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Quandles and Hurwitz Orbits

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Contents

Abstract

In this thesis we study quandles and Hurwitz orbits. A quandle is a self-distributive algebraic structure whose binary operation is like the conjugation in a group. The algebraic structure of quandles can be studied as sequences of permutations. The cycle structure of the permutations of an indecomposable quandle is well-behaved because the permutations of an indecomposable quandle are mutually conjugate and hence have the same cycle structure. We study the cycle structure of quandles with the main focus on a conjecture in [18] saying that any permutation of an indecomposable quandle has cycles whose cycle lengths divide the largest among them.

Hurwitz orbits are the orbits of a braid group action on the powers of a quandle. The Hurwitz orbits for the action of the braid group on three strands are used in [21] and [22] for the classification of certain Hopf algebras. This classification is based on a combinatorial invariant called a plague on the Hurwitz orbits. The immunity on a Hurwitz orbit is the quotient of the size of the minimal plague and the size of the Hurwitz orbit. An estimation on the immunity of the Hurwitz orbits is provided in [22] by using Schreier graphs of the Hurwitz orbit quotients and the weights on the Hurwitz orbits, where the weight on an Hurwitz orbit is defined by the cycle structure of that Hurwitz obit. In this study only few Schreier graphs of the Hurwitz orbit quotients with small cycles are considered. We introduce a new method to calculate plagues on the Hurwitz orbits for infinitely many Schreier graphs of the Hurwitz orbit quotients with all cycles. Our method is based on the posets of robust subgraphs of pointed Schreier graphs of the Hurwitz orbit quotients. By using this method we estimate the immunity on the Hurwitz orbits through a case-by-case analysis of infinitely many pointed Schreier graphs of the Hurwitz orbit quotients.

Introduction

Quandles are self-distributive algebraic structures with three axioms which are related to three Reidemeister moves of planner knot diagrams. *Racks* are a generalization of quandles. The binary operation of racks and quandles is like the conjugation in a group. The algebraic structure of racks and quandles can be studied as sequences of permutations. The cycle structure of the permutations of an indecomposable rack (resp. quandle) is well-behaved because the permutations of an indecomposable rack (resp. quandle) are mutually conjugate and hence have the same cycle structure. In [18], C. Hayashi observed another interesting property of the cycle structure of an indecomposable quandle and conjectured that the permutation of an indecomposable quandle has cycles whose cycle lengths divide the largest among them. In Chapter 1 of this thesis we will study the cycle structure of racks and quandles with the main focus on the Hayashi's conjecture. We will discuss the classes of indecomposable quandles for which Hayashi's conjecture is true. We will also provide the obstructions on the cycle structure of certain indecomposable racks and quandles, which are among the main results of this thesis.

Racks and the braid group action on the powers of a rack are useful for the classification of certain Hopf algebras and the solutions of the braid equation, see [2], [3], [15], [21], [22]. *Hurwitz orbits* are the orbit of a braid group action on the powers of a rack. The Hurwitz orbits for the action of the braid group on three strands are used in [21] and [22] for the classification of certain Hopf algebras. This classification is based on a combinatorial invariant of the Hurwitz orbits which is called a *plague*. The *immunity* of a Hurwitz orbit is the quotient of the size of the minimal plague and the size of the Hurwitz orbit. An estimation on the immunity of the Hurwitz orbits is provided in [22] by using labelled Schreier graphs of the Hurwitz orbit quotients and the weights of the Hurwitz orbits, where the weight of an Hurwitz orbit is defined by the cycle structure of the Hurwitz obits. In Chapter 2 we will recall the study about the Hurwitz orbits and the method to estimate the immunity of the Hurwitz orbits from [15], [21] and [22]. Note that in [22] only few Schreier graphs with small cycles are considered.

In Chapter 3 we will introduce a general method to calculate the plague and estimate the immunity of the Hurwitz orbits. With this method we can study infinitely many

Schreier graphs of the Hurwitz orbit quotients with all cycles. Our method is based on the posets of certain subgraphs called *robust subgraphs* of pointed Schreier graphs of the Hurwitz orbit quotients. We will estimate the immunity of the Hurwitz orbits through a case-by-case analysis of pointed Schreier graphs of the Hurwitz orbit quotients. With this analysis we will consider those Schreier graphs of the Hurwitz orbit quotients for which the immunity of the Hurwitz orbit is bounded above by a quarter. All the results proved in Chapter 1 and in Chapter 3 are claimed by the author as original. The summary of the main results is given in the Conclusion of the thesis.

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Chapter 1

Quandles and their Cycle Structure

1.1 Preliminaries

In this section we recall the basics of quandle theory. For further details on these topics we refer to [3], [25], [28].

Definition 1.1.1. A *rack* is a pair $(X, \triangleright)$, where X is a non-empty set and $\triangleright : X \times X \longrightarrow X$ is a binary operation such that

(R1) the map $\phi_x : X \longrightarrow X$, defined by $\phi_x(y) = x \triangleright y$, is bijective for all $x \in X$,

(R2) $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$ for all $x, y, z \in X$ (i.e., $\triangleright$ is self-distributive).

Definition 1.1.2. A rack X is called:

- a *quandle* if $x \triangleright x = x$ for all $x \in X$,
- a *crossed set* if X is a quandle and for all $x, y \in X$,

$$x \triangleright y = y \Leftrightarrow y \triangleright x = x,$$

- *braided* if X is a quandle and at least one of the equations

$$x \triangleright (y \triangleright x) = y, x \triangleright y = y,$$

  holds, for all $x, y \in X$,

Definition 1.1.3. A *subrack* (resp. subquandle) of a rack (resp. quandle) $(X, \triangleright)$ is a non-empty subset $Y \subseteq X$ such that $(Y, \triangleright)$ is also a rack (resp. quandle).

Observe that if X is a finite rack (resp. quandle) then any closed subset of X is a subrack (resp. subquandle). Indeed, every singleton subset of a quandle is a subquandle, but analogous statement is not true for a rack which is not a quandle.

Example 1.1.4. A group G is a quandle with

$$x \triangleright y = xyx^{-1}$$

for all $x, y \in G$. This quandle is called *conjugation quandle*. Note that the conjugation quandle is a crossed set. Similarly, the union of conjugacy classes in G is a crossed set.

Example 1.1.5. The *dihedral quandle* $\mathbb{D}_n$ of order n is defined on $\mathbb{Z}_n = \{0, 1, ..., n-1\}$ by

$$i \triangleright j = 2j - i \pmod{n}.$$

for all $i, j \in \mathbb{Z}_n$. Note that the dihedral quandle $\mathbb{D}_n$ is the subquandle of the conjugation quandle of the dihedral group of order $2n$, consisting of reflections. Observe also that $\mathbb{D}_3 \cong (1\ 2)^{S_3}$ is a braided rack, where $(1\ 2)^{S_3}$ is the conjugacy class of transpositions in the symmetric group S_3.

Example 1.1.6. Let A be an abelian group and $Aut(A)$ is the group of automorphisms of A. Let $g \in Aut(A)$ and $1 = id_A$. Then we have a quandle structure on A given by

$$x \triangleright y = (1 - g)(x) + g(y).$$

for all $x, y \in A$. This quandle is called the *affine quandle (or Alexander quandle)* associated to the pair (A, g) and is denoted by *Aff*(A, g).

In particular, let $\mathbb{F}_q$ be a finite field, where q is a power of a prime number p, and $\alpha \in \mathbb{F}_q \setminus \{0, 1\}$. Then we write *Aff*$(\mathbb{F}_q, \alpha)$, or simply *Aff*$(q, \alpha)$, for the affine quandle *Aff*(A, g), where $A = \mathbb{F}_q$ and g is the automorphism given by $y \longmapsto \alpha y$ for all $y \in F_q$. For $\alpha = 1$, the affine quandle *Aff*(q, α) is trivial. For $\alpha = -1$, we have the affine quandle *Aff*$(p, -1)$, which is isomorphic to the dihedral quandle $\mathbb{D}_p$.

Example 1.1.7. Let G be a group and α an automorphism on G. Let H be a subgroup of the fixed points of α in G and $G/H = \{gH : g \in G\}$. Then for any $g, f \in G$ we have a quandle structure on G/H given by

$$gH \triangleright fH = g\alpha(g^{-1}f)H.$$

This quandle is known as the *coset quandle* or *homogeneous quandle* and is denoted by (G, H, α). In particular, if G is a group with identity 1 and $H = \{1\}$, then the coset quandle (G, H, α) is called *twisted homogenous crossed set or principal quandle* and is written as (G, α) for short (see [3]).

1.2 Quandles and Groups

In this section we recall the definitions of some important groups associated to quandles from [15], [25].

Definition 1.2.1. The *enveloping group* of a quandle X is:

$$G_X = F(X)/\left\langle xyx^{-1} = x \triangleright y, x, y \in X\right\rangle,$$

where $F(X)$ is the free group generated by X. If X is finite, the *finite enveloping group* is:

$$\overline{G_X} = G_X/\left\langle x^{|\phi_x|}, x \in X\right\rangle,$$

where $|\phi_x|$ is the order of ϕ_x.

Definition 1.2.2. A quandle X is called *injective* if the canonical map $X \longrightarrow G_X$ (defined by $x \longrightarrow [x]$, where $[x]$ denotes the equivalence class in G_X of the generator x) is injective.

1.2.1 Automorphism Groups of Quandles

Let X be a quandle. From the quandle axioms (R1) and (R2), it follows that each ϕ_x is a quandle automorphism on X. We write the *full automorphism group* of a quandle X as $Aut(X)$. The $Aut(X)$ contains two important subgroups which are defined as follows.

Definition 1.2.3. The *inner group* of a quandle X is the group

$$Inn(X) = \langle \phi_x | x \in X\rangle.$$

The *transvection group* of a quandle X is the group

$$Trans(X) = \left\langle \phi_x\phi_y^{-1} | x, y \in X\right\rangle.$$

Observe that both $Inn(X)$ and $Trans(X)$ are normal subgroups of $Aut(X)$. A quandle X is called *faithful* if the map $X \longrightarrow Inn(X)$, defined by $x \longrightarrow \phi_x$, is injective.

The $Aut(X)$, $Inn(X)$ and $Trans(X)$ naturally act on a quandle X. Note that, throughout this thesis, we will only consider the left group actions. Recall that the action of $Aut(X)$ (resp. $Inn(X)$ and $Trans(X)$) on a quandle X is said to be *transitive* if for any two $y, z \in X$ there exists a ϕ_x in $Aut(X)$ (resp. $Inn(X)$ and $Trans(X)$) such that $\phi_x(y) = x \triangleright y = z$. The transitive actions of $Aut(X)$, $Inn(X)$ and $Trans(X)$ on a quandle X give rise to the following definitions.

Definition 1.2.4. A quandle X is called:

- *homogenous* if $Aut(X)$ acts transitively on X,
- *indecomposable or connected* if $Inn(X)$ acts transitively on X. Otherwise, X is called *decomposable*.

Remark 1.2.5. Any faithful quandle is injective but the converse is not true, (see [25], Lemma 2.10 and Example 2.12). Observe also that any faithful quandle is a crossed set (see [3]].

1.3 Some Classes of Connected Quandles

The problem of classification of connected quandles up to isomorphism has been attacked by many by studying special classes of quandles, starting with finite quandles. In this section we recall some known classes of connected quandles. For further details on classification of connected quandles we refer to [1], [3], [10], [13], [14], [24], and [31].

1.3.1 Small Connected Quandles

L. Vendramin calculated all connected quandles (upto isomorphism) of size at most 35 (see [31], Proposition 5.1). By using the theory of transitive groups, L. Vendramin also computed all isomorphism classes of indecomposable quandles up to size 47. The list of all indecomposable quandles of size $n \leq 47$ is available in a GAP package called *Rig* (see [16]). These small quandles are included in Rig as

$$SmallQuandle(n, q(n)), \text{ where } q(n) := \text{quandle number of size } n.$$

1.3.2 Galkin Quandles

Let A be an abelian group and $\mu : \mathbb{Z}_3 \to \mathbb{Z}, \tau : \mathbb{Z}_3 \to A$ be functions. If $\mu(0) = 2, \mu(1) = \mu(2) = -1$ and $\tau(0) = 0$, then we have quandle structure on $\mathbb{Z}_3 \times A$ given by

$$(x, a) \triangleright (y, b) = (2x - y, \tau(y - x) + \mu(y - x)a - b).$$

for all $x, y \in \mathbb{Z}_3$ and $a, b \in A$. This quandle is called the *Galkin quandle*. The Galkin quandle $\mathbb{Z}_3 \times A$ is connected (see [10]).

1.3.3 Connected Affine Quandles

The class of affine quandles consists of connected and decomposable quandles. An affine quandle *Aff*(A, g) is connected if and only if $1 - g$ is onto (see [24], Corollary 7.2). Note that the affine quandle *Aff*(q, α) has no nontrivial subquandles (see [2], Proposition 4.1). Therefore the affine quandle *Aff*(q, α) is indecomposable.

1.3.4 Connected Quandles of Prime Power Order

Let X be a connected quandle of order $q = p^n$, where p is a prime number. Then $Tran(X)$ is a $p-$group (see [6], Corollary 5.2). The connected quandles of size $q \in \{p, p^2, p^3\}$ are studied in [13],[14],[6]. Recall that the connected quandles of order p and p^2 are affine, principal and faithful, and a connected quandle of size p^3 is either principal, or isomorphic to a coset quandle (G, H, α), where G has order p^4.

1.3.5 Simple Quandles

A quandle X is called a *simple quandle* if it has no quotients except itself and the one-element quandle. An equivalent condition is that every homomorphism from the quandle is either constant or a monomorphism (see [25]). Recall that a simple quandle is connected and faithful (see [6]). Note also that a simple quandle of prime power order is affine (see [6], Proposition 5.9).

1.3.6 Cyclic and Primitive Quandles

A quandle X is said to be of *cyclic type* or *cyclic* if for each $x \in X$ the permutation φ_x acts on $X \setminus \{x\}$ as a cycle of length $|X| - 1$, where $|X|$ denotes the cardinality of X. It is easy to see that a quandle X of cyclic type with $|X| \geq 3$ is connected (see [32]). A quandle X is said to be *primitive* if $Inn(X)$ acts primitively on X, that is, $Inn(X)$ acts transitively on X and $Inn(X)$ preserves no nontrivial partition of X. Note that by using the theory of transitive groups and GAP database of primitive groups, L. Vendramin computed all isomorphism classes of primitive quandles up to size 2106.

1.4 Cycle Structure of Quandles

The algebraic structure of quandles can be studied by using a perspective based on the sequence of their permutations. The cycle structure of permutations of quandles is refer to as the cycle structure of quandles. In this section, we first recall some definitions and results from [6], [22] and [28] about the cycle structure of quandles. Next, we will discuss a conjecture about the cycle structure of connected quandles and see that for which families of connected quandles that conjecture is true. Finally, we will provide obstructions on the cycle structure of connected quandles, which are among the main results of this thesis.

Definition 1.4.1. Let X be a finite rack (resp. quandle) and $\phi_x : X \longrightarrow X$ is a permutation on X for any $x \in X$. Let $\phi_x = \sigma_1\sigma_2...\sigma_k$ be the decomposition of ϕ_x into product of disjoint cycles σ_i for $1 \leq i \leq k$. Let $\ell(\sigma_i) = \ell_i$ is the cycle length of σ_i. Then the list of all ℓ_i (a set with possible repeats) is called the *pattern* of ϕ_x. The *profile* of X is the sequence of patterns of all ϕ_x.

Remark 1.4.2. Since $(x \triangleright y) \triangleright (x \triangleright z) = x \triangleright (y \triangleright z)$ for all $x, y, z \in X$,

$$\phi_{x \triangleright y} = \phi_x \phi_y \phi_x^{-1}$$

By using this equation, any two automorphisms ϕ_x, ϕ_y of a finite connected rack (resp. quandle) X are mutually conjugate (see [15] and [28]). Since the conjugate permutations have the same cycle structure, any two permutations of a finite connected rack (resp. quandle) X have the same pattern. Therefore, the profile of a finite connected rack (resp. quandle) X is a constant sequence and the pattern of any automorphism of X can be considered as the profile of a finite connected rack (resp. quandle) X for short. We will write the automorphism of a connected rack (resp. quandle) X by φ.

Notation 1.4.3. We write the profile of a finite connected rack (resp. quandle) X as:

$$Profile(X) = 1^{m_0} \ell_1^{m_1} \ell_2^{m_2} ... \ell_k^{m_k},$$

where $1 < \ell_1 < \ell_2 < ... < \ell_k$, and $m_0, m_1, ..., m_k$ are the multiplicities of $1, \ell_1, \ell_2, ..., \ell_k$, respectively. For example:

$$Profile(SmallQuandle(42, 7)) = 1^2.2^2.3^4.6^4.$$

Note that, if $Profile(X) = 1^{m_0} \ell_1^{m_1} \ell_2^{m_2} ... \ell_k^{m_k}$ then,

$$|X| = m_0 + m_1 \ell_1 + ... + m_k \ell_k,$$

and for any $x \in X$,

$$| \ supp(\varphi_x) \ | = m_1 l_1 + ... + m_k l_k, \text{ where } supp(\varphi_x) := \{y \in X \mid \varphi_x(y) \neq y\}.$$

1.4.1 Hayashi's Conjecture on Quandles

In [18], C. Hayashi proposed a conjecture about the cycle structure of connected quandles, which we refer as the Hayashi's conjecture. Note that the Hayashi's conjecture on connected quandles can also be stated for connected racks. By using the notation of the profile of a finite connected rack (resp. quandle) X, we state the Hayashi's conjecture as follows.

Conjecture 1.4.4. Let X be a finite connected rack (resp. quandle) with

$$Profile(X) = 1^{m_0} \ell_1^{m_1} \ell_2^{m_2} ... \ell_k^{m_k}.$$

Then $\ell_i | \ell_k$ (i.e., ℓ_i divides ℓ_k) for any integer i with $1 \leq i \leq k-1$.

Remark 1.4.5. The cycle structure of an automorphism α of a finite group G is also studied recently, notably in [7], [17]. In these studies, a cycle σ of α is called a *regular cycle or orbit* if $\ell(\sigma) = ord(\alpha)$, where $ord(\alpha)$ denotes the order of α. Recall that $ord(\alpha)$ is the least common multiple (lcm) of the cycle lengths of α. Therefore, if α has a regular cycle σ, then all cycle lengths of α divide the largest cycle length $\ell(\sigma)$. In this way, the Hayashi's conjecture simply says that, for a connected rack (resp. quandle) X, any automorphism φ_x for $x \in X$ have a regular cycle. Since the automorphisms of certain connected racks depend on the automorphisms of underlying groups, therefore the results about the cycle structure of group automorphism can be used to verify the Hayashi's for certain connected racks (resp. quandles). We recall the following corollary from [7] which we will use later.

Corollary 1.4.6. *Any automorphism α of a finite nilpotent group has a regular cycle.*

Observations 1.4.7. Now we see that for which known families of connected quandles, the Hayashi's conjecture is true.

1. The Hayashi's conjecture is true for all connected quandles of size at most 47. This can be seen by inspection in Rig [16] with the following function:

$$Profile := CycleLengths(Permutations(q)[1], [1..n]),$$

where $q := SmallQuandle(n, q(n))$, with $q(n) :=$ quandle number of size n. For example:

$$Profile(SmallQuandle(42, 7)) = 1^2.2^2.3^4.6^4.$$

2. The Hayashi's conjecture is trivially true for a connected dihedral quandle $\mathbb{D}_n$. Since, for $i, j \in \mathbb{D}_n$, $\varphi_i(j) := 2j - i \pmod{n}$, and

$$\varphi_i = \prod_{j=1}^{[\frac{n-1}{2}]} (i + j \; i - j) \pmod{n}.$$

Hence, $Profile(\mathbb{D}_n) = 1^{m_0} \ell_1^{m_1}$, where $\ell_1 = 2$.

3. The Hayashi's conjecture is true for any Galkin quandle $\mathbb{Z}_3 \times A$ (see [10], Lemma 4.5), since $Profile(\mathbb{Z}_3 \times A) = 1^{m_0} \ell_1^{m_1} \ell_2^{m_2}$, where $\ell_1 = 2$ and $\ell_2 = 2k$, where k is the order of an element in the group A.

4. The Hayashi's conjecture is true for any connected affine quandle. Recall that, for an affine quandle *Aff*(A, α) we have:

$$\varphi_x(y) = x \triangleright y = (1 - \alpha)(x) + \alpha(y).$$

Now if we take $x = e$, the identity of A, then $\varphi_e(y) = \alpha(y)$. Hence, the cycle structure of φ_e is equal to the cycle structure of $\alpha \in Aut(A)$. Now, since an abelian group is nilpotent, the automorphism α of A has a regular cycle by Corollary 1.4.6. Therefore the Hayashi's conjecture is true for a finite connected affine quandle *Aff*(A, α).

5. Connected quandles of size p and p^2 are affine (see [13], [14]). Therefore, the Hayashi's conjecture is true for connected quandles of size p and p^2.

6. A connected quandle of size p^3 is either affine or isomorphic to a coset quandle (G, H, α), where the group G has order p^4 (see [6]). Recall that for the coset quandle (G, H, α),

$$gH \triangleright fH = g\alpha(g^{-1}f)H.$$

Therefore, $\varphi_H(fH) = \alpha(f)H$. This implies that the cycle structure of φ_H depends on the cycle structure of $\alpha \in Aut(A)$. Now since a group of prime power order is nilpotent, therefore by Corollary 1.4.6, the Hayashi's conjecture must be true for all connected quandles of size p^3.

7. The Hayashi's conjecture is true for simple quandles of prime power sizes which are affine (see [3]). Since, the primitive quandles are simple (see [29]), the Hayashi's conjecture is true for any primitive quandle of prime power size.

The results of [7] and [17] can be used to verify the Hayashi's conjecture for those families of connected racks and quandles whose automorphisms depend on the automorphisms of underlying groups. In the next section, we consider the Hayashi's conjecture more generally in order to see that to which extent it is true.

1.4.2 Obstruction on the Cycle Structure of Connected Quandles

In this section we provide the obstructions on the profiles of connected racks and quandles. These obstructions will be helpful to see that to which extent the Hayashi's conjecture is true. We begin with the following definition.

Definition 1.4.8. Let X be an indecomposable rack. For any subset Y of X, the subrack of X generated by Y is the smallest subrack of X containing Y.

For any subrack $Y \subseteq X$ let $Y^c = X \setminus Y$. Now we prove the following result.

Lemma 1.4.9. *Let Y be a subrack of X with $Y \neq X$. Then X is is generated by Y^c.*

Proof. Since Y is a subrack of X, we conclude that $Y \triangleright Y^c = Y^c$. Let

$$Z = \{y_1 \triangleright (y_2 \triangleright ... \triangleright (y_{n-1} \triangleright y_n)) \mid n \geq 1, y_1, ..., y_n \in Y^c\}.$$

Then $y \triangleright z \in Z$ for all $y \in Y^c, z \in Z$ by definition, and $y \triangleright z \in Z$ for all $y \in Y$ by the self-distributivity of $\triangleright$ and the Y-invariance of Y^c. Hence Z is a non-empty X-invariant subset of X, and therefore equal to X since X is indecomposable. □

Now by using the fact that the complement of a subrack of an indecomposable rack is not necessarily a subrack, we have the following lemma.

Lemma 1.4.10. *Let X be an indecomposable rack such that $X = Y \cup Z$, for two subracks Y and Z of X, then $X = Y$ or $X = Z$.*

Proof. Assume that $X \neq Y$, then X is generated by $Y^c \subseteq Z$ because of Lemma 1.4.9. Since Z is a subrack of X, one concludes that $X = Z$. □

Corollary 1.4.11. *Let $p, q \in \mathbb{Z}$ with $p, q \geq 2$. Let X be an indecomposable rack and $x \in X$,*

$$Y = \{y \in X \mid \varphi_x^p(y) = y\}, Z = \{z \in X \mid \varphi_x^q(z) = z\}.$$

Assume that $X = Y \cup Z$. Then $X = Y$ or $X = Z$.

Proof. By the self-distributivity of $\triangleright$, the sets Y and Z are subracks of X. Then the claim follows from Lemma 1.4.10. □

The Corollary 1.4.11 is useful to provide the following obstruction on the profile of an indecomposable rack.

Proposition 1.4.12. *There is no finite indecomposable rack X of profile $1^{m_0} l_1^{m_1} l_2^{m_2} ... l_k^{m_k}$ such that $lcm(l_1, l_2, ..., l_i)$ and $lcm(l_{i+1}, l_{i+2}, ..., l_k)$ do not divide each other.*

Proof. Let $p = lcm(l_1, l_2, ..., l_i)$, and $q = lcm(l_{i+1}, l_{i+2}, ..., l_k)$, such that $p, q \geq 2$ and p, q do not divide each other. Let $x \in X$, and

$$Y = \{y \in X \mid \varphi_x^p(y) = y\}, Z = \{z \in X \mid \varphi_x^q(z) = z\}.$$

By the self-distributivity of $\triangleright$, the sets Y and Z are subracks of X. Then $X = Y \cup Z$ by definition of p and q and, $X \neq Y$ and $X \neq Z$, a contradiction to Corollary 1.4.11. □

Remark 1.4.13. By [22](Proposition 3.2), there is no finite crossed set X with profile $1^{m_0} l_1^{m_1} l_2^{m_2} ... l_k^{m_k}$ such that $gcd(l_1 l_2 ... l_{k-1}, l_k) = 1$. Now by Corollary 1.4.11, it follows that there is no finite indecomposable rack X with profile $1^{m_0} l_1^{m_1} l_2^{m_2} ... l_k^{m_k}$ such that and $l_1 l_2 ... l_{k-1}$ and l_k do not divide each other. In particular there is no finite indecomposable rack X of profile $1^{m_0} l_1^{m_1} l_2^{m_2}$ with $l_1 \nmid l_2$. Therefore, it follows that the Hayashi's conjecture is true for all finite indecomposable racks and quandles with profiles $1^{m_0} l_1^{m_1} l_2^{m_2}$.

The next step is to look for finite indecomposable quandles with profiles $1^{m_0}l_1l_2l_3$. With this profile if $l_1, l_2 \mid l_3$ then the Hayashi's conjecture is true. If $l_i \nmid l_3$ for $i \in \{1,2\}$, or $l_i \nmid l_3$ and $l_j \mid l_3$ for distinct i, j in $\{1,2\}$, then we have further two cases to consider, namely, when $l_k \nmid lcm(l_{k+1}, l_{k+2}) \pmod 3$ and when $l_k \mid lcm(l_{k+1}, l_{k+2}) \pmod 3$ for a positive integer k. The case when $l_k \nmid lcm(l_{k+1}, l_{k+2}) \pmod 3$, is excluded by Proposition 1.4.12. The case with $l_k \mid lcm(l_{k+1}, l_{k+2}) \pmod 3$ can not be excluded by Proposition 1.4.12. In the next section we consider this case in details.

1.4.3 Obstruction on the Profile of Connected Crossed Sets

Let $m_0, m_1, m_2, m_3 \in \mathbb{N}$ and let $p_1, p_2, ..., p_r$ be pairwise distinct primes for a positive integer r. Let X be a finite indecomposable crossed set. Assume that the profile of X is $1^{m_0}l_1l_2l_3$ with,

$$l_1 = \prod_{i=1}^{r} p_i^{a_i}, l_2 = \prod_{i=1}^{r} p_i^{b_i} \text{ and } l_3 = \prod_{i=1}^{r} p_i^{c_i},$$

for non-negative integers a_i, b_i and c_i for all $1 \leq i \leq r$. Let $l_1 \nmid l_2,\ l_1, l_2 \nmid l_3$ and $l_k \mid lcm(l_{k+1}, l_{k+2}) \pmod 3$ for a positive integer k. We show that these assumptions lead to a contradiction when $l_3 \mid lcm(l_1, l_2)$. The proof will be similar when $l_1 \mid lcm(l_2, l_3)$ or $l_2 \mid lcm(l_1, l_3)$.

Since $lcm(l_1, l_2) = \prod_{i=1}^{r} p_i^{max(a_i,b_i)}$, therefore for all $1 \leq i \leq r$, $l_3 \mid lcm(l_1, l_2) \Leftrightarrow c_i \leq \max\{a_i, b_i\}$. If there exists i such that $a_i \leq b_i < c_i$, then $c_i \nleq \max\{a_i, b_i\}$ and hence $l_3 \nmid lcm(l_1, l_2)$. By using Proposition 1.4.12, with $p = lcm(l_1, l_2)$ and $q = l_3$, it follows that there exists no such X.

Now assume that for all i, $A = \{p_i | c_i = b_i > a_i\}$, $B = \{p_i | a_i = c_i > b_i\}$, $C = \{p_i | a_i = b_i > c_i\}$ and $D = \{p_i | a_i = b_i = c_i\}$, then $\{p_1, p_2, ..., p_r\} = A \cup B \cup C \cup D$. Let

$$\begin{aligned} p &:= \prod_{p_i \in C} p_i^{a_i} = \prod_{p_i \in C} p_i^{b_i}, \\ q &:= \prod_{p_i \in B} p_i^{a_i} = \prod_{p_i \in B} p_i^{c_i}, \\ r &:= \prod_{p_i \in A} p_i^{b_i} = \prod_{p_i \in A} p_i^{c_i}, \\ s &:= \prod_{p_i \in D} p_i^{a_i} = \prod_{p_i \in D} p_i^{b_i} = \prod_{p_i \in D} p_i^{c_i} \end{aligned}$$

Then $p, q, r > 1$ and p, q, r, s are pairwise coprime integers. Let

$$p' := \prod_{p_i \in C} p_i^{c_i}, q' := \prod_{p_i \in B} p_i^{b_i}, r' := \prod_{p_i \in A} p_i^{a_i},$$

are such that , $p' \mid p,\ q' \mid q$ and $r' \mid r$ then,

$$l_1 = \prod_{p_i \in C} p_i^{a_i} \prod_{p_i \in B} p_i^{a_i} \prod_{p_i \in A} p_i^{a_i} \prod_{p_i \in D} p_i^{a_i} = pqr's,$$
$$l_2 = \prod_{p_i \in C} p_i^{b_i} \prod_{p_i \in B} p_i^{b_i} \prod_{p_i \in A} p_i^{b_i} \prod_{p_i \in D} p_i^{b_i} = pq'rs,$$
$$l_3 = \prod_{p_i \in C} p_i^{c_i} \prod_{p_i \in B} p_i^{c_i} \prod_{p_i \in A} p_i^{c_i} \prod_{p_i \in D} p_i^{c_i} = p'qrs$$

Note that if $C = \emptyset$, then $p = 1 = p'$ and $l_1 = qr's, l_2 = q'rs, l_3 = qrs$. Since $r' \mid r$ and $q' \mid q$, $l_1 \mid l_3$ and $l_2 \mid l_3$. Therefore, we assume that $A \neq \emptyset$, $B \neq \emptyset$ and $C \neq \emptyset$. Observe that, if $s \neq 1$ then the number of moved points is equal to {the number of moved points for $s = 1$}. s. If we fix p, q, r, then there are finitely many choices for p', q', r', s. In particular, for $(p', q', r') = (1, 1, 1)$, we have $(l_1, l_2, l_3) = (pqs, prs, qrs)$. For example, for $(p, q, r) = (4, 3, 5)$ and $(p', q', r', s) = (1, 1, 1, 1)$ we have $(l_1, l_2, l_3) = (12, 15, 20)$, and for $(p', q', r') = (1, 1, 1)$ and $(p, q, r, s) = (4, 3, 5, 2)$ we have $(l_1, l_2, l_3) = (12, 15, 20)$. We consider such cases in details.

Let $x \in X$. For all $t \geq 1$, let $X_t = \{y \in X \mid \varphi_x^t(y) = y\}$ and let $X'_t = X_t \setminus X_1$ for all $t > 1$. Then X is the disjoint union of the non-empty sets $X_1, X'_{pqs}, X'_{prs}, X'_{qrs}$.

Lemma 1.4.14. *X_t is a non-empty subrack of X. In particular, $y \triangleright (X \setminus X_t) = X \setminus X_t$ for all $y \in X_t$.*

Proof. Since X is a crossed set, $x \in X_t$ and therefore, $X_t \neq \emptyset$. Let $y_1, y_2 \in X_t$, then $\varphi_1^t(y_1 \triangleright y_2) = \varphi_1^t(y_1) \triangleright \varphi_1^t(y_2) = y_1 \triangleright y_2$, which implies that X_t is a non-empty subrack of X. For all $y \in X_t$, $y \triangleright (X \setminus X_t) = y \triangleright X \setminus y \triangleright X_t = \varphi_y(X) \setminus \varphi_y(X_t) = X \setminus X_t$, since X is a rack and X_t is a subrack, and hence φ_y is a bijection on X and also on X_t for all $y \in X_t$. □

Lemma 1.4.15. *For all $y \in X'_{pqs}$, there exists $z \in X'_{prs}$, such that $y \triangleright z \neq z$.*

Proof. Suppose $y \triangleright z = z$ for all $z \in X'_{prs}$, then $y' \triangleright z = z$ and $z \triangleright y' = y'$ for all $z \in X'_{prs}$ and $y' \in X'_{pqs}$, since X is a crossed set and φ_x acts transitively on X'_{pqs}. Now, since $Y = X_{pqs}$ and X_{prs} are subracks of X, $Y = X_{pqs} \cup X_{prs}$ is a subrack of X with $Y \neq X$ and $Y \cup X_{qrs} = X$, which is a contradiction to Lemma 1.4.10. □

Lemma 1.4.16. *Let $y \in X'_{pqs}$ and $z \in X \setminus X_{pqs}$ with $y \triangleright z \neq z$. Let t be the smallest positive integer with $\varphi_y^t(z) = z$, then $t = prs$ or $t = qrs$.*

Proof. Since X is indecomposable, φ_y is the product of a $pqs-$, a $prs-$, and a $qrs-$ cycle. Let $\varphi_y = \sigma_1\sigma_2\sigma_3$, such that $supp(\sigma_i) \cap supp(\sigma_j) = \emptyset$ and $|supp(\sigma_i)| \in \{pqs, prs, qrs\}$ for all $1 \leq i, j \leq 3$ with $i \neq j$. Since $y \in X'_{pqs}$, therefore by Lemma 1.4.15, there exist $z \in X'_{prs} \cup X'_{qrs}$ such that $y \triangleright z \neq z$, which implies that $z \in supp(\sigma_i)$. Now by Lemma 1.4.14, $X \setminus X_{pqs} = X'_{prs} \cup X'_{qrs}$ is invariant under X_{pqs}, therefore $y \triangleright z \in supp(\sigma_i)$, and hence $supp(\sigma_i) \subseteq (X'_{prs} \cup X'_{qrs}) = X \setminus X_{pqs}$.

Since $y \in X_{pqs}$, and X_{pqs} is a subrack of X therefore we have $\varphi_x^{pqs}\varphi_y = \varphi_y\varphi_x^{pqs}$. Hence, $z \in supp(\sigma_i)$ with $\varphi_y^t(z) = z$ imply that $y \triangleright u \neq u$ and $\varphi_y^t(u) = u$ for $u = \varphi_x^{pqs}(z)$. Since all φ_x^{pqs}-orbits of X'_{prs} and of X'_{pqs} have $rs = gcd(prs, qrs)$ elements, the number of elements of $X \setminus X_{pqs}$ moved by φ_y is a multiple of rs. Therefore the elements of $X \setminus X_{pqs}$ moved by φ_y are contained in the prs- and the qrs-cycles of φ_y. □

Lemma 1.4.17. *Let $y \in X'_{pqs}$. Then there exist $z \in X'_{prs}$, $f \in X'_{qrs}$ such that $z \triangleright f = y$ or $f \triangleright z = y$.*

Proof. Let $Y = X_{prs} \cup X_{qrs}$. If Y is a subrack of X, then $X = Y \cup X_{pqs}$, which is a contradiction to Lemma 1.4.10. Hence there exist $z', f' \in Y$ such that $y' := z' \triangleright f' \in X \setminus Y = X'_{pqs}$. If z', f' are both in X_{prs} or X_{qrs}, (in particular if $z' \in X_1$ or $f' \in X_1$) then $z' \triangleright f' \in Y$ since X_{prs} and X_{qrs} are subracks of X. Thus $(z', f') \in X'_{prs} \times X'_{qrs} \cup X'_{qrs} \times X'_{prs}$. Then the claim of the lemma follows from the fact that φ_x acts transitively on X'_{pqs} and permutes both X'_{prs} and X'_{qrs}. □

Lemma 1.4.18. *Assume that X'_{prs} and X'_{qrs} are subracks of X. Then X'_{pqs} is not a subrack of X.*

Proof. Assume that X'_{pqs} is a subrack of X. Let $Y = X'_{pqs} \cup X'_{prs} \cup X'_{qrs}$. Lemma 1.4.14 implies that X'_{pqs} permutes $X \setminus X_{pqs} = X'_{prs} \cup X'_{qrs}$ and hence it permutes Y, since X'_{pqs} also permutes itself being a subrack of X. Similarly, being subracks of X and by Lemma 1.4.9, X'_{prs} and X'_{qrs} permute, respectively, X'_{prs}, $X \setminus X_{prs} = X'_{pqs} \cup X'_{qrs}$ and X'_{qrs}, $X \setminus X_{qrs} = X'_{pqs} \cup X'_{prs}$. Hence Y is a subrack of X. This is a contradiction to the fact that $X = Y \cup X_1, Y, X_1 \neq X$, and to Lemma 1.4.10. □

Lemma 1.4.19. *Let $y \in X'_{pqs}$. Then there exist $z \in X'_{prs}$ and $f \in X'_{qrs}$ with $y \triangleright z \in X'_{qrs}$, $y \triangleright f \in X'_{prs}$.*

Proof. Assume that $y \triangleright z \in X'_{prs}$, for all $z \in X'_{prs}$. Then $y' \triangleright z \in X'_{prs}$ for all $z \in X'_{prs}$ and $y \triangleright X'_{qrs} = X'_{qrs}$. By Lemma 1.4.15, $y \triangleright z \neq z$ and $y \triangleright f \neq f$, therefore the restrictions $\varphi_y|_{X'_{prs}}$ and $\varphi_y|_{X'_{qrs}}$ are not the identity. Therefore φ_y has at least two cycles consisting of elements of $X'_{prs} \cup X'_{qrs}$. By Lemma 1.4.16 these are the $prs-$ and $qrs-$ cycles of φ_y. Therefore the $prs-$ and $qrs-$ cycles of φ_y consist of the elements of X'_{prs} and the elements of X'_{qrs}, respectively.

If $z \triangleright y' \in X'_{pqs}$ for all $y' \in X'_{pqs}$, $z \in X'_{prs}$, then $Y = X_{pqs} \cup X_{prs}$ is a subrack of X, a contradiction to $X = Y \cup X_{qrs}$ and Lemma 1.4.10. Thus there exists $z \in X'_{prs}$ with $z \triangleright y \notin X'_{pqs}$. Then $z \triangleright y \notin X'_{qrs}$. Now z is in one of $prs-$ cycles of φ_y. Thus the $prs-$cycles of $\varphi_{z \triangleright y}$ consist of the elements of $z \triangleright X'_{prs} \subseteq X_{prs}$ and one of these elements is $z \in X'_{prs}$. Since $z \triangleright y \in X'_{qrs}$, the elements of the $prs-$cycles of $\varphi_{z \triangleright y}$ also belong to $X \setminus X_{qrs}$, and hence to X'_{prs}. Thus z permutes X'_{prs} and hence X'_{prs} is a subrack of X. By the same reason, X'_{qrs} is a subrack of X.

Consider now z is in one of $qrs-$ cycle of $\varphi_{z\triangleright y}$, which consist of the elements of $z \triangleright X'_{qrs} \subseteq X'_{pqs} \cup X'_{qrs}$. Since $z \triangleright y \in X'_{qrs}$ permutes $z \triangleright X'_{qrs}$ and it maps X'_{qrs} to X_{qrs} and X'_{pqs} to $X'_{pqs} \cup X'_{prs}$, respectively, we conclude that $z \triangleright X'_{qrs} \subseteq X'_{qrs}$ or $z \triangleright X'_{qrs} \subseteq X'_{pqs}$. If $z \triangleright X'_{qrs} \subseteq X'_{qrs}$, then the first part of the proof applied to z instead of y implies that X'_{pqs} is a subrack of X. This is a contradiction to Lemma 1.4.18.

Assume now that $z \triangleright X'_{qrs} \subseteq X'_{pqs}$. Since x acts transitively on X'_{prs} and on X'_{pqs}, we conclude that

$$X'_{prs} \triangleright X'_{qrs} = X'_{pqs}.$$

By applying X'_{pqs} to this equation and using the rack condition one concludes that

$$X'_{pqs} \triangleright X'_{pqs} = X'_{pqs} \triangleright (X'_{prs} \triangleright X'_{qrs}) \subseteq (X'_{pqs} \triangleright X'_{prs}) \triangleright (X'_{pqs} \triangleright X'_{qrs}) = (X'_{prs} \triangleright X'_{qrs}) = X'_{pqs}.$$

Again we conclude that X'_{pqs} is a subrack of X which is a contradiction to Lemma 1.4.18. □

Lemma 1.4.20. *Let $y \in X'_{pqs}$. Then $y \triangleright x \in X'_{pqs}$.*

Proof. Since $y, x \in X_{pqs}$, we conclude that $y \triangleright x \in X_{pqs}$. Assume that $y \triangleright x \notin X'_{pqs}$. Then $y \triangleright x \in X_1$. The $prs-$ and $qrs-$cycles of $\varphi_{y\triangleright x}$ consist of the elements of $y \triangleright X'_{prs}$ and $y \triangleright X'_{qrs}$, respectively. Since $y \in X'_{pqs}$, the $pqs-$ and $prs-$cycles of $\varphi_{y\triangleright x}$ contain together all elements of $X'_{prs} \cup X'_{qrs}$, by Lemma 1.4.9. Moreover, by conjugating with φ_x we conclude that if an element of X'_{prs} (or X'_{qrs}, respectively) is contained in a $qrs-$cycle (in a $prs-$cycle, respectively), then all of the elements of X'_{prs} (of X'_{qrs}, respectively) do so. Since $prs \neq qrs$, this is not possible. Hence φ_y permutes both X'_{prs} and X'_{qrs}. This is a contradiction to Lemma 1.4.19. □

Lemma 1.4.21. *Let $y \in X'_{pqs}$. Let $\varphi_y = \sigma_1\sigma_2\sigma_3$, such that $supp(\sigma_i) \cap supp(\sigma_j) = \emptyset$ and $|supp(\sigma_i)| \in \{pqs, prs, qrs\}$ for all $1 \leq i, j \leq 3$ with $i \neq j$. Then $supp(\sigma_1) \subseteq X_{pqs}$ and $supp(\sigma_2), supp(\sigma_3) \subseteq X'_{prs} \cup X'_{qrs}$.*

Proof. By Lemma 1.4.17 there exist $z \in X'_{prs}$ and $f \in X'_{qrs}$ such that $z \triangleright f = y$ or $f \triangleright z = y$. By interchanging 2 and 3 if necessary, we may assume that $z \triangleright f = y$.

Note that $f \triangleright x \neq x$ and $f \triangleright z \neq z$, since X is a crossed set. Moreover, since $f, x \in X_{qrs}$ and X_{qrs} is a subrack of X, and since $z \notin X_{qrs}$, the elements z and x are in different cycles of φ_f. Let $\varphi_f = (x...)(z...)(...)$ be the product of disjoint cycles. Then

$$\varphi_y = \varphi_{z\triangleright f} = \varphi_z\varphi_f\varphi_z^{-1} = (z \triangleright x...)(...).$$

Since $z \triangleright x \in X'_{prs}$ by Lemma 1.4.20 and $z \in X'_{prs}$, φ_y has two cycles containing elements of $X'_{prs} \cup X'_{qrs}$. These are the $prs-$ and $qrs-$cycles by Lemma 1.4.16. This implies the claim. $\square$

Proposition 1.4.22. *There is no finite indecomposable crossed set X with profile $1^{m_0}l_1l_2l_3$, where l_1, l_2, l_3 are defined as above, and $l_k \mid lcm(l_{k+1}, l_{k+2}) \pmod 3$ for positive integer k.*

Proof. Assume to the contrary that X is an indecomposable crossed set with the profile $1^{m_0}l_1l_2l_3$, where l_1, l_2, l_3 are such that $l_k \mid lcm(l_{k+1}, l_{k+2}) \pmod 3$ for positive integer k. Let X_t and X'_t be defined as above for all $t \geq 1$.

Let $y \in X'_{pqs}$ and $z \in X'_{prs}$, then $z \triangleright x \neq x$ and $\varphi_z^{prs}(x) = x$ by Lemma 1.4.20. Therefore $\varphi_{y \triangleright z}^{prs}(y \triangleright x) = y \triangleright x$. Moreover, $y \triangleright x \in X'_{pqs}$ by Lemma 1.4.20. Lemma 1.4.14 implies that $y \triangleright z \in X'_{prs} \cup X'_{qrs}$. If $y \triangleright z \in X'_{prs}$, then the entries of $prs-$cycle of $\varphi_{y \triangleright z}$ belong to X_{prs} by Lemma 1.4.21, in contradiction to $y \triangleright x \in X'_{pqs}$. Thus $y \triangleright z \in X'_{qrs}$, which implies that $y \triangleright X'_{prs} \subseteq X'_{qrs}$ and $y \triangleright X'_{qrs} \subseteq X'_{prs}$ by symmetry. This is impossible since $\mid X'_{prs} \mid = prs \neq qrs = \mid X'_{qrs} \mid$. $\square$

Remarks 1.4.23. Note that the Proposition 1.4.22 can not be further generalized to all indecomposable quandles. By Proposition 1.4.22, there is no indecomposable crossed set with profile $1^{m_0}l_1l_2l_3$, where $(\ell_1, \ell_2, \ell_3) = (pq, pr, qr)$ for pairwise distinct primes p, q, r. In particular, there is no indecomposable crossed set with profile $1^{m_0}l_1l_2l_3$, with $(\ell_1, \ell_2, \ell_3) = (6, 10, 15)$. By Remark 1.4.13 and Proposition 1.4.22, it follows that the Hayashi's conjecture is true for any connected crossed set X with $\varphi_x \in Aut(X)$ such that $supp(\varphi_x) \leq 31$.

Chapter 2

Racks and Hurwitz Orbits

In this chapter we recall the definitions and results, from [15], [21], [22], about the main objects of study of this thesis, namely, the Hurwitz orbits for the braid group action on powers of racks.

2.1 Hurwitz Action on Racks

Let n be a positive integer. The braid group on n strands is the following:

$$\mathbb{B}_n = \langle \sigma_1, \sigma_2, ..., \sigma_{n-1} \rangle / (\sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2, \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ if } |i-j| = 1).$$

The braid group B_1 is trivial and the braid group B_2 is infinite cyclic group isomorphic to the fundamental group of unknot. The braid group

$$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$$

is infinite non-abelian group isomorphic to the fundamental group of the trefoil knot. The center of B_3 is:

$$Z(\mathbb{B}_3) = \langle \Delta \rangle, \text{ where } \Delta = (\sigma_1\sigma_2)^3 = (\sigma_1\sigma_2\sigma_1)^2.$$

According to E. Brieskorn [8], A. Hurwitz in [23] studied implicitly an action of $\mathbb{B}_n$ on the n product of the conjugacy class X of a group, which is therefore called the *Hurwitz action* and is the following:

$$\sigma_i(x_1, x_2, ..., x_i, x_{i+1}, ..., x_n) = (x_1, x_2, ..., x_i x_{i+1} x_i^{-1}, x_i, ..., x_n),$$

for all $x_1, x_2, ..., x_n \in X$ and $i \in \{1, 2, ..., n-1\}$. Since the algebraic structure of racks and quandles is similar to conjugation in groups, the Hurwitz action can also be studied for racks and quandles. We recall the study of Hurwitz action on racks from [21].

Let X be a rack. The braid group $\mathbb{B}_n$ acts on X^n via the Hurwitz action:

$$\sigma_i(x_1, x_2, ..., x_i, x_{i+1}, ..., x_n) = (x_1, x_2, ..., x_i \triangleright x_{i+1}, x_i, ..., x_n),$$

for all $x_1, x_2, ..., x_n \in X$ and $i \in \{1, 2, ..., n-1\}$. For example, the Hurwitz action of the braid group $\mathbb{B}_3$ on X^3 is given by:

$$\sigma_1(x, y, z) = (x \triangleright y, x, z), \sigma_2(x, y, z) = (x, y \triangleright z, y)$$

for all $x, y, z \in X$. By the self-distributivity of $\triangleright$, the defining relation of $\mathbb{B}_3$ respects this action as follows:

$$\sigma_1\sigma_2\sigma_1(x, y, z) = \sigma_1\sigma_2(x \triangleright y, x, z) = \sigma_1(x \triangleright y, x \triangleright z, x) = ((x \triangleright y) \triangleright (x \triangleright z), x \triangleright y, x)$$
$$= (x \triangleright (y \triangleright z), x \triangleright y, x) = \sigma_2(x \triangleright (y \triangleright z), x, y) = \sigma_2\sigma_1(x, y \triangleright z, y) = \sigma_2\sigma_1\sigma_2(x, y, z)$$

2.1.1 Hurwitz Orbits

Let X be a rack. The orbit $\mathcal{O} = \mathcal{O}(x_1, ..., x_n) := \{\sigma(x_1, ..., x_n) : \sigma \in \mathbb{B}_n\}$ of the Hurwitz action on X^n is called the *Hurwitz orbit*. Note that, the rack X acts on itself via the map $\triangleright$, which can be extended to a canonical action of the enveloping group G_X of X. More generally, G_X acts on X^n diagonally:

$$g \triangleright (x_1, x_2, ..., x_n) = (g \triangleright x_1, g \triangleright x_2, ..., g \triangleright x_n),$$

for all $g \in G_X$. The diagonal action of G_X and the Hurwitz action of $\mathbb{B}_n$ on X^n commute. Two Hurwitz orbits $\mathcal{O}_1, \mathcal{O}_2 \subseteq X^n$ are called *conjugate* if there exists $g \in G_X$ such that the map $X^n \to X^n, \overline{x} \to g \triangleright \overline{x}$, induces a bijection $\mathcal{O}_1 \to \mathcal{O}_2$.

Two Hurwitz orbits $\mathcal{O}_1, \mathcal{O}_2 \subseteq X^n$ are called *isomorphic* if there exists a bijection $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ such that $\varphi(\sigma(\overline{x})) = \sigma(\varphi(\overline{x}))$ for all $\sigma \in \mathbb{B}_n, \overline{x} \in \mathcal{O}_1$. Clearly, conjugate Hurwitz orbits are isomorphic. By the definition of the enveloping group G_X if $(y_1, ..., y_n) \in \mathcal{O}(x_1, ..., x_n)$, for all $x_1, ..., x_n, y_1, ..., y_n \in X$, then $y_1y_2...y_n = x_1x_2...x_n$.

The Hurwitz orbits for the action of $\mathbb{B}_2$ on X^2 for a rack X are studied in [21]. In this study, the number (written by $\#$) of Hurwitz orbits of size $n \in \mathbb{N}_0$ is denoted by l_n, that is:

$$l_n = \#\{\mathcal{O}(i, j) | \#\mathcal{O}(i, j) = n\}$$

If $k_n = \#\{j \in X | \#\mathcal{O}(i, j) = n\}$ and d is the size of the rack, then $l_n = \frac{dk_n}{n}$ (see [15], Lemma 2.26.).

The Hurwitz orbits for the action of $\mathbb{B}_3$ on X^3 for a braided rack X are studied in [21]. If X is a braided rack and $\mathcal{O} \subseteq X^3$ a Hurwitz orbit, then $\#\mathcal{O} \in \{1, 3, 6, 8, 9, 12, 16, 24\}$ (see [21], Proposition 9). The Hurwitz orbits for the action of $\mathbb{B}_3$ on X^3 for arbitrary racks X are studied in [22].

Example 2.1.1. Let $X = \mathbb{D}_3 \cong (1\ 2)^{S_3} = \{(1\ 2), (1\ 3), (2\ 3)\}$. Then X^3 has three Hurwitz orbits of size 1 and three Hurwitz orbits of size 8.

2.2 From Racks and Hurwitz Orbits to Nichols Algebras

The study of racks and Hurwitz orbits is motivated by the classification problem of certain Hopf algebras which are known as Nichols algebras. We recall the definition of Nichols algebra and the idea of classification of certain Nichols algebras using Hurwitz orbits. For details we refer to [1], [2], [3], [15], [21], [22].

Definition 2.2.1. Let V be a vector space and $c \in Aut(V \otimes V)$. Then (V, c) is called a braided vector space if c is the solution of the braid equation in $Aut(V \otimes V \otimes V)$, that is:

$$(c \otimes id)(id \otimes c)(c \otimes id) = (id \otimes c)(c \otimes id)(id \otimes c)$$

Examples 2.2.2.

- Let $V = \langle x_1, x_2, ..., x_n \rangle$ and $\mathbb{C}$ is the field of complex numbers. Let $c(x_i \otimes x_j) = q_{ij}(x_j \otimes x_i)$ for $q_{ij} \in \mathbb{C}^{\times}$. Then (V, c) is a braided vector space.
- Let G be a group, $V = \mathbb{C}G$ and $c(g \otimes h) = (ghg^{-1} \otimes g)$. Then (V, c) is a braided vector space.
- Let X be a rack, $V = \mathbb{C}X$ and $c \in GL(V \otimes V)$ such that $c(x \otimes y) = (x \triangleright y \otimes x)$. Then (V, c) is a braided vector space.
- If X be a rack, $V = \mathbb{C}X$ and $c \in GL(V \otimes V)$. Let $q : X \times X \to \mathbb{C}^{\times}$ be a map (which is called a rack $2-$cocycle) such that:

  $$q(x, y \triangleright z)q(y, z) = q(x \triangleright y, x \triangleright z)q(x, z).$$

  Let $c(x \otimes y) = q(x, y)(x \triangleright y) \otimes x$. Then (V, c) is a braided vector space.

A braided vector space V gives a special type of algebra called the Nichols algebra $\mathfrak{B}(V)$. To define Nichols algebras we need a map $\mu : \mathbb{S}_n \to \mathbb{B}_n$, called the *Matsumoto section*, defined by $\tau_i \to \sigma_i$, where τ_i are the generators of the symmetric group $\mathbb{S}_n$. The Matsumoto section μ is a set-theoretical section that satisfies the following property: $\mu(xy) = \mu(x)\mu(y)$ if $\ell(xy) = \ell(x) + \ell(y)$, where ℓ is the usual length function.

Let (V, c) is a braided vector space and:

$$c_i := c_{i,i+1} = id_{V^{\otimes(i-1)}} \otimes c \otimes id_{V^{\otimes(n-i-1)}} \in Aut(V^{\otimes n}).$$

Then $c_1, ..., c_{n-1}$ satisfy the relations of the Braid group and hence $\rho : \mathbb{B}_n \to Aut(V^{\otimes n})$, defined by $\rho(\sigma_i) = c_i$, is a representation. Now the definition of Nichols algebra is the following.

Definition 2.2.3. Let V be a braided vector space over the field $\mathbb{K}$. Then the Nichols algebra of V is defined as:

$$\mathfrak{B}(V) = \mathbb{K} \oplus V \oplus \bigoplus_{n\geq 2} T^n(V)/(ker\mathfrak{S}_\mathfrak{n}),$$

where $\mathfrak{S}_\mathfrak{n} : V^{\otimes n} \to V^{\otimes n}$ is the map, called *quantum symmetrizer*, defined by:

$$\mathfrak{S}_\mathfrak{n} := \sum_{\sigma\in\mathbb{S}_n} \rho(\mu(\sigma))$$

where $\mathbb{S}_n$ is the symmetric group, ρ is the representation of $\mathbb{B}_n$ induced by c, and μ is the Matsumoto section.

A Nichols algebra $\mathfrak{B}(V)$ is said to be of *diagonal type* if there exists a basis $\{v_1, v_2, ..., v_n\}$ of V such that

$$c(v_i \otimes v_j) = q_{ij} v_j \otimes v_i \text{ for } q_{ij} \in \mathbb{K}^\times,$$

where $\mathbb{K}$ is the field. A fundamental problem in the theory of Nichols algebras is the classification of finite-dimensional Nichols algebras. I. Heckenberger classified finite-dimensional Nichols algebras of diagonal type in [19] and [20] by using Lie theoretic structures. The classification of finite-dimensional Nichols algebras of non-diagonal type is an open problem. So far only few finite-dimensional Nichols algebras of irreducible Yetter-Drinfeld modules over non-abelian groups are known. The calculations of finite-dimensional Nichols algebras of irreducible Yetter-Drinfeld modules over non-abelian groups are made by using the Hurwitz orbits for the action of braid groups $\mathbb{B}_2$ and $\mathbb{B}_3$ (see [15], [21], [22]). We recall the following facts from these papers.

- The quantum symmetrizers $\mathfrak{S}_\mathfrak{n}$ for $n \in \{2, 3\}$ are:

$$\mathfrak{S}_2 = 1 + c,$$

$$\mathfrak{S}_3 = 1 + c_{12} + c_{23} + c_{12}c_{23} + c_{23}c_{12} + c_{12}c_{23}c_{12}.$$

- Let $\mathbb{K}$ be a field. The Yetter-Drinfeld module over a group G is a $\mathbb{K}G$-modules with a left coaction $\delta : V \to \mathbb{K}G \otimes V$ satisfying the Yetter-Drinfeld condition. The category of Yetter-Drinfeld module over a group G is a braided monoidal category. Any Yetter-Drinfeld module V over G decomposes as

$$V = \bigoplus_{g\in G} V_g, \text{ where } V_g = \{v \in V | \delta(v) = g \otimes v\},$$

for all $g \in G$. Moreover, $hV_g = V_{hgh^{-1}}$ for all $g, h \in G$. The set $supp\ V = \{g \in G | V_g \neq 0\}$ is called the *support* of V. A Yetter-Drinfeld module V is a braided vector space with $c \in Aut(V \otimes V)$ defined by:

$$c(u \otimes v) = gv \otimes u \text{ for all } u \in V_g,\ g \in supp\ V, v \in V,$$

which satisfies the braid equation in $Aut(V \otimes V \otimes V)$. Note that the Yetter-Drinfeld modules can also be studied in terms of racks and $2-$cocycles. Two Yetter-Drinfeld modules V and W are said to be *bg-equivalent* if there exists a bijection $\varphi : supp\ V \to supp\ W$ and a linear isomorphism $\psi : V \to W$ such that

$$\psi(V_g) = W_{\varphi(g)}, \psi(gv) = \varphi(g)\psi(v)$$

for all $g, x \in supp\ V, v \in V$.

- For any Yetter-Drinfeld module V over a group G the Nichols algebra of V decomposes into the direct sum of the homogeneous components as follows:

$$\mathfrak{B}(V) = \bigoplus_{n\in\mathbb{N}_0} \mathfrak{B}_n(V),$$

where $\mathfrak{B}_0(V) = \mathbb{K}$, $\mathfrak{B}_1(V) = V$ and $\mathfrak{B}_n(V) = \mathbb{K}$ is a Yetter-Drinfeld submodule of $\mathfrak{B}(V)$ for all $n \in \mathbb{N}_0$. The Nichols algebra $\mathfrak{B}(V)$ of a Yetter-Drinfeld module V is called *elementary* if V is finite-dimensional, absolutely irreducible and $supp\ V$ generates the group G. Two Nichols algebras of Yetter-Drinfeld modules are called *bg-equivalent* if their degree one parts are bg-equivalent.

- The *Hilbert series* of a Nichols algebra $\mathfrak{B}(V)$ is the formal power series:

$$H_{\mathfrak{B}(V)}(t) := \sum_{n=0}^{\infty} (dim\mathfrak{B}_{\mathfrak{n}}(V))t^n.$$

The Hilbert series of the known Nichols algebras of Yetter-Drinfeld modules factorize into the product of polynomials of the form $1 + t^r + ... + t^{nr}$ with $r, n \geq 1$. Such an Hilbert series of a Nichols algebra is called *t-integral of depth two* if $r \leq 2$. Motivated by t-integrality of all known Hilbert series of Nichols algebras of group type, a classification program of finite-dimensional Nichols algebras over groups is started in [15]. In this study, the finite-dimensional Nichols algebras over groups with many quadratic relations are found, which corresponds to a factorization of the Hilbert series for $r = 1$. Subsequently, the Nichols algebras over groups which corresponds to a factorization of the Hilbert series for $r \leq 2$, are classified in [21]. In this classification, it is shown that for the Nichols algebra $\mathfrak{B}(V)$ of a Yetter-Drinfeld module V, with Hilbert series of t-integral of depth two, the following inequality is satisfied:

$$dim\ ker(1 + c_{12} + c_{12}c_{23}) \geq \tfrac{1}{3}dimV((dimV)^2 - 1).$$

If this inequality holds for V, then the Nichols algebras $\mathfrak{B}(V)$ is said to *has many cubic relations*. In [21] and [22], the authors intended to classify all Nichols algebras with many cubic relations and to prove that their Hilbert series are t-integral of depth two. If $supp\ V$ is a braided rack, then the claim is proved in [21], by using a combinatorial invariant, so called *plague*, on Hurwitz orbits of $(supp\ V)^3$. In the next step the general problem for arbitrary racks is attacked in [22], where an estimation on $dim\ ker(1 + c_{12} + c_{12}c_{23})$ is provided by a purely combinatorial method which is described as follows.

- Choose a subset Y of $(supp\ V)^3$ and take an element

$$\overline{\alpha} \in \bigoplus_{(x,y,z)\in Y} V_x \bigotimes V_y \bigotimes V_z.$$

Let $k(Y, \overline{\alpha})$ be the set of all $\alpha \in dim\ ker(1 + c_{12} + c_{12}c_{23})$ with projection $\overline{\alpha}$ to its homogeneous parts with degree in Y. If $x, y, z \in supp\ V$ and two of

$$\{(x, y, z), \sigma_2.(x, y, z), \sigma_1\sigma_2.(x, y, z)\}$$

are in Y, then for any $\alpha \in k(Y, \overline{\alpha})$ the summand with the third degree is uniquely determined. Thus $k(Y, \overline{\alpha}) \subseteq k(Y', \overline{\alpha}')$, where Y' is the union of Y and the third degree and $\overline{\alpha}'$ is the extension of $\overline{\alpha}$. This procedure of enlarging Y can be regarded as a cellular automaton on $(supp\ V)^3$. If $Y = (supp\ V)^3$, then $k(Y, \overline{\alpha}) = \overline{\alpha}$ or $k(Y, \overline{\alpha}) = \emptyset$. Hence, if a given subset $Y \subseteq (supp\ V)^3$ can be enlarged this way to $(supp\ V)^3$, then the projection of $dim\ ker(1 + c_{12} + c_{12}c_{23})$ to the sum of homogeneous parts of degree $(x, y, z) \in Y$ is injective. Thus an important question is the following: for a given Hurwitz orbit $\mathcal{O}$, provide (the size of) a smallest subset Y which can be enlarged by the above process to $\mathcal{O}$. The size of such a Y yields surprisingly often a sharp upper bound for $dim\ ker(1 + c_{12} + c_{12}c_{23})$. The quotient $|Y|/|O|$ is called as the *immunity* of $\mathcal{O}$.

2.3 Combinatorics on Hurwitz Orbits

In this section we recall the definitions of combinatorial objects, called quarantine and plague, on the Hurwitz orbit from [21]. These are graph theoretical structures closely related to those in the theory of bootstrap percolation (see [4], [5]). The plagues on Hurwitz orbits are used for estimating $dim\ ker(1 + c_{12} + c_{12}c_{23})$ in a process which is described in the last section. We also recall the definition of the immunity on a Hurwitz orbit.

2.3.1 Quarantine, Plague and Immunity on Hurwitz Orbits

Definition 2.3.1. A *quarantine* of $\mathcal{O}$ is a non-empty subset $Q \subseteq \mathcal{O}$ such that if any two of $(x, y, z), \sigma_2(x, y, z), \sigma_1\sigma_2(x, y, z)$ are in Q, then the third one is in Q.

If we indicate the triples in the Hurwitz orbit $\mathcal{O}$ by the circles (like $\bigcirc$), the action of σ_1 by a solid arrow (like $\rightarrow$), the action of σ_2 by a dashed arrow (like $\dashrightarrow$), and the triples in the quarantine Q by circles with a cross (like $\bigotimes$), then graphically, the rule defining a quarantine on $\mathcal{O}$ says that:

if two circles along a path consisting of a dashed arrow followed by a solid arrow are in Q, then the third circle is also in Q, as shown in Figure 2.3.1.

$$\left.\begin{array}{l} \bigotimes \dashrightarrow \bigotimes \longrightarrow \bigcirc \\ \bigotimes \dashrightarrow \bigcirc \longrightarrow \bigotimes \\ \bigcirc \dashrightarrow \bigotimes \longrightarrow \bigotimes \end{array}\right\} \Longrightarrow \bigotimes \dashrightarrow \bigotimes \longrightarrow \bigotimes$$

Figure 2.3.1. The rule defining a quarantine

Definition 2.3.2. A non-empty subset P of an Hurwitz orbit $\mathcal{O}$ is called *plague* if the smallest quarantine of $\mathcal{O}$ containing P is $\mathcal{O}$. If P is a plague of smallest possible size, then the *immunity* of $\mathcal{O}$ is defined as the quotient $|P|/|\mathcal{O}| \in \mathbb{Q} \cap (0, 1]$. The immunity of $\mathcal{O}$ is written as $imm(\mathcal{O})$.

In the case of braided racks, plagues can be computed manually, because there is only a small number of isomorphism types of Hurwitz orbits for braided racks (see [21], Proposition 9). We recall the following proposition from [21].

Proposition 2.3.3. *Let X be a braided rack and $\mathcal{O} \subseteq X^3$ a Hurwitz orbit. Then,*

$$imm(\mathcal{O}) = \begin{cases} 1 & \text{if } \#\mathcal{O} = 1, \\ 1/3 & \text{if } \#\mathcal{O} \in \{3, 6, 9, 12\}, \\ 3/8 & \text{if } \#\mathcal{O} = 8, \\ 5/16 & \text{if } \#\mathcal{O} = 16, \\ 7/24 & \text{if } \#\mathcal{O} = 24. \end{cases}$$

Example 2.3.4. Let $X = \mathbb{D}_3 \cong (1\ 2)^{S_3} = \{((2\ 3), (1\ 3), (1\ 2))\} = \{x_1, x_2, x_3\}$. Then X^3 has three Hurwitz orbits of size 1 and three Hurwitz orbits of size 8. Let the Hurwitz orbit of (x_1, x_2, x_3) is $\mathcal{O}(x_1, x_2, x_3) = \{A, B, C, D, E, F, G, H\}$. The orbit graph of this Hurwitz orbit $\mathcal{O}(x_1, x_2, x_3)$ is shown in Figure 2.3.4, where the elements of $\mathcal{O}(x_1, x_2, x_3)$ are indicated by black dots. Note that the set $\{A, D, H\}$ is a plague on $\mathcal{O}(x_1, x_2, x_3)$. Since $\{A, B, D, E, F, G\}$ and $\{B, C, D, E, G, H\}$ are quarantines, for any plague P of $\mathcal{O}(x_1, x_2, x_3)$ we have $P \cap \{C, H\} \neq \emptyset$ and $P \cap \{A, F\} \neq \emptyset$. Since none of $\{A, C\}, \{A, H\}, \{C, F\}, \{F, H\}$ is a plague, we get $imm(\mathcal{O}(x_1, x_2, x_3)) = 3/8$.

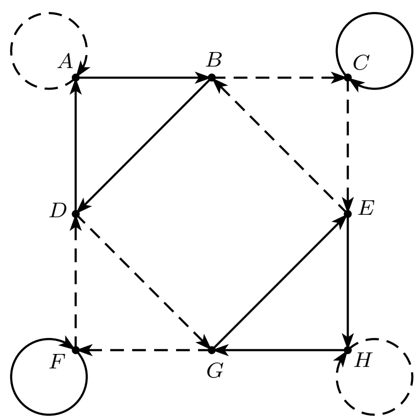

Figure 2.3.4. The Orbit graph of $\mathcal{O}(x_1, x_2, x_3)$

2.4 Hurwitz Orbit Quotients and their Coverings

The Hurwitz orbits under the action of the braid group $\mathbb{B}_3$ can be studied as coverings of the Hurwitz orbit quotients. In this section we recall the definitions and results about the Hurwitz orbit quotients and their coverings from [22]. We begin with the following two lemmas.

Lemma 2.4.1. *Let X be a rack and $\mathcal{O} \subseteq X^3$ a Hurwitz orbit. Then the map $\mathcal{O} \longrightarrow G_X, (x, y, z) \longmapsto xyz$ is constant.*

Lemma 2.4.2. *Let X be an injective rack, $\mathcal{O} \subseteq X^3$ a Hurwitz orbit and let $(x, y, z) \in \mathcal{O}$. Then* $|\, (x, y, z)^{\sigma_1} \cap (x, y, z)^{\sigma_2} \,| = 1$.

Let X be a rack and $\mathcal{O} \subseteq X^3$ a Hurwitz orbit. Define an equivalence relation on $\mathcal{O}$ by:

$$(x, y, z) \sim (x', y', z') \Leftrightarrow \Delta^m(x, y, z) = (x', y', z')$$

for some $m \in \mathbb{Z}$, and for all $(x, y, z), (x', y', z') \in \mathcal{O}$. Then $\sim$ is an equivalence relation. A *Hurwitz orbit quotient* is the set $\overline{\mathcal{O}}$ of equivalence classes of $\mathcal{O}$.

Let $x = \sigma_2^{-1}\sigma_1^{-1}Z(\mathbb{B}_3)$ and $y = \sigma_1\sigma_2\sigma_1 Z(\mathbb{B}_3)$. Then, by construction (see [26], Appendix A), we have

$$\langle x, y \mid x^3 = y^2 = 1\rangle \simeq \mathbf{PSL}(2, \mathbb{Z}) = \mathbb{B}_3/Z(\mathbb{B}_3)$$

Now, since the braid group $\mathbb{B}_3$ acts transitively on $\mathcal{O}$, the modular group $\mathbf{PSL}(2, \mathbb{Z})$ acts transitively on $\overline{\mathcal{O}}$, that is, $\overline{\mathcal{O}}$ is a homogeneous $\mathbf{PSL}(2, \mathbb{Z})$-space.

The finite homogeneous $\mathbb{B}_3$-spaces are studied in [22] as coverings of the homogeneous $\mathbf{PSL}(2, \mathbb{Z})$-spaces. We now recall the definitions and results about the coverings of the homogeneous $\mathbf{PSL}(2, \mathbb{Z})$-spaces.

Let $\overline{\mathcal{O}}$ be a homogeneous $\mathbf{PSL}(2, \mathbb{Z})$-space. Consider $\overline{\mathcal{O}}$ as $\mathbb{B}_3-$space on which $Z(\mathbb{B}_3)$ acts trivially. Then the covering of $\overline{\mathcal{O}}$ is defined as follows.

Definition 2.4.3. A covering of $\overline{\mathcal{O}}$ is a triple $(\pi, \mathcal{O}, \overline{\mathcal{O}})$, where $\pi : \mathcal{O} \to \overline{\mathcal{O}}$ is a surjective $\mathbb{B}_3$-equivariant map (i.e., $\pi(\sigma((x, y, z))) = \sigma\pi(x, y, z)$ for all $\sigma \in \mathbb{B}_3$, $(x, y, z) \in \mathcal{O}$) such that

$$\pi(x, y, z) = \pi(x', y', z') \text{ implies that } (x, y, z) = \Delta^m(x', y', z')$$

for some $m \in \mathbb{Z}$ and for all $(x, y, z), (x', y', z') \in \mathcal{O}$.

Note that a covering $(\pi, \mathcal{O}, \overline{\mathcal{O}})$ is finite if $\mathcal{O}$ (and hence $\overline{\mathcal{O}}$) is finite. Also, a covering $(\pi, \mathcal{O}, \overline{\mathcal{O}})$ is trivial if $\pi : \mathcal{O} \to \overline{\mathcal{O}}$ is bijective. For a covering $(\pi, \mathcal{O}, \overline{\mathcal{O}})$, the *fiber* of an element $v \in \overline{\mathcal{O}}$ is the subset $\pi^{-1}(v) \subseteq \mathcal{O}$. Following the notation of [22], we write $v[*]$ for the complete fiber $\pi^{-1}(v)$ over an element $v \in \overline{\mathcal{O}}$. Since $Z(\mathbb{B}_3) = \langle\Delta\rangle$, where $\Delta = (\sigma_1\sigma_2)^3 = (\sigma_1\sigma_2\sigma_1)^2$, the following lemma holds.

Lemma 2.4.4. *Let $(\pi, \mathcal{O}, \overline{\mathcal{O}})$ be a covering of a homogeneous $\mathbf{PSL}(2, \mathbb{Z})$-space $\overline{\mathcal{O}}$. Then $|\pi^{-1}(v)| = |\pi^{-1}(w)|$ for all points v, w of $\overline{\mathcal{O}}$.*

For a covering $(\pi, \mathcal{O}, \overline{\mathcal{O}})$, we write the size of the fibers by N, that is, $|\pi^{-1}(v)| = |\pi^{-1}(w)| = N$ for v, w of $\overline{\mathcal{O}}$. Now we recall the definitions of cycles in $\mathcal{O}$ and $\overline{\mathcal{O}}$, and the covering $(\pi, \mathcal{O}, \overline{\mathcal{O}})$ with simply intersecting cycles.

Definition 2.4.5. For $i \in \{1, 2\}$, a σ_i-*cycle* of a homogeneous $\mathbb{B}_3-$space $\mathcal{O}$ is a minimal non-empty subset $c_i \subseteq \mathcal{O}$ which is closed under the action of σ_i. A covering $(\pi, \mathcal{O}, \overline{\mathcal{O}})$ is said to be with *simply intersecting cycles* if any given σ_1-cycle c_1 and σ_2-cycle c_2 in $\mathcal{O}$ intersect at most once, i.e., $|c_1 \cap c_2| \leq 1$.

An $xy-$*cycle* in a homogeneous $\mathbf{PSL}(2, \mathbb{Z})$-space $\overline{\mathcal{O}}$ is a minimal non-empty subset $C_{xy} \subseteq \overline{\mathcal{O}}$ such that $xy.v \in C$ and $(xy)^{-1}.v \in C$ for all $v \in C$. Similarly, one can define $yx-$cycles C_{yx}. An $xy-$cycle containing a fixed element v of $\overline{\mathcal{O}}$ is written as $C_{xy}(v)$. Note that, for any covering $(\pi, \mathcal{O}, \overline{\mathcal{O}})$, the image of a σ_1-cycle in $\mathcal{O}$ is an $xy-$cycle in $\overline{\mathcal{O}}$, and the image of a σ_2-cycle in $\mathcal{O}$ is a $yx-$cycle in $\overline{\mathcal{O}}$.

Example 2.4.6. By Lemma 2.4.2, if X is an injective rack, then the Hurwitz orbit $\mathcal{O} \subseteq X^3$ has simply intersecting cycles. In particular, for $X = \mathbb{D}_3 \cong (1\ 2)^{S_3}$, any Hurwitz orbit $\mathcal{O}$ has simply intersecting cycles.

2.5 Schreier Graphs of Hurwitz Orbit Quotients

The Hurwitz orbits for the action of the braid group $\mathbb{B}_3$ on arbitrary racks are studied in [22] as coverings of their quotients which are homogeneous $\mathbf{PSL}(2, \mathbb{Z})$-spaces. These homogeneous $\mathbf{PSL}(2, \mathbb{Z})$-spaces can be presented in terms of Schreier coset graphs associated to the modular group $\mathbf{PSL}(2, \mathbb{Z})$ with respect to its generators x, y and its finite index subgroups. Recall that, finite homogeneous $\mathbf{PSL}(2, \mathbb{Z})$-spaces up to isomorphism are known to be in bijection with conjugacy classes of finite index subgroups of the modular group $\mathbf{PSL}(2, \mathbb{Z})$, which are intensively studied (see [30]). In this section we recall the study about the Schreier coset graphs of homogeneous $\mathbf{PSL}(2, \mathbb{Z})$-spaces from [22]. We first recall the definition of the Schreier graph from [11].

Definition 2.5.1. Let G be a group with a finite generating set $S = \{x_1, x_2, ..., x_d\}$. Let H be a subgroup of finite index n in G and $G/H = \{gH : g \in G\}$ be the set of left cosets of H in G. Then a *Schreier coset graph* or simply *Schreier graph* associated to G, with respect to S and H, is a graph whose vertices are the left cosets gH, and edges are of the form $(gH, x_i gH)$ for $x_i \in S$.

Example 2.5.2. The Schreier graph associated to the the modular group $\mathbf{PSL}(2, \mathbb{Z})$, with respect to a finite index subgroup H of $\mathbf{PSL}(2, \mathbb{Z})$ and the generators x and y of $\mathbf{PSL}(2, \mathbb{Z})$, is an oriented labelled graph whose vertices are the left $H-$cosets and edges are of the form (gH, xgH) and (gH, ygH). In the Schreier graphs for $\mathbf{PSL}(2, \mathbb{Z})$, an $x-$arrow points from any coset gH to the coset xgH and a $y-$edge points from any coset gH to the coset ygH. Since, $\mathbf{PSL}(2,\mathbb{Z}) = \langle x, y \mid x^3 = y^2 = 1\rangle$, the Schreier graph associated to $\mathbf{PSL}(2,\mathbb{Z})$ consists of oriented triangles of $x-$arrows (slid arrow) and double $y-$edges (dashed lines). Usually, instead of a double $y-$edge, a single edge or dashed line is displayed in the Schreier graph for $\mathbf{PSL}(2,\mathbb{Z})$. The fixed points of x are shown by solid loops or circles with an arrow on them and fixed points of y are shown by dashed loops or circle.

Note that in the interpretation as a homogeneous $\mathbf{PSL}(2,\mathbb{Z})$-space (in particular, the Hurwitz orbit quotient $\overline{\mathcal{O}}$), the vertices of the Schreier graph for $\mathbf{PSL}(2,\mathbb{Z})$ correspond to the points of the $\mathbf{PSL}(2,\mathbb{Z})$-space. The Schreier graph of a $\mathbf{PSL}(2,\mathbb{Z})$-space $\overline{\mathcal{O}}$ can also be used to display the covering $(\pi, \mathcal{O}, \overline{\mathcal{O}})$ as a graph. The *graph of the covering* $(\pi, \mathcal{O}, \overline{\mathcal{O}})$ is the *labelled Schreier graph* of the homogeneous $\mathbb{B}_3$-space $\mathcal{O}$ with respect to the generators $\sigma_2^{-1}\sigma_1^{-1}$ and $\sigma_1\sigma_2\sigma_1$ of $\mathbb{B}_3$. We recall the labelled Schreier graph of the homogeneous $\mathbb{B}_3$-space $\mathcal{O}$ from [22](Remark 4.8).

Remark 2.5.3. Let $(\pi, \mathcal{O}, \overline{\mathcal{O}})$ be a covering of a $\mathbf{PSL}(2,\mathbb{Z})$-space. Since $x = \sigma_2^{-1}\sigma_1^{-1}\mathbb{B}_3$ and $y = \sigma_1\sigma_2\sigma_1\mathbb{B}_3$, the generators $\sigma_2^{-1}\sigma_1^{-1}$ and $\sigma_1\sigma_2\sigma_1$ of $\mathbb{B}_3$ correspond to labeled $x-$ and $y-$edges, respectively, in the labeled Schreier graph. Since the covering is a homogeneous space and the sequence

$$Z(\mathbb{B}_3) \to \mathbb{B}_3 \to \mathbf{PSL}(2,\mathbb{Z})$$

is exact, the fiber $v[*]$ over any $v \in \overline{\mathcal{O}}$ consists of $\langle\Delta\rangle$-orbit. Now if we fix a point $v[0]$ in the fiber $v[*]$ then all other points of the fiber can be enumerated by $v[i] = \Delta^i v[0]$ for all $i \in \{0, 1, ..., N-1\}$, where N is the size of the fiber. Now by choosing a spanning tree of the Schreier graph of $\overline{\mathcal{O}}$ and the images of $v[0]$ along the arrows of the spanning tree, one can obtain the images of $v[i]$ for all i since Δ is central. The remaining arrows $v_i \to v_j$ (which are not on the spanning tree) in the graph of $\overline{\mathcal{O}}$ then have to obtain labels indicating the index shift in the fiber. For instance, a label s tells that $v_i[k]$ is mapped to $v_j[k+s \pmod N]$ for all k. Then, up to the choice of the spanning tree, any covering of $\overline{\mathcal{O}}$ is uniquely determined by the labels of the $x-$ and $y-$edges.

Observe that, since $\Delta = (\sigma_1\sigma_2)^3 = (\sigma_1\sigma_2\sigma_1)^2$, the sum of the labels in any $x-$triangle is -1 and the sum of the two labels of a $y-$edge is 1. The $y-$edges we interpret as double arrows and put the label of the arrow close to its destination. For any $xy-$cycle (or yx-cycle) C in $\overline{\mathcal{O}}$, the label of C is the sum of the labels of $x-$ and $y-$edges of the cycle.

Now we recall the following lemmas from [22], which are easy consequences of Remark 2.5.3 and the Definition 2.4.5 of a covering with simply intersecting cycles.

Lemma 2.5.4. *Let $\overline{\mathcal{O}}$ be a $PSL(2,\mathbb{Z})$-space and let $(\pi, \mathcal{O}, \overline{\mathcal{O}})$ be a covering of $\overline{\mathcal{O}}$ with simply intersecting cycles. Let v be a vertex of the graph of $\overline{\mathcal{O}}$.*

(a) *If there exists an x-loop on v with label a then* $3a \equiv -1 \pmod{N}$.

(b) *If there exists a y-loop on v with label a then* $2a \equiv 1 \pmod{N}$.

Lemma 2.5.5. *Let $\overline{\mathcal{O}}$ be a $PSL(2,\mathbb{Z})$-space and let $(\pi, \mathcal{O}, \overline{\mathcal{O}})$ be a covering of $\overline{\mathcal{O}}$ with simply intersecting cycles. Let $v \in \overline{\mathcal{O}}$ and $w \in C_{yx}(v) \cap C_{xy}(v)$. If $w \neq v$, then $(\pi, \mathcal{O}, \overline{\mathcal{O}})$ is not trivial.*

Lemma 2.5.6. *Let $\overline{\mathcal{O}}$ be a $PSL(2,\mathbb{Z})$-space and let $(\pi, \mathcal{O}, \overline{\mathcal{O}})$ be a covering of $\overline{\mathcal{O}}$ with simply intersecting cycles. Let $v \in \overline{\mathcal{O}}$ and $N = |\pi^{-1}(v)|$. Let $\lambda \in Z_N$ and $\mu \in Z_N$ be the labels of the xy- and yx-cycle containing v, respectively. Then $\langle \lambda \rangle \cap \langle \mu \rangle = 0$ as subgroups of Z_N.*

Lemma 2.5.7. *Let $\overline{\mathcal{O}}$ be a $PSL(2,\mathbb{Z})$-space and let $(\pi, \mathcal{O}, \overline{\mathcal{O}})$ be a covering of $\overline{\mathcal{O}}$ with simply intersecting cycles. Let $v \in \overline{\mathcal{O}}$ and assume that $xv = v$ or $yv = v$ and that $PSL(2, Z)v \neq \{v\}$. Then the labels of the xy- and yx-cycles containing v are 0.*

Lemma 2.5.8. *Let $\overline{\mathcal{O}}$ be a $PSL(2,\mathbb{Z})$-space and let $(\pi, \mathcal{O}, \overline{\mathcal{O}})$ be a covering of $\overline{\mathcal{O}}$ with simply intersecting cycles. Let $v, w \in \overline{\mathcal{O}}$, $N = |\pi^{-1}(v)|$ and assume that $v \not\equiv w$ and that v, w are on the same xy- and yx-cycle. Let λ and μ be the labels of the xy- and yx-path from v to w, respectively. Then $\lambda \not\equiv \mu \pmod{N}$.*

Corollary 2.5.9. *Let $\overline{\mathcal{O}}$ be a $PSL(2,\mathbb{Z})$-space and let $(\pi, \mathcal{O}, \overline{\mathcal{O}})$ be a covering of $\overline{\mathcal{O}}$ with simply intersecting cycles. Let $v, w \in \overline{\mathcal{O}}$ and assume that $v \neq w, xv = v, xw = w$ (or $yv = v, yw = w$) and that v, w are on the same xy- and yx-cycles. Then $\overline{\mathcal{O}}$ has no coverings with simply intersecting cycles.*

Proof. Assume to the contrary that $(\pi, \mathcal{O}, \overline{\mathcal{O}})$ is a covering of $\overline{\mathcal{O}}$ with simply intersecting cycles. Let $N = |\pi^{-1}(v)|$ and let a and b be the labels of the x-loops at v and w, respectively. By Lemma 2.5.4(b) we have $3a \equiv -1 \pmod{N}$ and $3b \equiv -1 \pmod{N}$ and hence $a \equiv b \pmod{N}$. Since $xv = v$ and $xw = w$, v and w are on the same xy- and yx-cycles, and the xy and yx-paths from v to w have the same labels. This is a contradiction to Lemma 2.5.8. ☐

2.6 Hurwitz Orbits and Cellular Automata

The graph theoretical structures of quarantine and plague defined on Hurwitz orbits are closely related to bootstrap percolation (see [4], [5]). These structures can be considered as cellular automata. The principle of the method to calculate these structure is well-known from the theory of cellular automata on groups (see [9]). The technique to calculate the plague and immunity on Hurwitz orbits in the language of cellular automata is formulated in [22]. We recall this formulation with some examples in this section.

Let G be a group acting transitively on a set Ω and let A be a set called an *alphabet*. Let A^Ω be the set of all functions from Ω to A. Then the cellular automaton over (G, Ω) is defined as follows.

Definition 2.6.1. Let S be a set, $(g_s)_{s\in S}$ be a family of elements in G, and $\mu : A^S \to A$ be a map. Then the map $\tau : A^\Omega \to A^\Omega$ such that

$$\tau(f)(w) = \mu((f(g_s.w))_{s\in S})$$

for all $f \in A^\Omega, w \in \Omega$, is called a cellular automaton over (G, Ω) with alphabet A. The infinite sequence $(\tau^n(f))$ for $n \geq 0$ is called the evolution of f.

A good interpretation of a cellular automaton over (G, Ω) is the following. For any $w \in \Omega$, consider the family of points $(g_s)_{s\in S}$ as the neighborhood of w. Then for any function $f \in A^\Omega$, the value of $\tau(f)$ at w is obtained from the values of f in the neighborhood of w according to the local defining rule determined by μ.

Note that the cellular automata to be considered here are with the alphabet $A = \mathbb{Z}_2$. For any function $f \in \mathbb{Z}_2^\Omega$ let

$$supp\ f = \{w \in \Omega | f(w) = 1\},$$

and the characteristic function of a set $I \subseteq \Omega$ is

$$\chi_I \in \mathbb{Z}_2^\Omega,\ x \to \begin{cases} 1 & \text{if } x \in I, \\ 0 & \text{if otherwise.} \end{cases}$$

Definition 2.6.2. Let τ be a cellular automaton over (G, Ω) with alphabet $\mathbb{Z}_2$. Then τ is said to be monotonic if

(1) $supp\ f \subseteq supp\ \tau(f)$ for all $f \in \mathbb{Z}_2^\Omega$, and

(2) $supp\ \tau(f) \subseteq supp\ \tau(g)$ for all $f, g \in \mathbb{Z}_2^\Omega$ with $supp\ f \subseteq supp\ g$.

Definition 2.6.3. Let τ be a monotonic cellular automaton over (G, Ω) with alphabet $\mathbb{Z}_2$. For any subsets $I, J \subseteq \Omega$ with $I \subseteq J$, the subset I is said to *spread* to J, if $J \subseteq (\tau^n(\chi_I))$ for some $n \in \mathbb{N}$. A subset $I \subseteq \Omega$ is a *quarantine* if $\tau(\chi_I) = \chi_I$. A subset $I \subseteq \Omega$ is a *plague* if the smallest quarantine containing I is Ω. The cardinality of a plague I is also called its size.

Note that if a subset I spreads to another subset J of Ω, then any subset $I' \subseteq \Omega$ with $I \subseteq I'$ spreads to J. Assume that Ω has only finitely many points. Then a subset I of Ω is a plague if and only if it spreads to Ω. In this case, any subset of Ω containing I is a plague.

Now we recall the plague on the Hurwitz orbits as a special case of a plague of the monotonic cellular automaton in the following example, which is the main example of interest.

Example 2.6.4. Let $G = \mathbb{B}_3$ and $\Omega = \mathcal{O} \subseteq X^3$, where $\mathcal{O}$ is an Hurwitz orbit under the action of $\mathbb{B}_3$ on X^3 for a rack X. Take $A = \mathbb{Z}_2$, $S = \{1, 2, ..., 7\}$ and

$$(g_s)_{s \in S} = (1, \sigma_2, \sigma_1\sigma_2, \sigma_2^{-1}\sigma_1^{-1}, \sigma_1^{-1}, \sigma_2^{-1}, \sigma_1) \in \mathbb{B}_3^7$$

Consider the configuration given in Figure 2.6.1, where the solid arrow indicates the action of σ_1, and the dashed arrow indicates the action of σ_2. Then $x_s = g_s.x_1$ for all $s \in S$. Define $\mu : A^7 \to A$ by

$$\mu(f_1, f_2, ..., f_7) = f_1 \vee f_2f_3 \vee f_4f_5 \vee f_6f_7 = 1 - (1 - f_1)(1 - f_2f_3)(1 - f_4f_5)(1 - f_6f_7),$$

where $f_1, f_2, ..., f_7 \in A$, and $\vee$ denotes logical or. Then the map τ defined by μ and $(g_s)_{s \in S}$ is a monotonic cellular automaton over $(\mathbb{B}_3, \mathcal{O})$. A plague of this automaton is literally the same as a plague defined in 2.3.2.

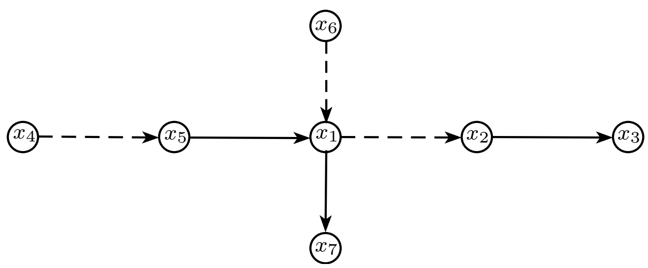

Figure 2.6.1. Neighbors of x_1

Now we recall an example of the cellular automata over $(Z, \mathbb{Z}_m)$, where the group $G = \mathbb{Z}$ acts transitively on $\Omega = \mathbb{Z}_m$ for $m \in \mathbb{N}_{\geq 2}$.

Example 2.6.5. Let $f \in \mathbb{Z}_2^{\mathbb{Z}_m}$, $r \in \mathbb{N}$, and $a_1, ..., a_r \in \mathbb{Z}_m \setminus \{0\}$. Let $A = \mathbb{Z}_2$, $S = \{0, 1, ..., r\}$, and $(g_s)_{s \in S} = (0, -a_1, -a_2, ..., -a_r) \in G^S$. Define $\mu : A^S \to A$ by

$$\mu(f_0, f_1, ..., f_7) = \begin{cases} 1 & \text{if } f_0 = 1 \text{ or } f_1 = f_2 = ... = f_r = 1, \\ 0 & \text{if otherwise.} \end{cases}$$

The map $\tau : \mathbb{Z}_2^{\mathbb{Z}_m} \to \mathbb{Z}_2^{\mathbb{Z}_m}$ defined by μ and $(g_s)_{s \in S}$ is then a monotonic cellular automaton. By definition,

$$supp\ \tau(f) \subseteq supp\ f \cup \{w \in \Omega | f(x - a_1) = ... = f(x - a_1) = 1\}$$

for all $f \in \mathbb{Z}_2^{\mathbb{Z}_m}$. The plagues for special cases of cellular automata over $(Z, \mathbb{Z}_m)$, which will be useful for studying the evolution process of plagues on the Hurwitz orbits, are the following.

Case 1. Let $r = 1$ and $a_1 = \lambda$. The cellular automaton is determined by the rule

$$supp\ \tau(f) \subseteq supp\ f \cup \{x \in \mathbb{Z}_m | f(x - \lambda) = 1\}.$$

Let $\Gamma = \langle \lambda \rangle$ and let I be a set of representatives for Ω/Γ. Then I is a plague.

Case 2. Let $\lambda \in \mathbb{Z}_m$, $r = 3$, $a_1 = 1, a_2 = \lambda + 1$, and $a_3 = -\lambda$. Let $\Gamma = \langle \lambda \rangle$ and let I be the union of a set of representatives for Ω/Γ with Γ. For example, $I = \langle \lambda \rangle \cup \{1, 2, ..., \lambda - 1\}$. Now if $supp\ f$ contains a coset $a + \Gamma$, where $a + 1 \in I$, then $supp\ f$ spreads to $a + 1 + \Gamma$. Thus I is a plague.

Case 3. Let $\lambda \in \Omega \setminus \{0, 1\}$, $r = 2, a_1 = \lambda, a_2 = \lambda - 1$. Let $I = \{0, 1, ..., (m-1)/2\}$ if m is odd, and $I = \{0, 1, ..., (m/2) - 1\}$ if m is even. Then I is a plague of size $\leq (m+1)/2$. It is in general not minimal, for example for $m \geq 3$, $\lambda = 2$, the set $\{0, 1\}$ is a plague.

Now we recall a formulation of the cellular automaton over $(\mathbb{B}_3, \mathcal{O})$ in terms of the generators $\sigma_2^{-1}\sigma_1^{-1}$ and $\sigma_1\sigma_2\sigma_1$ of $\mathbb{B}_3$ by using the Schreier graph of $PSL(2, \mathbb{Z})$-space $\overline{\mathcal{O}}$. Note that $x = \sigma_2^{-1}\sigma_1^{-1} \langle \Delta \rangle$ and $y = \sigma_1\sigma_2\sigma_1 \langle \Delta \rangle$ are the generators of $PSL(2, \mathbb{Z})$.

Let τ be a cellular automaton over $(\mathbb{B}_3, \mathcal{O})$. Let f be a $\mathbb{Z}_2-$valued function on $\mathcal{O}$ and let $P = supp\ f$. Let v be a point in the Hurwitz orbit quotient $\overline{\mathcal{O}}$ and $v[*]$ be a fiber over v of size N. Let I be a subset of $\mathbb{Z}_N$ and let $v[I] = \{v[i] | i \in I\}$ be the corresponding subset of $v[*]$. Now consider the following three neighboring subsets of $v[I]$

$$(\sigma_2^{-1}\sigma_1^{-1})^{-1}.v[I], \quad \sigma_1\sigma_2\sigma_1.v[I], \text{and} \quad (\sigma_2^{-1}\sigma_1^{-1}).v[I].$$

These three subsets are shown in Figure 2.6.2, and are denoted by $x_1[I - c]$, $x_2[I + a]$ and $x_3[I + b + 1]$, where $I + a = \{i + a | i \in I\}$. In this setting $v[I]$ is called a *pivot*.

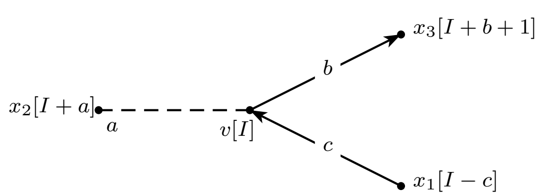

Figure 2.6.2. The cellular automaton on braid group orbits

Now by Example 2.6.4, if $P = supp\ f$ contains the subsets $x_1[I-c] = \sigma_1\sigma_2.v[I]$ and $\sigma_2.x_1[I-c] = x_2[I+a]$, then $supp\ \tau(f)$ contains

$$\sigma_1.x_2[I+a] = \sigma_1\sigma_2\sigma_1\sigma_2.v[I] = \sigma_1^{-1}\sigma_2^{-1}\Delta.v[I] = x_3[I+b+1].$$

Similarly, if any two of the neighboring subsets $x_1[I-c]$, $x_2[I+a]$, and $x_3[I+b+1]$ of $v[I]$ are contained in P, then the third is a subset of $supp\ \tau(f)$. Moreover, $supp\ \tau(f)$ is the smallest subset of $\mathcal{O}$ containing $supp\ f$ and all sets constructed this way for some point v and some subset $I \subseteq \mathbb{Z}_N$.

2.7 Weight and Immunity on Hurwitz Orbits

The immunity of an Hurwitz orbit is estimated in [22] by its weight which is defined by using the cycle structure at each point of that Hurwitz orbit. In this section we recall the definition of the weight of an Hurwitz orbit. We also recall a theorem and a conjecture from [22] about the estimation on immunity on the Hurwitz orbits. Following the same notation in [22], we write the homogeneous B_3-space by Σ and the homogeneous $PSL(2,\mathbb{Z})$-space by $\overline{\Sigma}$.

Definition 2.7.1. Let Σ be a homogeneous B_3-space and $v \in \Sigma$ such that v belongs to a σ_1-cycle of length i, and also to a σ_2-cycle of length j. Let $\omega : \Sigma \longrightarrow \mathbb{Q}$ be the map defined by $\omega(v) = \omega'_{ij}$, where ω'_{ij} is defined in [22] by the following matrix

$$(\omega'_{ij})_{i,j\geq 1} = \begin{pmatrix} 1 & 1/3 & 11/24 & 1/2 & 1/2 & \cdots \\ 1/3 & 1/3 & 1/3 & 1/3 & 1/3 & \cdots \\ 11/24 & 1/3 & 7/24 & 7/24 & 7/24 & \cdots \\ 1/2 & 1/3 & 7/24 & 1/4 & 1/4 & \cdots \\ 1/2 & 1/3 & 7/24 & 1/4 & 1/4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The *weight* of Σ is defined as

$$\omega(\Sigma) = \tfrac{1}{|\Sigma|} \sum_{v\in\Sigma} \omega(v)$$

The weight of an Hurwitz orbit provides a good upper bound for the immunity of that Hurwitz orbit which is given in [22](Theorem 6.3). We recall this theorem here.

Theorem 2.7.2. *Let Σ be a covering with simply intersecting cycles of finite homogeneous $PSL(2,\mathbb{Z})$-space $\overline{\Sigma}$. Assume that any xy-cycle of $\overline{\Sigma}$ has at most four elements. Then $imm(\Sigma) \leq \omega(\Sigma)$.*

Example 2.7.3. Let $X = \mathbb{D}_3 \cong (1\ 2)^{S_3} = \{x_1, x_2, x_3\}$. Let Σ be the Hurwitz orbit of (x_1, x_2, x_3), and let $\Sigma = \mathcal{O}(x_1, x_2, x_3) = \{A, B, C, D, E, F, G, H\}$. Let $\overline{\Sigma} = \overline{\mathcal{O}}(x_1, x_2, x_3) = \{\{D, E\}, \{B, G\}, \{A, H\}, \{C, F\}\} = \{v_1, v_2, v_3, v_4\}$. Then $(\pi, \Sigma, \overline{\Sigma})$ is a covering as shown in the Figure 2.7.1.

In this covering $N =| \pi^{-1}(v_k) |= 2$ for all $k \in \{1, 2, 3, 4\}$. The $xy-$cycles with labels are: $(v_1\ v_3\ v_2)$ with $a + b$ and (v_4) with $1 - b$. The $yx-$cycles with labels are: $(v_1\ v_2\ v_4)$ with $a + b$ and (v_3) with $1 - b$. By Lemma 2.5.4(a), it follows that $3a \equiv -1 \pmod N$ and $a + b = 0 \pmod N$ and hence $a = b = 1$. Therefore for $i \in \{0, 1\}$, $v_1[i], v_2[i]$ have two 3-cycles and $v_3[i], v_4[i]$ have cycles of lengths 1 and 3. Now by Definition 2.7.1,

$$\omega(\Sigma) = \frac{1}{|\Sigma|} \sum_{v \in \Sigma} \omega(v) = 1/8(2N(7/24) + 2N(11/24)) = 3/4.$$

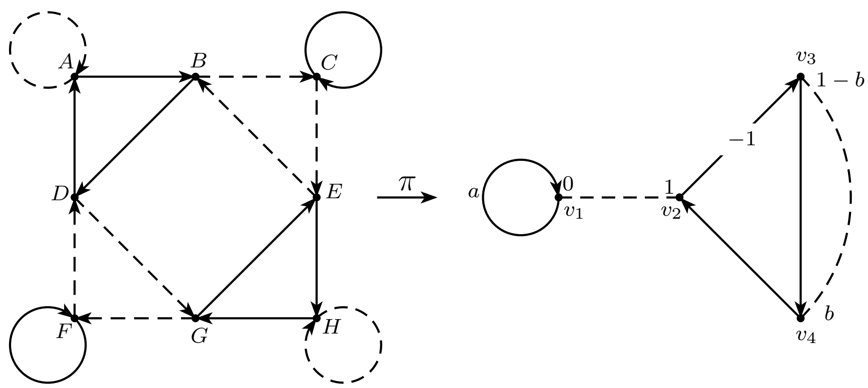

Figure 2.7.1. The Covering $(\pi, \Sigma, \overline{\Sigma})$

Note that $P = \{A, H, D\} = \{v_3[0], v_3[1], v_1[0]\}$ is a plague since with pivot $v_4[*] = \{v_4[0], v_4[1]\}$, P spreads to $v_2[*]$, with pivot $v_1[1]$, P spreads to $x.v_1[1] = v_1[1+1+1] = v_1[1]$, and with pivot $v_2[*]$, P spreads to $v_4[*]$. Hence $imm(\Sigma) = 3/8 < 3/4 = \omega(\Sigma)$.

Note that the assumption about the length of an $xy-$cycle in Theorem 2.7.2 can be replaced by a weaker assumption that $imm(\Sigma) \leq \omega(\Sigma)$ for all homogeneous B_3-spaces Σ. This weaker assumption is proposed as a following conjecture in [22].

Conjecture 2.7.4. Let Σ be a covering with simply intersecting cycles of finite homogeneous $PSL(2, \mathbb{Z})$-space $\overline{\Sigma}$. Then $imm(\Sigma) \leq \omega(\Sigma)$ holds for all homogeneous B_3-spaces Σ.

Note that by Theorem 2.7.2, the Conjecture 2.7.4 is true for the coverings Σ of finite homogeneous $PSL(2,\mathbb{Z})$-space $\overline{\Sigma}$ such that any xy-cycle of $\overline{\Sigma}$ has at most four elements. There are only 18 different finite homogeneous $PSL(2,\mathbb{Z})$-spaces $\overline{\Sigma}$ such that any xy-cycle of $\overline{\Sigma}$ has at most four elements (see [22], Proposition 4.3). In the next chapter we will show by a case-by-case analysis that the Conjecture 2.7.4 is true for the coverings Σ of infinitely many Schreier graphs of finite homogeneous $PSL(2,\mathbb{Z})$-spaces $\overline{\Sigma}$.

Chapter 3

Hurwitz Orbits and Pointed Schreier Graphs

In this chapter we introduce a new method to calculate the plague and estimate the immunity of the Hurwitz orbits. With this method we can consider infinitely many Schreier graphs of the Hurwitz orbit quotients. Our method is based on the posets of certain subgraphs of pointed Schreier graphs of the Hurwitz orbit quotients. By using this method we will estimate the immunity of the Hurwitz orbits through a case-by-case analysis of pointed Schreier graphs of the Hurwitz orbit quotients.

3.1 Pointed Schreier Graphs of Homogenous $PSL(2,\mathbb{Z})$-spaces

Let $\mathcal{G} = (V, E)$ be the Schreier graph of a finite homogenous $PSL(2,\mathbb{Z})$-space $\overline{\Sigma}$ of size n with the vertex set $V = V(\mathcal{G})$ and the edge set E. Then $\mathcal{G}$ consists of $x-$triangles and $y-$edges as described in Example 2.5.2. We denote the number of $x-$triangles by t and the number of $y-$edges of $\mathcal{G}$ by e. We call $\mathcal{G}$ with a distinguished vertex as a *pointed Schreier graph* of $\overline{\Sigma}$. We will write

$$\begin{aligned} V_x &:= \{v \in V(\mathcal{G}) | x(v) = v\}; V_x[*] := \{v[*] | v \in V_x\}, \\ V_y &:= \{v \in V(\mathcal{G}) | y(v) = v\}; V_y[*] := \{v[*] | v \in V_y\}, \\ V_{xy} &:= \{v \in V(\mathcal{G}) | xy(v) = v\}; V_{xy}[*] := \{v[*] | v \in V_{xy}\}. \end{aligned}$$

It is easy to see that V_x, V_y, and V_{xy} are mutually disjoint if $|V(\mathcal{G})| \geq 2$. In some cases we will denote the pointed Schreier graph $\mathcal{G}$ of a finite homogenous $PSL(2,\mathbb{Z})$-space $\overline{\Sigma}$ of size n explicitly by $\mathcal{G}_{n\{l_1,l_2,\ldots,l_k\}X}$ for a positive integer k, where $\{l_1, l_2, \ldots, l_k\}$ is a multiset of the lengths of xy-cycles (or yx-cycles) of $\overline{\Sigma}$, and X is a capital letter which serves as a further distinction if $\overline{\Sigma}$ has same multisets of the lengths of xy-cycles.

3.1.1 Robust Subgraphs of Pointed Schreier Graphs

Let t, j be integers with $0 \leq j \leq t$. Let $\mathcal{G}$ be a finite pointed Schreier graph of size n with t triangles and a distinguished vertex, say v_0. Let $\mathcal{H}_j$ is a connected subgraph of $\mathcal{G}$ with j triangles such that:

- v_0 belongs to $\mathcal{H}_j$,
- each $x-$edge of $\mathcal{H}_j$, which is not an $x-$loop, belongs to a triangle in $\mathcal{H}_j$,
- each $x-$loop belongs to $\mathcal{H}_j$,
- each vertex of $\mathcal{H}_j$ is either adjacent to itself through a $y-$loop or adjacent to another vertex of $\mathcal{H}_j$ through $y-$edge.

We call this subgraph $\mathcal{H}_j$ as a *robust subgraph.* We write $|V(\mathcal{H}_j)| = |\mathcal{H}_j| := n_j$. Note that $V(\mathcal{H}_t) = V(\mathcal{G})$. If $\mathcal{H}_0 = \mathcal{G}$ and $n = 1$, then v_0 is with both $x-$ and $y-$loops. If $\mathcal{H}_0 \neq \mathcal{G}$ then $\mathcal{H}_0$ is one of the following three types;

- $\mathcal{H}_0$ with a $y-$loop on v_0,
- $\mathcal{H}_0$ with an $x-$loop on v_0 and v_0 is adjacent to another vertex through $y-$edge,
- $\mathcal{H}_0$ with no $x-$loop, no $y-$loop and v_0 is adjacent to another vertex through $y-$edge,

as shown in Figure 3.1.1. A non-trivial robust subgraph $\mathcal{H}_1 \neq \mathcal{G}$, with no $x-$loop and with at most one $y-$loop, is one of the following three types as in Figure 3.1.1;

- $\mathcal{H}_1$ with a fixed point of xy and $n_1 = 4$,
- $\mathcal{H}_1$ with only one fixed point of y and $n_1 = 5$,
- $\mathcal{H}_1$ with no fixed point of xy, no fixed point of y and $n_1 = 6$.

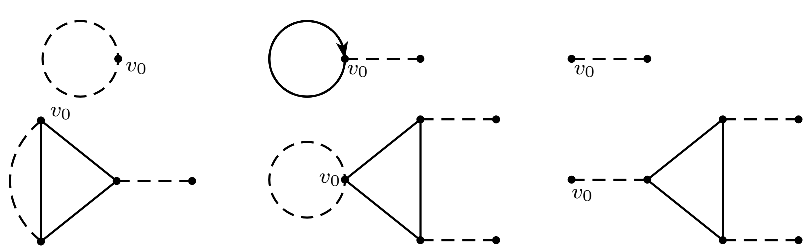

Figure 3.1.1. Small Robust Subgraphs

Let $\mathcal{H}$ be the family of all robust subgraphs $\mathcal{H}_j$ of a finite $PSL(2,\mathbb{Z})$-space. Then $\mathcal{H}$ is a finite partially ordered set by subgraph inclusion relation, $\mathcal{H}_i \preceq \mathcal{H}_j$, for all non-negative integers i, j with $0 \leq i \leq j \leq t$. We say that $\mathcal{H}_i$ is a *predecessor* of $\mathcal{H}_j$ (or equivalently $\mathcal{H}_i$ is covered by $\mathcal{H}_j$) if $\mathcal{H}_i \neq \mathcal{H}_j$ and there is no $\mathcal{H}_k \in \mathcal{H}$ such that $\mathcal{H}_i \preceq \mathcal{H}_k \preceq \mathcal{H}_j$ and $\mathcal{H}_i \neq \mathcal{H}_k \neq \mathcal{H}_j$. If $\mathcal{H}_i$ is a predecessor of $\mathcal{H}_j$ then we write $\mathcal{H}_i \prec \mathcal{H}_j$.

Let $e(\mathcal{H}_j)$ denote the number of $y-$edges of $\mathcal{H}_j$. Note that we will consider the $y-$loop at a fixed vertex of y as one $y-$edge. If $i, j \in \{0, 1, ..., t\}$ with $\mathcal{H}_i \prec \mathcal{H}_j$ and $m_j \in \{0, 1, 2\}$ such that $e(\mathcal{H}_j) = e(\mathcal{H}_i) + m_j$, then we write $\mathcal{H}_i \prec_{m_j} \mathcal{H}_j$. Since m_j varies with j, we write a sequence of predecessors in $\mathcal{H}$ as $\mathcal{H}_0 \prec_{m_1} \mathcal{H}_1 \prec_{m_2} ... \prec_{m_t} \mathcal{H}_t$, where $m_j \in \{0, 1, 2\}$ for $1 \leq j \leq t$. Note that if $\mathcal{H}_i \prec_{m_j} \mathcal{H}_j$ then $n_j = n_i + m$, where $m \in \{0, 1, 2, 3, 4\}$.

3.2 Coverings of Robust Subgraphs with Plague and Immunity

Let $(\pi, \Sigma, \overline{\Sigma})$ be a covering with simply intersecting cycles of a finite homogeneous $PSL(2,\mathbb{Z})$-space $\overline{\Sigma}$ of size n. Let $\mathcal{G}$ be the pointed Schreier graph of $\overline{\Sigma}$ and $\mathcal{H}_j$ be the robust subgraph of $\mathcal{G}$. For the map $\pi : \Sigma \to V(\mathcal{G})$, the covering of $\mathcal{H}_j$ is the set

$$\Sigma_{\mathcal{H}_j} := \pi^{-1}(V(\mathcal{H}_j)) = \bigcup_{v_i \in \mathcal{H}_j} v_i[*] \subseteq \Sigma,$$

such that $|\Sigma_{\mathcal{H}_j}| = |V(\mathcal{H}_j)|N = n_j N$.

For $1 \leq j \leq t$ let $J := \{j | \mathcal{H}_{j-1} \prec_2 \mathcal{H}_j\}$ and $v : J \to V(\mathcal{G})$ is a map such that $v(j) \in \mathcal{H}_j \backslash \mathcal{H}_{j-1}$ for all $j \in J$, where $\mathcal{H}_j \backslash \mathcal{H}_{j-1} := \{v \in \mathcal{H}_j | v \notin \mathcal{H}_{j-1}\}$. Let

$$P_J := \{v(j)[*] | j \in J\} \subset \Sigma_{\mathcal{H}_j}.$$

Note that if $v \in V_{xy}$ such that $v \in \mathcal{H}_j \setminus \mathcal{H}_{j-1}$, then $\mathcal{H}_{j-1} \prec_1 \mathcal{H}_j$ for all $1 \leq j \leq t$. Therefore $P_J \cap V_{xy}[*] = \emptyset$. However in general, $P_J \cap V_x[*] \neq \emptyset$ and $P_J \cap V_y[*] \neq \emptyset$.

If P is a plague on Σ then we write $P(\Sigma_{\mathcal{H}_j}) := P \cap \Sigma_{\mathcal{H}_j}$ for the plague on $\Sigma_{\mathcal{H}_j}$. If $P(\Sigma_{\mathcal{H}_j})$ consists of complete fibers then we write $|P(\Sigma_{\mathcal{H}_j})| := p_j N$. We write the immunity of $\Sigma_{\mathcal{H}_j}$ as $imm(\Sigma_{\mathcal{H}_j})$. If $P(\Sigma_{\mathcal{H}_j})$ consists of complete fibers then

$$imm(\Sigma_{\mathcal{H}_j}) \leq \tfrac{|P(\Sigma_{\mathcal{H}_j})|}{|\Sigma_{\mathcal{H}_j}|} = \tfrac{p_j N}{n_j N} = \tfrac{p_j}{n_j}.$$

Note that for $\mathcal{H}_i \prec_2 \mathcal{H}_j$, $n_j = n_i + m$ with $m \in \{2, 3, 4\}$ and $P(\Sigma_{\mathcal{H}_j}) = P(\Sigma_{\mathcal{H}_i}) \cup v(j)[*]$. If $\mathcal{H}_i \prec_2 \mathcal{H}_j$ and $n_j = n_i + 2$, then the jth triangle of $\mathcal{H}_j$ have two fixed points of y, and by Corollary 2.5.9, there will be no covering Σ with simply intersecting cycles. Therefore, for $\mathcal{H}_i \prec_2 \mathcal{H}_j$, $n_j = n_i + m$ with $m \in \{3, 4\}$.

For $\mathcal{H}_i \prec_1 \mathcal{H}_j$, $n_j = n_i + m$ with $m \in \{1, 2\}$. If $\mathcal{H}_i \prec_1 \mathcal{H}_j$ and $n_j = n_i + 1$, then $P(\Sigma_{\mathcal{H}_j}) = P(\Sigma_{\mathcal{H}_i})$. Therefore for $\mathcal{H}_i \prec_1 \mathcal{H}_j$ and $n_j = n_i + 1$,

$$imm(\Sigma_{\mathcal{H}_j}) \leq \frac{p_j}{n_j} = \frac{p_i}{n_i + 1},$$

If $\mathcal{H}_i \prec_1 \mathcal{H}_j$, $n_j = n_i + 2$, and $V_{xy} \cap (\mathcal{H}_j \setminus \mathcal{H}_i) = \emptyset$, then $P(\Sigma_{\mathcal{H}_j}) = P(\Sigma_{\mathcal{H}_i})$. Therefore

$$imm(\Sigma_{\mathcal{H}_j}) \leq \frac{p_j + 1}{n_j} = \frac{p_i}{n_i + 2},$$

For $\mathcal{H}_i \prec_0 \mathcal{H}_j$, $n_j = n_i$ and $P(\Sigma_{\mathcal{H}_j}) = P(\Sigma_{\mathcal{H}_i})$. Therefore for $\mathcal{H}_i \prec_0 \mathcal{H}_j$,

$$imm(\Sigma_{\mathcal{H}_j}) \leq \frac{p_j}{n_j} = \frac{p_i}{n_i}.$$

Now we prove that if the immunity of a robust subgraph $\mathcal{H}_i$, for some positive integer $i < t$, is bounded above by a quarter then the immunity of the robust subgraph $\mathcal{H}_{i+1}$ with $V_{xy} \cap (\mathcal{H}_{i+1} \setminus \mathcal{H}_i) = \emptyset$ is also bounded above by a quarter.

Lemma 3.2.1. *Let Σ be a covering with simply intersecting cycles of a finite homogeneous $PSL(2, \mathbb{Z})$-space $\overline{\Sigma}$. Let $\mathcal{G}$ be the pointed Schreier graph of $\overline{\Sigma}$ with t triangles. Assume that there exists a robust subgraph $\mathcal{H}_i$ for $1 < i < t$, such that $imm(\Sigma_{\mathcal{H}_i}) \leq 1/4$ and $V_{xy} \cap (\mathcal{H}_{i+1} \setminus \mathcal{H}_i) = \emptyset$. Then $imm(\Sigma_{\mathcal{H}_{i+1}}) \leq 1/4$.*

Proof. Suppose that $imm(\Sigma_{\mathcal{H}_i}) \leq \frac{p_i}{n_i} \leq 1/4$ and $V_{xy} \cap (\mathcal{H}_{i+1} \setminus \mathcal{H}_i) = \emptyset$. Therefore $p_i \leq \frac{n_i}{4}$. We have $\mathcal{H}_i \prec_{m_{i+1}} \mathcal{H}_{i+1}$ with $m_{i+1} \in \{0, 1, 2\}$ and $n_{i+1} = n_i + m$ with $m \in \{0, 1, 2, 3, 4\}$. More precisely, if $m_{i+1} = 0, m = 0$; if $m_{i+1} = 1, m \in \{1, 2\}$; and if $m_{i+1} = 2, m \in \{3, 4\}$. Now we discuss these cases in details.

For $m_{i+1} = 0, m = 0$. Therefore $p_{i+1} = p_i$ and $n_{i+1} = n_i$. Hence $imm(\Sigma_{\mathcal{H}_{i+1}}) \leq 1/4$. Next suppose that $m_{i+1} = 1$. Then $m \in \{1, 2\}$. For $m_{i+1} = 1$ and $m = 1$, we have $p_{i+1} = p_i$ and $n_{i+1} = n_i + 1$. Therefore,

$$imm(\Sigma_{\mathcal{H}_{i+1}}) \leq \tfrac{p_{i+1}}{n_{i+1}} = \tfrac{p_i}{n_i+1} < \tfrac{p_i}{n_i} \leq 1/4.$$

For $m_{i+1} = 1$ and $m = 2$, we have two possibilities, namely, when $V_{xy} \cap (\mathcal{H}_{i+1} \setminus \mathcal{H}_i) = \emptyset$ and when $V_{xy} \cap (\mathcal{H}_{i+1} \setminus \mathcal{H}_i) \neq \emptyset$. By hypothesis, we consider only the case when $m_{i+1} = 1$, $m = 2$ and $V_{xy} \cap (\mathcal{H}_{i+1} \setminus \mathcal{H}_i) = \emptyset$. In this case $p_{i+1} = p_i$ and $n_{i+1} = n_i + 2$. Therefore,

$$imm(\Sigma_{\mathcal{H}_{i+1}}) \leq \tfrac{p_{i+1}}{n_{i+1}} = \tfrac{p_i}{n_i+2} < \tfrac{p_i}{n_i} \leq 1/4.$$

Next suppose that $m_{i+1} = 2$ and $m = 3$. Then $V_y \cap (\mathcal{H}_{i+1} \setminus \mathcal{H}_i) \neq \emptyset$. Let $V_y \cap (\mathcal{H}_{i+1} \setminus \mathcal{H}_i) = \{v_{i+1}\}$ and let b_i is the label of $y-$loop on v_{i+1}, see Figure 3.2.1. Then, by Lemma 2.5.4(**b**), $2b_{i+1} \equiv 1 \pmod{N}$, where N is the size of any fiber. Note that, by Lemma 2.5.5, $N > 1$ since $xy-$ and $yx-$cycles contain two vertices which are adjacent to $(i+1)$th triangle. Since $2b_{i+1} \equiv 1 \pmod{N}$ and $N > 1$, $b_{i+1} \neq 0, 1$. Now from Figure 3.2.1, $P(\Sigma_{\mathcal{H}_i}) \cup v_{i+1}[0]$ spreads to $v_i[*]$. Hence $P(\Sigma_{\mathcal{H}_i}) \cup v_{i+1}[0]$ is a plague on $\Sigma_{\mathcal{H}_{i+1}}$. This implies that $p_{i+1}N = p_iN + 1$. Since $n_{i+1} = n_i + 3$, therefore,

$$imm(\Sigma_{\mathcal{H}_{i+1}}) \leq \tfrac{p_{i+1}}{n_{i+1}} = \tfrac{p_iN+1}{(n_i+3)N} \leq \tfrac{\frac{n_i}{4}N+1}{(n_i+3)N} = \tfrac{n_iN+4}{4(n_i+3)N} \leq 1/4.$$

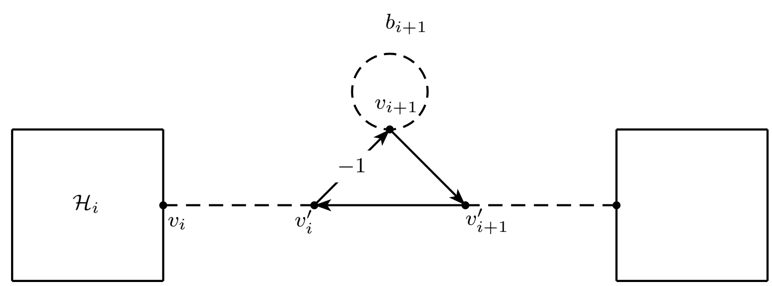

Figure 3.2.1

Finally, suppose that $m_{i+1} = 2$ and $m = 4$. Then we have $p_{i+1} = p_i + 1$ and $n_{i+1} = n_i + 4$. Therefore,

$$imm(\Sigma_{\mathcal{H}_{i+1}}) \leq \tfrac{p_{i+1}}{n_{i+1}} = \tfrac{p_i+1}{n_i+4} \leq \tfrac{\frac{n_i}{4}+1}{(n_i+4)N} = 1/4.$$

□

3.3 Estimation on the Immunity of Coverings of Pointed Schreier Graphs

In this section we provide an estimation on the immunity of coverings Σ with simply intersecting cycles of finite $PSL(2, \mathbb{Z})$-spaces $\overline{\Sigma}$. We discuss the pointed Schreier graphs $\mathcal{G}$ of $\overline{\Sigma}$ for which $imm(\Sigma) \leq 1/4$. We consider the pointed Schreier graphs $\mathcal{G}$ of $\overline{\Sigma}$ with following eight cases.

Case 1. when $V_x \neq \emptyset$ and $V_y = \emptyset = V_{xy}$.

Case 2. when $V_y \neq \emptyset$ and $V_x = \emptyset = V_{xy}$.

Case 3. when $V_{xy} \neq \emptyset$ and $V_x = \emptyset = V_y$.

Case 4. when $V_x = V_y = V_{xy} = \emptyset$.

Case 5. when $V_x \neq \emptyset \neq V_y$ and $V_{xy} = \emptyset$.

Case 6. when $V_x \neq \emptyset \neq V_{xy}$ and $V_y = \emptyset$.

Case 7. when $V_x \neq \emptyset$ and $V_y \neq \emptyset \neq V_{xy}$.

Case 8. when $V_x = \emptyset$ and $V_y \neq \emptyset \neq V_{xy}$.

3.3.1 Case 1. Pointed Schreier Graphs with $V_x \neq \emptyset$, $V_y = \emptyset = V_{xy}$

Suppose that Σ is a covering with simply intersecting cycles of finite homogeneous $PSL(2, \mathbb{Z})$-space $\overline{\Sigma}$. Let $\mathcal{G}$ be the pointed Schreier graph of $\overline{\Sigma}$ with t triangles and let $V_x \neq \emptyset$ and $V_y = \emptyset = V_{xy}$. Let $v_0 \in V_x$ is the distinguished vertex of $\mathcal{G}$. Then $\mathcal{H}_0$ is the robust subgraph of $\mathcal{G}$, consisting of v_0 and a vertex adjacent to it through $y-$edge. Let $\mathcal{H}_0 \prec_{m_1} \mathcal{H}_1 \prec_{m_2} ... \prec_{m_t} \mathcal{H}_t$ is the sequence of robust subgraphs of $\mathcal{G}$ with $v_0 \in \mathcal{H}_j$ for all $0 \leq j \leq t$. Note that for $t = 0$, $\mathcal{G}$ has no covering Σ with simply intersecting cycles (see [22], Section 7.2). Since $\mathcal{G}$ is with $V_x \neq \emptyset$ and $V_y = \emptyset = V_{xy}$, it is easy to see that there is one $\mathcal{G}$ with $t = 1$ and $|V_x| = 3$. However, by using Corollary 2.5.9, such a $\mathcal{G}$ has no covering Σ with simply intersecting cycles. By inspection one can see that the smallest given $\mathcal{G}$ which has a covering Σ with simply intersecting cycles is with $(n, t) = (8, 2)$. For the covering Σ of this $\mathcal{G}$, $imm(\Sigma) \leq 1/4$ (see [22], Section 7.12).

Motivated by the example of $\mathcal{G}$ with $(n, t) = (8, 2)$, we study the pointed Schreier graphs $\mathcal{G}$ with $V_x \neq \emptyset$ and $V_y = \emptyset = V_{xy}$ which have coverings Σ such that $imm(\Sigma) \leq 1/4$. There are infinitely many $\mathcal{G}$ which have coverings Σ such that $imm(\Sigma) \leq 1/4$. The plague on the coverings of such a $\mathcal{G}$ consists of complete fibers over certain points of $\mathcal{G}$. We see this in the following lemma. Recall that for $1 \leq j \leq t$, $J = \{j | \mathcal{H}_{j-1} \prec_2 \mathcal{H}_j\}$,

$v : J \to V(\mathcal{H}_j)$ is a map such that $v(j) \in \mathcal{H}_j \backslash \mathcal{H}_{j-1}$ for all $j \in J$, and $P_J := \{v(j)[*] | j \in J\} \subset \Sigma_{\mathcal{H}_j}$.

Lemma 3.3.1. *Let Σ be a covering with simply intersecting cycles of finite homogeneous $PSL(2,\mathbb{Z})$-space with $V_x \neq \emptyset$ and $V_y = \emptyset = V_{xy}$. Then $P = v_0[*] \cup P_J$ is a plague on $\Sigma_{\mathcal{H}_j}$ for all $1 \leq j \leq t$, where $v_0 \in V_x$.*

Proof. We prove the claim by induction on $1 \leq j \leq t$. For $j = 1$, we have $\mathcal{H}_0 \prec_2 \mathcal{H}_1$, since $V_x \neq \emptyset$ and $V_y = \emptyset = V_{xy}$. The robust subgraph $\mathcal{H}_1$ is shown in Figure 3.3.1. Since $j = 1$, $J = \{1\}$ and $v(j) = v(1) \in \{v_2, v_3, v_4, v_5\}$. If we take $v(1) = v_2$, then $P_J = \{v_2[*]\}$. Now by the following table $P = v_0[*] \cup P_J = \{v_0[*], v_2[*]\}$ is a plague on $\Sigma_{\mathcal{H}_1}$.

pivot	$v_0[*]$	$v_1[*]$	$v_3[*]$	$v_2[*]$
	$v_1[*]$	$v_3[*]$	$v_4[*]$	$v_5[*]$

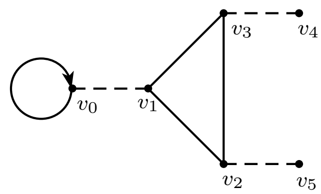

Figure 3.3.1. Robust Subgraph $\mathcal{H}_1$

Now suppose that the claim is true for $j = k < t$ and $P = v_0[*] \cup P_K$ is a plague on $\Sigma_{\mathcal{H}_k}$ for $2 \leq k < t$. For $k+1$, we have $\mathcal{H}_k \prec_{m_{k+1}} \mathcal{H}_{k+1}$ with $m_{k+1} \in \{0, 1, 2\}$. First, for $m_{k+1} = 0$, $P = v_0[*] \cup P_K$ spreads to $\Sigma_{\mathcal{H}_{k+1}}$. Next, for $m_{k+1} = 1$, $n_{k+1} = n_k + 2$. That is, $\mathcal{H}_{k+1}$ has two vertices, say, $v_{i'}$ and $v_{i'+1}$, which are not on $\mathcal{H}_k$. In this case $P = v_0[*] \cup P_K$ will also spread to the fibers $v_{i'}[*]$ and $v_{i'+1}[*]$. Finally, for $m_{k+1} = 2$, $n_{k+1} = n_k + 4$. That is $\mathcal{H}_{k+1}$ has four vertices which are not on $\mathcal{H}_k$. In this case $P = v_0[*] \cup P_K$ can not spread to $\Sigma_{\mathcal{H}_{k+1}}$. However, if we take the fiber over only one $v_{i'} \in \mathcal{H}_k \backslash \mathcal{H}_{k-1}$ then $P = v_0[*] \cup P_K \cup v_{i'}[*]$ will spread to $\Sigma_{\mathcal{H}_{k+1}}$. □

Now by using the plague discussed in Lemma 3.3.1, we prove the following proposition.

Proposition 3.3.2. *Let Σ be a covering with simply intersecting cycles of a finite homogeneous $PSL(2,\mathbb{Z})$-space $\overline{\Sigma}$ of size n. Let $\mathcal{G}$ be the pointed Schreier graph of $\overline{\Sigma}$ with t triangles and $V_x \neq \emptyset$ and $V_y = \emptyset = V_{xy}$. Assume that $\mathcal{G}$ contains at least one robust subgraph $\mathcal{H}_i$ for some $2 \leq i \leq t$ such that $\mathcal{H}_i \prec_1 \mathcal{H}_{i+1}$. Then $imm(\Sigma) \leq 1/4$.*

Proof. Let $\mathcal{H}_0 \prec_{m_1} \mathcal{H}_1 \prec_{m_2} \dots \prec_{m_t} \mathcal{H}_t$ is a sequence of robust subgraphs of $\mathcal{G}$ with $v_0 \in \mathcal{H}_j$ for all j. By Lemma 3.3.1, $P = v_0[*] \cup P_J$ is a plague on Σ for $P_J = \{v(j)[*] | j \in J\}$. Assume that, for all $1 \leq j \leq t$, $|\Sigma_{\mathcal{H}_i}| = n_j N$ and $|P(\Sigma_{\mathcal{H}_j})| = p_j N$, where N is the size of the fiber over any point of $\mathcal{H}_i$. Suppose that there exists no $0 \leq i' \leq i-1$ such that $\mathcal{H}_{i'} \prec_1 \mathcal{H}_{i'+1}$. Then we prove by induction on i' that

$$imm(\Sigma_{\mathcal{H}_{i'}}) \leq \tfrac{n_{i'}+2}{4n_{i'}},$$

where $n_{i'} = |V(\mathcal{H}_{i'})|$. For $i' = 0$, we have $\mathcal{H}_0$ with $n_0 = 2$. Note that $n_0 \neq 1$ as $V_y = \emptyset$. Now $P = v_0[*]$ is a plague on $\mathcal{H}_0$. Therefore, $imm(\Sigma_{\mathcal{H}_0)} \leq 1/2 = \frac{n_0+2}{4n_0}$. For $i' = 1$, we have $\mathcal{H}_1$ with $n_1 = 6$, and $n_1 \neq 4, 5$, since $V_y = \emptyset = V_{xy}$. Since $\mathcal{H}_0 \prec_2 \mathcal{H}_1$, $P = v_0[*] \cup v(1)[*]$ is a plague on $\mathcal{H}_1$ as in Figure 3.3.1. Therefore, $imm(\Sigma_{\mathcal{H}_1}) \leq 2N/6N = \frac{n_1+2}{4n_1}$. Next suppose that the claim is true for $i' = k$. Then,

$$imm(\Sigma_{\mathcal{H}_k}) \leq \tfrac{n_k+2}{4n_k} = \tfrac{\frac{1}{4}(n_k+2)}{n_k} = \tfrac{p_k}{n_k}.$$

Therefore, $p_k = \frac{1}{4}(n_k + 2)$. Now for $k+1$ and $m_{k+1} = 0$ we have $n_{k+1} = n_k$ and $p_{k+1}N = p_k N$. Therefore,

$$imm(\Sigma_{\mathcal{H}_{k+1}}) = imm(\Sigma_{\mathcal{H}_k}) \leq \tfrac{n_k+2}{4n_k}.$$

For $k+1$ and $m_{k+1} = 2$ we have $n_{k+1} = n_k + 4$ and $p_{k+1}N = (p_k + 1)N$. Therefore,

$$imm(\Sigma_{\mathcal{H}_{k+1}}) \leq \tfrac{p_{k+1}}{n_{k+1}} = \tfrac{(p_k)+1}{n_{k+1}} = \tfrac{\frac{1}{4}(n_k+2)+1}{n_{k+1}} = \tfrac{(n_k+6)}{4n_{k+1}} = \tfrac{(n_k+4)+2}{4n_{k+1}} = \tfrac{n_{k+1}+2}{4n_{k+1}}.$$

Now we show that $imm(\Sigma_{\mathcal{H}_j}) \leq 1/4$ for $i \leq j \leq t$. Since $\mathcal{H}_i \prec_1 \mathcal{H}_{i+1}$, $n_j = n_{i-1} + 2$ and $p_j N = p_{i-1}N$ for $j = i$. Therefore for $j = i$,

$$imm(\Sigma_{\mathcal{H}_j}) \leq \tfrac{p_j}{n_j} = \tfrac{p_{i-1}}{n_{i-1}+2} = \tfrac{\frac{1}{4}(n_{i-1}+2)}{n_{i-1}+2} = 1/4,$$

since $imm(\mathcal{H}_{i-1}) \leq \frac{p_{i-1}}{n_{i-1}} \leq \frac{n_{i-1}+2}{4n_{i-1}} = \frac{\frac{1}{4}(n_{i-1}+2)}{n_{i-1}}$. Now by Lemma 3.2.1, $imm(\Sigma_{\mathcal{H}_j}) \leq 1/4$ for $i < j \leq t$. Since $\Sigma_{\mathcal{H}_t} = \Sigma$, therefore $imm(\Sigma) \leq 1/4$. □

Example 3.3.3. For the pointed Schreier graph $\mathcal{G}$ in Figure 3.3.2, $P = v_0[*] \cup v_1[*] \cup v_2[*] \cup v_3[*]$ is a plague on the covering Σ of $\mathcal{G}$ with simply intersecting cycles. Therefore, $imm(\Sigma) \leq \frac{4N}{16N} = 1/4$.

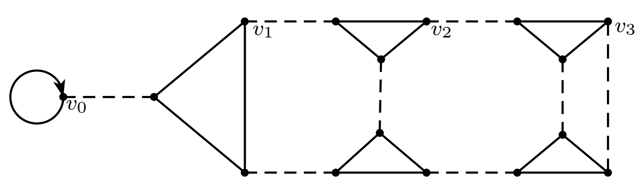

Figure 3.3.2

The next step is to look for the coverings Σ of those pointed Schreier graphs $\mathcal{G}$ with $V_x \neq \emptyset$ and $V_y = \emptyset = V_{xy}$ for which there is no robust subgraph $\mathcal{H}_i$ such that $\mathcal{H}_i \prec_1 \mathcal{H}_{i+1}$. It is easy to see that the smallest such pointed Schreier graph $\mathcal{G}$ has 3 triangles and 3 robust subgraphs such that $\mathcal{H}_0 \prec_{m_1} \mathcal{H}_1 \prec_{m_2} \mathcal{H}_2 \prec_{m_3} \mathcal{H}_3$ with $(m_1, m_2, m_3) = (2, 2, 0)$. We discuss this smallest pointed Schreier graph in the following two sections.

3.3.1.1 The Graph $\mathcal{G}_{10\{10\}}$.

Lemma 3.3.4. *Let Σ be a covering of $\mathcal{G}_{10\{10\}}$ in Figure 3.3.3 with simply intersecting cycles. Then $a + 2 \equiv 0 \pmod N$ and $N = 5$.*

Proof. The xy-cycles with their labels are: $(v_1\ v_3\ v_{10}\ v_7\ v_9\ v_4\ v_6\ v_8\ v_5\ v_2)$ with $a + 2$. The yx-cycles with their labels are: $(v_1\ v_2\ v_9\ v_6\ v_8\ v_3\ v_5\ v_{10}\ v_7\ v_4)$ with $a + 2$. By Lemma 2.5.5, $N > 1$, and by Lemma 2.5.7, $a + 2 \equiv 0 \pmod N$ which implies $3a \equiv -6 \pmod N$. Now as by Lemma 2.5.4(**a**) on v_1, $3a \equiv -1 \pmod N$, therefore $N = 5$. ☐

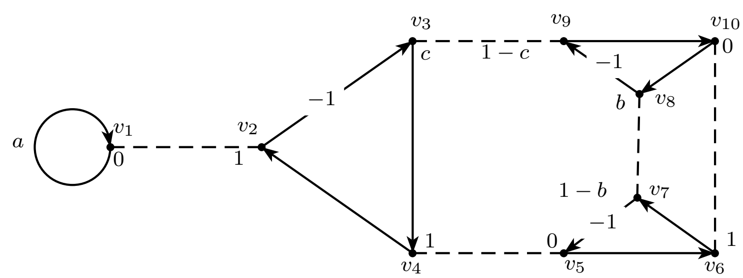

Figure 3.3.3. Schreier graph $\mathcal{G}_{10\{10\}}$ and its coverings

Lemma 3.3.5. *Let Σ be a covering of $\mathcal{G}_{10\{10\}}$ with simply intersecting cycles. Then $imm(\Sigma) \leq 1/4 = \omega(\Sigma)$.*

Proof. To prove that $\omega(\Sigma)) = 1/4$ observe that in every covering all cycles have length 10. From the following table it follows that $P = v_1[*] \cup v_3[*] \cup v_{10}[0]$ is a plague.

pivot	$v_1[*]$	$v_2[*]$	$v_3[*]$	$v_4[*]$	$v_6[0]$	$v_5[0]$	$v_{10}[0]$	$v_9[1]$	...
	$v_2[*]$	$v_4[*]$	$v_9[*]$	$v_5[*]$	$v_7[1]$	$v_6[1]$	$v_8[1]$	$v_{10}[1]$	...

Thus $imm(\Sigma) \leq 11/50 < 1/4 = \omega(\Sigma)$. ☐

3.3.1.2 The Graph $\mathcal{G}_{10\{5,\,3,\,2\}}$.

Lemma 3.3.6. *Let* Σ *be a covering of* $\mathcal{G}_{10\{5,\,3,\,2\}}$ *in Figure 3.3.4 with simply intersecting cycles. Then* $N > 1$ *and* $a - b - c + 1 \equiv \pmod{N}$.

Proof. The xy-cycles with their labels are: $(v_1\ v_3\ v_8\ v_5\ v_2)$ with $a-b-c+1$, $(v_4\ v_6\ v_{10})$ with c, $(v_7\ v_9)$ with $1+b$. The yx-cycles with their labels are: $(v_1\ v_2\ v_{10}\ v_7\ v_4)$ with $a-b-c+1$, $(v_3\ v_5\ v_9)$ with c, $(v_6\ v_8)$ with $1+b$. By Lemma 2.5.5, $N > 1$, and by Lemma 2.5.7, $a - b - c + 1 \equiv 0 \pmod{N}$. ☐

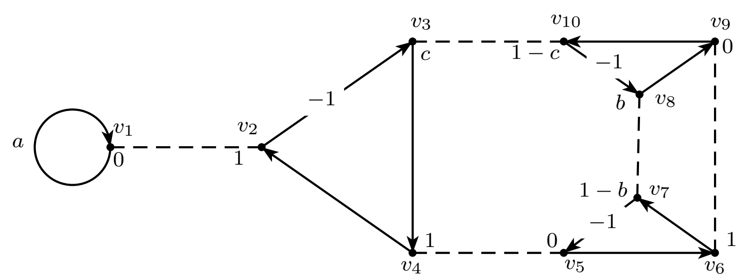

Figure 3.3.4. Schreier graph $\mathcal{G}_{10\{5,\,3,\,2\}}$ and its coverings

Lemma 3.3.7. *Let* Σ *be a covering of* $\mathcal{G}_{10\{5,\,3,\,2\}}$ *with simply intersecting cycles. Then* $imm(\Sigma) \leq \omega(\Sigma)$.

Proof. The cycle structure on each vertex of Σ is the following:

$$\begin{aligned} v_1[i], v_2[i] &: \text{two 5-cycles},\\ v_3[i], v_5[i], v_4[i], v_{10}[i] &: \text{cycles of length } 5 \text{ and } 3\,|\langle c\rangle|\,,\\ v_6[i], v_9[i] &: \text{cycles of length } 2\,|\langle 1+b\rangle| \text{ and } 3\,|\langle c\rangle|\,,\\ v_7[i], v_8[i] &: \text{cycles of length } 5 \text{ and } 2\,|\langle 1+b\rangle|\,, \end{aligned}$$

for all $i \in Z_N$. Using this cycle structure the weight of Σ is the following:

$$\omega(\Sigma) = \begin{cases} 3/10 & \text{if } 1+b \equiv c \equiv 0 \pmod{N},\\ 11/40 & \text{if } |\langle 1+b\rangle| \not\equiv 0 \pmod{N} \textit{ and } c \equiv 0 \pmod{N},\\ 17/60 & \text{if } |\langle 1+b\rangle| \equiv 0 \pmod{N} \textit{ and } c \not\equiv 0 \pmod{N},\\ 1/4 & \text{otherwise.} \end{cases}$$

Assume first that $1 + b \equiv c \equiv 0 \pmod{N}$. By Lemma 3.3.6, $a - b - c + 1 = 0$, and by Lemma 2.5.4(a) on v_1, $3a \equiv -1 \pmod{N}$. Therefore, $N = 5$ and $(a, b, c) = (3, 4, 0)$. Now from the following table it follows that $P = v_1[*] \cup v_3[*] \cup v_6[*]$ is a plague.

pivot	$v_1[*]$	$v_2[*]$	$v_4[*]$	$v_5[*]$	$v_7[*]$	$v_{10}[*]$	$v_3[*]$
	$v_2[*]$	$v_4[*]$	$v_5[*]$	$v_7[*]$	$v_8[*]$	$v_9[*]$	$v_{10}[*]$

In this case $imm(\Sigma) \leq 3/10 = \omega(\Sigma)$.

Next assume that $|\langle 1+b\rangle| \not\equiv 0 \pmod N$ *and* $c \equiv 0 \pmod N$, and I is a set of representatives of $Z_N/\langle 1+b\rangle$, then we claim that $P = v_1[*] \cup v_3[*] \cup v_8[I]$ is plague. We compute

pivot	$v_9[I]$	$v_7[I+1]$
	$v_6[I+1]$	$v_8[I+1+b]$

Now, by Example 2.6.5, it follows that P spreads to $v_8[*]$. Thus $P = v_1[*] \cup v_3[*] \cup v_8[I]$ is plague. In this case $imm(\Sigma) \leq 2N + |I|/10N \leq (2N + N/2)/10N = 1/4 \leq \omega(\Sigma)$.

Next assume that $|\langle 1+b\rangle| \equiv 0 \pmod N$ *and* $c \not\equiv 0 \pmod N$, and I is a set of representatives of $Z_N/\langle c\rangle$, then we claim that $P = v_5[*] \cup v_8[*] \cup v_3[I]$ is plague. We compute

pivot	$v_7[*]$	$v_9[*]$	$v_4[I]$	$v_3[I]$	$v_5[I]$	$v_6[I]$	$v_{10}[I]$
	$v_6[*]$	$v_{10}[*]$	$v_2[I+1]$	$v_4[I+1]$	$v_7[I+1]$	$v_9[I]$	$v_3[I+c]$

Now by Example 2.6.5 it follows that spreads to $v_3[*]$. Thus $P = v_5[*] \cup v_8[*] \cup v_3[I]$ is a plague. In this case $imm(\Sigma) \leq 2N + |I|/10N \leq (2N + N/2)/10N = 1/4 \leq \omega(\Sigma)$.

Finally, assume that $|\langle 1+b\rangle| \not\equiv 0 \pmod N$ *and* $c \not\equiv 0 \pmod N$. Then again $P = v_1[*] \cup v_3[*] \cup v_8[I]$ is plague. Therefore, $imm(\Sigma) \leq 2N + |I|/10N \leq (2N + N/2)/10N = 1/4 \leq \omega(\Sigma)$. □

3.3.1.3 The Fragments of Pointed Schreier Graphs

Let $\mathcal{G}$ be a pointed Schreier graph of a finite $PSL(2,\mathbb{Z})$-space with t triangles. Let $\mathcal{H}_j$ is a robust subgraph of $\mathcal{G}$ for $j \in \{0, 1, ..., t\}$. We define a *fragment* $\mathcal{F}$ of $\mathcal{G}$ as a part of $\mathcal{G}$ which is separated from $\mathcal{G}$ by its robust subgraph $\mathcal{H}_j$ for any $j \in \{0, 1, ..., t-1\}$. That is, $\mathcal{F} = \mathcal{G} \setminus \mathcal{H}_j$ for any $j \in \{0, 1, ..., t-1\}$. In this section we will discuss the fragments of pointed Schreier graphs $\mathcal{G}_{10\{10\}}$ and $\mathcal{G}_{10\{5,3,2\}}$ which are obtained by separating the robust subgraph $\mathcal{H}_0$. Such fragments has only one free site which can be used to glue them with robust subgraphs $\mathcal{H}_j$ of $\mathcal{G}$ for any $j \in \{0, 1, ..., t-1\}$. We write these fragments as

$$\mathcal{F}_1 := \mathcal{G}_{10\{10\}} \setminus \mathcal{H}_0(\mathcal{G}_{10\{10\}}) \text{ and } \mathcal{F}_2 := \mathcal{G}_{10\{10\}} \setminus \mathcal{H}_0(\mathcal{G}_{10\{5,3,2\}}).$$

Note that $|\mathcal{F}_1| = |\mathcal{F}_2| = 8$ and $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) has a free site for one vertex which can be used for gluing $\mathcal{F}_1$ with other subgraphs.

Let k be an integer with $0 \leq k < t$ and $\mathcal{H}_0 \prec_2 \mathcal{H}_1 \prec_2 \ldots \prec_2 \mathcal{H}_k$ is a sequence of a pointed Schreier graph $\mathcal{G}$. Then $\mathcal{H}_k$ have $k+1$ possible open $y-$edges (which have a vertex without $x-$edge on it). Note that the pointed Schreier graphs $\mathcal{G}$ with $V_x \neq \emptyset$, $V_y = \emptyset = V_{xy}$ and having no robust subgraph $\mathcal{H}_i$ such that $\mathcal{H}_i \prec_1 \mathcal{H}_{i+1}$ can be generated by gluing $k+1$ copies of $\mathcal{F}_1$ (resp. $\mathcal{F}_2$ or both $\mathcal{F}_1$ and $\mathcal{F}_2$) with $k+1$ possible open $y-$edges of $\mathcal{H}_k$. Therefore we write any such graph as $\mathcal{G} = Span(\mathcal{H}_k, \mathcal{F}_1, \mathcal{F}_2)$. Since there are infinitely many subgraphs $\mathcal{H}_k$ with $\mathcal{H}_0 \prec_2 \mathcal{H}_1 \prec_2 \ldots \prec_2 \mathcal{H}_{k-1}$ for $k \geq 3$, we get infinitely many pointed Schreier graphs $\mathcal{G} = Span(\mathcal{H}_k, \mathcal{F}_1, \mathcal{F}_2)$.

Note that $\mathcal{H}_k \prec_2 \mathcal{H}_{k+1}$. Therefore $k+1 \in J$ and $v(k+1)[*] \in P_J$. Assume that the label of $y-$edge at $v(k+1)$ is 0, like in the graph $\mathcal{G}_{10\{10\}}$. Let $P'_J := P_J \setminus v(k+1)[*]$. Then the set $P = v_0[*] \cup P' \cup v(k+1)[0]$ is a plague on the covering Σ of $\mathcal{G}$ which contains at least one copy of the fragment $\mathcal{F}_1$. If $|V_x| = 1$, then $t = 4k+3$ and $|\mathcal{G}| = 3t+1 = 12k+10$. Note that for $k = 0$, we have $\mathcal{G} = \mathcal{G}_{10\{10\}}$. For any covering Σ of $\mathcal{G} = Span(\mathcal{H}_k, \mathcal{F}_1, \mathcal{F}_2)$ and $\mathcal{G}$ has at least one copy of the fragment $\mathcal{F}_1$, we have:

$$imm(\Sigma) \leq \tfrac{(3k+2)N+1}{(12k+10)N} \leq 1/4.$$

Finally, we discuss the case when $\mathcal{G} = Span(\mathcal{H}_k, \mathcal{F}_2)$. We see that when $imm(\Sigma) \leq 1/4$ for any covering Σ of $\mathcal{G} = Span(\mathcal{H}_k, \mathcal{F}_2)$. Since $\mathcal{G} = Span(\mathcal{H}_k, \mathcal{F}_2)$ contains k copies of $\mathcal{F}_2$, any xy-cycle (resp. yx-cycle) have length ≥ 2. Therefore, by Definition 2.7.1, $(\omega'_{2j})_{j>1} = 1/3$ (resp. $(\omega'_{i2})_{i>1} = 1/3$). Note that there are $2k$ vertices on $xy-$ cycles and $2k$ vertices on $yx-$ cycles, which implies that the number of vertices on cycles of length two are $4k$ and the number of vertices on cycles of length ≥ 3 are $n-4k$. Therefore we have:

$$\omega(\Sigma) \geq \frac{\frac{4k}{3} + \frac{n-4k}{4}}{n} = \frac{1}{4} + \frac{k}{3n},$$

where $n = |\mathcal{G}| = 12k-2$. Let $v(k+1) \in P_J$ such that the label of $y-$edge at $v(k+1)$ is b. Let $P'_J := P_J \setminus v(k+1)[*]$. Then for $b = 0$, the set $P = v_x[*] \cup P'_J \cup v(k+1)[*]$ is a plague on the covering Σ of $\mathcal{G} = Span(\mathcal{H}_k, \mathcal{F}_2)$. For $b \neq 0$, $P = v_x[*] \cup P_{J-1} \cup v(k+1)[I]$ with $|I| \leq N/2$ is a plague on the covering Σ of $\mathcal{G} = \langle \mathcal{H}_{k-1}, \mathcal{F}_2 \rangle$, like in the graph $\mathcal{G}_{10\{5,\,3,\,2\}}$. Since $t = 12k+3$ and $|\mathcal{G}| = 3t+1 = 12k+10$, therefore for $b = 0$,

$$imm(\Sigma) \leq \tfrac{(3k+2)N+1}{(12k+10)N} \leq \omega(\Sigma)$$

and for $b \neq 0$,

$$imm(\Sigma) \leq \tfrac{(3k+2)N+N/2}{(12k+10)N} = 1/4.$$

3.3.2 Case 2. Pointed Schreier Graphs with $V_y \neq \emptyset$, $V_x = \emptyset = V_{xy}$

Let Σ be a covering with simply intersecting cycles of finite homogeneous $PSL(2,\mathbb{Z})$-space $\overline{\Sigma}$. Let $\mathcal{G}$ be the pointed Schreier graph of $\overline{\Sigma}$ with $V_y \neq \emptyset$ and $V_x = \emptyset = V_{xy}$. In this case we consider $v_0 \in V_y$ as a distinguished vertex of $\mathcal{G}$. Let t be the number of triangles of $\mathcal{G}$ and $\mathcal{H}_0 \prec_{m_1} \mathcal{H}_1 \prec_{m_2} \dots \prec_{m_t} \mathcal{H}_t$ is the sequence of robust subgraphs of $\mathcal{G}$ with $v_0 \in \mathcal{H}_j$ for all $0 \leq j \leq t$. We first prove the following lemma.

Lemma 3.3.8. *Let Σ be a covering with simply intersecting cycles of a finite homogeneous $PSL(2,\mathbb{Z})$-space with $V_y \neq \emptyset$ and $V_x = \emptyset = V_{xy}$. Then $P = v_0[*] \cup P_J$ is a plague on $\Sigma_{\mathcal{H}_j}$ for all $1 \leq j \leq t$, where $v_0 \in V_y$.*

Proof. We prove the claim by induction on $1 \leq j \leq t$. For $j = 1$, we have $\mathcal{H}_0 \prec_2 \mathcal{H}_1$, since $V_y \neq \emptyset$ and $V_x = \emptyset = V_{xy}$. The robust subgraph $\mathcal{H}_1$ is shown in the Figure 3.3.5. Since $j = 1$, $J = \{1\}$ and $v(j) = v(1) \in \{v_1, v_2, v_3, v_4\}$. If we take $v(1) = v_1$, then $P_J = \{v_1[*]\}$. Now by the following table $P = v_0[*] \cup P_J = \{v_0[*], v_1[*]\}$ is a plague on $\Sigma_{\mathcal{H}_1}$.

pivot	$v_0[*]$	$v_1[*]$	$v_2[*]$
	$v_2[*]$	$v_4[*]$	$v_3[*]$

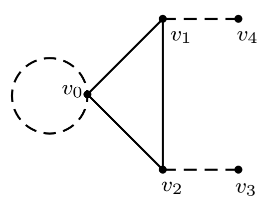

Figure 3.3.5. Robust Subgraph $\mathcal{H}_1$

Now suppose that the claim is true for $j = k < t$ and $P = v_0[*] \cup P_K$ is a plague on $\Sigma_{\mathcal{H}_k}$ for $2 \leq k < t$. For $k+1$, we have $\mathcal{H}_k \prec_{m_{k+1}} \mathcal{H}_{k+1}$ with $m_{k+1} \in \{0, 1, 2\}$. First, for $m_{k+1} = 0$, $P = v_0[*] \cup P_K$ spreads to $\Sigma_{\mathcal{H}_{k+1}}$. Next, for $m_{k+1} = 1$, $n_{k+1} = n_k + m$ with $m \in \{1, 2\}$. That is, $\mathcal{H}_{k+1}$ has one or two vertices, say, $v_{i'}$ and $v_{i'+1}$, which are not on $\mathcal{H}_k$. In both these cases $P = v_0[*] \cup P_K$ will also spread to the fibers $v_{i'}[*]$ and $v_{i'+1}[*]$. Finally, for $m_{k+1} = 2$, $n_{k+1} = n_k + m$ with $m \in \{3, 4\}$. That is $\mathcal{H}_{k+1}$ has three or four vertices which are not on $\mathcal{H}_k$. In both these cases $P = v_0[*] \cup P_K$ can not spread to $\Sigma_{\mathcal{H}_{k+1}}$. However, if we take the fiber over only one $v_{i'} \in \mathcal{H}_k \backslash \mathcal{H}_{k-1}$ then $P = v_0[*] \cup P_K \cup v_{i'}[*]$ will spread to $\Sigma_{\mathcal{H}_{k+1}}$. ◻

Now by using the plague discussed in Lemma 3.3.8, we prove the following results.

Lemma 3.3.9. *Let Σ be a covering with simply intersecting cycles of finite homogeneous $PSL(2,\mathbb{Z})$-space $\overline{\Sigma}$. Let $\mathcal{G}$ be the pointed Schreier graph of $\overline{\Sigma}$ with $V_y \neq \emptyset$ and $V_x = \emptyset = V_{xy}$. Assume that for any $1 < i \leq t$, $\mathcal{H}_i$ is a robust subgraph of $\mathcal{G}$ such that $\mathcal{H}_i \cap V_y = \{v_0\}$ and $\mathcal{H}_{i-1} \prec_{m_i} \mathcal{H}_i$ with $m_i \neq 1$. Then $imm(\Sigma_{\mathcal{H}_i}) \leq \frac{n_i+3}{4n_i}$.*

Proof. We prove the claim by induction on i. For $i = 1$, we have $n_1 = 5$ and $p_1 = 2$ since $P = v_0[*] \cup v_1[*]$ is a plague on $\Sigma_{\mathcal{H}_1}$ from Figure 3.3.5. Therefore,

$$imm(\Sigma_{\mathcal{H}_1}) \leq \frac{p_1}{n_1} = \frac{2}{5} = \frac{5+3}{4.5} = \frac{n_1+3}{4n_1}.$$

Now suppose that the claim is true for $i = k \geq 2$. That is, $imm(\Sigma_{\mathcal{H}_k}) \leq \frac{n_k+3}{4n_k}$. This implies that $p_k \leq \frac{n_k+3}{4}$. Now for $\mathcal{H}_{k+1}$ we have, $\mathcal{H}_k \prec_{m_{k+1}} \mathcal{H}_{k+1}$, where $m_{k+1} \in \{0,2\}$. For $m_{k+1} = 0$, we have $n_{k+1} = n_k$ and $p_{k+1} = p_k$. Therefore,

$$imm(\Sigma_{\mathcal{H}_{k+1}}) \leq \frac{p_{k+1}}{n_{k+1}} = \frac{p_k}{n_k} \leq \frac{n_k+3}{4n_k}.$$

Now for $m_{k+1} = 2$, we have $p_{k+1} = p_k + 1$ and $n_{k+1} = n_k + 4$ since $\mathcal{H}_i \cap V_y = \{v_0\}$ for any $1 < i \leq t$. Therefore,

$$imm(\Sigma_{\mathcal{H}_{k+1}}) \leq \frac{p_{k+1}}{n_{k+1}} = \frac{p_k+1}{n_k+4} \leq \frac{\frac{n_k+3}{4}+1}{n_k+4} = \frac{n_k+7}{4(n_k+4)} = \frac{n_{k+1}+3}{4n_{k+1}}.$$

□

Lemma 3.3.10. *Let Σ be a covering with simply intersecting cycles of finite homogeneous $PSL(2,\mathbb{Z})$-space $\overline{\Sigma}$. Let $\mathcal{G}$ be the pointed Schreier graph of $\overline{\Sigma}$ with $|V_y| = 1$ and $V_x = \emptyset = V_{xy}$. Assume that $\mathcal{G}$ contains at least two distinct robust subgraphs $\mathcal{H}_i$ and $\mathcal{H}_{i'}$ for $2 \leq i, i' \leq t$ such that $\mathcal{H}_{i-1} \prec_1 \mathcal{H}_i$ and $\mathcal{H}_{i'-1} \prec_1 \mathcal{H}_{i'}$. Then $imm(\Sigma) < 1/4$.*

Proof. Without lost of generality, suppose that $i < i'$ and $\mathcal{H}_i$ is the smallest robust subgraph with $\mathcal{H}_{i-1} \prec_1 \mathcal{H}_i$. Then by Lemma 3.3.9,

$$imm(\Sigma_{\mathcal{H}_{i-1}}) \leq \frac{n_{i-1}+3}{4n_{i-1}} = \frac{\frac{1}{4}(n_{i-1}+3)}{n_{i-1}} = \frac{p_{i-1}}{n_{i-1}}.$$

Therefore, $p_{i-1} = \frac{1}{4}(n_{i-1}+3)$. Now since $\mathcal{H}_{i-1} \prec_1 \mathcal{H}_i$, $|V_y| = 1$ and $V_{xy} = \emptyset$, therefore $n_i = n_{i-1} + 2$ and $p_i = p_{i-1}$. Therefore,

$$imm(\Sigma_{\mathcal{H}_i}) \leq \frac{p_i}{n_i} = \frac{p_{i-1}}{n_{i-1}+2} = \frac{n_{i-1}+3}{4(n_{i-1}+2)} = \frac{n_i+1}{4n_i}.$$

Now let j' be a positive integer such that $i+1 \leq j' \leq i'-1$. Then it is easy to see by induction on j' that $imm(\Sigma_{\mathcal{H}_{j'}}) \leq \frac{n_{j'}+1}{4n_{j'}}$. In particular,

$$imm(\Sigma_{\mathcal{H}_{i'-1}}) \leq \frac{n_{i'-1}+1}{4n_{i'-1}} = \frac{\frac{1}{4}(n_{i'-1}+1)}{n_{i'-1}} = \frac{p_{i'-1}}{n_{i'-1}}.$$

Therefore, $p_{i'-1} = \frac{1}{4}(n_{i'-1}+1)$. Now since $\mathcal{H}_{i'-1} \prec_1 \mathcal{H}_{i'}$, $|V_y| = 1$ and $V_{xy} = \emptyset$, therefore $n_{i'} = n_{i'-1} + 2$ and $p_{i'} = p_{i'-1}$. Therefore,

$$imm(\Sigma_{\mathcal{H}_{i'}}) \leq \frac{p_{i'}}{n_{i'}} = \frac{p_{i'-1}}{n_{i'-1}+2} = \frac{n_{i'-1}+1}{4(n_{i'-1}+2)} = \frac{n_{i'}-1}{4n_{i'}} < \frac{n_{i'}}{4n_{i'}} = 1/4.$$

Now the claim follows by Lemma 3.2.1.

□

Example 3.3.11. For the pointed Schreier graph $\mathcal{G}$ in Figure 3.3.6, $P = v_0[*] \cup v_1[*] \cup v_2[*] \cup v_3[*] \cup v_4[*]$ is a plague on the covering Σ of $\mathcal{G}$ with simply intersecting cycles. Therefore,

$$imm(\Sigma) \leq \frac{5N}{21N} = 1/4.$$

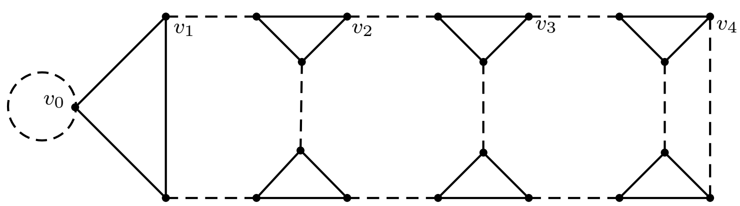

Figure 3.3.6

Now we discuss the pointed Schreier graphs $\mathcal{G}$ with $|V_y| \geq 2$ and $V_x = \emptyset = V_{xy}$. Let $k \geq 1$ and $V_y = \{v_0, v_{i_1}, v_{i_2}, ..., v_{i_k}\}$, where i_k are integers with $2 \leq i_k \leq t$ and $v_{i_k} \in \mathcal{H}_{i_k}$. More precisely, $v_{i_k} \in \mathcal{H}_{i_k} \setminus \mathcal{H}_{i_k-1}$ or equivalently, $V_{xy} \cap (\mathcal{H}_{i_k} \setminus \mathcal{H}_{i_k-1}) = \{v_{i_k}\}$ for $k \geq 1$.

Proposition 3.3.12. *Let Σ be a covering with simply intersecting cycles of finite homogeneous $PSL(2,\mathbb{Z})$-space $\overline{\Sigma}$. Let $\mathcal{G}$ be the pointed Schreier graph of $\overline{\Sigma}$ with $V_x = \emptyset = V_{xy}$ and $V_y = \{v_0, v_{i_1}, v_{i_2}, ..., v_{i_k}\}$ for $k \geq 1$. Assume that $\mathcal{G}$ contains at least one robust subgraph $\mathcal{H}_i$ for $2 \leq i \leq i_1 - 1$ such that $\mathcal{H}_{i-1} \prec_1 \mathcal{H}_i$. Then $imm(\Sigma) < 1/4$.*

Proof. Since $\mathcal{H}_i \cap V_y = \{v_0\}$ and $\mathcal{H}_{i-1} \prec_1 \mathcal{H}_i$, therefore by Lemma 3.3.9, $imm(\Sigma_{\mathcal{H}_i}) \leq \frac{n_i+1}{4n_i}$ for $2 \leq i \leq i_1 - 1$. Now for $\mathcal{H}_{i_1}$ there are two possibilities, namely, $\mathcal{H}_{i_1-1} \prec_1 \mathcal{H}_{i_1}$ and $\mathcal{H}_{i_1-1} \prec_2 \mathcal{H}_{i_1}$.

Suppose first that $\mathcal{H}_{i_1-1} \prec_1 \mathcal{H}_{i_1}$. Then $n_{i_1} = n_{i_1-1}+1$ and $p_{i_1} = p_{i_1-1} = \frac{1}{4}(n_{i_1-1}+1)$. Therefore,

$$imm(\Sigma_{\mathcal{H}_{i_1}}) \leq \frac{p_{i_1}}{n_{i_1}} = \frac{p_{i_1-1}}{n_{i_1-1}+1} = \frac{n_{i_1-1}+1}{4(n_{i_1-1}+1)} = 1/4$$

Now if $i_1 = t$, then we are done. If $i_1 < j \leq t$, then $imm(\mathcal{H}_j) \leq 1/4$ by Lemma 3.2.1, and hence $imm(\Sigma) < 1/4$.

Suppose next that $\mathcal{H}_{i_1-1} \prec_2 \mathcal{H}_{i_1}$. Then $n_{i_1} = n_{i_1-1} + 3$. Let $V_y \cap (\mathcal{H}_{i_1} \setminus \mathcal{H}_{i_1}) = \{v_{i_1}\}$ and let b_{i_1} is the label of $y-$loop on v_{i_1}. Then $2b_{i_1} \equiv 1 \pmod N$, and $b_{i_1} \neq 0, 1$ since $N > 1$. Note that $N > 1$ because one can see that $xy-$ and $yx-$cycles contain two vertices which are adjacent to (i_1)th triangle. Now from Figure 3.3.7, the set $P(\Sigma_{\mathcal{H}_{i_1}}) \cup v_{i_1}[0]$ spreads to $v_{i_1}[*]$. Hence $P(\Sigma_{\mathcal{H}_i}) \cup v_{i_1}[0]$ is a plague on $\Sigma_{\mathcal{H}_{i_1}}$. This implies that $p_{i_1}N = p_{i_1-1}N + 1$. Therefore,

$$imm(\Sigma_{\mathcal{H}_{i_1}}) \leq \frac{p_{i_1}}{n_{i_1}} = \frac{p_{i_1-1}N+1}{(n_{i_1-1}+3)N} = \frac{(\frac{n_{i_1-1}+1}{4})N+1}{(n_{i_1-1}+3)N} \leq 1/4,$$

for $N > 1$. Now $imm(\Sigma_{\mathcal{H}_j}) \leq 1/4$ for all $i_1 < j \leq t$ by Lemma 3.2.1, and hence $imm(\Sigma) < 1/4$.

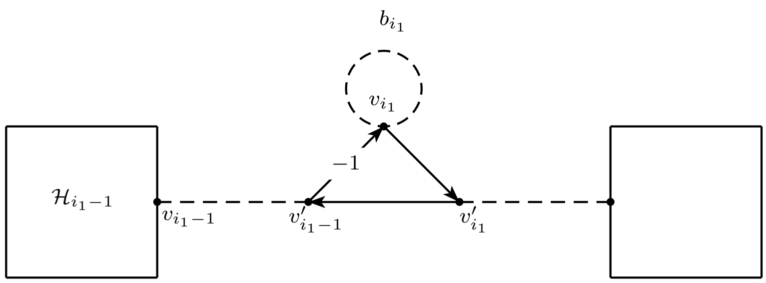

Figure 3.3.7

□

Example 3.3.13. For the pointed Schreier graph $\mathcal{G}$ in Figure 3.3.8, $P = v_0[*] \cup v_1[*] \cup v_2[0] \cup v_3[*]$ is plague on the covering Σ of $\mathcal{G}$ with simply intersecting cycles. Therefore,

$$imm(\Sigma) \leq \frac{3N+1}{15N} < 1/4.$$

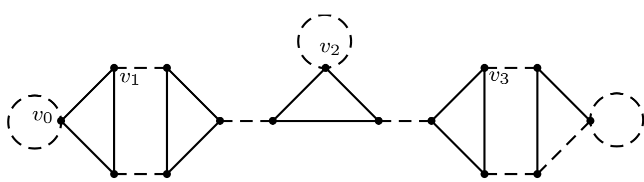

Figure 3.3.8

Remark 3.3.14. Note that the there are infinitely many pointed Schreier graphs $\mathcal{G}$ with $V_x \neq \emptyset$ and $V_y = \emptyset = V_{xy}$ which are not covered by Lemma 3.3.10 and Proposition 3.3.12. Generating all such graphs is complicated.

3.3.3 Case 3. Pointed Schreier Graphs with $V_x = \emptyset = V_y$, $V_{xy} \neq \emptyset$

Suppose that Σ is a covering with simply intersecting cycles of finite homogeneous $PSL(2,\mathbb{Z})$-space. Let $\mathcal{G}$ be the pointed Schreier graph with $V_x = \emptyset = V_y$ and $V_{xy} \neq \emptyset$ and a distinguished $v_0 \in V_{xy}$. Let t be the number of triangles of $\mathcal{G}$ and $\mathcal{H}_0 \prec_{m_1} \mathcal{H}_1 \prec_{m_2} \dots \prec_{m_t} \mathcal{H}_t$ is the sequence of robust subgraphs of $\mathcal{G}$ with $v_0 \in \mathcal{H}_j$ for all $0 \leq j \leq t$. Note that the robust subgraph $\mathcal{H}_0$ of $\mathcal{G}$ consists of v_0 and $v_1 = y(v_0)$. Since $V_x = \emptyset = V_y$, it is easy to see that there is no graph $\mathcal{G}$ for $t = 0, 1$. For $t = 2$ there is one $\mathcal{G}$ which is discussed in [22] (see Section 7.7). In this section we discuss the general method to calculate plagues, consisting of complete fibers, for pointed Schreier graphs with $V_x = \emptyset = V_y$ and $V_{xy} \neq \emptyset$ and $v_0 \in V_{xy}$. We first prove the following lemma.

Lemma 3.3.15. *Let Σ be a covering with simply intersecting cycles of finite homogeneous $PSL(2,\mathbb{Z})$-space with $V_{xy} \neq \emptyset$ and $V_x = \emptyset = V_y$. Then $P = v_0[*] \cup v_1[*] \cup P_J \cup V'_{xy}[*]$ is a plague on $\Sigma_{\mathcal{H}_j}$ for all $1 \leq j \leq t$, where $v_0 \in V_{xy}$, $v_1 = y(v_0)$ and $V'_{xy} = V_{xy} \setminus \{v_0\}$.*

Proof. We prove the claim by induction on $1 \leq j \leq t$. For $j = 1$, we have $\mathcal{H}_0 \prec_1 \mathcal{H}_1$. The robust subgraph $\mathcal{H}_1$ is shown in Figure 3.3.9. Now by the following table $P = v_0[*] \cup v_1[*]$ is a plague on $\Sigma_{\mathcal{H}_1}$.

pivot	$v_1[*]$	$v_2[*]$
	$v_2[*]$	$v_3[*]$

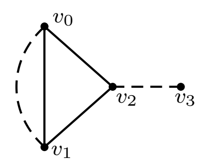

Figure 3.3.9. Robust Subgraph $\mathcal{H}_1$

Next suppose that the claim is true for $j = k < t$ and $P = v_0[*] \cup v_1[*] \cup P_K \cup V'_{xy}|_{\mathcal{H}_k}[*]$ is a plague on $\Sigma_{\mathcal{H}_k}$ for $2 \leq k < t$, where $K \subseteq J$ and $V'_{xy}|_{\mathcal{H}_k} \subseteq V'_{xy}$ are restricted subsets. Now for $k+1$, we have $\mathcal{H}_k \prec_{m_{k+1}} \mathcal{H}_{k+1}$ with $m_{k+1} \in \{0, 1, 2\}$. First, for $m_{k+1} = 0$, $P = v_0[*] \cup v_1[*] \cup P_K \cup V'_{xy}|_{\mathcal{H}_k}[*]$ spreads to $\Sigma_{\mathcal{H}_{k+1}}$. Next for $m_{k+1} = 1$, $n_{k+1} = n_k + 2$. That is, $\mathcal{H}_{k+1}$ has two vertices, say v_{k+1} and $v'_{k+1}(:= y(v_{k+1}))$, which are not on $\mathcal{H}_k$. If $V'_{xy}[*] \cap \{v_{k+1}, v'_{k+1}\} = \emptyset$, then $P = v_0[*] \cup v_1[*] \cup P_K \cup V'_{xy}|_{\mathcal{H}_k}[*]$ will also spread to the fibers $v_{k+1}[*]$ and $v'_{k+1}[*]$. Let $v_{k+1} \in V'_{xy}$, then $P = v_0[*] \cup v_1[*] \cup P_K \cup V'_{xy}|_{\mathcal{H}_k}[*] \cup v_{k+1}[*]$ will spread to $\mathcal{H}_{k+1}$ Finally, for $m_{k+1} = 2$, $n_{k+1} = n_k + 4$. That is $\mathcal{H}_{k+1}$ has four vertices which are not on $\mathcal{H}_k$. In this case case $P = v_0[*] \cup v_1[*] \cup P_K \cup V'_{xy}|_{\mathcal{H}_k}[*]$ can not spread to $\Sigma_{\mathcal{H}_{k+1}}$. However, if we take the fiber over only one $v_{k+1} \in \mathcal{H}_{k+1} \backslash \mathcal{H}_k$ then $P = v_0[*] \cup v_1[*] \cup P_K \cup V'_{xy}|_{\mathcal{H}_k}[*] \cup v_{k+1}[*]$ will spread to $\Sigma_{\mathcal{H}_{k+1}}$. ∎

Now by using the plague discussed in Lemma 3.3.15, we prove the following results.

Lemma 3.3.16. *Let Σ be a covering with simply intersecting cycles of finite homogeneous $PSL(2,\mathbb{Z})$-space whose Schreier graph $\mathcal{G}$ is with t triangles and with $V_x = \emptyset = V_y$ and $V_{xy} \neq \emptyset$. Assume that $\mathcal{H}_i$ is a robust subgraph of $\mathcal{G}$ for $1 < i \leq t$, such that $\mathcal{H}_i \cap V_{xy} = \{v_0\}$ and $\mathcal{H}_{i-1} \prec_{m_i} \mathcal{H}_i$, where $m_i \in \{0, 2\}$. Then $imm(\Sigma_{\mathcal{H}_i}) \leq \frac{n_i+4}{4n_i}$.*

Proof. Let $\mathcal{H}_0 \prec_{m_1} \mathcal{H}_1 \prec_{m_2} \dots \prec_{m_t} \mathcal{H}_t$ is a finite sequence of robust subgraphs $\mathcal{H}_j$ of $\mathcal{G}$ with a distinguished vertex $v_0 \in V_{xy}$ for all $1 \leq j \leq t$. We prove, by induction on i, that $imm(\Sigma_{\mathcal{H}_i}) \leq \frac{n_i+4}{4n_i}$. For $i = 1$, we have $n_1 = 4$ and $p_1 = 2$, since $P = v_0[*] \cup v_1[*]$ is a plague on $\Sigma_{\mathcal{H}_1}$ from Figure 3.3.9. Therefore,

$$imm(\Sigma_{\mathcal{H}_1}) \leq \frac{p_1}{n_1} = \frac{2}{4} = \frac{4+4}{4.4} = \frac{n_1+4}{4n_1}.$$

Now suppose that the claim is true for $i = k \geq 2$. That is, $imm(\Sigma_{\mathcal{H}_k}) \leq \frac{p_k}{n_k} = \frac{n_k+4}{4n_k}$. This implies that $p_k = \frac{n_k+4}{4}$. Now for $\mathcal{H}_{k+1}$ we have, $\mathcal{H}_k \prec_{m_{k+1}} \mathcal{H}_{k+1}$, where $m_{k+1} \in \{0, 2\}$. For $m_{k+1} = 0$, we have $n_{k+1} = n_k$ and $p_{k+1} = p_k$. Therefore,

$$imm(\Sigma_{\mathcal{H}_{k+1}}) \leq \frac{p_{k+1}}{n_{k+1}} = \frac{p_k}{n_k} = \frac{n_k+4}{4n_k} = \frac{n_{k+1}+4}{4n_{k+1}}.$$

For $m_{k+1} = 2$, we have $p_{k+1} = p_k + 1$ and $n_{k+1} = n_k + 4$ since $\mathcal{H}_i \cap V_{xy} = \{v_0\}$. Therefore,

$$imm(\Sigma_{\mathcal{H}_{k+1}}) \leq \frac{p_{k+1}}{n_{k+1}} = \frac{p_k+1}{n_k+4} \frac{\frac{n_k+4}{4}+1}{n_k+4} = \frac{n_k+8}{4(n_k+4)} = \frac{n_{k+1}+4}{4n_{k+1}}.$$

□

Lemma 3.3.17. *Let Σ be a covering with simply intersecting cycles of finite homogeneous $PSL(2,\mathbb{Z})$-space $\overline{\Sigma}$. Let $\mathcal{G}$ be the pointed Schreier graph of $\overline{\Sigma}$ with t triangles, $|V_{xy}| = 1$, $V_x = \emptyset = V_{xy}$ and a distinguished vertex $v_0 \in V_{xy}$. Assume that $\mathcal{G}$ contains at least two distinct robust subgraphs $\mathcal{H}_i$ and $\mathcal{H}_{i'}$ for $2 \leq i, i' \leq t$ such that $\mathcal{H}_{i-1} \prec_1 \mathcal{H}_i$ and $\mathcal{H}_{i'-1} \prec_1 \mathcal{H}_{i'}$. Then $imm(\Sigma) \leq 1/4$.*

Proof. Without lost of generality, suppose that $i < i'$ and $\mathcal{H}_i$ is the smallest robust subgraph with $\mathcal{H}_{i-1} \prec_1 \mathcal{H}_i$. Since $|V_{xy}| = 1$ and $V_x = \emptyset = V_{xy}$, therefore it is easy to see that the smallest possible robust subgraph $\mathcal{H}_i$ with $\mathcal{H}_{i-1} \prec_1 \mathcal{H}_i$ is for $i = 3$. Now by Lemma 3.3.16,

$$imm(\Sigma_{\mathcal{H}_{i-1}}) \leq \frac{n_{i-1}+4}{4n_{i-1}} = \frac{\frac{1}{4}(n_{i-1}+4)}{n_{i-1}} = \frac{p_{i-1}}{n_{i-1}}.$$

Therefore, $p_{i-1} = \frac{1}{4}(n_{i-1} + 4)$. Since $\mathcal{H}_{i-1} \prec_1 \mathcal{H}_i$, $|V_{xy}| = 1$ and $V_y = \emptyset$, therefore $n_i = n_{i-1} + 2$ and $p_i = p_{i-1}$, and

$$imm(\Sigma_{\mathcal{H}_i}) \leq \frac{p_i}{n_i} = \frac{p_{i-1}}{n_{i-1}+2} = \frac{n_{i-1}+4}{4(n_{i-1}+2)} = \frac{n_i+2}{4n_i}.$$

Now let j' be a positive integer such that $i+1 \leq j' \leq i'-1$. Then it is easy to see by induction on j' that $imm(\mathcal{H}_{j'}) \leq \frac{n_{j'}+2}{4n_{j'}}$. In particular,

$$imm(\Sigma_{\mathcal{H}_{i'-1}}) \leq \frac{n_{i'-1}+2}{4n_{i'-1}} = \frac{\frac{1}{4}(n_{i'-1}+2)}{n_{i'-1}} = \frac{p_{i'-1}}{n_{i'-1}}.$$

Therefore, $p_{i'-1} = \frac{1}{4}(n_{i'-1}+2)$. Now since $\mathcal{H}_{i'-1} \prec_1 \mathcal{H}_{i'}$, $|V_{xy}| = 1$ and $V_y = \emptyset$, therefore $n_{i'} = n_{i'-1} + 2$ and $p_{i'} = p_{i'-1}$. Therefore,

$$imm(\Sigma_{\mathcal{H}_{i'}}) \leq \frac{p_{i'}}{n_{i'}} = \frac{p_{i'-1}}{n_{i'-1}+2} = \frac{n_{i'-1}+2}{4(n_{i'-1}+2)} = \frac{n_{i'}-1}{4n_{i'}} = 1/4.$$

Now the claim follows by Lemma 3.2.1, since $|V_{xy}| = 1$.

□

Example 3.3.18. For the pointed Schreier graph $\mathcal{G}$ in Figure 3.3.10, $P = v_0[*] \cup v_1[*] \cup v_2[*] \cup v_3[*] \cup v_4[*] \cup v_5[*]$ is plague on the covering Σ of $\mathcal{G}$ with simply intersecting cycles. Therefore,

$$imm(\Sigma) \leq \frac{6N}{24N} = 1/4.$$

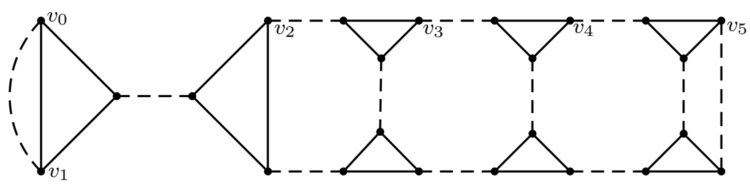

Figure 3.3.10

Now we discuss the pointed Schreier graphs $\mathcal{G}$ with $|V_{xy}| \geq 2$ and $V_x = \emptyset = V_y$. Let $k \geq 1$ and $V_{xy} = \{v_0, v_{i_1}, v_{i_2}, ..., v_{i_k}\}$, where i_k are integers with $2 \leq i_k \leq t$ and $v_{i_k} \in \mathcal{H}_{i_k}$. More precisely, $v_{i_k} \in \mathcal{H}_{i_k} \setminus \mathcal{H}_{i_k-1}$ or equivalently, $V_{xy} \cap (\mathcal{H}_{i_k} \setminus \mathcal{H}_{i_k-1}) = \{v_{i_k}\}$ for $k \geq 1$. Note that $\mathcal{H}_{i_k-2} \prec_{m_{i_k-1}} \mathcal{H}_{i_k-1} \prec_1 \mathcal{H}_{i_k}$, where $m_{i_k-1} \in \{1,2\}$. We consider the pointed Schreier graphs $\mathcal{G}$ with $m_{i_k-1} \in \{1,2\}$. Observe that if $\mathcal{H}_{i_k-2} \prec_1 \mathcal{H}_{i_k-1} \prec_2 \mathcal{H}_{i_k}$ then $n_{i_k} = n_{i_k-2} + 8$.

Remark 3.3.19. For certain pointed Schreier graph with $V_x = \emptyset = V_y$ and $V_{xy} \neq \emptyset$ it is also possible to calculate the plague on $\Sigma_{\mathcal{H}_j}$ which is smaller than the plague $P = v_0[*] \cup v_1[*] \cup P_J \cup V'_{xy}[*]$ which is discussed in Lemma 3.3.15. Assume that for $2 < i_1 < t$ there exists a robust subgraph $\mathcal{H}_{i_1}$ such that $V'_{xy} \cap (\mathcal{H}_{i_1} \setminus \mathcal{H}_{i_1-1}) = \{v_{i_1}\}$ and $\mathcal{H}_{i_1-2} \prec_2 \mathcal{H}_{i_1-1} \prec_1 \mathcal{H}_{i_1}$, as shown in Figure 3.3.11. From this figure we calculate:

pivot	$v'_{i_1}[*]$	$v'_{i_1-1}[*]$	$v_{i_1-2}[*]$	$v''_{i_1-1}[*]$
	$v''_{i_1-1}[*]$	$v_{i_1-1}[*]$	$v'_{i_1-1}[*]$	$v'_{i_1}[*]$

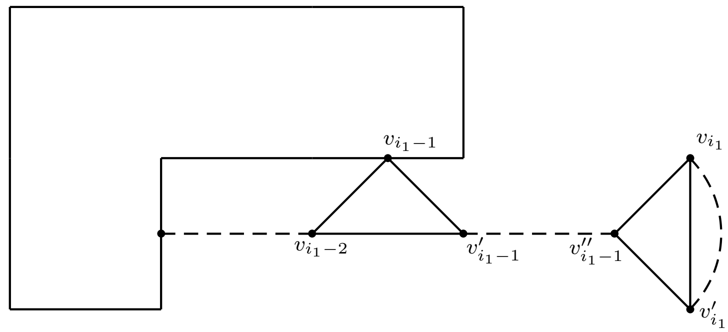

Figure 3.3.11

This implies that $P(\Sigma_{\mathcal{H}_{i_1-2}}) \cup v_{i_1}[*]$ is a plague on $\Sigma_{\mathcal{H}_{i_1}}$. Therefore, $p_{i_1} = p_{i_1-2} + 1$. Since $\mathcal{H}_{i_1-2} \prec_2 \mathcal{H}_{i_1-1} \prec_1 \mathcal{H}_{i_1}$, therefore, $n_{i_1} = n_{i_1-2} + 6$. Now if $imm(\Sigma_{\mathcal{H}_{i_1-2}}) \leq 1/4$, then

$$imm(\Sigma_{\mathcal{H}_{i_1}}) \leq \frac{p_{i_1}}{n_{i_1}} = \frac{p_{i_1-2}+1}{n_{i_1-2}+6} \leq \frac{\frac{n_{i_1-2}}{4}+1}{(n_{i_1-2}+6)} = \frac{n_{i_1-2}+4}{4(n_{i_1-2}+6)} < 1/4.$$

Note also that, if $i_2 = i_1 + 1$ and $\mathcal{H}_{i_1-2} \prec_2 \mathcal{H}_{i_1-1} \prec_1 \mathcal{H}_{i_1} \prec_1 \mathcal{H}_{i_2}$, then $n_{i_2} = n_{i_1-2} + 8$ and $P(\Sigma_{\mathcal{H}_{i_1-2}}) \cup v_{i_1}[*] \cup v_{i_2}[*]$ is a plague on $\Sigma_{\mathcal{H}_{i_2}}$. Therefore, $p_{i_1} = p_{i_1-2} + 2$, and hence $imm(\Sigma_{\mathcal{H}_{i_2}}) \leq 1/4$.

Proposition 3.3.20. *Let Σ be a covering with simply intersecting cycles of finite homogeneous $PSL(2,\mathbb{Z})$-space $\overline{\Sigma}$. Let $\mathcal{G}$ be the pointed Schreier graph of $\overline{\Sigma}$ with $V_x = \emptyset = V_y$ and $V_{xy} = \{v_0, v_{i_1}, v_{i_2}, ..., v_{i_k}\}$ for $k \geq 1$. Assume that $\mathcal{G}$ contains at least two distinct robust subgraphs $\mathcal{H}_i$ and $\mathcal{H}_{i'}$ for $2 \leq i, i' < i_1 - 2$ such that $\mathcal{H}_{i-1} \prec_1 \mathcal{H}_i$ and $\mathcal{H}_{i'-1} \prec_1 \mathcal{H}_{i'}$. Then $imm(\Sigma) \leq 1/4$.*

Proof. Since $\mathcal{H}_{i_1-2} \cap V_{xy} = \{v_0\}$ and $\mathcal{H}_{i-1} \prec_1 \mathcal{H}_i$ and $\mathcal{H}_{i'-1} \prec_1 \mathcal{H}_{i'}$ for $2 \leq i, i' < i_1 - 2$, therefore by Lemma 3.3.17, $imm(\Sigma_{\mathcal{H}_{i_1-2}}) \leq \frac{p_{i_1-2}}{n_{i_1-2}} \leq 1/4$. Now we consider two cases here. First assume that $\mathcal{H}_{i_1-2} \prec_1 \mathcal{H}_{i_1-1} \prec_1 \mathcal{H}_{i_1}$. Then $n_{i_1} = n_{i_1-2}+4$, and $p_{i_1} = p_{i_1-2}+1$ since $P(\Sigma_{\mathcal{H}_{i_1-2}}) \cup v_{i_1}[*]$ is a plague on $\Sigma_{\mathcal{H}_{i_1}}$. Therefore,

$$imm(\Sigma_{\mathcal{H}_{i_1}}) \leq \frac{p_{i_1}}{n_{i_1}} = \frac{p_{i_1-2}+1}{n_{i_1-2}+4} \leq \frac{\frac{n_{i_1-2}}{4}+1}{(n_{i_1-2}+4)} = \frac{n_{i_1-2}+4}{4(n_{i_1-2}+4)} = 1/4.$$

Now assume that $\mathcal{H}_{i_1-2} \prec_2 \mathcal{H}_{i_1-1} \prec_1 \mathcal{H}_{i_1}$. Then by Remark 3.3.19, $imm(\Sigma_{\mathcal{H}_{i_1}}) \leq 1/4$. Repeating this process for all i_k it follows that $imm(\Sigma_{\mathcal{H}_{i_1}}) \leq 1/4$. Hence $imm(\Sigma_{\mathcal{H}_j}) \leq 1/4$ for all $1 \leq j \leq t$.

□

Example 3.3.21. For the pointed Schreier graph $\mathcal{G}$ in Figure 3.3.12, $P = v_0[*] \cup v_1[*] \cup v_2[*] \cup v_3[*] \cup v_4[*] \cup v_5[*]$ is plague on the covering Σ of $\mathcal{G}$ with simply intersecting cycles. Therefore, $imm(\Sigma) \leq \frac{6N}{24N} = 1/4$.

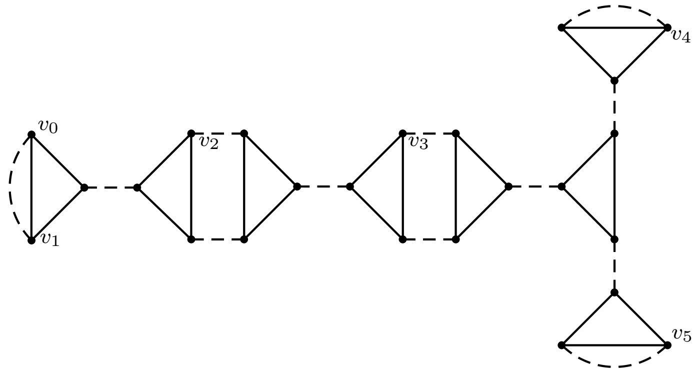

Figure 3.3.12

3.3.4 Case 4. Pointed Schreier Graphs with $V_x = V_y = V_{xy} = \emptyset$

Suppose that Σ is a covering with simply intersecting cycles of a finite homogeneous $PSL(2,\mathbb{Z})$-space whose Schreier graph $\mathcal{G}$ is with t triangles and with $V_x = V_y = V_{xy} = \emptyset$. For such graphs any vertex can be considered as a distinguished vertex v_0, that is, $v_0 \in V(\mathcal{G})$. Let $y(v_0) = v_1$. Then $\mathcal{H}_0$ will consist of v_0 and v_1. Let $\mathcal{H}_0 \prec_{m_1} \mathcal{H}_1 \prec_{m_2} \ldots \prec_{m_t} \mathcal{H}_t$ is the sequence of robust subgraphs of $\mathcal{G}$ with $v_0 \in \mathcal{H}_j$ for all j. Note that for $t \in \{0,1\}$, there is no given graph. The given graphs $\mathcal{G}$ with $t \in \{2,4,6,8\}$ and $|C_{xy}| \leq 4$ are discussed in [22] (see Sections 7.10, 7.15-7.18). Among these graphs, the graph with $t = 8$ triangles has coverings whose immunity can be calculated by using plague with complete fibers. We discuss such graphs in this section. The general method to calculate plagues consisting of complete fibers for graphs with $V_x = V_y = V_{xy} = \emptyset$ is explained in the following lemma.

Lemma 3.3.22. *Let Σ be a covering with simply intersecting cycles of finite homogeneous $PSL(2,\mathbb{Z})$-space whose Schreier graph $\mathcal{G}$ is with $V_x = V_y = V_{xy} = \emptyset$. Then $P = v_0[*] \cup v_1[*] \cup P_J$ is a plague on $\Sigma_{\mathcal{H}_j}$ for all $1 \leq j \leq t$, where $v_0 \in V(\mathcal{G})$ and $v_1 = y(v_0)$.*

Proof. The claim follows by induction on $1 \leq j \leq t$. We see the claim only for $j = 1$. Note that $\mathcal{H}_0$ consists of v_0 and v_1 and $\mathcal{H}_0 \prec_2 \mathcal{H}_1$, since $V_y = V_{xy} = \emptyset$. The robust subgraph $\mathcal{H}_1$ is shown in Figure 3.3.13. Since $j = 1$, $J = \{1\}$ and $v(j) = v(1) \in \{v_2, v_3, v_4, v_5\}$. If we take $v(1) = v_2$, then $P_J = \{v_2[*]\}$. Now by the following table $P = v_0[*] \cup v_1[*] \cup P_J = \{v_0[*], v_1[*], v_2[*]\}$ is a plague on $\Sigma_{\mathcal{H}_1}$.

pivot	$v_1[*]$	$v_2[*]$	$v_3[*]$
	$v_3[*]$	$v_5[*]$	$v_4[*]$

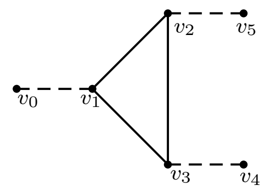

Figure 3.3.13. Robust Subgraph $\mathcal{H}_1$

□

Now by using the plague discussed in Lemma 3.3.22, we prove the following results.

Lemma 3.3.23. *Let Σ be a covering with simply intersecting cycles of finite homogeneous $PSL(2,\mathbb{Z})$-space $\overline{\Sigma}$. Let $\mathcal{G}$ be the pointed Schreier graph of $\overline{\Sigma}$ with $V_x = V_y = V_{xy} = \emptyset$. Assume that for any $1 \leq i \leq t$, $\mathcal{H}_i$ is a robust subgraph of $\mathcal{G}$ such that $\mathcal{H}_{i-1} \prec_{m_i} \mathcal{H}_i$ with $m_i \neq 1$. Then $imm(\Sigma_{\mathcal{H}_i}) \leq \frac{n_i+6}{4n_i}$.*

Proof. We prove the claim by induction on i. For $i = 1$, we have $n_1 = 6$ and $p_1 = 3$, since $P = v_0[*] \cup v_1[*] \cup v_2[*]$ is a plague on $\Sigma_{\mathcal{H}_1}$ from Figure 3.3.13. Therefore,

$$imm(\Sigma_{\mathcal{H}_1}) \leq \frac{p_1}{n_1} = \frac{3}{6} = \frac{6+6}{4.6} = \frac{n_1+6}{4n_1}.$$

Now suppose that the claim is true for $i = k \geq 2$. That is, $imm(\Sigma_{\mathcal{H}_k}) \leq \frac{n_k+6}{4n_k}$. This implies that $p_k = \frac{n_k+6}{4}$. For $\mathcal{H}_{k+1}$ we have, $\mathcal{H}_k \prec_{m_{k+1}} \mathcal{H}_{k+1}$, where $m_{k+1} \in \{0, 2\}$. For $m_{k+1} = 0$, we have $n_{k+1} = n_k$ and $p_{k+1} = p_k$. Therefore,

$$imm(\Sigma_{\mathcal{H}_{k+1}}) \leq \frac{p_{k+1}}{n_{k+1}} = \frac{p_k}{n_k} = \frac{n_k+6}{4n_k} = \frac{n_{k+1}+6}{4n_{k+1}}.$$

Now for $m_{k+1} = 2$, we have $p_{k+1} = p_k + 1$ and $n_{k+1} = n_k + 4$. Therefore,

$$imm(\Sigma_{\mathcal{H}_{k+1}}) \leq \frac{p_{k+1}}{n_{k+1}} = \frac{p_k+1}{n_k+4} = \frac{\frac{n_k+6}{4}+1}{n_k+4} = \frac{n_k+10}{4(n_k+4)} = \frac{n_{k+1}+6}{4n_{k+1}}.$$

□

Proposition 3.3.24. *Let Σ be a covering with simply intersecting cycles of finite homogeneous $PSL(2,\mathbb{Z})$-space $\overline{\Sigma}$. Let $\mathcal{G}$ be the pointed Schreier graph of $\overline{\Sigma}$ with t triangles and $V_x = V_y = V_{xy} = \emptyset$. Assume that $\mathcal{G}$ contains at least three distinct robust subgraphs $\mathcal{H}_i$, $\mathcal{H}_{i'}$ and $\mathcal{H}_{i''}$ for $2 \leq i, i', i'' \leq t$ such that $\mathcal{H}_{i-1} \prec_1 \mathcal{H}_i$, $\mathcal{H}_{i'-1} \prec_1 \mathcal{H}_{i'}$ and $\mathcal{H}_{i''-1} \prec_1 \mathcal{H}_{i''}$. Then $imm(\Sigma) < 1/4$.*

Proof. Without lost of generality, suppose that $i < i' < i''$ and $\mathcal{H}_i$ is the smallest robust subgraph with $\mathcal{H}_{i-1} \prec_1 \mathcal{H}_i$. Then by Lemma 3.3.23,

$$imm(\Sigma_{\mathcal{H}_{i-1}}) \leq \frac{n_{i-1}+6}{4n_{i-1}} = \frac{\frac{1}{4}(n_{i-1}+3)}{n_{i-1}} = \frac{p_{i-1}}{n_{i-1}}.$$

Therefore, $p_{i-1} = \frac{1}{4}(n_{i-1}+6)$. Now since $\mathcal{H}_{i-1} \prec_1 \mathcal{H}_i$, and $V_y = V_{xy} = \emptyset$, therefore $n_i = n_{i-1}+2$ and $p_i = p_{i-1}$. Therefore,

$$imm(\mathcal{H}_i) \leq \frac{p_i}{n_i} = \frac{p_{i-1}}{n_{i-1}+2} = \frac{n_{i-1}+6}{4(n_{i-1}+2)} = \frac{n_i+4}{4n_i}.$$

Next it is easy to see (by induction) that

$$imm(\mathcal{H}_{i'-1}) \leq \frac{n_{i'-1}+4}{4n_{i'-1}} = \frac{\frac{1}{4}(n_{i'-1}+4)}{n_{i'-1}} = \frac{p_{i'-1}}{n_{i'-1}}.$$

Therefore, $p_{i'-1} = \frac{1}{4}(n_{i'-1}+4)$. Now since $\mathcal{H}_{i'-1} \prec_1 \mathcal{H}_{i'}$, and $V_y = V_{xy} = \emptyset$, therefore $n_{i'} = n_{i'-1}+2$ and $p_{i'} = p_{i'-1}$. Therefore,

$$imm(\mathcal{H}_{i'}) \leq \frac{p_{i'}}{n_{i'}} = \frac{p_{i'-1}}{n_{i'-1}+2} = \frac{n_{i'-1}+4}{4(n_{i'-1}+2)} = \frac{n_{i'}+2}{4n_{i'}}.$$

Again it is easy to see (by induction) that

$$imm(\mathcal{H}_{i''-1}) \leq \frac{n_{i''-1}+2}{4n_{i''-1}} = \frac{\frac{1}{4}(n_{i''-1}+2)}{n_{i''-1}} = \frac{p_{i''-1}}{n_{i''-1}}.$$

Therefore, $p_{i''-1} = \frac{1}{4}(n_{i''-1}+2)$. Now since $\mathcal{H}_{i''-1} \prec_1 \mathcal{H}_{i''}$, and $V_y = V_{xy} = \emptyset$, therefore $n_{i''} = n_{i''-1}+2$ and $p_{i''} = p_{''-1}$. Hence,

$$imm(\mathcal{H}_{i''}) \leq \frac{p_{i''}}{n_{i''}} = \frac{p_{i''-1}}{n_{i''-1}+2} = \frac{n_{i''-1}+2}{4(n_{i''-1}+2)} = 1/4.$$

Now the claim follows by Lemma 3.2.1.

□

Example 3.3.25. For the pointed Schreier graph $\mathcal{G}$ in Figure 3.3.14, $P = v_0[*] \cup v_1[*] \cup v_2[*] \cup v_3[*] \cup v_4[*] \cup v_5[*]$ is plague on the covering Σ of $\mathcal{G}$ with simply intersecting cycles. Therefore,

$$imm(\Sigma) \leq \frac{6N}{24N} = 1/4.$$

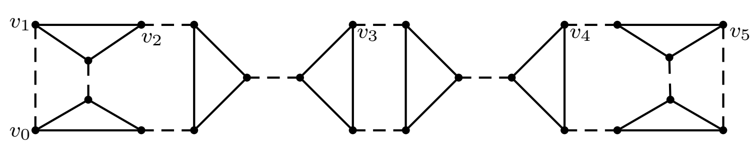

Figure 3.3.14

3.3.5 Case 5. Pointed Schreier Graphs with $V_x \neq \emptyset \neq V_y$, $V_{xy} = \emptyset$

In this section we study the coverings Σ with simply intersecting cycles of the finite homogeneous $PSL(2,\mathbb{Z})$-spaces $\overline{\Sigma}$ such that the pointed Schreier graphs $\mathcal{G}$ of $\overline{\Sigma}$ are with $V_x \neq \emptyset \neq V_y$ and $V_{xy} = \emptyset$. For the pointed Schreier graphs $\mathcal{G}$ with $V_x \neq \emptyset \neq V_y$ and $V_{xy} = \emptyset$ we will consider $v_0 \in V_x$ as a distinguished vertex. Therefore the robust subgraph $\mathcal{H}_j$ of $\mathcal{G}$ will be with $v_0 \in V_x$ for any $0 \leq j \leq t$, where t is the number of triangles of $\mathcal{G}$. We write $V_y = \{v_{i_1}, v_{i_2}, ..., v_{i_k}\}$, where $v_{i_k} \in \mathcal{H}_{i_k} \setminus \mathcal{H}_{i_k-1}$ for positive integers $i_1, i_2, ..., i_k$ with $1 \leq i_1 < i_2 < ... < i_k \leq t$. Note that $\mathcal{H}_{i_k-1} \prec_{m_{i_k}} \mathcal{H}_{i_k}$ with $m_{i_k} \in \{1,2\}$.

Remark 3.3.26.

(1). It is easy to see that for pointed Schreier graphs $\mathcal{G}$ with $V_x \neq \emptyset \neq V_y$, $V_{xy} = \emptyset$ and $|V(\mathcal{G})| \geq 2$, $|C_{xy}(v_0) \cap C_{yx}(v_0)| > 1$, where $C_{xy}(v_0)$ and $C_{yx}(v_0)$ are the $xy-$ and $yx-$cycles of $\mathcal{G}$ containing $v_0 \in V_x$. Therefore, by Lemma 2.5.5, $N > 1$, where N is the size of fiber over any point of $\mathcal{G}$. This implies that every covering Σ with simply intersecting cycles of $\mathcal{G}$ with $V_x \neq \emptyset \neq V_y$, $V_{xy} = \emptyset$ and $|V(\mathcal{G})| \geq 2$ is non-trivial.

(2). If a_0 is the label of an $x-$loop at $v_0 \in V_x$, then $3a_0 \equiv -1 \pmod N$, by Lemma 2.5.4**(a)**. If b_k is the label of $y-$loop at $v_{i_k} \in V_y$, then $2b_k \equiv 1 \pmod N$, by Lemma 2.5.4**(b)**, and $b_k \neq 0, 1$, since $N > 1$. From $3a_0 \equiv -1 \pmod N$ and $2b_k \equiv 1 \pmod N$, it follows that N is not a multiple of 3 and N is odd.

(3). If $V_x \neq \emptyset \neq V_y$ and $V_{xy} = \emptyset$, then $V_y \cap \mathcal{H}_{i_1-1} = \emptyset$. Therefore, by Proposition 3.3.2,

$$imm(\mathcal{H}_{i_1-1}) \leq \begin{cases} \frac{n_{i_1-1}+2}{4n_{i_1-1}}, & \text{if there is no } \mathcal{H}_i \text{ for } 2 \leq i \leq i_1 - 1 \text{ such that } \mathcal{H}_i \prec_1 \mathcal{H}_{i+1}, \\ 1/4, & \text{if there is a } \mathcal{H}_i \text{ for } 2 \leq i \leq i_1 - 1 \text{ such that } \mathcal{H}_i \prec_1 \mathcal{H}_{i+1}. \end{cases}$$

Now we discuss the coverings Σ of $\overline{\Sigma}$ whose pointed Schreier graphs $\mathcal{G}$ are with $V_x \neq \emptyset \neq V_y$ and $V_{xy} = \emptyset$ such that $imm(\Sigma) \leq 1/4$.

Lemma 3.3.27. *Let Σ be a covering with simply intersecting cycles of a finite homogeneous $PSL(2,\mathbb{Z})$-space $\overline{\Sigma}$. Let $\mathcal{G}$ be the pointed Schreier graph of $\overline{\Sigma}$ such that $V_x \neq \emptyset$, $V_{xy} = \emptyset$ and $V_y = \{v_{i_1}\}$, where $v_{i_1} \in \mathcal{H}_{i_1} \setminus \mathcal{H}_{i_1-1}$, and $\mathcal{H}_{i_1-1} \prec_1 \mathcal{H}_{i_1}$. Assume further that there exists at least one robust subgraph $\mathcal{H}_i$ for $i \neq i_1$ such that $\mathcal{H}_{i-1} \prec_1 \mathcal{H}_i$. Then $imm(\Sigma) \leq 1/4$.*

Proof. Let t be the number of triangles of $\mathcal{G}$ and $\mathcal{H}_0 \prec_{m_1} \mathcal{H}_1 \prec_{m_2} ... \prec_{m_t} \mathcal{H}_t$ is a finite sequence of robust subgraphs $\mathcal{H}_j$ of $\mathcal{G}$. Now choose $i_1 = t$. Then $V_y \cap \mathcal{H}_{t-1} = \emptyset$ and $1 \leq i \leq t-1$. Since there exist a robust subgraph $\mathcal{H}_i$ for $i \neq i_1$ such that $\mathcal{H}_{i-1} \prec_1 \mathcal{H}_i$, therefore by Remark 3.3.26(3), we have $imm(\Sigma_{\mathcal{H}_{t-1}}) \leq \frac{p_{t-1}}{n_{t-1}} \leq 1/4$. Therefore $p_{t-1} \leq \frac{n_{t-1}}{4}$. Since $\mathcal{H}_{i_1-1} \prec_1 \mathcal{H}_{i_1}$, we have $n_t = n_{t-1} + 1$ and $p_t = p_{t-1}$. Therefore,

$$imm(\Sigma) = imm(\Sigma_{\mathcal{H}_t}) \leq \frac{p_t}{n_t} = \frac{p_{t-1}}{n_{t-1}+1} < \frac{p_{t-1}}{n_{t-1}} \leq 1/4.$$

□

Lemma 3.3.28. *Let Σ be a covering with simply intersecting cycles of finite homogeneous $PSL(2,\mathbb{Z})$-space $\overline{\Sigma}$. Let $\mathcal{G}$ be the pointed Schreier graph of $\overline{\Sigma}$ such that $V_x \neq \emptyset$, $V_{xy} = \emptyset$ and $V_y = \{v_{i_1}\}$, where $v_{i_1} \in \mathcal{H}_{i_1} \setminus \mathcal{H}_{i_1-1}$ and $\mathcal{H}_{i_1-1} \prec_2 \mathcal{H}_{i_1}$. Then $imm(\Sigma) \leq 1/4$.*

Proof. Let t be the number of triangles of $\mathcal{G}$ and $\mathcal{H}_0 \prec_{m_1} \mathcal{H}_1 \prec_{m_2} \ldots \prec_{m_t} \mathcal{H}_t$ is a finite sequence of robust subgraphs $\mathcal{H}_j$ of $\mathcal{G}$. By Remark 3.3.26(3), we have:

$$imm(\Sigma_{\mathcal{H}_{i_1-1}}) \leq \frac{p_{i_1-1}}{n_{i_1-1}} \leq \frac{n_{i_1-1}+2}{4n_{i_1-1}}.$$

Therefore $p_{i_1-1} \leq \frac{n_{i_1-1}+2}{4}$. Since $\mathcal{H}_{i_1-1} \prec_2 \mathcal{H}_{i_1}$, $n_{i_1} = n_{i_1-1}+3$. Now from Figure 3.3.15, we have the following table:

pivot	$v_{i_1}[-b_{i_1}]$	$v_{i_1-1}[1-b_{i_1}]$
	$v'_{i_1}[1-b_{i_1}]$	$v_{i_1}[1-b_{i_1}]$

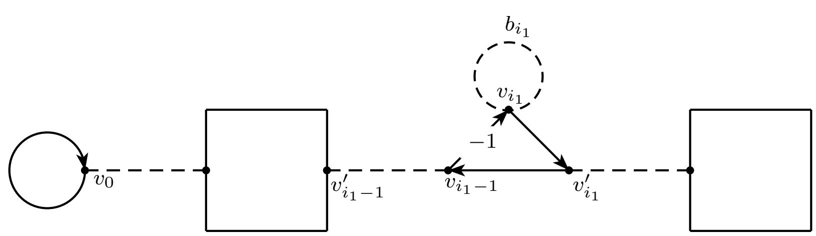

Figure 3.3.15

This implies that $P(\mathcal{H}_{i_1}) = P(\mathcal{H}_{i_1-1}) \cup v_{i_1}[0]$ spreads to $v_{i_1}[*]$, and hence $P(\mathcal{H}_{i_1}) = P(\mathcal{H}_{i_1-1}) \cup v_{i_1}[0]$ is a plague on $\Sigma_{\mathcal{H}_{i_1}}$. Therefore $p_{i_1}N = p_{i_1-1}N + 1 \leq \frac{(n_{i_1-1}+2)N+4}{4}$, and for $N \geq 5$,

$$imm(\Sigma_{\mathcal{H}_{i_1}}) \leq \frac{p_{i_1}N}{n_{i_1}N} = \frac{p_{i_1-1}N+1}{(n_{i_1}+3)N} \leq \frac{(n_{i_1-1}+2)N+4}{4(n_{i_1-1}+3)N} < 1/4.$$

Now, by Lemma 3.2.1, $imm(\Sigma_{\mathcal{H}_j}) \leq 1/4$ for all $i_1 < j \leq t$ and hence $imm(\Sigma) \leq 1/4$.

□

Proposition 3.3.29. *Let Σ be a covering with simply intersecting cycles of a finite homogeneous $PSL(2,\mathbb{Z})$-space $\overline{\Sigma}$. Let $\mathcal{G}$ be the pointed Schreier graph of $\overline{\Sigma}$ such that $V_x \neq \emptyset$, $V_{xy} = \emptyset$, $|V_y| \geq 2$ and a distinguished vertex $v_0 \in V_x$. Then $imm(\Sigma) \leq 1/4$.*

Proof. Let t be the number of triangles of $\mathcal{G}$ and $\mathcal{H}_0 \prec_{m_1} \mathcal{H}_1 \prec_{m_2} \dots \prec_{m_t} \mathcal{H}_t$ is a finite sequence of robust subgraphs $\mathcal{H}_j$ of $\mathcal{G}$. Let $V_y = \{v_{i_1}, v_{i_2}, \dots, v_{i_k}\}$, where $v_{i_k} \in \mathcal{H}_{i_k} \setminus \mathcal{H}_{i_k-1}$ for positive integers $i_1, i_2, \dots, i_k$ with $i_1 < i_2 < \dots < i_k$. By Remark 3.3.26(3), we have:

$$imm(\Sigma_{\mathcal{H}_{i_1-1}}) \leq \frac{p_{i_1-1}}{n_{i_1-1}} \leq \frac{n_{i_1-1}+2}{4n_{i_1-1}}.$$

Therefore $p_{i_1-1} \leq \frac{n_{i_1-1}+2}{4}$. Now we have $\mathcal{H}_{i_1-1} \prec_{m_{i_1}} \mathcal{H}_{i_1}$ with $m_{i_1} \in \{1, 2\}$. If $m_{i_1} = 2$, then by Lemma 3.3.28, $imm(\Sigma_{\mathcal{H}_{i_1}}) \leq 1/4$. Therefore, by Lemma 3.2.1, $imm(\Sigma_{\mathcal{H}_j}) \leq 1/4$ for all $i_1 < j \leq t$ and hence $imm(\Sigma) \leq 1/4$.

If $m_{i_1} = 2$, then $n_{i_1} = n_{i_1-1} + 1$ and $P(\mathcal{H}_{i_1}) = P(\mathcal{H}_{i_1-1})$. Therefore $p_{i_1} = p_{i_1-1}$ and

$$imm(\Sigma_{\mathcal{H}_{i_1}}) \leq \frac{p_{i_1}}{n_{i_1}} \leq \frac{n_{i_1-1}+2}{4(n_{i_1-1}+1)}.$$

This implies that $p_{i_1} = p_{i_1-1} \leq \frac{n_{i_1-1}+2}{4}$. Next by induction it follows that

$$imm(\Sigma_{\mathcal{H}_{i_2-1}}) \leq \frac{n_{i_1-1}+2}{4(n_{i_1-1}+1)}.$$

Therefore, $p_{i_2-1} \leq \frac{n_{i_1-1}+2}{4}$. Now if $\mathcal{H}_{i_2-1} \prec_2 \mathcal{H}_{i_1}$, then by Lemma 3.3.28, it follows that $imm(\Sigma_{\mathcal{H}_{i_2}}) \leq 1/4$ and hence, by Lemma 3.2.1, $imm(\Sigma_{\mathcal{H}_j}) \leq 1/4$ for all $i_1 < j \leq t$.

Suppose $\mathcal{H}_{i_2-1} \prec_1 \mathcal{H}_{i_1}$. Then it is possible to choose $i_2 = i_1 + 1$ so that we have the sequence $\mathcal{H}_{i_1-1} \prec_1 \mathcal{H}_{i_1} \prec_1 \mathcal{H}_{i_2} = \mathcal{H}_{i_1-1} \prec_1 \mathcal{H}_{i_2-1} \prec_1 \mathcal{H}_{i_2}$. This implies that $n_{i_2} = n_{i_1} + 1 = n_{i_1-1} + 2$ and $P(\mathcal{H}_{i_2}) = P(\mathcal{H}_{i_1}) = P(\mathcal{H}_{i_1-1})$, that is, $p_{i_2} = p_{i_1-1}$. Therefore, $imm(\Sigma_{\mathcal{H}_{i_2}}) \leq \frac{p_{i_2}}{n_{i_2}} = \frac{p_{i_1-1}}{n_{i_1-1}+2} \leq \frac{n_{i_1-1}+2}{4(n_{i_1-1}+2)} = 1/4$. Hence, by Lemma 3.2.1, $imm(\Sigma_{\mathcal{H}_j}) \leq 1/4$ for all $i_1 < j \leq t$ and hence $imm(\Sigma) \leq 1/4$.

□

Example 3.3.30. For the pointed Schreier graph $\mathcal{G}$ in Figure 3.3.16, $P = v_0[*] \cup v_1[*] \cup v_2[*] \cup v_3[*]$ is a plague on the covering Σ of $\mathcal{G}$ with simply intersecting cycles. Therefore, $imm(\Sigma) \leq \frac{4N}{16N} = 1/4$.

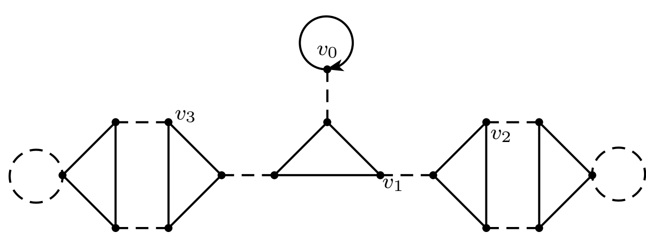

Figure 3.3.16

3.3.6 Case 6. Pointed Schreier Graphs with $V_x \neq \emptyset \neq V_{xy}$, $V_y = \emptyset$

In this section we study the coverings Σ with simply intersecting cycles of the finite homogeneous $PSL(2,\mathbb{Z})$-spaces $\overline{\Sigma}$ such that the pointed Schreier graphs $\mathcal{G}$ of $\overline{\Sigma}$ are with $V_x \neq \emptyset \neq V_{xy}$ and $V_y = \emptyset$. For the pointed Schreier graphs $\mathcal{G}$ with $V_x \neq \emptyset \neq V_{xy}$ and $V_y = \emptyset$ we consider $v_0 \in V_x$ as a distinguished vertex. Therefore the robust subgraph $\mathcal{H}_j$ of $\mathcal{G}$ will be with $v_0 \in V_x$ for any $0 \leq j \leq t$, where t is the number of triangles of $\mathcal{G}$.

Remark 3.3.31. We write $V_{xy} = \{v_{i_1}, v_{i_2}, ..., v_{i_k}\}$, where $v_{i_k} \in \mathcal{H}_{i_k} \setminus \mathcal{H}_{i_k-1}$ for positive integers $i_1, i_2, ..., i_k$ with $i_1 < i_2 < ... < i_k$. If $V_x \neq \emptyset \neq V_{xy}$ and $V_y = \emptyset$, then $V_{xy} \cap \mathcal{H}_{i_1-1} = \emptyset$. Therefore, by Proposition 3.3.2,

$$imm(\mathcal{H}_{i_1-1}) \leq \begin{cases} \frac{n_{i_1-1}+2}{4n_{i_1-1}}, & \text{if there is no } \mathcal{H}_i \text{ for } 2 \leq i \leq i_1-1 \text{ such that } \mathcal{H}_i \prec_1 \mathcal{H}_{i+1}, \\ 1/4, & \text{if there is a } \mathcal{H}_i \text{ for } 2 \leq i \leq i_1-1 \text{ such that } \mathcal{H}_i \prec_1 \mathcal{H}_{i+1}. \end{cases}$$

Note that $\mathcal{H}_{i_k-1} \prec_{m_{i_k-1}} \mathcal{H}_{i_k-1} \prec_1 \mathcal{H}_{i_k}$ for all i_k, where $m_{i_k-1} \in \{1,2\}$.

Now we discuss the coverings Σ of $\overline{\Sigma}$ whose pointed Schreier graphs $\mathcal{G}$ are with $V_x \neq \emptyset \neq V_{xy}$ and $V_y = \emptyset$ such that $imm(\Sigma) \leq 1/4$.

Proposition 3.3.32. *Let Σ be a covering with simply intersecting cycles of finite homogeneous $PSL(2,\mathbb{Z})$-space $\overline{\Sigma}$. Let $\mathcal{G}$ be the pointed Schreier graph of $\overline{\Sigma}$ with $V_x \neq \emptyset$, $V_y = \emptyset$ and $V_{xy} = \{v_{i_1}, v_{i_1}, ..., v_{i_k}\}$, such that $2 \leq i_1 < i_2 < ... < i_k \leq t$ and $v_{i_k} \in \mathcal{H}_{i_k} \setminus \mathcal{H}_{i_k-1}$ for $k \geq 1$. Let $\mathcal{H}_{i_1-2} \prec_1 \mathcal{H}_{i_1-1} \prec_1 \mathcal{H}_{i_1}$. Assume further that there exists al least one robust subgraph $\mathcal{H}_i$ for $2 \leq i < i_1 - 2$ such that $\mathcal{H}_{i-1} \prec_1 \mathcal{H}_i$. Then $imm(\Sigma) \leq 1/4$.*

Proof. Let t be the number of triangles of $\mathcal{G}$ and $\mathcal{H}_0 \prec_{m_1} \mathcal{H}_1 \prec_{m_2} ... \prec_{m_t} \mathcal{H}_t$ is a finite sequence of robust subgraphs $\mathcal{H}_j$ of $\mathcal{G}$. Then $V_{xy} \cap \mathcal{H}_{i_1-2} = \emptyset$. Since $\mathcal{H}_{i-1} \prec_1 \mathcal{H}_i$ for $2 \leq i < i_1 - 2$, therefore by Remark 3.3.31, we have

$$imm(\Sigma_{\mathcal{H}_{i_1-2}}) \leq \frac{p_{i_1-2}}{n_{i_1-2}} \leq 1/4.$$

Therefore $p_{i_1-2} \leq \frac{n_{i_1-2}}{4}$. Now since $\mathcal{H}_{i_1-2} \prec_1 \mathcal{H}_{i_1-1} \prec_1 \mathcal{H}_{i_1}$, we have $n_{i_1} = n_{i_1-2} + 4$ and $p_{i_1} = p_{i_1-2} + 1$. Therefore,

$$imm(\Sigma_{\mathcal{H}_{i_1}}) \leq \frac{p_{i_1}}{n_{i_1}} = \frac{p_{i_1-2}+1}{n_{i_1-1}+2} \leq \frac{\frac{n_{i_1-2}}{4}+1}{n_{i_1-2}+4} = 1/4.$$

Now if $i_1 = t$, then we are done. If $i_1 < t$, then by Lemma 3.2.1 and Remark 3.3.19, we have $imm(\Sigma_{\mathcal{H}_j}) \leq 1/4$ for all $i_1 < j \leq t$. Hence $imm(\Sigma) \leq 1/4$.

□

Example 3.3.33. For the pointed Schreier graph $\mathcal{G}$ in Figure 3.3.17, $P = v_0[*] \cup v_1[*] \cup v_2[*] \cup v_3[*]$ is a plague on the covering Σ of $\mathcal{G}$ with simply intersecting cycles. Therefore, $imm(\Sigma) \leq \frac{4N}{16N} = 1/4$.

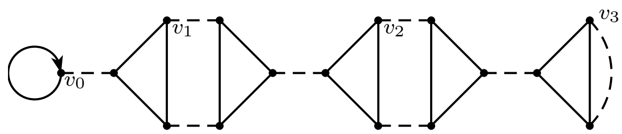

Figure 3.3.17

Proposition 3.3.34. *Let Σ be a covering with simply intersecting cycles of finite homogeneous $PSL(2,\mathbb{Z})$-space $\overline{\Sigma}$. Let $\mathcal{G}$ be the pointed Schreier graph of $\overline{\Sigma}$ with $V_x \neq \emptyset$, $V_y = \emptyset$ and $V_{xy} = \{v_{i_1}, v_{i_1}, ..., v_{i_k}\}$, such that $2 \leq i_1 < i_2 < ... < i_k \leq t$, $i_2 \neq i_1 + 1$ and $v_{i_k} \in \mathcal{H}_{i_k} \setminus \mathcal{H}_{i_k - 1}$ for $k \geq 1$. Let $\mathcal{H}_{i_1-2} \prec_2 \mathcal{H}_{i_1-1} \prec_1 \mathcal{H}_{i_1}$. Then $imm(\Sigma) \leq 1/4$.*

Proof. Let t be the number of triangles of $\mathcal{G}$ and $\mathcal{H}_0 \prec_{m_1} \mathcal{H}_1 \prec_{m_2} ... \prec_{m_t} \mathcal{H}_t$ is a finite sequence of robust subgraphs $\mathcal{H}_j$ of $\mathcal{G}$. Then $V_{xy} \cap \mathcal{H}_{i_1-2} = \emptyset$. Therefore by Remark 3.3.31, we have

$$imm(\Sigma_{\mathcal{H}_{i_1-2}}) \leq \tfrac{p_{i_1-2}}{n_{i_1-2}} \leq \tfrac{n_{i_1-2}+2}{4n_{i_1-2}}.$$

Therefore $p_{i_1-2} \leq \frac{n_{i_1-2}+2}{4}$. Since $\mathcal{H}_{i_1-2} \prec_2 \mathcal{H}_{i_1-1} \prec_1 \mathcal{H}_{i_1}$, $n_{i_1} = n_{i_1-2} + 6$, and $p_{i_1} = p_{i_1-2} + 1$ by Remark 3.3.19. Therefore,

$$imm(\Sigma_{\mathcal{H}_{i_1}}) \leq \tfrac{p_{i_1}}{n_{i_1}} = \tfrac{p_{i_1-2}+1}{n_{i_1-2}+6} \leq \tfrac{\frac{n_{i_1-2}}{4}+1}{n_{i_1-2}+6} < 1/4.$$

Now if $i_1 = t$, then we are done. If $i_1 < t$, then by Lemma 3.2.1 and Remark 3.3.19, we have $imm(\Sigma_{\mathcal{H}_j}) \leq 1/4$ for all $i_1 < j \leq t$. Hence $imm(\Sigma) \leq 1/4$.

□

Example 3.3.35. For the pointed Schreier graph $\mathcal{G}$ in Figure 3.3.18, $P = v_0[*] \cup v_1[*] \cup v_2[*] \cup v_3[*]$ is a plague on the covering Σ of $\mathcal{G}$ with simply intersecting cycles. Therefore,

$$imm(\Sigma) \leq \tfrac{4N}{16N} = 1/4.$$

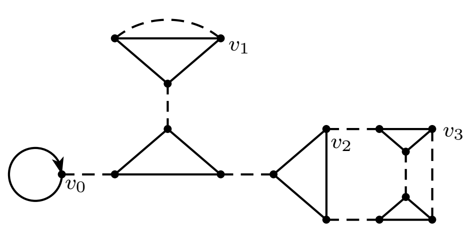

Figure 3.3.18

Remark 3.3.36. Suppose that $\mathcal{G}$ be a pointed Schreier graph of $\overline{\Sigma}$ with $V_x \neq \emptyset$, $V_y = \emptyset$ and $V_{xy} = \{v_{i_1}, v_{i_1}, ..., v_{i_k}\}$, such that $2 \leq i_1 < i_2 < ... < i_k \leq t$, $i_2 = i_1 + 1$, $v_{i_k} \in \mathcal{H}_{i_k} \setminus \mathcal{H}_{i_k - 1}$ for $k \geq 1$, and $\mathcal{H}_{i_1-2} \prec_2 \mathcal{H}_{i_1-1} \prec_1 \mathcal{H}_{i_1}$. Then it is easy to see, for example in Figure 3.3.19, that $n_{i_2} \leq n_{i_1-2}+12$ and $p_{i_2} = p_{i_1-2}+3$, where $p_{i_1-2} \leq \frac{n_{i_1-2}+2}{4}$. Therefore,

$$imm(\Sigma_{\mathcal{H}_{i_2}}) \leq \frac{p_{i_2}}{n_{i_2}} = \frac{p_{i_1-2}+3}{n_{i_1-2}+12} \leq \frac{\frac{n_{i_1-2}+2}{4}+3}{n_{i_1-2}+12} = \frac{n_{i_1-2}+14}{4(n_{i_1-2}+12)} \nleq 1/4.$$

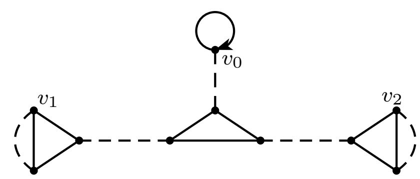

Figure 3.3.19

3.3.7 Case 7. Pointed Schreier Graphs with $V_x \neq \emptyset$, $V_y \neq \emptyset \neq V_{xy}$

In this section we study the coverings Σ with simply intersecting cycles of finite homogeneous $PSL(2,\mathbb{Z})$-space $\overline{\Sigma}$ such that the pointed Schreier graphs $\mathcal{G}$ of $\overline{\Sigma}$ are with $V_x \neq \emptyset$ and $V_y \neq \emptyset \neq V_{xy}$. For the pointed Schreier graphs $\mathcal{G}$ with $V_x \neq \emptyset$ and $V_y \neq \emptyset \neq V_{xy}$ we will consider $v_0 \in V_x$ as a distinguished vertex. Therefore the robust subgraph $\mathcal{H}_j$ of $\mathcal{G}$ will be with $v_0 \in V_x$ for any $0 \leq j \leq t$, where t is the number of triangles of $\mathcal{G}$. The plague on $\mathcal{H}_j$ is $v_0[*] \cup P_J \cup V_{xy}[*]$. We write $V_y = \{v_{i_1}, v_{i_1}, ..., v_{i_k}\}$, such that $1 \leq i_1 < i_2 < ... < i_k \leq t$ for $k \geq 1$, and $V_{xy} = \{v_{j_1}, v_{j_2}, ..., v_{j_{k'}}\}$, such that $2 \leq j_1 < j_2 < ... < j_{k'} \leq t$ for $k' \geq 1$.

Proposition 3.3.37. *Let Σ be a covering with simply intersecting cycles of finite homogeneous $PSL(2,\mathbb{Z})$-space $\overline{\Sigma}$. Let $\mathcal{G}$ be the pointed Schreier graph of $\overline{\Sigma}$ with $V_x \neq \emptyset$, $V_y = \{v_{i_1}, v_{i_2}, ..., v_{i_k}\}$ for $k \geq 1$, and $V_{xy} = \{v_{j_1}, v_{j_2}, ..., v_{j_{k'}}\}$ where $2 \leq j_1 < j_2 < ... < j_{k'} \leq t$ for $k' \geq 1$. Assume further that $\mathcal{G}$ contains at least one robust subgraph $\mathcal{H}_i$ for $2 \leq i \leq j_1 - 1$ such that $\mathcal{H}_{i-1} \prec_1 \mathcal{H}_i$ and $V_y \cap (\mathcal{H}_i \setminus \mathcal{H}_{i-1}) = \emptyset$. Then $imm(\Sigma) \leq 1/4$.*

Proof. Since there exists at least one robust subgraph $\mathcal{H}_i$ for $2 \leq i \leq j_1 - 1$ such that $\mathcal{H}_{i-1} \prec_1 \mathcal{H}_i$ and $V_y \cap (\mathcal{H}_i \setminus \mathcal{H}_{i-1}) = \emptyset$, therefore by Propositions 3.3.2, 3.3.29, and Lemmas 3.3.27, 3.3.28, $imm(\Sigma_{\mathcal{H}_{j_1-2}}) \leq \frac{p_{j_1-2}}{n_{j_1-2}} \leq 1/4$. This implies that $p_{j_1-2} \leq \frac{n_{j_1-2}}{4}$. Now we have $\mathcal{H}_{j_1-2} \prec_{m_{j_1-1}} \mathcal{H}_{j_1-1} \prec_1 \mathcal{H}_{j_1}$ with $m_{j_1-1} \in \{1,2\}$. If $\mathcal{H}_{j_1-2} \prec_1 \mathcal{H}_{j_1-1} \prec_1 \mathcal{H}_{j_1}$, then $n_{j_1} = n_{j_1-2} + 4$ and $p_{j_1} = p_{j_1-2} + 1$ since $P(\Sigma_{\mathcal{H}_{j_1-2}}) \cup v_{j_1}[*]$ is a plague on $\Sigma_{\mathcal{H}_{j_1}}$. Therefore,

$$imm(\Sigma_{\mathcal{H}_{j_1}}) \leq \frac{p_{j_1}}{n_{j_1}} = \frac{p_{j_1-2}+1}{n_{j_1-2}+4} \leq \frac{\frac{n_{j_1-2}}{4}+1}{n_{j_1-2}+4} = 1/4.$$

If $\mathcal{H}_{j_1-2} \prec_1 \mathcal{H}_{j_1-1} \prec_1 \mathcal{H}_{j_1}$, then $n_{j_1} \leq n_{j_1-2} + 6$. Now consider Figure 3.3.20, where $v_{j_1} \in V_{xy}$ such that $v_{j_1} \in \mathcal{H}_{j_1} \setminus \mathcal{H}_{j_1-1}$ and $\mathcal{H}_{j_1-2} \prec_2 \mathcal{H}_{j_1-1} \prec_1 \mathcal{H}_{j_1}$.

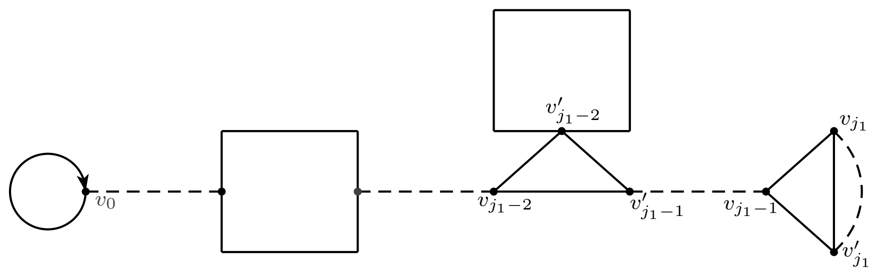

Figure 3.3.20

Now $P(\Sigma_{\mathcal{H}_{j_1-2}}) \cup v_{j_1}[*]$ is a plague on $P(\Sigma_{\mathcal{H}_{j_1}})$ by the following table.

pivot	$v'_{j_1}[*]$	$v'_{j_1-1}[*]$	$v_{j_1-2}[*]$	$v_{j_1-1}[*]$
	$v_{j_1-1}[*]$	$v''_{j_1-1}[*]$	$v'_{j_1-1}[*]$	$v'_{j_1}[*]$

Therefore, $p_{j_1} = p_{j_1-2} + 1 \leq \frac{n_{j_1-2}}{4} + 1 = \frac{n_{j_1-2}+4}{4}$. Thus,

$$imm(\mathcal{H}_{j_1}) \leq \frac{p_{j_1}}{n_{j_1}} \leq \frac{n_{j_1-2}+4}{(n_{j_1-2}+6)} \leq 1/4.$$

Now if $j_1 = t$, then we are done. If $j_1 < t$, then by Lemma 3.2.1 and Remark 3.3.19, $imm(\mathcal{H}_j) \leq 1/4$ for $j_1 < j \leq t$. Hence $imm(\Sigma) < 1/4$. □

Example 3.3.38. For the pointed Schreier graph $\mathcal{G}$ in Figure 3.3.21, $P = v_0[*] \cup v_1[*] \cup v_2[*]$ is a plague on the covering Σ of $\mathcal{G}$ with simply intersecting cycles. Therefore,

$$imm(\Sigma) \leq \frac{3N}{12N} = 1/4.$$

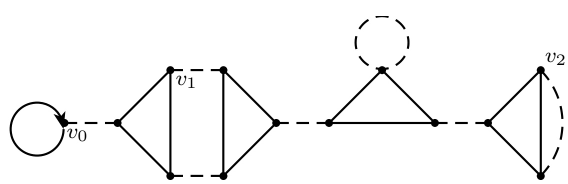

Figure 3.3.21

Proposition 3.3.39. *Let Σ be a covering with simply intersecting cycles of finite homogeneous $PSL(2,\mathbb{Z})$-space $\overline{\Sigma}$. Let $\mathcal{G}$ be the pointed Schreier graph of $\overline{\Sigma}$ with $V_x \neq \emptyset$, $V_y = \{v_{i_1}, v_{i_2}, ..., v_{i_k}\}$ and $V_{xy} = \{v_{j_1}, v_{j_2}, ..., v_{j_{k'}}\}$ where $1 \leq i_1 < i_2 < ... < i_k \leq t$ and $2 \leq j_1 < j_2 < ... < j_{k'} \leq t$ for $k \geq 1$ and $k' \geq 1$. Assume further that $\mathcal{H}_{i_1-1} \prec_2 \mathcal{H}_{i_1}$ and $j_1 \neq i_1 + 1$. Then $imm(\Sigma) \leq 1/4$.*

Proof. If $\mathcal{H}_{i_1-1} \prec_2 \mathcal{H}_{i_1}$ and $j_1 \neq i_1 + 1$. Then, by Lemma 3.3.28, $imm(\Sigma_{\mathcal{H}_{i_1}}) \leq 1/4$. Since $j_1 \neq i_1 + 1$, therefore by Lemma 3.2.1 and Remark 3.3.19, $imm(\Sigma_{\mathcal{H}_{j_1}}) \leq 1/4$. Now if $j_1 = t$, then we are done. If $j_1 < t$ then again by Lemma 3.2.1 and Remark 3.3.19, $imm(\mathcal{H}_j) \leq 1/4$ for $j_1 < j \leq t$. Hence $imm(\Sigma) < 1/4$. ☐

Example 3.3.40. For the pointed Schreier graph $\mathcal{G}$ in Figure 3.3.22, $P = v_0[*] \cup v_1[0] \cup v_2[*] \cup v_3[*]$ is a plague on the covering Σ of $\mathcal{G}$ with simply intersecting cycles. Therefore, $imm(\Sigma) \leq \frac{3N+1}{13N} \leq 1/4$, for $N \geq 5$.

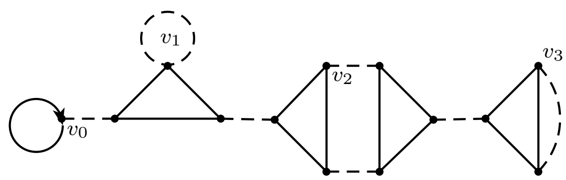

Figure 3.3.22

Proposition 3.3.41. *Let Σ be a covering with simply intersecting cycles of finite homogeneous $PSL(2,\mathbb{Z})$-space $\overline{\Sigma}$. Let $\mathcal{G}$ be the pointed Schreier graph of $\overline{\Sigma}$ with $V_x \neq \emptyset$, $V_y = \{v_{i_1}, v_{i_2}, ..., v_{i_k}\}$ and $V_{xy} = \{v_{j_1}, v_{j_2}, ..., v_{j_{k'}}\}$ where $1 \leq i_1 < i_2 < ... < i_k \leq t$ and $2 \leq j_1 < j_2 < ... < j_{k'} \leq t$ for $k \geq 1$ and $k' \geq 1$. Assume further that $\mathcal{H}_{j_1-2} \prec_2 \mathcal{H}_{j_1-1}$ and $j_2 \neq j_1 + 1$. Then $imm(\Sigma) \leq 1/4$.*

Proof. If $\mathcal{H}_{j_1-2} \prec_2 \mathcal{H}_{j_1-1}$ and $j_2 \neq j_1 + 1$, then by Lemma , $imm(\Sigma_{\mathcal{H}_{j_1}}) \leq 1/4$. Now by Lemma 3.2.1 and Remark 3.3.19, $imm(\Sigma_{\mathcal{H}_{i_1}}) \leq 1/4$. Now if $i_1 = t$, then we are done. If $i_1 < t$ then again by Lemma 3.2.1 and Remark 3.3.19, $imm(\mathcal{H}_j) \leq 1/4$ for $i_1 < j \leq t$. Hence $imm(\Sigma) < 1/4$. ☐

Example 3.3.42. For the pointed Schreier graph $\mathcal{G}$ in Figure 3.3.23, $P = v_0[*] \cup v_1[*] \cup v_2[*]$ is a plague on the covering Σ of $\mathcal{G}$ with simply intersecting cycles. Therefore,

$$imm(\Sigma) \leq \frac{4N}{16N} = 1/4.$$

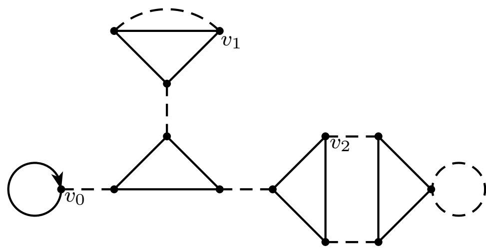

Figure 3.3.23

3.3.8 Case 8. Pointed Schreier Graphs with $V_x = \emptyset$, $V_y \neq \emptyset \neq V_{xy}$

In this section we study the coverings Σ with simply intersecting cycles of finite homogeneous $PSL(2, \mathbb{Z})$-space $\overline{\Sigma}$ such that the pointed Schreier graphs $\mathcal{G}$ of $\overline{\Sigma}$ are with $V_x = \emptyset$ and $V_y \neq \emptyset \neq V_{xy}$. For such pointed Schreier graphs we consider $v_0 \in V_y$ as a distinguished vertex. Therefore the robust subgraph $\mathcal{H}_j$ of $\mathcal{G}$ will be with $v_0 \in V_y$ for any $0 \leq j \leq t$, where t is the number of triangles of $\mathcal{G}$. The plague on $\mathcal{H}_j$ is $v_0[*] \cup P_J \cup V_{xy}[*]$. We write $V_y = \{v_0, v_{i_1}, v_{i_1}, ..., v_{i_k}\}$, such that $1 < i_1 < i_2 < ... < i_k \leq t$ for $k \geq 1$, and $V_{xy} = \{v_{j_1}, v_{j_2}, ..., v_{j_{k'}}\}$, such that $2 \leq j_1 < j_2 < ... < j_{k'} \leq t$ for $k' \geq 1$.

Proposition 3.3.43. *Let Σ be a covering with simply intersecting cycles of finite homogeneous $PSL(2, \mathbb{Z})$-space $\overline{\Sigma}$. Let $\mathcal{G}$ be the pointed Schreier graph of $\overline{\Sigma}$ with $V_x = \emptyset$, $V_y = \{v_0\}$ and $V_{xy} = \{v_{j_1}, v_{j_2}, ..., v_{j_{k'}}\}$ such that $2 \leq j_1 < j_2 < ... < j_{k'} \leq t$ for $k' \geq 1$. Assume that $\mathcal{G}$ contains least two distinct robust subgraphs $\mathcal{H}_i$ and $\mathcal{H}_{i'}$ for $2 \leq i < j_1 - 1$ and $2 \leq i' < j_1 - 1$ such that $\mathcal{H}_{i-1} \prec_1 \mathcal{H}_i$ and $\mathcal{H}_{i'-1} \prec_1 \mathcal{H}_{i'}$. Then $imm(\Sigma) < 1/4$.*

Proof. By Lemma 3.3.10 and Proposition 3.3.12, $imm(\mathcal{H}_{j_1-1}) \leq 1/4$. Now we have $\mathcal{H}_{j_1-2} \prec_{m_{j_1-1}} \mathcal{H}_{j_1-1} \prec_1 \mathcal{H}_{j_1}$ with $m_{j_1-1} \in \{1, 2\}$. In both cases, $imm(\mathcal{H}_{j_1}) \leq 1/4$, by Lemma 3.2.1 and Remark 3.3.19,. Now if $j_1 = t$, then we are done. If $j_1 < t$ then again by Lemma 3.2.1 and Remark 3.3.19, $imm(\mathcal{H}_j) \leq 1/4$ for $j_1 < j \leq t$. Hence $imm(\Sigma) < 1/4$. ☐

Example 3.3.44. For the pointed Schreier graph $\mathcal{G}$ in Figure 3.3.24, $P = v_0[*] \cup v_1[*] \cup v_2[*] \cup v_3[*] \cup v_4[*]$ is a plague on the covering Σ of $\mathcal{G}$ with simply intersecting cycles. Therefore, $imm(\Sigma) \leq \frac{5N}{21N} < 1/4$.

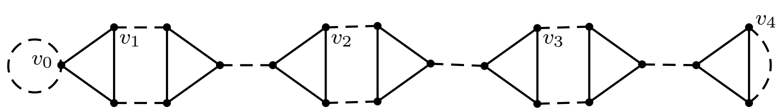

Figure 3.3.24

Proposition 3.3.45. *Let Σ be a covering with simply intersecting cycles of finite homogeneous $PSL(2,\mathbb{Z})$-space $\overline{\Sigma}$. Let $\mathcal{G}$ be the pointed Schreier graph of $\overline{\Sigma}$ with $V_x = \emptyset$, $V_y = \{v_0, v_{i_1}, v_{i_2}, ..., v_{i_k}\}$ for $k \geq 1$, and $V_{xy} = \{v_{j_1}, v_{j_2}, ..., v_{j_{k'}}\}$ such that $2 \leq j_1 < j_2 < ... < j_{k'} \leq t$ for $k' \geq 1$. Let $\mathcal{H}_{j_1-2} \prec_2 \mathcal{H}_{j_1-1}$. Assume further that $\mathcal{G}$ contains at least one robust subgraph $\mathcal{H}_i$ for $2 \leq i \leq j_1 - 1$ such that $\mathcal{H}_{i-1} \prec_1 \mathcal{H}_i$. Then $imm(\Sigma) \leq 1/4$.*

Proof. Since there exists at least one robust subgraph $\mathcal{H}_i$ for $2 \leq i \leq j_1 - 1$ such that $\mathcal{H}_{i-1} \prec_1 \mathcal{H}_i$, therefore by Lemma 3.3.10, $imm(\Sigma_{\mathcal{H}_{j_1-2}}) \leq \frac{p_{j_1-2}}{n_{j_1-2}} \leq \frac{n_{j_1-2}+1}{4n_{j_1-2}}$. Hence $p_{j_1-2} \leq \frac{n_{j_1-2}+1}{4}$. Now consider the following Figure 3.3.25, where $v_{j_1} \in V_{xy}$ such that $v_{j_1} \in \mathcal{H}_{j_1} \setminus \mathcal{H}_{j_1-1}$ and $\mathcal{H}_{j_1-2} \prec_2 \mathcal{H}_{j_1-1} \prec_1 \mathcal{H}_{j_1}$.

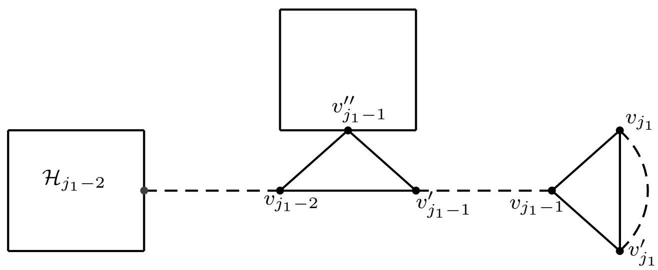

Figure 3.3.25

Now $P(\Sigma_{\mathcal{H}_{j_1-2}}) \cup v_{j_1}[*]$ is a plague on $P(\Sigma_{\mathcal{H}_{j_1}})$ by following table.

pivot	$v'_{j_1}[*]$	$v'_{j_1-1}[*]$	$v_{j_1-2}[*]$	$v_{j_1-1}[*]$
	$v_{j_1-1}[*]$	$v''_{j_1-1}[*]$	$v'_{j_1-1}[*]$	$v'_{j_1}[*]$

Therefore, $p_{j_1} = p_{j_1-2} + 1 \leq \frac{n_{j_1-2}+1}{4} + 1 = \frac{n_{j_1-2}+5}{4}$, and $n_{j_1} \leq n_{j_1-2} + 6$. Thus, $imm(\mathcal{H}_{j_1}) \leq \frac{p_{j_1}}{n_{j_1}} \leq \frac{n_{j_1-2}+5}{(n_{j_1-2}+6)} \leq 1/4$. Now if $j_1 = t$, then we are done. If $j_1 < t$, then by Lemma 3.2.1 and Remark 3.3.19, $imm(\mathcal{H}_j) \leq 1/4$ for $j_1 < j \leq t$. Hence $imm(\Sigma) \leq 1/4$. □

Example 3.3.46. For the pointed Schreier graph $\mathcal{G}$ in Figure 3.3.26, $P = v_0[*] \cup v_1[*] \cup v_2[*]$ is a plague on the covering Σ of $\mathcal{G}$ with simply intersecting cycles. Therefore, $imm(\Sigma) \leq \frac{3N}{12N} = 1/4$.

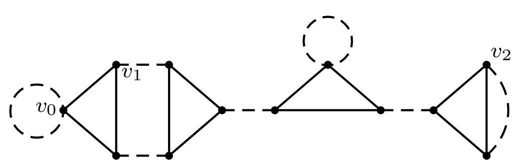

Figure 3.3.26

Conclusion

In this thesis we have studied two conjectures, namely, the Hayashi's Conjecture 1.4.4 on quandles, and a Conjecture 2.7.4 on Hurwitz orbits. The Hayashi's conjecture says that all permutations of an indecomposable quandle has cycles whose cycle lengths divide the largest among them. We first observed (see Observations 1.4.7) that for which known families of connected quandles the Hayashi's conjecture is true. Similarly, one can see wether the Hayashi's conjecture is true for other families of connected quandles, like primitive quandles. We have also provided the obstructions on the profiles of connected quandles (see Section 1.4.2). By these obstructions we have shown that the Hayashi's conjecture is true for all connected quandles with profile $1^{m_0}\ell_1^{m_1}\ell_2^{m_2}$ and also for all connected crossed sets with profile $1^{m_0}\ell_1\ell_2\ell_3$. By Observations 1.4.7 and the obstructions on the profiles of connected quandles in Section 1.4.2, it seems that the Hayashi's conjecture will not be true for all connected racks and quandles.

The Conjecture 2.7.4 on Hurwitz orbits says that the immunity on any Hurwitz orbit Σ with simply intersecting cycles is bounded above by the weight on Σ, that is, $imm(\Sigma) \leq \omega(\Sigma)$. It is known, by Theorem 2.7.2 of [22], that $imm(\Sigma) \leq \omega(\Sigma)$ for coverings Σ of few finite homogeneous $PSL(2,\mathbb{Z})$-spaces $\overline{\Sigma}$ with any $xy-$cycle of length at most 4. In Chapter 3 we have shown that $imm(\Sigma) \leq \omega(\Sigma)$ for coverings Σ of infinitely many Schreier graphs of finite homogeneous $PSL(2,\mathbb{Z})$-spaces with all $xy-$cycles. Our method is based on the posets of robust subgraphs of pointed Schreier graphs $\mathcal{G}$ (see Section 2.7.4) of finite homogeneous $PSL(2,\mathbb{Z})$-spaces. By using this method we did a case-by-case analysis of infinitely many pointed Schreier graphs $\mathcal{G}$ of finite homogeneous $PSL(2,\mathbb{Z})$-spaces $\overline{\Sigma}$ and their coverings Σ with simply intersecting cycles. In Case 1 (Section 3.3.1) of this analysis we first described the method (in Lemma 3.3.1) to calculate the plague on a covering Σ of $PSL(2,\mathbb{Z})$-space $\overline{\Sigma}$ whose pointed Schreier graph $\mathcal{G}$ is with $V_x \neq \emptyset$ and $V_y = \emptyset = V_{xy}$. By using this plague we have shown (in Proposition 3.3.2), when $imm(\Sigma) \leq 1/4 \leq \omega(\Sigma)$. In the next step we searched for all those $\mathcal{G}$ which are not covered by Proposition 3.3.2. We observed (in Section 3.3.1.3) that all such $\mathcal{G}$ are generated or spanned by the robust subgraphs $\mathcal{H}_k$ and two fragments, namely, $\mathcal{F}_1 = \mathcal{G}_{10\{10\}} \setminus \mathcal{H}_0(\mathcal{G}_{10\{10\}})$ and $\mathcal{F}_2 = \mathcal{G}_{10\{10\}} \setminus \mathcal{H}_0(\mathcal{G}_{10\{5,\,3,\,2\}})$. Finally in Case 1, we showed that $imm(\Sigma) \leq 1/4 \leq \omega(\Sigma)$ for any covering Σ of $\mathcal{G} = Span(\mathcal{H}_k, \mathcal{F}_1, \mathcal{F}_2)$.

Hence in Case 1 we found that $imm(\Sigma) \leq 1/4 \leq \omega(\Sigma)$ for all coverings Σ with simply intersecting cycles of all $PSL(2,\mathbb{Z})$-spaces $\overline{\Sigma}$ such that the pointed Schreier graphs $\mathcal{G}$ is with $V_x \neq \emptyset$ and $V_y = \emptyset = V_{xy}$.

In Case 2 (Section 3.3.2) of our analysis, we first described the method (in Lemma 3.3.8) to calculate the plague on a covering Σ of $PSL(2,\mathbb{Z})$-space $\overline{\Sigma}$ whose pointed Schreier graph $\mathcal{G}$ is with $V_y \neq \emptyset$ and $V_x = \emptyset = V_{xy}$. By using this plague we have shown (in Lemma 3.3.8 and Proposition 3.3.12), when $imm(\Sigma) \leq 1/4 \leq \omega(\Sigma)$. In Case 3 (Section 3.3.3) of our analysis, we first described the method (in Lemma 3.3.15) to calculate the plague on a covering Σ of $PSL(2,\mathbb{Z})$-space $\overline{\Sigma}$ whose pointed Schreier graph $\mathcal{G}$ is with $V_{xy} \neq \emptyset$ and $V_x = \emptyset = V_y$. By using this plague we have shown (in Lemma 3.3.17 and Proposition 3.3.20), when $imm(\Sigma) \leq 1/4 \leq \omega(\Sigma)$. In Case 4 (Section 3.3.4) of our analysis, again we first described the method (in Lemma 3.3.22) to calculate the plague on a covering Σ of $PSL(2,\mathbb{Z})$-space $\overline{\Sigma}$ whose pointed Schreier graph $\mathcal{G}$ is with $V_x = V_y = V_{xy} = \emptyset$. By using this plague we have shown (in Proposition 3.3.24), when $imm(\Sigma) \leq 1/4 \leq \omega(\Sigma)$. In the remaining cases (Case 5-Case 8), we have used the results from Case 1-4 in order to show that when $imm(\Sigma) \leq 1/4 \leq \omega(\Sigma)$. Unlike the Case 1, it is difficult to generate all $\mathcal{G}$ in Cases 2-8. However, by our method it seems that the inequality $imm(\Sigma) \leq \omega(\Sigma)$ is true for any covering Σ with simply intersecting cycles of all $\mathcal{G}$ in Cases 2-8.

Zusammenfassung

In dieser Arbeit untersuchen wir Quandles und Hurwitz-Bahnen. Quandles sind selbst distributive algebraische Strukturen mit drei Axiomen, die mit drei Reidemeister-Bewegungen von Knotendiagrammen verwandt sind. Racks sind eine Verallgemeinerung von Quandles. Die Verknüpfung eines Quandles ist wie die Konjugation in einer Gruppe. Die algebraische Struktur von Quandles kann als Folgen von Permutationen untersucht werden. Die Zykel-Struktur der Permutationen eines unzerlegbaren Racks (bzw. Quandles) verhält sich gut, weil die Permutationen eines unzerlegbaren Racks (bzw. Quandles) zueinander konjugiert sind und daher die gleiche Zykel-Struktur haben. In [18], beobachtet C. Hayashi eine weitere interessante Eigenschaft der Zykel-Struktur eines unzerlegbaren Quandles und vermutet, dass die Permutation eines unzerlegbaren Quandles Zykel hat, deren Zykellängen die größte unter ihnen teilen. In Kapitel 1 dieser Arbeit untersuchen wir die Zykel-Struktur von Quandles mit dem Schwerpunkt auf der Vermutung von Hayashi. In Abschnitt 1.4.7 diskutieren wir die Klassen unzerlegbarer Quandles, für die die Vermutung von Hayashi wahr ist. In Abschnitt 1.4.2 geben wir Einschränkungen für die Zykel-Struktur bestimmter unzerlegbarer Quandles, die zu den wichtigsten Ergebnissen dieser Arbeit gehören.

Racks und die Wirkung der Zopfgruppe auf den Potenzen von Racks sind für die Klassifizierung bestimmter Hopfalgebren nützlich (siehe [2], [3] [15], [21], [22]). Hurwitz-Bahnen sind die Bahnen einer Wirkung der Zopfgruppe auf der Potenz eines Racks. Die Hurwitz–Bahnen für die Wirkung der Zopfgruppe auf drei Strängen werden in [21] und [22] für die Klassifizierung bestimmter Hopfalgebren verwendet. Diese Klassifizierung basiert auf einer kombinatorischen Invarianten der Hurwitz-Bahnen, die Plage genannt wird. Die Immunität einer Hurwitz-Bahn ist der Quotient aus der Größe der minimalen Plage und der Größe der Hurwitz-Bahn. Eine Abschätzung über die Immunität der Hurwitz-Bahnen wird in [22] gegeben mit Hilfe von markierten Schreier-Graphen der Hurwitz-Bahn-Quotienten und den Gewichten der Hurwitz-Bahnen, wobei das Gewicht einer Hurwitz-Bahn durch ihre Zykel-Struktur definiert ist. In Kapitel 2 erinnern wir an die Ergebnisse zu Hurwitz-Bahnen und das Verfahren zur Abschätzung der Immunität der Hurwitz-Bahnen aus [15], [21] und [22]. Man beachte, dass in [22] nur wenige Schreier-Graphen mit kleinen Zykeln betrachtet werden, für die gezeigt wird, dass die

Immunität der Hurwitz-Bahnen nach oben durch ihre Gewichte beschränkt ist.

In Kapitel 3 stellen wir eine neue Methode vor, um die Plage einer Hurwitz-Bahn zu berechen und die Immunität einer Hurwitz-Bahn abzuschätzen. Mit dieser Methode kann man unendlich viele Schreier-Graphen der Hurwitz-Bahn-Quotienten mit alle Zyklen studieren. Unsere Methode basiert auf den posets bestimmter Teilgraphen, genannt robuste Subgraphen, von punktierten Schreier-Graphen der Hurwitz-Bahn-Quotienten. Wir schätzen die Immunität der Hurwitz-Bahnen durch eine Fall-zu-Fall-Analyse von punktierten Schreier-Graphen der Hurwitz-Bahn-Quotienten. Mit dieser Analyse betrachten wir die Schreier-Graphen der Hurwitz-Bahn-Quotienten, für die die Immunität der Hurwitz-Bahn von oben durch ein Viertel beschränkt ist. Alle in Kapitel 1 und Kapitel 3 bewiesenen Resultate werden vom Autor als originell behauptet.

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Erklärung

Ich versichere, dass ich meine Dissertation

Quandles und Hurwitz-Bahnen

selbständig, ohne unerlaubte Hilfe angefertigt und mich dabei keiner anderen als der von mir ausdrücklich bezeichneten Quellen und Hilfen bedient habe.

Die Dissertation wurde in der jetzigen oder einer ähnlichen Form noch bei keiner anderen Hochschule eingereicht und hat noch keinen sonstigen Prüfungszwecken gedient.

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- Modular Group Acting on Real Quadratic Fields, on 5.1.2008.

at http://www.uni-marburg.de/fb12/mathematik/arbeitsgebiete/algebraische_lietheorie/koalg.

- Certain Triangle Groups as Homomorphic Images of PGL(2, Z), on 21.2.2013.
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