Dissertation

Besov Regularity of Solutions to Navier-Stokes Equations

Frank Eckhardt

2016

Besov Regularity of Solutions to Navier-Stokes Equations

Dissertation

zur
Erlangung des akademischen Grades
Doktor der Naturwissenschaften
(Dr. rer. nat.)

vorgelegt
dem Fachbereich Mathematik und Informatik
der
Philipps–Universität Marburg

von
Frank Eckhardt

geboren am 15. September 1986

in Bad Berleburg

Vom Fachbereich Mathematik und Informatik
der Philipps-Universität Marburg (Hochschulkennziffer: 1180)
als Dissertation angenommen am: 12. 12. 2016

Erstgutachter: **Prof. Dr. Stephan Dahlke**, Philipps-Universität Marburg

Zweitgutachterin: **Prof. Dr. Dorothee Haroske**, Friedrich-Schiller-Universität Jena

Tag der mündlichen Prüfung: 19. 12. 2016

I dedicate this thesis to my sister Kathrin.

Acknowledgement

I would like to express my gratitude to everyone who has supported me by writing this dissertation. First of all I want to thank my supervisor Prof. Dr. Stephan Dahlke. He gave me the chance to be his doctoral student and he supported and encouraged me a lot while working on this thesis. Also I want to thank Prof. Dr. Dorothee D. Haroske for accepting to be the second referee of this thesis. A special thanks goes to Dr. Petru A. Cioica-Licht for many fruitful discussions and a lot of helpful advices.

I also want to thank the workgroup Numerical Analysis for a very friendly and nice atmosphere. Working with you was a pleasure for me.

My deep gratitude goes to Stefan Bösner who shared his wisdom with me as a mentor for more than six years now. You have inspired and encouraged me during difficult time when I needed words of encouragement. Thank you for your time, support and patience.

Without my parents Ulrich and Doris and my sister Kathrin it would have been impossible to finish this thesis. Thank you for staying with me, encouraging me and loving me unconditionally. Furthermore I want to say thank you to a lot of wonderful friends who supported me in the past years: Alan Shekho, Christian Bardtke, Christoph Wahl, Dago Nguessan, Daniel and Tamara Mombartz, Erika Höschele, Giulliana Tessarin, Hanna Fett, Joy Dunne, Juliane Welzl, Kathrin Schönwald, Lisa Herbst, Lisa Schmidt, Matthias Havemann, Silvana Sanchez, Urszula Włośniewska and Wen Wu.

The most of all I want to thank Jesus Christ for saving me and giving me life.

Abstract

This thesis is concerned with the regularity of solutions to Navier-Stokes and Stokes equation on domains with point singularities, namely polyhedral domains contained in $\mathbb{R}^3$ and general bounded Lipschitz domains in $\mathbb{R}^d$, $d \geq 3$ with connected boundary. The Navier-Stokes equations provide a mathematical model of the motion of a fluid. These Navier-Stokes equations form the basis for the whole world of computational fluid dynamics, and therefore they are considered as maybe the most important PDEs known so far. We consider the stationary (Navier-)Stokes equations. The study the Besov regularity of the solution in the scale $B^s_\tau(L_\tau(\Omega))^d$, $1/\tau = s/d + 1/2$ of Besov spaces. This scale is the so-called adaptivity scale. The parameter s determines the approximation order of adaptive numerical wavelet schemes and other nonlinear approximation methods when the error is measured in the L_2-norm. In contrast to this the convergence order of linear schemes is determined by the classical L_2-Sobolev regularity.

In many papers the Besov regularity of the solution to various operator equations/partial differential equations was investigated. The proof of Besov regularity in the adaptivity scale was in many contributions performed by combining weighted Sobolev regularity results with characterizations of Besov spaces by wavelet expansions. Choosing a suitable wavelet basis the coefficients of the wavelet expansion of the solution can be estimated by exploiting the weighted Sobolev regularity of the solution, such that a certain Besov regularity can be established. This technique was applied for the Stokes system in all papers which are part of this thesis. For achieving Besov regularity for Navier-Stokes equation we used a fixed point argument. We formulate the Navier-Stokes equation as a fixed point equation and therefore regularity results for the corresponding Stokes equation can be transferred to the non-linear case.

In the first paper *Besov regularity for the Stokes and the Navier-Stokes system in polyhedral domains* we considered the stationary Stokes- and the Navier-Stokes equations in polyhedral domains. Exploiting weighted Sobolev estimates for the solution we proved that the Besov regularity of the solutions to these equations exceed their Sobolev regularity. In the second paper *Besov Regularity for the Stationary Navier-Stokes Equation on Bounded Lipschitz Domains* we have investigated the stationary (Navier-)Stokes equations on bounded Lipschitz domain. Based on weighted Sobolev estimates again we could establish a Besov regularity result for the solution to the Stokes system. By applying Banach's fixed point theorem we transferred these results to the non-linear Navier-Stokes equation. In order to apply the fixed point theorem we had to require small data and small Reynolds number.

Zusammenfassung

In der vorliegenden Arbeit beschäftigen wir uns mit der Regularität von Lösungen zu Navier-Stokes- und Stokes-Gleichungen auf Gebieten mit Randsingularitäten. Mit Hilfe der Navier-Stokes-Gleichungen lassen sich die Ausbreitung von Fluiden mathematisch modellieren. Sie bilden die Grundlage der gesamten Strömungsmechanik und gelten daher als eine der wichtigsten partiellen Differentialgleichungen überhaupt. Wir betrachten stationäre, d.h. zeitunabhängige (Navier-)Stokes-Gleichungen in polyhedralen Gebieten im $\mathbb{R}^3$ und in allgemeinen beschränkten Lipschitz-Gebieten mit zusammenhängenden Rand im $\mathbb{R}^d$, $d \geq 3$. Wir bestimmen die Regularität s in der Skala von Besov-Räumen $B^s_\tau(L_\tau(\Omega))^d$, $1/\tau = s/d+1/2$. Diese Skala ist die sogenannte Adaptivitäts-Skala. Der Glattheitsparameter s bestimmt die Konvergenzordnung von bestimmten adaptiven, numerischen Wavelet-Verfahren, sowie von anderen nicht linearen Approximationsmethoden. Die Konvergenzordnung von linearen Verfahren wird dagegen durch die klassische L_2-Sobolev-Regularität der Lösung bestimmt.

In zahlreichen Arbeiten wurde die Besov-Regularität in der Adaptivitäts-Skala von Lösungen verschiedener Operatorgleichunge/partiellen Differentialgleichungen untersucht. Dabei wurden Resultate über gewichtete Sobolev-Regularität verwendet, um die Koeffizienten einer Wavelet-Entwicklung der Lösung geeignet abzuschätzen. Diese Beweisidee beruht auf der Charakterisierung der Besov-Räume durch Wavelets. Diese Technik wurde in dieser Arbeit verwendet, um Besov-Regularität für die Lösungen der (Navier-)Stokes-Gleichungen auf polyhedralen Gebieten, sowie der Stokes-Gleichung auf Lipschitz-Gebieten zu beweisen. Um Besov-Regularität für die Navier-Stokes-Gleichung auf Lipschitz-Gebieten zu etablieren, wurde ein Fixpunktargument angewendet: Die Navier-Stokes-Gleichung lässt sich als Fixpunktproblem formulieren, so dass sich die nicht lineare Gleichung als lineare Gleichung mit modifizierte rechter Seite auffassen lässt. Die Regularitätsaussagen folgen dann aus den entsprechenden Aussagen für die Stokes-Gleichung.

In dem ersten Paper *Besov regularity for the Stokes and the Navier-Stokes system in polyhedral domains* haben wir die Regularität der Lösungen der stationären (Navier-)Stokes-Gleichungen in polyhedralen Gebieten untersucht. Unter Zuhilfenahme von gewichteten Sobolev-Regularitätsaussagen für die Lösung konnten wir Besov-Regularitätsresultate beweisen, die zeigen, dass die Besov-Regularität die Sobolev-Regularität der Lösung tatsächlich übertrifft. In der zweiten Arbeit *Besov Regularity for the Stationary Navier-Stokes Equation on Bounded Lipschitz Domains* haben wir die Besov-Regularität der Lösung von (Navier-)Stokes-Gleichungen in beschränkten Lipschtz-Gebieten untersucht. Genau wie bei der Untersuchung in polyhedralen Gebieten, wurden hier gewichtete Sobolev-Abschätzungen verwendet, um Besov-Regularität der Lösung für die Stokes-Gleichung zu zeigen. Um entsprechende Aussagen für die Navier-Stokes-Gleichung zu zeigen, haben wir den Banach'schen Fixpunktsatz angewandt. Um die Existenz eines Fixpunktes garantieren zu können, sind Bedingungen an das Gebiet, die Norm der rechten Seite, sowie der Reynolds-Zahl zu stellen.

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1 Introduction

Partial differential equations (PDE) are a powerful tool for modelling natural phenomena. Consequently, the research field of PDEs was one of the key areas in the past century and also in recent years. The theoretical study of existence, uniqueness and regularity in suitable function spaces of a solution to PDEs were main aspects of mathematical research. Since an analytic description of the solution is available only in rare cases, one is forced to develop numerical schemes for the constructive approximation of the solution. Therefore the analysis of efficient, numerical schemes for solving PDEs were promoted. In this thesis we are concerned with the famous Navier-Stokes equation on a bounded domain Ω contained in $\mathbb{R}^d$, $d \geq 3$.

Navier-Stokes equations

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $T > 0$. The Navier-Stokes equations

$$\begin{aligned} u_t - \Delta u + \nu u \cdot (\nabla u) + \nabla \pi = f \quad &\text{on } \Omega \times (0,T), \\ \text{div } u = 0 \quad &\text{on } \Omega \times (0,T), \end{aligned}$$

form the basis for the mathematical description of fluid mechanics. By Δ we denote the Laplace operator, ∇ stands for the gradient, further we put

$$u \cdot \nabla u = \sum_{i=1}^{d} u_i \cdot \frac{\partial u}{\partial x_i}.$$

The quantity $\nu > 0$ denotes the Reynolds number that describes the viscosity of the fluid. The field $u = (u_1, ..., u_d)$ describes the velocity of the fluid, the term π denotes the pressure, the right hand side f describe the exterior force. We give a short overview of the physically derivation of the Navier-Stokes equations as it is displayed in [67]. We start by the second equation div $u = 0$. We consider the mapping

$$\Phi : \Omega \times [0, \infty) \to \Omega,$$

which maps for a particle, which is localized in $x = \Phi(x, 0)$ for $t = 0$, the point (x, t) to the position $\Phi(x, t)$ for $t \in (0, \infty)$. The velocity of the fluid is given by

$$u(\Phi(x,t), t) = \frac{\partial}{\partial t} \Phi(x, t).$$

We define for $t \in (0, \infty)$ the set

$$\Omega_t := \Phi(\Omega_0, t) = \{\Phi(x, t) \in \Omega : x \in \Omega_0\},$$

where $\Omega_0 \subset \Omega$ is an arbitrary subdomain. With $r : \Omega \times [0, \infty) \to \mathbb{R}$ we denote the density of the fluid, then the mass is given by

$$m(t) := \int_{\Omega_t} r(x,t)\mathrm{d}x.$$

The transport theorem, see [67, Chapter 2.1] and the references therein, yields (we assume, that the smoothness assumptions on r are fulfilled):

$$\frac{\partial}{\partial t} m(t) = \int_{\Omega_t} \left(\frac{\partial}{\partial t} r + \mathrm{div}\ (r \cdot u) \right) (x,t)\mathrm{d}x.$$

The law of conservation of mass yields

$$\int_{\Omega_t} \left(\frac{\partial}{\partial t} r + \mathrm{div}\ (r \cdot u) \right) (x,t)\mathrm{d}x = 0.$$

The principle of localization says, if the integrand is smooth and the domain Ω_0 is choosen arbitrary, then we have

$$\frac{\partial}{\partial t} r + \mathrm{div}\ (r \cdot u) = 0. \tag{1.0.1}$$

This equation is called *equation of continuity*. Considering incompressible motions, i.e. r is constant, then the equation of continuity is given by

$$\mathrm{div}\ u = 0. \tag{1.0.2}$$

Fluids with property (1.0.2) are called *solenoidal*. The first equation in the Navier-Stokes equation is based on the conservation of momentum. It says that

$$\frac{\partial}{\partial t} \int_{\Omega_t} (ru)(x,t)\mathrm{d}x = F_V(t) + F_R(t), \tag{1.0.3}$$

where

$$F_V(t) := \int_{\Omega_t} (r \cdot f_v)(x,t)\mathrm{d}x$$

is the force which depends on the given external force field f_v. The term

$$F_R(t) := \int_{\partial\Omega_t} (\sigma \cdot n)(x,t)dS$$

models the boundary force. The vector n stand for the outward unit vector from $\partial\Omega_t$, the matrix σ is the stress tensor

$$\sigma := \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)_{i,j=1,\ldots,d}.$$

Considering the left side of (1.0.3) componentwise and after applying the transport theorem we find

$$\frac{\partial}{\partial t} \int_{\Omega_t} (ru_j)(x,t)\mathrm{d}x = \int_{\Omega_t} \left(\frac{\partial}{\partial t}(ru_j) + \mathrm{div}\ (ru_j u) \right) (x,t)\mathrm{d}x.$$

For incompressible fluids we get

$$\frac{\partial}{\partial t}\int_{\Omega_t}(ru_j)(x,t)\mathrm{d}x = \int_{\Omega_t}\left(r\frac{\partial}{\partial t}u_j + r\mathrm{div}\ (u_j u)\right)(x,t)\mathrm{d}x.$$

Using the divergence theorem, equation (1.0.3) reads as follows:

$$\int_{\Omega_t}\left(r\frac{\partial}{\partial t}u_j + r\mathrm{div}\ (u_j u)\right)(x,t)\mathrm{d}x = \int_{\Omega_t}((r\cdot f_j) + \mathrm{div}\ \sigma_j)(x,t)\mathrm{d}x,$$

consequently

$$r\frac{\partial}{\partial t}u_j + r\mathrm{div}\ (u_j u) = r\cdot f_j + \mathrm{div}\ \sigma_j.$$

We find

$$r\frac{\partial}{\partial t}u + r(u\cdot\nabla)u = rf + \mathrm{div}\ \sigma.$$

The term div σ can be expressed in terms of the gradient $\nabla\pi$, and the Laplace operator Δ applied to the velocity field u and the Reynolds number $\nu > 0$. For fluids with constant density, this yields in the equation

$$\frac{\partial u}{\partial t} + \nu\cdot(u\cdot\nabla)u - \Delta u + \nabla\pi = f.$$

We add boundary conditions for u and initial values for $t = 0$ in order to achieve a well-posed mathematical problem. In this thesis we only consider Dirichlet boundary conditions. The non stationary Navier-Stokes equation is then given by

$$\begin{aligned} u_t - \Delta u + \nu u\cdot(\nabla u) + \nabla\pi &= f \quad \text{on } \Omega\times(0,T),\\ \mathrm{div}\ u &= 0 \quad \text{on } \Omega\times(0,T),\\ u &= g \quad \text{on } \partial\Omega\times(0,T),\\ u(x,0) &= u_0 \quad \text{on } \Omega \end{aligned} \tag{NavSt1}$$

for $T > 0$. The linearized version of (NavSt1) is

$$\begin{aligned} u_t - \Delta u + \nabla\pi &= f \quad \text{on } \Omega\times(0,T),\\ \mathrm{div}\ u &= 0 \quad \text{on } \Omega\times(0,T),\\ u &= g \quad \text{on } \partial\Omega\times(0,T),\\ u(x,0) &= u_0 \quad \text{on } \Omega \end{aligned} \tag{St1}$$

the time-dependent Stokes equation. We achieve this from neglecting the nonlinear term $u\cdot(\nabla u)$. From a physically point of view the Stokes equation is a limit case of (NavSt1) for very tough fluids. In this thesis we are only concerned with the stationary case of (NavSt1):

$$\begin{aligned} -\Delta u + \nu u\cdot(\nabla u) + \nabla\pi &= f \quad \text{on } \Omega,\\ \mathrm{div}\ u &= 0 \quad \text{on } \Omega,\\ u &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{NavSt2}$$

and its linearized version:

$$\begin{aligned} -\Delta u + \nabla\pi &= f \quad \text{on } \Omega,\\ \mathrm{div}\ u &= 0 \quad \text{on } \Omega,\\ u &= g \quad \text{on } \partial\Omega. \end{aligned} \tag{St2}$$

In the second chapter (see also [40]) we also consider the generalized case div $u = h$.

We first have to deal with the question, what we mean by a solution to (St2). A natural approach would be, to consider classical solutions, i.e. a pair $(u, \pi) \in \mathcal{C}^2(\Omega)^d \times \mathcal{C}^1(\Omega)$, such that (u, π) fulfills (St2). Since in many relevant cases, there exists no classical solution to (St2), it is necessary to consider a different concept. To this end we discuss the basic idea of developing a weak formulation for the stationary Stokes equation. Assume u and π to be a classical solution of (St2). Multiplying the first equation in (St2) by $\varphi \in \mathcal{C}_0^\infty(\Omega)$, the set of test functions on Ω, and integration by parts yields

$$\int_\Omega \sum_{i,j=1}^{d} (\nabla u)_{ij} (\nabla \varphi)_{ij} (x) \, \mathrm{d}x = f(\varphi) - \int_\Omega \sum_{i=1}^{d} \pi(x) \frac{\partial \varphi_i(x)}{\partial x_i} \, \mathrm{d}x. \tag{1.0.4}$$

Thus, every classical solution fulfills (1.0.4) for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$. As mentionted above, in some settings it can be shown, that there is no classical solution. In these situations there may exists a pair (u, π) which fulfills (1.0.4) in the weak sense, as we explain now. We therefore look for a solution u in the Sobolev space $H^1(\Omega)^d$. In the case $g = 0$ in (St1), we require $u \in H_0^1(\Omega)^d$. Assume $f \in H^{-1}(\Omega)^d$. Then $u \in H^1(\Omega)^d$ fulfills

$$\int_\Omega \sum_{i,j=1}^{d} (\nabla u)_{ij} (\nabla \psi)_{ij} (x) \, \mathrm{d}x = f(\psi) \tag{1.0.5}$$

for all

$$\psi \in D_0^{1,2}(\Omega) := \overline{\{v \in \mathcal{C}_0^\infty(\Omega) : \text{div } v = 0\}}^{|\cdot|_{1,2}},$$

where $|v|_{1,2}^2 := \sum_{|\alpha|=1} \int_\Omega |D^\alpha v(x)|^2 \mathrm{d}x$ (which is a norm on $\mathcal{C}_0^\infty(\Omega)$), if and only if there exists a $\pi \in L_2(\Omega)$ such that (u, π) fulfills (1.0.4) for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$, see [43, Lemma IV.1.1]. According to this result we define the following:

Definition: We call u a *weak solution* to (St2) if the following conditions are satisfied:

(i) $u \in H^1(\Omega)^d$.

(ii) u is weakly divergence free in Ω.

(iii) u satisfies the boundary condition $u|_{\partial\Omega} = g$ in the trace sense.

(iv) u fulfills (1.0.5) for all $\psi \in D_0^{1,2}(\Omega)$.

We call π the corresponding pressure.

For the third point we refer to [38] for trace theorems on Lipschitz domains. Since div $u = 0$ we have due to the divergence theorem the natural compatibility condition

$$\int_{\partial\Omega} g \cdot n d\sigma = 0. \tag{1.0.6}$$

Based on this definition of a solution, we have indeed a well-posed problem: Requiring (1.0.6) in the above situation for $g \in H^{1/2}(\partial\Omega)^d$ we know, that there exists a unique solution $u \in H^1(\Omega)^d$ and a corresponding pressure field $\pi \in L_2(\Omega)$, see e.g. [43, Theorem

IV.1.1]. For (NavSt2) the weak formulation can be derived in a similar way. We call $u \in H^1(\Omega)^d$ a *(weak) solution* of (NavSt2) if u is divergence free, $u = g$ on the boundary $\partial\Omega$ (in a trace sense) and u fulfills the equation

$$\int_\Omega \sum_{i,j=1}^{d} (\nabla u)_{ij} (\nabla \varphi)_{ij} \, \mathrm{d}x + \nu \int_\Omega (u \cdot (\nabla u)) \, \varphi \, \mathrm{d}x = f(\varphi) - \int_\Omega \sum_{i=1}^{d} \pi \, \frac{\partial \varphi_i}{\partial x_i} \, \mathrm{d}x$$

for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$ with a suitable $\pi \in L_2(\Omega)$. See [43, Chapter IX] for more details. For existence results for Navier-Stokes equations see for instance [43], [52] and [62].

Having a well-posed problem, i.e. existence and uniqueness of the solution are ensured, the next issue is the study of properties of the solution. One important property is the regularity of the solution. In the case of Navier-Stokes equations we address the question how regular the motion of the fluid depending on the properties of the given data and the underlaying domain is. While in the case of classical solutions the regularity measured in classical Hölder spaces is of interest, we investigate the L_2-Sobolev regularity of the velocity field u and the corresponding pressure field π. We a priori know $u \in H^1(\Omega)^d$ and $\pi \in L_2(\Omega)$, but possibly the L_2-Sobolev regularity is higher. It turns out, that the Sobolev regularity depends on the regularity of the domain. Assuming Ω is a smooth domain, an increasing Sobolev regularity of f and g leads to an increasing Sobolev regularity of u and the corresponding pressure field π, see e.g. [1], [43, Theorem IX.5.1] (interior regularity), [60]. This conclusion is no longer true on domains with singularities, e.g. polyhedral domains or general Lipschitz domains: If Ω is only assumed to be a bounded Lipschitz domain a higher Sobolev regularity of f and g does not guarantee a higher Sobolev regularity for the solution. This is due to boundary singularities, which can cause higher derivatives to blow up near the boundary. These singularities therefore diminish the Sobolev regularity. For the case of general bounded Lipschitz domains and suitable right-hand side f and boundary data g results have been proven, which provide a Sobolev regularity of $3/2$ for the solution u and $1/2$ for the pressure term to the stationary (Navier-)Stokes equation, see for instance [3], [42] and [57]. Similar results for the spatial Sobolev regulartiy for the non-stationary equations were proven in [3], [34]. To the best of our knowledge there is no result which assures higher Sobolev regularity on Lipschitz domains, even if the given data are assumed to be smoother. The fact, that the Sobolev regularity is limited on domains with singularities, leads to the natural question which regularity results can be established in weighted Sobolev spaces. In the weighted Sobolev norm the (weak) derivatives are multiplied by the distance to the singularity (or to the boundary) to the power of a certain parameter. The hope is that these weights compensate the growing derivatives near the boundary, such that the weighted norm is finite. For stationary Stokes equations the weighted Sobolev regularity has been studied in [56] on polyhedral domains and in [3, 41] for Lipschitz domains.

Adaptive Wavelet schemes and Besov regularity

To study Besov regularity of the solution to Navier-Stokes equations is motivated by the connection of Besov regularity and the convergence rate of adaptive numerical wavelet schemes. Let us first briefly discuss the construction of wavelets. Consider a domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$. By $L_2(\Omega)$ we denote the space of quadratically Lebesgue-integrable functions.

The aim is now to construct a wavelet (Riesz-)basis $\{\psi_\lambda : \lambda \in \Lambda\}$ for the Hilbert space $L_2(\Omega)$ with the following properties:

- The wavelets ψ_λ have compact support.
- They fulfill smoothness assumptions: $\psi_\lambda \in \mathcal{C}^r(\Omega)$ for a suitable $r \in \mathbb{N}$.
- The vanishing moment property is fulfilled:

$$\int_{\text{supp}\ \psi_\lambda} x^\alpha\, \psi_\lambda(x)\, dx = 0 \quad \text{for all } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \le r.$$

Exploiting these facts give the following statements, see [17].

- Weighted sequence norms of coefficients of wavelet decomposition allow to characterize certain smoothness spaces as Besov- and Sobolev spaces.
- The representation of a wide class of operators in the wavelet basis is nearly diagonal.
- The vanishing moments of wavelets remove the smooth part of a function.

The construction of a Riesz basis can be done by means of a *multiresolution analysis*, i.e., a sequence $(V_j)_{j\ge j_0}$ of closed linear subspaces of $L_2(\Omega)$ such as

$$V_j \subset V_{j+1} \quad \text{for all } j \ge j_0, \qquad \overline{\bigcup_{j\ge j_0} V_j}^{\|\cdot\|_{L_2(\Omega)}} = L_2(\Omega).$$

We assume that there are so-called scaling functions $\{\phi_\lambda : \lambda \in I_j\}$ which form a Riesz basis of V_j. By using the concept of multiresolution analysis, it is possible to construct a biorthogonal basis. Therefore we assume, that there exists a Riesz basis $\{\tilde{\phi}_\lambda : \lambda \in I_j\}$ for a second sequence of approximation spaces $(\tilde{V}_j)_{j\ge j_0}$ with the following property. We consider the complements W_j and $\tilde{W}_j$, which satisfy the biorthogonality condition, i.e.

$$V_{j+1} = V_j \oplus W_j, W_j \perp \tilde{V}_j, \quad \tilde{V}_{j+1} = \tilde{V}_j \oplus \tilde{W}_j, \tilde{W}_j \perp V_j.$$

Based on the scaling functions $\{\phi_i\}_{i\in I}$ and $\{\tilde{\phi}_i\}_{i\in I}$, we can construct a Riesz basis $\{\psi_\lambda : \lambda \in \Lambda_j\}$ of W_j and $\{\tilde{\psi}_\lambda : \lambda \in \Lambda_j\}$ of $\tilde{W}_j$. Following the notation in [10] we write $\Lambda_{j_0-1} := I_{j_0}$ and denote the scaling functions spanning V_{j_0} also by ψ_λ, $\lambda \in \Lambda_{j_0-1}$. Assuming that the scaling functions fulfill further regularity- and approximation properties, then

$$\{\psi_\lambda : \lambda \in \Lambda\}, \quad \Lambda := \bigcup_{j\ge j_0-1} \Lambda_j$$

form a Riesz basis of $L_2(\Omega)$. We call this basis a *wavelet Riesz basis*. The Riesz basis $\{\tilde{\psi}_\lambda : \lambda \in \Lambda\}$ is called *biorthogonal basis*. The construction of wavelets for $L_2(\mathbb{R}^d)$ with properties as mentioned above can be found for instance in [10, Chapter 2], [14] and [32]. Since wavelets can be used to design numerical schemes for solving operator equations on bounded domains Ω, as we explain more detailed below, it it desirable to construct wavelets for $L_2(\Omega)$. It has been spend much effort in the construction of such wavelet basis on domains with singularities, including polygonal and polyhedral domains, see e.g. [5, 6] and [29–31].

As already mentioned above, wavelets can be used to characterize function spaces, for instance Sobolev spaces and Besov spaces. Let us display this more explicit by using an example: Consider the Besov space $B^s_q(L_p(\mathbb{R}^d))$ with $p \in (0,\infty)$, $q \in (0,\infty]$ and $s > \max\{0, d\,(1/p-1)\}$. Assume that the generator and the wavelets system $\{\phi_k\}_{k\in\mathbb{Z}^d}$, $\{\psi_{i,j,k}\}_{(i,j,k)\in\{1,\ldots,2^d-1\}\times\mathbb{N}_0\times\mathbb{Z}^d}$ and the dual basis $\{\tilde{\phi}_k\}_{k\in\mathbb{Z}^d}$, $\{\tilde{\psi}_{i,j,k}\}_{(i,j,k)\in\{1,\ldots,2^d-1\}\times\mathbb{N}_0\times\mathbb{Z}^d}$ fulfill certain technical assumptions. Further, we assume that there exists a dual Riesz basis satisfying the same requirements. Then a locally integrable function $f : \mathbb{R}^d \to \mathbb{R}$ is in the Besov space if, and only if,

$$f = \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\phi}_k\rangle\, \phi_k + \sum_{i=1}^{2^d-1} \sum_{j\in\mathbb{N}_0} \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\psi}_{i,j,k}\rangle\, \psi_{i,j,k}$$

(convergence in $\mathcal{D}'(\mathbb{R}^d)$) with

$$\left(\sum_{k\in\mathbb{Z}^d} \left|\langle f, \widetilde{\phi}_k\rangle\right|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{2^d-1} \sum_{j\in\mathbb{N}_0} 2^{j\left(s+d\left(\frac{1}{2}-\frac{1}{p}\right)\right)q} \left(\sum_{k\in\mathbb{Z}^d} \left|\langle f, \widetilde{\psi}_{i,j,k}\rangle\right|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} < \infty. \qquad (1.0.7)$$

A proof can be found in [10, Theorem 3.7.7]. Equation (1.0.7) states, that the decay of the coefficients of the wavelets decomposition of a measurable function gives information about its Besov regularity.

Wavelets become a very important tool in applied mathematics. They are used for instance in image/signal analysis, see e.g. [7], [55]: Wavelets can be used to construct very efficient compression schemes for images. An image can be modelled by a function $f \in L_2(Q)$, where Q is the unit cube in $\mathbb{R}^2$. The approach is now to approximate f by the linear combination of a suitable selection of wavelet basis elements. The task is to choose this selection in a proper way. A natural approach is to consider all wavelet coefficients up to a fixed refinement level of the underlying multiresolution analysis. We call this a *linear approximation*. The quality of the linear approximation depends on the L_2-Sobolev regularity of f, see [7] for more information. An alternative approach is called *hard thresholding strategy*. Roughly speaking we only choose wavelets such that the corresponding coefficients are large enough, i.e. their absolut value exceeds a fixed value. This kind of approximation is called *nonlinear approximation*. The approximation rate of the nonlinear approximation is determined by the Besov regularity of the solution, where the regularity is measured in the scale $B^s_\tau(L_\tau(Q))$, $1/\tau = s/2 + 1/2$. See [7] for details.

Furthermore wavelets can be applied for denoising, i.e. we start with a measurement of corrupted wavelet coefficients and our goal is to find a approximation of the original signal. For details we refer again to [7]. Furthermore wavelets are used for pre-conditioning, tomography and in geophysics and meteorology. An overview of possible applications can be found in [54].

Next we discuss an important application, which basically motivates our investigation of Besov regularity: Wavelets became a very powerful tool for solving operator equations. Let us discuss this by considering a general elliptic operator equation. By $\mathring{H}^1(\Omega)$ we denote the closure of the set of all infinitely differentiable functions with compact support in Ω

with respect to the $L_2(\Omega)$-Sobolev norm $\|\cdot\|_{H^1(\Omega)}$. Let $H^{-1}(\Omega)$ be the dual space of $\mathring{H}^1(\Omega)$. Further we write

$$a : \mathring{H}^1(\Omega) \times \mathring{H}^1(\Omega) \to \mathbb{R}$$

for a continuous, symmetric and elliptic bilinear form. In this setting we know

$$\frac{1}{C}\|u\|_{\mathring{H}^1(\Omega)} \leq a(u,u) \leq C \cdot \|u\|_{\mathring{H}^1(\Omega)}, \quad u \in \mathring{H}^1(\Omega)$$

for a finite constant $C > 0$. The operator

$$A : \mathring{H}^1(\Omega) \to H^{-1}(\Omega),\ u \mapsto a(u, \cdot)$$

is an isomorphism. Consequently, the equation

$$Au = f, \quad f \in H^{-1}(\Omega)$$

has an unique solution $u \in \mathring{H}^1(\Omega)$. Obviously, this equation is equivalent to the variational formulation

$$a(u,v) = f(v), \quad v \in \mathring{H}^1(\Omega). \tag{1.0.8}$$

Since this solution is not known explicitly, one has to develop numerical schemes to construct an approximation of the solution. The approach is now to discretize (1.0.8) and then to solve finite linear equations systems. One way to discretize (1.0.8) is to use a Galerkin method, i.e. we consider a nested sequence $(S_m)_{m\geq 0}$ of finite dimensional subspaces of $\mathring{H}^1(\Omega)$. Then we solve the problem

$$a(u_m, v_m) = f(v_m), \quad v_m \in S_m. \tag{1.0.9}$$

If the solution u_m is sufficiently close to the solution of (1.0.9) we end, otherwise we consider (1.0.9) in S_{m+1}. One of the main questions is how to construct the sequence $(S_m)_{m\geq 0}$ and in which way we can update the space S_{m+1} (space refinement) if the solution in S_m is not close enough to the solution. In the wavelet setting, a direct approach is to choose the subspaces of a suitable multiresolution analysis $(V_j)_{j\geq 0}$, i.e.

$$S_{m(j)} := V_j,$$

where $m(j) := \left|\bigcup_{i\geq j_0-1}^{j} \Lambda_i\right|$. Obviously the space refinement is a priori fixed and therefore independent of the current approximation. This kind of approximation schemes is called *linear schemes*. The practical advantage of this strategy is the easy implementation. Also determining the convergence rate of these kind of schemes is quite simple. The approximation error for uniform schemes as suggested above is defined by

$$E_m(u) := \inf_{u_m \in S_m} \|u - u_m\|_{L_2(\Omega)}.$$

Under certain technical conditions for the wavelets we find the following: There is a $r \in \mathbb{N}$ depending on the wavelet basis such that for all $\alpha \in [0, r]$ holds:

$$u \in H^\alpha(\Omega) \Longrightarrow E_m(u) \leq C \cdot m^{-\alpha/d},$$

i.e. the convergence rate of linear schemes is determined by the L_2-Sobolev regularity. One can also show the converse implication:

$$E_m(u) \leq C \cdot m^{-\alpha/d},\ m = m(j), \quad \text{for all } j \geq j_0 \Longrightarrow u \in H^{\alpha'}(\Omega),\ \alpha' < \alpha.$$

See [35] for detailed information. As already discussed above, the guaranteed L_2-Sobolev regularity of solutions to Navier-Stokes equations on domains with singularities is limited, even if the right-hand is smooth. Therefore the convergence rate of linear schemes as discussed above is limited as well. A way out is the use of adaptivity, as we briefly explain in the following. There might be areas of the domain in which the approximation is already close to the solution and other parts of the domain, where the approximation is still poor. Consequently it is reasonable to improve the approximation only in those parts were the approximation is far from the exact solution. Therefore one needs to develop an *a posteriori error estimator*, which estimates the local error of the recent result. Additionally one has to invent an updating strategy, i.e. how to update the subspace S_{m+1}. The strong analytic properties of wavelets make it possible to construct adaptive schemes based on wavelets as described above, i.e. to construct a posteriori error estimator and a adaptive refinement strategy, see e.g. [17, Section 3.2.1, Section 3.2.2]. Having such an adaptive strategy it is a hard task to proof convergence of this scheme and to determine the convergence rate. Furthermore the implementation of these schemes is much more difficult than the implementation of linear schemes. Thus, before developing an adaptive scheme, it is desirable to check if adaptivity really pays out. Meaning we have to analyze whether it is possible to improve the convergence rate of uniform schemes. To this end we consider the error of the *best N-term wavelet approximation.* We consider the manifold

$$\Sigma_N := \left\{ \sum_{\lambda \in \Lambda_0} c_\lambda \cdot \psi_\lambda : |\Lambda_0| = N,\ c_\lambda \in \mathbb{R} \right\}$$

and the approximation error

$$\sigma_N(u) := \inf_{u_N \in \Sigma_N} \|u - u_N\|_{L_2(\Omega)}.$$

Obviously, the convergence rate of best N-term approximation is a upper bound for the convergence rate of any numerical scheme based on $\{\psi_\lambda : \lambda \in \Lambda\}$. Therefore, best N-term approximation serves as a benchmark for numerical wavelet schemes. The quantity $\sigma_N(u)$ is connected to the Besov regularity of the target function. For the error of the best N-term approximation we have

$$u \in B^s_\tau(L_\tau(\Omega)), \quad \frac{1}{\tau} = \frac{s}{d} + \frac{1}{2} \Longrightarrow \sigma_N(u) \leq C \cdot N^{-s/d},$$

see [35]. In order to justify the use of adaptive schemes we have to ensure:

- It exists $s > 0$ such that $\sigma_N(u) \leq C \cdot N^{-s/d}$ for a constant C independent on $N \in \mathbb{N}$.
- $s > \alpha_{\max}(u) := \sup\{\alpha \geq 0 : \forall m \in \mathbb{N} : E_m \leq C \cdot m^{-\alpha/d}\}$.

Therefore we are in this thesis concerned with the question

$$s > \alpha_{\max}?$$

In recent years there were successfully developed several adaptive wavelet schemes for diverse problems. These schemes use a suitable wavelet basis. In many cases the convergence rate of these schemes reach the convergence rate of best N-Term approximation. The starting point were papers [2, 11]. They designed convergent, numerical adaptive wavelet schemes to solve elliptic operator equations. For nonlinear problems we refer to [13]. Beside that adaptive wavelet methods were used to solve integral equations, see e.g. [28, 48]. Moreover saddle point problems were addressed, see [19, 24]. For the stochastic Poisson equation an adative wavelet algorithm was developed, see [9].

When using a wavelet basis one has to deal with a number of difficulties: One is usually faced with relatively high condition numbers, the smoothness assumptions on wavelets are hard to ensure and the existing constructions of wavelet basis are not easy to implement. To this end, a weaker concept can be used. Instead of wavelet bases one uses wavelet frames. A collection $\mathcal{F} = \{f_n\}_{n\in\mathcal{N}}$ of elements of a Hilbert space $(\mathcal{H}, (\cdot,\cdot)_{\mathcal{H}})$ is called a frame for $\mathcal{H}$ if there exists two constants $0 < C_1 \leq C_2 < \infty$ such that

$$C_1\|f\|_{\mathcal{H}}^2 \leq \sum_{n\in\mathcal{N}} |(f, f_n)_{\mathcal{H}}|^2 \leq C_2\|f\|_{\mathcal{H}}^2, \quad f \in \mathcal{H}.$$

The construction of a frame can be performed in the following way: One has to construct an overlapping partition $\Omega = \bigcup_i \Omega_i$ of the underlaying domain, where Ω_i is images of the unite cube under a diffeomorphism. In a second step one can transfer a suitable wavelet basis for the unit cube to Ω_i. Finally, collecting everything together, gives a wavelet frame on Ω. See [61] for details. In the past years, several methods for the solution of linear equations using wavelet frames have been developed and analyzed, see e.g. [21–23, 68]. Also for nonlinear equations, results have been obtained. See [50], [53].

Besov regularity: State of the art

In recent years there were many partial differential equations studied concerning the Besov regularity of their solution. For many equations it was possible to show that the Besov regularity of the solution is indeed higher than its Sobolev regularity. We can not mention all results here, but we discuss these results which are related to this thesis.

In [20] the Besov regularity of the solution to the Dirichlet problem for harmonic functions and for the Poisson equation in Lipschitz domains was investigated. The main result states, that a harmonic function v on a bounded Lipschitz domain with $v \in B_p^\lambda(L_p(\Omega))$, $1 < p < \infty$, $\lambda > 0$ is contained in $B_\tau^\alpha(L_\tau(\Omega))$, $1/\tau = \alpha/d + 1/p$, $0 < \alpha < \lambda \cdot d/(d-1)$. Note, that $B_p^s(L_p(\Omega)) = W^s(L_p(\Omega))$ for all $p \in (1,\infty)$ and $s \in (0,\infty)\backslash\mathbb{N}$ in the sense of equivalent norms, where $W^s(L_p(\Omega))$ denotes the L_p-Sobolev space. Since α is strictly larger than λ, the Besov regularity for a harmonic function exceeds its L_p-Sobolev regularity. This statement was used to prove regularity assertions for the Dirichlet problem [20, Eq. (1.2)] and the Laplace's equations [20, Eq. (4.3)], using additionally Sobolev regularity results proven in [49]. The paper [20] was the first contribution, in which the technique of wavelet characterization of function spaces for proving Besov regularity was applied. This approach turned out to be quite profitable and was used in further papers. We will discuss more details of this technique later on. One main ingredient for estimating the wavelet coefficients is a weighted Sobolev estimate for harmonic functions. A Besov regularity result already implies a weighted norm estimate for arbitrary large smoothness parameter

$k \in \mathbb{N}$ of the form

$$\|\rho(x)^{k-\beta} \cdot |\nabla^k v(x)|\|_{L_p(\Omega)} \leq C \cdot \|v\|_{B_p^\beta(L_p(\Omega))}, \quad \rho(x) := \mathrm{dist}(x, \partial\Omega), \tag{1.0.10}$$

for $p \in [1, \infty]$, $\beta > 0$ and $k > \beta$. By $\nabla^k v(x)$ we denote the vector of all derivatives of v of order k, the norm $|\cdot|$ denotes the euclidean length. The proof of this estimate uses the mean value property of harmonic functions. Since this property is a special feature of harmonic functions, it is in general not possible to prove such an estimate for non harmonic functions. This leads to the fact, that in other results, parameters related to the weighted Sobolev regularity occur in the bound of the Besov regularity parameter, see e.g. (1.0.12), (1.0.16).

In [26] the Besov regularity of the Poisson equation in smooth and polyhedral cones was studied. This is related to this thesis since we study (Navier-)Stokes equation in polyhedral domains. These domains are a generalization of polyhedral cones, which are defined by

$$\mathcal{K} = \{x \in \mathbb{R}^3 : x = \rho_0(x) \cdot \omega(x),\ 0 < \rho_0(x) < \infty, \omega(x) \in \mathcal{O}\},$$

where $\mathcal{O}$ is a curvilinear polygon on the unit sphere bounded by the arcs $\gamma_1, ..., \gamma_n$. Suppose that the boundary $\partial\mathcal{K}$ consists of the vertex $x = 0$, the edges $M_1, ..., M_n$ and the faces $\Gamma_j := \{x : x/|x| \in \gamma_j\}$, $j = 1, ..., n$. The angle at edge M_j will be denoted by θ_j. Furthermore we define for $x \in \mathcal{K}$ the function $r_j(x) := \mathrm{dist}(x, M_j)$. For $r_0 > 0$ we define the truncated cone $\mathcal{K}_0 := \{x \in \mathcal{K} : |x| < r_0\}$.

The proof of Besov regularity is again performed by estimating the wavelet coefficients of the decomposition in a proper way. An important tool for doing this are weighted Sobolev estimates corresponding to the weighted Sobolev norm as defined next. The corresponding weighted Sobolev space on $\mathcal{K}$ is for $l \in \mathbb{N}_0$, $\beta \in \mathbb{R}$, $\vec{\delta} = (\delta_1, ..., \delta_n) \in \mathbb{R}^n$, $\delta_j > -1$ defined by the norm

$$\|w\|_{W^{l,2}_{\beta,\vec{\delta}}(\mathcal{K})} := \left(\int_{\mathcal{K}} \sum_{|\alpha| \leq l} \rho_0(x)^{2(\beta - l + |\alpha|)} \prod_{k=1}^{n} \left(\frac{r_k(x)}{\rho_0(x)} \right)^{2\delta_k} |D^\alpha w(x)|^2 \mathrm{d}x \right)^{1/2}.$$

A definition of trace spaces $W^{l-1/2,2}_{\beta,\vec{\delta}}(\Gamma_j)$ can be found in [56]. It was proven (see [26, Theorem 3.1]), that the Besov regularity of the unique solution to

$$-\Delta u = f \quad \text{in } \mathcal{K},\ \frac{\partial u}{\partial n_j} = \Gamma_j,\ j = 1, ..., n \tag{1.0.11}$$

for $f \in W^{l-2,2}_{\beta,\vec{\delta}}(\mathcal{K}) \cap L_2(\mathcal{K})$, $g_j \in W^{l-3/2,2}_{\beta,\vec{\delta}}(\Gamma_j)$ is contained in

$$B^s_\tau(L_\tau(\mathcal{K}_0)), \quad 1/\tau = s/3 + 1/2,\ s < \min\left(l, 3/2 \cdot \alpha_0, 3 \cdot (l - |\delta|)\right), \tag{1.0.12}$$

where $|\delta| = \delta_1 + ... + \delta_n$. The number $\alpha_0 > 3/2$ is a value depending on $\mathcal{K}_0$, such that the solution of (1.0.11) is contained in $H^\alpha(\mathcal{K}_0)$ for all $\alpha < \alpha_0$.

In [27] the question of Besov regularity to nonlinear elliptic partial differential equations in a bounded Lipschitz domain was addressed. They considered equations of the form

$$-\Delta u(x) + g(x, u(x)) = f(x) \quad \text{in } \Omega, \quad u(x) = 0 \quad \text{on } \partial\Omega.$$

The (in general) nonlinear function g has to fulfill some smoothness assumptions and certain growth conditions, see [27, Section 3.2]. The above nonlinear problem was reformulate as a fixed point problem defined in the scale of Besov spaces $B^s_\tau(L_\tau(\Omega))$, $1/\tau = s/d + 1/2$. Since the above scale includes quasi-Banach spaces, Banach's fixed point theorem is not applicable. They used a fixed point theorem as stated in [58, Chapter 6.3.1]. The growth condition on g are needed to achieve estimates of the form [58, Chapter 6.3.1, Eq. (4)], which are needed to apply the fixed point theorem.

There are already positive results concerning the Besov regularity of the Stokes system. For the stationary Stokes problem on a polygonal domain contained in $\mathbb{R}^2$ was established a result which states that for body force $f \in H^m(\Omega)^2$, $\operatorname{div} u = h \in H^{m+1}(\Omega)$ one has $u \in B^s_\tau(L_\tau(\Omega))^2$, $s < m+2$, $\pi \in B^s_\tau(L_\tau(\Omega))$, $s < m+1$, $1/\tau = s/2 + 1/2$, see [16]. The proof is performed by splitting the solution into a sum of two components

$$u = u_I + u_B, \quad \pi = \pi_I + \pi_B$$

by using a suitable truncation function. The first summands u_I and π_I, respectively, belong to the functions in the interior of the domain. The regularity in these parts are achieved by applying regularity theory for smooth domains. The second summands u_B and π_B belong to the sector parts, where the point singularities of the domain are located. These parts are estimated by using the technique of wavelet characterization of Besov spaces. In [57] the Besov regularity of the Stokes equation in general bounded Lipschitz domain contained in $\mathbb{R}^d$, $d \geq 2$ was investigated. The authors used boundary integral methods to establish their results, which do not solely cover the scale of Besov spaces $B^s_\tau(L_\tau(\Omega))$, $1/\tau = s/d + 1/2$. In this specific scale, we could (partly) improve their results, see Remark 3.3.4 for a detailed discussion.

Discussion of the results in this thesis

Both contributions to this thesis (see Chapter 2, Chapter 3 and [40, 41]) address the question: Which Besov regularity possesses the solution u to a (Navier-)Stokes equation in the scale

$$B^\alpha_\tau(L_\tau(\Omega)), \quad \frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{2} \tag{1.0.13}$$

and is it higher than its L_2-Sobolev regularity? As mentioned above, this positive result in this direction was proven in numerous works for several problems. Therefore, it was conjectured, that this is also true for Navier-Stokes equations on domains with non-smooth boundary. Here we give an overview of the results that we have achieved. One main feature our results have in common is that they are proven by exploiting weighted norm estimates. These weighted spaces, i.e. especially the weights, depend on the shape of the underlaying domain.

In the second chapter we consider the stationary Stokes equation (St2) and Navier-Stokes equation (NavSt2) on a polyhedral domain, where we also include the case $\operatorname{div} u = h \neq 0$. The basic type of a polyhedral domain is a polyhedral cone with vertex at the origin as defined above. General polyhedral domains $\mathcal{G}$ are usually defined by means of diffeomorphism which maps the domain local to a polyhedral cone (see [56, Chapter 8.1] for details):

(i) The boundary $\partial\mathcal{G}$ consists of smooth open two-dimensional manifolds Γ_j ($j = 1, ..., N$), smooth curves M_k ($k = 1, ..., n$) and vertices $x^{(1)}, ..., x^{(d')}$.

(ii) For every $\xi \in M_k$ there exist a neighborhood U_ξ and a diffeomorphism κ_ξ wich maps $\mathcal{G} \cap U_\xi$ onto $D_\xi \cap B_1$ where D_ξ is a dihedron and B_1 is the unit ball.

(iii) For every vertex $x^{(i)}$ there exists a neighbourhood U_i and a diffeomorphism κ_i mapping $\mathcal{G} \cap U_i$ onto $\mathcal{K}_i \cap B_1$ where $\mathcal{K}_i$ is a polyhedral cone with vertex at the origin.

We only consider the cases of domains with

$$\kappa_i : \mathcal{G} \cap U_i \to \mathcal{K}_i \cap B_i, x \mapsto x + b, \tag{1.0.14}$$

where b is a vector in $\mathbb{R}^3$ independent of x. We will explain later on, why this restriction comes into play.

For $l \in \mathbb{N}_0$, $\beta = (\beta_1, ..., \beta_{d'}) \in \mathbb{R}^{d'}$ and $\delta := (\delta_1, ..., \delta_n) \in \mathbb{R}^n$ with $\delta_k > -1$ for $k = 1, ..., n$ we define the weighted Sobolev space $W^{l,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})$ to be the closure of the set $\mathcal{C}_0^\infty(\overline{\mathcal{G}} \backslash \{x^{(1)}, ..., x^{(d')}\})$ with respect to the norm

$$\|u\|_{W^{l,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})} = \left(\sum_{j=1}^{d'} \int_{G \cap U_j} \sum_{|\alpha| \leq l} \rho_j(x)^{2(\beta_j - l + |\alpha|)} \prod_{k \in X_j} \left(\frac{r_k(x)}{\rho_j(x)} \right)^{2\delta_k} |D^\alpha u(x)|^2 \mathrm{d}x \right)^{1/2}.$$

The set X_j denotes the collection of all indices k such that $x^{(j)}$ is an end point of the edge M_k.

Sobolev regularity results were proven from M. Dauge, see [33]. She showed that for $(f, h) \in L_2(\mathcal{G})^3 \cap H^{\alpha_0}(\mathcal{G})$, $0 < \alpha_0 < 1/2$ the solution (u, π) is contained in $H^{\alpha_0+1}(\mathcal{G})^3 \times H^{\alpha_0}(\mathcal{G})$. As already pointed out above, a higher Sobolev regularity of the given data f and h do not guarantee a higher Sobolev regularity of the solution to stationary (Navier-) Stokes equations. This observation motivates to consider these equations in weighted Sobolev spaces. In [56] was proven, that if the given data of the Stokes equation are contained in a weighted Sobolev space with suitable weight parameters, the solution u and the corresponding pressure term π are contained in the corresponding weighted space with increased smoothness parameter:

$$(f, h) \in W^{l-2,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})^3 \times W^{l-1,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G}) \Longrightarrow (u, \pi) \in W^{l,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})^3 \times W^{l-1,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G}), \quad l \geq 2. \tag{1.0.15}$$

The discussions above suggest that weighted norm estimates can be used to establish Besov regularity in the scale $B^s_\tau(L_\tau(\mathcal{G}))$, $1/\tau = s/3 + 1/2$. The result (1.0.15) indicates, that indeed a positive result can be expected. Thus, the idea is to prove an embedding of (weighted) Sobolev spaces into Besov spaces. Indeed, we can show the following result:

$$W^{l,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})^3 \cap H^{s_0}(\mathcal{G})^3 \times W^{l-1,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G}) \cap H^{t_0}(\mathcal{G}) \hookrightarrow B^{s_1}_{\tau_1}(L_{\tau_1}(\mathcal{G}))^3 \times B^{s_2}_{\tau_2}(L_{\tau_2}(\mathcal{G})), \ \frac{1}{\tau_i} = \frac{s_i}{3} + \frac{1}{2},$$

$i = 1, 2$ for

$$s_1 < \min\left(l, 3/2 \cdot s_0, 3 \cdot (l - |\vec{\delta}|)\right), \ s_2 < \min\left(l - 1, 3/2 \cdot t_0, 3 \cdot (l - 1 - |\vec{\delta}|)\right), \tag{1.0.16}$$

see Remark 2.2.2 in this thesis. Consequently, valid results, which ensures that the solution u and the corresponding pressure term have certain (weighted) Sobolev regularity, immediately lead to a Besov regularity result.

The embedding is proven by estimating the coefficients of the wavelet decomposition. The norm equivalence (1.0.7) is the basis for this approach. Let us briefly discuss the basic ideas. We consider one component of $(v_1, v_2, v_3, v_4) \in W^{l,2}_{\beta,\vec{\delta}}(\mathcal{G})^3 \cap H^{s_0}(\mathcal{G})^3 \times W^{l-1,2}_{\beta,\vec{\delta}}(\mathcal{G}) \cap H^{t_0}(\mathcal{G})$ and denote it by v. We use the diffeomorphism defined in (1.0.14) in order to transform v to the corresponding truncated polyhedral cones $\mathcal{K}_i$. In order to keep notation simple we denote the transformed v also by v. We cannot consider general diffeomorphism κ_i since we first prove Besov regularity in the polyhedral cones. For general diffeomorphism κ_i we cannot guarantee $u \circ \kappa_i^{-1} \in B^s_\tau(L_\tau(\mathcal{G} \cap U_i))$ for $u \in B^s_\tau(L_\tau(\mathcal{K} \cap B_i))$ for parameters s and τ as considered in the regularity results of this paper. We know by assumption, that v is contained in a L_2-Sobolev space. We use Rychkov's extension operator, which simultaneously extends Sobolev- and Besov spaces on a Lipschitz domain to the corresponding spaces on the whole euclidean plane, independent of the defining smoothness and metric parameters, see [59]. Since we assume, that v has a certain L_2-Sobolev regularity, we can extend v to the whole euclidean plane, such that the extension has the same regularity. Since this operator is continuous, it is sufficient to estimate the Besov regularity of the extension. This is performed by using the equivalent norm as displayed in (1.0.7). We only have to consider those wavelets, whose support have non empty intersection with the underlying domain. Basically, since the underlying domain is bounded, the first summand in (1.0.7) is bounded. The main effort must be spend in the treatment of the second summand. The proof is quite technical, therefore we only discuss the basic ideas without going into detail. Since the wavelets are assumed to be compact supported, there exists a cube Q such that

$$Q_{j,k} := 2^{-j}k + 2^{-j}Q, \quad j \in \mathbb{N}_0, k \in \mathbb{Z}^3$$

contains the supports of $\tilde{\psi}_{i,j,k}$, $\psi_{i,j,k}$ for all $i \in \{1, ..., 7\}$. We split the estimate into two parts.

1. We start by estimating the coefficients corresponding to the interior wavelets, i.e., we estimate those coefficients $\langle v, \tilde{\psi}_{i,j,k} \rangle$ such that $Q_{j,k}$ is contained in the cone. We do this by considering two cases. First we give consideration to those wavelets, whose support can not be arbitrary close to the origin. The corresponding wavelet coefficients are estimated by using a Whitney type estimate, see [37]. There exists a polynomial P of degree less or equal to $l-1$ such that

$$\|\varphi - P\|_{L_p(Q_{j,k})} \leq C \cdot |Q_{j,k}|^{l/n} \cdot |\varphi|_{W^l(L_p(Q_{j,k}))}, \quad \varphi \in W^l(L_p(Q_{j,k})). \qquad (1.0.17)$$

The vanishing moment property of wavelets allow to exploit (1.0.17) to estimate the coefficients $\langle v, \tilde{\psi}_{i,j,k} \rangle$ by the Sobolev half norm of v:

$$|\langle v, \tilde{\psi}_{i,j,k} \rangle| \leq C \cdot 2^{-lj} |v|_{W^l(L_2(Q_{j,k}))}.$$

Roughly speaking, in a next step we insert suitable weights corresponding to the weighted Sobolev space as displayed above, and therefore the corresponding coefficients are bounded by the weighted Sobolev norm. Doing this, the first factor, depending on parameters of the weighted Sobolev space, remains to be estimated. The conditions (1.0.16) are sufficient to ensure boundedness of the sum over all corresponding indices of these factors.

In a second step we have to estimate all coefficients $\langle v, \tilde{\psi}_{i,j,k}\rangle$, such that $Q_{j,k}$ can be arbitrary close to the origin: For $j \geq 0$ we consider all indices (i, j, k) such that

$$0 < \operatorname{dist}(Q_{j,k}, 0) < 2^{-j}.$$

Due to the Lipschitz character of $\mathcal{K}_0$, the cardinality of this index set is bounded by 2^{2j}. Using this fact allows us to estimate the sum over all corresponding indices by $3/2 \cdot s_0$ and $3/2 \cdot t_0$, respectively.

2. In the last part we estimate the coefficients $\langle v, \tilde{\psi}_{i,j,k}\rangle$ such that $Q_{j,k}$ has non-empty intersection with the boundary. The cardinality of the set of all corresponding indices is again bounded by 2^{2j}, so we can argue as above.

In the third chapter we consider the stationary Stokes problem (St2) and the Navier-Stokes system (NavSt2) on a bounded Lipschitz domain with connected boundary contained in $\mathbb{R}^d$, $d \geq 3$. For the sake of completeness we recall the definition of a domain with Lipschitz boundary:

A bounded domain Ω with boundary $\partial\Omega$ is called a Lipschitz domain if for every $x \in \partial\Omega$ there exists a neighbourhood U of x and a bijective mapping $\phi_x : U \to B_1(0) := \{z \in \mathbb{R}^d : \|z\|_2 < 1\}$ such that ϕ_x and ϕ_x^{-1} are Lipschitz continuous and

$$\begin{aligned}
\phi_x(U \cap \Omega) &= \{z \in B_1(0), z_d > 0\},\\
\phi_x(U \cap \partial\Omega) &= \{z \in B_1(0) : z_d = 0\},\\
\phi_x(U \backslash \overline{\Omega}) &= \{z \in B_1(0), z_d < 0\}.
\end{aligned}$$

In contrast to polyhedral domains, singularities in Lipschitz domains can possibly occur everywhere on the boundary. Therefore, the weight in the definition of the weighted Sobolev spaces on Lipschitz domain, as we use them in this thesis, consists of the distance to the boundary of the domain. For $m \in \mathbb{N}_0$, $\alpha > 0$ and $p \in [1, \infty)$ the weighted Sobolev space is defined as

$$W^m_\alpha(L_p(\Omega)) := \left\{ f \in L_p(\Omega) : \|f\|^p_{W^m_\alpha(L_p(\Omega))} := \|f\|^p_{L_p(\Omega)} + \int_\Omega \rho(x)^\alpha |\nabla^m f(x)|^p_{\ell_p} \mathrm{d}x < \infty \right\},$$

where $|\nabla^m f|_{\ell_p}$ is the ℓ_p-norm of the vector $\nabla^m f$ and $\rho(x) := \operatorname{dist}(x, \partial\Omega)$, $x \in \Omega$. We were able to prove, that the solution u to the Stokes equation (St2) and the corresponding pressure term π fulfill

$$u \in B^{s_1}_{\tau_1}(L_{\tau_1}(\Omega))^d, \quad \frac{1}{\tau_1} = \frac{s_1}{d} + \frac{1}{2}, \quad 0 < s_1 < \min\left\{\frac{3}{2} \cdot \frac{d}{d-1}, 2\right\},$$

and

$$\pi \in B^{s_2}_{\tau_2}(L_{\tau_2}(\Omega)), \quad \frac{1}{\tau_2} = \frac{s_2}{d} + \frac{1}{2}, \quad 0 < s_2 < \frac{1}{2} \cdot \frac{d}{d-1},$$

if the right hand side $f \in L_2(\Omega)^d$ and the boundary data fulfill $g \in H^1(\partial\Omega)^d$. The bound 2 for the Besov regularity for u corresponds to the weighted Sobolev regularity of u, see

Proposition 3.3.2 in this thesis. A result in [3] yields that the solution u to (St2) with $f = 0$ fulfill $u \in W_1^2(L_2(\Omega))$ if $g \in H^1(\partial\Omega)^d$. We showed $(u,\pi) \in W_1^2(L_2(\Omega))^d \times W_1^1(L_2(\Omega))$ for the solution u and the corresponding pressure π even for general $f \in L_2(\Omega)^d$. Exploiting additionally Sobolev regularity results for the Stokes equation, see Proposition 3.3.1 (the proof is based on results in [3] and [42]) we can prove the statement concerning the Besov regularity of the solution, see Theorem 3.3.3 in this thesis. This result is based on the following embedding for $\alpha_0 > 0$, $\alpha > 0$, $\gamma \in \mathbb{N}$ and $\alpha < 2\gamma$:

$$H^{\alpha_0}(\Omega) \cap W_\alpha^\gamma(L_2(\Omega)) \hookrightarrow B_\tau^s(L_\tau(\Omega)), \quad \frac{1}{\tau} = \frac{s}{d} + \frac{1}{2},$$

if $0 < s < \min\{\frac{2\gamma-\alpha}{2}\cdot\frac{d}{d-1}, \alpha_0\cdot\frac{d}{d-1}, \gamma\}$. Again we see the dependence of the Besov smoothness parameter on the Sobolev regularity and the parameter of the weighted Sobolev spaces. The factor $d/(d-1)$ also occurs in [40], which is equal to $3/2$ for a subspace of $\mathbb{R}^3$. The proof of the above embedding uses again the wavelet characterization of Besov spaces by wavelets.

In order to treat the nonlinear Navier-Stokes equation we use Banach's fixed point theorem. Let us give a sketch of the proof: We reduce the Navier-Stokes equation to the Stokes equation with modified right-hand side. To this end we consider the linear solution operator of (St2)

$$\begin{aligned} L := L_{t,p,\Omega}\colon Y_{t,p,\Omega} &\to X_{t,p,\Omega} \\ (f,g) &\mapsto L(f,g) := (u,\pi), \end{aligned}$$

for suitable function spaces $Y_{t,p,\Omega}$ and $X_{t,p,\Omega}$, see Section 3.2 and Section 3.4 for details. The function u is the unique solution to (St2) with body force f and boundary value g, and π is the corresponding pressure. The operator is well-defined due to results proven in [57]. We consider the nonlinear operator

$$N : X_{t,p,\Omega} \to Y_{t,p,\Omega} \cap (L_2(\Omega)^d \times H^1(\partial\Omega)^d),\ N(u,\pi) := (f - \nu u\cdot\nabla u, g)$$

for fixed $(f,g) \in Y_{t,p,\Omega} \cap (L_2(\Omega)^d \times H^1(\partial\Omega)^d)$. This operator is well-defined, see Proof of Theorem 3.4.1 in this thesis. Consequently, the operator

$$T := L\circ N : X_{t,p,\Omega} \to X_{t,p,\Omega},\ \ (u,\pi) \mapsto L(f - \nu u\cdot\nabla u, g)$$

is also well-defined. Obviously, a fixed point of T is a solution to (NavSt2). The existence of a fixed point was proven by using Banach's fixed point theorem. The restriction (3.4.4), see Section 3.4 in this thesis, is needed to ensure, that T is a contraction on the subspace

$$A := \{(v,q) \in X_{r,p,\Omega}\colon \|L_{t,p,\Omega}\|\cdot\nu\cdot C_{t,p,\Omega}\cdot\|(v,q)\|_{X_{t,p,\Omega}} \le 1/2\},$$

where $C_{t,p,\Omega} > 0$ is a finite constant, see also Remark 3.4.2, (ii) in this thesis. We further note that

$$X_{t,p,\Omega} \hookrightarrow H^1(\Omega)^d \times (L_2(\Omega)/\mathbb{R}_\Omega)\,,$$

holds, i.e. any fixed point is a solution to the Stokes equation with modified right hand side $f - \nu\cdot u\cdot\nabla u \in L_2(\Omega)^d$. The desired Besov regularity results

$$u \in B_{\tau_1}^{s_1}(L_{\tau_1}(\Omega))^d, \quad \frac{1}{\tau_1} = \frac{s_1}{d} + \frac{1}{2}, \quad 0 < s_1 < \min\left\{\frac{3}{2}\cdot\frac{d}{d-1}, 2\right\}, \tag{1.0.18}$$

and

$$\pi \in B^{s_2}_{\tau_2}(L_{\tau_2}(\Omega)), \quad \frac{1}{\tau_2} = \frac{s_2}{d} + \frac{1}{2}, \quad 0 < s_2 < \frac{1}{2} \cdot \frac{d}{d-1}, \tag{1.0.19}$$

follow by applying Theorem 3.3.3 in this thesis. Different to the approach in [27] as described above, we do not apply the fixed point theorem directly in the scale of Besov spaces $B^s_\tau(L_\tau(\Omega))$, $1/\tau = s/d + 1/2$. Since the space $X_{t,p,\Omega}$ is a Banach space, the application of Banach's fixed point theorem is possible and it turned out to be profitable.

We oberserv in all results, that under some technical conditions, the Besov regularity of the solution to (Navier-)Stokes equations is higher than its Sobolev regularity. In summary we conclude that the development of adaptive wavelet schemes for solving (Navier-)Stokes equation on polyhedral domains and Lipschitz domains is completely justified.

2 Besov Regularity for the Stokes and the Navier-Stokes System in Polyhedral Domains

Author: Frank Eckhardt
Journal: Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM) **95** (2015), no. 2, 1161–1173.
Abstract: In this paper we study the regularity of solutions to the Stokes and the Navier-Stokes system in polyhedral domains contained in $\mathbb{R}^3$. We consider the scale $B^s_\tau(L_\tau)$, $1/\tau = s/3 + 1/2$ of Besov spaces which determines the approximation order of adaptive numerical wavelet schemes and other nonlinear approximation methods. We show that the regularity in this scale is large enough to justify the use of adaptive methods. The proofs of the main results are performed by combining regularity results in weighted Sobolev spaces with characterizations of Besov spaces by wavelet expansions.
Subject Classification: 30H25, 35B65, 42C40, 46E35, 65T60, 76D07.
Key Words: Stokes system, Besov spaces, weighted Sobolev spaces, wavelets, characterization of function spaces, nonlinear and adaptive approximation.

2.1 Introduction

In this paper we are concerned with the $3D$-Navier-Stokes system

$$\begin{aligned} -\nu\Delta u + \sum_{j=1}^{3} u_j \frac{\partial u}{\partial x_j} + \nabla p &= f \quad \text{in } \mathcal{G} \\ \operatorname{div} u &= g \quad \text{in } \mathcal{G} \\ u &= 0 \quad \text{on } \Gamma_j,\ j = 1, ..., N \end{aligned}$$

and the $3D$-Stokes system

$$\begin{aligned} -\Delta u + \nabla p &= f \quad \text{in } \mathcal{G} \\ \operatorname{div} u &= g \quad \text{in } \mathcal{G} \\ u &= 0 \quad \text{on } \Gamma_j,\ j = 1, ..., N \end{aligned}$$

on a polyhedral domain $\mathcal{G} \subset \mathbb{R}^3$ where Γ_j are the faces of the domain. The Navier-Stokes equations and its linearized version, the Stokes equations, describe the motion of a viscous fluid. Here $\Delta := \sum_{k=1}^{3} \frac{\partial^2}{\partial^2 x_i}$ is the Laplace operator and by $\nabla := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)^T$ we denote the gradient. As usual, $u(\cdot) = (u_1(\cdot), u_2(\cdot), u_3(\cdot))$ denotes the velocity field and p stands for the pressure field. Our aim is to prove regularity results for each component of the solution (u, p) in the specific scale of Besov spaces $B^s_\tau(L_\tau(\mathcal{G}))$, $1/\tau = s/3 + 1/2$ (see [64, Chapter 2 and 3] for definition of Besov spaces). This specific scale comes into play when studying the convergence rate of adaptive numerical schemes. We will explain the relationship very briefly in the following. Let us for the sake of simplicity assume $g = 0$, then the weak formulation of the Stokes problem is given by

$$a(u, v) + b(p, v) = f(v) \text{ for all } v \in H^1_0(\mathcal{G})^3,$$
$$b(q, u) = 0 \text{ for all } q \in L_{2,0}(\mathcal{G})$$

with

$$a(u, v) := \int_{\mathcal{G}} \sum_{i,j=1}^{3} \frac{\partial u_i}{\partial x_j} \frac{\partial v_j}{\partial x_j} \mathrm{dx},$$

$$b(p, v) := -\int_{\mathcal{G}} p(x)(\mathrm{div} v)(x) \mathrm{d}x$$

and

$$f(v) := \int_{\mathcal{G}} \langle f, v \rangle \, \mathrm{d}x.$$

$H^1_0(\mathcal{G})$ is the closure of $\mathcal{C}^\infty_0(\mathcal{G})$ with respect to the $H^1(\mathcal{G})$-Sobolev norm and $L_{2,0}(\mathcal{G}) := \{p \in L_2(\mathcal{G}) : \int_{\mathcal{G}} p(x)\mathrm{d}x = 0\}$. For detailed definition of Sobolev spaces see [64, Chapter 2 and 3]. To treat the equation numerically we use the Galerkin approach, i.e. we consider a nested sequence $\{S_j \times \tilde{S}_j\}_{j \geq 0}$ of finite dimensional linear subspaces of $H^1_0(\mathcal{G})^3 \times L_{2,0}(\mathcal{G})$ such that the union is dense in $H^1_0(\mathcal{G})^3 \times L_{2,0}(\mathcal{G})$. This leads to the problems

$$a(u_j, v) + b(p_j, v) = f(v) \text{ for all } v \in S_j,$$

$$b(q, u_j) = 0 \text{ for all } q \in \tilde{S}_j.$$

In many cases, the approximation spaces S_j and $\tilde{S}_j$ are constructed by means of a uniform grid refinement strategy. This kind of approximation is called *linear approximation*. It is well-known that the performance usually depends on the Sobolev regularity of the solution. For details we refer to [18], [35], [44] and [47]. However, in practice, due to singularities at the boundary of the domain, this Sobolev regularity might not be very high and therefore the approximation rate of uniform schemes drops down. In this setting, the use of adaptive strategies seems to be reasonable. Roughly speaking, an adaptive scheme corresponds to nonuniform grid refinement where the underlaying space is only refined in regions where the current approximation is still far away from the exact solution. In this paper we are in particular interested in adaptive wavelet algorithms. In this setting, an adaptive scheme can be interpreted as a nonlinear approximation scheme, and for that reason best n-term approximation serves as a benchmark for adaptive strategies (see [11], [18] for further information): Instead of linear spaces one uses the nonlinear manifold $\mathcal{M}_n$ of all functions

$$S = \sum_{\lambda \in \Lambda} c_\lambda \psi_\lambda, \quad |\Lambda| \leq n,$$

where $\{\psi_\lambda\}_{\lambda\in\mathcal{J}}$ is a suitable wavelet basis. We define the approximation error

$$\sigma_n(u)_{L_2} := \inf_{S\in\mathcal{M}_n} ||u-S||_{L_2}.$$

In contrast to linear approximation schemes, the order of convergence for best n-term wavelet approximation does not depend on the Sobolev regularity, but on the Besov smoothness, i.e.

$$\sum_{n=1}^{\infty}[n^{s/3}\sigma_n(u)_{L_2}]^\tau \frac{1}{n} < \infty \Longleftrightarrow u \in B^s_\tau(L_\tau),\ 1/\tau = s/3+1/2,$$

see [35], [36] for further details. As suggested above this shows that it is profitable to use adaptive schemes if the Besov regularity of the solution in this specific scale is higher than the Sobolev regularity. It is known that in smooth domains the Sobolev regularity of the solution increases if the Sobolev regularity of f and g increase (see e.g. [47] for details for the Stokes system). If the domain is only Lipschitz, this conclusion is no longer true due to singularities at the boundary (see Proposition 2.5.1), but there is some hope that these singularities do not influence the Besov smoothness in the scale $1/\tau = s/d + 1/2$. Indeed there are already some positive results in this direction for a large class of partial differential equations: In [16] it was shown that the Besov regularity of the 2D-Stokes system in a polygonal domain is under some technical conditions higher than the Sobolev regularity. In [20] the Besov regularity of the solution to the Dirichlet problem for harmonic functions and for the Poisson equation in Lipschitz domains was investigated. A result which is similar to our main statement was proven in [26] for Poisson equation. In many cases these results are proven by using the characterization of Besov spaces by means of weighted sequence norms of coefficients related to the wavelet decomposition of the solution. Similar to the investigation in [26] we estimate the wavelet coefficient of the solution by exploiting regularity results related to weighted Sobolev spaces introduced by Maz'ya and Rossmann (see [56, Chapter 10 and 11]). Furthermore there are also results for nonlinear partial differential equations, see [27]. In this paper we consider the Navier-Stokes system and the Stokes system on a polyhedral domain where singularities at the vertices and on the edges might occur. To prove regularity results we need certain weighted Sobolev spaces which take these singularities into account. We denote these spaces by $W^{l,2}_{\vec{\beta},\vec{\delta}}$, for details see Section 2.2 and Section 2.3. In this paper we establish a result which shows that under certain technical conditions the Besov regularity to the solution of the Navier-Stokes respectively the Stokes problem is higher than the Sobolev regularity if additionally the parameter l is not so small: For suitable values of l the Besov regularity is at least 3/2 times higher than the Sobolev regularity. For details, we refer to Theorem 2.2.1, Theorem 2.3.1 and Theorem 2.3.2, respectively.

This paper is organized as follows: In the second section we state and prove a result for the Navier-Stokes system on a polyhedral domain. In the third section we show analog results for the Stokes system. As mentioned above we use weighted Sobolev estimates. In Section 4 we discuss some norm estimates for the solution of the considered Navier-Stokes and Stokes equations. In Appendix 5, we discuss the Sobolev regularity and results for weighted Sobolev regularity of the solution as far as they are needed for our purposes. In the last section we recall the definition of Besov and Sobolev spaces and explain the connection between the Besov regularity of a distribution and the decay of its wavelet coefficients.

2.2 Besov regularity for the Navier-Stokes System in polyhedral domains

In this section we state and prove the main result of this paper: We will show that under some technical assumptions the Besov regularity of the solution to

$$\begin{aligned} -\nu\Delta u + \sum_{j=1}^{3} u_j \frac{\partial u}{\partial x_j} + \nabla p = f \quad &\text{in } \mathcal{G} \\ \text{div } u = g \quad &\text{in } \mathcal{G} \\ u = 0 \quad &\text{on } \Gamma_j,\ j = 1, ..., N \end{aligned} \tag{2.2.1}$$

in the scale $1/\tau = s/3 + 1/2$ is 3/2 times higher than its Sobolev regularity. We consider the Navier-Stokes equation on polyhedral domains. The basic type of a polyhedral domain is a polyhedral cone. Define $\rho(x) := |x|$. Let

$$\mathcal{K} = \{x \in \mathbb{R}^3 : x = \rho(x) \cdot \omega(x),\ \ 0 < \rho(x) < \infty, \omega(x) \in \Omega\} \tag{2.2.2}$$

be a polyhedral cone with vertex at the origin where Ω is a curvilinear polygon on the unit sphere bounded by the arcs $\gamma_1, ..., \gamma_d$. Suppose that the boundary $\partial\mathcal{K}$ consists of the vertex $x = 0$, the edges $M_1, ..., M_d$ and the faces $\Gamma_j := \{x : x/|x| \in \gamma_j\}$, $j = 1, ..., d$. The angle at edge M_j will be denoted by θ_j. Furthermore we define for $x \in \mathcal{K}$ the function $r_j(x) := \text{dist}(x, M_j)$. By $\mathcal{K}_0$ we denote an arbitrary truncated cone, i.e. there exists a positive real number r_0 such that

$$\mathcal{K}_0 = \{x \in \mathcal{K} : |x| < r_0\}.$$

Our technique requires regularity assertions in weighted Sobolev spaces. Following Maz'ya and Rossmann we define these spaces for cones (see [56, Chapter 7] for details): Let l be a nonnegative integer, $\beta \in \mathbb{R}$ and $\vec{\delta} = (\delta_1, ..., \delta_d) \in \mathbb{R}^d$, $\delta_j > -1$ for $j = 1, ..., d$. We define the space $W^{l,2}_{\beta,\vec{\delta}}(\mathcal{K})$ as the closure of the set $\mathcal{C}_0^\infty(\overline{\mathcal{K}}\backslash\{0\})$ with respect to the norm

$$||u||_{W^{l,2}_{\beta,\vec{\delta}}(\mathcal{K})} := \left(\int_{\mathcal{K}} \sum_{|\alpha| \le l} \rho(x)^{2(\beta - l + |\alpha|)} \prod_{k=1}^{d} \left(\frac{r_k(x)}{\rho(x)} \right)^{2\delta_k} |D^\alpha u(x)|^2 \mathrm{d}x \right)^{1/2}.$$

General polyhedral domains are usually defined by means of diffeomorphism which maps the domain local to a polyhedral cone (see [56, Chapter 8.1] for details):

(i) The boundary $\partial\mathcal{G}$ consists of smooth open two-dimensional manifolds Γ_j ($j = 1, ..., N$), smooth curves M_k ($k = 1, ..., d$) and vertices $x^{(1)}, ..., x^{(d')}$.

(ii) For every $\xi \in M_k$ there exist a neighborhood U_ξ and a diffeomorphism κ_ξ which maps $\mathcal{G} \cap U_\xi$ onto $D_\xi \cap B_1$ where D_ξ is a dihedron and B_1 is the unit ball.

(iii) For every vertex $x^{(i)}$ there exist a neighborhood U_i and a diffeomorphism κ_i mapping $\mathcal{G} \cap U_i$ onto $\mathcal{K}_i \cap B_1$ where $\mathcal{K}_i$ is a polyhedral cone with vertex at the origin.

We will restrict ourselves to the case of

$$\kappa_j : \mathcal{G} \cap U_j \to \mathcal{K}_j \cap B_j, x \mapsto x + b,$$

where is a vector in $\mathbb{R}^3$ independent of x.

Now we recall the definition of weighted Sobolev spaces corresponding to polyhedral domains (see again [56, Chapter 8] for details). We put

$$r_k(x) := \mathrm{dist}(x, M_k),\ k = 1, ..., d,$$

$$\rho_j(x) := \mathrm{dist}(x, x^{(j)}).\ j = 1, ..., d'.$$

With X_j we denote the set of indices k such that $x^{(j)}$ is an end point of the edge M_k. Let $U_1, ..., U_{d'}$ be domains in $\mathbb{R}^3$ such that

$$U_1 \cup ... \cup U_{d'} \supset \overline{G} \text{ and } \overline{U_j} \cap \overline{M_k} = \emptyset \text{ if } k \notin X_j.$$

For $l \in \mathbb{N}_0$, $\beta = (\beta_1, ..., \beta_{d'}) \in \mathbb{R}^{d'}$ and $\delta := (\delta_1, ..., \delta_d) \in \mathbb{R}^d$ with $\delta_k > -1$ for $k = 1, ..., d$ we define the weighted Sobolev space $W^{l,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})$ to be the closure of the set $\mathcal{C}_0^\infty(\overline{\mathcal{G}} \backslash \{x^{(1)}, ..., x^{(d')}\})$ with respect to the norm

$$||u||_{W^{l,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})} = \left(\sum_{j=1}^{d'} \int_{G \cap U_j} \sum_{|\alpha| \le l} \rho_j(x)^{2(\beta_j - l + |\alpha|)} \prod_{k \in X_j} \left(\frac{r_k(x)}{\rho_j(x)} \right)^{2\delta_k} |D^\alpha u(x)|^2 \mathrm{d}x \right)^{1/2}.$$

In our case we consider polyhedral domains for which we can find a partition of unity $\{\sigma_j\}_{j=1}^{d'}$ related to the domain decomposition $\mathcal{G} = \bigcup_{j=1}^{d'} \mathcal{G} \cap U_j$ which fulfills

$$||\sigma_j v||_{B^s_p(L_p(G \cap U_j))} \lesssim ||v||_{B^s_p(L_p(G))},\ \ 1/p = s/3 + 1/2, \tag{2.2.3}$$

uniformly for all $v \in B^s_p(L_p(G))$. The symbol $\lesssim$ means that the estimate is true up to a constant. In many cases the condition (2.2.3) is fulfilled. For example investigations for the L-shaped domain can be found in [21, Section 4.2]. Let us introduce a further notation: By $|\vec{\delta}|$ we denote the sum of all δ_k. Now we can formulate and prove the following result:

Theorem 2.2.1. *It exists a countable set $E \subset \mathbb{C}$ such that for all $\vec{\beta} \in \mathbb{R}^{d'}$, $\vec{\delta} \in \mathbb{R}^d$ with*

$$\beta^* := \max_{j=1,...,d'} \beta_j < 1,$$

$$Re\ \lambda \neq 1/2 - \beta_j \quad \textit{for all } \lambda \in E \tag{2.2.4}$$

and

$$\max(0, 1 - \mu_k) < \delta_k < 1,\ \ k = 1, ..., d,$$

where $\mu_k = \pi/\theta_k$ if $\theta_k < \pi$ and μ_k is the minimum of all solutions of $\mu \sin(\theta_k) + \sin(\mu\theta_k) = 0$ if $\theta_k > \pi$, the following holds: If $(f, g) \in W^{0,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})^3 \times W^{1,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G}) \cap H^{1+\varepsilon}(\mathcal{G})$, $\varepsilon > 0$, g fulfills the compatibility condition

$$g|_{M_k} = 0, \;\; k = 1, ..., d$$

and if a solution (u, p) *of (2.2.1) is contained in* $H^{s_0}(\mathcal{G})^3 \times H^{t_0}(\mathcal{G})$ *then*

$$u \in B^{s_1}_{\tau_1}(L_{\tau_1}(\mathcal{G}))^3, \;\; \frac{1}{\tau_1} = \frac{s_1}{3} + \frac{1}{2}, \;\; s_1 < \min\Big(2, 3/2 \cdot s_0, 3 \cdot (2 - |\vec{\delta}|)\Big), \tag{2.2.5}$$

$$p \in B^{s_2}_{\tau_2}(L_{\tau_2}(\mathcal{K}_0)), \;\; \frac{1}{\tau_2} = \frac{s_2}{3} + \frac{1}{2}, \;\; s_2 < \min\Big(1, 3/2 \cdot t_0, 3 \cdot (1 - |\vec{\delta}|)\Big) \tag{2.2.6}$$

Proof. : To prove the theorem we will study each component of the solution $(u, p) = (u_1, u_2, u_3, p)$ to (2.2.1) separately. Let v be one of the functions u_1, u_2, u_3 or p, respectively. Moreover we define

$$\mu := \begin{cases} 2 & v = u_i \text{ for } i = 1, 2 \text{ or } 3 \\ 1 & v = p \end{cases}$$

and

$$\alpha := \begin{cases} s_0 & v = u_i \text{ for } i = 1, 2 \text{ or } 3 \\ t_0 & v = p \end{cases}. \tag{2.2.7}$$

From Proposition 2.5.6 we obtain $v \in W^{\mu,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})$. Using the transformation $\kappa_j = \cdot + b$ introduced in the beginning of this section we define the function

$$v_j := v \circ \kappa_j^{-1} : \mathcal{K}_j \cap B_j \to \mathbb{R}.$$

For the sake of notation simplicity we denote v_j by v, $\mathcal{K}_j$ by $\mathcal{K}$ and $\mathcal{K}_j \cap B_j$ by $\mathcal{K}_0$. We obtain

$$\left(\int_{\mathcal{K}_0} \sum_{|\alpha| \le \mu} \rho(x)^{2(\beta - l + |\alpha|)} \prod_{k=1}^{d} \left(\frac{r_k(x)}{\rho(x)} \right)^{2\delta_k} |D^\alpha v(x)|^2 \mathrm{d}x \right)^{1/2} < \infty \text{ and } v \in H^\alpha(\mathcal{K}_0), \tag{2.2.8}$$

with the abbreviation $\beta := \beta_j$. The proof uses the characterizations of Besov spaces by wavelet expansions. Therefore we estimate the wavelet coefficients of v in order to show that the equivalent quasi-norm as outlined in Proposition 2.6.1 is bounded. We make the following agreements concerning the wavelet characterization of Besov spaces on $\mathbb{R}^3$: For the sake of simplicity we associate to each dyadic cube $I := 2^{-j}k + 2^{-j}[0, 1]^3$ the functions

$$\eta_I := \tilde{\psi}_{i,j,k}, \;\; j \in \mathbb{N}_0, \;\; k \in \mathbb{Z}^3, \; i = 1, ..., 7,$$

see Section 2.6 for details. Note that we disregard the dependence on i. By η_I^* we denote the corresponding element of the primal basis. Because the supports of the wavelets are assumed to be compact there exists a cube Q centered at the origin such that

$$Q(I) := 2^{-j}k + 2^{-j}Q$$

contains the support of η_I and η_I^* for all I. We will prove the result in three steps: In a first step we will estimate the coefficients $|\langle v, \eta_I \rangle|$ for with $Q(I)$ is contained in the truncated cone and the distance from $Q(I)$ to the origin is not to small. We will specify this later. In a second step we look for the coefficients for which $Q(I)$ is contained in $\mathcal{K}_0$ but $Q(I)$ can be located arbitrarily close to the origin. In the last step we consider the coefficients for which the intersection of $Q(I)$ and the boundary of $\mathcal{K}_0$ is not empty.

step 1: We start by estimating the coefficients $|\langle v, \eta_I\rangle|$ with $Q(I) \subset \mathcal{K}_0$. We put

$$\rho_I := \operatorname{dist}(Q(I), 0)$$

and

$$r_I := \min_{j=1,\dots,d} \min_{x \in Q(I)} r_j(x).$$

For $j \in \mathbb{N}_0$ consider the set of indices:

$$\Lambda_j := \{I : Q(I) \subset \mathcal{K}_0,\ \ 2^{-3j} \leq |I| \leq 2^{-3j+2}\}.$$

Then we define a subset of Λ_j for $k \in \mathbb{N}$:

$$\Lambda_{j,k} := \{I \in \Lambda_j : k2^{-j} \leq \rho_I < (k+1)2^{-j}\}.$$

Further we put for $m \in \mathbb{N}$

$$\Lambda_{j,k,m} := \{I \in \Lambda_{j,k} : m2^{-j} \leq r_I < (m+1)2^{-j}\}.$$

We observe the following facts:

- There exists a general number C such that
$$\Lambda_{j,k} = \emptyset, k > C2^j. \tag{2.2.9}$$
- For the cardinality $|\Lambda_{j,k}|$ of $\Lambda_{j,k}$ holds
$$|\Lambda_{j,k}| \lesssim k^2,\ \ k \in \mathbb{N}. \tag{2.2.10}$$
- It holds
$$|\Lambda_{j,k,m}| \lesssim m,\ \ m \in \mathbb{N}. \tag{2.2.11}$$

In every case the constant is independent of j, k and m. Recall that

$$|v|_{W^\mu(L_2(Q(I)))} := \left(\int_{Q(I)} |\nabla^\mu v(x)|^2 \mathrm{d}x\right)^{1/2},$$

which is well defined because of (2.2.8). The vector space of polynomials of order at most μ is finite dimensional so there exists a polynomial P_I such that

$$||v - P_I||_{L_2(Q(I))} = \inf\left\{||v - P||_{L_2(Q(I))} : P \text{ is a polynomial of degree } \leq \mu\right\}.$$

The vanishing moment property of wavelets, see Subsection 2.6, Hölder's inequality and a classical Whitney-estimate (see [37, Theorem 3.4]) lead to

$$
\begin{aligned}
|\langle v, \eta_I \rangle| &\le ||v - P_I||_{L_2(Q(I))} ||\eta_I||_{L_2(Q(I))} \\
&\lesssim |I|^{\mu/3} \cdot |v|_{W^\mu(L_2(Q(I)))}.
\end{aligned}
$$

For $I \in \Lambda_j$ we obtain

$$
|\langle v, \eta_I \rangle| \lesssim 2^{-\mu j} |v|_{W^\mu(L_2(Q(I)))}.
$$

Let $0 < \tau < 2$. Summing up over $I \in \Lambda_{j,k}$ yields

$$
\begin{aligned}
\sum_{I \in \Lambda_{j,k}} |\langle v, \eta_I \rangle|^\tau &\lesssim \sum_{I \in \Lambda_{j,k}} 2^{-\mu j \tau} \left(\int_{Q(I)} |\nabla^\mu v(x)|^2 \mathrm{d}x \right)^{\tau/2} \\
&\lesssim \sum_{I \in \Lambda_{j,k}} 2^{-\mu j \tau} r_I^{-\tau|\vec{\delta}|} \rho_I^{-\tau(\beta - |\vec{\delta}|)} \left(\int_{Q(I)} \rho^{2(\beta - |\vec{\delta}|)} \left(\prod_{\nu=1}^d r_\nu^{\delta_\nu} \right)^2 |\nabla^\mu v(x)|^2 \mathrm{d}x \right)^{\tau/2}.
\end{aligned}
$$

We define

$$
v_I := \int_{Q(I)} \rho^{2(\beta - |\vec{\delta}|)} \left(\prod_{\nu=1}^d r_\nu^{\delta_\nu} \right)^2 |\nabla^\mu v(x)|^2 \mathrm{d}x.
$$

Now we focus on the coefficients belonging to $\Lambda_{j,k,m}$. We now have to consider the cases $\beta > |\vec{\delta}|$ and $|\vec{\delta}| \ge \beta$ separately. If $\beta - |\vec{\delta}| > 0$ we can conclude $\rho_I^{-\tau(\beta - |\vec{\delta}|)} \lesssim (k2^{-j})^{-\tau(\beta - |\vec{\delta}|)}$. Otherwise we get $\rho_I^{-\tau(\beta - |\vec{\delta}|)} \lesssim ((k+1)2^{-j})^{-\tau(\beta - |\vec{\delta}|)}$. We will only discuss the case $\beta > |\vec{\delta}|$ in detail. The second case can be treated analogously. Using Hölder's inequality with $q = 2/\tau$, $q' = 2/(2-\tau)$ results in

$$
\begin{aligned}
\sum_{I \in \Lambda_{j,k,m}} |\langle v, \eta_I \rangle|^\tau &\lesssim 2^{-\mu \tau j} (k2^{-j})^{-\tau(\beta - |\vec{\delta}|)} \left(\sum_{I \in \Lambda_{j,k,m}} r_I^{-\tau|\vec{\delta}| \frac{2}{2-\tau}} \right)^{\frac{2-\tau}{2}} \cdot \left(\sum_{I \in \Lambda_{j,k,m}} v_I \right)^{\frac{\tau}{2}} \\
&\lesssim 2^{-\mu \tau j} (k2^{-j})^{-\tau(\beta - |\vec{\delta}|)} \left(\sum_{I \in \Lambda_{j,k,m}} (m2^{-j})^{-\tau|\vec{\delta}| \frac{2}{2-\tau}} \right)^{\frac{2-\tau}{2}} \left(\sum_{I \in \Lambda_{j,k,m}} v_I \right)^{\frac{\tau}{2}}
\end{aligned}
$$

Together with (2.2.11) we obtain

$$
\sum_{I \in \Lambda_{j,k,m}} |\langle v, \eta_I \rangle|^\tau \lesssim 2^{\tau j(\beta - \mu)} k^{-\tau(\beta - |\vec{\delta}|)} m^{-\tau|\vec{\delta}| + \frac{2-\tau}{2}} \left(\sum_{I \in \Lambda_{j,k,m}} v_I \right)^{\frac{\tau}{2}}.
$$

We continue by using the fact that there are of order k sets $\Lambda_{j,k,m}$ in each layer $\Lambda_{j,k}$. Together with Hölders inequality, this gives

$$
\sum_{I \in \Lambda_{j,k}} |\langle v, \eta_I \rangle|^\tau \lesssim 2^{j\tau(\beta - \mu)} k^{-\tau(\beta - |\vec{\delta}|)} \left(\sum_{m=1}^{Ck} m^{-\tau|\vec{\delta}| \frac{2}{2-\tau} + 1} \right)^{\frac{2-\tau}{2}} \left(\sum_{I \in \Lambda_{j,k}} v_I \right)^{\frac{\tau}{2}}. \tag{2.2.12}
$$

Note, that the constant C only depends on $\mathcal{K}_0$. Together with

$$\left(\sum_{m=1}^{Ck} m^{-\tau|\vec{\delta}|\frac{2}{2-\tau}+1}\right)^{\frac{2-\tau}{2}} \lesssim \begin{cases} k^{-\tau|\vec{\delta}|+2-\tau} & 2 > \tau(1+|\vec{\delta}|), \\ (\log(1+k))^{\frac{2-\tau}{2}} & 2 = \tau(1+|\vec{\delta}|), \\ 1 & 2 < \tau(1+|\vec{\delta}|). \end{cases}$$

we obtain from (2.2.12)

$$\sum_{I\in\Lambda_{j,k}} |\langle v,\eta_I\rangle|^\tau \lesssim 2^{j\tau(\beta-\mu)} \left(\sum_{I\in\Lambda_{j,k}} v_I\right)^{\frac{\tau}{2}} \times \begin{cases} k^{-\tau(\beta+1)+2} & 2 > \tau(1+|\vec{\delta}|), \\ k^{-\tau(\beta-|\vec{\delta}|)}(\log(1+k))^{\frac{2-\tau}{2}} & 2 = \tau(1+|\vec{\delta}|), \\ k^{-\tau(\beta-|\vec{\delta}|)} & 2 < \tau(1+|\vec{\delta}|). \end{cases}$$

To simplify the notation we denote these functions of k in the second line by a_k. Employing (2.2.9) and Hölder's inequality we get

$$\sum_{I\in\Lambda_j} |\langle v,\eta_I\rangle|^\tau \lesssim 2^{j\tau(\beta-\mu)} \left(\sum_{k=1}^{C2^j} a_k^{\frac{2}{2-\tau}}\right)^{\frac{2-\tau}{2}} \left(\sum_{I\in\Lambda_j} v_I\right)^{\frac{\tau}{2}}.$$

From (2.2.8) we conclude that the last factor is bounded. To complete the estimate we have to sum with respect to $j \in \mathbb{N}_0$: We first consider the sum $\sum_{k=1}^{C2^j} a_k^{\frac{2}{2-\tau}}$ and derive estimates depending on β. Then we study the convergence of

$$\sum_{j\geq 0} \left(2^{j\tau(\beta-\mu)} \left(\sum_{k=1}^{C2^j} a_k^{\frac{2}{2-\tau}}\right)^{\frac{2}{2-\tau}} \right).$$

More detailed we get the following cases:

$$\begin{aligned} 3\left(\tfrac{1}{\tau}-\tfrac{1}{2}\right) &< \mu && \text{if } \tau(1+|\vec{\delta}|) < 2 && \text{and} && \beta < 3\left(\tfrac{1}{\tau}-\tfrac{1}{2}\right), \\ \beta &< \mu && \text{if } \tau(1+|\vec{\delta}|) < 2 && \text{and} && \beta \geq 3\left(\tfrac{1}{\tau}-\tfrac{1}{2}\right), \\ \tfrac{3}{2}|\vec{\delta}| &< \mu && \text{if } \tau(1+|\vec{\delta}|) = 2 && \text{and} && \beta < \tfrac{3}{2}|\vec{\delta}|, \\ \beta &< \mu && \text{if } \tau(1+|\vec{\delta}|) = 2 && \text{and} && \beta \geq \tfrac{3}{2}|\vec{\delta}|, \\ \tfrac{1}{\tau}-\tfrac{1}{2} &< \mu-|\vec{\delta}| && \text{if } \tau(1+|\vec{\delta}|) > 2 && \text{and} && \tfrac{1}{\tau}-\tfrac{1}{2} > \beta-|\vec{\delta}|, \\ \beta &< \mu && \text{if } \tau(1+|\vec{\delta}|) > 2 && \text{and} && \tfrac{1}{\tau}-\tfrac{1}{2} \leq \beta-|\vec{\delta}|. \end{aligned}$$

Now we want to derive from these six cases sufficient conditions for $s := 3\left(\frac{1}{\tau}-\frac{1}{2}\right)$ such that

$$v^* := \sum_{j\in\mathbb{N}_0}\sum_{k\in\mathbb{N}}\sum_{I\in\Lambda_{j,k}} \langle v,\eta_I\rangle\,\eta_I^*$$

belongs to $B^s_\tau(L_\tau(\mathbb{R}^3))$. We find that $\beta < \mu$ is necessary in all six cases. First we consider the case $|\vec{\delta}| < \frac{2}{3}\mu$. If we require

$$s < \mu$$

we can conclude (depending on the value of $\tau(1+|\vec{\delta}|)$) from the first, the third respectively the fifth case the convergence of the above series if additionally $s > \beta$ is fulfilled. For the regularity result this is no relevant restriction. Next we look for the case $|\vec{\delta}| \geq \frac{2}{3}\mu$. From the fifth case again we conclude

$$\beta - |\vec{\delta}| < s < 3(\mu - |\vec{\delta}|).$$

Since we have already found $s < \mu$ we actually obtain from $|\vec{\delta}| \geq \frac{2}{3}\mu$ the condition $s < \frac{3}{2}|\vec{\delta}|$. But $|\vec{\delta}| \geq \frac{2}{3}\mu$ implies $3(\mu - |\vec{\delta}|) \leq 3/2|\vec{\delta}|$. Finally we have found the second restriction in (2.2.5), (2.2.6).

step 2: In the next step we have to estimate the coefficients in

$$\Lambda_{j,0} := \{I \in \Lambda_j : 0 < \rho_I < 2^{-j}\}.$$

If $\Lambda_{j,0}$ is empty, there is nothing to do. Otherwise we argue as follows. From the Lipschitz character of $\mathcal{K}_0$ follows $|\Lambda_{j,0}| \lesssim 2^{2j}, \quad j \in \mathbb{N}_0$. For $0 < q < 2$ we obtain with Hölder's inequality and by summing up over $j \in \mathbb{N}_0$:

$$\sum_{j\geq 0} 2^{j(s+3(1/2-1/q))q} \sum_{I\in\Lambda_{j,0}} |\langle v, \eta_I\rangle|^q \lesssim ||v||^q_{B_q^{s+\frac{1}{2}-\frac{1}{q}}(L_2(\mathbb{R}^3))} \lesssim ||v||^q_{B_2^{s+\frac{1}{2}-\frac{1}{q}-\varepsilon}(L_2(\mathbb{R}^3))},$$

where $\varepsilon > 0$ can be chosen arbitrarily small, see [64, Chapter 2.3]. We choose s and q such that

$$s := \frac{3\alpha}{2} \quad \text{and} \quad \frac{1}{q} := \frac{s}{3} + \frac{1}{2}, \quad \text{i.e. } s = 3\left(\frac{1}{q} - \frac{1}{2}\right),$$

see (2.2.7) for the definition of α. We obtain $\alpha = \frac{2}{q} - 1$, i.e. $\alpha > 0$ is insured. Additionally we get $\alpha = s + \frac{1}{2} - \frac{1}{q}$. That means $||v||^q_{B_2^{s+\frac{1}{2}-\frac{1}{q}}(L_2(\mathbb{R}^3))} < \infty$. We get that

$$v^{**} := \sum_{j\geq 0} \sum_{I\in\Lambda_{j,0}} \langle v, \eta_I\rangle \, \eta_I^*$$

belongs to $B_q^{3/2\alpha}(L_q(\mathbb{R}^3))$.

step 3: Finally we have to estimate the coefficients for which the supports of the appendant wavelets intersect with the boundary of the truncated cone. More precisely, we consider the set

$$\Lambda_j^{\#} := \{I|\, Q(I) \cap \partial\mathcal{K}_0 \neq \emptyset, \; 2^{-3j} \leq |I| \leq 2^{-3j+2}\}, \quad j \in \mathbb{N}_0.$$

Since $\mathcal{K}_0$ is a bounded Lipschitz domain there exists a linear and bounded extension operator

$$\mathcal{E} : H^{\alpha}(\mathcal{K}_0) \to H^{\alpha}(\mathbb{R}^3).$$

which is simultaneously a bounded operator from $B^s_q(L_p(\mathcal{K}_0))$ to $B^s_q(L_p(\mathbb{R}^3))$ not depending on s, p and q. We refer to [59] for further details. We define

$$v^{\#} := \sum_{j=0}^{\infty} \sum_{I \in \Lambda_j^{\#}} \langle \mathcal{E}v, \eta_I \rangle \, \eta_I^*.$$

We recognize that

$$|\Lambda_j^{\#}| \lesssim 2^{2j}, \quad j \in \mathbb{N}_0.$$

So we can argue as in step 2, this yields:

$$||v^{\#}||^q_{B_q^{3/2\alpha}(L_q(\mathbb{R}^3))} \lesssim ||\mathcal{E}v||^q_{B_2^{\alpha}(L_2(\mathbb{R}^3))} \lesssim ||v||^q_{B_2^{\alpha}(L_2(\mathcal{K}_0))} \lesssim ||v||^q_{H^{\alpha}(\mathcal{K}_0)}.$$

We end with summing up the functions v^*, v^{**} and $v^{\#}$ and obtain a function belonging to $B^s_\tau(L_\tau(\mathbb{R}^3))$ where $s < (\mu, 3/2 \cdot \alpha, 3 \cdot (\mu - |\vec{\delta}|))$ and $1/\tau = s/3 + 1/2$. This shows that $v \in B^s_\tau(L_\tau(\mathcal{K}_0))$. Using the translation invariance of Besov spaces and property (2.2.3) ends the proof.

□

Remark 2.2.2. (i) If we consider arbitrary functions (u, p) in $\left[W^{l,2}_{\beta,\vec{\delta}}(\mathcal{K}) \cap H^{s_0}(\mathcal{K}_0)\right]^3 \times W^{l-1,2}_{\beta,\vec{\delta}}(\mathcal{K}) \cap H^{t_0}(\mathcal{K}_0)$, $l \geq 2$, $\beta \in \mathbb{R}, \beta < l-1$, $\vec{\delta} \in \mathbb{R}^d$ we achieve by applying the arguments in the proof of Theorem 2.2.1 the estimate

$$||u||_{B^{s_1}_{\tau_1}(L_{\tau_1}(\mathcal{K}_0))^3} + ||p||_{B^{s_2}_{\tau_2}(L_{\tau_2}(\mathcal{K}_0))} \lesssim$$
$$||u||_{W^{l,2}_{\beta,\vec{\delta}}(\mathcal{K})^3} + ||u||_{H^{s_0}(\mathcal{K}_0)^3} + ||p||_{W^{l-1,2}_{\beta,\vec{\delta}}(\mathcal{K})} + ||p||_{H^{t_0}(\mathcal{K}_0)}.$$

Note that this estimate is true independent of problem (2.2.1). That means we have a continuous embedding from $\left[W^{l,2}_{\beta,\vec{\delta}}(\mathcal{K}) \cap H^{s_0}(\mathcal{K}_0)\right]^3 \times W^{l-1,2}_{\beta,\vec{\delta}}(\mathcal{K}) \cap H^{t_0}(\mathcal{K}_0)$ into $B^{s_1}_{\tau_1}(L_{\tau_1}(\mathcal{K}_0))^3 \times B^{s_2}_{\tau_2}(L_{\tau_2}(\mathcal{K}_0))$. We will use this embedding in Section 2.4 in order to show norm estimates for the solution of the Navier-Stokes and the Stokes system on polyhedral domains.

(ii) It can be shown that the set E in the theorem is the set of eigenvalues of the operator pencil related to (2.2.1). It is known that E consists of isolated points, see [51], [56] for details. Therefore by a minor modification of $\vec{\beta}$, condition (2.2.4) is satisfied and our arguments in the proof below also work with this minor modification. That shows that condition (2.2.4) is not as restrictive as it seems to be.

(iii) Assume that the weight $\vec{\delta} \in \mathbb{R}^d$ is chosen such that $|\vec{\delta}| < 2/3$ then the Besov regularity of u is bounded by the minimum of 2 and $3/2 \cdot s_0$ and the Besov regularity of p is bounded by the minimum of 1 and $3/2 \cdot t_0$. Therefore according to our motivation explained in the introduction the use of adaptive schemes is justified if $s_0 < 4/3$ and $t_0 < 2/3$. These bounds for the Sobolev regularity depend on the smoothness index related to the weighted Sobolev spaces: If Proposition 2.5.6 held for a higher smoothness index l (as it does for the Stokes problem, see Proposition 2.5.4) then the use of adaptive schemes would be justified even for higher Sobolev regularity of the solution of (2.2.1): In this case we get that the Besov smoothness s_1 of u is bounded by

$$\min\left(l, \frac{3}{2} \cdot s_0, 3 \cdot (l - |\vec{\delta}|)\right),$$

and the Besov smoothness s_2 of p is bounded by

$$\min\left(l - 1, \frac{3}{2} \cdot t_0, 3 \cdot (l - (|\vec{\delta}| + 1))\right).$$

This can be derived from the arguments in the proof of Theorem 2.2.1, see also part (i) of this remark.

Proposition 2.5.5 yields a result about the Sobolev regularity for the solution u of problem (2.2.1). For p we only know that it is contained in $L_2(\mathcal{G})$ so we can not achieve a result for the Besov regularity. Applying Proposition 2.5.5 together with Theorem 2.2.1 we get the following result.

Corollary 2.2.3. *It exists a countable set $E \subset \mathbb{C}$ such that for all $\vec{\beta} \in \mathbb{R}^{d'}$, $\vec{\delta} \in \mathbb{R}^d$ with*

$$\beta^* := \max_{j=1,...,d'} \beta_j < 1,$$

$$Re\ \lambda \neq 1/2 - \beta_j \quad \textit{for all } \lambda \in E$$

and

$$\max(0, 1 - \mu_k) < \delta_k < 1, \ \ k = 1, ..., d,$$

where $\mu_k = \pi/\theta_k$ if $\theta_k < \pi$ and μ_k is the minimum of all solutions of $\mu \sin(\theta_k) + \sin(\mu\theta_k) = 0$ if $\theta_k > \pi$, the following holds: If $(f, g) \in W^{0,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})^3 \times W^{1,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G}) \cap H^{1+\varepsilon}(\mathcal{G})$, $\varepsilon > 0$, g fulfills the compatibility condition

$$g|_{M_k} = 0, \ \ k = 1, ..., d$$

and the functional defined in (2.5.2) fulfills $F \in \mathcal{H}^*$ and

$$||F||_{\mathcal{H}^*} + ||g||_{L_2(\mathcal{G})}$$

is sufficiently small then a solution (u, p) of problem (2.2.1) satisfies

$$u \in B^s_\tau(L_\tau(\mathcal{G}))^3, \ \ \frac{1}{\tau} = \frac{s}{3} + \frac{1}{2}, \ \ s < \min\left(3/2, 3 \cdot (2 - |\vec{\delta}|)\right).$$

Remark 2.2.4. (i) From Proposition 2.5.5 we conclude that u is unique on the set of all functions with H^1-norm less than a certain positive ε and p is unique up to a constant.

(ii) Regarding our explanation in Remark 2.2.2, (iii) we see since $s_0 = 1 < 4/3$ the use of adaptive wavelet schemes to determine the solution u is justified.

2.3 Besov regularity for the Stokes system in polyhedral domains

In this section we study the Besov regularity of the stationary Stokes system on polyhedral domains. We start with the investigation for the Stokes system on polyhedral cones and then we use these results to prove analog results for the general case of a polyhedral domain. The Stokes equations on a polyhedral cone are defined by

$$\begin{aligned} -\Delta u + \nabla p &= f \text{ in } \mathcal{K}, \\ \text{div } u &= g \text{ in } \mathcal{K}, \\ u &= 0 \text{ on } \Gamma_j,\ j = 1, ..., d, \end{aligned} \tag{2.3.1}$$

where $\Gamma_j,\ j = 1, ..., d$ are the faces of the cone. Using the estimate in Remark 2.2.2, Proposition 2.5.1 and Proposition 2.5.2 we immediately achieve the following regularity result:

Theorem 2.3.1. *Fix an integer $l \geq 2$ and a real number $0 < \alpha_0 < 0.5$. It exists a countable set $E \subset \mathbb{C}$ such that for all $\beta \in \mathbb{R}$, $\vec{\delta} \in (\mathbb{R}\backslash\mathbb{Z})^d$ with*

$$\beta < l - 1,$$

$$Re\lambda \neq l - \beta - \frac{3}{2} \quad \text{for all } \lambda \in E \tag{2.3.2}$$

and

$$\max(0, l - 1 - \mu_k) < \delta_k < l - 1, \quad k = 1, ..., d,$$

where $\mu_k = \pi/\theta_k$ if $\theta_k < \pi$ and μ_k is the minimum of all solutions of $\mu \sin(\theta_k) + \sin(\mu\theta_k) = 0$ if $\theta_k > \pi$, the following holds: If $(f, g) \in \left[W^{l-2,2}_{\beta,\vec{\delta}}(\mathcal{K}) \cap L_2(\mathcal{K}_0)\right]^3 \times W^{l-1,2}_{\beta,\vec{\delta}}(\mathcal{K}) \cap H^{\alpha_0}(\mathcal{K}_0)$ and g fulfills in case of $\delta_k < l - 2$ the compatibility condition

$$g|_{M_k} = 0$$

then the unique solution (u, p) of problem (2.3.1) satisfies

$$u \in B^{s_1}_{\tau_1}(L_{\tau_1}(\mathcal{K}_0))^3, \quad \frac{1}{\tau_1} = \frac{s_1}{3} + \frac{1}{2}, \ \ s_1 < \min\left(l, \frac{3}{2} \cdot (\alpha_0 + 1), 3 \cdot (l - |\vec{\delta}|)\right),$$

$$p \in B^{s_2}_{\tau_2}(L_{\tau_2}(\mathcal{K}_0)), \quad \frac{1}{\tau_2} = \frac{s_2}{3} + \frac{1}{2}, \ \ s_2 < \min\left(l - 1, \frac{3}{2} \cdot \alpha_0, 3 \cdot (l - (|\vec{\delta}| + 1))\right)$$

Next we investigate the Stokes System on a polyhedral domain:

$$\begin{aligned} -\Delta u + \nabla p &= f \ \text{ in } \mathcal{G} \\ \text{div } u &= g \ \text{ in } \mathcal{G} \\ u &= 0 \ \text{ on } \Gamma_j,\ j = 1, ..., N. \end{aligned} \tag{2.3.3}$$

In the proof of Theorem 2.2.1 we have reduced the problem on a general polyhedral domain to a polyhedral cone. Using these arguments once again we obtain together with Theorem 2.3.1 the following result:

Theorem 2.3.2. *Fix an integer $l \geq 2$ and a real number $0 < \alpha_0 < 0.5$. It exists a countable set $E \subset \mathbb{C}$ such that for all $\vec{\beta} \in \mathbb{R}^{d'}$, $\vec{\delta} \in \mathbb{R}^d$ with*

$$\beta^* := \max_{j=1,\ldots,d'} \beta_j < l-1, \tag{2.3.4}$$

$$Re\ \lambda \neq l - \beta_j - \frac{3}{2} \quad \text{for all } \lambda \in E$$

and

$$\max\left(0, l-1-\frac{\pi}{\theta_k}\right) < \delta_k < l-1, \quad k = 1, ..., d$$

the following holds: If $(f,g) \in \left[W^{l-2,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G}) \cap L_2(\mathcal{G})\right]^3 \times W^{l-1,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G}) \cap H^{\alpha_0}(\mathcal{G})$ and g fulfills in case of $\delta_k < l-2$ the compatibility condition

$$g|_{M_k} = 0, \ \ k = 1, ..., d$$

then the unique solution (u,p) of problem (2.3.3) satisfies

$$u \in B^{s_1}_{\tau_1}(L_{\tau_1}(\mathcal{G}))^3, \quad \frac{1}{\tau_1} = \frac{s_1}{3} + \frac{1}{2}, \ \ s_1 < \min\left(l, \frac{3}{2}\cdot(\alpha_0+1), 3\cdot(l-|\vec{\delta}|)\right), \tag{2.3.5}$$

$$p \in B^{s_2}_{\tau_2}(L_{\tau_2}(\mathcal{G})), \quad \frac{1}{\tau_2} = \frac{s_2}{3} + \frac{1}{2}, \ \ s_2 < \min\left(l-1, \frac{3}{2}\cdot\alpha_0, 3\cdot(l-(|\vec{\delta}|+1))\right) \tag{2.3.6}$$

Remark 2.3.3. (i) In the case of Theorem 2.3.1 and Theorem 2.3.2 we have a unique solution in $H^1(\mathcal{G})^3 \times L_2(\mathcal{G})$. M. Dauge could show (see [33]) that under the conditions formulated in Proposition 2.5.1 the unique solution (u,p) is contained in $H^{\alpha_0+1}(\mathcal{G})^3 \times H^{\alpha_0}(\mathcal{G})$, i.e. the solution (u,p) found in Theorem 2.3.1 and Theorem 2.3.2 respectively has Sobolev regularity $\alpha_0 + 1$ or α_0 respectively. Therefore for valid parameters l and $\vec{\delta}$ the Besov regularity in the specific scale we are interested in is 3/2 times higher than its Sobolev regularity. Consequently the use of adaptive schemes is justified also in this case.

(ii) Remark 2.2.2, (ii) applies analogously for set set E in Theorem 2.3.1, Theorem 2.3.2.

2.4 Norm estimates for Navier-Stokes and Stokes equations on polyhedral domains

Analyzing the convergence of adaptive wavelet schemes we observe that the error of the approximation can be estimated by a term in which the Besov norm of the exact solution occurs. A typical estimate is of the form

$$\sigma_{m.t}(v) \leq C||v||_{B^s_q(L_q(\Omega))} m^{-\frac{s-t}{d}},$$

where $\sigma_{m,t}$ denotes the error of the best m-term approximation measured in the H^t-norm. We refer to [18] and [35] for details. Therefore it is worthwile to ensure that the Besov norm of the exact solution can be estimated by terms depending only on the right hand side of the partial differential equation. Otherwise we may have a certain convergence rate but this is only asymptotic and for practice it is not applicable. We will derive norm estimates by exploiting Remark 2.2.2. We begin with the Stokes system on a polyhedral cone.

Theorem 2.4.1. *If the assumptions of Theorem 2.3.1 are fulfilled we obtain*

$$||u||_{B^{s_1}_{\tau_1}(L_{\tau_1}(\mathcal{K}_0))^3} + ||p||_{B^{s_2}_{\tau_2}(L_{\tau_2}(\mathcal{K}_0))} \lesssim \tag{2.4.1}$$

$$\left(||F||_{\mathcal{H}^*_{l-1-\beta}} + ||g||_{V^{0,2}_{\beta-l+1}(\mathcal{K})} + ||f||_{W^{l-2,2}_{\beta,\delta}(\mathcal{K})^3} + ||g||_{W^{l-1,2}_{\beta,\delta}(\mathcal{K})} + ||f||_{L_2(\mathcal{K}_0)^3} + ||g||_{H^{\alpha}(\mathcal{K}_0)}\right).$$

Proof:
From Remark 2.5.3 we obtain an estimate for the weighted Sobolev norm and the usual Sobolev norm which occurs on the right side of the estimate in Remark 2.2.2. So, if the assumptions in Theorem 2.3.1 are fulfilled the solution of the Stokes problem on a polyhedral cone fulfills the stated estimate. □

For the Stokes system on a polyhedral domain as considered in Section 2.2 and Section 2.3 we obtain a norm estimate by reducing it to the estimate (2.4.1). Using the notation $\varphi_j := \kappa_j^{-1} : \mathcal{K}_j \cap B_j \to G \cap U_j$ we find by exploiting (2.2.3):

$$\begin{aligned}
&||u||_{B^{s_1}_{\tau_1}(L_{\tau_1}(\mathcal{G}))^3} + ||p||_{B^{s_2}_{\tau_2}(L_{\tau_2}(\mathcal{G}))} \\
&\lesssim \sum_{j=1}^{d'} \left(||u||_{B^{s_1}_{\tau_1}(L_{\tau_1}(\mathcal{G}\cap U_j))^3} + ||p||_{B^{s_2}_{\tau_2}(L_{\tau_2}(\mathcal{G}\cap U_j))}\right) \\
&\lesssim \sum_{j=1}^{d'} \left(||u\circ\varphi_j||_{B^{s_1}_{\tau_1}(L_{\tau_1}(\mathcal{K}_j\cap B_j))^3} + ||p\circ\varphi_j||_{B^{s_2}_{\tau_2}(L_{\tau_2}(\mathcal{K}_j\cap B_j))}\right)
\end{aligned}$$

Then (2.4.1) leads to

$$\begin{aligned}
&||u||_{B^{s_1}_{\tau_1}(L_{\tau_1}(\mathcal{G}))^3} + ||p||_{B^{s_2}_{\tau_2}(L_{\tau_2}(\mathcal{G}))} \\
&\lesssim \sum_{j=1}^{d'} (||u\circ\varphi_j||_{W^{l,2}_{\beta_j,\vec{\delta}}(\mathcal{K}_j)} + ||u\circ\varphi_j||_{H^{\alpha_0+1}(\mathcal{K}_j\cap B_j)} \\
&\qquad + ||p\circ\varphi_j||_{W^{l-1,2}_{\beta_j,\vec{\delta}}(\mathcal{K}_j)} + ||p\circ\varphi_j||_{H^{\alpha_0}(\mathcal{K}_j\cap B_j)}) \\
&\lesssim \sum_{j=1}^{d'} (||F||_{\mathcal{H}^*_{l-1-\beta_j}} + ||g\circ\varphi_j||_{V^{0,2}_{\beta_j-l+1}(\mathcal{K}_j)} + ||f\circ\varphi_j||_{W^{l-2,2}_{\beta_j,\vec{\delta}}(\mathcal{K}_j)^3} + ||g\circ\varphi_j||_{W^{l-1,2}_{\beta_j,\vec{\delta}}(\mathcal{K}_j)}) \\
&\qquad + ||f||_{L_2(\mathcal{G})^3} + ||g||_{H^{\alpha_0}(\mathcal{G})}.
\end{aligned}$$

We finally get

Theorem 2.4.2. *If the assumptions of Theorem 2.3.2 are fulfilled we obtain*

$$\begin{aligned}
&||u||_{B^{s_1}_{\tau_1}(L_{\tau_1}(\mathcal{G}))^3} + ||p||_{B^{s_2}_{\tau_2}(L_{\tau_2}(\mathcal{G}))} \\
&\lesssim \sum_{j=1}^{d'} \left(||F||_{\mathcal{H}^*_{l-1-\beta_j}} + ||g \circ \varphi_j||_{V^{0,2}_{\beta_j - l+1}(\mathcal{K}_j)} \right) \\
&\quad + ||f||_{W^{l-2,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})^3} + ||f||_{L_2(\mathcal{G})^3} + ||g||_{W^{l-1,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})} + ||g||_{H^{\alpha_0}(\mathcal{G})}.
\end{aligned}$$

In a last step we want to deduce an estimate for the Navier-Stokes system which we have investigated in Corollary 2.2.3. Especially this means we only obtain an estimate for the Besov norm of u. In the proof of existence of a weak solution for the Navier Stokes equation (see [56, Theorem 11.2.1]) we see that the solution exists on a set with bounded H^1-Norm. Hence it exists a constant $\eta > 0$ such that for the solution u of problem (2.2.1) we have

$$||u||_{H^1(\mathcal{G})^3} < \eta.$$

From [56, Lemma 8.1.1, Theorem 11.2.8] we conclude for arbitrary small $\varepsilon > 0$

$$||u||_{W^{2,2}_{\vec{\delta},\vec{\beta}}(\mathcal{G})} + ||p||_{W^{1,2}_{\vec{\delta},\vec{\beta}}(\mathcal{G})} \lesssim$$

$$(1+\eta+\eta^2)\left(\sum_{j=1}^{d'} \left(||F||_{\mathcal{H}^*_{l-1-\beta_j}} + ||g \circ \varphi_j||_{V^{0,2}_{\beta_j - l+1}(\mathcal{K}_j)} \right) + ||f||_{W^{0,2+\varepsilon}_{\vec{\delta},\vec{\beta}}(\mathcal{G})} + ||g||_{W^{1,2+\varepsilon}_{\vec{\delta},\vec{\beta}}(\mathcal{G})}\right) \\ +\eta^3 ||u||_{W^{2,2}_{\vec{\delta},\vec{\beta}}(\mathcal{G})}.$$

If we assume $\eta \in (0,1)$ we obtain with $\mu := \frac{1+\eta+\eta^2}{1-\eta^3}$ the estimate

$$\begin{aligned}
&||u||_{W^{2,2}_{\vec{\delta},\vec{\beta}}(\mathcal{G})^3} \\
&\lesssim \mu \left(\sum_{j=1}^{d'} \left(||F||_{\mathcal{H}^*_{l-1-\beta_j}} + ||g \circ \varphi_j||_{V^{0,2}_{\beta_j - l+1}(\mathcal{K}_j)} \right) + ||f||_{W^{0,2+\varepsilon}_{\vec{\delta},\vec{\beta}}(\mathcal{G})} + ||g||_{W^{1,2+\varepsilon}_{\vec{\delta},\vec{\beta}}(\mathcal{G})} \right).
\end{aligned}$$

Exploiting Remark 2.2.2 we achieve

Theorem 2.4.3. *If the assumptions of Corollary 2.2.3 are fulfilled then*

$$\begin{aligned}
&||u||_{B^{s}_{\tau}(L_{\tau}(\mathcal{G}))^3} \\
&\lesssim \mu \left(\sum_{j=1}^{d'} \left(||F||_{\mathcal{H}^*_{l-1-\beta_j}} + ||g \circ \varphi_j||_{V^{0,2}_{\beta_j - l+1}(\mathcal{K}_j)} \right) + ||f||_{W^{0,2+\varepsilon}_{\vec{\delta},\vec{\beta}}(\mathcal{G})} + ||g||_{W^{1,2+\varepsilon}_{\vec{\delta},\vec{\beta}}(\mathcal{G})} \right) + \eta.
\end{aligned}$$

2.5 Appendix A - Sobolev and weighted Sobolev regularity of solutions of the Stokes and the Navier-Stokes system

In this section we state several results which play a fundamental role in the proof of the main theorems. First of all we recall a result concerning the Sobolev regularity for the Stokes system on a polyhedral domain, see [33, Theorem 9.20].

Proposition 2.5.1. *Let $\mathcal{G} \subset \mathbb{R}^3$ be a polyhedral domain. Consider problem (2.3.3). Assume that $(f, g) \in L_2(\mathcal{G})^3 \times H^{\alpha_0}(\mathcal{G})$ for $0 < \alpha_0 < 0.5$. Furthermore let g fulfill the compatibility condition*

$$\int_{\mathcal{G}} g(x)dx = 0. \tag{2.5.1}$$

Then there exists a unique solution $(u, p) \in H^{\alpha_0+1}(\mathcal{G})^3 \times H^{\alpha_0}(\mathcal{G})$.

Of course this theorem is true for the special case that $\mathcal{G}$ is a polyhedral cone. Next we cite a regularity result for solutions of the Stokes System in weighted Sobolev spaces defined for polyhedral cones, see [56, Theorem 10.3.2].

Proposition 2.5.2. *Let $\mathcal{K}$ be a polyhedral cone as defined in (2.2.2). Suppose $(f, g) \in W^{l-2,2}_{\beta,\vec{\delta}}(\mathcal{K})^3 \times W^{l-1,2}_{\beta,\vec{\delta}}(\mathcal{K})$ where $l \geq 2$ is an integer. Then there exists a countable set $E \subset \mathbb{C}$ such that the following holds. If $\beta \in \mathbb{R}$ and the vector $\vec{\delta} \in (\mathbb{R}\backslash\mathbb{Z})^d$ are chosen such that*

$$Re\ \lambda \neq l - \beta - \frac{3}{2} \quad \text{for all } \lambda \in E$$

and

$$\max(0, l - 1 - \mu_k) < \delta_k < l - 1, \quad k = 1, ..., d,$$

where $\mu_k = \pi/\theta_k$ if $\theta_k < \pi$ and μ_k is the minimum of all solutions of $\mu \sin(\theta_k) + \sin(\mu\theta_k) = 0$ if $\theta_k > \pi$ and if $\delta_k < l - 2$ we claim $g|_{M_k} = 0$, then there exists a uniquely determined solution of (2.3.1)

$$(u, p) \in W^{l,2}_{\beta,\vec{\delta}}(\mathcal{K})^3 \times W^{l-1,2}_{\beta,\vec{\delta}}(\mathcal{K})$$

Remark 2.5.3. Following Maz'ya and Rossmann (see [56, Chapter 10.2, 10.3]) we use the notation $V^{l,2}_{\beta}(\mathcal{K}) := W^{l,2}_{\beta,0}(\mathcal{K})$ and define the space

$$\mathcal{H}_\beta := \left\{u \in V^{1,2}_\beta(\mathcal{K})^3 : u = 0 \text{ on } \Gamma_j,\ j = 1, ..., d\right\}.$$

If the assumptions from Proposition 2.5.2 are fulfilled the functional

$$F(v) := \int_{\mathcal{K}} (f + \nabla g) \cdot v\mathrm{d}x$$

defines a linear and continuous mapping on $\mathcal{H}_{l-1-\beta}$. The solution (u, p) found in Proposition 2.5.2 fulfills

$$||u||^2_{W^{l,2}_{\beta,\vec{\delta}}(\mathcal{K})^3} + ||p||^2_{W^{l-1,2}_{\beta,\vec{\delta}}(\mathcal{K})} \lesssim \left(||F||^2_{\mathcal{H}^*_{l-1-\beta}} + ||g||^2_{V^{0,2}_{\beta-l+1}} + ||f||^2_{W^{l-2,2}_{\beta,\delta}} + ||g||^2_{W^{l-1,2}_{\beta,\delta}(\mathcal{K})}\right).$$

Moreover we obtain from [33] the estimate

$$||u||_{H^{\alpha_0+1}(\mathcal{K}_0)^3} + ||p||_{H^{\alpha_0}(\mathcal{K}_0)} \lesssim ||f||_{L_2(\mathcal{K}_0)^3} + ||g||_{H^{\alpha_0}(\mathcal{K}_0)}.$$

Analogously to Proposition 2.5.2 we cite a result for general polyhedral domains (see [56, Theorem 11.1.5]):

Proposition 2.5.4. *Let $\mathcal{G}$ be a polyhedral domain. Consider problem (2.3.1). Let $(u,p) \in H^1(\mathcal{G})^3 \times L_2(\mathcal{G})$ be a solution. Suppose $(f,g) \in W^{l-2,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})^3 \times W^{l-1,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})$ where $l \geq 2$ is an integer and $g|_{M_k} = 0$ for $k = 1,...,d$. Then there exists a countable set $E \subset \mathbb{C}$ such that the following holds: If $\vec{\beta} \in \mathbb{R}^{d'}$ and $\vec{\delta} \in \mathbb{R}^d$ are chosen such that*

$$Re\lambda \neq l - \beta_j - 3/2 \text{ for all } \lambda \in E,\ j = 1,...,d'$$

and

$$\max(0, l-1-\mu_k) < \delta_k < l-1,\ k = 1,...,d,$$

where $\mu_k = \pi/\theta_k$ if $\theta_k < \pi$ and μ_k is the minimum of all solutions of $\mu\sin(\theta_k) + \sin(\mu\theta_k) = 0$ if $\theta_k > \pi$, and if $\delta_k < l-2$ we claim $g|_{M_k} = 0$, then $(u,p) \in W^{l,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})^3 \times W^{l-1,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})$.

For the Navier-Stokes system (2.2.1) we state the following result concerning the Sobolev regularity (see [56, Theorem 11.2.1]). Therefore we consider the functional

$$F : \mathcal{H} := \{u \in H^1(\mathcal{G})^3 : u|_{\Gamma_j} = 0 \text{ für } j = 1,...,N\} \to \mathbb{R}$$

$$F(v) = \int_{\mathcal{G}} (f(x) + \nabla g(x)) \cdot v(x) \mathrm{d}x \tag{2.5.2}$$

Proposition 2.5.5. *Let $g \in L_2(\mathcal{G})$. Assume that*

$$||F||_{\mathcal{H}^*} + ||g||_{L_2(\mathcal{G})}$$

is sufficiently small. Then there exists a solution $(u,p) \in H^1(\mathcal{G})^3 \times L_2(\mathcal{G})$ of (2.2.1). Here u is unique on the set of all functions with norm less than a certain positive ε, p is unique up to a constant.

We cite the analog to Proposition 2.5.4 for the nonlinear problem (2.2.1) in the case $l = 2$, see [56, Theorem 11.2.8].

Proposition 2.5.6. *Let $(u,p) \in H^1(\mathcal{G})^3 \times L_2(\mathcal{G})$ be a solution of the problem (2.2.1), where $g \in W^{1,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G}) \cap H^{1+\varepsilon}(\mathcal{G})$, $\varepsilon > 0$, $g|_{M_k} = 0$ for $k = 1,...,d$ and $F \in \mathcal{H}^*$ with given $f \in W^{0,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})^3$. Then there exists a countable set $E \subset \mathbb{C}$ such that the following holds: If $\vec{\beta} \in \mathbb{R}^{d'}$ and $\vec{\delta} \in \mathbb{R}^d$ are chosen such that*

$$Re\ \lambda \neq 1/2 - \beta_j \quad \text{for all } \lambda \in E$$

and that

$$\max(0, 1-\mu_k) < \delta_k < 1,\ \ k = 1,...,d,$$

where $\mu_k = \pi/\theta_k$ if $\theta_k < \pi$ and μ_k is the minimum of all solutions of $\mu\sin(\theta_k) + \sin(\mu\theta_k) = 0$ if $\theta_k > \pi$, then $u \in W^{2,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})^3$ and $p \in W^{1,2}_{\vec{\beta},\vec{\delta}}(\mathcal{G})$

2.6 Appendix B - Function spaces and wavelets

In this section we impose the notations concerning the wavelets. Moreover we state the result which provides a characterization of the Besov spaces by the coefficients of the wavelet expansion. For the construction of wavelets see, e.g., [32]. Let φ be a compactly supported scaling function of sufficiently high regularity and let ψ, $i = 1, ..., 2^n - 1$ be corresponding wavelets. More detailed we require for some $N > 0$ and $r \in \mathbb{N}$:

- $\operatorname{supp} \varphi, \operatorname{supp} \psi_i \subset [-N, N]$, $i = 1, ..., 2^n - 1$.
- $\varphi, \psi_i \in \mathcal{C}^r(\mathbb{R}^n)$, $i = 1, ..., 2^n - 1$.
- The wavelets have the vanishing moments property:

  $$\int x^\alpha \psi_i(x) dx = 0$$

  for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq r$, $i = 1, ..., 2^n - 1$.
- We use the standard abbreviations $\varphi_k(x) := \varphi(x - k)$ and $\psi_{i,j,k}(x) := 2^{jn/2}\psi_i(2^j x - k)$. We assume that

  $$\{\varphi_k,\ \psi_{i,j,k} : (i, j, k) \in \{1, ..., 2^n - 1\} \times \mathbb{N}_0 \times \mathbb{Z}^n\}$$

  is a Riesz basis in $L_2(\mathbb{R}^n)$.

Further, the dual Riesz basis should fulfill the same requirements, i.e. there exist functions $\tilde{\varphi}$ and $\tilde{\psi}_i$, $i = 1, ..., 2^n - 1$, such that

- $\langle \tilde{\varphi}_k, \psi_{i,j,k} \rangle = \left\langle \tilde{\psi}_{i,j,k}, \varphi_k \right\rangle = 0$, $\langle \tilde{\varphi}_k, \varphi_l \rangle = \delta_{k,l}$, $\left\langle \tilde{\psi}_{i,j,k}, \psi_{u,v,l} \right\rangle = \delta_{i,u}\delta_{j,v}\delta_{k,l}$,
- $\operatorname{supp} \tilde{\varphi}, \operatorname{supp} \tilde{\psi}_i \subset [-N, N]$, $i = 1, ..., 2^n - 1$.
- $\tilde{\varphi}, \tilde{\psi}_i \in \mathcal{C}^r(\mathbb{R}^n)$, $i = 1, ..., 2^n - 1$ and $\tilde{\psi}_i$ fulfill the vanishing moment property, $i = 1, ..., 2^n - 1$.

Next we state a result which allows to prove Besov regularity by estimating the coefficients of the wavelet expansion. More detailed, it holds:

Proposition 2.6.1. *Let $s \in \mathbb{R}$ and $0 < p, q \leq \infty$. Suppose*

$$r > \max\left(s, n \max\left(0, \frac{1}{p} - 1\right) - s\right).$$

Then $B^s_q(L_p(\mathbb{R}^n))$ is the collection of all tempered distributions f such that f is representable as

$$f = \sum_{k \in \mathbb{Z}^n} a_k \varphi_k + \sum_{i=1}^{2^n-1} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^n} a_{i,j,k} \psi_{i,j,k}$$

with

$$||f||^*_{B^s_q(L_p(\mathbb{R}^n))} := \left(\sum_{k\in\mathbb{Z}^n} |a_k|^p\right)^{1/p} + \left(\sum_{i=1}^{2^n-1}\sum_{j=0}^{\infty} 2^{j(s+n(1/2-1/p))q}\left(\sum_{k\in\mathbb{Z}^n} |a_{i,j,k}|^p\right)^{q/p}\right)^{1/q} < \infty$$

if $q < \infty$ and

$$||f||^*_{B^s_\infty(L_p(\mathbb{R}^n))} := \left(\sum_{k\in\mathbb{Z}^n} |a_k|^p\right)^{1/p} + \sup_{i=1,\ldots,2^n-1}\sup_{j\geq 0} 2^{j(s+n(1/2-1/p))}\left(\sum_{k\in\mathbb{Z}^n} |a_{i,j,k}|^p\right)^{1/p} < \infty.$$

The representation is unique and

$$a_k = \langle f, \tilde{\varphi}_k\rangle \text{ and } a_{i,j,k} = \left\langle f, \tilde{\psi}_{i,j,k}\right\rangle$$

hold. Further $J : f \mapsto \{\langle f, \tilde{\varphi}_k\rangle, \left\langle f, \tilde{\psi}_{i,j,k}\right\rangle\}$ is an isomorphic map of $B^s_q(L_p(\mathbb{R}^n))$ onto the sequence space (equipped with the quasi-norm $||\cdot||^*_{B^s_q(L_p(\mathbb{R}^n))}$), i.e. $||\cdot||^*_{B^s_q(L_p(\mathbb{R}^n))}$ may serve as an equivalent quasi-norm on $B^s_q(L_p(\mathbb{R}^n))$.

A proof of this proposition can be found in [63].

Acknowledgement: I would like to thank Ms. Monique Dauge for several helpful remarks. Furthermore I want to thank the reviewers for there feedback which helped to improve the article.

3 Besov Regularity for the Stationary Navier-Stokes Equation on Bounded Lipschitz Domains

Authors: Frank Eckhardt, Petru A. Cioica-Licht, Stephan Dahlke
Journal: Applicable Analysis (accepted for publication)
Abstract: We use the scale $B^s_\tau(L_\tau(\Omega))$, $1/\tau = s/d + 1/2$, $s > 0$, to study the regularity of the stationary Stokes equation on bounded Lipschitz domains $\Omega \subset \mathbb{R}^d$, $d \geq 3$, with connected boundary. The regularity in these Besov spaces determines the order of convergence of nonlinear approximation schemes. Our proofs rely on a combination of weighted Sobolev estimates and wavelet characterizations of Besov spaces. By using Banach's fixed point theorem, we extend this analysis to the stationary Navier-Stokes equation with suitable Reynolds number and data, respectively.
2010 Mathematics Subject Classification: Primary: 35B65, 35Q30, 76D05, 76D07; Secondary: 42C40, 46E35, 65T60.
Key Words: Stokes equation, Navier-Stokes equation, Besov space, weighted Sobolev estimate, wavelet, nonlinear approximation, fixed point theorem.

3.1 Introduction

The Navier-Stokes equations provide a mathematical model of the motion of a fluid and form the basis for the theory of computational fluid dynamics. Due to their relevance, the Navier-Stokes equations have been very intensively studied over the centuries, hence the amount of literature is enormous and can clearly not be discussed in detail here. Let us just refer to [15, 43, 62, 66] for an overview. An analytic description of the solution is only available in rare cases, so that numerical schemes for the constructive approximation of the solutions are needed. Once again, the deluge of literature cannot be comprehensively presented here; let us just refer to [4, 44, 62] and the references therein.

In this paper, we consider an important special case: The incompressible, steady-state, viscous Navier-Stokes equation given by

$$\begin{aligned} -\Delta u + \nu u \cdot (\nabla u) + \nabla \pi &= f \quad \text{on } \Omega, \\ \operatorname{div} u &= 0 \quad \text{on } \Omega, \\ u &= g \quad \text{on } \partial\Omega; \end{aligned} \tag{NAST}$$

Ω denotes a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 3$, with connected boundary, u is the velocity field of the fluid, π denotes the pressure, f is the given body force, g is a prescribed velocity field, and $\nu > 0$ denotes the Reynolds number that describes the viscosity of the fluid. We are concerned with the regularity analysis of solutions to (NAST). In particular, we study the smoothness of solutions in the specific scale

$$B^s_\tau(L_\tau(\Omega)), \quad \frac{1}{\tau} = \frac{s}{d} + \frac{1}{2}, \quad s > 0, \tag{$*$}$$

of Besov spaces.

The motivation for our analysis can be described as follows. Also for the stationary Navier-Stokes equations (NAST), an analytic description of the solution is usually not possible, so that once again efficient numerical schemes are needed. A first natural idea would be to employ classical nonadaptive schemes. These methods correspond to a uniform space refinement strategy, i.e., the underlying degrees of freedom are uniformly distributed and do not depend on the shape of the unknown solutions. As a rule of thumb, the convergence order of such a nonadaptive, uniform scheme depends on the regularity of the object one wants to approximate as measured in the classical Sobolev scale, see, e.g., [35, 47]. If the boundary of the underlying domain and the data of the equation are smooth enough, then also sufficiently high Sobolev smoothness of the solutions to (NAST) can be expected, since this is the case for the linearized equation (SP), see [1]. However, on a general Lipschitz domain, boundary singularities may occur that significantly diminish the Sobolev smoothness, so that the convergence order of uniform schemes drops down. In these cases, adaptive algorithms suggest themselves. Essentially, adaptive algorithms are tricky updating strategies. Based on an a posteriori error estimator, additional degrees of freedom are only spent in regions where the numerical approximation is still far away from the exact solution. Therefore, in each step, the current distribution of the degrees of freedom depends strongly on the unknown solution. Although the idea of adaptivity seems to be convincing, one principle problem remains. Adaptive schemes are very hard to design, to analyze, and to implement. Therefore, a rigorous mathematical foundation that indicates that adaptivity really pays is highly desirable. Our line of attack to give a reasonable answer is based on the following observation. Given a dictionary $\{\psi_\lambda\}_{\lambda\in\mathcal{J}}$ of functions, the best we can expect is that an adaptive scheme realizes the convergence order of best N-term approximation with respect to this dictionary. In this sense, best N-term approximation serves as the benchmark for adaptive algorithms. In best N-term approximation, one does not approximate by linear spaces but by nonlinear manifolds of the form

$$S_N := \Big\{ g \mid g = \sum_{\lambda\in\Lambda} c_\lambda \psi_\lambda,\ |\Lambda| = N,\ c_\lambda \in \mathbb{R} \Big\},$$

i.e., one collects all functions g for which the expansion with respect to $\{\psi_\lambda\}_{\lambda\in\mathcal{J}}$ has at most N nonvanishing coefficients. In many cases, e.g., if the dictionary consists of a wavelet basis, adaptive algorithms that indeed realize the convergence order of best N-term wavelet approximation schemes are known to exist, see, e.g., [11, 12]. These relationships in mind, the following question arises: what is the order of convergence of best N-term approximation, and is it higher than the order of nonadaptive, uniform schemes? For then, the development of adaptivity would be completely justified. It is well-known that in many settings, e.g., for the wavelet case, the order of approximation (in L_2) that can be achieved by best N-term approximation exactly depends on the smoothness of the object under consideration in the so-called *adaptivity scale* $(*)$, i.e,

$$u \in B^s_\tau(L_\tau(\Omega)), \quad \frac{1}{\tau} = \frac{s}{d} + \frac{1}{2} \iff \sum_{N=1}^{\infty} [N^{s/d}\sigma_N(u)]^\tau \frac{1}{N} < \infty,\ \sigma_N(u) := \inf_{g\in S_N} \|u - g\|_{L_2},$$

see, e.g, [35,36] as well as [25] for similar relationships for approximations with respect to other norms. Consequently, in order to decide the question whether adaptivity pays in the context of Navier-Stokes equations, a rigorous analysis of the regularity of the solutions to (NAST) in the scale (∗) is needed. If this regularity is higher than the classical L_2-Sobolev smoothness of the solutions under consideration, then adaptivity pays in the sense that there is indeed the possibility that adaptive methods exhibit higher convergence rates than their uniform alternatives.

This paper consists of two parts. In the first part, we study the linear version of (NAST), i.e., the stationary Stokes problem

$$\begin{aligned} -\Delta u + \nabla\pi &= f \quad \text{on } \Omega, \\ \operatorname{div} u &= 0 \quad \text{on } \Omega, \\ u &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{SP}$$

on an arbitrary bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 3$, with connected boundary.

For this class of problems, some positive results concerning Besov regularity in the scale (∗) already exist. In [16], the Stokes equations on a polygonal domain in $\mathbb{R}^2$ have been studied. The proofs were based on decompositions of the solutions into regular and singular parts. In [40], these results have been generalized to polyhedral domains, whereat specific Kondratiev spaces have been employed. For the case of general Lipschitz domains we are interested in here, first results have been derived by Mitrea and Wright in [57]. Their proofs are based on the well-known concept of layer potentials. In this paper, we improve the results of [57] in the following sense.

Our analysis shows that, other than conjectured in [57, p. 9], the results for the Besov smoothness in the scale (∗) obtained therein, are not sharp for higher dimension $d \geq 4$. Our proof technique is completely different to the one used in [57]. We first show regularity results in weighted Sobolev spaces, and then we prove that these spaces can be embedded into the Besov spaces corresponding to the adaptivity scale (∗), which gives the desired results.

The second part of this paper is concerned with the Besov regularity of solutions to (NAST). To the best of our knowledge, no regularity result in the scale (∗) has been obtained so far. We tackle this task by re-writing (NAST) as a fixed point problem. For semilinear elliptic partial differential equations, this strategy has already been successfully applied in [27]. Nevertheless, there is an important difference. In [27], the fixed point theorems were directly applied to the quasi-Banach spaces in the adaptivity scale (∗), whereas here we study the problem first in classical Besov spaces which enables us to reduce everything to the case of the Stokes problem with modified right-hand side. Then the desired Besov regularity results in the scale (∗) follows from the corresponding results for the Stokes problem. This approach has the advantage that certain admissibility problems that arise in the context of quasi-Banach spaces can be avoided. Moreover, the application of the Banach fixed point theorem guarantees uniqueness of the solution in a suitable, small ball. To the best of our knowledge, the non-standard fixed point arguments used in [27] only provide the existence of a solution. We show that by proceeding this way we indeed obtain the desired result, in the sense that also for (NAST) the Besov regularity of the solutions is higher than the standard Sobolev smoothness, so that the use of adaptivity is again completely justified.

This paper is organized as follows. In Section 2, we fix notation and briefly recall some basic concepts that are used in the sequel, in particular, concerning function spaces and their wavelet characterization. Section 3 is devoted to the Stokes problem. First of all, in Section 3.1, we discuss (SP) in weighted Sobolev spaces. We generalize regularity results obtained by [3] for the homogeneous Stokes equations to the inhomogeneous case. Then, in Section 3.2, we prove that the weighted Sobolev spaces under consideration intersected with classical (unweighted) Sobolev spaces can be embedded into the Besov spaces from the adaptivity scale (∗) up to a certain smoothness s that depends on the Sobolev and the weighted Sobolev regularity. A combination of these two facts provides our Besov regularity result. In Section 4, we discuss the stationary Navier-Stokes equations (NAST). We use Banach's fixed point theorem to reduce the problem to the Stokes case. Then, we apply the results derived in Section 3.

3.2 Preliminaries

3.2.1 Notations

In this paper $G \subseteq \mathbb{R}^d$, $d \geq 2$, stands for an arbitrary (not necessarily bounded) domain. By $\mathcal{D}'(G)$ we denote the space of Schwartz distributions on G. For $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}_0^d$, we write $D^\alpha f := \frac{\partial^{|\alpha|} f}{\partial^{\alpha_d} x_d \dots \partial^{\alpha_1} x_1}$ for the corresponding derivative of $f \in \mathcal{D}'(G)$, where $|\alpha| := \alpha_1 + \ldots + \alpha_d$; $D^0 f := f$. For $m \in \mathbb{N}_0$, $\nabla^m f := \{D^\alpha f : |\alpha| = m\}$ is the set of all m^{th} order derivatives of f and is identified with an $\mathbb{R}^n$-valued distribution, $n = \binom{d+m-1}{m}$. $\nabla := \nabla^1$ denotes the gradient and $\Delta := \sum_{i=1}^d \frac{\partial^2}{\partial x_i \partial x_i}$ is the Laplace operator. For $p \in [1, \infty)$ and $m \in \mathbb{N}_0$, $W^m(L_p(G))$ is the classical Sobolev space consisting of all (equivalence classes of) measurable functions $f \colon G \to \mathbb{R}$ such that $\|f\|_{W^m(L_p(G))} := \left(\sum_{|\alpha| \leq m} \|D^\alpha f\|^p_{L_p(G)}\right)^{1/p}$ is finite. For $p \in (1, \infty)$ and fractional $s \in (0, \infty) \setminus \mathbb{N}$, we define the Sobolev space $W^s(L_p(G))$ to be the Besov space $B^s_p(L_p(G))$, as defined below (see Definition 3.2.1). We write $\mathring{W}^s(L_p(G))$ for the closure with respect to the Sobolev norm $\|\cdot\|_{W^s(L_p(G))}$ of the space $\mathcal{C}_0^\infty(G)$ of infinitely differentiable functions with compact support within G. For negative $s < 0$, $W^s(L_p(G))$ is defined as the dual space of $\mathring{W}^{-s}(L_{p'}(G))$, where $1/p + 1/p' = 1$. If $p = 2$ we use the common notations $H^s(G) := W^s(L_2(G))$ and $\mathring{H}^s(G) := \mathring{W}^s(L_2(G))$, $s \in \mathbb{R}$. By making slight abuse of notation, we sometimes use the same abbreviations for $\mathbb{R}^d$-valued (generalized) functions. Moreover, we use the common notation

$$u \cdot (\nabla v) := \sum_{i=1}^d u_i \frac{\partial v}{\partial x_i},$$

for d-dimensional (generalized) functions u and v and write $\mathbb{R}_G$ for the set of all real-valued constant functions on a domain G.

Throughout, we denote by Ω a bounded Lipschitz domain contained in $\mathbb{R}^d$, $d \geq 3$, which in some of the central statements is assumed to have connected boundary. We set

$$\rho(x) := \rho(x, \partial\Omega), \quad x \in \Omega,$$

and define the weighted Sobolev space $W^m_\alpha(L_p(\Omega))$ for $m \in \mathbb{N}_0$, $\alpha > 0$ and $p \in [1, \infty)$ as

$$W^m_\alpha(L_p(\Omega)) := \left\{ f \in L_p(\Omega) : \|f\|^p_{W^m_\alpha(L_p(\Omega))} := \|f\|^p_{L_p(\Omega)} + \int_\Omega \rho(x)^\alpha |\nabla^m f(x)|^p_{\ell_p} \mathrm{d}x < \infty \right\},$$

where $|\nabla^m f|_{\ell_p}$ is the ℓ_p-norm of the vector $\nabla^m f$. For $p \in (0,\infty)$ and $\max\{0,(d-1)(1/p-1)\} < s < 1$ we define the Besov spaces $B^s_p(L_p(\partial\Omega))$ as they were introduced in [57, Chapter 2.5]. We further introduce the subspace

$$B^{s,0}_p(L_p(\partial\Omega)) := \Big\{g \in B^s_p(L_p(\partial\Omega)) : \int_{\partial\Omega} g\cdot \mathbf{n}\,\mathrm{d}\sigma = 0\Big\},$$

where $\mathbf{n}$ is the outward unit normal to $\partial\Omega$. We norm this space with the norm inherited from $B^s_p(L_p(\partial\Omega))$. Further we put $W^s(L_p(\partial\Omega)) := B^s_p(L_p(\partial\Omega))$ for $p \in [1,\infty)$ and $s \in (0,1)$. The space $W^1(L_2(\partial\Omega))$ is defined analogously, see for example [3]. For $p = 2$ we also write $H^s(\partial\Omega) := W^s(L_2(\partial\Omega))$, $s \in (0,1]$.

For arbitrary normed spaces $E_1,...,E_n$, $n \in \mathbb{N}$, the Cartesian product $E_1 \times \ldots \times E_n$ is endowed with the norm

$$\|(e_1,...,e_n)\|_{E_1\times\ldots\times E_n} := \sum_{i=1}^n \|e_i\|_{E_i}, \quad (e_1,...,e_n) \in E_1 \times \ldots \times E_n;$$

we write shorthand E^n if $E_j = E$ for all $j = 1,\ldots,n$. The intersection of two normed spaces $(E_1, \|\cdot\|_{E_1})$ and $(E_2, \|\cdot\|_{E_2})$ is normed by

$$\|f\|_{E_1\cap E_2} := \|f\|_{E_1} + \|f\|_{E_2}, \qquad f \in E_1 \cap E_2.$$

If $E_1 \subseteq E_2$ and there exists a constant $C \in (0,\infty)$, such that

$$\|f\|_{E_2} \le C\|v\|_{E_1}, \quad v \in E_1,$$

then we write $E_1 \hookrightarrow E_2$ and say that E_1 is embedded in E_2. Quotient spaces $E/E_0 := \{x + E_0 \colon x \in E\}$ of a normed space $(E, \|\cdot\|_E)$ and a subspace $E_0 \subseteq E$ are endowed with the usual norm

$$\|f\|_{E/E_0} := \inf_{g\in E_0} \|f+g\|_E,$$

where we make use of the common abuse of notation to write simply f instead of the equivalence class $f + E_0$. Throughout, the letter C denotes a finite positive constant that may differ from one appearance to another, even in the same chain of inequalities.

3.2.2 Besov spaces and wavelet decompositions

In this section we present the definition of Besov spaces and describe their wavelet characterization. Our standard references in this context are [10], [39] and [64].

We introduce the Besov spaces $B^s_q(L_p(G))$ by using the common Fourier-analytical approach. Therefore, we fix an arbitrary function $\varphi_0 \in \mathcal{C}^\infty_0(\mathbb{R}^d)$ with

$$\varphi_0(x) = 1 \text{ if } |x| \le 1 \quad \text{and} \quad \varphi_0(x) = 0 \text{ if } |x| \ge 3/2,$$

and define for $k \in \mathbb{N}$,

$$\varphi_k(x) := \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x) \quad \text{for} \quad x \in \mathbb{R}^d,$$

to obtain a smooth dyadic resolution of unity on $\mathbb{R}^d$, i.e., $\varphi_k \in \mathcal{C}^\infty_0(\mathbb{R}^d)$ for all $k \in \mathbb{N}_0$, and

$$\sum_{k=0}^\infty \varphi_k(x) = 1 \quad \text{for all} \quad x \in \mathbb{R}^d.$$

We write $\mathfrak{F}$ for the Fourier transform on the space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions. Recall that $\mathfrak{F}^{-1}(\varphi_k\mathfrak{F}f)$ is an entire analytic function for arbitrary $f \in \mathcal{S}'(\mathbb{R}^d)$ and $k \in \mathbb{N}_0$.

Definition 3.2.1. Let $\{\varphi_k\}_{k\in\mathbb{N}_0} \subseteq \mathcal{C}_0^\infty(\mathbb{R}^d)$ be a resolution of unity as described above. Furthermore, let $0 < p, q < \infty$ and $s \in \mathbb{R}$.

(i) The *Besov space* $B^s_q(L_p(\mathbb{R}^d))$ is defined by

$$B^s_q(L_p(\mathbb{R}^d)) := \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \colon \|f\|_{B^s_q(L_p(\mathbb{R}^d))} := \left(\sum_{k=0}^{\infty} 2^{ksq} \|\mathfrak{F}^{-1}\left[\varphi_k \mathfrak{F} f\right]\|^q_{L_p(\mathbb{R}^d)} \right)^{1/q} < \infty \right\}.$$

(ii) Let $G \subseteq \mathbb{R}^d$ be an arbitrary domain. Then, the *Besov space* $B^s_q(L_p(G))$ is defined by

$$B^s_q(L_p(G)) := \left\{ f \in \mathcal{D}'(G) \colon \text{there exists } g \in B^s_q(L_p(\mathbb{R}^d)) : g|_G = f \right\}.$$

It is equipped with the (quasi-)norm

$$\|f\|_{B^s_q(L_p(G))} := \inf \left\{ \|g\|_{B^s_q(L_p(\mathbb{R}^d))} : g \in B^s_q(L_p(\mathbb{R}^d)), g|_G = f \right\}, \quad f \in B^s_q(L_p(G)).$$

Remark 3.2.2. Besides the definition given above, Besov spaces $B^s_q(L_p(G))$ of positive smoothness $s > 0$ are frequently defined by means of iterated differences, see for example [64]. The two definitions coincide in the sense of equivalent norms for the range of parameters $s > \max\{0, d \cdot (1/p - 1)\}$, provided, e.g., G is a bounded Lipschitz domain or $G = \mathbb{R}^d$, see, e.g., [39, Theorem 3.18] and [64, Theorem 2.5.12], respectively.

In order to present a characterization of Besov spaces on $\mathbb{R}^d$ in terms of wavelets, we fix the following setting. Let ϕ be a scaling function of tensor product type on $\mathbb{R}^d$ and let ψ_i, $i = 1, \dots, 2^d - 1$, be corresponding multivariate mother wavelets such that, for a given $r \in \mathbb{N}$ and some cube Q centered at the origin, the following locality, smoothness and vanishing moment conditions hold. For all $i = 1, \dots, 2^d - 1$,

$$\text{supp } \phi, \text{ supp } \psi_i \subseteq Q, \tag{3.2.1}$$

$$\phi, \, \psi_i \in \mathcal{C}^r(\mathbb{R}^d), \tag{3.2.2}$$

$$\int_{\mathbb{R}^d} x^\alpha \, \psi_i(x) \, dx = 0 \qquad \text{for all } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \le r. \tag{3.2.3}$$

For the dyadic shifts and dilations of the scaling function and the corresponding wavelets we use the abbreviations

$$\phi_k(x) := \phi(x - k), \; x \in \mathbb{R}^d, \qquad \text{for } k \in \mathbb{Z}^d, \text{ and} \tag{3.2.4}$$

$$\psi_{i,j,k}(x) := 2^{jd/2} \psi_i(2^j x - k), \; x \in \mathbb{R}^d, \quad \text{for } (i,j,k) \in \{1, \dots, 2^d - 1\} \times \mathbb{N}_0 \times \mathbb{Z}^d, \tag{3.2.5}$$

and assume that

$$\left\{ \phi_k, \psi_{i,j,k} \; : \; (i,j,k) \in \{1, \dots, 2^d - 1\} \times \mathbb{N}_0 \times \mathbb{Z}^d \right\}$$

is a Riesz basis of $L_2(\mathbb{R}^d)$. Further, we assume that there exists a dual Riesz basis satisfying the same requirements. That is, there exist functions $\widetilde{\phi}$ and $\widetilde{\psi}_i$, $i = 1, \dots, 2^d - 1$, such that conditions (3.2.1), (3.2.2) and (3.2.3) hold if ϕ and ψ_i are replaced by $\widetilde{\phi}$ and $\widetilde{\psi}_i$, and such that the biorthogonality relations

$$\langle \widetilde{\phi}_k, \psi_{i,j,k} \rangle = \langle \widetilde{\psi}_{i,j,k}, \phi_k \rangle = 0 \, , \quad \langle \widetilde{\phi}_k, \phi_l \rangle = \delta_{k,l}, \quad \langle \widetilde{\psi}_{i,j,k}, \psi_{u,v,l} \rangle = \delta_{i,u} \, \delta_{j,v} \, \delta_{k,l} \, ,$$

are fulfilled. Here we use analogous abbreviations to (3.2.4) and (3.2.5) for the dyadic shifts and dilations of $\widetilde{\phi}$ and $\widetilde{\psi}_i$, and $\delta_{k,l}$ denotes the Kronecker symbol. We refer to [10, Chapter 2] for the construction of biorthogonal wavelet bases, see also [14] and [32].

Such a wavelet basis at hand, it is possible to characterize Besov spaces by the decay of the wavelet coefficients in the following way. A proof can be found in [10, Theorem 3.7.7].

Proposition 3.2.3. *Let $p, q \in (0, \infty)$ and $s > \max\{0, d(1/p - 1)\}$. Choose $r \in \mathbb{N}$ such that $r > s$ and construct a biorthogonal wavelet Riesz basis as described above. Then a locally integrable function $f : \mathbb{R}^d \to \mathbb{R}$ is in the Besov space $B^s_q(L_p(\mathbb{R}^d))$ if, and only if,*

$$f = \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\phi}_k\rangle\, \phi_k + \sum_{i=1}^{2^d-1} \sum_{j\in\mathbb{N}_0} \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\psi}_{i,j,k}\rangle\, \psi_{i,j,k}$$

(convergence in $\mathcal{D}'(\mathbb{R}^d)$) with

$$\left(\sum_{k\in\mathbb{Z}^d} \left|\langle f, \widetilde{\phi}_k\rangle\right|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{2^d-1} \sum_{j\in\mathbb{N}_0} 2^{j\left(s+d\left(\frac{1}{2}-\frac{1}{p}\right)\right)q} \left(\sum_{k\in\mathbb{Z}^d} \left|\langle f, \widetilde{\psi}_{i,j,k}\rangle\right|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} < \infty, \qquad (3.2.6)$$

and (3.2.6) is an equivalent (quasi-)norm for $B^s_q(L_p(\mathbb{R}^d))$.

A short computation shows that the Besov spaces $B^s_\tau(L_\tau(\mathbb{R}^d))$, with $1/\tau = s/d + 1/2$, $s > 0$, admit the following characterization.

Proposition 3.2.4. *Let $s > 0$ and $\tau \in \mathbb{R}$ such that $1/\tau = s/d + 1/2$. Choose $r \in \mathbb{N}$ such that $r > s$ and construct a biorthogonal wavelet Riesz basis as described above. Then a locally integrable function $f : \mathbb{R}^d \to \mathbb{R}$ is in the Besov space $B^s_\tau(L_\tau(\mathbb{R}^d))$ if, and only if,*

$$f = \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\phi}_k\rangle\, \phi_k + \sum_{i=1}^{2^d-1} \sum_{j\in\mathbb{N}_0} \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\psi}_{i,j,k}\rangle\, \psi_{i,j,k}$$

(convergence in $\mathcal{D}'(\mathbb{R}^d)$) with

$$\left(\sum_{k\in\mathbb{Z}^d} \left|\langle f, \widetilde{\phi}_k\rangle\right|^\tau\right)^{\frac{1}{\tau}} + \left(\sum_{i=1}^{2^d-1} \sum_{j\in\mathbb{N}_0} \sum_{k\in\mathbb{Z}^d} \left|\langle f, \widetilde{\psi}_{i,j,k}\rangle\right|^\tau\right)^{\frac{1}{\tau}} < \infty\,, \qquad (3.2.7)$$

and (3.2.7) is an equivalent (quasi-)norm for $B^s_\tau(L_\tau(\mathbb{R}^d))$.

3.3 The stationary Stokes equation

3.3.1 The stationary Stokes equation in (weighted) Sobolev spaces

In this section we collect the relevant results known so far concerning existence, uniqueness, and (weighted) Sobolev regularity of the solution to the stationary Stokes equation

$$\begin{aligned} -\Delta u + \nabla\pi &= f \quad \text{on } \Omega, \\ \operatorname{div} u &= 0 \quad \text{on } \Omega, \\ u &= g \quad \text{on } \partial\Omega, \end{aligned} \qquad \text{(SP)}$$

on an arbitrary bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 3$, with connected boundary. Since we require div $u = 0$, we have to make sure that the prescribed velocity field g satisfies the compatibility condition

$$\int_{\partial\Omega} g \cdot \mathbf{n} \, \mathrm{d}\sigma = 0, \tag{3.3.1}$$

where $\mathbf{n}$ is the outward unit normal to $\partial\Omega$. Following, for instance, [43, Chapter IV.1] we call $u \in H^1(\Omega)^d$ a *(weak) solution* of (SP) if u is divergence free, satisfies $u = g$ on the boundary $\partial\Omega$ (in a trace sense) and fulfills the equation

$$\int_\Omega \sum_{i,j=1}^{d} (\nabla u)_{ij} (\nabla \varphi)_{ij} (x) \, \mathrm{d}x = f(\varphi) - \int_\Omega \sum_{i=1}^{d} \pi(x) \frac{\partial \varphi_i(x)}{\partial x_i} \, \mathrm{d}x \tag{3.3.2}$$

for arbitrary $\varphi \in \mathcal{C}_0^\infty(\Omega)$ with a suitable pressure $\pi \in L_2(\Omega)$. The existence and uniqueness of such a solution $u \in H^1(\Omega)^d$ of (SP) can be guaranteed for arbitrary $f \in H^{-1}(\Omega)^d$ and $g \in H^{1/2}(\partial\Omega)^d$ satisfying (3.3.1), see, e.g., [43, Theorem IV.1.1]. The corresponding pressure $\pi \in L_2(\Omega)$ is only unique up to a constant. In what follows, whenever we speak about "the corresponding pressure" to a solution u, we mean any of the corresponding pressures. Up to a certain degree, the solution to (SP) gets smoother, if the right hand side f and the boundary value g are assumed to be more regular, see [1] for instance. If we use classical Sobolev spaces to measure the smoothness, the following proposition can be proven. Due to the linear structure of (SP), the statement follows from [3, Theorem 2.12] together with [3, Theorem 2.2], which relies on the results proven in [42].

Proposition 3.3.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 3$, with connected boundary. Furthermore, let $f \in L_2(\Omega)^d$ and let $g \in H^1(\partial\Omega)^d$ fulfill the condition* (3.3.1)*. Then there exists a unique solution $u \in H^{3/2}(\Omega)^d$ to the Stokes equation* (SP) *with corresponding pressure $\pi \in H^{1/2}(\Omega)$. Moreover, the estimate*

$$\|u\|_{H^{3/2}(\Omega)^d} + \inf_{c \in \mathbb{R}} \|\pi + c\|_{H^{1/2}(\Omega)} \leq C \left(\|f\|_{L_2(\Omega)^d} + \|g\|_{H^1(\partial\Omega)^d} \right) \tag{3.3.3}$$

holds with a constant $C \in (0, \infty)$ that depends only on d and Ω.

If Ω is only assumed to be a bounded Lipschitz domain with connected boundary, we cannot guarantee higher regularity of the solution to (SP) in the classical Sobolev spaces, even if we assume the body force f and prescribed velocity field g to be smoother than required above. This is due to boundary singularities, which can cause the second derivatives of the solution to blow up near the boundary and can therefore diminish its Sobolev regularity. This effect is already known for a long time from the theory of elliptic equations on polygonal and polyhedral domains as well as on general bounded Lipschitz domains, see, e.g., [45, 46, 49]. However, we can capture the bad behavior of the solution at the boundary by using appropriate powers of the distance $\rho(x) := \rho(x, \partial\Omega)$ of a point $x \in \Omega$ to the boundary $\partial\Omega$. In particular, the following holds.

Proposition 3.3.2. *Given the setting of Proposition 3.3.1, the estimate*

$$\int_\Omega \rho(x) \cdot |\nabla^2 u(x)|^2 \, dx + \int_\Omega \rho(x) \cdot |\nabla \pi(x)|^2 \, dx \leq C \left(\|f\|^2_{L_2(\Omega)^d} + \|g\|^2_{H^1(\partial\Omega)^d} \right) \tag{3.3.4}$$

holds with a constant $C \in (0, \infty)$ that depends only on d and Ω.

Proof. Estimate (3.3.4) has been proven in [3, Section 2] for (SP) with zero body force. I.e., for the solution $\bar{u} \in H^1(\Omega)^d$ of the homogeneous boundary value problem

$$\begin{aligned} -\Delta \bar{u} + \nabla \bar{\pi} &= 0 \quad \text{on } \Omega, \\ \operatorname{div} \bar{u} &= 0 \quad \text{on } \Omega, \\ \bar{u} &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{3.3.5}$$

with corresponding pressure $\bar{\pi} \in L_2(\Omega)$, we have

$$\int_\Omega \rho(x) \cdot |\nabla^2 \bar{u}(x)|^2 \,\mathrm{d}x + \int_\Omega \rho(x) \cdot |\nabla \bar{\pi}(x)|^2 \,\mathrm{d}x \leq C \, \|g\|^2_{H^1(\partial\Omega)^d}. \tag{3.3.6}$$

In order to extend this estimate to general body forces $f \in L_2(\Omega)^d$, we argue as follows. Let $\widetilde{\Omega} \subseteq \mathbb{R}^d$ be a bounded $\mathcal{C}^\infty$-domain containing the closure of Ω. Furthermore, let $\mathcal{E} \colon L_2(\Omega) \to L_2(\widetilde{\Omega})$ be a bounded extension operator, e.g., take Rychkov's extension operator from [59] combined with the restriction operator onto $\widetilde{\Omega}$. Since the boundary of $\widetilde{\Omega}$ is smooth, the stationary Stokes equation on $\widetilde{\Omega}$ with zero Dirichlet boundary condition and body force $\mathcal{E}f \in L_2(\widetilde{\Omega})$, i.e.,

$$\begin{aligned} -\Delta \widetilde{u} + \nabla \widetilde{\pi} &= \mathcal{E}f \quad \text{on } \widetilde{\Omega}, \\ \operatorname{div} \widetilde{u} &= 0 \quad \text{on } \widetilde{\Omega}, \\ \widetilde{u} &= 0 \quad \text{on } \partial\widetilde{\Omega}, \end{aligned}$$

has a unique solution $\widetilde{u} \in H^2(\widetilde{\Omega})^d$ with pressure $\widetilde{\pi} \in H^1(\widetilde{\Omega})$, which fulfill the estimate

$$\|\widetilde{u}\|_{H^2(\widetilde{\Omega})^d} + \|\widetilde{\pi}\|_{H^1(\widetilde{\Omega})} \leq C \, \|\mathcal{E}f\|_{L_2(\widetilde{\Omega})^d} \tag{3.3.7}$$

see, e.g., [1, Theorem 3]. Due to the boundedness of the domain Ω and of the extension operator $\mathcal{E}$, this yields

$$\int_\Omega \rho(x) \cdot |\nabla^2 \widetilde{u}(x)|^2 \,\mathrm{d}x + \int_\Omega \rho(x) \cdot |\nabla \widetilde{\pi}(x)|^2 \,\mathrm{d}x \leq C \, \|f\|^2_{L_2(\Omega)^d}. \tag{3.3.8}$$

The linear structure of the Stokes equation (SP) allows us to split its solution u into $u = \widetilde{u}|_\Omega + \bar{u} - u_0$ with corresponding pressure $\pi = \widetilde{\pi}|_\Omega + \bar{\pi} - \pi_0$, where $u_0 \in H^1(\Omega)^d$ solves

$$\begin{aligned} -\Delta u_0 + \nabla \pi_0 &= 0 \quad \text{on } \Omega, \\ \operatorname{div} u_0 &= 0 \quad \text{on } \Omega, \\ u_0 &= \widetilde{u}|_{\partial\Omega} \quad \text{on } \partial\Omega, \end{aligned}$$

with corresponding pressure $\pi_0 \in L_2(\Omega)$. Such a solution u_0 exists by [43, Theorem IV.1.1], since $\widetilde{u} \in H^2(\widetilde{\Omega})$, so that $\widetilde{u}|_{\partial\Omega} \in H^1(\partial\Omega)^d$ due to classical results on traces in Sobolev spaces, see, e.g., [38]; note that $\widetilde{u}|_{\partial\Omega}$ verifies the compatibility condition (3.3.1), since

$$\int_{\partial\Omega} \widetilde{u}|_{\partial\Omega} \cdot \mathbf{n} \,\mathrm{d}\sigma = \int_\Omega \operatorname{div} \widetilde{u}(x) \,\mathrm{d}x = 0,$$

due to a proper generalization of Gauss' theorem (see [43, Exercise II.4.3]), which can be proven by [43, Lemma II.4.1,Theorem II.3.3, Theorem II.4.1]. Moreover, the pair (u_0, π_0) verifies

$$\int_\Omega \rho(x) \cdot |\nabla^2 u_0(x)|^2 \,\mathrm{d}x + \int_\Omega \rho(x) \cdot |\nabla \pi_0(x)|^2 \,\mathrm{d}x \leq C \|\widetilde{u}|_{\partial\Omega}\|^2_{H^1(\partial\Omega)^d},$$

due to the corresponding estimate from [3, Section 2] already used above to obtain (3.3.6). Thus, since $\widetilde{u}$ verifies (3.3.7),

$$\|\widetilde{u}|_{\partial\Omega}\|_{H^1(\partial\Omega)^d} \leq C\|\widetilde{u}\|_{H^2(\widetilde{\Omega})^d} \leq C\|\mathcal{E}f\|_{L_2(\widetilde{\Omega})^d} \leq C\|f\|_{L_2(\Omega)^d},$$

so that

$$\int_\Omega \rho(x) \cdot |\nabla^2 u_0(x)|^2 \,\mathrm{d}x + \int_\Omega \rho(x) \cdot |\nabla \pi_0(x)|^2 \,\mathrm{d}x \leq C\|f\|^2_{L_2(\Omega)^d}.$$

Since all the constants used in this proof depend only on d and Ω, the last estimate, together with (3.3.6) and (3.3.8) proves the assertion. □

3.3.2 Besov regularity for the stationary Stokes equation

In this section we prove the following main result concerning the Besov regularity in the scale $(*)$ of the solution to the Stokes equation (SP) and of the corresponding pressure.

Theorem 3.3.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 3$, with connected boundary. Let u be the unique solution of* (SP) *with $f \in L_2(\Omega)^d$ and $g \in H^1(\partial\Omega)^d$ fulfilling additionally* (3.3.1)*, and let π be the corresponding pressure. Then*

$$u \in B^{s_1}_{\tau_1}(L_{\tau_1}(\Omega))^d, \quad \frac{1}{\tau_1} = \frac{s_1}{d} + \frac{1}{2}, \quad 0 < s_1 < \min\left\{\frac{3}{2}\cdot\frac{d}{d-1}, 2\right\}, \tag{3.3.9}$$

and

$$\pi \in B^{s_2}_{\tau_2}(L_{\tau_2}(\Omega)), \quad \frac{1}{\tau_2} = \frac{s_2}{d} + \frac{1}{2}, \quad 0 < s_2 < \frac{1}{2}\cdot\frac{d}{d-1}. \tag{3.3.10}$$

Moreover, for this range of parameters, the estimate

$$\|u\|_{B^{s_1}_{\tau_1}(L_{\tau_1}(\Omega))^d} + \inf_{c\in\mathbb{R}} \|\pi + c\|_{B^{s_2}_{\tau_2}(L_{\tau_2}(\Omega))} \leq C\left(\|f\|_{L_2(\Omega)^d} + \|g\|_{H^1(\partial\Omega)^d}\right) \tag{3.3.11}$$

holds with a constant $C \in (0, \infty)$ that depends only on Ω, d, s_1, s_2, τ_1, and τ_2.

Before we prove this statement, we want to emphasize its significance for the question raised in the introduction, whether adaptivity pays or not for the numerical treatment of the Stokes equation. Moreover, we relate our result to what is already known about the Besov regularity of the Stokes equation in the scale $(*)$.

Remark 3.3.4. (i) If we only assume that the boundary of the underlying domain Ω is Lipschitz (and connected), then to the best of our knowledge the Sobolev regularity result presented in Proposition 3.3.1 is sharp, i.e. for arbitrary $\varepsilon > 0$, there exists a bounded Lipschitz domain Ω_ε and a function $f \in L_2(\Omega_\varepsilon)^d$, such that the solution u to (SP) with $g = 0$ fails to have L_2-Sobolev regularity of order $3/2 + \varepsilon$. However, Theorem 3.3.3 shows that for arbitrary Lipschitz domains we can go beyond 3/2 in the scale $(*)$ of Besov spaces. For $d = 3$ we can choose any s_1 less than 2, whereas for $d \geq 4$ the bound is given by $3/2 \cdot d/(d-1)$, which is strictly greater than 3/2. As already mentioned in the introduction, this justifies the usage of adaptive numerical methods for the Stokes equation in the sense that in this situation they can have a higher convergence rate than their classical uniform alternatives. The same is true for the pressure π, since its Besov regularity in the scale $(*)$ is $d/(d-1)$ times higher than its worst case Sobolev regularity.

(ii) To the best of our knowledge, the most far reaching results concerning the Besov regularity of the solution u to (SP) on general bounded Lipschitz domains (and for the corresponding pressure π), have been obtained in [57]. Using boundary integral methods, the authors undertake a detailed analysis of the Besov regularity of the boundary value problem for the Stokes equation. Among others, for arbitrary dimensions $d \geq 2$, they determine corresponding ranges of smoothness and integrability parameters allowing for implications of the type

$$f \in B_q^{s-2}(L_p(\Omega))^d, g \in B_q^{s-1/p}(L_p(\partial\Omega))^d \quad \Longrightarrow \quad u \in B_q^s(L_p(\Omega))^d, \pi \in B_q^{s-1}(L_p(\Omega))$$

for Lipschitz domains $\Omega \subseteq \mathbb{R}^d$, see [57, Theorem 10.15]. However, these ranges of admissible parameters depend on the degree of roughness of the boundary $\partial\Omega$, which is described by a value $\varepsilon = \varepsilon(\Omega) \in (0, 1]$: the smaller the ε, the rougher the underlying domain and the smaller the admissible range $\mathcal{R}_{d,\varepsilon}$. Theorem 3.3.3 supports the claim from [57] that the results therein are sharp for low dimensions $d = 2, 3$. However, for higher dimensions $d \geq 4$, on general bounded Lipschitz domains with connected boundary, i.e., if we do not make any further assumptions on the smoothness of the boundary, we obtain higher regularity in the scale $(*)$ than what is possible to extract from [57]. In detail, we have the following relationship between the two results.

- If $d \geq 4$ and $\varepsilon \leq 1/2 \cdot 1/(d-1)$, i.e., if the underlying domain Ω is rough, the results in [57] do not imply a Besov regularity higher than 3/2 for the solution and 1/2 for the corresponding pressure. Even choosing the integrability parameter less than 2, does not help. However, we can get higher with our result and reach any Besov regularity $s_1 < 3/2 \cdot d/(d-1)$ in the scale $(*)$ for the solution and $s_2 < 1/2 \cdot d/(d-1)$ for the pressure. Our findings are depicted in Figure 3.1 by means of a so called DeVore-Triebel diagram. A point $(1/\tau, s)$ in the first quadrant stands for the Besov space $B_\tau^s(L_\tau(\Omega))^d$. In particular, the ray with slope d starting in $(1/2, 0)$ represents the scale $(*)$ of Besov spaces. Due to Theorem 3.3.3, the regularity of the solution to (SP) climbs up this scale until it (almost) reaches the smoothness $s^* = 3/2 \cdot d/(d-1)$. Firstly, we observe, as already discussed in the first part of this remark, that the Besov regularity in the scale $(*)$ is higher than the Sobolev regularity. Secondly this Besov regularity is more than we can extract from [57]. If we do not impose any further assumption on the Lipschitz character of the domain, the results in [57] merely guarantee that the solution is contained in those Besov spaces that correspond to the dark shaded area (and to any point below and to its right until reaching the line $\{(1/\tau, s) \colon 1/\tau = s/d + 1\}$ due to standard embeddings), see, e.g., [8, Theorem 2.61]. The only possibility to enlarge the admissible range of parameters by using [57] is to impose more regularity on the boundary $\partial\Omega$ of

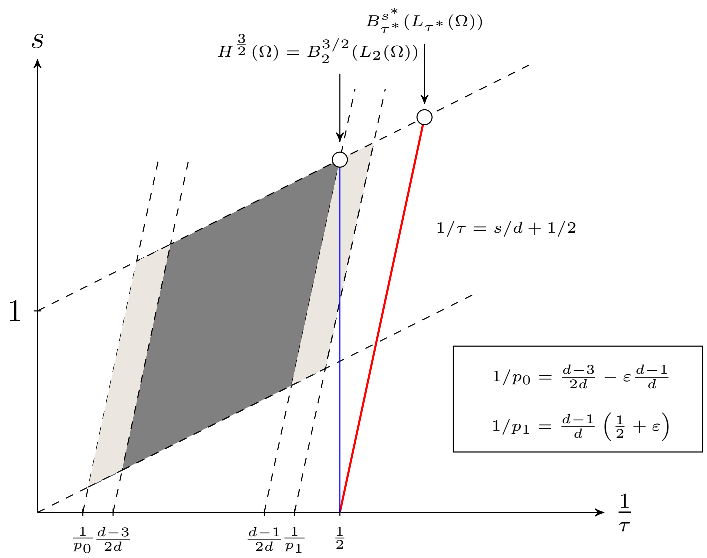

Figure 3.1: Besov regularity for the solution u to (SP) achieved by exploiting [57], versus the results in Theorem 3.3.3, illustrated in a DeVore/Triebel diagram ($d \geq 4$).

the domain, i.e., to relax the conditions on $\varepsilon = \varepsilon(\Omega)$. In Figure 3.1, this adds the light shaded area to the range covered by [57], see, in particular, Theorem 10.15 therein. However, if $\varepsilon \leq 1/2 \cdot 1/(d-1)$, this area does not include all Besov spaces from the scale $(*)$ with smoothness parameter less than s^*.

- If $d \geq 4$ and $\varepsilon > 1/2 \cdot 1/(d-1)$, i.e., if the boundary $\partial\Omega$ is smooth enough, then, the same regularity as in Theorem 3.3.3 can be established by exploiting [57, Theorem 10.15]; with slightly weaker assumptions on f and g.
- If $d = 3$, Theorem 3.3.3 is covered by [57, Theorem 10.15], which guarantees that, even under weaker assumptions on the data, the solution $u \in B^{2-\delta}_p(L_p(\Omega))$ for arbitrary small $\delta > 0$ and suitable $p = p(\delta) > 1$. Consequently, u has Besov regularity of any order $s_1 < 2$ in the scale $(*)$ due to standard embeddings of Besov spaces. If $\varepsilon > 1/2$, then under slightly stronger assumptions on the data, any regularity up to $9/4$ is possible due to [57, Theorem 10.15]. The pressure has the corresponding regularity $s_2 < 1$ and $s_2 < 5/4$, respectively.

As already pointed out, the Sobolev regularity of the solution to the Stokes equation is limited by $3/2$ in a worst case scenario. However, we know from Proposition 3.3.2 that we can still guarantee integrability of the second derivatives if multiplied by a proper power of the distance to the boundary. These relationships will be the most important ingredients for the proof of the following embedding, which, together with Proposition 3.3.2, proves Theorem 3.3.3 as we will explain at the end of the Section. Recall that

$$W^m_\alpha(L_p(\Omega)) = \left\{ f \in L_p(\Omega) : \|f\|^p_{W^m_\alpha(L_p(\Omega))} = \|f\|^p_{L_p(\Omega)} + \int_\Omega \rho(x)^\alpha |\nabla^m f(x)|^p_{\ell_p} \mathrm{d}x < \infty \right\},$$

for $m \in \mathbb{N}_0$, $\alpha > 0$ and $p \in (1, \infty)$.

Theorem 3.3.5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 3$. Let $\alpha_0 > 0$, $\alpha > 0$, and $\gamma \in \mathbb{N}$ with $\alpha < 2\gamma$. Then,*

$$H^{\alpha_0}(\Omega) \cap W^{\gamma}_{\alpha}(L_2(\Omega)) \hookrightarrow B^s_\tau(L_\tau(\Omega)), \qquad \frac{1}{\tau} = \frac{s}{d} + \frac{1}{2},$$

for all

$$0 < s < \min\left\{\frac{2\gamma - \alpha}{2} \cdot \frac{d}{d-1}, \alpha_0 \cdot \frac{d}{d-1}, \gamma\right\}.$$

Proof. The proof can essentially be performed by following the line of [20]. For reader's convenience, we briefly discuss the basic steps. Let us fix s and τ as stated in the theorem. We choose a suitable wavelet basis

$$\left\{\phi_k, \psi_{i,j,k} : (i,j,k) \in \{1, ..., 2^d - 1\} \times \mathbb{N}_0 \times \mathbb{Z}^d\right\}$$

of $L_2(\mathbb{R}^d)$ satisfying the assumptions from Section 3.2.2 with $r > \gamma$. In particular, this means that there exists a cube Q centered at the origin such that for $(j,k) \in \mathbb{N}_0 \times \mathbb{Z}^d$ the cube

$$Q_{j,k} := 2^{-j}k + 2^{-j}Q$$

contains the support of $\psi_{i,j,k}$ for all $i \in \{1, ..., 2^d - 1\}$ and $\operatorname{supp} \phi_k \subseteq Q_{0,k}$; remember that the supports of the corresponding dual basis fulfill the same requirements. Fix $v \in H^{\alpha_0}(\Omega) = B^{\alpha_0}_2(L_2(\Omega))$. Since Ω is assumed to have a Lipschitz continuous boundary, there exists a linear bounded extension operator $\mathcal{E}\colon B^{\alpha_0}_2(L_2(\Omega)) \to B^{\alpha_0}_2(L_2(\mathbb{R}^d))$, see, e.g., [59]. Due to Proposition 3.2.3, we have the following wavelet expansion

$$\mathcal{E}v = \sum_{k \in \mathbb{Z}^d} \langle \mathcal{E}v, \widetilde{\phi}_k\rangle\, \phi_k + \sum_{i=1}^{2^d-1} \sum_{j \in \mathbb{N}_0} \sum_{k \in \mathbb{Z}^d} \langle \mathcal{E}v, \widetilde{\psi}_{i,j,k}\rangle\, \psi_{i,j,k}$$

for the extended v. If we restrict to the scaling functions and wavelets associated to those cubes $Q_{j,k}$ that have a non-empty intersection with Ω, i.e., if we consider only the indexes from

$$\Lambda := \left\{(i,j,k) \in \{1, ..., 2^d - 1\} \times \mathbb{N}_0 \times \mathbb{Z}^d : Q_{j,k} \cap \Omega \neq \emptyset\right\}$$

and

$$\Gamma := \{k \in \mathbb{Z}^d : Q_{0,k} \cap \Omega \neq \emptyset\},$$

then

$$v = \sum_{k \in \Gamma} \langle \mathcal{E}v, \widetilde{\phi}_k\rangle\, \phi_k + \sum_{(i,j,k) \in \Lambda} \langle \mathcal{E}v, \widetilde{\psi}_{i,j,k}\rangle\, \psi_{i,j,k}$$

still holds on Ω. Consequently, due to Proposition 3.2.4 and since Besov spaces on domains are defined via restriction, in order to prove the theorem, it is enough to show that

$$\left(\sum_{k \in \Gamma} |\langle \mathcal{E}v, \widetilde{\phi}_k\rangle|^\tau\right)^{\frac{1}{\tau}} \leq C\|v\|_{H^{\alpha_0}(\Omega)} \tag{3.3.12}$$

and

$$\left(\sum_{(i,j,k) \in \Lambda} |\langle \mathcal{E}v, \widetilde{\psi}_{i,j,k}\rangle|^\tau\right)^{\frac{1}{\tau}} \leq C\big(\|v\|_{H^{\alpha_0}(\Omega)} + \|v\|_{W^{\gamma}_{\alpha}(L_2(\Omega))}\big), \tag{3.3.13}$$

with appropriate constants $C \in (0,\infty)$ that do not depend on v. For simplicity, we make slight abuse of notation and write v instead of $\mathcal{E}v$ in the sequel.

We start with (3.3.12). The index set Γ is finite due to the boundedness of the underlying domain, so that a simple application of Jensen's inequality, followed by an application of Proposition 3.2.3 together with the boundedness of the extension operator and the equivalence of the norms $\|\cdot\|_{H^{\alpha_0}(\Omega)}$ and $\|\cdot\|_{B_2^{\alpha_0}(L_2(\Omega))}$ on $B_2^{\alpha_0}(L_2(\Omega))$, yields

$$\sum_{k\in\Gamma}|\langle v,\widetilde{\phi}_k\rangle|^\tau \le C\bigg(\sum_{k\in\Gamma}|\langle v,\widetilde{\phi}_k\rangle|^2\bigg)^{\frac{\tau}{2}} \le C\|v\|^\tau_{B_2^{\alpha_0}(L_2(\mathbb{R}^d))} \le C\|v\|^\tau_{B_2^{\alpha_0}(L_2(\Omega))} \le C\|v\|^\tau_{H^{\alpha_0}(\Omega)}.$$

The constants above as well as all the other constants appearing in this proof do not depend on v.

To prove the second estimate (3.3.13), we split the sum on the left hand side into two parts and consider those coefficients that are related to wavelets with support in the interior of Ω isolated from those associated with wavelets that might have support on the boundary $\partial\Omega$ or outside of Ω. By using the notations

$$\begin{aligned}
\rho_{j,k} &:= \rho(Q_{j,k},\partial\Omega),\\
\Lambda_j &:= \{(i,l,k)\in\Lambda : l=j\},\\
\Lambda_{j,m} &:= \{(i,j,k)\in\Lambda_j : m\cdot 2^{-j}\le\rho_{j,k}<(m+1)\cdot 2^{-j}\},\\
\Lambda_j^0 &:= \Lambda_j\backslash\Lambda_{j,0},\\
\Lambda^0 &:= \bigcup_{j\in\mathbb{N}_0}\Lambda_j^0,
\end{aligned}$$

for $j,m\in\mathbb{N}_0$, $k\in\mathbb{Z}^d$, this splitting can be written as

$$\sum_{(i,j,k)\in\Lambda}|\langle v,\widetilde{\psi}_{i,j,k}\rangle|^\tau = \sum_{(i,j,k)\in\Lambda^0}|\langle v,\widetilde{\psi}_{i,j,k}\rangle|^\tau + \sum_{(i,j,k)\in\Lambda\backslash\Lambda^0}|\langle v,\widetilde{\psi}_{i,j,k}\rangle|^\tau =: I + I\!I. \qquad (3.3.14)$$

To estimate I, we exploit a Whitney type estimate (see [37, Theorem 3.4]), which guarantees that that for every fixed $(i,j,k)\in\Lambda^0$, there exists a polynomial $P_{j,k}$ of total degree less than γ, and a finite constant C that does not depend on j or k, such that

$$\|v-P_{j,k}\|_{L_2(Q_{j,k})} \le C\,2^{-j\gamma}\bigg(\int_{Q_{j,k}}|\nabla^\gamma v(x)|^2_{\ell_2}\,\mathrm{d}x\bigg)^{1/2};$$

note that the integral is finite, since $(i,j,k)\in\Lambda^0$. Consequently, since $\widetilde{\psi}_{i,j,k}$ is orthogonal to every polynomial of total degree less than γ,

$$\begin{aligned}
|\langle v,\widetilde{\psi}_{i,j,k}\rangle| &= |\langle v-P_{j,k},\widetilde{\psi}_{i,j,k}\rangle|\\
&\le \|v-P_{j,k}\|_{L_2(Q_{j,k})}\|\widetilde{\psi}_{i,j,k}\|_{L_2(Q_{j,k})}\\
&\le C\,2^{-j\gamma}\bigg(\int_{Q_{j,k}}|\nabla^\gamma v(x)|^2_{\ell_2}\,\mathrm{d}x\bigg)^{1/2}\\
&\le C\,2^{-j\gamma}\rho_{j,k}^{-\alpha/2}\bigg(\int_{Q_{j,k}}|\rho(x)^{\alpha/2}\nabla^\gamma v(x)|^2_{\ell_2}\,\mathrm{d}x\bigg)^{1/2}\\
&=: C\,2^{-j\gamma}\rho_{j,k}^{-\alpha/2}\,\mu_{j,k}.
\end{aligned}$$

Fix $j \in \mathbb{N}_0$. Exploiting Hölder's inequality with parameters $2/\tau > 1$ and $2/(2-\tau)$, we obtain

$$\begin{aligned}\sum_{(i,j,k)\in\Lambda_j^0} |\langle v, \widetilde{\psi}_{i,j,k}\rangle|^\tau &\leq C \sum_{(i,j,k)\in\Lambda_j^0} 2^{-j\tau\gamma} \rho_{j,k}^{-\alpha\tau/2}\, \mu_{j,k}^\tau \\ &\leq C \left(\sum_{(i,j,k)\in\Lambda_j^0} \mu_{j,k}^2 \right)^{\frac{\tau}{2}} \left(\sum_{(i,j,k)\in\Lambda_j^0} 2^{-\frac{2j\tau\gamma}{2-\tau}} \rho_{j,k}^{-\frac{\alpha\tau}{2-\tau}} \right)^{\frac{2-\tau}{2}}. \end{aligned} \tag{3.3.15}$$

The first factor on the right hand side can be estimated by a constant times $\|v\|^\tau_{W^\gamma_\alpha(L_2(\Omega))}$, which is bounded by assumption (see, e.g., the proof of [8, Theorem 4.7] for details). In order to estimate the second factor we use the Lipschitz character of Ω, which guarantees that

$$|\Lambda_{j,m}| \leq C\, 2^{j(d-1)} \quad \text{for all} \quad j, m \in \mathbb{N}_0. \tag{3.3.16}$$

Moreover, the boundedness of Ω yields $\Lambda_{j,m} = \emptyset$ for all $j, m \in \mathbb{N}_0$ with $m \geq C \cdot 2^j$. The constant C in both estimates does not depend on j or m. Consequently, we obtain

$$\begin{aligned}\left(\sum_{(i,j,k)\in\Lambda_j^0} 2^{-\frac{2j\tau\gamma}{2-\tau}} \rho_{j,k}^{-\frac{\alpha\tau}{2-\tau}} \right)^{\frac{2-\tau}{2}} &\leq C \left(\sum_{m=1}^{C2^j} \sum_{(i,j,k)\in\Lambda_{j,m}} 2^{-\frac{2j\tau\gamma}{2-\tau}} \rho_{j,k}^{-\frac{\alpha\tau}{2-\tau}} \right)^{\frac{2-\tau}{2}} \\ &\leq C \left(\sum_{m=1}^{C2^j} \sum_{(i,j,k)\in\Lambda_{j,m}} 2^{-\frac{2j\tau\gamma}{2-\tau}} (m\, 2^{-j})^{-\frac{\alpha\tau}{2-\tau}} \right)^{\frac{2-\tau}{2}} \\ &\leq C \left(2^{j\left(d-1-\frac{(2\gamma-\alpha)\tau}{2-\tau}\right)} + 2^{j\left(d-\frac{2\gamma\tau}{2-\tau}\right)} \right)^{\frac{2-\tau}{2}}. \end{aligned} \tag{3.3.17}$$

If $\alpha\tau/(2-\tau) > 1$, the last estimate follows from the convergence of the harmonic series. For $0 < \alpha\tau/(2-\tau) \leq 1$, it can be obtained by estimating the integral $\int_1^{C2^j} t^{-\frac{\alpha\tau}{2-\tau}}\,\mathrm{d}t$ properly. Summing up over $j \in \mathbb{N}_0$ in (3.3.15) and using (3.3.17), leads to

$$\sum_{(i,j,k)\in\Lambda^0} |\langle v, \widetilde{\psi}_{i,j,k}\rangle|^\tau \leq C \|v\|^\tau_{W^\gamma_\alpha(L_2(\Omega))} \sum_{j\in\mathbb{N}_0} \left(2^{j\left(d-1-\frac{(2\gamma-\alpha)\tau}{2-\tau}\right)} + 2^{j\left(d-\frac{2\gamma\tau}{2-\tau}\right)} \right)^{\frac{2-\tau}{2}}.$$

Obviously, the sums on the right hand side converge if, and only if, $0 < s < \min\left\{\gamma, \frac{2\gamma-\alpha}{2}\frac{d}{d-1}\right\}$. Therefore, since this is part of our assumptions,

$$\sum_{(i,j,k)\in\Lambda^0} |\langle v, \widetilde{\psi}_{i,j,k}\rangle|^\tau \leq C \|v\|^\tau_{W^\gamma_\alpha(L_2(\Omega))}. \tag{3.3.18}$$

In the last step we have to estimate the second term *II* from (3.3.14). To this end we use Hölder's inequality together with (3.3.16) to verify that for every $j \in \mathbb{N}_0$,

$$\sum_{(i,j,k)\in\Lambda_{j,0}} |\langle v, \widetilde{\psi}_{i,j,k}\rangle|^\tau \leq C\, 2^{j(d-1)\frac{2-\tau}{2}} \left(\sum_{(i,j,k)\in\Lambda_{j,0}} |\langle v, \widetilde{\psi}_{i,j,k}\rangle|^2 \right)^{\frac{\tau}{2}},$$

where $C \in (0,\infty)$ is a constant independent of j. Summing up over all $j \in \mathbb{N}_0$ and using the relationship $1/\tau = s/d + 1/2$ yields

$$\begin{aligned}
\sum_{(i,j,k)\in\Lambda\setminus\Lambda^0} |\langle v, \widetilde{\psi}_{i,j,k}\rangle|^\tau &= \sum_{j\in\mathbb{N}_0} \sum_{(i,j,k)\in\Lambda_{j,0}} |\langle v, \widetilde{\psi}_{i,j,k}\rangle|^\tau \\
&\leq C \sum_{j\in\mathbb{N}_0} \left(2^{j(d-1)\frac{2-\tau}{2}} \left(\sum_{(i,j,k)\in\Lambda_{j,0}} |\langle v, \widetilde{\psi}_{i,j,k}\rangle|^2 \right)^{\frac{\tau}{2}} \right) \\
&= C \sum_{j\in\mathbb{N}_0} \left(2^{j\frac{d-1}{d}\cdot s\cdot\tau} \left(\sum_{(i,j,k)\in\Lambda_{j,0}} |\langle v, \widetilde{\psi}_{i,j,k}\rangle|^2 \right)^{\frac{\tau}{2}} \right) \\
&\leq C\, \|v\|^\tau_{B_\tau^{\frac{d-1}{d}s}(L_2(\mathbb{R}^d))},
\end{aligned}$$

where the last estimate is due to Proposition 3.2.3. Since $B_2^{\alpha_0}(L_2(\mathbb{R}^d)) \hookrightarrow B_\tau^{\frac{d-1}{d}s}(L_2(\mathbb{R}^d))$ for arbitrary $0 < s < \alpha_0 \frac{d}{d-1}$, see, e.g., [64, Proposition 2.3.2.2], we obtain

$$\sum_{(i,j,k)\in\Lambda\setminus\Lambda^0} |\langle v, \widetilde{\psi}_{i,j,k}\rangle|^\tau \leq C\, \|v\|^\tau_{B_\tau^{\alpha_0}(L_2(\mathbb{R}^d))} \leq C\, \|v\|^\tau_{B_\tau^{\alpha_0}(L_2(\Omega))},$$

where we used the boundedness of the extension operator for the last estimate. Consequently, since $H^{\alpha_0}(\Omega) = B_2^{\alpha_0}(L_2(\Omega))$ with equivalent norms,

$$\sum_{(i,j,k)\in\Lambda\setminus\Lambda^0} |\langle v, \widetilde{\psi}_{i,j,k}\rangle|^\tau \leq C\, \|v\|^\tau_{H^{\alpha_0}(\Omega)}.$$

This estimate together with (3.3.18) prove that (3.3.13) holds and, therefore, so does our assertion. □

Proof of Theorem 3.3.3. Due to Proposition 3.3.1 and Proposition 3.3.2, the statement follows from a straightforward application of the embedding obtained above in Theorem 3.3.5. □

3.4 Besov regularity for the stationary Navier-Stokes equation

In this section we extend our analysis to the stationary Navier-Stokes equation

$$\begin{aligned}
-\Delta u + \nu u \cdot (\nabla u) + \nabla \pi &= f \quad \text{on } \Omega, \\
\operatorname{div} u &= 0 \quad \text{on } \Omega, \qquad\qquad \text{(NAST)} \\
u &= g \quad \text{on } \partial\Omega;
\end{aligned}$$

$\nu > 0$ denotes the Reynolds number and, as before, g is assumed to fulfil (3.3.1) for compatibility reasons. Following the lines of the previous sections, we understand (NAST) in a weak sense and call $u \in H^1(\Omega)^d$ a *(weak) solution* of (NAST) if u is divergence free, satisfies $u = g$ on the boundary $\partial\Omega$ (in a trace sense) and fulfils the equation

$$\int_\Omega \sum_{i,j=1}^d (\nabla u)_{ij} (\nabla \varphi)_{ij}\, \mathrm{d}x + \nu \int_\Omega (u \cdot (\nabla u))\, \varphi\, \mathrm{d}x = f(\varphi) - \int_\Omega \sum_{i=1}^d \pi \frac{\partial \varphi_i}{\partial x_i}\, \mathrm{d}x$$

for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$ with a suitable $\pi \in L_2(\Omega)$. See [43, Chapter IX] for more details.

Our goal is to establish existence of a solution to (NAST) with high regularity in the scale $(*)$ of Besov spaces. To this end we exploit what we proved in the previous section about the regularity of the Stokes equation together with a fixed point argument. Our approach is based on the following basic observation: Assume that $u \in H^1(\Omega)^d$ is a solution to (NAST) with $f \in L_2(\Omega)^d$ and $g \in H^1(\partial\Omega)^d$. Then, if we could guarantee that $u \cdot (\nabla u) \in L_2(\Omega)^d$, the solution $u \in H^1(\Omega)^d$ to (NAST) would actually be a solution to (SP) with body force $f - u \cdot (\nabla u) \in L_2(\Omega)^d$ (instead of f) and prescribed velocity field $g \in H^1(\partial\Omega)^d$. As a consequence, u and the corresponding pressure π would have the Besov regularity in the scale $(*)$ guaranteed by Theorem 3.3.3. Of course, $u \in H^1(\Omega)^d$ is not a sufficient condition for $u \cdot (\nabla u) \in L_2(\Omega)^d$. However, the latter would hold if we could additionally guarantee that our solution is essentially bounded. This would definitively be fulfilled if we require $u \in B^t_p(L_p(\Omega))^d$ for some $p > 1$ and $t \notin \mathbb{N}$ with $t > d/p$, since in this case $B^t_p(L_p(\Omega)) = W^t(L_p(\Omega)) \hookrightarrow L_\infty(\Omega)$ due to Sobolev's embedding theorem. A solution to (SP) with this property can be obtained by exploiting the results from [57]. Theorem 10.15 therein guarantees, among others, that the Stokes problem (SP) with body force $f \in B^{t-2}_p(L_p(\Omega))$ and boundary condition $g \in B^{t-1/p}_p(L_p(\partial\Omega))^d$ fulfilling (3.3.1) has a unique solution $u \in B^t_p(L_p(\Omega))^d$ with corresponding pressure $\pi \in B^{t-1}_p(L_p(\Omega))$—provided the parameters p and t are within an admissible range $\mathcal{R}_{d,\varepsilon}$ that depends on the dimension d and on the Lipschitz character of the underlying domain Ω, which is expressed by the quantity $\varepsilon = \varepsilon(\Omega) \in (0,1]$, see also Remark 3.3.4. Moreover, there exists a finite constant $C > 0$, such that

$$\|u\|_{B^t_p(L_p(\Omega))^d} + \inf_{c\in\mathbb{R}} \|\pi + c\|_{B^{t-1}_p(L_p(\Omega))} \leq C\big(\|f\|_{B^{t-2}_p(L_p(\Omega))^d} + \|g\|_{B^{t-1/p}_p(L_p(\partial\Omega))^d}\big).$$

For $d = 3$, there always exists a pair (p,t) with $t > d/p$ and $t \notin \mathbb{N}$ within the range $\mathcal{R}_{3,\varepsilon}$ of parameters admissible in [57, Theorem 10.15], since $\mathcal{R}_{3,\varepsilon}$ covers all $p > 2$ and $t \in \mathbb{R}$ such that

$$\max\left\{\frac{3}{p}, 1\right\} < t < \min\left\{\frac{3}{p} + \varepsilon,\ 1 + \frac{1}{p}\right\}. \tag{3.4.1}$$

This is also the case for $d \geq 4$ if the underlying domain is smooth enough, i.e., if we assume that the quantity $\varepsilon = \varepsilon(\Omega)$ describing the Lipschitz character of Ω fulfils

$$\varepsilon > \frac{d-3}{2(d-1)}.$$

Then the corresponding range $\mathcal{R}_{d,\varepsilon}$ in [57, Theorem 10.15] covers all $p > d-1$ and $t \in \mathbb{R}$ such that

$$\max\left\{\frac{d}{p}, 1\right\} < t < \min\left\{\frac{d}{p} + (d-1)\varepsilon - \frac{d-3}{2},\ 1 + \frac{1}{p}\right\}. \tag{3.4.2}$$

Thus, for these ranges of parameters [57, Theorem 10.15] guarantees that the linear solution operator of the Stokes problem (SP),

$$\begin{aligned} L := L_{t,p,\Omega}\colon B^{t-2}_p(L_p(\Omega))^d \times B^{t-1/p,0}_p(L_p(\partial\Omega))^d &\to B^t_p(L_p(\Omega))^d \times \big(B^{t-1}_p(L_p(\Omega))/\mathbb{R}_\Omega\big) \\ (f,g) &\mapsto L(f,g) := (u,\pi), \end{aligned}$$

where u is the unique solution to (SP) with body force f and boundary value g, and π is the corresponding pressure, is well-defined and bounded (see Section 3.2.1 for notation). We denote its operator norm by

$$\|L_{t,p,\Omega}\| := \sup_{(f,g)\in Y\setminus\{0\}} \frac{\|L_{t,p,\Omega}(f,g)\|_{X_{t,p,\Omega}}}{\|(f,g)\|_{Y_{t,p,\Omega}}},$$

where

$$X := X_{t,p,\Omega} := B^t_p(L_p(\Omega))^d \times \left(B^{t-1}_p(L_p(\Omega))/\mathbb{R}_\Omega\right)$$

and

$$Y := Y_{t,p,\Omega} := B^{t-2}_p(L_p(\Omega))^d \times B^{t-1/p,0}_p(L_p(\partial\Omega))^d.$$

Using this notation, we can present our main result concerning the regularity of the solution to (NAST) in the scale $(*)$ of Besov spaces.

Theorem 3.4.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 3$, with connected boundary. Assume that the quantity $\varepsilon = \varepsilon(\Omega) \in (0,1]$ from [57] describing the Lipschitz character of Ω fulfils*

$$\varepsilon > \frac{d-3}{2(d-1)}. \tag{3.4.3}$$

Let $p > d-1$ and $t \in \mathbb{R}$ satisfy (3.4.1) *for $d = 3$ and* (3.4.2) *for $d \geq 4$, respectively. Fix*

$$f \in L_2(\Omega)^d \cap B^{t-2}_p(L_p(\Omega))^d$$

and

$$g \in H^1(\partial\Omega)^d \cap B^{t-1/p}_p(L_p(\partial\Omega))^d \quad \textit{with} \quad \int_{\partial\Omega} g\cdot\mathbf{n}\, d\sigma = 0.$$

Then there exists a finite constant $C = C_{t,p,\Omega} > 0$ such that, if

$$C_{t,p,\Omega}\cdot\nu\cdot\left(\|f\|_{B^{t-2}_p(L_p(\Omega))^d} + \|g\|_{B^{t-1/p}_p(L_p(\partial\Omega))^d}\right) < \frac{1}{4\cdot\|L_{t,p,\Omega}\|^2}, \tag{3.4.4}$$

then (NAST) *has at least one solution $u \in H^{3/2}(\Omega)^d$ with corresponding pressure $\pi \in H^{1/2}(\Omega)$ which satisfy*

$$u \in B^{s_1}_{\tau_1}(L_{\tau_1}(\Omega))^d, \quad \frac{1}{\tau_1} = \frac{s_1}{d} + \frac{1}{2}, \quad 0 < s_1 < \min\left\{\frac{3}{2}\cdot\frac{d}{d-1}, 2\right\}, \tag{3.4.5}$$

and

$$\pi \in B^{s_2}_{\tau_2}(L_{\tau_2}(\Omega)), \quad \frac{1}{\tau_2} = \frac{s_2}{d} + \frac{1}{2}, \quad 0 < s_2 < \frac{1}{2}\cdot\frac{d}{d-1}, \tag{3.4.6}$$

respectively. The pair (u,π) is unique in $A_{1/2} := \{(v,q) \in X_{r,p,\Omega} \colon \|L_{t,p,\Omega}\|\cdot\nu\cdot C_{t,p,\Omega}\cdot \|(v,q)\|_{X_{t,p,\Omega}} \leq 1/2\}$.

Remark 3.4.2. (i) Note that the restriction (3.4.3) is empty if $d = 3$, i.e., the theorem holds for any arbitrary Lipschitz domain $\Omega \subseteq \mathbb{R}^3$ with connected boundary.

(ii) A close look to the proof below reveals that the constant $C = C_{t,p,\Omega} > 0$ in the statement of Theorem 3.4.1 can be chosen to be the product of the embedding constants of the embeddings in (3.4.7) and (3.4.8).

(iii) The solution u to (NAST) and the corresponding pressure π determined in Theorem 3.4.1 have L_2-Sobolev regularity 3/2 and 1/2, respectively. To the best of our knowledge there is no result which guarantees that a solution u to (NAST) and a corresponding pressure term π have a higher L_2-Sobolev regularity in the given setting. However, their Besov regularity in the scale $(*)$ is strictly higher than 3/2 and 1/2, respectively, see (3.4.5) and (3.3.10). Therefore, the usage of adaptive wavelet schemes for solving (NAST) is justified in the sense described in the introduction, see also Remark 3.3.4.

Proof of Theorem 3.4.1. Fix the parameters p and t as well as the body force f and the velocity field g as required in our assumptions. We prove that the mapping

$$\begin{aligned} T := T^{f,g,\nu}_{t,p,\Omega} : X_{t,p,\Omega} &\to X_{t,p,\Omega} \\ (u,\pi) &\mapsto L_{t,p,\Omega}(f - \nu u \cdot (\nabla u), g), \end{aligned}$$

has a fixed point $(u,\pi) \in X_{t,p,\Omega}$ consisting of a solution u to (NAST) and the corresponding pressure π, both of them having the asserted properties. To this end, we first need some preparations.

First of all we check that the operator T is well-defined. Since $p > 2$, $1 < t < 2$, and Ω is bounded, the embeddings

$$L_p(\Omega) \hookrightarrow B^{t-2}_p(L_p(\Omega)) \cap L_2(\Omega) \quad \text{and} \quad B^t_p(L_p(\Omega)) \hookrightarrow W^1(L_p(\Omega)), \tag{3.4.7}$$

hold, since the classical embeddings for Besov and Triebel-Lizorkin spaces, as they can be found, e.g., in [64, Proposition 2.3.2/2], can be carried over to the case of bounded Lipschitz domains (definition via restriction) and the Sobolev spaces $W^k(L_p(\Omega))$, $k \in \mathbb{N}_0$, can be described in terms of Triebel-Lizorkin spaces, see, e.g., [65, Proposition 1.122(i)]. Moreover, since $t > d/p$, $t \notin \mathbb{N}$, Sobolev's embedding theorem yields

$$B^t_p(L_p(\Omega)) = W^t(L_p(\Omega)) \hookrightarrow L_\infty(\Omega), \tag{3.4.8}$$

see, e.g., [58, Chapter 2.2.4]. These embeddings imply that $v \cdot (\nabla v) \in B^{t-2}_p(L_p(\Omega))^d \cap L_2(\Omega)^d$ whenever $v \in B^t_p(L_p(\Omega))^d$, since

$$\begin{aligned} \|v \cdot (\nabla v)\|_{B^{t-2}_p(L_p(\Omega))^d} + \|v \cdot (\nabla v)\|_{L_2(\Omega)^d} &\le C\, \|v \cdot (\nabla v)\|_{L_p(\Omega)^d} \\ &= C\, \Big\| \sum_{i=1}^d v_i \frac{\partial v}{\partial x_i} \Big\|_{L_p(\Omega)^d} \\ &\le C\, \|v\|_{L_\infty(\Omega)^d} \|v\|_{W^1(L_p(\Omega))^d} \\ &\le C\, \|v\|^2_{B^t_p(L_p(\Omega))^d}, \end{aligned}$$

with a finite constant $C =: C_{t,p,\Omega}$, which is the product of the embedding constants of the embeddings above. As a consequence, the operator

$$\begin{aligned} N := N^{f,g,\nu}_{t,p,\Omega} \colon X_{t,p,\Omega} &\to Y_{t,p,\Omega} \cap \big(L_2(\Omega)^d \times H^1(\partial\Omega)^d\big) \\ (u,\pi) &\mapsto (f - \nu u \cdot (\nabla u), g) \end{aligned}$$

is well-defined, and, therefore, so is $T = L \circ N \colon X_{t,p,\Omega} \to X_{t,p,\Omega}$.

Secondly, we prove the existence of a fixed point of T. A similar calculation as above shows that for $(u,\pi),(\tilde{u},\tilde{\pi})\in X_{t,p,\Omega}$,

$$\begin{aligned}
&\|T_{t,p,\Omega}(u,\pi)-T_{t,p,\Omega}(\tilde{u},\tilde{\pi})\|_{X_{t,p,\Omega}}\\
&\qquad\le \|L_{t,p,\Omega}\|\cdot\nu\cdot\|u\cdot(\nabla u)-\tilde{u}\cdot(\nabla\tilde{u})\|_{B^{t-2}_p(L_p(\Omega))^d}\\
&\qquad\le \|L_{t,p,\Omega}\|\cdot\nu\cdot C_{t,p,\Omega}\cdot\max\left\{\|u\|_{B^t_p(L_p(\Omega))^d},\|\tilde{u}\|_{B^t_p(L_p(\Omega))^d}\right\}\cdot\|u-\tilde{u}\|_{B^t_p(L_p(\Omega))^d}.
\end{aligned}$$

Thus, if we can find $\lambda<1$ such that

$$(u,\pi)\in A_\lambda:=\left\{(v,q)\in X_{r,p,\Omega}\colon \|L_{t,p,\Omega}\|\cdot\nu\cdot C_{t,p,\Omega}\cdot\|(v,q)\|_{X_{t,p,\Omega}}\le\lambda\right\}$$

implies $T(u,\pi)\in A_\lambda$, then T is a contraction on a the closed ball $A_\lambda\subseteq X$, so that we can obtain the fixed point we are seeking for by applying Banach's fixed point theorem. Assume $(u,\pi)\in A_\lambda$ for some $\lambda\in(0,1)$. Then,

$$\begin{aligned}
&\|T_{t,p,\Omega}(u,\pi)\|_{X_{t,p,\Omega}}\\
&\qquad\le\|L_{t,p,\Omega}\|\left(\|f\|_{B^{t-2}_p(L_p(\Omega))^d}+\|g\|_{B^{t-1/p}_p(L_p(\partial\Omega))^d}+\nu\,\|u\cdot(\nabla u)\|_{B^{t-2}_p(L_p(\Omega))^d}\right)\\
&\qquad\le\|L_{t,p,\Omega}\|\left(\|f\|_{B^{t-2}_p(L_p(\Omega))^d}+\|g\|_{B^{t-1/p}_p(L_p(\partial\Omega))^d}+\nu\,C_{t,p,\Omega}\,\|u\|^2_{B^t_p(L_p(\Omega))^d}\right)\\
&\qquad\le\|L_{t,p,\Omega}\|\left(\|f\|_{B^{t-2}_p(L_p(\Omega))^d}+\|g\|_{B^{t-1/p}_p(L_p(\partial\Omega))^d}+\frac{\lambda^2}{\nu\,C_{t,p,\Omega}\,\|L_{t,p,\Omega}\|^2}\right).
\end{aligned}$$

Moreover,

$$\|L_{t,p,\Omega}\|\left(\|f\|_{B^{t-2}_p(L_p(\Omega))^d}+\|g\|_{B^{t-1/p}_p(L_p(\partial\Omega))^d}+\frac{\lambda^2}{\nu\,C_{t,p,\Omega}\,\|L_{t,p,\Omega}\|^2}\right)\le\frac{\lambda}{\nu\,C_{t,p,\Omega}\,\|L_{t,p,\Omega}\|}$$

if, and only if,

$$\nu\,C_{t,p,\Omega}\,\|L_{t,p,\Omega}\|^2\left(\|f\|_{B^{t-2}_p(L_p(\Omega))^d}+\|g\|_{B^{t-1/p}_p(L_p(\partial\Omega))^d}\right)\le-\lambda^2+\lambda.$$

The right hand side is positive on $(0,1)$ and attains its maximum $1/4$ at $\lambda=1/2$. Thus, $\lambda_0=1/2$ does the job, so that, if (3.4.4) is assumed to hold, then we have a fixed point $(u,\pi)\in A_{1/2}\subseteq X_{t,p,\Omega}$.

Finally, we note that

$$X_{t,p,\Omega}\hookrightarrow H^1(\Omega)^d\times(L_2(\Omega)/\mathbb{R}_\Omega)\,,$$

due to the embeddings discussed at the beginning of the proof. Moreover, $u\cdot(\nabla u)\in L_2(\Omega)$, as shown above. Therefore, any fixed point $(u,\pi)\in X_{t,p,\Omega}$ consists, by definition, of a solution $u\in H^1(\Omega)^d$ to the Stokes problem (SP) with body force $(f-\nu u\cdot(\nabla u))\in L_2(\Omega)$ (instead of f) and prescribed velocity field $g\in H^1(\partial\Omega)^d$ together with its corresponding pressure $\pi\in L_2(\Omega)$. Consequently, $u\in H^{3/2}(\Omega)^d$ and $\pi\in H^{1/2}(\Omega)$ due to Proposition 3.3.1 and they have the asserted Besov regularity in the scale $(*)$ due to Theorem 3.3.3. □

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Erklärung

Ich versichere, dass ich meine Dissertation

Besov Regularity of Solutions to Navier-Stokes Equations

selbstständig, ohne unerlaubte Hilfe angefertigt und mich dabei keiner anderen als der von mir ausdrücklich bezeichneten Quellen und Hilfen bedient habe.

Die Dissertation wurde in der jetzigen oder einer ähnlichen Form noch bei keiner anderen Hochschule eingereicht und hat noch keinen sonstigen Prüfungszwecken gedient.

Frank Eckhardt