kl} \\
&\equiv_2 t_{lj}t_{jk;il} + t_{ik}t_{ij;kl},
\end{aligned}$$

the last equation following from Lemma 6.11. □

To continue we need to apply the derivations to $[\![t_{ij;kl}, t_{ik}]\!]$. To reduce the number of cases we need to consider we observe

$$\begin{aligned}
[\![t_{ij;kl}, t_{ik}]\!] &= t_{ij;kl}t_{ik} + t_{lj}t_{ji;lk} \\
&= t_{lj}t_{ji}t_{lk} + t_{ij}t_{lk}t_{ki} + t_{jl}t_{lk}t_{ji} + t_{kl}t_{ji}t_{ik} \\
&\equiv_1 t_{ij}(t_{il}t_{lk} + t_{lk}t_{ki}) + t_{jk}t_{jl}t_{ji} + t_{kl}(t_{kj}t_{ji} + t_{ji}t_{ik}) + t_{li}t_{lj}t_{lk} \\
&\equiv_1 t_{ij}t_{ik}t_{il} + t_{jk}t_{jl}t_{ji} + t_{kl}t_{ki}t_{kj} + t_{li}t_{lj}t_{lk},
\end{aligned}$$

which implies

$$[\![t_{ij;kl}, t_{ik}]\!] \equiv_1 [\![t_{jk;li}, t_{jl}]\!] \equiv_1 [\![t_{kl;ij}, t_{ki}]\!] \equiv_1 [\![t_{li;jk}, t_{lj}]\!]. \tag{6.2}$$

LEMMA 6.13. *Let* $1 \le i, j, k, l \le n-1$ *with* $\#\{i, j, k, l\} = 4$. *Then*

(1) $d_{ik}([\![t_{ij;kl}, t_{ik}]\!]) \equiv_2 t_{ik;jl}t_{ij;kl} + t_{jk;il}$.
(2) $d_{ij}([\![t_{ij;kl}, t_{ik}]\!]) \equiv_1 0$.

PROOF. (1) We compute using Lemma 6.12

$$\begin{aligned}
d_{ik}([\![t_{ij;kl}, t_{ik}]\!]) &= d_{ik}(t_{ij;kl}t_{ik}) - d_{ik}(t_{jl}t_{ij;kl}) \\
&= d_{ik}(t_{ij;kl})t_{ik} + t_{kj;il} - t_{jl}d_{ik}(t_{ij;kl}) \\
&\equiv_2 (t_{jk;il}t_{ik} + t_{ik}t_{ij;kl})t_{ik} + t_{kj;il} - t_{jl}(t_{lj}t_{jk;il} + t_{ik}t_{ij;kl}) \\
&\equiv_2 t_{ik}t_{jl}t_{ij;kl} + t_{jk;il} - t_{jl}t_{ik}t_{ij;kl} \\
&= t_{ik;jl}t_{ij;kl} + t_{jk;il}.
\end{aligned}$$

(2) From Lemma 6.11 we know $[\![t_{ij;kl}, t_{ik}]\!] \equiv_1 d_{ij}(t_{ki}t_{il} + t_{il}t_{lk} + t_{lk}t_{ki})$. The claim now is an immediate consequence of Lemma 6.7.

□

We are done for $n \le 5$ and can give a complete description of $\mathcal{T}_{n-1}$ in terms of generators an relations. We will use this in Section 6.3 to calculate the dimension of $\mathcal{D}_n$ for $n \le 5$.

COROLLARY 6.14. *Let $n \leq 5$. The defining relations of $\mathcal{T}_{n-1}$ are*

$$t_{ij}^2 - 1, \qquad (I_0)$$
$$t_{ij}t_{jk} + t_{jk}t_{ki} + t_{ki}t_{ij}, \qquad (I_1)$$
$$[\![t_{ij;kl}, t_{ik}]\!], \qquad (I_2)$$
$$t_{ik;jl}t_{ij;kl} - t_{il;jk}, \qquad (I_3)$$

with $i, j, k, l \in \{1, \ldots, n-1\}$ pairwise distinct.

PROOF. We obtained all generators of I_2 in Lemma 6.11. Due to (6.2) we found all new generators of I_3 in Lemma 6.13. Moreover, using the Lemmas 6.6, 6.7, and 6.8 and similar arguments as in Lemma 6.13(2) it is easy to see that $d(V)I_3 \subseteq I_3$. $\square$

6.3. Calculating the Dimension for $n \leq 5$. In this section we calculate the dimensions of the full Fomin-Kirillov algebra for $n = 3, 4$, and 5. The cases $n = 3, 4$ were already treated with a different method in [**MS00**], which does not appear to work for larger n. For the case $n = 5$ to the best of our knowledge the dimension was until now only known through Groebner basis calculations (see [**GV16**] for some history).

PROPOSITION 6.15. *Let $3 \leq n \leq 4$. Then*

(1) *$\mathcal{T}_{n-1} \cong \mathcal{D}_{n-1}$ as algebras.*
(2) *$\dim \mathcal{E}_3 = \dim_{\Bbbk} \mathcal{D}_3 = 12$.*
(3) *$\dim \mathcal{E}_4 = \dim_{\Bbbk} \mathcal{D}_4 = 576$.*

PROOF. (1)We see directly from the defining relations $\mathcal{T}_{n-1}$ in Corollary 6.14 that $\mathcal{T}_{n-1} \cong \mathcal{D}_{n-1}$ for $n = 3, 4$.

(2),(3) Follows directly from $\dim_{\Bbbk} \mathcal{D}_2 = 2$, Proposition 6.15(1) and Corollary 4.4. $\square$

For $n = 5$ the situation is a bit more complicated. If we assign degree 1 to the generators t_{ij} and degree 0 to the generators $t_{ij;kl}$ of $\mathcal{T}_4$, it is clear from Corollary 6.14 and Equation (6.1) that we have a surjective linear map

$$\mathcal{E}_4 \otimes A \to \operatorname{gr} \mathcal{T}_4, \qquad (6.3)$$

where A is the subalgebra of $\mathcal{T}_4$ generated by $t_{12;34}, t_{13,24}$, and $t_{14,23}$.

We give an upper bound for the dimension of A.

LEMMA 6.16. *We have $\dim_{\Bbbk} A \leq 5$.*

Proof. Let $1 \leq i, j, k, l \leq 4$ with $\#\{i, j, k, l\} = 4$. We observe using the relations from Corollary 6.14

$$\begin{aligned} t_{ij;kl}^2 &= t_{ij;kl} t_{il;kj} t_{ik;lj} \\ &= t_{ik;lj} t_{ik;lj} \\ &= t_{ik;lj}^2 \end{aligned}$$

and as a consequence

$$\begin{aligned} t_{ij;kl}^3 &= t_{ik;lj}^2 t_{ij;kl} \\ &= t_{ik;lj} t_{il;kj} \\ &= -t_{ij;kl}. \end{aligned}$$

Due to $t_{ik;jl} t_{ij;kl} = t_{il;jk}$ and $t_{ij;kl} = -t_{ji;kl} = t_{ji;lk}$ this implies that

$$\{1, t_{12;34}, t_{13;24}, t_{14;23}, t_{12;34}^2\}$$

spans A as a $\Bbbk$-vector space. □

It now follows that

$$\dim_\Bbbk \mathcal{D}_5 = 5 \dim_\Bbbk \varphi_5(\mathcal{D}_5) = 5 \dim_\Bbbk \mathcal{D}_4 \dim_\Bbbk \mathcal{T}_4 \leq 5^2 \cdot 576^2 \leq 8294400.$$

Here in the first equation we used Corollary 4.4, in the second Proposition 6.3 and in the third Equation (6.3) and Lemma 6.16.

We want to show that this estimate is exact. For this we need a bit of extra work. First, we show that in fact $t_{ij;kl} \neq 0$.

Lemma 6.17. *Let $1 \leq i, j, k, l \leq n-1$ with $\#\{i, j, k, l\} = 4$. Then*

$$t_{ij;kl} \neq 0$$

in $\mathcal{T}_{n-1}$.

Proof. Recall that

$$\begin{aligned} t_{ij;kl} &= t_{ij} t_{kl} - t_{kl} t_{ij} \\ &= y_{ni} y_{nj} y_{ni} y_{nk} y_{nl} y_{nk} - y_{nk} y_{nl} y_{nk} y_{ni} y_{nj} y_{ni} \in \mathcal{D}_n. \end{aligned}$$

Due to Proposition 2.2 we can act on $\mathcal{E}_n$ with this element. First observe that

$$y_{nk} y_{nl} y_{nk} y_{ni} y_{nj} y_{ni}(x_{ni}) \in \mathcal{E}_n^6,$$

due to $x_{ni}(x_{ni} x_{nj} x_{ni} x_{nk} x_{nl} x_{nk}) = 0 \in \mathcal{E}_n$. On the other hand,

$$y_{ni} y_{nj} y_{ni} y_{nk} y_{nl} y_{nk}(x_{ni}) = x_{ni}(x_{nk} x_{nl} x_{nk} x_{ni} x_{nj} x_{ni}) + \tilde{x}$$

for some $\tilde{x} \in \mathcal{E}_n^6$. Hence, it suffices to show that

$$x_{ni} x_{nk} x_{nl} x_{nk} x_{ni} x_{nj} x_{ni} \neq 0 \in \mathcal{E}_n.$$

This however is easy: An elementary calculation shows

$$\partial_{ni}^* \partial_{nk}^* \partial_{nl}^* \partial_{nk}^* \partial_{ni}^* \partial_{nj}^* \partial_{ni}^* (x_{ni} x_{nk} x_{nl} x_{nk} x_{ni} x_{nj} x_{ni}) = 1$$

and we are done. □

Next we turn $\mathcal{T}_4$ into a left $\mathcal{E}_4\#\Bbbk\mathbb{S}_4$-comodule algebra.

PROPOSITION 6.18. *The assignment*

$$\rho(t_{ij}) = x_{ij}\otimes 1 + (ij)\otimes t_{ij}$$

for $1\le i,j\le 4$ *turns* $\mathcal{T}_4$ *into a left* $\mathcal{E}_4\#\Bbbk\mathbb{S}_4$*-comodule algebra.*

PROOF. We first need to check that this assignment is compatible with the defining relations of $\mathcal{T}_4$. Let $1\le i,j,k,l\le 4$ with $\#\{i,j,k,l\}=4$. First observe

$$\begin{aligned}\rho(t_{ij})\rho(t_{ij}) &= (x_{ij}\otimes 1 + (ij)\otimes t_{ij})(x_{ij}\otimes 1 + (ij)\otimes t_{ij})\\ &= x_{ij}^2\otimes 1 + x_{ij}(ij)\otimes t_{ij} + (ij)x_{ij}\otimes t_{ij} + 1\otimes t_{ij}^2\\ &= x_{ij}(ij)\otimes t_{ij} + x_{ji}(ij)\otimes t_{ij} + 1\otimes 1\\ &= 1\otimes 1\\ &= \rho(1),\end{aligned}$$

where we used $x_{ij}^2=0, t_{ij}^2=1$, and $x_{ji}=-x_{ij}$.

Next, consider

$$\begin{aligned}\rho(t_{ij})\rho(t_{jk}) &= (x_{ij}\otimes 1 + (ij)\otimes t_{ij})(x_{jk}\otimes 1 + (jk)\otimes t_{jk})\\ &= x_{ij}x_{jk}\otimes 1 + x_{ij}(jk)\otimes t_{jk} + (ij)x_{jk}\otimes t_{ij} + (ijk)\otimes t_{ij}t_{jk}\\ &= x_{ij}x_{jk}\otimes 1 + x_{ij}(jk)\otimes t_{jk} + x_{ik}(ij)\otimes t_{ij} + (ijk)\otimes t_{ij}t_{jk}.\end{aligned}$$

This immediately implies $\rho(t_{ij})\rho(t_{jk})+\rho(t_{jk})\rho(t_{ki})+\rho(t_{ki})\rho(t_{ij})=0$ if we use the corresponding relations in $\mathcal{E}_4$ and $\mathcal{T}_4$ and $x_{ij}=-x_{ji}$.

For the remaining two relations consider first

$$\begin{aligned}\rho(t_{ij})\rho(t_{kl}) &= (x_{ij}\otimes 1 + (ij)\otimes t_{ij})(x_{kl}\otimes 1 + (kl)\otimes t_{kl})\\ &= x_{ij}x_{kl}\otimes 1 + x_{ij}(kl)\otimes t_{kl} + (ij)x_{kl}\otimes t_{ij} + (ij)(kl)\otimes t_{ij}t_{kl}\\ &= x_{ij}x_{kl}\otimes 1 + x_{ij}(kl)\otimes t_{kl} + x_{kl}(ij)\otimes t_{ij} + (ij)(kl)\otimes t_{ij}t_{kl}.\end{aligned}$$

This immediately implies

$$[\![\rho(t_{ij}),\rho(t_{kl})]\!] = (ij)(kl)\otimes t_{ij;kl}, \tag{6.4}$$

by using $x_{ij}x_{kl}=x_{kl}x_{ij}$. From this we see directly that the last relation of Corollary 6.14 is satisfied. Using this formula it is also trivial to check the second to last relation of Corollary 6.14, we will skip this.

We have now checked that the assignment gives a well defined algebra map $\mathcal{T}_4\to\mathcal{E}_4\#\Bbbk\mathbb{S}_4\otimes\mathcal{T}_4$. We still need to check coassociativity and the counit property. It suffices to check this on the generators, there it is trivial. □

Now we can show a lower bound for the dimension.

LEMMA 6.19. *We have* $\dim_{\Bbbk} \mathcal{T}_4 \geq 5 \cdot 576 = 2880$.

PROOF. Here we shall write $x = t_{12;34}, y = t_{13;24}$, and $z = t_{14;23}$. Choose a basis $(x_u)_{1 \leq u \leq 576}$ of $\mathcal{E}_4$ consisting of monomials. For each $1 \leq u \leq 576$ there is a $m \geq 0$ and $s_1, \dots, s_m \in T$ such that $x_u = x_{s_1} \cdots x_{s_m}$. For $1 \leq u \leq 576$ let $t_u = t_{s_1} \cdots t_{s_m} \in \mathcal{T}_4$. We show that

$$\{t_u, t_u x, t_u y, t_u z, t_u x^2 | 1 \leq u \leq 576\} \subseteq \mathcal{T}_4$$

is linearly independent over $\Bbbk$. Indeed, for $1 \leq u \leq 576$ and $w \in \{1, x, y, z, x^2\}$ let $\lambda_u^w \in \Bbbk$ not all zero such that

$$\sum_{w \in \{1,x,y,z,x^2\}} \sum_{u=1}^{576} \lambda_u^w t_u w = 0. \tag{6.5}$$

Let K be the maximal length of a monomial t_u that appears in (6.5) nontrivially. Due to $x^2 = y^2 = z^2$ our situation is symmetric in x, y, z. Thus due to $x^3 = -x$ we may assume (by multiplying (6.5) with x if necessary) that the there is a $1 \leq v \leq 576$ with $\deg t_v = \deg x_v = K$ such that $\lambda_v^x \neq 0$. By Lemma 6.17 we have $x \neq 0$. Since the $\mathbb{S}_4$-degrees of $y, z, x^2, 1$ and x differ, those elements are not equal to x. Now from Equation (6.4) in the proof of Proposition 6.18 and the definition of ρ it follows that

$$\sum_{u=1, \deg x_u = K}^{576} \lambda_u^x x_u + \tilde{x} = 0 \in \mathcal{E}_4,$$

for some $\tilde{x} \in \mathcal{E}_4^{K-1}$. Since $\mathcal{E}_4$ is graded this implies

$$\sum_{u=1, \deg x_u = K}^{576} \lambda_u^x x_u = 0.$$

This implies $\lambda_u^x = 0$ if $\deg x_u = K$ since the x_u are linearly independent in $\mathcal{E}_4$. This however is a contradiction to what we have seen above. $\square$

As a corollary we finally have.

COROLLARY 6.20. *The dimension of* $\mathcal{E}_5$ *is* 8294400.

6.4. Description of I for $n = 6$. In this section we consider the case $n = 6$. Before we start let us remark that we *do not* currently understand the algebra $\mathcal{T}_5$, and the results of this section are incomplete.

Before we continue we introduce some notation. Let $i, j, k, l, m \in \{1, \dots, 5\}$ with $\#\{i, j, k, l, m\} = 5$ and let

$$\begin{aligned} t_{ijm;kl} &= [\![t_{ij;kl}, t_{jm}]\!] \text{ and} \\ u_{jm;kl} &= [\![t_{ijm;kl}, t_{im}]\!]. \end{aligned}$$

The element in the first row has degree $(i\,j\,m)(k\,l)$, the element in the second row has degree $(j\,m)(k\,l)$. We will collect some elementary equations involving these elements later.

Again as preparation we calculate the value of the derivation that we have not considered in Lemma 6.12 on $t_{ij,kl}$.

LEMMA 6.21. *Let* $1 \leq i,j,k,l,m \leq 5$ *with* $\#\{i,j,k,l,m\} = 5$. *Then*

$$d_{ij}(t_{jk;lm}) = t_{ij}t_{jk;lm} - t_{jk}t_{ki;lm} - t_{jki;lm}.$$

PROOF. Compute

$$\begin{aligned} d_{ij}(t_{jk;lm}) &= d_{ij}(t_{jk})t_{lm} - t_{lm}d_{ij}(t_{jk}) \\ &= t_{jk}t_{ik}t_{lm} - t_{lm}t_{jk}t_{ik} \\ &= t_{jk}t_{ik;lm} + t_{jk;lm}t_{ik} + t_{ji}t_{jk;lm} + t_{ij}t_{jk;lm} \\ &= t_{ij}t_{jk;lm} - t_{jk}t_{ki;lm} - t_{jki;lm}. \end{aligned}$$

□

Using this we can easily evaluate the last derivation at $[\![t_{ij;kl}, t_{ik}]\!]$. Also we evaluate the remaining derivation at $t_{ji}t_{ik} + t_{ik}t_{kj} + t_{kj}t_{ji}$.

LEMMA 6.22. *Let* $1 \leq i,j,k,l,m \leq 5$ *with* $\#\{i,j,k,l,m\} = 5$. *Then*

(1) $d_{lm}(t_{ji}t_{ik} + t_{ik}t_{kj} + t_{kj}t_{ji}) = 0$.
(2) $d_{lm}([\![t_{ij;kl}, t_{ik}]\!]) \equiv_2 [\![t_{lkm;ij}, t_{ik}]\!] + t_{kl;jm}t_{ij;km}$.

PROOF. (1) This is clear since $d_{ij}(t_{kl}) = 0$ if $\#\{i,j,k,l\} = 4$ by definition.

(2) Compute

$$\begin{aligned} d_{lm}([\![t_{ij;kl}, t_{ik}]\!]) &= d_{lm}(t_{ij;kl})t_{ik} - d_{lm}(t_{jl})t_{ij;kl} - t_{jm}d_{lm}(t_{ij;kl}) \\ &= -d_{ml}(t_{lk;ij})t_{ik} - t_{jl}t_{jm}t_{ij;kl} + t_{jm}d_{ml}(t_{lk;ij}) \\ &= -(t_{ml}t_{lk;ij} - t_{lk}t_{km;ij} - t_{lkm;ij})t_{ik} \\ &\qquad + t_{jl}t_{jm}t_{kl;ij} + t_{jm}(t_{ml}t_{lk;ij} - t_{lk}t_{km;ij} - t_{lkm;ij}) \\ &\equiv_2 -t_{ml}t_{jl}t_{lk;ij} + t_{lk}t_{jm}t_{km;ij} + t_{lkm;ij}t_{ik} \\ &\qquad - t_{lj}t_{jm}t_{kl;ij} - t_{jm}t_{ml}t_{kl;ij} - t_{jm}t_{lk}t_{km;ij} - t_{jm}t_{lkm;ij} \\ &\equiv_1 t_{lk;jm}t_{km;ij} + [\![t_{lkm;ij}, t_{ik}]\!]. \end{aligned}$$

□

We saw earlier that

$$[\![t_{ij;kl}, t_{ik}]\!] = t_{ij}t_{ik}t_{il} + t_{jk}t_{jl}t_{ji} + t_{kl}t_{ki}t_{kj} + t_{li}t_{lj}t_{lk},$$

using this it is easy to show that

$$\begin{aligned}
[\![t_{lkm;ij}, t_{ik}]\!] + t_{kl;jm} t_{ij;km} &\equiv_2 d_{lm}([\![t_{ij;kl}, t_{ik}]\!]) \\
&\equiv_1 t_{ij}t_{ik}t_{il}t_{im} + t_{jk}t_{jl}t_{jm}t_{ji} \\
&\qquad + t_{kl}t_{km}t_{ki}t_{kj} + t_{lm}t_{li}t_{lj}t_{lk} + t_{mi}t_{mj}t_{mk}t_{ml}.
\end{aligned}$$

As did (6.2) this symmetry greatly reduces the number of derivations we need to evaluate at $[\![t_{lkm;ij}, t_{ik}]\!] + t_{kl;jm}t_{ij;km}$. We are now prepared to show the following.

Corollary 6.23. *The defining relations of $\mathcal{T}_5$ are*

$$\begin{aligned}
&t_{ij}^2 - 1, && (I_0) \\
&t_{ij}t_{jk} + t_{jk}t_{ki} + t_{ki}t_{ij}, && (I_1) \\
&[\![t_{ij;kl}, t_{ik}]\!], && (I_2) \\
&t_{ik;jl}t_{ij;kl} - t_{il;jk}, && (I_3) \\
&[\![t_{lkm;ij}, t_{ik}]\!] + t_{kl;jm}t_{ij;km}, && (I_3) \\
&d_{ik}([\![t_{lkm;ij}, t_{ik}]\!] + t_{kl;jm}t_{ij;km}), && (I_4) \\
&d_{il}d_{ik}([\![t_{lkm;ij}, t_{ik}]\!] + t_{kl;jm}t_{ij;km}), && (I_5) \\
&d_{jl}d_{ik}([\![t_{lkm;ij}, t_{ik}]\!] + t_{kl;jm}t_{ij;km}), && (I_5) \\
&d_{ij}d_{il}d_{ik}([\![t_{lkm;ij}, t_{ik}]\!] + t_{kl;jm}t_{ij;km}), && (I_6) \\
&d_{ik}d_{ij}d_{il}d_{ik}([\![t_{lkm;ij}, t_{ik}]\!] + t_{kl;jm}t_{ij;km}), && (I_7) \\
&d_{ij}d_{ik}d_{ij}d_{il}d_{ik}([\![t_{lkm;ij}, t_{ik}]\!] + t_{kl;jm}t_{ij;km}), && (I_8) \\
&d_{im}d_{ij}d_{ik}d_{ij}d_{il}d_{ik}([\![t_{lkm;ij}, t_{ik}]\!] + t_{kl;jm}t_{ij;km}), && (I_9)
\end{aligned}$$

with $i, j, k, l, m \in \{1, \dots, 5\}$ pairwise distinct.

Proof. Elementary calculations similar to Corollary 6.14 using the Lemmas 6.6, 6.7, 6.8, and similar arguments as in Lemma 6.13(2). □

We now know that the process of getting relations by applying derivations ends after finitely many steps. Of course, we still desire a more explicit description. The remainder of this section is dedicated towards that. For the remainder of this section $1 \le i, j, k, l, m \le 5$ are always assumed to be pairwise distinct natural numbers.

To prepare we collect some equations concerning the elements $t_{ijm;kl}$ and $u_{jm;kl}$ which were introduced earlier.

Trivially, we have

$$t_{ijm;kl} = -t_{ijm;lk}, \quad u_{jm;kl} = -u_{jm;lk}.$$

Moreover,

$$\begin{aligned}
t_{ijm;kl}+t_{jmi;kl}+t_{mij;kl} =& t_{ij;kl}t_{jm}-t_{im}t_{ij;kl}\\
&+t_{jm;kl}t_{mi}-t_{ji}t_{jm;kl}+t_{mi;kl}t_{ij}-t_{mj}t_{mi;kl}\\
=&-t_{kl}t_{ij}t_{jm}-t_{im}t_{ij}t_{kl}-t_{kl}t_{jm}t_{mi}\\
&-t_{ji}t_{jm}t_{kl}-t_{kl}t_{mi}t_{ij}-t_{mj}t_{mi}t_{kl}\\
\equiv_1& 0. \qquad (6.6)
\end{aligned}$$

And as an easy consequence already using the upcoming Lemma 6.24(1),(2)

$$\begin{aligned}
0 &\equiv_1 [\![t_{ijm;kl},t_{ij}]\!]+[\![t_{jmi;kl},t_{ij}]\!]+[\![t_{mij;kl},t_{ij}]\!]\\
&\equiv_1 -u_{im;kl}-u_{mi;kl},
\end{aligned}$$

and hence

$$u_{im;kl}\equiv_1 -u_{mi;kl}. \tag{6.7}$$

As further preparation we collect the commutation relations for the $t_{ijm;kl}$.

LEMMA 6.24. *The following hold.*

(1) $[\![t_{ijm;kl},t_{ij}]\!]\equiv_1 -u_{im;kl}$.
(2) $[\![t_{ijm;kl},t_{jm}]\!]\equiv_0 0$.
(3) $[\![t_{ijm;kl},t_{im}]\!]\equiv_1 -u_{mj;kl}$.
(4) $[\![t_{ijm;kl},t_{ik}]\!]\equiv_2 t_{ij;kl}t_{jm;ik}-t_{im;jl}t_{ij;kl}$.
(5) $[\![t_{ijm;kl},t_{jk}]\!]\equiv_3 t_{ij;lm}t_{kl;jm}$.
(6) $[\![t_{ijm;kl},t_{mk}]\!]\equiv_3 t_{im;kl}t_{ij;mk}$.
(7) $[\![t_{ijm;kl},t_{kl}]\!]\equiv_0 t_{ij;kl}t_{jm;kl}-t_{mi;kl}t_{ij;kl}$.

PROOF. (1) Compute

$$\begin{aligned}
[\![t_{ijm;kl},t_{ij}]\!] &= t_{ijm;kl}t_{ij}-t_{jm}t_{ijm;kl}\\
&= t_{ij;kl}t_{jm}t_{ij}-t_{im}t_{ij;kl}t_{ij}-t_{jm}t_{ijm;kl}\\
&\equiv_1 t_{ij;kl}(-t_{ji}t_{im}-t_{im}t_{mj})-t_{mi}t_{ij}t_{ij;kl}-t_{jm}t_{ijm;kl}\\
&= t_{ij}t_{ji;kl}t_{im}+(t_{jim;kl}+t_{jm}t_{ji;kl})t_{mj}\\
&\qquad -t_{mi}t_{ij}t_{ij;kl}-t_{jm}t_{ijm;kl}\\
&= t_{ij}(t_{jim;kl}+t_{jm}t_{ji;kl})+t_{jim;kl}t_{mj}\\
&\qquad -t_{mi}t_{ij}t_{ij;kl}+t_{jm}t_{im}t_{ij;kl}\\
&\equiv_1 [\![t_{jim;kl},t_{mj}]\!]\\
&= -u_{im;kl}.
\end{aligned}$$

(2) We have

$$\begin{aligned} t_{ijm;kl}t_{jm} &= t_{ij;kl}t_{jm}^2 - t_{im}t_{ij;kl}t_{jm} \\ &\equiv_0 t_{mi}(-t_{im}t_{ij;kl} + t_{ij;kl}t_{jm}) \\ &= t_{mi}t_{ijm;kl}. \end{aligned}$$

(3) Follows from the definition and (6.7).

(4) Compute

$$\begin{aligned} [\![t_{ijm;kl}, t_{ik}]\!] &= (t_{ij;kl}t_{jm} - t_{im}t_{ij;kl})t_{ik} - t_{jl}t_{ijm;kl} \\ &= t_{ij;kl}(t_{jm;ik} + t_{ik}t_{jm}) - t_{im}t_{ij;kl}t_{ik} - t_{jl}t_{ijm;kl} \\ &\equiv_2 t_{ij;kl}t_{jm;ik} + t_{jl}t_{ij;kl}t_{jm} - t_{im}t_{jl}t_{ij;kl} - t_{jl}t_{ijm;kl} \\ &= t_{ij;kl}t_{jm;ik} + t_{jl}t_{im}t_{ij;kl} - t_{im}t_{jl}t_{ij;kl} \\ &= t_{ij;kl}t_{jm;ik} - t_{im;jl}t_{ij;kl}. \end{aligned}$$

(5) See Corollary 6.23.

(6) Follows immediately from (4) and (5) using (6.6).

(7) Compute

$$\begin{aligned} t_{ijm;kl}t_{kl} &= t_{ij;kl}t_{jm}t_{kl} - t_{im}t_{ij;kl}t_{kl} \\ &= t_{ij;kl}t_{jm;kl} + t_{ij;kl}t_{kl}t_{jm} + t_{im}t_{kl}t_{ij;kl} \\ &\equiv_0 t_{ij;kl}t_{jm;kl} - t_{kl}t_{ij;kl}t_{jm} + (t_{im;kl} + t_{kl}t_{im})t_{ij;kl} \\ &= t_{ij;kl}t_{jm;kl} + t_{lk}t_{ijm;kl} + t_{im;kl}t_{ij;kl}. \end{aligned}$$

□

Before we start with the commutation relations for $u_{jm;kl}$ we consider the following helpful formula.

$$\begin{aligned} t_{ikl;jm}t_{ij;km} &= t_{ik;jm}t_{kl}t_{ij;km} - t_{il}t_{ik;jm}t_{ij;km} \\ &= t_{ik;jm}(t_{ij;km}t_{ml} + t_{kml;ij}) - t_{il}t_{ik;jm}t_{ij;km} \\ &\equiv_3 t_{ik;jm}t_{kml;ij} + t_{im;jk}t_{ml} - t_{il}t_{im;jk} \\ &= t_{ik;jm}t_{kml;ij} + t_{iml;jk}. \end{aligned} \tag{6.8}$$

As a consequence of (6.8) we observe that the element $u_{jm;kl}$ can already be expressed using the notations introduced before:

$$\begin{aligned} t_{ilm;kj}t_{kim;lj} &= t_{ilm;kj}(t_{ki;lj}t_{im} - t_{km}t_{ki;lj}) \\ &= -t_{ilm;kj}t_{ik;lj}t_{im} + t_{ilm;kj}t_{mk}t_{ki;lj} \\ &\equiv_3 -(t_{il;kj}t_{ljm;ik} - t_{ijm;lk})t_{im} + (t_{ij}t_{ilm;kj} + t_{im;kj}t_{il;mk})t_{ki;lj} \\ &\equiv_3 t_{il;kj}(t_{lk}t_{ljm;ik} + t_{lm;ik}t_{lj;mi}) + t_{ji}t_{ijm;lk} + u_{jm;lk} \\ &\qquad - t_{ij}(t_{il;kj}t_{ljm;ik} - t_{ijm;lk}) + t_{im;kj}t_{il;mk}t_{ki;lj} \\ &\equiv_1 u_{jm;lk} + t_{il;kj}t_{lm;ik}t_{lj;mi} + t_{im;kj}t_{il;mk}t_{ki;lj}, \end{aligned}$$

or

$$u_{jm;kl} \equiv_3 -t_{ilm;kj}t_{kim;lj} + t_{il;kj}t_{lm;ik}t_{lj;mi} + t_{im;kj}t_{il;mk}t_{ki;lj}. \tag{6.9}$$

We also have commutation relations for $u_{jm;kl}$. Check (6.10) in the paragraph of I_5 later for a completion of this list.

LEMMA 6.25. *The following hold.*

(1) $[\![u_{jm;kl}, t_{im}]\!] \equiv_0 0$.
(2) $[\![u_{jm;kl}, t_{jk}]\!] \equiv_3 t_{ijm;kl}t_{im;jk} + t_{ij;ml}t_{ijm;kl} + t_{ij;ml}t_{jmi;kl}$.
(3) $[\![u_{jm;kl}, t_{jm}]\!] \equiv_3 t_{il;jm}t_{kj;lm}t_{kim;jl} - t_{im;kj}t_{il;mk}t_{ljm;ki} - t_{ilm;kj}t_{km;jl}t_{ki;mj} - t_{jkm;il}t_{lm;ik}t_{lj;mi}$.

PROOF. (1) Compute

$$\begin{aligned} u_{jm;kl}t_{im} &= t_{ijm;kl}t_{im}^2 - t_{ji}t_{ijm;kl}t_{im} \\ &\equiv_0 t_{ijm;kl} - t_{ji}(u_{jm;kl} + t_{ji}t_{ijm;kl}) \\ &\equiv_0 t_{ij}u_{jm;kl}. \end{aligned}$$

(2) We compute using Lemma 6.24(5)

$$\begin{aligned} u_{jm;kl}t_{jk} &= t_{ijm;kl}t_{im}t_{jk} - t_{ji}t_{ijm;kl}t_{jk} \\ &\equiv_3 t_{ijm;kl}t_{im;jk} + t_{ijm;kl}t_{jk}t_{im} - t_{ji}(t_{ml}t_{ijm;kl} - t_{ij;ml}t_{jm;kl}) \\ &\equiv_3 t_{ijm;kl}t_{im;jk} + (t_{ml}t_{ijm;kl} - t_{ij;ml}t_{jm;kl})t_{im} \\ &\qquad + t_{ij}t_{ml}t_{ijm;kl} - t_{ij}t_{ij;ml}t_{jm;kl} \\ &= t_{ijm;kl}t_{im;jk} + t_{ml}(t_{ji}t_{ijm;kl} + u_{jm;kl}) + t_{ij;ml}(t_{ji}t_{jm;kl} + t_{jmi;kl}) \\ &\qquad + t_{ij}t_{ml}t_{ijm;kl} - t_{ij}t_{ij;ml}t_{jm;kl} \\ &= t_{ijm;kl}t_{im;jk} + t_{ij;ml}t_{ijm;kl} + t_{ml}u_{jm;kl} + t_{ij;ml}t_{jmi;kl}. \end{aligned}$$

(3) Use (6.9) and compute

$$\begin{aligned} u_{jm;kl}t_{jm} &\equiv_3 (t_{ilm;kj}t_{kim;jl} + t_{il;kj}t_{lm;ik}t_{lj;mi} - t_{im;kj}t_{il;mk}t_{lj;ki})t_{jm} \\ &\equiv_3 t_{ilm;kj}t_{lk}t_{kim;jl} - t_{ilm;kj}t_{km;jl}t_{ki;mj} + t_{il;kj}t_{lm;ik}t_{li}t_{lj;mi} \\ &\qquad - t_{im;kj}t_{il;mk}t_{lm}t_{lj;ki} - t_{im;kj}t_{il;mk}t_{ljm;ki} \\ &\equiv_3 t_{mj}t_{ilm;kj}t_{kim;jl} + t_{il;jm}t_{kj;lm}t_{kim;jl} - t_{ilm;kj}t_{km;jl}t_{ki;mj} \\ &\qquad - t_{jk;il}t_{km}t_{lm;ik}t_{lj;mi} - t_{mj}t_{im;kj}t_{il;mk}t_{lj;ki} - t_{im;kj}t_{il;mk}t_{ljm;ki} \\ &\equiv_3 t_{mj}t_{ilm;kj}t_{kim;jl} + t_{il;jm}t_{kj;lm}t_{kim;jl} - t_{ilm;kj}t_{km;jl}t_{ki;mj} \\ &\qquad - t_{jm}t_{jk;il}t_{lm;ik}t_{lj;mi} - t_{jkm;il}t_{lm;ik}t_{lj;mi} \\ &\qquad - t_{mj}t_{im;kj}t_{il;mk}t_{lj;ki} - t_{im;kj}t_{il;mk}t_{ljm;ki} \\ &\equiv_3 t_{mj}u_{jm;kl} + t_{il;jm}t_{kj;lm}t_{kim;jl} \\ &\qquad - t_{ilm;kj}t_{km;jl}t_{ki;mj} - t_{jkm;il}t_{lm;ik}t_{lj;mi} - t_{im;kj}t_{il;mk}t_{ljm;ki}. \end{aligned}$$

□

Similar to before, we need to evaluate our derivations on the elements $t_{ijm;kl}$.

LEMMA 6.26. *The following hold.*

(1) $d_{ij}(t_{ijm;kl}) = t_{ij}t_{ijm;kl} + t_{im}t_{jim;kl} - u_{jm;kl}$.
(2) $d_{im}(t_{ijm;kl}) \equiv_1 t_{im}t_{ijm;kl} + t_{jm}t_{mji;kl} + t_{ij;kl}$.
(3) $d_{jm}(t_{ijm;kl}) \equiv_1 t_{jm}t_{ijm;kl} - t_{ij}t_{imj;kl} - t_{im;kl} - u_{im;kl}$.
(4) $d_{ik}(t_{ijm;kl}) \equiv_1 t_{ik}t_{ijm;kl} - t_{kjm;il}t_{ik} + t_{jk;il}t_{ik;jm}$.
(5) $d_{jk}(t_{ijm;kl}) \equiv_1 t_{jk}t_{ijm;kl} + t_{ikm;jl}t_{kj} - t_{im;jk}t_{ij;kl}$.
(6) $d_{mk}(t_{ijm;kl}) \equiv_1 t_{mk}t_{ijm;kl} + t_{ijk;ml}t_{km} - t_{ji;mk}t_{jm;kl} - t_{ik;ml}t_{mk;ij}$.
(7) $d_{kl}(t_{ijm;kl}) \equiv_1 0$

PROOF. (1) By the definitions we have

$$\begin{aligned}
d_{ij}(t_{ijm;kl}) &= t_{ji;kl}d_{ij}(t_{jm}) - d_{ij}(t_{im})t_{ij;kl} \\
&= t_{ji;kl}t_{jm}t_{im} - t_{im}t_{mj}t_{ij;kl} \\
&= -(t_{ijm;kl} + t_{im}t_{ij;kl})t_{im} - t_{im}t_{mj}t_{ij;kl} \\
&= -(u_{jm;kl} + t_{ji}t_{ijm;kl}) \\
&\qquad + t_{im}(t_{jim;kl} + t_{jm}t_{ji;kl}) - t_{im}t_{mj}t_{ij;kl} \\
&= -u_{jm;kl} + t_{ij}t_{ijm;kl} + t_{im}t_{jim;kl}.
\end{aligned}$$

(2) We use Lemma 6.24(2) and Lemma 6.21 to obtain

$$\begin{aligned}
d_{im}(t_{ijm;kl}) &= d_{im}(t_{ij;kl})t_{jm} + t_{mj;kl}d_{im}(t_{jm}) \\
&\qquad - d_{im}(t_{im})t_{ij;kl} + t_{im}d_{im}(t_{ij;kl}) \\
&\equiv_1 (-t_{mi}t_{ij;kl} + t_{ij}t_{jm;kl} + t_{ijm;kl})t_{jm} + t_{mj;kl}t_{jm}t_{ij} \\
&\qquad - t_{ij;kl} + t_{im}(-t_{mi}t_{ij;kl} + t_{ij}t_{jm;kl} + t_{ijm;kl}) \\
&= -t_{mi}(t_{ijm;kl} + t_{im}t_{ij;kl}) - t_{ij}t_{jm}t_{jm;kl} \\
&\qquad + t_{mi}t_{ijm;kl} + t_{jm}t_{mj;kl}t_{ji} \\
&\qquad + t_{im}t_{ij}t_{jm;kl} + t_{im}t_{ijm;kl} \\
&= t_{ij;kl} - t_{ij}t_{jm}t_{jm;kl} + t_{jm}(t_{mji;kl} + t_{mi}t_{mj;kl}) \\
&\qquad - t_{mi}t_{ij}t_{jm;kl} + t_{im}t_{ijm;kl} \\
&\equiv_1 t_{im}t_{ijm;kl} + t_{jm}t_{mji;kl} + t_{ij;kl}.
\end{aligned}$$

(3) Follows immediately from (1) and (2) using (6.6).

(4) Compute using Lemma 6.12(2)

$$\begin{aligned}
d_{ik}(t_{ijm;kl}) &= d_{ik}(t_{ij;kl})t_{jm} - d_{ik}(t_{im})t_{ij;kl} - t_{km}d_{ik}(t_{ij;kl}) \\
&\equiv_1 (t_{jk;il}t_{ik} + t_{ik}t_{ij;kl})t_{jm} - t_{im}t_{mk}t_{ij;kl} \\
&\quad - t_{km}(t_{jk;il}t_{ik} + t_{ik}t_{ij;kl}) \\
&= t_{jk;il}t_{ik;jm} + t_{jk;il}t_{jm}t_{ik} + t_{ik}(t_{ijm;kl} + t_{im}t_{ij;kl}) \\
&\quad - t_{im}t_{mk}t_{ij;kl} - t_{km}t_{jk;il}t_{ik} - t_{mk}t_{ki}t_{ij;kl} \\
&= t_{jk;il}t_{ik;jm} - (t_{kjm;il} + t_{km}t_{kj;il})t_{ik} + t_{ik}t_{ijm;kl} - t_{ki}t_{im}t_{ij;kl} \\
&\quad - t_{im}t_{mk}t_{ij;kl} - t_{km}t_{jk;il}t_{ik} - t_{mk}t_{ki}t_{ij;kl} \\
&\equiv_1 t_{jk;il}t_{ik;jm} - t_{kjm;il}t_{ik} + t_{ik}t_{ijm;kl}.
\end{aligned}$$

(5) Compute using Lemma 6.12(2)

$$\begin{aligned}
d_{jk}(t_{ijm;kl}) &= d_{jk}(t_{ij;kl})t_{jm} + t_{ik;jl}d_{jk}(t_{jm}) - t_{im}d_{jk}(t_{ij;kl}) \\
&\equiv_1 -(t_{ik;jl}t_{jk} + t_{jk}t_{ji;kl})t_{jm} + t_{ik;jl}t_{jm}t_{mk} \\
&\quad + t_{im}(t_{ik;jl}t_{jk} + t_{jk}t_{ji;kl}) \\
&\equiv_1 t_{ik;jl}t_{km}t_{kj} + t_{jk}t_{ij;kl}t_{jm} + t_{im}t_{ik;jl}t_{jk} - t_{im}t_{jk}t_{ij;kl} \\
&= t_{ikm;jl}t_{kj} + t_{jk}(t_{ijm;kl} + t_{im}t_{ij;kl}) - t_{im}t_{jk}t_{ij;kl} \\
&= t_{jk}t_{ijm;kl} + t_{ikm;jl}t_{kj} - t_{im;jk}t_{ij;kl}.
\end{aligned}$$

(6) Follows immediately from (4) and (5) using (6.6).
(7) Follows immediately from Lemma 6.12(1). □

Since we only need to evaluate derivations of $u_{jm;kl}$ once, we will just compute the derivation when it is needed.

We now proceed to finally giving a more explicit description of the relations of $\mathcal{T}_5$. We do this by successively applying the derivations on $[\![t_{lkm;ij}, t_{ik}]\!] + t_{kl;jm}t_{ij;km}$ and simplifying with the already obtained relations. We check Corollary 6.23 to see which derivations need to be applied. As mentioned earlier we do not understand the structure of the algebra and even failed to write the last relation in a somewhat presentable form.

Note that we will not explicitly mention using the lemmas regarding the commutation relations and the evaluations of derivations.

I_4. Observe

$$
\begin{aligned}
&d_{ik}([\![t_{lkm;ij}, t_{ik}]\!] + t_{kl;jm}t_{ij;km})\\
&\equiv_2 d_{ik}d_{lm}([\![t_{ij;kl}, t_{ik}]\!])\\
&\equiv_2 d_{lm}d_{ik}([\![t_{ij;kl}, t_{ik}]\!])\\
&\equiv_3 d_{lm}(t_{ik;jl}t_{ij;kl} - t_{il;jk})\\
&= -d_{ml}(t_{lj;ik})t_{ij;kl} - t_{ik;jm}d_{ml}(t_{lk;ij}) - d_{ml}(t_{li;jk})\\
&\equiv_1 -(t_{ml}t_{lj;ik} - t_{lj}t_{jm;ik} - t_{ljm;ik})t_{ij;kl}\\
&\qquad - t_{ik;jm}(t_{ml}t_{lk;ij} - t_{lk}t_{km;ij} - t_{lkm;ij})\\
&\qquad - (t_{ml}t_{li;jk} - t_{li}t_{im;jk} - t_{lim;jk})\\
&= -t_{ml}t_{ik;jl}t_{ij;kl} + t_{lj}t_{jm;ik}t_{ij;kl} + t_{ljm;ik}t_{ij;kl}\\
&\qquad + t_{jm;ik}t_{ml}t_{lk;ij} - t_{ik;jm}t_{kl}t_{km;ij} + t_{ik;jm}t_{lkm;ij}\\
&\qquad + t_{ml}t_{il;jk} + t_{li}t_{im;jk} + t_{lim;jk}\\
&\equiv_3 t_{lj}t_{jm;ik}t_{ij;kl} + t_{ljm;ik}t_{ij;kl} + (t_{jml;ik} + t_{jl}t_{jm;ik})t_{lk;ij}\\
&\qquad - (t_{ikl;jm} + t_{il}t_{ik;jm})t_{km;ij} + t_{ik;jm}t_{lkm;ij}\\
&\qquad + t_{li}t_{im;jk} + t_{lim;jk}\\
&\equiv_3 -t_{mlj;ik}t_{ij;kl} + t_{ikl;jm}t_{ij;km} + t_{ik;jm}t_{lkm;ij} + t_{lim;jk}\\
&\equiv_3 -t_{mlj;ik}t_{ij;kl} + t_{ik;jm}t_{kml;ij} + t_{iml;jk} + t_{ik;jm}t_{lkm;ij} + t_{lim;jk}\\
&\equiv_1 -t_{mlj;ik}t_{ij;kl} - t_{ik;jm}t_{mlk;ij} - t_{mli;jk}
\end{aligned}
$$

In the second to last step we used (6.8).

I_5, first part. We calculate

$$
\begin{aligned}
&d_{il}d_{ik}([\![t_{lkm;ij}, t_{ik}]\!] + t_{kl;jm}t_{ij;km})\\
&\equiv_4 d_{il}(-t_{mlj;ik}t_{ij;kl} - t_{ik;jm}t_{mlk;ij} - t_{mli;jk})\\
&= d_{li}(t_{mlj;ik})t_{ij;kl} + t_{mij;lk}d_{il}(t_{ij;lk})\\
&\qquad + d_{li}(t_{ik;jm})t_{mlk;ij} + t_{lk;jm}d_{li}(t_{mlk;ij}) + d_{li}(t_{mli;jk})\\
&\equiv_2 (t_{li}t_{mlj;ik} + t_{mij;lk}t_{il} - t_{mj;li}t_{ml;ik})t_{ij;kl}\\
&\qquad + t_{mij;lk}(t_{il}t_{ij;lk} + t_{jk}t_{ik;jl})\\
&\qquad + (t_{li}t_{ik;jm} - t_{ik}t_{kl;jm} - t_{ikl;jm})t_{mlk;ij}\\
&\qquad + t_{lk;jm}(t_{li}t_{mlk;ij} + t_{mik;lj}t_{il} - t_{mk;li}t_{ml;ij})\\
&\qquad + t_{li}t_{mli;jk} - t_{ml}t_{mil;jk} - t_{mi;jk} - u_{mi;jk}\\
&= t_{li}t_{mlj;ki}t_{lk;ji} - t_{mj;li}t_{ml;ik}t_{ij;kl}\\
&\qquad - t_{mij;kl}t_{jk}t_{ik;jl}\\
&\qquad + t_{li}t_{ik;jm}t_{mlk;ij} - t_{ik}t_{kl;jm}t_{mlk;ij} - t_{ikl;jm}t_{mlk;ij}\\
&\qquad - t_{kl;jm}t_{li}t_{mlk;ij} + t_{lk;jm}t_{mik;lj}t_{il} - t_{lk;jm}t_{mk;li}t_{ml;ij}
\end{aligned}
$$

$$
\begin{aligned}
&+ t_{li}t_{mli;jk} - t_{ml}t_{mil;jk} - t_{mi;jk} - u_{mi;jk}\\
\equiv_4\ & t_{li}(-t_{mj;ki}t_{mlk;ji} - t_{mli;jk}) - t_{mj;li}t_{ml;ik}t_{ij;kl}\\
&- (t_{ml}t_{mij;kl} + t_{mj;kl}t_{mi;jk})t_{ik;jl}\\
&+ t_{li}t_{ik;jm}t_{mlk;ij} - t_{ik}t_{kl;jm}t_{mlk;ij} - t_{ikl;jm}t_{mlk;ij}\\
&- (t_{ki}t_{kl;jm} + t_{kli;jm})t_{mlk;ij}\\
&+ t_{lk;jm}(t_{kj}t_{mik;lj} - t_{mi;kj}t_{ik;lj}) - t_{lk;jm}t_{mk;li}t_{ml;ij}\\
&+ t_{li}t_{mli;jk} - t_{ml}t_{mil;jk} - t_{mi;jk} - u_{mi;jk}\\
\equiv_4\ & -t_{mj;li}t_{ml;ik}t_{ij;kl} - t_{ml}(-t_{mj;kl}t_{mik;jl} - t_{mil;jk})\\
&- t_{mj;kl}t_{mi;jk}t_{ik;jl} + t_{lik;jm}t_{mlk;ij}\\
&+ t_{lm}t_{lk;jm}t_{mik;lj} - t_{lk;jm}t_{mi;kj}t_{ik;lj}\\
&- t_{lk;jm}t_{mk;li}t_{ml;ij} + u_{im;jk} - t_{ml}t_{mil;jk} + t_{im;jk}\\
\equiv_4\ & -t_{mj;li}t_{ml;ik}t_{ij;kl} + t_{lik;jm}t_{mlk;ij}\\
&- t_{lk;jm}t_{mk;li}t_{ml;ij} + u_{im;jk} + t_{im;jk}\\
\equiv_3\ & -t_{mj;li}t_{ml;ik}t_{ij;kl}\\
&- (u_{jk;im} + t_{li;mj}t_{ik;lm}t_{ij;kl} + t_{lk;mj}t_{li;km}t_{ml;ij})\\
&- t_{lk;jm}t_{mk;li}t_{ml;ij} + u_{im;jk} + t_{im;jk}\\
=\ & u_{im;jk} + t_{im;jk} - u_{jk;im}.
\end{aligned}
$$

We used (6.9) in the second to last step.

Observe that due to Lemma 6.25(1) this implies

$$
[\![u_{jm;kl}, t_{ik}]\!] \equiv_5 -t_{lki;jm}. \tag{6.10}
$$

I_5, second part. We consider

$$
\begin{aligned}
&d_{jl}d_{ik}([\![t_{lkm;ij}, t_{ik}]\!] + t_{kl;jm}t_{ij;km})\\
&\equiv_4 d_{jl}(-t_{mlj;ik}t_{ij;kl} - t_{ik;jm}t_{mlk;ij} - t_{mli;jk})\\
&= d_{lj}(t_{mlj;ik})t_{ij;kl} - t_{mjl;ik}d_{jl}(t_{ji;lk})\\
&\quad - d_{lj}(t_{jm;ik})t_{mlk;ij} - t_{ik;lm}d_{lj}(t_{mlk;ji}) + d_{lj}(t_{mli;jk})\\
&\equiv_2 (t_{lj}t_{mlj;ik} - t_{ml}t_{mjl;ik} - t_{mj;ik} - u_{mj;ik})t_{ij;kl}\\
&\quad - t_{mjl;ik}(t_{il;jk}t_{jl} + t_{jl}t_{ji;lk})\\
&\quad - (t_{lj}t_{jm;ik} - t_{jm}t_{ml;ik} - t_{jml;ik})t_{mlk;ij}\\
&\quad - t_{ik;lm}(t_{lj}t_{mlk;ji} + t_{mjk;li}t_{jl} - t_{mk;lj}t_{ml;ji})\\
&\quad + t_{lj}t_{mli;jk} + t_{mji;lk}t_{jl} - t_{mi;lj}t_{ml;jk}\\
&= t_{lj}t_{mlj;ki}t_{lk;ji} - t_{ml}t_{mjl;ik}t_{ji;lk} - t_{mj;ik}t_{ij;kl}\\
&\quad - u_{mj;ik}t_{ij;kl} + t_{mjl;ki}t_{jk;li}t_{jl} - t_{lm}t_{mjl;ik}t_{ji;lk}\\
&\quad - t_{lj}t_{jm;ik}t_{mlk;ij} + t_{jm}t_{ml;ik}t_{mlk;ij} - t_{jml;ik}t_{mlk;ji}
\end{aligned}
$$

$$
\begin{aligned}
&- t_{ml;ik}t_{lj}t_{mlk;ji} - t_{ik;lm}t_{mjk;li}t_{jl} + t_{ik;lm}t_{mk;lj}t_{ml;ji}\\
&+ t_{lj}t_{mli;jk} + t_{mji;lk}t_{jl} - t_{mi;lj}t_{ml;jk}\\
\equiv_4{}& t_{lj}(-t_{mj;ki}t_{mlk;ji} - t_{mli;jk}) - t_{mj;ik}t_{ij;kl}\\
&- u_{mj;ik}t_{ij;kl} + (-t_{ml;ki}t_{mjk;li} - t_{mji;lk})t_{jl}\\
&- t_{lj}t_{jm;ik}t_{mlk;ij} + t_{jm}t_{ml;ik}t_{mlk;ij} - t_{jml;ik}t_{mlk;ji}\\
&- (t_{mj}t_{ml;ik} + t_{mlj;ik})t_{mlk;ji} - t_{ik;lm}t_{mjk;li}t_{jl}\\
&+ t_{ik;lm}t_{mk;lj}t_{ml;ji}\\
&+ t_{lj}t_{mli;jk} + t_{mji;lk}t_{jl} - t_{mi;lj}t_{ml;jk}\\
={}& -t_{mj;ik}t_{ij;kl} - u_{mj;ik}t_{ij;kl} + t_{ljm;ik}t_{mlk;ji}\\
&+ t_{ik;lm}t_{mk;lj}t_{ml;ji} - t_{mi;lj}t_{ml;jk}\\
\equiv_5{}& u_{ik;mj}t_{ij;kl} + t_{ljm;ik}t_{mlk;ji}\\
&+ t_{ik;lm}t_{mk;lj}t_{ml;ji} - t_{mi;lj}t_{ml;jk}
\end{aligned}
$$

I_6. We first compute

$$
\begin{aligned}
d_{ij}(u_{im;jk}) &= d_{ij}(t_{lim;jk}t_{lm} - t_{il}t_{lim;jk})\\
&= d_{ij}(t_{lim;jk})t_{lm} - d_{ij}(t_{il})t_{lim;jk} - t_{jl}d_{ij}(t_{lim;jk})\\
&\equiv_2 (t_{ij}t_{lim;jk} + t_{ljm;ik}t_{ji} - t_{lm;ij}t_{li;jk})t_{lm} - t_{il}t_{lj}t_{lim;jk}\\
&\quad - t_{jl}(t_{ij}t_{lim;jk} + t_{ljm;ik}t_{ji} - t_{lm;ij}t_{li;jk})\\
&\equiv_1 (-t_{lj}t_{ji} - t_{il}t_{lj} - t_{ji}t_{il})t_{lim;jk} + t_{ij}u_{im;jk}\\
&\qquad + t_{lm;ij}(-t_{li;jk}t_{lm} + t_{im}t_{li;jk})\\
&\qquad + t_{ljm;ik}(t_{ji}t_{lm} - t_{lm}t_{ji}) + u_{jm;ik}t_{ji}\\
&\equiv_1 t_{ij}u_{im;jk} + t_{lm;ij}t_{ilm;jk} + t_{ljm;ik}t_{ji;lm} + u_{jm;ik}t_{ji},
\end{aligned}
$$

and obtain

$$
\begin{aligned}
&d_{ij}d_{il}d_{ik}([\![t_{lkm;ij}, t_{ik}]\!] + t_{kl;jm}t_{ij;km})\\
&\equiv_5 d_{ij}(u_{im;jk} + t_{im;jk} - u_{jk;im})\\
&= d_{ij}(u_{im;jk}) + d_{ji}(u_{jk;im}) + d_{ij}(t_{im;jk})\\
&\equiv_1 t_{ij}u_{im;jk} + t_{lm;ij}t_{ilm;jk} + t_{ljm;ik}t_{ji;lm} + u_{jm;ik}t_{ji}\\
&\qquad + t_{ji}u_{jk;im} + t_{lk;ji}t_{jlk;im} + t_{lik;jm}t_{ij;lk} + u_{ik;jm}t_{ij}\\
&\qquad + t_{mj;ik}t_{ij} + t_{ij}t_{im;jk}\\
&\equiv_5 t_{lm;ij}t_{ilm;jk} + t_{ljm;ik}t_{ji;lm} + t_{lk;ji}t_{jlk;im} + t_{lik;jm}t_{ij;lk}.
\end{aligned}
$$

Moreover we calculate

$$
\begin{aligned}
&t_{ljm;ik}t_{ji;lm} + t_{lik;jm}t_{ij;lk} \\
&= t_{ljm;ik}t_{ji}t_{lm} - t_{ljm;ik}t_{lm}t_{ji} + t_{lik;jm}t_{ij}t_{lk} - t_{lik;jm}t_{lk}t_{ij} \\
&\equiv_3 (t_{mk}t_{ljm;ik} - t_{lj;mk}t_{jm;ik})t_{lm} - (t_{jl}t_{ljm;ik} + u_{jm;ik})t_{ji} \\
&\qquad + (t_{km}t_{lik;jm} - t_{li;km}t_{ik;jm})t_{lk} - (t_{il}t_{lik;jm} + u_{ik;jm})t_{ij} \\
&\equiv_3 t_{mk}(t_{jl}t_{ljm;ik} + u_{jm;ik}) + t_{lj;mk}(t_{jl}t_{jm;ik} + t_{jml;ik}) \\
&\qquad - t_{jl}(t_{mk}t_{ljm;ik} - t_{lj;mk}t_{jm;ik}) \\
&\qquad + t_{km}(t_{il}t_{lik;jm} + u_{ik;jm}) + t_{li;km}(t_{il}t_{ik;jm} + t_{ikl;jm}) \\
&\qquad - t_{il}(t_{km}t_{lik;jm} - t_{li;km}t_{ik;jm}) + (u_{jm;ik} - u_{ik;jm})t_{ij} \\
&\equiv_0 t_{mk;jl}t_{ljm;ik} + t_{lj;mk}t_{jml;ik} + t_{km;il}t_{lik;jm} + t_{li;km}t_{ikl;jm} \\
&\qquad + t_{mk}(u_{jm;ik} - u_{ik;jm}) + (u_{jm;ik} - u_{ik;jm})t_{ij} \\
&\equiv_5 -t_{mk;jl}t_{mlj;ik} - t_{km;il}t_{kli;jm} + t_{mk}t_{ik;jm} + t_{ik;jm}t_{ij} \\
&\equiv_2 -t_{mk;jl}t_{mlj;ik} - t_{km;il}t_{kli;jm},
\end{aligned}
$$

and so finally obtain

$$
\begin{aligned}
&d_{ij}d_{il}d_{ik}([\![t_{lkm;ij}, t_{ik}]\!] + t_{kl;jm}t_{ij;km}) \\
&\quad\equiv_5 t_{lm;ij}t_{ilm;jk} + t_{lk;ji}t_{jlk;im} - t_{mk;jl}t_{mlj;ik} - t_{km;il}t_{kli;jm} \\
&\quad= t_{lm;ij}t_{ilm;jk} + t_{lk;ji}t_{jlk;im} + t_{lj;mk}t_{mlj;ki} + t_{li;km}t_{kli;mj}.
\end{aligned}
$$

I_7. Observe

$$
\begin{aligned}
&d_{ik}d_{ij}d_{il}d_{ik}([\![t_{lkm;ij}, t_{ik}]\!] + t_{kl;jm}t_{ij;km}) \\
&\equiv_6 d_{ik}(t_{lm;ij}t_{ilm;jk} + t_{lk;ji}t_{jlk;im} - t_{mk;jl}t_{mlj;ik} - t_{km;il}t_{kli;jm}) \\
&\equiv_1 d_{ki}(t_{ij;lm})t_{ilm;jk} + t_{lm;jk}d_{ik}(t_{ilm;kj}) \\
&\qquad - d_{ik}(t_{ij;kl})t_{jlk;im} + t_{li;jk}d_{ki}(t_{lkj;im}) + t_{li;jk}d_{ki}(t_{kjl;im}) \\
&\qquad + d_{ik}(t_{km;lj})t_{mlj;ki} + t_{im;jl}d_{ik}(t_{mlj;ik}) \\
&\qquad + d_{ik}(t_{il;km})t_{kli;jm} + t_{lk;im}d_{ki}(t_{kli;jm}) \\
&\equiv_1 (t_{ki}t_{ij;lm} - t_{ij}t_{jk;lm} - t_{ijk;lm})t_{ilm;jk} \\
&\qquad + t_{lm;jk}(t_{ik}t_{ilm;kj} + t_{lk;ij}t_{ik;lm} - t_{klm;ij}t_{ik}) \\
&\qquad + (t_{li;jk}t_{ki} + t_{ik}t_{ij;kl})t_{jlk;im} \\
&\qquad + t_{li;jk}(t_{ki}t_{lkj;im} + t_{lij;km}t_{ik} - t_{lj;ki}t_{lk;im}) \\
&\qquad + t_{li;jk}(t_{ki}t_{kjl;im} + t_{ji;km}t_{ki;jl} + t_{ijl;km}t_{ik}) \\
&\qquad + (t_{ik}t_{km;lj} - t_{km}t_{mi;lj} - t_{kmi;lj})t_{mlj;ki} \\
&\qquad + (t_{lk;im}t_{ik} + t_{ik}t_{il;km})t_{kli;jm} \\
&\qquad + t_{lk;im}(t_{ki}t_{kli;jm} + t_{li}t_{ilk;jm} + t_{kl;jm})
\end{aligned}
$$

$$
\begin{aligned}
&\equiv_3 t_{li;jk}t_{ki}(t_{lkj;im}+t_{kjl;im}+t_{jlk;im})\\
&\qquad+(t_{ki}t_{ij;lm}-t_{ij}t_{jk;lm}-t_{ijk;lm})t_{ilm;jk}\\
&\qquad+t_{ji}t_{jk;lm}t_{ilm;kj}+t_{jki;lm}t_{ilm;kj}+t_{lm;jk}t_{lk;ij}t_{ik;lm}\\
&\qquad+t_{mk}t_{lm;jk}t_{klm;ij}+t_{lm;jk}t_{kl;ij}t_{lm;ki}-t_{lm;jk}t_{km;lj}t_{kl;ij}\\
&\qquad+t_{ik}t_{ij;kl}t_{jlk;im}-t_{km}t_{kj;li}t_{jli;km}\\
&\qquad-t_{kjm;li}t_{jli;km}-t_{li;jk}t_{ji;km}t_{jl;ik}\\
&\qquad-t_{li;jk}t_{lj;ki}t_{lk;im}+t_{li;jk}t_{ji;km}t_{ki;jl}\\
&\qquad+(t_{ik}t_{km;lj}-t_{km}t_{mi;lj}-t_{kmi;lj})t_{mlj;ki}\\
&\qquad+t_{ml}t_{lk;im}t_{kli;jm}+t_{ik}t_{il;km}t_{kli;jm}\\
&\qquad+t_{lm}t_{lk;im}t_{kli;jm}+t_{km}t_{lk;im}t_{ilk;jm}+t_{lk;im}t_{kl;jm}\\
&\equiv_1 t_{ki}(t_{ij;lm}t_{ilm;jk}-t_{ij;kl}t_{jlk;im}-t_{km;lj}t_{mlj;ki}-t_{il;km}t_{kli;jm})\\
&+t_{km}(-t_{lm;jk}t_{klm;ij}-t_{kj;li}t_{jli;km}-t_{mi;lj}t_{mlj;ki}+t_{lk;im}t_{ilk;jm})\\
&-t_{ijk;lm}t_{ilm;jk}+t_{jki;lm}t_{ilm;kj}-t_{lm;jk}t_{km;lj}t_{kl;ij}\\
&-t_{kjm;li}t_{jli;km}-t_{li;jk}t_{lj;ki}t_{lk;im}-t_{kmi;lj}t_{mlj;ki}+t_{lk;im}t_{kl;jm}\\
&\equiv_6 t_{kij;lm}t_{ilm;jk}+t_{kjm;li}t_{jli;mk}+t_{kmi;lj}t_{mlj;ik}\\
&+t_{lk;im}t_{lk;mj}+t_{lk;mj}t_{lk;ji}+t_{lk;ji}t_{lk;im}.
\end{aligned}
$$

I_8. We consider

$$
\begin{aligned}
&d_{ij}d_{ik}d_{ij}d_{il}d_{ik}([\![t_{lkm;ij},t_{ik}]\!]+t_{kl;jm}t_{ij;km})\\
&\equiv_7 d_{ij}(t_{kij;lm}t_{ilm;jk}+t_{kjm;li}t_{jli;mk}+t_{kmi;lj}t_{mlj;ik}\\
&\qquad+t_{lk;im}t_{lk;mj}+t_{lk;mj}t_{lk;ji}+t_{lk;ji}t_{lk;im})\\
&=d_{ij}(t_{kij;lm})t_{ilm;jk}+t_{kji;lm}d_{ij}(t_{ilm;jk})\\
&\qquad+d_{ji}(t_{kjm;il})t_{jli;mk}-t_{kim;lj}d_{ji}(t_{jli;mk})\\
&\qquad-d_{ij}(t_{kmi;jl})t_{mlj;ik}-t_{kmj;li}d_{ji}(t_{mlj;ik})\\
&\qquad+d_{ji}(t_{im;lk})t_{lk;mj}+t_{lk;jm}d_{ij}(t_{jm;lk})\\
&\qquad+d_{ij}(t_{jm;lk})t_{lk;ji}+t_{lk;mi}d_{ij}(t_{lk;ji})\\
&\qquad-d_{ij}(t_{lk;ij})t_{lk;im}-t_{lk;ji}d_{ji}(t_{im;lk})\\
&\equiv_1 (t_{ij}t_{kij;lm}-t_{ki}t_{kji;lm}-t_{kj;lm}-u_{kj;lm})t_{ilm;jk}\\
&\qquad+t_{kji;lm}(t_{lj;ik}t_{ij;lm}-t_{jlm;ik}t_{ij}+t_{ij}t_{ilm;jk})\\
&\qquad+(t_{ji}t_{kjm;il}+t_{kim;jl}t_{ij}-t_{km,ji}t_{kj;il})t_{jli;mk}\\
&\qquad-t_{kim;lj}(t_{ji}t_{jli;mk}+t_{li}t_{ilj;mk}+t_{jl;mk})\\
&\qquad-(t_{ij}t_{kmi;jl}+t_{kmj;il}t_{ji}+t_{mk;ij}t_{mi;jl}-t_{kj;il}t_{ij;km})t_{mlj;ik}\\
&\qquad-t_{kmj;li}(t_{ji}t_{mlj;ik}+t_{mli;jk}t_{ij}+t_{lm;ji}t_{lj;ik}-t_{mi;jk}t_{ji;ml})
\end{aligned}
$$

$$
\begin{aligned}
&\quad + (t_{ji}t_{im;lk} - t_{im}t_{mj;lk} - t_{imj;lk})t_{lk;mj}\\
&\quad + t_{lk;jm}(t_{ij}t_{jm;lk} - t_{jm}t_{mi;lk} - t_{jmi;lk})\\
&\quad + (t_{ij}t_{jm;lk} - t_{jm}t_{mi;lk} - t_{jmi;lk})t_{lk;ji}\\
&\quad - t_{lk;ji}(t_{ji}t_{im;lk} - t_{im}t_{mj;lk} - t_{imj;lk})\\
&\equiv_3 (t_{ij}t_{kij;lm} - t_{ki}t_{kji;lm} - t_{kj;lm} - u_{kj;lm})t_{ilm;jk}\\
&\quad + t_{kji;lm}t_{lj;ik}t_{ij;lm} + t_{kji;lm}(t_{lk}t_{jlm;ik} + t_{jl;ik}t_{lm;ji} - t_{jm;lk}t_{jl;ik})\\
&\quad + t_{ki}t_{kji;lm}t_{ilm;jk}\\
&\quad + t_{ji}t_{kjm;il}t_{jli;mk} - t_{km;ji}t_{kj;il}t_{jli;mk}\\
&\quad + t_{mj}t_{kim;lj}t_{ilj;mk} - t_{ki;mj}t_{im;lj}t_{ilj;mk} - t_{kim;lj}t_{jl;mk}\\
&\quad - (t_{ij}t_{kmi;jl} + t_{mk;ij}t_{mi;jl} - t_{kj;il}t_{ij;km})t_{mlj;ik}\\
&\quad - t_{kmj;li}t_{mk}t_{mli;jk} - t_{kmj;li}t_{mi;jk}t_{ml;ij}\\
&\quad - t_{kmj;li}(t_{lm;ji}t_{lj;ik} - t_{mi;jk}t_{ji;ml})\\
&\quad + (t_{ji}t_{im;lk} - t_{im}t_{mj;lk} - t_{imj;lk})t_{lk;mj}\\
&\quad - t_{mi}t_{mj;lk}t_{jm;lk} - t_{mji;lk}t_{jm;lk}\\
&\quad - t_{mj}t_{lk;jm}t_{mi;lk} - t_{lk;jm}t_{jmi;lk}\\
&\quad + (t_{ij}t_{jm;lk} - t_{jm}t_{mi;lk} - t_{jmi;lk})t_{lk;ji}\\
&\quad - t_{ij}t_{lk;ji}t_{im;lk} + t_{lk;ji}t_{imj;lk}\\
&\quad - t_{jm}t_{ji;lk}t_{mj;lk} - t_{jim;lk}t_{mj;lk}\\
&= t_{ij}(t_{kij;lm}t_{ilm;jk} + t_{kjm;li}t_{jli;mk} + t_{kmi;lj}t_{mlj;ik}\\
&\quad + t_{lk;im}t_{lk;mj} + t_{lk;mj}t_{lk;ji} + t_{lk;ji}t_{lk;im})\\
&\quad + t_{mj}(t_{kim;lj}t_{ilj;mk} + t_{lk;jm}t_{lk;mi} + t_{lk;mi}t_{lk;ij} + t_{lk;ij}t_{lk;jm}\\
&\quad + t_{kji;lm}t_{jlm;ik} + t_{kmj;li}t_{mli;jk})\\
&\quad - t_{kj;lm}t_{ji;kl}t_{jlm;ik} + t_{ki;jm}t_{kj;lm}t_{jlm;ik} - u_{kj;li}t_{mli;jk}\\
&\quad - t_{kj;lm}t_{ilm;jk} - u_{kj;lm}t_{ilm;jk} - t_{kji;lm}t_{jm;lk}t_{jl;ik}\\
&\quad - t_{km;ji}t_{kj;il}t_{jli;mk} - t_{ki;mj}t_{im;lj}t_{ilj;mk} - t_{kim;lj}t_{jl;mk}\\
&\quad - t_{mk;ij}t_{mi;jl}t_{mlj;ik} + t_{kj;il}t_{ij;km}t_{mlj;ik} - t_{kmj;li}t_{lm;ji}t_{lj;ik}\\
&\quad - t_{lk;jm}t_{jmi;lk} - t_{jmi;lk}t_{lk;ji} + t_{lk;ji}t_{imj;lk}\\
&\quad + (-t_{imj;lk}t_{lk;mj} - t_{mji;lk}t_{lk;mj} + t_{jim;lk}t_{lk;mj}).
\end{aligned}
$$

Furthermore we compute

$$
\begin{aligned}
t_{ijm;kl}t_{ik;ml} &\equiv_1 -t_{jmi;lk}t_{ml;ik} - t_{mij;lk}t_{ml;ik}\\
&\equiv_4 t_{ji;lk}t_{jml;ik} + t_{jmk;il} - t_{mi;lk}t_{ikj;ml} + t_{mkj;il}\\
&\equiv_1 t_{ji;lk}t_{jml;ik} - t_{mi;lk}t_{ikj;ml} - t_{kjm;il},
\end{aligned}
$$

which yields

$$\begin{aligned}
&t_{kmj;li}t_{ml;ji}t_{lj;ik}\\
&\equiv_4 (-t_{kj;li}t_{kml;ji} - t_{kmi;jl})t_{lj;ik}\\
&= t_{kj;li}t_{kml;ij}t_{ki;lj} - t_{kmi;jl}t_{lj;ik}\\
&\equiv_4 t_{kj;li}(t_{mk;ji}t_{mlj;ki} - t_{lk;ji}t_{kim;lj} - t_{iml;kj}) - t_{kmi;jl}t_{lj;ik}\\
&\equiv_3 t_{kj;li}t_{mk;ji}t_{mlj;ki} + t_{ki;lj}t_{kim;lj} - t_{kj;li}t_{iml;kj} - t_{kmi;jl}t_{lj;ik},
\end{aligned}$$

and similarly

$$\begin{aligned}
&- t_{kji;lm}t_{kl;jm}t_{jl;ik}\\
&\equiv_3 -(-t_{kj;lm}t_{jmi;lk} - t_{kmi;jl})t_{jl;ik}\\
&\equiv_4 t_{kj;lm}(t_{mj;kl}t_{mik;jl} - t_{ij;kl}t_{jlm;ik} - t_{lmi;jk}) + t_{kmi;jl}t_{jl;ik}\\
&\equiv_3 t_{km;lj}t_{mik;jl} - t_{kj;lm}t_{ij;kl}t_{jlm;ik} - t_{kj;lm}t_{lmi;jk} + t_{kmi;jl}t_{jl;ik}.
\end{aligned}$$

Thus we get

$$\begin{aligned}
&d_{ij}d_{ik}d_{ij}d_{il}d_{ik}([\![t_{lkm;ij}, t_{ik}]\!] + t_{kl;jm}t_{ij;km})\\
&\equiv_7 t_{ki;jm}(t_{kj;lm}t_{jlm;ik} + t_{im;lj}t_{ilj;mk})\\
&\qquad + t_{km;ji}(t_{kj;li}t_{jli;mk} + t_{mi;lj}t_{mlj;ik})\\
&\qquad - u_{lm;kj}t_{ilm;jk} + t_{lm;kj}t_{lmi;jk}\\
&\qquad - u_{li;kj}t_{mli;jk} + t_{li;kj}t_{lim;jk}\\
&\qquad + t_{kim;jl}t_{jl;mk} + t_{lj;ki}t_{kim;jl}\\
&\qquad + t_{jmi;lk}t_{lk;ij} + t_{kl;jm}t_{jmi;lk}\\
&\qquad + t_{lj;mk}t_{mik;jl} + 2t_{kmi;jl}t_{jl;ik}\\
&\qquad + t_{kl;ij}t_{imj;lk} + 2t_{jim;lk}t_{lk;mj}.
\end{aligned}$$

Moreover we have

$$\begin{aligned}
t_{ij;kl}t_{ijm;kl} &= t_{ij;kl}t_{ij;kl}t_{jm} - t_{ij;kl}t_{im}t_{ij;kl}\\
&= t_{ij;kl}^2t_{jm} - t_{jm}t_{ij;kl}^2 + t_{jim;kl}t_{ij;kl},
\end{aligned}$$

and thus

$$t_{kl;jm}t_{jmi;lk} + t_{lm;kj}t_{lmi;jk} \equiv_3 t_{mji;lk}t_{kl;jm} + t_{mli;jk}t_{lm;kj}$$

because of $t_{kl;jm}^2 \equiv_3 t_{lm;kj}^2$. In the same way obtain

$$t_{li;kj}t_{lim;jk} + t_{lj;ki}t_{kim;jl} \equiv_3 t_{ikm;lj}t_{jl;ki} + t_{ilm;kj}t_{li;jk}.$$

This finally yields

$$
\begin{aligned}
&d_{ij}d_{ik}d_{ij}d_{il}d_{ik}(\llbracket t_{lkm;ij}, t_{ik}\rrbracket + t_{kl;jm}t_{ij;km})\\
&\equiv_7 t_{ki;jm}(t_{kj;lm}t_{jlm;ik} + t_{im;lj}t_{ilj;mk})\\
&\qquad + t_{km;ji}(t_{kj;li}t_{jli;mk} + t_{mi;lj}t_{mlj;ik})\\
&\qquad - u_{lm;kj}t_{ilm;jk}\\
&\qquad - u_{li;kj}t_{mli;jk}\\
&\qquad + t_{mji;lk}t_{kl;jm} + t_{mli;jk}t_{lm;kj}\\
&\qquad + t_{ikm;lj}t_{jl;ki} + t_{ilm;kj}t_{li;jk}\\
&\qquad + t_{kim;jl}t_{jl;mk} + t_{jmi;lk}t_{lk;ij}\\
&\qquad + t_{lj;mk}t_{mik;jl} + t_{kmi;jl}t_{jl;ik}\\
&\qquad - t_{mik;jl}t_{jl;ik} - t_{ikm;jl}t_{jl;ik}\\
&\qquad + t_{kl;ij}t_{imj;lk} + t_{jim;lk}t_{lk;mj}\\
&\qquad - t_{imj;lk}t_{lk;mj} - t_{mji;lk}t_{lk;mj}\\
&\equiv_3 t_{ki;jm}(t_{kj;lm}t_{jlm;ik} + t_{im;lj}t_{ilj;mk})\\
&\qquad + t_{km;ji}(t_{kj;li}t_{jli;mk} + t_{mi;lj}t_{mlj;ik})\\
&\qquad - u_{lm;kj}t_{ilm;jk} + t_{kim;jl}t_{jl;mk} + t_{mli;jk}t_{lm;kj} + t_{jim;lk}t_{lk;mj}\\
&\qquad - u_{li;kj}t_{mli;jk} + t_{kmi;jl}t_{jl;ik} + t_{ilm;jk}t_{li;kj} + t_{jmi;lk}t_{lk;ij}\\
&\qquad + t_{mik;jl}t_{ik;jl} + t_{km;lj}t_{mik;jl}\\
&\qquad + t_{imj;lk}t_{mj;lk} + t_{ji;kl}t_{imj;lk}
\end{aligned}
$$

I_9. We currently do not have a presentable version of the final relation.

We collect the results.

COROLLARY 6.27. *The defining relations of $\mathcal{T}_5$ are*

$$t_{ij}^2 - 1, \tag{I_0}$$

$$t_{ij}t_{jk} + t_{jk}t_{ki} + t_{ki}t_{ij}, \tag{I_1}$$

$$\llbracket t_{ij;kl}, t_{ik}\rrbracket, \tag{I_2}$$

$$t_{ik;jl}t_{ij;kl} - t_{il;jk}, \tag{I_3}$$

$$\llbracket t_{lkm;ij}, t_{ik}\rrbracket + t_{kl;jm}t_{ij;km}, \tag{I_3}$$

$$t_{mlj;ik}t_{ij;kl} + t_{ik;jm}t_{mlk;ij} + t_{mli;jk}, \tag{I_4}$$

$$u_{im;jk} + t_{im;jk} - u_{jk;im}, \tag{I_5}$$

$$u_{ik;mj}t_{ij;kl} + t_{ljm;ik}t_{mlk;ji} + t_{ik;lm}t_{mk;lj}t_{ml;ji} - t_{mi;lj}t_{ml;jk}, \tag{I_5}$$

$$t_{lm;ij}t_{ilm;jk} + t_{lk;ji}t_{jlk;im} + t_{lj;mk}t_{mlj;ki} + t_{li;km}t_{kli;mj}, \tag{I_6}$$

$$t_{kij;lm}t_{ilm;jk} + t_{kjm;li}t_{jli;mk} + t_{kmi;lj}t_{mlj;ik}$$

$$+ t_{lk;im}t_{lk;mj} + t_{lk;mj}t_{lk;ji} + t_{lk;ji}t_{lk;im}, \qquad (I_7)$$

$$d_{ij}(t_{kij;lm}t_{ilm;jk} + t_{kjm;li}t_{jli;mk} + t_{kmi;lj}t_{mlj;ik} + t_{lk;im}t_{lk;mj} + t_{lk;mj}t_{lk;ji} + t_{lk;ji}t_{lk;im}), \qquad (I_8)$$

$$d_{im}d_{ij}(t_{kij;lm}t_{ilm;jk} + t_{kjm;li}t_{jli;mk} + t_{kmi;lj}t_{mlj;ik} + t_{lk;im}t_{lk;mj} + t_{lk;mj}t_{lk;ji} + t_{lk;ji}t_{lk;im}), \qquad (I_9)$$

with $i, j, k, l, m \in \{1, \ldots, 5\}$ *pairwise distinct.*

The "nicest" version we have of the second to last relation is

$$\begin{aligned}
&t_{ki;jm}(t_{kj;lm}t_{jlm;ik} + t_{im;lj}t_{ilj;mk}) \\
&\quad + t_{km;ji}(t_{kj;li}t_{jli;mk} + t_{mi;lj}t_{mlj;ik}) \\
&\quad - u_{lm;kj}t_{ilm;jk} + t_{kim;jl}t_{jl;mk} + t_{mli;jk}t_{lm;kj} + t_{jim;lk}t_{lk;mj} \\
&\quad - u_{li;kj}t_{mli;jk} + t_{kmi;jl}t_{jl;ik} + t_{ilm;jk}t_{li;kj} + t_{jmi;lk}t_{lk;ij} \\
&\quad + t_{mik;jl}t_{ik;jl} - t_{km;jl}t_{mik;jl} \\
&\quad + t_{imj;kl}t_{mj;kl} - t_{ji;kl}t_{imj;kl}.
\end{aligned}$$

Observe that the second and fourth lines are merely the first respectively third lines with the roles of i and m swapped. Similarly, we get the sixth line from the fifth by acting with $(im)(jk)$.

7. Examples for the Trivial Component

In this section we consider $(\mathcal{D}_S)_{()}$ for the graphs D_n and the circle. We will often take $\mathbb{C}$ as our ground field for convenience.

In Section 9.1 we will very briefly discuss the same object for a few select graphs over the field $\mathbb{F}_2$.

7.1. D_n**.** We will use very well known statements on Clifford algebras in this section, see for example [**Bou59**].

Assume $\operatorname{char} \Bbbk \neq 2$ and $n \geq 4$. Let

$$S = D_n = \{(1,3), (i, i+1) \,|\, 2 \leq i \leq n-1\}.$$

We wish to determine the structure of $(\mathcal{D}_S)_{()}$.

For this let $X = -(y_{13}y_{23}y_{13}y_{34})^2$. Furthermore, for any $4 \leq i \leq n$ let $w_i = \prod_{j=4}^{i-1} y_{j,j+1} = y_{45}y_{56}\cdots y_{i-1,i}$ and put $f_i = w_i^{-1}Xw_i$, for $4 \leq i \leq n$. We have the following lemma.

Lemma 7.1. (1) *The elements* f_i *with* $4 \leq i \leq n$ *generate* $(\mathcal{D}_S)_{()}$ *as an algebra.*
(2) $f_i^2 + f_i + 1 = 0$ *for all* $4 \leq i \leq n$.
(3) $(f_jf_i)^2 = -1$ *for all* $4 \leq i, j \leq n$ *with* $i \neq j$.

Since the proof of this statement is very similar (albeit not identical since we don't divide out -1 here) to the treatment in Section 9.3 we will skip it here.

Let $G = (g_{ij})_{1\leq i,j\leq n-3} \in \Bbbk^{(n-3)\times(n-3)}$ with $g_{ii} = -3$ and $g_{ij} = -1$ for $1 \leq i,j \leq n-3$ with $i \neq j$. Note that the matrix G is non degenerate if and only if $n-1 \neq 0$ in $\Bbbk$.

LEMMA 7.2. *The trivial component $(\mathcal{D}_S)_{()}$ is as an algebra isomorphic to a quotient of the Clifford algebra associated to the Gramian matrix G.*

PROOF. For $1 \leq i \leq n-3$ let $X_i = 2f_{i+3} + 1$. Then by Lemma 7.1 we obviously have

$$\begin{aligned} X_i^2 &= -3, \text{ and} \\ X_iX_j + X_jX_i &= -2. \end{aligned}$$

Since $X_1, \ldots X_{n-3}$ generate $(\mathcal{D}_S)_{()}$ as an algebra ($\operatorname{char}\Bbbk \neq 2$) the claim follows. $\square$

For convenience we restrict ourselves to $\Bbbk = \mathbb{C}$.

PROPOSITION 7.3. *We have*

$$(\mathcal{D}_S(\mathbb{C}))_{()} \cong \begin{cases} \mathbb{C}^{2^m\times 2^m} & \text{if } 2m = n-3, \\ \mathbb{C}^{2^m\times 2^m} \times \mathbb{C}^{2^m\times 2^m} & \text{if } 2m+1 = n-3. \end{cases}$$

as $\mathbb{C}$-algebras. In particular, $\dim_{\mathbb{C}} \mathcal{D}_S(\mathbb{C}) = n!2^{n-3}$.

PROOF. Follows immediately from the classification of complex Clifford algebras, Lemma 7.2, and the fact that $(\mathcal{D}_{D_n}(\mathbb{C}))_{()}$ is a subalgebra of $(\mathcal{D}_{D_{n+1}}(\mathbb{C}))_{()}$. $\square$

7.2. The Circle. The treatment here is related to [**BLM13**, Sec. 7]. Let $n \geq 3$. Let

$$S = \{(i,i+1),(n,1) \mid 1 \leq i \leq n-1\}.$$

Then S is a circle. As we did in Section 3.2 we will adopt some convenient notation. We will allow the indices of our elements $y_{ij} \in \mathcal{D}_S$ to range over all integers and then consider them modulo n. Furthermore, we will write y_i for $y_{i,i+1}$ for $i \in \mathbb{Z}$. Recall the notation $m_i = y_iy_{i+1}\cdots y_{i-2}$ for $i \in \mathbb{Z}$.

We wish to investigate the subalgebra $(\mathcal{D}_S)_{()}$. For this introduce the notation

$$T_j = m_jm_1^{-1}$$

for $2 \leq j \leq n$.

It follows from the results later in Section 9.4 that the elements $T_2, \ldots, T_n$ generate $(\mathcal{D}_S)_{()}$ as an algebra. Later in Corollary 9.15 we will also see that the elements $T_2, \ldots, T_n$ commute. Furthermore it follows from Lemma 9.13 that we can turn $(\mathcal{D}_S)_{()}$ into a $\Bbbk\mathbb{S}_{n-1}$-module algebra by letting $\sigma(T_j) = T_{\sigma(j)}$ for $\sigma \in \mathbb{S}_{n-1}$, $2 \leq j \leq n$ (for convenience let $\mathbb{S}_{n-1}$ operate on $2, \ldots, n$).

Let us assume in the next proposition that $\Bbbk = \mathbb{C}$. We show that $(\mathcal{D}_S)_{()}$ is a deformation of the coinvariant algebra. For this recall Proposition 3.13 and our notation for the elementary symmetric polynomials.

We will only need to use the dimension of the coinvariant algebra, for this see for example [**Art44**].

PROPOSITION 7.4. *Let $I \subseteq \mathbb{C}[x_1, \ldots, x_{n-1}]$ be the ideal generated by $e_m - (-1)^m$ for $1 \leq m \leq n-1$. Then*

$$(\mathcal{D}_S(\mathbb{C}))_{()} \cong \mathbb{C}[x_1, \ldots, x_{n-1}]/I$$

as algebras. In particular, $\dim_{\mathbb{C}} \mathcal{D}_S(\mathbb{C}) = n!(n-1)!$.

PROOF. Observe that with the natural filtration the associated graded algebra of $\mathbb{C}[x_1, \ldots, x_{n-1}]/I$ is a quotient of the coinvariant algebra. Since the coinvariant algebra has dimension $(n-1)!$ this implies

$$\dim_{\mathbb{C}} \mathbb{C}[x_1, \ldots, x_{n-1}]/I \leq (n-1)!.$$

Let now $J \subseteq \mathbb{C}[y_1, \ldots, y_{n-1}]$ be the ideal in $\mathbb{C}[y_1, \ldots, y_{n-1}]$ such that $\mathbb{C}[y_1, \ldots, y_{n-1}]/J \cong (\mathcal{D}_S)_{()}$ by mapping y_j to T_{j+1} for $1 \leq j \leq n-1$. By Proposition 3.13 we have $I \subseteq J$. Hence $e_m - (-1)^m \in J$ for $1 \leq m \leq n-1$, so it follows by Vieta's formulas that the variety of J is contained in

$$\{(\zeta^{k_1}, \ldots, \zeta^{k_{n-1}}) \mid 1 \leq k_1, \ldots, k_{n-1} \leq n-1 \text{ pairwise distinct}\}, \tag{7.1}$$

where ζ is a primitive n-th root of unity. On the other hand, due to the statements above each permutation of $y_1, \ldots, y_{n-1}$ yields an algebra isomorphism of $\mathbb{C}[y_1, \ldots, y_{n-1}]/J$. Hence for each element in the variety of J any permutation of the entries yields again an element in the variety of J. However, we have $(\mathcal{D}_S)_{()} \neq 0$ and thus the variety of J is non empty by the Nullstellensatz. Applying all permutations in $\mathbb{S}_{n-1}$ to one element of (7.1) in the variety of J yields $(n-1)!$ distinct elements in the variety of J. Hence $\dim_{\mathbb{C}} \mathbb{C}[y_1, \ldots, y_{n-1}]/J = (n-1)!$ and therefore $\mathbb{C}[y_1, \ldots, y_{n-1}]/J \cong \mathbb{C}[y_1, \ldots, y_{n-1}]/I$, which is the claim. $\square$

CHAPTER 4

Groups Attached to Fomin-Kirillov Algebras

In this chapter we attach a group $\mathcal{G}_S$ to a Fomin-Kirillov algebra $\mathcal{E}_S$. It turns out that finiteness of the group is related to finite dimensionality of the Fomin-Kirillov algebra.

In Section 8 we give the definition, basic properties, and discuss some relations of the group, which for $n \leq 4$ turn out to be defining relations.

In Section 9 we consider examples. Encouraged by the results we suggest a tentative strategy to tackle infinite dimensionality of $\mathcal{E}_6$ based on [**FLZ01**]. The most important tool for the results in this section are the algorithms outlined in [**BHLGO15**].

8. Groups Attached to Subsets of T

8.1. Definition and Basic Properties. For $S \subseteq T$ we define the group $\tilde{\mathcal{G}}_S$ to be the subgroup of $\mathrm{Aut}_{\Bbbk}(\mathcal{E}_S)$ generated by the elements $y_s \in \mathcal{D}_S$ for $s \in S$ and $-\mathrm{id}$. We will often consider the quotient $\mathcal{G}_S$ of $\tilde{\mathcal{G}}_S$ by the normal subgroup generated by $-\mathrm{id}$. For $s = (i, j) \in S$ we will write $z_{ij} = z_s$ for the class in $\mathcal{G}_S$ represented by y_s.

The following lemma follows directly from the definitions.

LEMMA 8.1. *Let $R, S \subseteq T$ with $R \subseteq S$. Then $\tilde{\mathcal{G}}_R$ is a subgroup of $\tilde{\mathcal{G}}_S$. The inclusion map induces an injective group homomorphism $\mathcal{G}_R \to \mathcal{G}_S$.*

The following proposition allows us to relate finite dimensionality of $\mathcal{E}_S$ to finiteness of $\mathcal{G}_S$.

PROPOSITION 8.2.
(1) *If $\mathcal{E}_S$ is an infinite dimensional vector space, then $\mathcal{G}_S$ is an infinite group.*
(2) *Assume $\#\Bbbk < \infty$. If $\mathcal{E}_S$ is a finite dimensional vector space, then $\mathcal{G}_S$ is a finite group.*

PROOF. (1) If $\mathcal{E}_S$ is an infinite dimensional vector space there is a monomial $x_k = x_{s_1} \cdots x_{s_k}, s_1, \ldots, s_k \in S$ with $x_k \neq 0$ for any $k \in \mathbb{N}$. Let $y_k = y_{s_k} \cdots y_{s_1} \in \tilde{\mathcal{G}}_S$. Then $y_k(1) = x_k + \tilde{x}_k$ where $\tilde{x}_k$ is of degree at most $k - 1$. This implies $y_k \neq y_l$ if $k \neq l$. Hence $\tilde{\mathcal{G}}_S$ and thus also $\mathcal{G}_S$ are infinite.

(2) If $\mathcal{E}_S$ is a finite dimensional vector space, $\mathrm{Aut}_{\Bbbk}(\mathcal{E}_S)$ is a finite group. Hence $\tilde{\mathcal{G}}_S$ is finite as a subgroup of a finite group. □

Note that the groups defined above in principle depend on the ground field $\Bbbk$. In fact, if $\mathrm{char}\,\Bbbk = 2$ then $\tilde{\mathcal{G}}_S = \mathcal{G}_S$ for all S. However, in all the examples we considered we were not able to find a dependency of $\mathcal{G}_S$ on the ground field $\Bbbk$. This gives rise to the following conjectures.

CONJECTURE 8.3. *The group $\mathcal{G}_S$ does not depend on the ground field $\Bbbk$.*

If this conjecture turns out to be true, 8.2 will immediately imply the following.

CONJECTURE 8.4. *The algebra $\mathcal{E}_S$ is finite dimensional if and only if $\mathcal{G}_S$ is a finite group.*

There is a homomorphism from $\mathcal{G}_S$ to the symmetric group $\mathbb{S}_n$.

PROPOSITION 8.5. *Let $S \subseteq T$. There is a unique group homomorphism $\tilde{\mathcal{G}}_S \to \mathbb{S}_n$ mapping $y_s \mapsto s$ for $s \in S$ and $-\mathrm{id}$ to 1. This homomorphism factorizes through $\mathcal{G}_S$. This homomorphism is surjective if and only if S is a connected graph on n vertices.*

PROOF. Uniqueness is clear, since $\tilde{\mathcal{G}}_S$ is generated by y_s, $s \in S$, and $-\mathrm{id}$. We need to show that this assignment in fact defines a homomorphism. To this end let $y = y_{s_1} \cdots y_{s_k} = \mathrm{id}$ or $-\mathrm{id}$ with $s_1, \dots, s_k \in S$. Then $y(1) = 1$ or -1. On the other hand, $y(1)$ is a $\mathbb{S}_n$-homogeneous element of $\mathcal{E}_S$ of degree $s_k \cdots s_1$ by Remark 2.3. This implies $s_k \cdots s_1 = 1$ and thus $s_1 \cdots s_k = 1$, hence we indeed have a well defined homomorphism. That the homomorphism factorizes through $\mathcal{G}_S$ is obvious, as is the last claim. □

We will denote the homomorphism $\mathcal{G}_S \to \mathbb{S}_n$ from Proposition 8.5 by γ. We will denote the kernel of γ by $\mathcal{N}_S$.

REMARK 8.6. The *modified shift groups* considered in [**Loc13**] are essentially the same as our groups $\tilde{\mathcal{G}}_S$ if we take S to be the complete graph on $3, 4$, or 5 vertices. Analogues of Proposition 8.2 and Proposition 8.5 in this setting were already proved there. See also Remark 2.3.

8.2. Some Relations in $\mathcal{G}_S$. We derive some relations among the elements y_t that follow from Lemma 2.4. If our graph happens to have 4 vertices or fewer, the relations given in this section are in fact all the defining relations of the group.

LEMMA 8.7. (1) *Let $i, j \in \{1, \dots, n\}$. If $i \neq j$ then $y_{ij}^2 = 1$.*

(2) *Let* $i, j, k, l \in \{1, \dots, n\}$ *with* $\#\{i, j, k, l\} = 4$. *Then*

$$(y_{ij} y_{kl})^2 = 1.$$

(3) *Let* $i, j, k \in \{1, \dots, n\}$ *with* $\#\{i, j, k\} = 3$. *Then*

$$(y_{ij} y_{jk})^3 = 1, \qquad (y_{ji} y_{jk})^3 = -1.$$

(4) *Let* $i, j, k \in \{1, \dots, n\}$ *with* $\#\{i, j, k\} = 3$. *Then*

$$(y_{ij} y_{jk} y_{ik})^2 = 1.$$

(5) *Let* $i, j, k, l \in \{1, \dots, n\}$ *with* $\#\{i, j, k, l\} = 4$. *Then*

$$(y_{ij} y_{ik} y_{il})^4 = -1.$$

(6) *Let* $i, j, k, l \in \{1, \dots, n\}$ *with* $\#\{i, j, k, l\} = 4$. *Then*

$$(y_{ij} y_{jk} y_{kl} y_{li})^3 = -1.$$

PROOF. (1) and (2) were proved in Lemma 2.4.
(3) and (4) were proved in Lemma 3.1.
(5)

$$\begin{aligned} y_{ik} y_{il} y_{ij} &= y_{ik}(y_{ij} y_{jl} + y_{jl} y_{li}) \\ &= (y_{ij} y_{jk} + y_{jk} y_{ki}) y_{jl} + y_{jl} y_{ik} y_{li} \\ &= y_{ij} y_{jk} y_{jl} - y_{jk} y_{jl} y_{ik} - y_{jl} y_{ik} y_{il}. \end{aligned}$$

We multiply this equation with y_{ij} from the left and with $y_{ik} y_{il}$ from the right. Since

$$y_{ik} y_{il} y_{ik} y_{il} = -y_{il} y_{ik} y_{il}^2 = -y_{il} y_{ik}$$

by Lemma 8.7(1), we conclude that

$$\begin{aligned} (y_{ij} y_{ik} y_{il})^2 &= y_{ij} y_{ik} y_{il} y_{ij} y_{ik} y_{il} \\ &= y_{jk} y_{jl} y_{ik} y_{il} - y_{ij} y_{jk} y_{jl} y_{il} + y_{ij} y_{jl} y_{il} y_{ik} \\ &= -y_{jk} y_{ki} y_{jl} y_{il} - y_{ij} y_{jk} y_{jl} y_{il} + y_{ij} y_{jl} y_{il} y_{ik} \\ &= y_{ki} y_{ij} y_{jl} y_{il} + y_{ij} y_{jl} y_{il} y_{ik}. \end{aligned} \tag{8.1}$$

By interchanging l and j, and using Lemma 8.7(2) and (8.1) we obtain that

$$\begin{aligned} -(y_{il} y_{ik} y_{ij})^2 &= -y_{il} y_{ik} y_{ij} y_{il} y_{ik} y_{ij} \\ &= -y_{ki} y_{il} y_{lj} y_{ij} - y_{il} y_{lj} y_{ij} y_{ik} \\ &= -y_{ki} y_{ij} y_{lj} y_{il} - y_{ij} y_{lj} y_{il} y_{ik} \\ &= y_{ki} y_{ij} y_{jl} y_{il} + y_{ij} y_{lj} y_{il} y_{ik} \\ &= (y_{ij} y_{ik} y_{il})^2. \end{aligned}$$

This the claim.
(6) See Example 5.3. □

COROLLARY 8.8. *Let* $i, j, k, l \in \{1, \dots n\}$. *If* $\#\{i, j, k, l\} = 4$ *then* $(y_{ij}y_{ik}y_{ij}y_{il})^2 = (y_{il}y_{ij}y_{il}y_{ik})^2$.

PROOF. Using Lemma 8.7(1),(3) we conclude that

$$\begin{aligned} y_{ij}y_{ik}y_{ij}y_{il}(y_{ij}y_{ik}y_{ij})y_{il} &= -y_{ij}y_{ik}y_{ij}y_{il}y_{ik}y_{ij}y_{ik}y_{il} \\ &= -y_{ij}(y_{ik}y_{ij}y_{il}y_{ik}y_{ij}y_{il})y_{il}y_{ik}y_{il} \\ &= (y_{ij}y_{il}y_{ij})y_{ik}y_{il}y_{ij}(y_{ik}y_{il}y_{ik})y_{il} \\ &= y_{il}y_{ij}y_{il}y_{ik}y_{il}y_{ij}y_{il}y_{ik}(y_{il}y_{il}) \\ &= y_{il}y_{ij}y_{il}y_{ik}y_{il}y_{ij}y_{il}y_{ik}. \end{aligned}$$

This completes the proof. □

These relations in $\tilde{\mathcal{G}}_S$ imply relations in $\mathcal{G}_S$. Recall that for $s \in S$ we write z_s for the class in $\mathcal{G}_S$ represented by y_s.

LEMMA 8.9. *The generators* z_t *of* $\mathcal{G}_T$, *where* $t \in T$, *satisfy the following relations.*

(1) *Let* $i, j \in \{1, \dots, n\}$. *If* $i \neq j$ *then* $z_{ij}^2 = 1$.
(2) *Let* $i, j, k, l \in \{1, \dots, n\}$ *with* $\#\{i, j, k, l\} = 4$. *Then*
$$(z_{ij}z_{kl})^2 = 1.$$
(3) *Let* $i, j, k \in \{1, \dots, n\}$ *with* $\#\{i, j, k\} = 3$. *Then*
$$(z_{ij}z_{ik})^3 = 1.$$
(4) *Let* $i, j, k \in \{1, \dots, n\}$ *with* $\#\{i, j, k\} = 3$. *Then*
$$(z_{ij}z_{jk}z_{ki})^2 = 1.$$
(5) *Let* $i, j, k, l \in \{1, \dots, n\}$ *with* $\#\{i, j, k, l\} = 4$. *Then*
$$(z_{ij}z_{ik}z_{il})^4 = 1.$$
(6) *Let* $i, j, k, l \in \{1, \dots, n\}$ *with* $\#\{i, j, k, l\} = 4$. *Then*
$$(z_{ij}z_{jk}z_{kl}z_{li})^3 = 1.$$

8.3. $n \leq 4$**.** We consider connected graphs on at most 4 vertices. In this subsection we assume $\operatorname{char} \Bbbk \in \{0, 2\}$. Our work here is largely computational and was done using GAP [**GAP16**].

For Groebner basis computations we used the package GBNP [**CK16**] implemented in GAP [**GAP16**].

PROPOSITION 8.10. *Let* S *be a graph on at most* 4 *vertices. Then the relations in Lemma 8.9 that apply are defining relations of* $\mathcal{G}_S$.

PROOF. Our proof of this is computational. For each graph S we can use computer calculations to obtain the order of the group given by the relations of Lemma 8.9. On the other hand, we have a description of $\mathcal{E}_S$ by generators and relations and can use Proposition 2.6 to get a presentation of $\mathcal{G}_S$ (or $\tilde{\mathcal{G}}_S$) as a matrix group and then calculate its order. Since the orders agree in all cases we are done. $\square$

REMARK 8.11. If we have 5 or more vertices we know that the relations given in Lemma 8.9 do not suffice. For example, the relation

$$(z_{12}z_{23}z_{34}z_{45}z_{51})^4 = 1,$$

which we have seen in Example 5.6, is needed if the graph contains a circle on 5 vertices.

We do not have a complete description of the groups $\mathcal{G}_S$ in terms of generators and relations if $n \geq 5$.

We get another proposition with a computational proof.

PROPOSITION 8.12. *Let S be a graph on at most 4 vertices. Then $\mathcal{G}_S$ is a solvable group.*

Of course we already know that the opposite is true for connected graphs on 5 or more vertices due to Proposition 8.5.

9. Examples

In this section we consider examples.

We focus our attention on computing the kernel $\mathcal{N}_S$ of the homomorphism $\mathcal{G}_S \to \mathbb{S}_n$ from Proposition 8.5.

While the computational results are summarized in the Appendix, we also mention what we consider to be the most important examples in Subsection 9.1.

In the end we determine the group $\mathcal{N}_S$ for the graph D_n (9.3) and approximate it for the circle (9.4).

9.1. $n \geq 5$**.** In this section we collect our results on the structure of $\mathcal{N}_S$ for some examples on 5 or more vertices. In the Appendix one can find a summary of the results. In this section we want to briefly discuss the examples where sporadic simple groups (or something close) occur. We use ATLAS [**CCN$^+$85**] notation.

For all the results in this section we work with $\Bbbk = \mathbb{F}_2$. We want to briefly explain our strategy. We consider examples where have a description of $\mathcal{E}_S$ by generators and relations over $\mathbb{F}_2$. Proposition 2.6 then allows us to obtain a presentation of $\mathcal{N}_S$ as a matrix group with entries in $\mathbb{F}_2$. We then use the methods for composition trees outlined in

[**BHLGO15**] which are implemented in MAGMA [**BCP97**] to obtain our results.

For Groebner basis computations we used the package GBNP [**CK16**] implemented in GAP [**GAP16**].

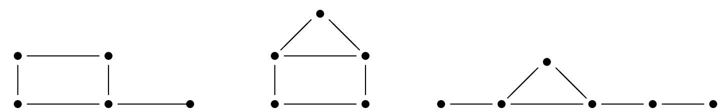


FIGURE 1. The graphs S where $\mathcal{N}_S$ is close to a sporadic simple group.

PROPOSITION 9.1. *Let $S = K_{4,5}$, the graph to the left in Figure 1. Then*

(1) $(\mathcal{D}_S)_{()} \cong \mathbb{F}_4^{6\times 6}$ *as $\mathbb{F}_2$-algebras.*
(2) $\mathcal{N}_S \cong J_2$ *as groups, where J_2 denotes the Hall-Janko group of order* $604800 = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7$.

PROOF. Computational.

(1) can be checked using the MEATAXE [**Par84**] as implemented in GAP [**GAP16**].

(2) can be checked as outlined above. □

Similarly we have.

PROPOSITION 9.2. *Let S be the graph in the middle in Figure 1. Then*

(1) $(\mathcal{D}_S)_{()} \cong \mathbb{F}_2^{24\times 24}$ *as $\mathbb{F}_2$-algebras.*
(2) $\mathcal{N}_S \cong Co_1$ *as groups, where Co_1 denotes the Conway group of order* $4157776806543360000 = 2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$.

PROPOSITION 9.3. *Let $S = H_6$, the graph to the right in Figure 1. Then*

(1) $(\mathcal{D}_S)_{()} \cong \mathbb{F}_4^{12\times 12}$ *as $\mathbb{F}_2$-algebras.*
(2) $\mathcal{N}_S \cong 3.Suz$ *as groups, where Suz denotes the Suzuki group of order* $448345497600 = 2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$.

REMARK 9.4. We do not know a proof of the statements in this section that works without using computers. However, we believe one should consider the results presented here in the context of [**FLZ01**]. There, the absolutely irreducible subgroups G of some $GL_n(\mathbb{F}_{2^k})$ which are generated by the conjugacy class g^G of some element $g \in G$ of order 3 with some technical assumption are classified. The groups described in this section appear naturally in exactly this way.

A basic strategy to prove that $\mathcal{E}_6$ is infinite dimensional would be to search for a graph S on 6 vertices where finite dimensionality of $\mathcal{E}_S$ would imply that $\mathcal{N}_S$ falls into this classification aswell. Then one could try to rule out all the possibilties in the classification in a case by case analysis.

Note however that even for several examples on 5 vertices the group $\mathcal{N}_S$ doesn't seem to fall into this framework.

REMARK 9.5. It is not always the case that the groups $\mathcal{N}_S$ are close to simple groups. For example, if we take S to be the star on 5 vertices the order of $\mathcal{N}_S$ is $2^{50}3^4$ and the group is solvable by Burnside's theorem. Check also the Appendix for all the Chief series we were able compute.

9.2. A_n**.** Assume that $n \geq 2$. Let

$$S = A_n = \{(i, i+1) \,|\, 1 \leq i \leq n-1\}.$$

Then there is a unique group homomorphism $f : \mathbb{S}_n \to \mathcal{G}_S$ such that $f(s) = z_s$ for all $s \in S$. Indeed, consider the presentation of $\mathbb{S}_n$ as a Coxeter group with generators $s \in S$. By Lemma 8.9, the generators z_s, $s \in S$, satisfy appropriate Coxeter relations, such that the map f is a well-defined surjective group homomorphism. The composition of this map with the group homomorphism $\mathcal{G}_S \to \mathbb{S}_n$ in Proposition 8.5 is the identity on the generators. Therefore f is also injective, hence an isomorphism. In particular, $\mathcal{N}_S = 1$.

9.3. D_n**.** In this section we will denote with $\triangleright$ the left adjoint action in $\mathcal{G}_S$. Assume that $n \geq 4$. Let

$$S = D_n = \{(1,3), (i, i+1) \,|\, 2 \leq i \leq n-1\}.$$

We determine $\mathcal{N}_S$. Let $X = (z_{13}z_{23}z_{13}z_{34})^2$. Then $X = (z_{23}z_{13}z_{23}z_{34})^2$ by Lemma 8.9(3), and

$$X = (z_{34}z_{13}z_{34}z_{23})^2 = (z_{34}z_{23}z_{34}z_{13})^2$$

by Corollary 8.8. We conclude that

$$\begin{aligned} X^2 &= (z_{13}z_{23}z_{13}z_{34})^2(z_{34}z_{13}z_{34}z_{23})^2 \\ &= z_{13}z_{23}z_{13}z_{34}z_{13}z_{23}z_{34}z_{23}z_{34}z_{13}z_{34}z_{23} \\ &= z_{13}z_{23}z_{13}z_{34}z_{13}z_{34}z_{23}z_{13}z_{34}z_{23} \\ &= z_{13}z_{23}z_{34}z_{13}z_{23}z_{13}z_{34}z_{23} \\ &= z_{13}z_{23}z_{34}z_{23}z_{13}z_{23}z_{34}z_{23}. \end{aligned}$$

Since $z_{23}z_{34}z_{23} = z_{34}z_{23}z_{34}$, we conclude that $X^2 = X^{-1}$. Moreover,

$$\begin{aligned} z_{13}Xz_{13} &= z_{13}(z_{13}z_{23}z_{13}z_{34})^2 z_{13} = (z_{23}z_{13}z_{34}z_{13})^2 = X^{-1}, \\ z_{23}Xz_{23} &= z_{23}(z_{23}z_{13}z_{23}z_{34})^2 z_{23} = (z_{13}z_{23}z_{34}z_{23})^2 = X^{-1}, \\ z_{34}Xz_{34} &= z_{34}(z_{34}z_{13}z_{34}z_{23})^2 z_{34} = (z_{13}z_{34}z_{23}z_{34})^2 = X^{-1}. \end{aligned}$$

The degree four summands of $y_{13}y_{23}y_{13}y_{34}(1)$ and $y_{34}y_{13}y_{23}y_{13}(1)$ are non-zero since $\partial_{34}(x_{34}x_{13}x_{23}x_{13}) = x_{13}x_{23}x_{13} \neq 0$ in $\mathcal{E}_4$. Since they have different $\mathbb{S}_n$-degrees this implies $y_{13}y_{23}y_{13}y_{34} \neq \pm y_{34}y_{13}y_{23}y_{13}$ and hence $X \neq 1$.

It is clear that $z_{13}z_{23}z_{13}, z_{23}, z_{34}, \dots, z_{n-1,n}$ generate $\mathcal{G}_S$. The homomorphism in Proposition 8.5 maps $z_{13}z_{23}z_{13}$, z_{23}, $z_{34}, \dots, z_{n-1,n}$ to the transpositions $(1\,2)$, $(2\,3)$, $(3\,4), \dots, (n-1\,n)$, respectively. Moreover, the generators $z_{13}z_{23}z_{13}, z_{23}, z_{34}, \dots, z_{n-1,n}$ satisfy all appropriate Coxeter relations except for $((z_{13}z_{23}z_{13})z_{34})^2 = 1$. Thus the kernel $\mathcal{N}_S$ of the homomorphism in Proposition 8.5 is the smallest normal subgroup of $\mathcal{G}_S$ containing X.

To better understand $\mathcal{N}_S$ let us first prove the following technical lemma.

Lemma 9.6. *In $\mathcal{G}_S$ the relation*

$$z_{34}z_{45}Xz_{45}z_{34} = X^{-1}z_{45}Xz_{45}$$

holds.

Proof. We first record that

$$\begin{aligned} Xz_{34}z_{45}X &= z_{13}z_{23}z_{13}z_{34}(z_{13}z_{23}z_{13}z_{45}z_{13}z_{23}z_{13})z_{34}z_{13}z_{23}z_{13}z_{34} \\ &= z_{13}z_{23}z_{13}(z_{34}z_{45}z_{34})z_{13}z_{23}z_{13}z_{34} \\ &= (z_{13}z_{23}z_{13}z_{45})z_{34}(z_{45}z_{13}z_{23}z_{13})z_{34} \\ &= z_{45}(z_{13}z_{23}z_{13}z_{34}z_{13}z_{23}z_{13})z_{45}z_{34} \\ &= z_{45}Xz_{34}z_{45}z_{34}. \end{aligned}$$

From this we easily conclude that

$$\begin{aligned} z_{34}z_{45}Xz_{45}z_{34} &= z_{34}z_{45}X(z_{45}z_{34}z_{45})z_{45} \\ &= z_{34}(z_{45}Xz_{34}z_{45}z_{34})z_{45} \\ &= (z_{34}Xz_{34})z_{45}Xz_{45} \\ &= X^{-1}z_{45}Xz_{45}, \end{aligned}$$

which is the claim. $\square$

For any $4 \leq i \leq n$ let $w_i = \prod_{j=4}^{i-1} z_{j,j+1} = z_{45}z_{56}\cdots z_{i-1,i}$. We will use the previous Lemma to derive some formulas for the adjoint action $\triangleright$ on the elements $f_i = w_i^{-1}Xw_i$, $4 \leq i \leq n$.

LEMMA 9.7. *Let* $4 \le i \le n$. *Then*

(1) *Let* $j \in \{1,2\}$. *Then* $z_{j3} \triangleright f_i = f_i^{-1}$.
(2) $z_{34} \triangleright f_i = f_4^{-1} f_i$ *if* $i > 4$, *and*
$z_{34} \triangleright f_4 = f_4^{-1}$.
(3) $z_{i,i+1} \triangleright f_i = f_{i+1}$, *if* $i < n$,
$z_{i-1,i} \triangleright f_i = f_{i-1}$, *if* $i > 4$, *and*
$z_{j,j+1} \triangleright f_i = f_i$ *if* $4 \le j < n$ *and* $j \ne i, i-1$.

PROOF. (1) Since z_{j3} commutes with w_i it is clear from the statements above that

$$z_{j3} \triangleright f_i = z_{j3} w_i^{-1} X w_i z_{j3} = w_i^{-1} z_{j3} X z_{j3} w_i = w_i^{-1} X^2 w_i = f_i^{-1}.$$

(2) If $4 < i$ the element $w_i' = z_{45} w_i = z_{56} \cdots z_{i-1,i}$ commutes with z_{34} and X. In this case we therefore obtain from Lemma 9.6

$$\begin{aligned} z_{34} \triangleright f_i &= z_{34} f_i z_{34} \\ &= z_{34} (w_i')^{-1} z_{45} X z_{45} w_i' z_{34} \\ &= (w_i')^{-1} (z_{34} z_{45} X z_{45} z_{34}) w_i' \\ &= (w_i')^{-1} X^{-1} z_{45} X z_{45} w_i' \\ &= X^{-1} w_i^{-1} X w_i \\ &= f_4^{-1} f_i. \end{aligned}$$

The equation $z_{34} \triangleright f_4 = z_{34} X z_{34} = X^{-1} = f_4^{-1}$ has been proved above already.

(3) It is clear from the definition that $w_i z_{i,i+1} = w_{i+1}$ if $i < n$, therefore $z_{i,i+1} \triangleright f_i = f_{i+1}$. Similarly $w_i z_{i-1,i} = w_{i-1}$ and therefore $z_{i-1,i} \triangleright f_i = f_{i-1}$ is clear if $i > 4$. If $j > i$ the element $z_{j,j+1}$ clearly commutes with both X and w_i, hence we obtain $z_{j,j+1} \triangleright f_i = f_i$. If $j < i-1$ we use the braid relation among $z_{j+1,j+2}$ and $z_{j,j+1}$ to see $w_i z_{j,j+1} = z_{j+1,j+2} w_i$ which implies the last claim. □

COROLLARY 9.8. *The normal subgroup* $\mathcal{N}_S$ *of* $\mathcal{G}_S$ *is as a group generated by the elements* f_i, $4 \le i \le n$.

In the next lemma we determine relations between the elements f_i, $4 \le i \le n$.

LEMMA 9.9. *In the group* $\mathcal{G}_S$ *the following relations hold.*

(1) $f_i^3 = 1$ *for all* $4 \le i \le n$.
(2) $(f_j f_i)^2 = 1$ *for all* $4 \le i, j \le n$ *with* $i \ne j$.

PROOF. (1) We already know $X^2 = X^{-1}$. Thus the statement follows immediately since each f_i is a conjugate of X.

(2) We only show $(f_j f_i)^2 = 1$ for $j > i$, the other case follows by conjugating with z_{13} or z_{23}. First by Lemma 9.7 observe that for $k \geq 2$

$$\begin{aligned} z_{i+k-1,i+k} \triangleright (f_{i+k-1} f_i)^2 &= ((z_{i+k-1,i+k} \triangleright f_{i+k-1}) \cdot (z_{i+k-1,i+k} \triangleright f_i))^2 \\ &= (f_{i+k} f_i)^2, \end{aligned}$$

hence it suffices to show the claim for $j = i + 1$. For $i \geq 5$ we have similarly

$$z_{i-1,i} z_{i,i+1} \triangleright (f_i f_{i-1})^2 = (f_{i+1} f_i)^2,$$

so it suffices to show the claim for $j = 5, i = 4$. For this observe $(z_{23} z_{34})^3 = 1$ and obtain that $(z_{23} z_{34})^3 \triangleright f_5 = f_5$. Again using 9.7 we get

$$\begin{aligned} f_5 &= (z_{23} z_{34})^3 \triangleright f_5 \\ &= z_{23} z_{34} z_{23} z_{34} z_{23} \triangleright f_4^{-1} f_5 \\ &= z_{23} z_{34} z_{23} z_{34} \triangleright f_4 f_5^{-1} \\ &= z_{23} z_{34} z_{23} \triangleright f_4^{-1} f_5^{-1} f_4 \\ &= z_{23} z_{34} \triangleright f_4 f_5 f_4^{-1} \\ &= z_{23} \triangleright f_4^{-1} f_4^{-1} f_5 f_4 \\ &= f_4 f_4 f_5^{-1} f_4^{-1} \end{aligned}$$

and hence $f_5 = f_4^{-1} f_5^{-1} f_4^{-1}$. Thus $(f_5 f_4)^2 = 1$ which concludes the proof. □

In order to identify the group $\mathcal{N}_S$ we use a result of Carmichael.

LEMMA 9.10. [**Car56**, Ch. VII,§IV] *Let $n \geq 3$ and let G be the group given by generators g_i, $3 \leq i \leq n$, and relations g_i^3, $1 \leq i \leq 3$, and $(g_i g_j)^2$, $3 \leq i < j \leq n$. Then there is an isomorphism $\varphi : G \to \mathbb{A}_n$ with $\varphi(g_i) = (1\,2\,i)$ for all $3 \leq i \leq n$.*

Using the Lemma above we can determine the structure of $\mathcal{N}_S$.

PROPOSITION 9.11. *The group $\mathcal{N}_S$ is isomorphic to the alternating group $\mathbb{A}_{n-1}$.*

PROOF. By the relations from Lemma 9.9 and Lemma 9.10 we know that $\mathcal{N}_S$ is isomorphic to a quotient of $\mathbb{A}_{n-1}$. Further $\mathcal{N}_S$ is not trivial since $X \neq 1$. For $n \neq 5$ this implies the claim since in this case $\mathbb{A}_{n-1}$ is simple. For $n = 5$ the claim follows since we can interpret the group $\mathcal{N}_{D_5}$ as a subgroup of $\mathcal{N}_{D_6}$ by Lemma 8.1. □

REMARK 9.12. Recall the notation

$$D'_n = \{(1,2)\} \cup D_n$$

from Section 5.5. We have seen that $\varphi_1(\mathcal{D}_{D_n})$ is as an algebra isomorphic to $\mathcal{D}_{D'_{n-1}}$. Since of course

$$(\mathcal{D}_S)_{()} = (\varphi_1(\mathcal{D}_S))_{()}$$

for all $S \subseteq T$ we can deduce

$$\mathcal{N}_{D'_{n-1}} \cong \mathcal{N}_{D_n} \cong \mathbb{A}_{n-1}.$$

9.4. The Circle. In this section we will again denote with $\triangleright$ the left adjoint action in $\mathcal{G}_S$.

Let $n \geq 3$. Let

$$S = \{(i, i+1), (n,1) \mid 1 \leq i \leq n-1\}.$$

Then S is a circle. We will use the same conventions and notations as we did in the latter part of Section 3.2 and Section 7.2. We write z_j for the class of $\mathcal{G}_S$ represented by $y_j = y_{j,j+1}$ for $j \in \mathbb{Z}$. Abusing notation T_j will denote the class in $\mathcal{G}_S$ represented by T_j for $2 \leq j \leq n$, as will m_i denote the class in $\mathcal{G}_S$ represented by m_i for $i \in \mathbb{Z}$. Note that everything done in this section would also work if we replaced $\mathcal{G}_S$ by $\tilde{\mathcal{G}}_S$, the only difference being that we would need to add the extra generator -1 to the generators of our normal subgroup. This observation is important for Section 7.2.

Note that

$$T_2 = (z_2 z_3 \cdots z_{n-1} z_n z_{n-1} \cdots z_2) z_1$$

by the definition from Section 7.2.

The group $\mathcal{G}_S$ is generated by $T_2, z_2, \ldots, z_n$. Those generators are mapped to $(), (2,3), (3,4), \ldots, (n,1)$ under the homomorphism of Proposition 8.5. Since $z_2, \ldots, z_n$ fulfill the appropriate Coxeter relations for $\mathbb{S}_n$ it follows that the element T_2 generates $\mathcal{N}_S$ as a normal subgroup of $\mathcal{G}_S$.

Let us evaluate the adjoint action on the elements $T_2, \ldots, T_n$.

LEMMA 9.13. *Let $1 \leq j \leq n$ and $1 < k < n$. The following hold.*

(1) $z_1 \triangleright T_2 = T_2^{-1}$ *and* $z_1 \triangleright T_j = T_j T_2^{-1}$ *if* $j > 2$.
(2) $z_n \triangleright T_n = T_n^{-1}$ *and* $z_n \triangleright T_j = T_j T_n^{-1}$ *if* $j < n$.
(3) $z_k \triangleright T_j = T_{j^{(k,k+1)}}$.

PROOF. We rely on Lemma 3.8.

(1) Observe

$$\begin{aligned} z_1 \triangleright T_2 &= z_1 m_2 m_1^{-1} z_1 \\ &= m_1 z_n z_n m_2^{-1} \\ &= T_2^{-1}. \end{aligned}$$

Similarly for $j > 2$

$$\begin{aligned} z_1 T_j &= z_1 m_j m_1^{-1} z_1 \\ &= m_j z_n z_n m_2^{-1} \\ &= (m_j m_1^{-1})(m_1 m_2^{-1}) \\ &= T_j T_2^{-1}. \end{aligned}$$

(2) The situation is symmetric in 2 and n.
(3) Follows with a similar argument as above from the equations

$$\begin{aligned} &z_k m_k = m_{k+1} z_{k-1}, \quad z_k m_{k+1} = m_k z_{k-1}, \\ &z_k m_j = m_j z_{k-1} \text{ for } j \notin \{k, k+1\}, \\ &m_1^{-1} z_k = z_{k-1} m_1^{-1}. \end{aligned}$$

□

Corollary 9.14. *$\mathcal{N}_S$ is as a group generated by $T_2, \ldots, T_n$.*

Corollary 9.15. *The elements $T_2, \ldots, T_n$ commute pairwise.*

Proof. To show $T_j T_k = T_k T_j$ use the definition to expand T_k as a product of z's and then use Lemma 9.13 repeatedly to commute those with T_j. □

We use the relations from Proposition 3.13 to approximate the order of T_j.

Lemma 9.16. *We have $T_j^n = 1$ for $2 \le j \le n$.*

Proof. Clearly in $\mathcal{D}_S[X]$ by Proposition 3.13

$$\begin{aligned} \prod_{j=2}^{n} (X - T_j) &= X^{n-1} + \sum_{i=1}^{n-1} (-1)^i e_i(T_2, \ldots, T_n) X^{n-i-1} \\ &= X^{n-1} + \sum_{i=1}^{n-1} (-1)^{2i} X^{n-i-1}. \end{aligned}$$

So T_j is a zero of $X^{n-1} + X^{n-2} + \cdots + X + 1$, hence $T_j^n = 1$ for all $1 \le j \le n-1$. □

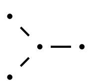
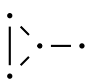
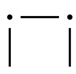
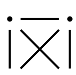
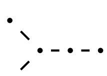
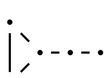
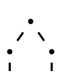
We now have a partial result on the structure of $\mathcal{N}_S$.

PROPOSITION 9.17. *The group $\mathcal{N}_S$ is a quotient of* $(C_n)^{n-2}$.

PROOF. The generators $T_2, \ldots, T_n$ commute and satisfy $T_j^n = 1$ for all $2 \leq j \leq n$ by Lemma 9.16. Furthermore, $T_2 T_3 \cdots T_n = 1$ by Proposition 3.13. The claim follows. $\square$

We conjecture that the group $\mathcal{N}_S$ is in fact isomorphic to $(C_n)^{n-2}$, and we know that this is the case for $n \leq 5$ and $\operatorname{char} \Bbbk \in \{0, 2\}$.

Appendix

S	$\mathcal{N}_S$	$\#\mathcal{N}_S$	$\approx$	$\dim_{\Bbbk} \mathcal{E}_S$
	1	1	1	6
	C_3	3	3	12
	1	1	1	24
	C_3	3	3	48
	$\mathbb{A}_4$	$2^2 \cdot 3$	12	96
	C_4^2	2^4	16	144
	9.5	$2^6 \cdot 3$	192	288
	9.6	$2^{10} \cdot 3$	3072	576
	1	1	1	120
	$\mathbb{A}_4$	$2^2 \cdot 3$	12	480
	$\mathbb{A}_5$	$2^2 \cdot 3 \cdot 5$	60	960
	C_5^3	5^3	125	2880
	9.7	$2^{10} \cdot 3$	3072	2880
	J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	604800	8640

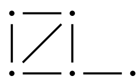
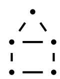
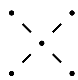
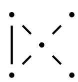
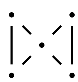
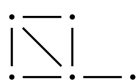
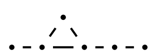
	$G_2(4)$	$2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$	251596800	17280
	Co_1	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$	$4 \cdot 10^{18}$	69120
	9.8	$2^{50} \cdot 3^4$	$9 \cdot 10^{16}$	14400
	9.9	$2^{60} \cdot 3^5 \cdot 5^5$	$9 \cdot 10^{23}$	28800
	9.10	$2^{75} \cdot 3^{10} \cdot 5^5$	$7 \cdot 10^{30}$	57600
	9.11	$2^{60} \cdot 3^{15} \cdot 5^{10} \cdot 7^5 \cdot 13^5$	$1 \cdot 10^{42}$	86400
	$3.Suz$	$2^{13} \cdot 3^8 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	$1 \cdot 10^{12}$	207360

TABLE 1

Here we collect the chief series we were able to compute as described in Section 9.1. Refer to Table 1 for the numbering.

9.5.

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G
|  Cyclic(3)
*
|  Cyclic(2) (2 copies)
*
|  Cyclic(2) (2 copies)
*
|  Cyclic(2)
*
|  Cyclic(2)
1
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9.6.

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G
|  Cyclic(3)
*
|  Cyclic(2) (2 copies)
*
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| Cyclic(2) (2 copies)
*
| Cyclic(2) (2 copies)
*
| Cyclic(2)
*
| Cyclic(2)
*
| Cyclic(2)
*
| Cyclic(2)
1
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And on 5 vertices, here we work with $\operatorname{char} \Bbbk = 2$.

9.7.

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G
| Cyclic(3)
*
| Cyclic(2) (2 copies)
*
| Cyclic(2) (2 copies)
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| Cyclic(2) (2 copies)
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| Cyclic(2)
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| Cyclic(2)
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| Cyclic(2)
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| Cyclic(2)
1
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9.8.

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G
| Cyclic(3)
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| Cyclic(3)
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| Cyclic(3)
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| Cyclic(3)
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*
|  Cyclic(2) (2 copies)
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|  Cyclic(2) (2 copies)
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9.9.

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G
|  Alternating(5)
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|  Alternating(5)
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|  Alternating(5)
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*
|  Alternating(5)
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|  Cyclic(2) (4 copies)
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|  Cyclic(2) (4 copies)
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|  Cyclic(2) (4 copies)
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|  Cyclic(2) (4 copies)
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9.10.

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G
|  Alternating(6)
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|  Alternating(6)
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|  Cyclic(2) (4 copies)
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9.11.

```
G
|  G(2, 4)
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|  G(2, 4)
*
|  G(2, 4)
*
|  G(2, 4)
*
|  G(2, 4)
1
```

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Anlage

Deutsche Zusammenfassung

Die Fomin-Kirillov-Algebra $\mathcal{E}_n$ ist eine quadratische Algebra, erzeugt von den Kanten des vollständigen Graphens auf n Knoten, die ursprünglich in [**FK98**] eingeführt wurde um den Schubert-Kalkül zu untersuchen. Später wurde festgestellt ([**MS00**], vergleiche auch [**GV16**] für geschichtliche Bemerkungen), dass es sich bei der Fomin-Kirillov-Algebra $\mathcal{E}_n$ für $n = 3, 4, 5$ um eine Nichols-Algebra zu einer nichtabelschen Gruppe handelt. Es wird vermutet dass dies auch für $n > 5$ zutrifft.

Bis heute sind so elementare Eigenschaften wie die Dimension von $\mathcal{E}_6$ als Vektorraum unbekannt.

Kürzlich wurde in [**BLM13**] die Klasse der Beispiele zu "hyperbolischen" Unteralgebren $\mathcal{E}_S$, erzeugt von den Kanten von Teilgraphen des vollständigen Graphen, erweitert. Diese Algebren haben einige interesante Eigenschaften (zum Beispiel "schöne" Hilbertreihen), für unsere Zwecke sind sie aber hauptsächlich von Interesse weil sie mehr Spielraum geben um neue Methoden zur Untersuchung von Fomin-Kirillov-Algebren zu entwickeln und zu testen.

Das Ziel dieser Arbeit ist es, neue Methoden zum Studium von Fomin-Kirillov-Algebren zu entwickeln. Insbesondere interessieren wir uns für die Berechnung ihrer Dimension als Vektorraum beziehungsweise für die Entscheidung ihrer Endlichdimensionalität.

Unser Werkzeug sind *deformierte* Fomin-Kirillov-Algebren, die wir mit $\mathcal{D}_S$ bezeichnen. Im zweiten Kapitel führen wir diese Algebren ein und stellen sie in Zusammenhang zu den ursprünglichen Fomin-Kirillov-Algebren. Im Kontext der Deformationen ist zu beachten dass die Fomin-Kirillov-Algebren als *nicht* koszul bekannt sind [**Roo99**].

Es ist zu bemerken dass allgemeinere Deformationen bereits in [**FK98**] betrachtet wurden, allerdings zu einem anderen Zweck und ohne Betrachtung des Zusammenhangs zwischen deformierter und undeformierter Algebra.

Im dritten und vierten Kapitel nutzen wir unsere Deformation und die Resultate des zweiten Kapitels um neue Methoden einzuführen.

Im dritten Kapitel untersuchen wir Unteralgebren von deformierten Fomin-Kirillov-Algebren die wir zu gewissen Untergruppen der symmetrischen Gruppe zuordnen. Diese Unteralgebren von $\mathcal{D}_S$ verhalten sich einigermaßen vernünftig - sie scheinen verwandt mit Fomin-Kirillov-Algebren $\mathcal{E}_{S'}$ gehörend zu Graphen auf weniger Knoten als S. Damit können wir die Dimensionen von einigen Algebren $\mathcal{E}_S$ per Hand neu berechnen, was vorher zum größten Teil nur mit Gröbner Basen möglich war. Der wichtigste Fortschritt ist unsere Berechnung der Dimension von $\mathcal{E}_5$ ohne Computer-Rechnungen, was nach unserem Wissen bislang nicht möglich war. Unser Vorgehen dabei ist ähnlich zu dem in [**FP00**]. Wir wenden unsere Methode auch auf $\mathcal{E}_6$ an. Unsere Resultate dort sind nur partiell, sie scheinen jedoch einigermaßen vielversprechend und geben einen Ansatz für weitere Untersuchungen.

Im vierten Kapitel assoziieren wir zu Fomin-Kirillov-Algebren dann Gruppen. Für Nichols-Algebren wurden diese Gruppen bereits in [**Loc13**] eingeführt. Es stellt sich heraus dass Endlichkeit der assoziierten Gruppe in Verbindung steht zur Endlichdimensionalität der zugehörigen Fomin-Kirillov-Algebra.

Unsere ursprüngliche Motivation diese Gruppen zu untersuchen war unsere Suche nach einem Beweis unserer Vermutung dass es sich bei den deformierten Fomin-Kirillov-Algebren um halbeinfache Algebren handelt. Zu unserer Überraschung stellte sich aber heraus, dass die Struktur der Gruppen selbst interessant scheint. Insbesondere treten die alternierende Gruppe, einige sporadische Gruppen, und eine Gruppe von Lie-Typ auf. In der Art in der diese Gruppen auftreten passen sie in natürlicher Weise in die Klassifikation in [**FLZ01**]. Basierend auf dieser Beobachtung geben wir eine prinzipielle Strategie um das Problem der Unendlichdimensionalität von $\mathcal{E}_6$ anzugehen.

Die meisten unserer Beispiele im vierten Kapitel basieren auf direkten Computer-Rechnungen, insbesondere auf den in [**BHLGO15**] beschriebenen Algorithmen. Eine theoretische Erklärung der auftretenden Gruppenstrukturen ist eine aus unserer Sicht hochinteressante Fragestellung.