Philipps Universität Marburg

# A Family of Right Coideal Subalgebras of $U_q(\mathfrak{sl}_{n+1})$

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# Contents

# Introduction

In the late eighties Jimbo ([Jim86]) and Drinfel'd ([Dri86]) independently introduced the quantum groups that nowadays play an important role in the theory not only of Lie algebras but of Hopf algebras, too. It took no long time until there arose generalisations concerning certain properties. In 1987 Woronowicz introduced compact quantum groups/compact matrix pseudogroups ([Wor87]). By using $*$-algebras he was able to construct a Haar state which has a bunch of properties common with the classical Haar measure—and he consequently called it Haar measure. E.g. Podleś ([Pod87]) used this work for the construction of homogeneous space for $S_qU(2)$.

In 1995 Noumi and Sugitani introduced a new method for the construction of quantum symmetric pairs ([NS95]): Instead of regarding a homogeneous space as a invariant space of a Hopf subalgebra they suggested swapping to coideal subalgebras. The method of construction was finding a solution for suitable reflection equations. They gave (right) coideal subalgebras for all types of riemannian symmetric spaces except AIII. Later on followed a publication of Dijkhuizen-Noumi ([DN98]) where they worked with a coideal for type AIII. Letzter contributed right coideal subalgebras for this type, too ([Let97]). The latter author presented additionally a universal approach ([Let99]) instead of case by case solutions for the different types of riemannian symmetric pairs. Other publications in that direction followed like [Let02], [Let03], [KL08] and [Kol08].

Kharchenko initiated a programme with the goal of classifying homogeneous right coideal subalgebras, that are right coideal subalgebras that contain $U^0$, the subalgebra generatad by all group-like elements, of $U_q(\mathfrak{g})$ with $\mathfrak{g}$ being a complex semisimple lie algebra. He gave a classification for $U_q^+(\mathfrak{so}_{2n+1})$ ([Kha11]) and the same classification together with Sagahon for $U_q(\mathfrak{sl}_{n+1})$ ([KS08]). This classification of right coideal subalgebras for all types was then done Heckenberger-Kolb ([HK12]). As Heckenberger-Schneider ([HS13, Theorem 7.13]) they linked the homogeneous right coideal subalgebras to the corresponding Weyl group and due to this link the homogeneous right coideal subalgebras can be described via PBW-type base elements. In a second publication ([HK11]) they classified the right coideal subalgebras of the Borel part, whose intersection with $U^0$ is a Hopf algebra. Again it turned out that these right coideal subalgebras are linked to the Weyl group—and can be constructed with PBW-type base elements and characters of certain subalgebras. These characters produce a deformation of $U^+[w]$.

Mainly based on the last named publication this work contributes another family of right coideal subalgebras for type $A_n$. While the works of Dijkhuizen, Letzter and Noumi focus on symmetric spaces and $*$-structures, that is dropped since homogeneous spaces are even then interesting if they lack this symmetric or geometric property. The approach shall be algebraic and one condition that is motivated by the work of Müller-Schneider ([MS99]) is the request of having reductive right coideal subalgebras, since this makes sure that the homogeneous space has quite good algebraic properties. As consequence the here presented family is quite big in the sense that it is in big parts a Hopf algebra and is quite similar to $U_q(\mathfrak{sl}_r) \otimes U_q(\mathfrak{sl}_{n-r+1})$. Having the generators of the quantum enveloping algebras in correspondence with the simple roots of the root system there is an unused simple root: $\alpha_r$.

Denote with $\mathfrak{A}_{n,r} \subset U_q(\mathfrak{sl}_{n+1})$ the right coideal subalgebra. Let $\Lambda$ be the weight lattice of $U_q(\mathfrak{sl}_{n+1})$. We call a $U_q(\mathfrak{sl}_{n+1})$-module spherical if the set of $\mathfrak{A}_{n,r}$-invariants is at most one-dimensional. Theorem 3.21 and Proposition 3.22 yield:

*Let $\lambda \in \Lambda$ be a dominant weight and let let $L(\lambda)$ be the simple, finite dimensional $U_q(\mathfrak{sl}_{n+1})$-module of highest weight $\lambda$. Then $L(\lambda)$ is a semisimple $\mathfrak{A}_{n,r}$-module. If $L(\lambda)$ is spherical as $U_q(\mathfrak{sl}_r) \otimes U_q(\mathfrak{sl}_{n-r+1})$-module, then it is spherical as $\mathfrak{A}_{n,r}$-module.*

In his work [Krä79] Krämer showed that $U_q(\mathfrak{sl}_r) \otimes U_q(\mathfrak{sl}_{n-r+1})$ is sphercial whenever $r \neq n-r+1$. (In fact he showed that for the Lie Group $SU(r) \times SU(n-r) \subset SU(n+1)$—which is for us all the same.)

The algebra $\mathfrak{A}_{2,2}$ has an attractive representation theory as Propositions 3.4 and 3.5 show—that is comparable to the representation theory of $U_q(\mathfrak{sl}_2)$. Explicitly the generators are $G_2 = K_2^{-1}(E_2 + \zeta \cdot 1)$, $G_{12} = K_1^{-1}K_2^{-1}(E_2E_1 - q^{-1}E_1E_2 + \zeta(q-q^{-1})E_1)$, $(K_1^2K_2)^{\pm 1}$ and $F_1$ with $\zeta$ being a nonzero element of the ground field $k$.

*For every pair $s \in \mathbb{N}_0$ and $\kappa \in k, \kappa \neq 0$ there is a unique simple $\mathfrak{A}_{2,2}$-module of dimension $s+1$ such that $G_2$ and $K_1^2K_2$ are simultaneously diagonalisable operators. Every finite dimensional simple $\mathfrak{A}_{2,2}$-module with the last named property is determined by a pair of above and every finite dimensional module with named property decomposes into simple modules.*

In the first chapter the notation and most of the necessary facts about quantum enveloping algebras and there Hopf dual the quantum coordinate ring are given.

The second chapter quotes at its beginning the main theorems used for the construction of named publication [HK11]. Then, for the deformation a suitable element of the Weyl group, namely the longest word of the subsystem $A_{r-1} \times A_{n-r}$ is chosen. The deformation via the the character than produces an element $G_{\beta_r^r} = K_{\alpha_r}^{-1}(E_{\alpha_r} + \zeta \cdot 1)$ ($\zeta$ invertible element of the ground field) that is going to play the role of a $K_{\alpha_r}$—but it is not an element of $U^0$. This forces a deformation on the roots $\alpha_{r-1}$ and $\alpha_{r+1}$ giving elements $G_{\beta_{r-1}^r}$ and $G_{\beta_r^{r+1}}$. Adding a negative part this finally gives a relation

$$G_{\beta_{r-1}^r}F_{\alpha_{r-1}} = qF_{\alpha_{r-1}}G_{\beta_{r-1}^r} + G_{\beta_r^r} - \zeta K_{\alpha_{r-1}}^{-2}K_{\alpha_r}^{-1}$$

and a corresponding for $r+1$ instead of $r-1$. The element $G_{\beta_{r-1}^r}$ should be regarded as a deformation of $E_{\alpha_r}$. Nevertheless is derives from $E_{\alpha_r}E_{\alpha_{r-1}} - q^{-1}E_{\alpha_{r-1}}E_{\alpha_r}$. The right coideal subalgebra is denoted as done above by $\mathfrak{A}_{n,r}$. After the computation of deformation is done some algebraic properties are derived like the existence of a PBW-type base and a grading.

In the third chapter the representation theory is accomplished under the assumption that $G_{\beta_r^r}$ operates as diagonal operator—which is true for all simple $U_q(\mathfrak{sl}_{n+1})$-modules (Proposition 3.1). The reductive property is shown in general for small cases, i.e. $\mathfrak{A}_{2,2}$ and $\mathfrak{A}_{3,2}$. In both cases the methods are similar to that of $U_q(\mathfrak{sl}_2)$ resp. $U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$. In the general case the property of being reductive is linked the property of $U_q(\mathfrak{sl}_r) \otimes U_q(\mathfrak{sl}_{n-r+1})$ being reductive—this link is also used to show the spherical property.

The fourth chapter is dedicated to the homogeneous space of $\mathfrak{A}_{2,2}$. In this special case invariant vectors are explicitly calculated. Letting aside the $*$-structure it is shown that homogeneous space is isomorphic to the homogeneous space of Dijkhuizen-Noumi—using their method with which they constructed their homogeneous space via quantum spheres. In the last section of the chapter the case $\zeta = 0$ is considered. In this case the representation theory of $\mathfrak{A}_{2,2}$ no longer works: neither is $G_{\beta_2^2}$ a diagonal operator nor is the right coideal subalgebra reductive. With elementary method using weight diagrams it is shown how that the space of invariants of a simple $U_q(\mathfrak{sl}_3)$-module is still at most one-dimensional.

The last chapters starts with type $B_2$. Seeing analogues to type $A_2$ generalisations concerning the algebras $\mathfrak{A}_{2,2}$ and $\mathfrak{B}_2$ and the programme enabled by the in this work presented method are formulated and conjectures postulated.

# Chapter 1

# Preliminaries

## Quantum Enveloping Algebras of Semisimple Lie Algebras

The definitions and stated properties of the defined objects are, if no other citation is given, taken from [Jan96, Chapter 4]. Let $\mathfrak{g}$ be a complex finite dimensional semisimple Lie-Algebra. Let $\Phi$ be the root system attached to a fixed Cartan subalgebra. Let $\Pi$ be a basis of $\Phi$, i.e. the set of simple roots. Let $W$ be the Weyl group of $\mathfrak{g}$ and let $l$ be the length function on $W$. There exists a $W$-invariant scalar product $(\cdot,\cdot)$ on $\operatorname{span}_{\mathbb{R}}(\alpha|\alpha\in\Pi)$ such that $(\alpha,\alpha)=2$ for short roots. Set $Q=\operatorname{span}_{\mathbb{Z}}(\Pi)$, this is the root lattice.

Let $k$ be an algberaic closed field of characteristic 0. Fix throughout this work $q\in k$, $q\neq 0$ and $q$ not a root of unity. Set for each $\alpha\in\Pi$

$$q_\alpha = q^{(\alpha,\alpha)/2}.$$

The algebra $U=U_q(\mathfrak{g})$ is the $k$-algebra generated by $F_\alpha$, $E_\alpha$, $K_\alpha$ and $K_\alpha^{-1}$ for $\alpha\in\Pi$ subject to the relations

$$\text{(R1)}\qquad K_\alpha K_\alpha^{-1} = K_\alpha^{-1}K_\alpha = 1,$$

$$\text{(R2)}\qquad K_\alpha K_\beta = K_\beta K_\alpha,$$

$$\text{(R3)}\qquad K_\alpha E_\beta K_\alpha^{-1} = q^{(\alpha,\beta)}E_\alpha,$$

$$\text{(R4)}\qquad K_\alpha F_\beta K_\alpha^{-1} = q^{-(\alpha,\beta)}F_\alpha,$$

$$\text{(R5)}\qquad E_\alpha F_\beta - F_\beta E_\alpha = \delta_{\alpha\beta}\frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}},$$

with $\delta_{\alpha\beta}$ the Kronecker delta; and the quantum Serre relations, with $a_{\alpha\beta}=2(\alpha,\beta)/(\alpha,\alpha)$, for which we use Gaussian binomial coefficients that can be found briefly in e.g. [Jan96, Chapter 0]

$$\text{(R6)}\qquad \sum_{k=0}^{1-a_{\alpha\beta}} (-1)^k \begin{bmatrix} 1-a_{\alpha\beta} \\ k \end{bmatrix}_\alpha E_\alpha^{1-a_{\alpha\beta}-k}E_\beta E_\alpha^k = 0,$$

$$\text{(R7)}\qquad \sum_{k=0}^{1-a_{\alpha\beta}} (-1)^k \begin{bmatrix} 1-a_{\alpha\beta} \\ k \end{bmatrix}_\alpha F_\alpha^{1-a_{\alpha\beta}-k}F_\beta F_\alpha^k = 0.$$

Let $U^-, U^+$ and $U^0$ be the subalgebras generated by $\{F_\alpha|\alpha\in\Pi\}$, $\{E_\alpha|\alpha\in\Pi\}$ resp. $\{K_\alpha, K_\alpha^{-1}|\alpha\in\Pi\}$. Let $\lambda=\sum_{\alpha\in\Pi} m_\alpha\alpha\in Q$, i.e. $m_\alpha\in\mathbb{Z}$. Since $U^0$ is commutative

$$K_\lambda = \prod_{\alpha\in\Pi} K_\alpha^{m_\alpha}$$

is well-defined. This definition is also compatible with the scalar product on the root lattice, such that holds

$$K_\lambda E_\alpha K_{-\lambda} = q^{(\lambda,\alpha)} E_\alpha \quad \text{and} \quad K_\lambda F_\alpha K_{-\lambda} = q^{-(\lambda,\alpha)} F_\alpha.$$

There exist automorphisms on $U_q(\mathfrak{g})$ that satisfy the braid relations of the Weyl group. Precisely: For each $\alpha \in \Pi$ exists an automorphism $T_\alpha$ such that for all $\beta \in \Pi$ holds

$$\begin{aligned} T_\alpha T_\beta T_\alpha &= T_\beta T_\alpha T_\beta, && (m=3)\\ T_\alpha T_\beta T_\alpha T_\beta &= T_\beta T_\alpha T_\beta T_\alpha, && (m=4)\\ T_\alpha T_\beta T_\alpha T_\beta T_\alpha T_\beta &= T_\beta T_\alpha T_\beta T_\alpha T_\beta T_\alpha, && (m=6) \end{aligned}$$

where $m$ is the order of $s_\alpha s_\beta$, c.f [Jan96, Section 8.14-8.16]. Let $w \in W$ and $s_{\alpha_1} \cdots s_{\alpha_t}$ be a reduced expression of $w$. Define $T_w = T_{s_{\alpha_1} \cdots s_{\alpha_t}}$, this is independent of the choice of the reduced expression. For two elements $w, w' \in W$ with $l(ww') = l(w) + l(w')$ then holds $T_{ww'} = T_w T_{w'}$. With the help of these automorphisms it is possible to define ›quantum root vectors‹ which lead to a PBW-type basis of $U_q(\mathfrak{g})$: Let $w \in W$ with reduced expression $w = s_{\alpha_1} \cdots s_{\alpha_t}$ and set $\beta_i = s_{\alpha_1} \cdots s_{\alpha_{i-1}}(\alpha_i)$ for $1 \le i \le t$. Then the

$$E_{\beta_i} = T_{s_{\alpha_1} \cdots s_{\alpha_{i-1}}}(E_{\alpha_i})$$

are ›quantum root vectors‹ of $w$. Define

$$U^+[w] = \langle E_{\beta_1}, \ldots, E_{\beta_t} \rangle.$$

The subalgebra $U^+[w]$ has by [CKP95] the ordered basis

$$\{E_{\beta_1}^{m_1} \cdots E_{\beta_t}^{m_t} \mid m_1, \ldots, m_t \in \mathbb{N}_0\}.$$

Similarly one defines $U^-[w]$. There is a unique automorphism $\omega$ of $U$ sucht that $\omega(E_\alpha) = F_\alpha$, $\omega(F_\alpha) = E_\alpha$ and $\omega(K_\alpha) = \omega(K_\alpha^{-1})$. This map is an involution. Let $\mu \in Q$, $u \in U$ of weight $\mu$ and $\alpha \in \Pi$, then holds

$$T_{s_\alpha}(\omega(u)) = (-q_\alpha)^{-\langle \mu, \alpha^\vee \rangle} \omega(T_{s_\alpha}(u)).$$

From this equation follows the more general formula: Let $w \in W$ and $w(\mu) - \mu = \sum_{\alpha \in \Pi} m_\alpha \alpha$, then holds

$$T_w(\omega(u)) = \Big( \prod_{\alpha \in \Pi} (-q_\alpha)^{m_\alpha} \Big) \omega(T_w(u)). \tag{1.1}$$

Note also that $U^-[w] = \omega(U^+[w])$.

There exists a coproduct $\Delta$, an antipode $S$ and a counit $\epsilon$ on $U$ that give a unique Hopf algebra structure determined by

$$\begin{aligned} \Delta(K_\alpha) &= K_\alpha \otimes K_\alpha, & \epsilon(K_\alpha) &= 1, & S(K_\alpha) &= K_\alpha^{-1},\\ \Delta(E_\alpha) &= K_\alpha \otimes E_\alpha + E_\alpha \otimes 1, & \epsilon(E_\alpha) &= 0, & S(E_\alpha) &= -K_\alpha^{-1} E_\alpha,\\ \Delta(F_\alpha) &= 1 \otimes F_\alpha + F_\alpha \otimes K_\alpha^{-1}, & \epsilon(F_\alpha) &= 0, & S(F_\alpha) &= -F_\alpha K_\alpha \end{aligned}$$

for all $\alpha \in \Pi$. We are going to use the Sweedler notation, that is $\Delta(u) = u_{(1)} \otimes u_{(2)}$.

Let $M$ be a finite dimensional $U_q(\mathfrak{g})$-module and let $\mu \in \Lambda$. Then

$$M_\mu = \{\, m \in M \mid K_\alpha m = q^{(\mu,\alpha)} m \quad \text{for all } \alpha \in \Pi \,\}$$

is called weight space of weight $\mu$. The module $M$ decomposes into the direct sum of its weight spaces. Concerning modules we follow the notation of [Jan96, Chapter 5].

The representation theory of $\mathfrak{g}$ and its quantum enveloping algebra $U_q(\mathfrak{g})$ is the same in the following sense: As every dominant weight $\lambda$ defines a unique simple module $L^{\mathfrak{g}}(\lambda)$ of $\mathfrak{g}$ it defines a unique simple module $L(\lambda)$ of $U_q(\mathfrak{g})$ of the same dimension (in particular, the weight spaces of both modules do have the same dimensions and the Weyl character formula holds) as every finite dimensional module decomposes into simple modules, i.e. is semisimple.

With help of the antipode the dual (vector) space $M^*$ is turned into a $U$-module via

$$u \cdot f(m) = f(S(u)m) \quad \text{for all } u \in U,\ f \in M^* \text{ and } m \in M,$$

while the coproduct $\Delta$ turns the tensor product of two finite dimensional modules $M, N$ into a $U$-module via

$$u(m \otimes n) = u_{(1)} m \otimes u_{(2)} n \quad \text{for } u \in U,\ m \in M \text{ and } n \in N,$$

with $\Delta(u) = u_{(1)} \otimes u_{(2)}$. Of course, $M \oplus N$ and $k$ (via $\epsilon$) are $U$-modules, too. The two following isomorphisms for $M, N$ being finite dimensional $U_q(\mathfrak{g})$-modules and $\lambda$ a dominant weight exist:

$$\begin{aligned} &M \otimes N \simeq N \otimes M, \\ &L(\lambda)^* \simeq L(-w_0\lambda), \end{aligned}$$

where $w_0$ is the longest element in the Weyl group of $\mathfrak{g}$. In particular the first isomorphism will be of some interest later. We shall come back to it in the next section and finish this section with two definitions.

Definition 1.1 *Let $H$ be a Hopf algebra with coproduct $\Delta$ and let $\mathfrak{R}$ be subalgebra. If $\Delta(\mathfrak{R}) \subset \mathfrak{R} \otimes H$, then $\mathfrak{R}$ is called* right coideal subalgebra*.*

Definition 1.2 *Let $H$ be a Hopf algebra with counit $\epsilon$ and $\mathfrak{R} \subset H$ be a right coideal subalgebra and $M$ be a finite dimensional, simple $H$-module.*

1. *A vector $v \in M$ is $\mathfrak{R}$-*invariant *if $rv = \epsilon(r)v$ for all $r \in \mathfrak{R}$.*
2. *If $M$ is semisimple as $\mathfrak{R}$-module and the set of $\mathfrak{R}$-invariant vectors in $M$ is at most one dimensional, then $M$ is called* spherical*.*
3. *If every simple, finite dimensional $H$-module is a spherical $\mathfrak{R}$-module, then $\mathfrak{R}$ is called* right coideal spherical subalgebra*.*

## Quantum Coordinate Ring

Let $M$ be a finite dimensional $U$-module and let $M^*$ be its dual vector space. Let $m \in M$ and $f \in M^*$. The linear operator $c_{f,m} \in U^*$ given by $c_{f,m}(u) = f(um)$ is called *matrix coefficient (of $M$)*.

Denote with $\mathcal{A}_q = \mathcal{A}_q(\mathfrak{g})$ the set of all matrix coefficients of finite dimensional $U$-modules. This is a Hopf algebra, cf. [Jos95, Section 1.4] or [Swe69]. This Hopf algebra is considered as the quantum coordinate ring or ring of regular functions, cf. [KS98, Chapter 3].

The Hopf algebra structure is given as follows: Let $M$ and $M'$ be finite dimensional $U$-modules, and $m_1, \ldots, m_r$ a basis of $M$ with dual basis $f_1, \ldots, f_r$. Let $m \in M$, $m' \in M'$, $f \in M^*$ and $f' \in (M')^*$. Then the structure extends from these equations on the generators:

$$\begin{aligned} c_{f,m} + c_{f',m'} &= c_{f+f',m+m'}, \\ c_{f,m}c_{f',m'} &= c_{f\otimes f',m\otimes m'}, \\ S(c_{f,m}) &= c_{f,m} \circ S_U, \\ \Delta(c_{f,m}) &= \sum_{i=1}^{r} c_{f,m_i} \otimes c_{f_i,m}. \end{aligned}$$

The unital element is $\epsilon$.

There are natural left and right actions of $U$ on $\mathcal{A}_q$. Let $u \in U$ and let $c \in \mathcal{A}_q$ with $\Delta(c) = c_{(1)} \otimes c_{(2)}$. The actions are given by

$$\begin{aligned} u \cdot c &= c_{(1)}c_{(2)}(u), \\ c \cdot u &= c_{(1)}(u)c_{(2)}. \end{aligned}$$

Using both operations, this turns $\mathcal{A}_q(\mathfrak{g})$ into a $(U, U)$-bimodule. Then $\mathcal{A}_q(\mathfrak{g})$ has the following decomposition that is also known as Peter-Weyl decomposition.

PROPOSITION 1.3 (Peter-Weyl decomposition) *As $(U, U)$-bimodule the algebra $\mathcal{A}_q(\mathfrak{g})$ has the decomposition*

$$\mathcal{A}_q(\mathfrak{g}) = \bigoplus_{\substack{\lambda\in\Lambda, \\ \lambda \text{ dominant}}} L(\lambda)^* \otimes L(\lambda). \tag{1.2}$$

Originally this decomposition comes from Lie groups and uses the so called Haar measure. If $H$ is compact quantum group, then there is a Haar measure on $H$ (to be precise: a state, as sketched in the introduction). It has the same principal property as in the classical case: The decomposition (1.2) is orthogonal subject to this Haar measure, c.f. [Wor87] and for a broader survey [MD98]. In the above case it is completely algebraic ([Swe69]).

Let $\mathfrak{R}$ be a spherical right coideal subalgebra of $U$ and let $Q^{\mathfrak{R}}$ be the set of dominant weights such that $L(\lambda)$ has a $\mathfrak{R}$-invariant vector. The set $\mathcal{A}_q^{\mathfrak{R}}$ of invariants of $\mathcal{A}_q$ under $\mathfrak{R}$ decomposes as

$$\mathcal{A}_q^{\mathfrak{R}} = \{c \in \mathcal{A}_q \mid r \cdot c = \epsilon(r)c \text{ for all } r \in \mathfrak{R}\ \} = \bigoplus_{\lambda\in Q^{\mathfrak{R}}} L(\lambda)^*$$

due to the Peter-Weyl decomposition. The fact that $\mathfrak{R}$ is a right coideal ensures that $\mathcal{A}_q^{\mathfrak{R}}$ is a subalgebra of $\mathcal{A}_q$.

Let us have a description of $\mathcal{A}_q = \mathcal{A}_q(\mathfrak{sl}_{n+1})$ in terms of generators and relations. In the classical case the coordinate ring $\mathcal{A}(\mathfrak{sl}_{n+1})$ of $\mathfrak{sl}_{n+1}$ is the polynomial ring in indeterminates $X_{ij}$ with $1 \leq i, j \leq n+1$ and the relation $\det(X_{ij})_{1\leq i,j\leq n+1} = 1$. A sketch of reasons: Every simple module is a submodule of some tensor power of the standard representation $V$, the $X_{ij}$ are the matrix coefficients of $V$, so they generate $\mathcal{A}(\mathfrak{sl}_{n+1})$. The determinate comes from the fact, that $\wedge^n(V)$ is the trivial representation. That the $X_{ij}$ commute comes from the fact, that the action of $\mathfrak{sl}_{n+1}$ on tensor products is ›symmetric‹: If $M$ and $M'$ are finite dimensional modules, then the isomorphism between $M \otimes M'$ and $M' \otimes M$ is the flip, i.e. the map that sends $m \otimes m'$ to $m' \otimes m$. In terms of algebraic geometry the point of view might be different. In particular, there is no need for a multiplication, an affine algebraic variety is sufficient. The coordinate ring (or ring of regular functions, which might be different in definition, but are the same object) is given as the polynomial ring in certain indeterminates divided out the defining ideal of the variety, see e.g. [Har06, Ch. 1] for a discussion and proofs. Having this in mind, the Hopf algebra of matrix coefficients of a quantum enveloping algebra can be considered as non-commutative coordinate ring.

Let $A = (a_{ij})$ be a $n \times n$-matrix. The quantum determinant of $A$ is given by

$$\det_q(A) = \sum_{\sigma\in\mathfrak{S}_n} (-q)^{l(\sigma)} a_{\sigma(1)1} \cdots a_{\sigma(n)n}.$$

In quantum case the quantum coordinate ring $\mathcal{A}_q$ is generated by indeterminates $t_{ij}$ ($1 \leq i, j \leq n+1$) subject to relations given by the $R$-matrix relations of the standard representation and the quantum determinant of $(t_{ij})_{1\leq i,j\leq n+1}$ that we simply denote with $\det_q$ being equals to 1, i.e. $\det_q = 1$. Before giving an explicit description let us informally describe the reasons: As in the classical case, the standard representation $V$ generates all finite dimensional simple modules. One defines a quantum exterior algebra $\wedge_q^k(V)$ (c.f. [APW91, Section 12.5]) - in the case $k = n$ it is one dimensional, which leads to the quantum determinant. The $R$-matrices for a quantum enveloping algebra play the role of the flip, i.e. they represent the isomorphism between $M \otimes M'$ and $M' \otimes M$. A description of the $R$-matrices and the isomorphism can be found in [Jan96, Chapter 7]. A proof for the description of $\mathcal{A}_q$ can be found in the appendix of [APW91]. The following explicit relations are taken from [APW91].

Let $1 \leq i, j, k, l \leq n+1$. Then the following relations are holding between the $t_{ij}$'s.

$$\begin{aligned}
t_{ij}t_{kj} &= qt_{kj}t_{ij} && (i < k)\\
t_{ij}t_{il} &= qt_{il}t_{ij} && (j < l)\\
t_{ij}t_{kl} &= t_{kl}t_{ij} && (i < k \text{ and } j > l)\\
t_{ij}t_{kl} &= t_{kl}t_{ij} + (q-q^{-1})t_{il}t_{kj} && (i < k \text{ and } j < l)\\
\det_q &= \sum_{\sigma\in\mathfrak{S}_{n+1}} (-q^{l(\sigma)} t_{\sigma(1)1} \cdots t_{\sigma(n+1)n+1} = 1
\end{aligned}$$

There is an important property of $\mathcal{A}_q$ for which a proof can be found in [LS93], who use some techniques/results that were presented in [AST91].

Theorem 1.4 *The algebra $\mathcal{A}_q$ has no zero divisors.*

# Chapter 2

# A Family of Right Coideal Subalgebras for $A_n$

## Preparation

The construction is as follows: In a first step we define a right coideal subalgebra of the borel part of $U_q(\mathfrak{sl}_{n+1})$ that will be a deformation of $U^+[w]$ for a certain element of the Weylgroup together with a certain subalgebra of $U^0$. Theorem 2.17 from [HK11] will ensure that this yields a right coideal subalgebra. In a second step $U^-[w']$ for a $w'$ not to much different from $w$ is added, this involves some explicit calculations that will be used later anyway.

Let us recap the Theorem 2.17 (2) of [HK11] and the notation necessary for its formulation. Let $A$ be an associative, unital algebra over the field $k$. A map $\varphi\colon A \to k$ is a character of $A$ if $\varphi$ is a homomorphism of algebras and $\varphi(1) = 1$.

Let $w \in W$ be an element of the Weyl group and $\varphi$ be a character of $U^+[w]$. Then

$$\operatorname{supp}\varphi = \{\beta \in Q_+ \mid \exists u \in U^+[w]_\beta \text{ s.t. } \varphi(u) \neq 0\}$$

is the support of $\varphi$. The set is closed under addition, this follows immediately from the facts that $\varphi$ is a homomorphism and that $U$ has no zero-divisors.

The map $\psi\colon U^+ \to S(U^+)$ given by

$$\psi(u_\beta) = q^{-(\beta,\beta)/2} u_\beta K_\beta^{-1} \quad \text{for } \beta \in \Lambda_+ \text{ and } u_\beta \in U_\beta^+$$

is an algebra isomorphism and the image of $U^+[w]$ under $\psi$ is $S(U^+) \cap U^+[w]U^0$ by [HK11, Lemma 2.11].

Theorem 2.1 (Thm. 2.17(2) of [HK11]) *Let $w \in W$, $\varphi$ be a character of $U^+[w]$ and $L \subset (\operatorname{supp}\varphi)^\perp$ be a subgroup. Then*

$$C(w, \varphi, L) := T_L\{\varphi\Big(\psi^{-1}(u_{(1)})\Big)u_{(2)} \mid u \in \psi(U^+[w])\}$$

*is a right coideal subalgebra of $U^{\geq 0}$ such that $C(w, \varphi, L) \cap U^0$ is a Hopf algebra.*

Moreover the theorem states, that the right coideal algebras obtained exhaust all right coideal subalgebras with the additional property stated last in the theorem and this exhaustion is injective. In their article, Heckenberger and Kolb determined also the possible characters of $U^+[w]$.

Let $t = l(w)$ and $\alpha_1, \dots, \alpha_t \in \Pi$ such that $w = s_{\alpha_1} \cdots s_{\alpha_t}$, which is then by definition a reduced expression. Define $\beta_i = s_{\alpha_1} \cdots s_{\alpha_{i-1}}(\alpha_i)$ for all $1 \leq i \leq t$. Then holds

$$\Phi_w^+ := \{\beta \in \Phi^+ \mid w^{-1}\beta \in \Phi^-\} = \{\beta_i \mid 1 \leq i \leq t\}$$

by [Hum90, Section 5.6]. In particular holds $l(w) = |\Phi_w^+|$. Let $\Theta \subset \Phi^+$ be a subset of pairwise orthogonal roots. Such a set is called *orthogonal*. Define

$$w_\Theta = (\prod_{\beta\in\Theta} s_\beta)w.$$

This element is well defined since two reflections $s_\beta$ and $s_\gamma$ commute whenever $(\beta,\gamma) = 0$, i.e. $\beta$ and $\gamma$ are orthogonal. Define now

$$T^w = \{\Theta \subset \Phi_w^+ \mid \Theta \text{ is orthogonal and } l(w_\Theta) = l(w) - |\Theta|\}$$

and denote for two sets $A$ and $B$ the set of maps from $A$ to $B$ with $\mathrm{map}(A, B)$.

THEOREM 2.2 (Thm. 3.18 of [HK11]) *There is a bijection*

$$\Psi\colon \{(\Theta, f) \mid \Theta \in T^w, f \in \mathrm{map}(\Theta, k^*)\} \to \mathrm{Char}(U^+[w])$$

*uniquely determined by*

$$\Psi(\Theta, f)(E_\beta) = \begin{cases} f(\beta) & \textit{if } \beta \in \Theta, \\ 0 & \textit{otherwise.} \end{cases}$$

They also gave an inverse map which we shall not need and omit it therefore.

## Construction

Fix $n \in \mathbb{N}$, $n \geq 2$. Let $\alpha_1, \ldots, \alpha_n$ be the simple roots of $A_n$ in standard order to be found in [Bou81, Planche I], i.e. $\alpha_i$ corresponds to the $i$-th node in the Dynkin diagram of type $A_n$ read from left to right. Let $s_{\alpha_i}$ be the corresponding reflection. We have

$$s_{\alpha_i}(\alpha_j) = \begin{cases} -\alpha_i & i = j, \\ \alpha_i + \alpha_j & |i-j| = 1, \\ \alpha_j & \text{else}, \end{cases}$$

$$s_{\alpha_i}(\alpha_i + \alpha_{i\pm1}) = \alpha_{i\pm1}.$$

Define for $1 \leq i \leq j \leq n$

$$s_i^{(j)} = s_{\alpha_i} \cdots s_{\alpha_j}$$

and for convenience set $s_i^{(j)}$ as the identity whenever $i > j$. For $1 \leq i \leq k < j \leq n$ we have then

$$s_i^{(j)}(\alpha_k) = \alpha_{k+1}$$

and for $1 \leq i < j \leq n$

$$s_i^{(j-1)}(\alpha_j) = \alpha_i + \ldots + \alpha_j.$$

Define for $1 \leq i \leq k \leq n$ some roots $\beta_i^k$:

$$\beta_i^k = \alpha_i + \ldots + \alpha_k.$$

These are $\frac{n(n+1)}{2}$ distinct positive roots, in fact, these are all positive roots, since $|\Phi^+| = \frac{n(n+1)}{2}$ ([Bou81, Planche I]).

LEMMA 2.3 *Let $1 \leq i \leq k < j \leq n$. Then holds*

$$s_i^{(j)} s_i^{(k-1)}(\alpha_k) = s_{i+1}^{(k)}(\alpha_{k+1}) = \beta_{i+1}^{k+1}.$$

*Proof.* We shall just use the braid relations of $W$ and the equations $s_{\alpha_k} s_{\alpha_{k-1}}(\alpha_k) = \alpha_k$ and $s_{\alpha_g}(\alpha_h) = \alpha_h$ whenever $g - h \notin \{1, 0, -1\}$.

$$\begin{aligned}
& s_i^{(j)} s_i^{(k-1)}(\alpha_k) \\
&= s_{\alpha_i} s_{\alpha_{i+1}} \cdots s_{\alpha_{j-1}} s_{\alpha_j} s_{\alpha_i} \cdots s_{\alpha_{k-1}}(\alpha_k) \\
&= \underbrace{s_{\alpha_i} s_{\alpha_{i+1}} s_{\alpha_i}}_{s_{\alpha_{i+1}} s_{\alpha_i} s_{\alpha_{i+1}}} \overbrace{s_{\alpha_{i+2}} s_{\alpha_{i+1}}}^{s_{\alpha_{i+2}} s_{\alpha_{i+1}} s_{\alpha_{i+2}}} \underbrace{s_{\alpha_{i+3}} s_{\alpha_{i+2}}}_{s_{\alpha_{i+3}} s_{\alpha_{i+2}} s_{\alpha_{i+3}}} \overbrace{\cdots}^{s_{\alpha_{i+4}} s_{\alpha_{i+3}} s_{\alpha_{i+4}}} \underbrace{\cdots \cdots}_{s_{\alpha_{k-1}} s_{\alpha_{k-2}} s_{\alpha_{k-1}}} \overbrace{s_{\alpha_{k-2}} s_{\alpha_k} s_{\alpha_{k-1}}}^{s_{\alpha_k} s_{\alpha_{k-1}} s_{\alpha_k}} s_{\alpha_{k+1}}(\alpha_k) \\
&= s_{\alpha_{i+1}} s_{\alpha_i} s_{\alpha_{i+2}} s_{\alpha_{i+1}} s_{\alpha_{i+3}} s_{\alpha_{i+2}} s_{\alpha_{i+4}} \cdots s_{\alpha_k} s_{\alpha_{k-1}} s_{\alpha_k} s_{\alpha_{k+1}}(\alpha_k) \\
&= s_{\alpha_{i+1}} s_{\alpha_i} s_{\alpha_{i+2}} s_{\alpha_{i+1}} s_{\alpha_{i+3}} s_{\alpha_{i+2}} s_{\alpha_{i+4}} \cdots s_{\alpha_k}(\alpha_{k+1}) \\
&= s_{\alpha_{i+1}} s_{\alpha_{i+2}} s_{\alpha_{i+3}} s_{\alpha_{i+4}} \cdots s_{\alpha_k}(\alpha_{k+1}) \\
&= \beta_{i+1}^{k+1}.
\end{aligned}$$

□

Note that

$$s_i^{(j+1)} s_i^{(j)} s_i^{(k-1)}(\alpha_k) = s_{\alpha_i} s_{i+1}^{(j+1)} s_{i+1}^{(k)}(\alpha_{k+1}) = s_{\alpha_i} s_{i+2}^{(k+1)}(\alpha_{k+2}) = s_{i+1}^{(k+1)}(\alpha_{k+2}).$$

So an iterated application of Lemma 2.3 finally gives

$$\beta_i^k = s_i^{(k-1)}(\alpha_k) = s_1^{(n)} s_1^{(n-1)} \cdots s_1^{(n-i+2)} s_1^{(k-i)}(\alpha_{k-i+1}) \quad (1 \leq i \leq k \leq n),$$

which shows that $s_1^{(n)} \cdots s_1^{(1)}$ is the longest element of the Weyl group. This is a reduced expression since the length of an element of the Weyl group equals the amount of positive roots send to negative ones, c.f. [Hum90, Section 1.8].

Fix another integer $r \in \mathbb{N}$ with $1 \leq r \leq n$. Denote with $w_a^{(r)}$ resp. $w_b^{(r)}$ the longest element of the subgroup generated by $s_{\alpha_1}, \ldots, s_{\alpha_{r-1}}$ resp. $s_{\alpha_{r+1}}, \ldots, s_{\alpha_n}$ where we use the convention that $w_a^{(r)}$ is the identity for $r = 1$ and similarly for $w_b^{(r)}$. Set $w^+ = s_{\alpha_r} w_a^{(r)} w_b^{(r)}$ and $w^- = w_a^{(r)} w_b^{(r)}$. Associated to $w^-$ we find the following roots:

$$\begin{aligned}
\beta_i^k &= s_1^{(r-1)} s_1^{(r-2)} \cdots s_1^{(r-i+1)} s_1^{(k-i)}(\alpha_{k-i+1}) = s_1^{(k-1)}(\alpha_k) && (1 \leq i \leq k < r), \\
\beta_i^k &= s_{r+1}^{(n)} s_{r+1}^{(n-1)} \cdots s_{r+1}^{(n-i+r+2)} s_{r+1}^{(k-i+r)}(\alpha_{k-i+r+1}) = s_i^{(k-1)}(\alpha_k) && (r < i \leq k \leq n)
\end{aligned}$$

and associated to $w^+$ we find the following:

$$\begin{aligned}
&\beta_r^r = \alpha_r, && \\
&\beta_i^r = s_{\alpha_r} s_1^{(r-1)} s_1^{(r-2)} \cdots s_1^{(r-i+1)} s_1^{(r-i-1)}(\alpha_{r-i}) = s_i^{(r-1)}(\alpha_r) && (1 \le i < r), \\
&\beta_i^k = s_{\alpha_r} s_1^{(r-1)} s_1^{(r-2)} \cdots s_1^{(r-i+1)} s_1^{(k-i)}(\alpha_{k-i+1}) = s_i^{(k-1)}(\alpha_k) && (1 \le i \le k < r-1), \\
&\beta_r^k = s_{\alpha_r} s_{r+1}^{(k-1)}(\alpha_k) = s_r^{(k-1)}(\alpha_k) && (r < k \le n), \\
&\beta_i^k = s_{\alpha_r} s_{r+1}^{(n)} \cdots s_{r+1}^{(n-i+r+2)} s_{r+1}^{(k-i+r)}(\alpha_{k-i+r+1}) = s_i^{(k-1)}(\alpha_k) && (r+1 < i \le k \le n).
\end{aligned}$$

Let us define for $\beta_i^k$ with $1 \le i \le k \le n$

$$E_{\beta_i^k} = T_{s_{\alpha_i}} \cdots T_{s_{\alpha_{k-1}}}(E_{\alpha_k}).$$

From the proof of Lemma 2.3 follows that $E_{\beta_i^k}$ is the root vector for the root $\beta_i^k$ coming from $w^-$ in our chosen presentation. We will chose $U^-[w^-]$ as negative part for the family of right coideal subalgebras. However, let us have a look first on $U^+[w^-]$ for using it as starting point for the deformation of $U^+[w^+]$. These elements $E_{\beta_i^k}$ satisfy the recursion $E_{\beta_k^k} = E_{\alpha_k}$ and $E_{\beta_i^{k+1}} = E_{\beta_i^k} E_{\alpha_{k+1}} - q^{-1} E_{\alpha_{k+1}} E_{\beta_i^k}$, which follows from the definition of $T_{s_{\alpha_i}}(E_{\alpha_j})$ on $U_q(\mathfrak{sl}_{n+1})$. With this recursion we compute $\Delta(E_{\beta_i^k})$.

$$\Delta(E_{\beta_i^k}) = K_{\beta_i^k} \otimes E_{\beta_i^k} + E_{\beta_i^k} \otimes 1 + (1 - q^{-2}) \sum_{l=i}^{k-1} E_{\beta_i^l} K_{\beta_{l+1}^k} \otimes E_{\beta_{l+1}^k} \tag{2.1}$$

The formula is certainly true for $i - k = 0$, so assume $i - k \ge 1$, then

$$\begin{aligned}
&\Delta(E_{\beta_i^{k+1}}) \\
&\quad = \Delta(E_{\beta_i^k} E_{\alpha_{k+1}} - q^{-1} E_{\alpha_{k+1}} E_{\beta_i^k}) \\
&\quad = \Big( K_{\beta_i^k} \otimes E_{\beta_i^k} + E_{\beta_i^k} \otimes 1 + (1 - q^{-2}) \sum_{l=i}^{k-1} E_{\beta_i^l} K_{\beta_{l+1}^k} \otimes E_{\beta_{l+1}^k} \Big) \\
&\qquad\quad \cdot \Big( K_{\alpha_{k+1}} \otimes E_{\alpha_{k+1}} + E_{\alpha_{k+1}} \otimes 1 \Big) \\
&\quad\; - q^{-1} \Big( K_{\alpha_{k+1}} \otimes E_{\alpha_{k+1}} + E_{\alpha_{k+1}} \otimes 1 \Big) \\
&\qquad\quad \cdot \Big( K_{\beta_i^k} \otimes E_{\beta_i^k} + E_{\beta_i^k} \otimes 1 + (1 - q^{-2}) \sum_{l=i}^{k-1} E_{\beta_i^l} K_{\beta_{l+1}^k} \otimes E_{\beta_{l+1}^k} \Big) \\
&\quad = K_{\beta_i^{k+1}} \otimes E_{\beta_i^{k+1}} + E_{\beta_i^{k+1}} \otimes 1 \\
&\quad\; + \Big( E_{\beta_i^k} K_{\alpha_{k+1}} - q^{-1} K_{\alpha_{k+1}} E_{\beta_i^k} \Big) \otimes E_{\alpha_{k+1}} \\
&\quad\; + (1 - q^{-2}) \sum_{l=i}^{k-1} \Big( E_{\beta_i^l} K_{\beta_{l+1}^k} K_{\alpha_{k+1}} \otimes E_{\beta_{l+1}^k} E_{\alpha_{k+1}} - q^{-1} K_{\alpha_{k+1}} E_{\beta_i^l} K_{\beta_{l+1}^k} \otimes E_{\alpha_{k+1}} E_{\beta_{l+1}^k} \Big) \\
&\quad\; + \Big( K_{\beta_i^k} E_{\alpha_{k+1}} - q^{-1} E_{\alpha_{k+1}} K_{\beta_i^k} \Big) \otimes E_{\beta_i^k} \\
&\quad\; + (1 - q^{-2}) \sum_{l=i}^{k-1} \Big( E_{\beta_i^l} K_{\beta_{l+1}^k} E_{\alpha_{k+1}} - q^{-1} E_{\alpha_{k+1}} E_{\beta_i^l} K_{\beta_{l+1}^k} \Big) \otimes E_{\beta_{l+1}^k}
\end{aligned}$$

$$= K_{\beta_i^{k+1}} \otimes E_{\beta_i^{k+1}} + E_{\beta_i^{k+1}} \otimes 1 + (1-q^{-2}) \sum_{l=i}^{k} E_{\beta_i^l} K_{\beta_{l+1}^{k+1}} \otimes E_{\beta_{l+1}^{k+1}}.$$

Clearly whenever $r \geq 2$ there appear other elements than the above $E_{\beta_i^k}$, namely the root vectors for $E_{\beta_i^r}$ with $1 \leq i < r$ are $T_{\alpha_r}(E_{\beta_i^{r-1}}) = E_{\alpha_r} E_{\beta_i^{r-1}} - q^{-1} E_{\beta_i^{r-1}} E_{\alpha_r}$. Their co-product computes as

$$\begin{aligned}
&\Delta\Big(T_{\alpha_r}(E_{\beta_i^{r-1}})\Big) \\
&= \Delta(E_{\alpha_r} E_{\beta_i^{r-1}} - q^{-1} E_{\beta_i^{r-1}} E_{\alpha_r}) \\
&= \Big(E_{\alpha_r} \otimes 1 + K_{\alpha_r} \otimes E_{\alpha_r}\Big) \\
&\qquad \cdot \Big(K_{\beta_i^{r-1}} \otimes E_{\beta_i^{r-1}} + E_{\beta_i^{r-1}} \otimes 1 + (1-q^{-2}) \sum_{l=i}^{r-2} E_{\beta_i^l} K_{\beta_{l+1}^{r-1}} \otimes E_{\beta_{l+1}^{r-1}}\Big) \\
&\quad - q^{-1}\Big(K_{\beta_i^{r-1}} \otimes E_{\beta_i^{r-1}} + E_{\beta_i^{r-1}} \otimes 1 + (1-q^{-2}) \sum_{l=i}^{r-2} E_{\beta_i^l} K_{\beta_{l+1}^{r-1}} \otimes E_{\beta_{l+1}^{r-1}}\Big) \\
&\qquad \Big(E_{\alpha_r} \otimes 1 + K_{\alpha_r} \otimes E_{\alpha_r}\Big) \\
&= K_{\beta_i^r} \otimes \Big(E_{\alpha_r} E_{\beta_i^{r-1}} - q^{-1} E_{\beta_i^{r-1}} E_{\alpha_r}\Big) + \Big(E_{\alpha_r} E_{\beta_i^{r-1}} - q^{-1} E_{\beta_i^{r-1}} E_{\alpha_r}\Big) \otimes 1 \\
&\quad + \Big(E_{\alpha_r} K_{\beta_i^{r-1}} - q^{-1} K_{\beta_i^{r-1}} E_{\alpha_r}\Big) \otimes E_{\beta_i^{r-1}} \\
&\quad + \Big(K_{\alpha_r} E_{\beta_i^{r-1}} - q^{-1} E_{\beta_i^{r-1}} K_{\alpha_r}\Big) \otimes E_{\alpha_r} \\
&\quad + (1-q^{-2}) \sum_{l=i}^{r-2} \Big(E_{\alpha_r} E_{\beta_i^l} K_{\beta_{l+1}^{r-1}} - q^{-1} E_{\beta_i^l} K_{\beta_{l+1}^{r-1}} E_{\alpha_r}\Big) \otimes E_{\beta_{l+1}^{r-1}} \\
&\quad + (1-q^{-2}) \sum_{l=i}^{r-2} \Big(K_{\alpha_r} E_{\beta_i^l} K_{\beta_{l+1}^{r-1}} \otimes E_{\alpha_r} E_{\beta_{l+1}^{r-1}} - q^{-1} E_{\beta_i^l} K_{\beta_{l+1}^{r-1}} K_{\alpha_r} \otimes E_{\beta_{l+1}^{r-1}} E_{\alpha_r}\Big) \\
&= K_{\beta_i^r} \otimes T_{\alpha_r}(E_{\beta_i^{r-1}}) + T_{\alpha_r}(E_{\beta_i^{r-1}}) \otimes 1 + (1-q^{-2}) E_{\alpha_r} K_{\beta_i^{r-1}} \otimes E_{\beta_i^{r-1}} \\
&\quad + (1-q^{-2})^2 \sum_{l=i}^{r-2} E_{\beta_i^l} E_{\alpha_r} K_{\beta_{l+1}^{r-1}} \otimes E_{\beta_{l+1}^{r-1}} \\
&\quad + (1-q^{-2}) \sum_{l=i}^{r-2} E_{\beta_i^l} K_{\beta_{l+1}^{k}} \otimes T_{\alpha_r}(E_{\beta_{l+1}^{r-1}}).
\end{aligned}$$

The subalgebra $U^+[w^+]$ has then the following PBW-type base:

$$\{E_{\beta_r^r}^{m_r^r} E_{\beta_1^1}^{m_1^1} \cdots E_{\beta_1^{r-1}}^{m_1^{r-1}} T_{\alpha_r}(E_{\beta_1^{r-1}})^{m_1^r} \cdots T_{\alpha_r}(E_{\beta_{r-1}^{r-1}})^{m_{r-1}^r} E_{\beta_r^{r+1}}^{m_r^{r+1}} \cdots E_{\beta_r^n}^{m_r^n} \cdots E_{\beta_n^n}^{m_n^n} | m_i^k \in \mathbb{N}_0\} \tag{2.2}$$

Let us now apply the deformation. Set $\Theta = \{\alpha_r\}$, clearly $\Theta \in T^w$ and fix a $\zeta \in k$, $\zeta \neq 0$. By Theorem 2.2 is the restriction on $U^+[w^+]$ of the map $\varphi\colon U^+ \to k^*$ given by

$$\varphi(E_{\alpha_i}) = \begin{cases} q\zeta & \text{for } i = r, \\ 0 & \text{else} \end{cases}$$

a character of $U^+[w^+]$. Indeed, for $\beta_i^k \in \Phi_{w^+}^+$ we have $\varphi(E_{\beta_i^k}) = 0$ whenever $i \neq r$ or $k \neq r$ while $\varphi(E_{\beta_r^r}) = \varphi(E_{\alpha_r}) = q\zeta$. Up from now we use the restriction of $\varphi$, still denoting it with $\varphi$. The support of $\varphi$ is given by the set $\{n\alpha_r \mid n \in \mathbb{N}_0\}$. Its orthogonal complement is spanned by

$$\begin{aligned}
&\alpha_1 + 2\alpha_2, \alpha_3, \ldots, \alpha_n && (r = 1),\\
&\alpha_1, \ldots, \alpha_{r-2}, 2\alpha_{r-1} + \alpha_r, \alpha_r, \alpha_r + 2\alpha_{r+1}, \alpha_{r+1}, \ldots, \alpha_n && (1 < r < n),\\
&\alpha_1, \ldots, \alpha_{n-2}, 2\alpha_{n-1} + \alpha_n && (r = n).
\end{aligned}$$

Set $L = (\operatorname{supp}\varphi)^\perp$. We will now give the PBW-type base of the algebra $C(w^+, \varphi, L)$. From (2.1) and $(\beta_i^k, \beta_i^k) = 2$ follows

$$\Delta \circ \psi(E_{\beta_i^k}) = q^{-1} \cdot 1 \otimes E_{\beta_i^k} K_{\beta_i^k}^{-1} + E_{\beta_i^k} K_{\beta_i^k}^{-1} \otimes K_{\beta_i^k}^{-1} + (1 - q^{-2}) \sum_{l=i}^{k-1} q^{-1} E_{\beta_i^l} K_{\beta_i^l}^{-1} \otimes E_{\beta_{l+1}^k} K_{\beta_{l+1}^k}^{-1}$$

so that we get under the contraction on the first tensor with $q^{-1}(\varphi \circ \psi^{-1})$ for $\beta_i^k$ with $1 \le i \le k \le r-1$ and $r+1 \le i \le k \le n$.

$$K_{\beta_i^k}^{-1} E_{\beta_i^k}.$$

These elements do not depend on $\zeta$. We find

$$\begin{aligned}
\Delta \circ \psi(T_{\alpha_r}(E_{\beta_i^{r-1}})) &= q^{-1} \cdot 1 \otimes T_{\alpha_r}(E_{\beta_i^{r-1}}) K_{\beta_i^r}^{-1} + T_{\alpha_r}(E_{\beta_i^{r-1}}) K_{\beta_i^{r-1}}^{-1} \otimes K_{\beta_i^r}^{-1}\\
&+ (1 - q^{-2}) q^{-1} E_{\alpha_r} K_{\alpha_r}^{-1} \otimes E_{\beta_i^{r-1}} K_{\beta_i^r}^{-1}\\
&+ (1 - q^{-2})^2 \sum_{l=i}^{r-2} q^{-1} E_{\beta_i^l} E_{\alpha_r} K_{\beta_i^l}^{-1} \otimes E_{\beta_{l+1}^{r-1}} K_{\beta_i^r}^{-1}\\
&+ (1 - q^{-2}) \sum_{l=i}^{r-2} q^{-1} E_{\beta_i^l} K_{\beta_i^l}^{-1} \otimes T_{\alpha_r}(E_{\beta_{l+1}^{r-1}}) K_{\beta_i^r}^{-1}.
\end{aligned}$$

Under the contraction the two sums vanisch for every $i < r$. Since $\zeta$ is fixed we simply write $G_{\beta_i^k}$ instead of $G^{(\zeta)}_{\beta_i^k}$. The contraction yields the elements

$$\begin{aligned}
G_{\beta_i^r} &= K_{\beta_i^r}^{-1}\Big(T_{\alpha_r}(E_{\beta_i^{r-1}}) + \zeta(q - q^{-1}) E_{\beta_i^{r-1}}\Big) \quad (1 \le i < r),\\
G_{\beta_r^r} &= K_{\alpha_r}^{-1}(E_{\alpha_r} + \zeta \cdot 1),\\
G_{\beta_r^k} &= K_{\beta_r^k}^{-1}\Big(E_{\beta_r^k} + \zeta(q - q^{-1}) E_{\beta_{r+1}^k}\Big) \qquad (r < k \le n).
\end{aligned}$$

Recall that the map $\big(q^{-1}(\varphi \circ \psi^{-1}) \otimes \mathrm{id}\big)\Delta\psi$ is an isomorphism of algebras [HK11, Lemma 2.11 & 2.14]. As conclusion we have indeed a PBW-type base (or at least a deformation of it) coming from $U^+[w^+]$ of $C(w^+, \varphi, \emptyset)$. It suffices to to replace some elements in equation (2.2):

$$\{G_{\beta_r^r}^{m_r^r} E_{\beta_1^1}^{m_1^1} \cdots E_{\beta_1^{r-1}}^{m_1^{r-1}} G_{\beta_1^r}^{m_1^r} \cdots G_{\beta_{r-1}^r}^{m_{r-1}^r} G_{\beta_r^{r+1}}^{m_r^{r+1}} \cdots G_{\beta_r^n}^{m_r^n} \cdots E_{\beta_n^n}^{m_n^n} \mid m_i^k \in \mathbb{N}_0\}.$$

For $\mu \in L$ we have $(\mu, \beta_i^r) = (\mu, \beta_i^{r-1})$ and so $K_\mu G_{\beta_i^r} = q^{(\mu, \beta_i^r)} G_{\beta_i^r} K_\mu$ for $1 \le i < r$ and for $r < k \le n$ holds $(\mu, \beta_r^k) = (\mu, \beta_{r+1}^k)$, so that $K_\mu G_{\beta_r^k} = q^{(\mu, \beta_r^k)} G_{\beta_r^k} K_\mu$. In particular we can extend the map $\big(q^{-1}(\varphi \circ \psi^{-1}) \otimes \mathrm{id}\big)\Delta\psi$ as an isomorphism of algebras from $U^+[w^+] \to C(w^+, \varphi, \emptyset)$ to $T_L U^+[w^+] \to C(w^+, \varphi, L)$ by setting $\psi(K_\mu) = K_\mu$ and $\varphi(K_\mu) = 1$. We can now transfer the restriction of the multiplication map (cf. [Jan96, Thm. 4.21])

$$T_L \otimes U^+[w^+] \to T_L U^+[w^+], \quad u_1 \otimes u_2 \mapsto u_1 u_2$$

to the algebra $C(w^+, \varphi, L)$ and it thus has the basis

$$\{K_\mu G_{\beta_r^r}^{m_r^r} E_{\beta_1^1}^{m_1^1} \cdots E_{\beta_1^{r-1}}^{m_1^{r-1}} G_{\beta_1^r}^{m_1^r} \cdots G_{\beta_{r-1}^r}^{m_{r-1}^r} G_{\beta_r^{r+1}}^{m_r^{r+1}} \cdots G_{\beta_r^n}^{m_r^n} \cdots E_{\beta_n^n}^{m_n^n} \mid \mu \in L, m_i^k \in \mathbb{N}_0\}.$$

Now we add $U^-[w^-]$, so we define

$$\mathfrak{A} = \mathfrak{A}_{n,r} = \mathfrak{A}_{n,r}^{(\zeta)} = U^-[w^-]C(w^+, \varphi, L).$$

Since $U^-[w^-]$ and $C(w^+, \varphi, L)$ are right coideal subalgebras, so is $\mathfrak{A}$ a right coideal. It remains to show that $\mathfrak{A}$ is an algebra.

Using the results of the next section, we state:

THEOREM 2.4 *$\mathfrak{A}$ is a right coideal subalgebra.*

This is suggested by [HK12] since $\mathfrak{A}$ is almost homogeneous. Let us thus check, that $U^-[w^-]U^0U^+[w^+]$ is a homogeneous right coideal subalgebra. Using the notation of that work we have to find elements $x, u \in W$ and $J \subset \Pi \cap x\Pi$ with $u^{-1} \leq_R x$ such that $w^- = uw_J$ and $w^+ = uw_Jx$ where $\leq_R$ is the weak order on $W$ and $w_J$ the longest element in the subgroup generated by $J$. Set $x = s_{\alpha_1+\ldots+\alpha_n}$, $u = s_1^{(r-1)} s_{r+1}^{(n)} = s_{\alpha_1} \cdots s_{\alpha_{r-1}} s_{\alpha_{r+1}} \cdots s_{\alpha_n}$ and $J = \{s_{\alpha_2}, \ldots, s_{\alpha_{n-1}}\} \setminus \{s_{\alpha_r}\}$. Since

$$s_{\alpha_1+\ldots+\alpha_n}(\alpha_i) = \begin{cases} -(\alpha_2 + \ldots + \alpha_n) & i = 1, \\ -(\alpha_1 + \ldots + \alpha_{n-1}) & i = n, \\ \alpha_i & \text{else} \end{cases}$$

follows $J = \Pi \cap x\Pi$ and in $l(x) = 2n - 1$. Since $s_{\alpha_n} \cdots s_{\alpha_2}(\alpha_1) = \alpha_1 + \ldots + \alpha_n$ we have $x = u^{-1} s_{\alpha_1} u$ by [Hum90, Prop 1.2] and $l(u) = n - 1$ so follows $u^{-1} \leq_R x$. Thus the triple $(x, u, J) \in B(W)$. By definition have we $w^- = uw_J$. We have $w^-(\alpha_r) = \alpha_1 + \ldots + \alpha_n$, so we can write $x = (w^-)^{-1} s_{\alpha_1} w^-$, thus having $uw_Jx = w_0 w_0^{-1} s_{\alpha_1} w_0 = s_{\alpha_1} w_0 = w^+$.

For the remainder of this section let us examine some more structural properties of $\mathfrak{A}$. Define for $1 \leq j \leq l \leq n$ the roots $\gamma_j^l = \beta_j^l$. Define

$$F_{\gamma_j^l} = T_{s_{\alpha_j}} \cdots T_{s_{\alpha_{l-1}}}(F_{\alpha_l}).$$

The elements with index $\gamma_j^l$ for $1 \leq j \leq l < r$ and $r < j \leq l \leq n$ form a PBW-type basis for $U^-[w^-]$. So we get an ordered basis for $\mathfrak{A}$:

$$\{F_{\gamma_1^1}^{\tilde{m}_1^1} \cdots F_{\gamma_{r-1}^{r-1}}^{\tilde{m}_{r-1}^{r-1}} F_{\gamma_{r+1}^{r+1}}^{\tilde{m}_{r+1}^{r+1}} \cdots F_{\gamma_n^n}^{\tilde{m}_n^n} K_\mu G_{\beta_r^r}^{m_r^r} E_{\beta_1^1}^{m_1^1} \cdots E_{\beta_1^{r-1}}^{m_1^{r-1}} G_{\beta_1^r}^{m_1^r} \cdots G_{\beta_{r-1}^r}^{m_{r-1}^r} G_{\beta_r^{r+1}}^{m_r^{r+1}} \cdots G_{\beta_r^n}^{m_r^n} \cdots E_{\beta_n^n}^{m_n^n} \\ \mid \mu \in L, \tilde{m}_i^k, m_i^k \in \mathbb{N}_0\}.$$

(2.3)

Since for all $1 \leq i \leq n$ with $i \leq r-2$ or $i \geq r+2$ the elements $K_{\alpha_i}, K_{\alpha_i}^{-1}, E_{\alpha_i}F_{\alpha_i}$ are in $\mathfrak{A}$, which generate a $U_q(\mathfrak{sl}_{r-1})$ and a $U_q(\mathfrak{sl}_{n-r})$, there is a natural embedding of $U_q(\mathfrak{sl}_{r-1}) \otimes U_q(\mathfrak{sl}_{n-r}) \hookrightarrow \mathfrak{A}$. This means that we have to verify that $G_{\beta_i^r}F_{\gamma_j^l}, G_{\beta_r^k}F_{\gamma_j^l}$ and $E_{\beta_i^k}F_{\gamma_j^{r-1}}, E_{\beta_i^k}F_{\gamma_{r+1}^l}$ are in $\mathfrak{A}$ for the corresponding indices $i, j, k, l$.

As conclusion of the ordered basis in (2.3) we get a triangular decomposition similar to that of $U_q(\mathfrak{g})$ in general and for their homogeneous right coideal subalgebras, for the latter see the work of Kharchenko [Kha10]. Consequently set $\mathfrak{A}^- = U^-[w^-]$ and $\mathfrak{A}^+ = \langle G_{\beta_{r-1}^r}, G_{\beta_r^{r+1}}, E_{\alpha_1}, \ldots, E_{\alpha_{r-2}}, E_{\alpha_{r+2}}, \ldots, E_{\alpha_n} \rangle$ as borel parts and set as torus $\mathfrak{A}^0 = \langle G_{\beta_r^r}, K_\mu \mid \mu \in L \rangle$.

PROPOSITION 2.5 *The multiplication map*

$$\mathfrak{A}^- \otimes \mathfrak{A}^0 \otimes \mathfrak{A}^+ \longrightarrow \mathfrak{A}, \qquad c_1 \otimes c_2 \otimes c_3 \longmapsto c_1c_2c_3$$

*is an isomorphism of vector spaces.*

The subalgebra $\mathfrak{A}$ is not graded by the induced grading of $U_q(\mathfrak{sl}_{n+1})$, the reason is very simpel: The elements $G_{\beta_i^r}$ resp. $G_{\beta_r^k}$ are not homogeneous, being a sum of weight vectors of weight $\beta_i^r$ and $\beta_i^{r-1}$ resp. $\beta_r^k$ and $\beta_{r-1}^k$ (well, this is not true for $G_{\beta_r^r}$: it decomposes into a component of weight $\alpha_r$ and 0.) But since the difference between $\beta_i^r$ and $\beta_i^{r-1}$ resp. $\beta_r^k$ and $\beta_{r-1}^k$ is $\alpha_r$, the inhomogeneous generators of $\mathfrak{A}$ become homogeneous by ›ignoring‹ $\alpha_r$.

Concretely realised: Set $P = \mathbb{Z}\Big(\Pi \setminus \{\alpha_r\}\Big)$ and define on $U_q(\mathfrak{sl}_{n+1})$ a $P$-grading by $\deg(E_{\alpha_i}) = \alpha_i$, $\deg(F_{\alpha_i}) = -\alpha_i$ and $\deg(K_{\alpha_i}) = 0$ for $1 \leq i \leq n$ with $i \neq r$, and $\deg(E_{\alpha_r}) = \deg(F_{\alpha_r}) = \deg(K_{\alpha_r}) = 0$. This grading is well-defined since all defining relations of $U_q(\mathfrak{sl}_{n+1})$ are homogeneous subject to $P$. All generators of $\mathfrak{A}$ are homogeneous with respect to $P$, so $P$ is a grading on $\mathfrak{A}$. For each $\mu \in P$ the space $\mathfrak{A}_\mu$ is non-trivial.

Let $\mu \in P$ and $c \in \mathfrak{A}_\mu$ and let $\nu \in L$, then

$$\begin{aligned} K_\nu c &= q^{(\nu,\mu)} c K_\nu, \\ G_{\beta_r^r} c &= q^{-(\alpha_r,\mu)} c G_{\beta_r^r}. \end{aligned} \tag{2.4}$$

The coefficients appearing do only depend on the element of $\mathfrak{A}^0$ and $\mu$. Since the sublattice of $Q$ generated by $L \cup \{\alpha_r\}$ has rang $n$ and $(\cdot,\cdot)$ is a scalar product we have

$$\mathfrak{A}_\mu = \{c \in \mathfrak{A} \mid K_\nu c = q^{(\nu,\mu)} c K_\nu \text{ for } \mu \in L \text{ and } G_{\beta_r^r} c = q^{-(\alpha_r,\mu)} c G_{\beta_r^r}\} \tag{2.5}$$

and

$$\mathfrak{A} = \oplus_{\mu \in P} \mathfrak{A}_\mu.$$

Thus it makes sense to denote $\mathfrak{A}_\mu$ as a weight space of weight $\mu$ even though there is not really a good action of $\mathfrak{A}$ on itself. The lack of good actions has the reason that $G_{\beta_r^r}$ does not have an inverse element (neither in $\mathfrak{A}$ nor in $U_q(\mathfrak{sl}_{n+1})$) and the induced ad-action of $G_{\beta_r^r}$ is not diagonal. We set $P^+ = \mathbb{N}_0\Big(\Pi \setminus \{\alpha_r\}\Big)$ and $P^- = -P^+$. The monoids correspond to $\mathfrak{A}^+$ resp. $\mathfrak{A}^-$.

## Commutation Rules between $E_{\beta_i^k}$ and $F_{\gamma_j^l}$

It is clear that $F_{\gamma_j^l}$ and $E_{\beta_i^k}$ commute whenever $j > k$ or $i > l$. Let us summarise some recursion formulas for the $E_{\beta_i^k}$ and $F_{\gamma_j^l}$.

LEMMA 2.6 *Let $i, k, r, j, l, s \in \mathbb{N}$, then the following recursions hold:*

$$E_{\beta_i^k} = E_{\beta_i^{k-1}} E_{\alpha_k} - q^{-1} E_{\alpha_k} E_{\beta_i^{k-1}} \qquad (1 \le i < k \le n), \tag{2.6}$$

$$E_{\beta_i^k} = E_{\alpha_i} E_{\beta_{i+1}^k} - q^{-1} E_{\beta_{i+1}^k} E_{\alpha_i} \qquad (1 \le i < k \le n), \tag{2.7}$$

$$E_{\beta_i^k} = E_{\beta_i^{s-1}} E_{\beta_s^k} - q^{-1} E_{\beta_s^k} E_{\beta_i^{s-1}} \qquad (1 \le i < s \le k \le n). \tag{2.8}$$

*Proof.* Equation (2.6) is just the recursion given in the paragraph right before equation (2.1). Equations (2.7) and (2.8) follow by induction on $k - i$: For $k - i = 1$ they are both just (2.6) resp. (2.7). Now let $k - i > 1$ and note that $E_{\alpha_i} E_{\alpha_k} = E_{\alpha_k} E_{\alpha_i}$, then

$$\begin{aligned} E_{\beta_i^k} &= E_{\beta_i^{k-1}} E_{\alpha_k} - q^{-1} E_{\alpha_k} E_{\beta_i^{k-1}} \\ &= E_{\alpha_i} E_{\beta_{i+1}^{k-1}} E_{\alpha_k} - q^{-1} E_{\alpha_k} E_{\alpha_i} E_{\beta_{i+1}^{k-1}} - q^{-1} E_{\beta_{i+1}^{k-1}} E_{\alpha_i} E_{\alpha_k} + q^{-2} E_{\alpha_i} E_{\beta_{i+1}^{k-1}} E_{\alpha_k} \\ &= E_{\alpha_i} E_{\beta_{i+1}^k} - q^{-1} E_{\beta_{i+1}^k} E_{\alpha_i}. \end{aligned}$$

and (where we may assume $s < k$, otherwise it is just (2.6))

$$\begin{aligned} E_{\beta_i^k} &= E_{\beta_i^{k-1}} E_{\alpha_k} - q^{-1} E_{\alpha_k} E_{\beta_{ii}^{k-1}} \\ &= E_{\beta_i^{s-1}} E_{\beta_s^{k-1}} E_{\alpha_k} - q^{-1} E_{\beta_s^{k-1}} E_{\beta_i^{s-1}} E_{\alpha_k} - q^{-1} E_{\alpha_k} E_{\beta_i^{s-1}} E_{\beta_s^{k-1}} + q^{-2} E_{\alpha_k} E_{\beta_s^{k-1}} E_{\beta_i^{s-1}} \\ &= E_{\beta_i^{s-1}} \Big( E_{\beta_s^{k-1}} E_{\alpha_k} - q^{-1} E_{\alpha_k} E_{\beta_s^{k-1}} \Big) - q^{-1} \Big( E_{\beta_s^{k-1}} E_{\alpha_k} - q^{-1} E_{\alpha_k} E_{\beta_s^{k-1}} \Big) E_{\beta_i^{s-1}} \\ &= E_{\beta_i^{s-1}} E_{\beta_s^k} - q^{-1} E_{\beta_s^k} E_{\beta_i^{s-1}}. \end{aligned}$$

□

Let $1 \le i \le n$ and $\mu \in Q_+$ and $u \in U_\mu^+$ such that $F_{\alpha_i} u = u F_{\alpha_i}$. Then

$$\Big(E_{\alpha_i} u - q^{-1} u E_{\alpha_i}\Big) F_{\alpha_i} = F_{\alpha_i} \Big(E_{\alpha_i} u - q^{-1} u E_{\alpha_i}\Big) + \frac{(1 - q^{-(\mu,\alpha_i)-1}) K_{\alpha_i} - (1 - q^{(\mu,E_{\alpha_i})-1}) K_{\alpha_i}^{-1}}{q - q^{-1}} \tag{2.9}$$

This is a consequence of the defining relations of $U_q(\mathfrak{sl}_{n+1})$. With the help of this formula we compute the equation of the following lemma.

LEMMA 2.7 *For $1 \le i < k \le n$ holds*

$$E_{\beta_i^k} F_{\alpha_i} = F_{\alpha_i} E_{\beta_i^k} - q^{-1} K_{\alpha_i}^{-1} E_{\beta_{i+1}^k}.$$

*Proof.* The Lemma follows from equation (2.7) of Lemma 2.6 and equation (2.9) with $u = E_{\beta_{i+1}^k}$ and $\mu = \beta_{i+1}^k = \alpha_{i+1} + \ldots + \alpha_k$. So $(\mu, \alpha_i) = -1$. □

LEMMA 2.8 *1. Let* $1 \leq i \leq n$ *and* $k, l \geq i$*, then holds*

$$E_{\beta_i^k} F_{\gamma_i^l} = \begin{cases} F_{\gamma_i^l} E_{\beta_i^k} + \left[K, \beta_i^k\right] & (k = l), \\ F_{\gamma_i^l} E_{\beta_i^k} + q F_{\gamma_{k+1}^l} K_{\beta_i^k} & (k < l), \\ F_{\gamma_i^l} E_{\beta_i^k} - q^{-1} K_{\beta_i^l}^{-1} E_{\beta_{l+1}^k} & (k > l). \end{cases}$$

*2. Let* $1 \leq i < j \leq n$ *and let* $k, l \geq j$*, then holds*

$$E_{\beta_i^k} F_{\gamma_j^l} = \begin{cases} F_{\gamma_j^l} E_{\beta_i^k} + K_{\beta_j^k} E_{\beta_i^{j-1}} & (k = l), \\ F_{\gamma_j^l} E_{\beta_i^k} - (1 - q^2) F_{\gamma_{k+1}^l} K_{\beta_j^k} E_{\beta_i^{j-1}} & (k < l), \\ F_{\gamma_j^l} E_{\beta_i^k} & (k > l). \end{cases}$$

*3. Let* $1 \leq j < i \leq n$ *and* $k, l \geq i$*, then holds*

$$E_{\beta_i^k} F_{\gamma_j^l} = \begin{cases} F_{\gamma_j^l} E_{\beta_i^k} - F_{\gamma_j^{i-1}} K_{\beta_i^k}^{-1} & (k = l), \\ F_{\gamma_j^l} E_{\beta_i^k} & (k < l), \\ F_{\gamma_j^l} E_{\beta_i^k} + (1 - q^{-2}) F_{\gamma_j^{i-1}} K_{\beta_i^l}^{-1} E_{\beta_{l+1}^k} & (k > l). \end{cases}$$

*Proof.* In the computations of the proof we shall apply a special case of formula (1.1). Let $1 \leq i < j \leq n$, set $v = s_{\alpha_j} \cdots s_{\alpha_{l-1}}, \mu = \alpha_l$ and $u = E_{\alpha_l}$. Then $v(\mu) - \mu = \alpha_j + \ldots + \alpha_{l-1}$ and $\prod_{\alpha \in \Pi} (-q_\alpha)^{m_\alpha} = (-q)^{l-j}$ so that we can rewrite $F_{\gamma_j^l}$:

$$\begin{aligned} F_{\gamma_j^l} &= T_{s_{\alpha_j}} \cdots T_{s_{\alpha_{j-1}}} (\omega(E_{\alpha_j})) \\ &= (-q)^{l-j} \omega(T_{s_{\alpha_j}} \cdots T_{s_{\alpha_{l-1}}} (E_{\alpha_j})) \\ &= (-q)^{l-j} \omega(E_{\beta_j^l}). \end{aligned}$$

Since $\omega$ is an involution we obtain $E_{\beta_j^l} = (-q)^{j-l} \omega(F_{\gamma_j^l})$. For Part 1. we compute with the definitions of $E_{\beta_i^k}, F_{\gamma_j^l}$ and Lemma 2.7. For $k = l$ follows immediately:

$$E_{\beta_i^k} F_{\gamma_i^k} = T_{\alpha_i} \cdots T_{\alpha_{k-1}} (E_{\alpha_k} F_{\alpha_k}) = F_{\gamma_i^k} E_{\beta_i^k} + \left[K, \beta_i^k\right].$$

For $k > l$:

$$\begin{aligned} E_{\beta_i^k} F_{\gamma_i^l} &= T_{\alpha_i} \cdots T_{\alpha_{l-1}} \left(E_{\beta_l^k} F_{\alpha_l}\right) \\ &= T_{\alpha_i} \cdots T_{\alpha_{l-1}} \left(F_{\alpha_l} E_{\beta_l^k}\right) - q^{-1} T_{\alpha_i} \cdots T_{\alpha_{l-1}} \left(K_{\alpha_l}^{-1} E_{\beta_{l+1}^k}\right) \\ &= F_{\gamma_i^l} E_{\beta_i^k} - q^{-1} K_{\beta_i^l}^{-1} E_{\beta_{l+1}^k}. \end{aligned}$$

For $k < l$ we can either compute directly deducing the corresponding formula in Lemma 2.7 or using the involution $\omega$ and the case $k > l$:

$$\begin{aligned}
E_{\beta_i^k}F_{\gamma_i^l} &= q^{i-k+l-i}\omega\Big(F_{\gamma_i^k}E_{\beta_i^l}\Big)\\
&= q^{i-k+l-i}\omega\Big(E_{\beta_i^l}F_{\gamma_i^k}\Big) + \omega\Big(K^{-1}_{\beta_i^k}\Big)q^{l-(k+1)}\omega\Big(E_{\beta_{k+1}^l}\Big)\\
&= F_{\gamma_i^l}E_{\beta_i^k} + K_{\beta_i^k}F_{\gamma_{k+1}^l}\\
&= F_{\gamma_i^l}E_{\beta_i^k} + qF_{\gamma_{k+1}^l}K_{\beta_i^k}.
\end{aligned}$$

For Part 2. we use the result of Part 1. For $k = l$:

$$\begin{aligned}
E_{\beta_i^k}F_{\gamma_j^k} &= E_{\beta_i^{j-1}}E_{\beta_j^k}F_{\gamma_j^k} - q^{-1}E_{\beta_j^k}F_{\gamma_j^k}E_{\beta_i^{j-1}}\\
&= F_{\gamma_j^k}E_{\beta_j^{j-1}}E_{\beta_j^k} - q^{-1}F_{\gamma_j^k}E_{\beta_j^k}E_{\beta_i^{j-1}} + E_{\beta_i^{j-1}}\Big[K,\beta_j^k\Big] - q^{-1}\Big[K,\beta_j^k\Big]E_{\beta_i^{j-1}}\\
&= F_{\gamma_j^k}E_{\beta_i^k} + K_{\beta_j^k}E_{\beta_i^{j-1}}.
\end{aligned}$$

For $k < l$:

$$\begin{aligned}
E_{\beta_i^k}F_{\gamma_j^l} &= E_{\beta_i^{j-1}}E_{\beta_j^k}F_{\gamma_j^l} - q^{-1}E_{\beta_j^k}F_{\gamma_j^l}E_{\beta_i^{j-1}}\\
&= F_{\gamma_j^l}E_{\beta_i^k} + qE_{\beta_i^{j-1}}F_{\gamma_{k+1}^l}K_{\beta_j^k} - F_{\gamma_{k+1}^l}K_{\beta_j^k}E_{\beta_i^{j-1}}\\
&= F_{\gamma_j^l}E_{\beta_i^k} - (1-q^2)F_{\gamma_{k+1}^l}K_{\beta_j^k}E_{\beta_i^{j-1}}.
\end{aligned}$$

For $k > l$:

$$\begin{aligned}
E_{\beta_i^k}F_{\gamma_j^l} &= E_{\beta_i^{j-1}}E_{\beta_j^k}F_{\gamma_j^l} - q^{-1}E_{\beta_j^k}F_{\gamma_j^l}E_{\beta_i^{j-1}}\\
&= F_{\gamma_j^l}E_{\beta_i^k} - q^{-1}E_{\beta_i^{j-1}}K^{-1}_{\beta_j^l}E_{\beta_{l+1}^k} + q^{-2}K^{-1}_{\beta_j^{l+1}}E_{\beta_{l+1}^k}E_{\beta_i^{j-1}}\\
&= F_{\gamma_j^l}E_{\beta_i^k}.
\end{aligned}$$

For Part 3 we are going to apply the involution $\omega$ and the proven Part 2. of the Lemma. For $k = l$:

$$\begin{aligned}
E_{\beta_i^k}F_{\gamma_j^k} &= q^{i-k+k-j}\omega\Big(F_{\gamma_i^k}E_{\beta_j^k}\Big) = F_{\gamma_j^k}E_{\beta_i^k} - q^{i-j}\omega\Big(K_{\beta_i^k}E_{\beta_j^{i-1}}\Big)\\
&= F_{\gamma_j^k}E_{\beta_i^k} - qK^{-1}_{\beta_i^k}F_{\gamma_j^{i-1}}\\
&= F_{\gamma_j^k}E_{\beta_i^k} - F_{\gamma_j^{i-1}}K^{-1}_{\beta_i^k}.
\end{aligned}$$

For $k < l$:

$$E_{\beta_i^k}F_{\gamma_j^l} = q^{i-k+l-j}\omega\Big(F_{\gamma_i^k}E_{\beta_j^l}\Big) = q^{i-k+l-j}\omega\Big(E_{\beta_j^l}F_{\gamma_j^l}\Big) = F_{\gamma_j^l}E_{\beta_i^k}.$$

For $k > l$:

$$\begin{aligned}
E_{\beta_i^k}F_{\gamma_j^l} &= (-q)^{i-k+l-j}\omega\Big(F_{\gamma_i^k}E_{\beta_j^l}\Big)\\
&= F_{\gamma_j^l}E_{\beta_i^k} - (1-q^2)(-q)^{(l+1)-k}\omega\Big(F_{\gamma_{l+1}^k}\Big)\omega\Big(K_{\beta_i^l}\Big)(-q)^{i-1-j}\omega\Big(E_{\beta_j^{i-1}}\Big)\\
&= F_{\gamma_j^l}E_{\beta_i^k} - (1-q^2)E_{\beta_{l+1}^k}K^{-1}_{\beta_i^l}F_{\gamma_j^{i-1}}\\
&= F_{\gamma_j^l}E_{\beta_i^k} + (1-q^{-2})F_{\gamma_j^{i-1}}K^{-1}_{\beta_i^l}E_{\beta_{l+1}^k}.
\end{aligned}$$

$\square$

As stated at the beginning of this section, we have to compute certain $G_{\beta_i^k} F_{\gamma_j^l}$ and certain $E_{\beta_i^k} F_{\gamma_j^l}$. For the the latter that is a direct consequence of the above Lemma. Let us summarise the formulas for $G_{\beta_i^k} F_{\gamma_j^l}$ in the following Proposition.

PROPOSITION 2.9 *1. Assume that $r \geq 2$, $1 \leq i < r$ and $1 \leq j \leq l < r$. Then holds*

$$G_{\beta_r^r} F_{\gamma_j^l} = q^{-\delta_{r,l+1}} F_{\gamma_j^l} G_{\beta_r^r},$$

$$G_{\beta_i^r} F_{\gamma_j^{r-1}} = \begin{cases} qF_{\gamma_i^{r-1}} G_{\beta_i^r} + G_{\beta_r^r} - \zeta K_{\beta_i^{r-1}}^{-2} K_{\alpha_r}^{-1} & (i = j), \\ F_{\gamma_j^{r-1}} G_{\beta_i^r} + G_{\beta_r^r} K_{\beta_i^{j-1}} E_{\beta_i^{j-1}} & (i < j), \\ F_{\gamma_j^{r-1}} G_{\beta_i^r} - \zeta(1 - q^{-2}) F_{\gamma_j^{i-1}} K_{\beta_i^{r-1}}^{-2} K_{\alpha_r}^{-1} & (i > j), \end{cases}$$

*and for $l < r - 1$*

$$G_{\beta_i^r} F_{\gamma_j^l} = \begin{cases} qF_{\gamma_i^l} G_{\beta_i^r} - q^{-1} G_{\beta_{l+1}^r} & (i = j), \\ F_{\gamma_j^l} G_{\beta_i^r} & (i < j), \\ F_{\gamma_i^j} G_{\beta_i^r} + q^{-2}(q - q^{-1}) F_{\gamma_j^{i-1}} G_{\beta_{l+1}^r} & (i > j). \end{cases}$$

*2. Assume that $r \leq n - 1$ and $r < j \leq k, l$. Then holds*

$$G_{\beta_r^r} F_{\gamma_j^l} = q^{-\delta_{r,j-1}} F_{\gamma_j^l} G_{\beta_r^r},$$

$$G_{\beta_r^h} F_{\gamma_{r+1}^l} = \begin{cases} qF_{\gamma_{r+1}^k} G_{\beta_r^k} + G_{\beta_r^r} - \zeta K_{\alpha_r}^{-1} K_{\beta_{r+1}^k}^{-2} & (k = l), \\ F_{\gamma_{r+1}^l} G_{\beta_r^k} + (q - q^{-1}) F_{\gamma_{k+1}^l} G_{\beta_r^r} & (k < l), \\ F_{\gamma_{r+1}^l} G_{\beta_r^k} - \zeta(1 - q^{-2}) K_{\alpha_r}^{-1} K_{\beta_{r+1}^l}^{-2} K_{\beta_{l+1}^k}^{-1} E_{\beta_{l+1}^k} & (k > l), \end{cases}$$

*and for $j > r + 1$*

$$G_{\beta_r^k} F_{\gamma_j^l} = \begin{cases} qF_{\gamma_j^k} G_{\beta_r^k} + G_{\beta_r^{j-1}} & (k = l), \\ F_{\gamma_j^l} G_{\beta_r^k} + (1 - q^{-1}) F_{\gamma_{k+1}^l} G_{\beta_r^{j-1}} & (k < l), \\ F_{\gamma_j^l} G_{\beta_r^k} & (k > l). \end{cases}$$

# Chapter 3

# Representations of $\mathfrak{A}_{n,r}$

In order to analyse the behaviour of $\mathfrak{A} = \mathfrak{A}_{n,r}$ on a finite dimensional (simple) $U_q(\mathfrak{sl}_{n+1})$-module, in this chapter we classify all simple, finite dimensional $\mathfrak{A}$-modules on which $G_{\beta_r^r}$ and $K_\mu, \mu \in L$ act diagonally. As it turns out, these modules are, up to an additional parameter—a non-zero scalar—in one-to-one correspondence with the finite dimensional, simple modules of $U_q(\mathfrak{sl}_r) \otimes U_q(\mathfrak{sl}_{n-r+1})$. After the classification we show a semisimplicity theorem for simple, finite dimensional $U_q(\mathfrak{sl}_{n+1})$-modules considered as $\mathfrak{A}$-modules. Since we have an embedding of $U_q(\mathfrak{sl}_{r-1}) \otimes U_q(\mathfrak{sl}_{n-r}) \hookrightarrow \mathfrak{A}$ we know the representation theory of a big part of $\mathfrak{A}$, e.g. that $K_{\alpha_1}, \dots, K_{\alpha_{r-2}}, K_{\alpha_{r+2}} \dots K_{\alpha_n}$ operate as diagonal operators. After analysing the representations of the ›missing‹ part we show that we can ›glue‹ both parts together via $U^-[w^-]$, which has embeddings in $\mathfrak{A}$ and $U_q(\mathfrak{sl}_r) \otimes U_q(\mathfrak{sl}_{n-r+1})$. Throughout this chapter we shall also often use $K^2_{\alpha_{r-1}} K_{\alpha_r}$ and $K_{\alpha_r} K^2_{\alpha_{r+1}}$, however, depending on $r$ one of them might not be well defined—let use the convention of non-existence of $K^2_{\alpha_{r-1}} K_{\alpha_r}$ resp. $K_{\alpha_r} K^2_{\alpha_{r+1}}$ whenever $r = n$ resp. $r = 1$. We will use the equivalent convention for $G_{\beta^r_{r-1}}$ and $G_{\beta_r^{r+1}}$.

Let us start with the motivation for the demand on $G_{\beta_r^r}$ and $K_\mu, \mu \in L$ being diagonalisable operators.

Proposition 3.1 *Let $\lambda \in \Lambda$ be a dominant weight and $L(\lambda)$ the simple $U_q(\mathfrak{sl}_{n+1})$-module of highest weight $\lambda$. Then $G_{\beta_r^r}$ is diagonalisable on $L(\lambda)$.*

*Proof.* We introduce a relation $\sim$ on the set of weights of $L(\lambda)$. Let $\mu = \lambda - \sum_{i=1}^n s_i \alpha_i$ and $\mu' = \lambda - \sum_{i=1}^n s_i' \alpha_i$ for suitable $s_i, s_i' \in \mathbb{N}_0$. Define

$$\mu \sim \mu' \iff (\alpha_r, s_{r-1}\alpha_r + s_r\alpha_r + s_{r+1}\alpha_{r+1}) = (\alpha_r, s'_{r-1}\alpha_r + s'_r\alpha_r + s'_{r+1}\alpha_{r+1}).$$

Let $[\mu]$ be the equivalence class of $\mu$ with respect to $\sim$ and $S$ the set of equivalence classes. Define on the set of equivalence classes the following linear ordering:

$$[\mu] \prec [\mu'] \iff (\alpha_r, s_{r-1}\alpha_r + s_r\alpha_r + s_{r+1}\alpha_{r+1}) < (\alpha_r, s'_{r-1}\alpha_r + s'_r\alpha_r + s'_{r+1}\alpha_{r+1}).$$

This definition is independent of the choice of representatives. Set $L(\lambda)_{[\mu]} = \oplus_{\nu \in [\mu]} L(\lambda)_\nu$, then of course $L(\lambda) = \oplus_{[\mu] \in S} L(\lambda)_{[\mu]}$. Let $\mathcal{B}$ be a basis with respect to this decomposition. Since $G_{\beta_r^r} = K^{-1}_{\alpha_r}(E_{\alpha_r} + \zeta \cdot 1)$ follows for $[\mu]$

$$G_{\beta_r^r} L(\lambda)_{[\mu]} \subset L(\lambda)_{[\mu]} \oplus L(\lambda)_{[\mu+\alpha_r]}.$$

We have $[\mu] \prec [\mu + \alpha_r]$. Let $t = |S|$, then we get the chain

$$[\mu_1] \prec [\mu_2] \prec \cdots \prec [\mu_t]$$

and

$$A = \begin{pmatrix} \zeta q^{(\alpha_r,\mu_1)} \cdot \mathrm{id} & * & \cdots\cdots & & & * \\ 0 & \zeta q^{(\alpha_r,\mu_2)} \cdot \mathrm{id} & * & \cdots & & * \\ 0 & 0 & \zeta q^{(\alpha_r,\mu_3)} \cdot \mathrm{id} & * & \cdots & * \\ 0 & \cdots & 0 & \ddots & \cdots & * \\ 0 & \cdots & \cdots & 0 & \ddots & * \\ 0 & \cdots & \cdots & \cdots & 0 & \zeta q^{(\alpha_r,\mu_t)} \cdot \mathrm{id} \end{pmatrix}$$

is the matrix of $G_{\beta_r^r}$ on $\mathcal{B}$ with respect the ordering $\prec$ using notation of block matrices. This matrix is diagonalisable since the minimal polynomial splits into linear factors, each occurring with multiplicity 1. □

Let $\lambda \in \Lambda$ be a dominant weight and let $\mu \in L$. It is well known that $K_\mu$ acts as diagonal operator on $L(\lambda)$. On the other hand follows from $(\alpha_r, \mu) = 0$ that $K_\mu G_{\beta_r^r} = G_{\beta_r^r} K_\mu$. With the above Proposition follows that $G_{\beta_r^r}$ and all$K_\mu, \mu \in L$ are simultaneously diagonalisable on $L(\lambda)$. This motivates the following definition:

DEFINITION 3.2 *Let $M$ be an $\mathfrak{A}$-module. $M$ is* simultaneously diagonalisable *if $G_{\beta_r^r}$ and $K_\mu$ for all $\mu \in L$ are diagonalisable on $M$ and $G_{\beta_r^r}$ has no eigenvalue equals to* 0.

As one analyses first the modules for $U_q(\mathfrak{sl}_2)$ and uses the results for the general case $U_q(\mathfrak{g})$ we shall do the same thing for $\mathfrak{A}$. So let us take a look at the case $n = 2$.

## The Case $n = 2$

Of course there are two different subalgebras $\mathfrak{A}_{2,1}^{(\zeta)}$ and $\mathfrak{A}_{2,2}^{(\zeta)}$ depending on $r$. From the automorphism that interchanges $\alpha_1$ and $\alpha_2$ in the Dynkindiagram $A_2$ follows immediately that $\mathfrak{A}_{2,1}^{(\zeta)}$ and $\mathfrak{A}_{2,2}^{(\zeta)}$ are isomorphic. Let us hence set $\mathfrak{A}_2 = \mathfrak{A}_{2,2}^{(\zeta)}$ and rename some elements.

$$\begin{aligned} F_1 &= F_{\alpha_1}, \\ G_2 &= G_{\beta_2^2} = K_{\alpha_2}^{-1}(E_{\alpha_2} + \zeta \cdot 1), \\ G_{12} &= G_{\beta_1^2} = K_{\alpha_1}^{-1} K_{\alpha_2}^{-1}(E_{\alpha_2}E_{\alpha_1} - q^{-1}E_{\alpha_1}E_{\alpha_2} + \zeta(q - q^{-1})E_{\alpha_1}), \\ K_{12}^{\pm 1} &= (K_{\alpha_1}^2 K_{\alpha_2})^{\pm 1}. \end{aligned}$$

Now $\mathfrak{A}_2$ is generated by these elements as a subalgebra of $U_q(\mathfrak{sl}_3)$, the following equations are a full set of relations on $\mathfrak{A}_2$ since $\mathfrak{A}_2$ has an ordered basis.

$$\begin{aligned} G_2 K_{12} &= K_{12} G_2, \\ G_2 F_1 &= q^{-1} F_1 G_2, \\ G_2 G_{12} &= q G_{12} G_2, \\ K_{12} F_1 &= q^{-3} F_1 K_{12}, \\ K_{12} G_{12} &= q^3 G_{12} K_{12}, \\ G_{12} F_1 &= q F_1 G_{12} + G_2 - \zeta K_{12}^{-1}. \end{aligned}$$

For the classification we use Verma-modules for $\mathfrak{A}_2$, using the technique for $U_q(\mathfrak{sl}_2)$ in [Jan96, Chapter 2]. To compute the action of $G_{12}$ we use

$$G_{12}F_1^s = q^s F_1^s G_{12} + \frac{1-q^{2s}}{1-q^2} F_1^{s-1}\Big(q^{-(s-1)}G_2 - q^{s-1}\zeta K_{12}^{-1}\Big) \tag{3.1}$$

which follows from the above listed relations by induction.

For each pair $\eta, \kappa \in k$ with $\eta, \kappa \neq 0$ there is an infinite dimensional $\mathfrak{A}_2$-module $M(\eta,\kappa)$ with basis $m_0, m_1, m_2, \dots$ such that for all $i \geq 0$

$$G_2 m_i = \eta q^{-i} m_i, \tag{3.2}$$

$$K_{12} m_i = \kappa q^{-3i} m_i, \tag{3.3}$$

$$F_1 m_i = m_{i+1}, \tag{3.4}$$

$$G_{12} m_i = \begin{cases} 0, & \text{if } i = 0, \\ \dfrac{1-q^{2i}}{1-q^2}\Big(q^{-(i-1)}\eta - q^{i-1}\zeta\kappa^{-1}\Big) m_{i-1}, & \text{if } i \geq 1. \end{cases} \tag{3.5}$$

Let $M$ be a simultaneously diagonalisable $\mathfrak{A}_2$-module and $m \in M$ satisfying $G_{12}m = 0$, $G_2 m = \eta m$ and $K_{12}m = \kappa m$. Then there is a unique homomorphism of $\mathfrak{A}_2$-modules $f\colon M(\eta,\kappa) \to M$ with $f(m_0) = m$. With the help of this universal property the simple, finite dimensional, simultaneously diagonalisable $\mathfrak{A}_2$-module will be classified.

Proposition 3.3 *The module $M(\eta,\kappa)$ contains exactly one proper submodule if and only if $\eta\kappa = \zeta q^{2(s-1)}$ holds for some integer $s \geq 1$, otherwise it is simple.*

*Proof.* Let $M'$ be a nonzero submodule of $M(\eta,\kappa)$. Since $M(\eta,\kappa)$ is the sum of its common eigenspaces, so is $M'$, i.e. $M'$ is spanned by the $m_i$ contained in $M'$. Choose $j \geq 0$ minimal with $m_j \in M'$. From (3.4) follows that $M'$ is generated by all $m_i$ with $i \geq j$. If $j$ is zero, then $M(\eta,\kappa) = M'$. If $j > 0$,then follows by (3.5) $G_{12}m_j = 0$, thus follows $\eta\kappa = \zeta q^{2(j-1)}$.

For $\eta\kappa \neq \zeta q^{2(s-1)}$ for all $s \geq 1$ the module $M(\eta,\kappa)$ is simple. If there is a $s \geq 1$ with $\eta\kappa = \zeta q^{2(s-1)}$, then there is at most one submodule different from 0 and $M(\eta,\kappa)$ and $G_{12}m_s = 0$, so we have a unique homomorphism $\varphi\colon M(\eta q^{-s}, \kappa q^{-3s}) \to M(\eta,\kappa)$ with $\varphi(m_0) = m_s$. The $\mathfrak{A}_2$-module $M(\eta q^{-s}, \kappa q^{-3s})$ is simple. □

Proposition 3.4 *For each $\eta \in k^*$ and $s \geq 0$ exists a simple simultaneously diagonalisable $\mathfrak{A}_2$-module of dimension $s+1$ and a basis $m_0, \dots, m_s$ such that*

$$G_2 m_i = \eta q^{-i},$$

$$K_{12} m_i = \eta^{-1}\zeta q^{2s-3i} m_i,$$

$$F_1 m_i = \begin{cases} m_{i+1}, & \textit{if } i < s, \\ 0, & \textit{if } i = s, \end{cases}$$

$$G_{12} m_i = \begin{cases} 0, & \textit{if } i = 0, \\ \dfrac{1-q^{2i}}{1-q^2}\Big(q^{-(i-1)}\eta - q^{-2s+i-1}\eta^{-1}\Big) m_{i-1}, & \textit{if } i \geq 1. \end{cases}$$

*Denote this module by $L(\eta,s)$. Each simple simultaneously diagonalisable $\mathfrak{A}_2$-module of dimension $s+1$ is isomorphic to $L(\eta,s)$ for a proper $\eta \neq 0$.*

*Proof.* The existence follows from the discussion above: Choose $M(\eta, \zeta\eta^{-1}q^{2s})$ and divide out its single proper submodule.

If $M$ is a simple simultaneously diagonalisable $\mathfrak{A}_2$-module of dimension $s+1$, denote with $\eta$ the eigenvalue of $G_2$ such that $q\eta$ is no eigenvalue of $G_2$. Pick a nonzero $m \in M_\eta$ that is simultaneously diagonalised, then $m$ is also an eigenvector of $K_{12}$ (with eigenvalue $\kappa$) and $G_{12}m = 0$. The universal property implies that there is a nonzero homomorphism $\varphi\colon M(\eta,\kappa) \to M$. Since $M$ is simple, $\varphi$ is surjective. Since $M$ has dimension $s+1$, $\varphi$ is an isomorphism and we have $\eta\kappa = \zeta q^{2s}$, i.e. $\kappa = \eta^{-1}\zeta q^{2s}$. □

PROPOSITION 3.5 *Each finite dimensional, simultaneously diagonalisable $\mathfrak{A}_2$-module is semisimple.*

*Proof.* Let $M$ be a finite dimensional non-simple, simultaneously diagonalisable $\mathfrak{A}_2$-module. We may assume that for $M$ holds the short exact sequence

$$0 \longrightarrow L(\eta, s) \xrightarrow{\iota} M \xrightarrow{\pi} L(\eta', s') \longrightarrow 0.$$

We will show that this sequence splits. For each eigenvalue $\nu$ of $G_2$ derives an exact sequence

$$0 \longrightarrow L(\eta, s)_\nu \xrightarrow{\iota} M_\nu \xrightarrow{\pi} L(\eta', s')_\nu \longrightarrow 0. \tag{3.6}$$

Choose $\bar{v} \in L(\eta', s')$, $\bar{v} \neq 0$ such that $G_{12}\bar{v} = 0$, i.e. $\bar{v} \in L(\eta', s')_{\eta'}$. Choose $v \in M_{\eta'}$ with $\pi(v) = \bar{v}$. If $G_{12}v = 0$ then $v$ spans by the universal property a simple submodule different from $L(\eta, s)$ and thus the sequence splits.

Assume now that $G_{12}v \neq 0$. Because every eigenspace of a simple simultaneously diagonalisable $\mathfrak{A}_2$-module is one-dimensional, there are just two possibilities for the dimension of $L(\eta, s)_{\eta'}$:

1. $\dim L(\eta, s)_{\eta'} = 1$. Then $\dim M_{\eta'} = 2$ since $\dim L(\eta', s')_{\eta'} = 1$ and the sequence (3.6) is exact. Since $v \notin L(\eta, s)$, there is a $w \in L(\eta, s)_{\eta'}$ such that $v, w$ are linearly independent. Since $\dim M_{\eta' q} = 1$, there is a scalar $t \in k$ such that $G_{12}v = tG_{12}w$ and thus there is $G_{12}(v - tw) = 0$ and $v - tw$ generates a simple submodule of $M$ different from $L(\eta, r)$.

2. $\dim L(\eta, s)_{\eta'} = 0$. Then $\dim M_{\eta'} = 1$ and there is a $w \in \dim L(\eta, s)_{\eta' q}$, $w \neq 0$ such that $G_{12}v = w$. Then there is
$$\begin{aligned} G_2 w &= q^{-s}\eta w \\ &= G_2 G_{12} v = q G_{12} G_2 v = q\eta' w \end{aligned}$$
thus $\eta = \eta' q^{s+1}$ and therefore the action of $K_{12}$ on $L(\eta, s)$ is determined by $\eta^{-1}\zeta q^{2s} = (\eta')^{-1}\zeta q^{s-1}$. On the other hand follows from
$$\begin{aligned} K_{12} w &= (\eta')^{-1}\zeta q^{s-1-3s} w = (\eta')^{-1}\zeta q^{-2s-1} w \\ &= K_{12} G_{12} v = q^3 G_{12} K_{12} v = (\eta')^{-1}\zeta q^3 q^{2s'} w = (\eta')^{-1}\zeta q^{2s'+3} w \end{aligned}$$
and therefore $q^{2(s+s')+2} = 1$ which yields $s + s' = -2$ since $q$ is not a root of unity, so this is a contradiction, since both, $s$ and $s'$ were chosen non-negative.

□

## The Case $n = 3$

In this section we will have a look at $\mathfrak{A}_{3,2}^{(\zeta)}$ which appears in the more general case instead of $\mathfrak{A}_2$ whenever $r \neq 1$ and $r \neq n$. Its representation theory is very likely to that of $U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$—with an additional parameter coming from $G_2$. The proofs are more or less the same as in the section before. We define additionally

$$\begin{aligned}
F_3 &= F_{\alpha_3}, \\
G_{23} &= G_{\beta_2^3} = K_{\alpha_2}^{-1} K_{\alpha_3}^{-1} (E_{\alpha_2} E_{\alpha_3} - q^{-1} E_{\alpha_3} E_{\alpha_2} - \zeta (q - q^{-1}) E_{\alpha_3}), \\
K_{23}^{\pm 1} &= (K_{\alpha_2} K_{\alpha_3}^2)^{\pm 1}
\end{aligned}$$

and get the following list of relations

$$\begin{aligned}
G_2 K_{12} &= K_{12} G_2, & G_2 K_{23} &= K_{23} G_2, \\
G_2 F_1 &= q^{-1} F_1 G_2, & G_2 F_3 &= q^{-1} F_3 G_2, \\
G_2 G_{12} &= q G_{12} G_2, & G_2 G_{23} &= q G_{23} G_2, \\
K_{12} F_1 &= q^{-3} F_1 K_{12}, & K_{12} F_3 &= q F_3 K_{12}, \\
K_{12} G_{12} &= q^3 G_{12} K_{12}, & K_{12} G_{23} &= q^{-1} G_{23} K_{12}, \\
K_{23} F_1 &= q F_1 K_{23}, & K_{23} F_3 &= q^{-3} F_3 K_{23}, \\
K_{23} G_{12} &= q^{-1} G_{12} K_{23}, & K_{23} G_{23} &= q^3 G_{23} K_{23}, \\
G_{12} F_1 &= q F_1 G_{12} + G_2 + \zeta K_{12}, & G_{12} F_3 &= q^{-1} F_3 G_{12}, \\
G_{23} F_1 &= q^{-1} F_1 G_{23}, & G_{23} F_3 &= q F_3 G_{23} + G_2 - \zeta K_{23}, \\
G_{12} G_{23} &= G_{23} G_{12}, & F_1 F_3 &= F_3 F_1.
\end{aligned}$$

For each $\eta, \kappa_1, \kappa_2$ there is an infinite dimensional $\mathfrak{A}_3$-module $M(\eta, \kappa_1, \kappa_2)$ with basis $m_{ij}$ with $(i, j) \in \mathbb{N}_0^2$ such that

$$G_2 m_{ij} = \eta q^{-(i+j)} m_{ij}, \tag{3.7}$$
$$K_{12} m_{ij} = \kappa_1 q^{-3i+j} m_{ij}, \tag{3.8}$$
$$K_{23} m_{ij} = \kappa_2 q^{i-3j} m_{ij}, \tag{3.9}$$
$$\begin{aligned}
F_1 m_{ij} &= m_{i+1,j}, \\
F_3 m_{ij} &= m_{i,j+1}, \\
G_{12} m_{ij} &= \begin{cases} 0, & \text{if } i = 0, \\ q^{-j} \dfrac{1 - q^{2i}}{1 - q^2} \Big( q^{-(i-1)} \eta - q^{i-1} \zeta \kappa_1^{-1} \Big) m_{i-1,j}, & \text{if } i \geq 1, \end{cases} \\
G_{23} m_{ij} &= \begin{cases} 0, & \text{if } j = 0, \\ q^{-i} \dfrac{1 - q^{2j}}{1 - q^2} \Big( q^{-(j-1)} \eta - q^{j-1} \zeta \kappa_2^{-1} \Big) m_{i,j-1}, & \text{if } j \geq 1. \end{cases}
\end{aligned}$$

This module has a universal property. Let $M$ be a simultaneously diagonalisable $\mathfrak{A}_3$-module and $m \in M$ such that $G_{12} m = G_{23} m = 0$, $G_2 m = \eta m$, $K_{12} m = \kappa_1 m$ and

$K_{23}m = \kappa_2 m$, then there is a unique homomorphism of $\mathfrak{A}_3$-modules $f\colon M(\eta,\kappa_1,\kappa_2) \to M$ with $f(m_{0,0}) = m$.

As it is immediately clear that each weight space of $M(\eta,\kappa)$ has dimension one, this is not a priori clear for $M(\eta,\kappa_1,\kappa_2)$. A short calculation will clarify that this is indeed true. Choose $i,j,k,l \in \mathbb{N}_0$ such that $m_{ij}$ and $m_{kl}$ do have the same weight. From equations (3.7), (3.8) and (3.9) follows the equation system

$$\begin{aligned} i + j &= k + l \\ -3i + j &= -3k + l \\ i - 3j &= k - 3l \end{aligned} \tag{3.10}$$

and a short computation shows that indeed $i = k$ and $j = l$. In fact one does not need the third equation, we shall abuse that in the proof of Proposition 3.8.

The following theorems are very similar to those in the section before, the main differences appear in the proofs—we shall also rely on the theory of case $n = 2$.

PROPOSITION 3.6 *Let $\eta \in k^*$ and $s,t \in \mathbb{N}$ and set $\kappa_1 = \eta^{-1}\zeta q^{2(s-1)}$ and $\kappa_2 = \eta^{-1}\zeta q^{2(t-1)}$. Then contains the module $M = M(\eta,\kappa_1,\kappa_2)$ exactly one maximal submodule.*

*Proof.* As $M$ is the sum of its common eigenspaces, so is every submodule and thus generated by the contained $m_{ij}$. For each submodule $M' \subset M$ there is by Proposition 3.3 a pair $(i,j) \in \mathbb{N}^2$ such that $G_{12}m_{ij} = G_{23}m_{ij} = 0$. Set

$$T = \Big\{ (i,j) \in \mathbb{N}^2\} \mid G_{12}m_{ij} = G_{23}m_{ij} = 0 \Big\}.$$

We will show that $|T| = 1$. With the assumed $\kappa_1$ and $\kappa_2$ it follows immediately from the universal property that $m_{st}$ spans a proper submodule, so $|T| \geq 1$.

Assume that $|T| \geq 2$ and take $(i,j),(k,l) \in T$. We may assume without loss of generality that $i < k$. If $j = l$ then $m_{0,j}$ spans an $\mathfrak{A}_2$-submodule that contains by the universal property two submodules (generated by $m_{i,k}$ and $m_{j,k}$) which is a contradiction to Proposition 3.3. If $j < l$ or $j > l$, then $G_{23}m_{i,j} = G_{23}m_{i,l} = 0$ since $G_{12}$ and $G_{23}$ commute. Again we have a contradiction to Proposition 3.3. □

PROPOSITION 3.7 *Let $\eta \in k^*$ and let $s,t \geq 0$ be integers. Then there exists a simultaneously diagonalisable $\mathfrak{A}_3$-module of dimension $(s+1)(t+1)$ and basis elements $m_{ij}$ with $0 \leq i \leq s$ and $0 \leq j \leq t$ such that*

$$\begin{aligned} G_2 m_{ij} &= \eta q^{-(i+j)} m_{ij}, \\ K_{12} m_{ij} &= \eta^{-1}\zeta q^{2s-3i+j} m_{ij}, \\ K_{23} m_{ij} &= \eta^{-1}\zeta q^{2t+i-3j} m_{ij}, \\ F_1 m_{ij} &= \begin{cases} m_{i+1,j}, & \text{if } i < s, \\ 0, & \text{if } i = s, \end{cases} \\ F_3 m_{ij} &= \begin{cases} m_{1,j+1}, & \text{if } j < t, \\ 0, & \text{if } j = t, \end{cases} \end{aligned}$$

$$G_{12}m_{ij} = \begin{cases} 0, & \textit{if } i = 0, \\ q^{-j}\dfrac{1-q^{2i}}{1-q^2}\left(q^{-(i-1)}\eta - q^{-2s+i-1}\zeta\kappa_1^{-1}\right)m_{i-1,j}, & \textit{if } i \geq 1, \end{cases}$$

$$G_{23}m_{ij} = \begin{cases} 0, & \textit{if } j = 0, \\ q^{-i}\dfrac{1-q^{2j}}{1-q^2}\left(q^{-(j-1)}\eta - q^{-2t+j-1}\zeta\kappa_2^{-1}\right)m_{i,j-1}, & \textit{if } j \geq 1. \end{cases}.$$

*Denote this module by $L(\eta, s, t)$. Every finite dimensional simple simultaneously diagonalisable $\mathfrak{A}_3$-module is ismorphic to a $L(\eta, r, s)$ for suitable $\eta \in k^*$ and $s, t \in \mathbb{N}_0$.*

*Proof.* The existence follows from Proposition 3.6. Take $M(\eta, \eta^{-1}\zeta q^{2s}, \eta^{-1}\zeta q^{2s})$ and divide out its unique maximal submodule.

If $M$ is a finite dimensional simple simultaneously diagonalisable $\mathfrak{A}_3$-module, then there is an eigenvalue $\eta$ of $G_2$ on $M$ such that $q\eta$ is no eigenvalue. Pick a non-zero element simultaneously diagonalised $m$ of the eigenspace of the eigenvalue $\eta$. It holds $G_{12}m = G_{23}m = 0$, so that there are integers $s, t \geq 1$ such that $m$ spans two $\mathfrak{A}_2$-modules, one on the first index of dimension $s+1$ and the other one on the second of dimension $t+1$, so that by Proposition 3.4 follows that $K_{12}m = \eta^{-1}\zeta q^{2s}$ and $K_{23}m = \eta^{-1}\zeta q^{2t}$. From the universal property of $M(\eta, \eta^{-1}\zeta q^{2s}, \eta^{-1}\zeta q^{2t})$ follows then the last part of the Proposition. □

PROPOSITION 3.8 *Each finite dimensional, simultaneously diagonalisable $\mathfrak{A}_3$-module is semisimple.*

*Proof.* Let $M$ be a finite dimensional, non-simple, simultaneously diagonalisable $\mathfrak{A}_3$-module. Without loss of generality we may assume that for $M$ holds the exact sequence

$$0 \longrightarrow L(\eta, s, t) \xrightarrow{\iota} M \xrightarrow{\pi} L(\eta', s', t') \longrightarrow 0.$$

As in the case $n = 2$ we show that this exact sequence splits. There are natural embeddings of $\mathfrak{A}_{1,2}$ and $\mathfrak{A}_{2,2}$ into $\mathfrak{A}_3$. Choose $\bar{v} \in L(\eta', s', t')$ such that

$$G_{12}\bar{v} = G_{23}\bar{v} = 0$$

and choose a common eigenvector $v \in M$ such that $\pi(v) = \bar{v}$. Since $M$ is a finite dimensional simultaneously diagonalisable $\mathfrak{A}_{1,2}$-module we may assume that $G_{12}v = 0$ by Proposition 3.5 and $v \notin L(\eta, s, t)$. We distinguish two cases

1. $G_{23}v = 0$. Then $v$ spans a simple $\mathfrak{A}_3$-module different from $L(\eta, s, t)$.

2. $G_{23}v \neq 0$. Let $k \in \mathbb{N}$ such that $(G_{23})^k v \neq 0$. Since $G_{12}$ and $G_{23}$ commute follows with the universal property that $(G_{23})^k v$ spans $L(\eta, s, t)$. Using again Proposition 3.5—this time for $\mathfrak{A}_{2,2}$—follows

   $$w = F_3^k (G_{23})^k v = 0$$

   and we can write

   $$v = v' + w$$

with $v' \notin L(\eta, s, t)$. Since $G_{12}w = 0$, follows $G_{12}v' = 0$. Either $G_{23}v' = 0$—then $v'$ generates a simple module different from $L(\eta, s, t)$—or $G_{23}v' \neq 0$. In the latter case we proceed as before which yields vectors $v''$ and $w'$ with $v' = v'' + w'$ and $v', v''$ and $w$ are linearly independent and all being common eigenvectors of the same eigenspace. On the other hand is each common eigenspace of a simple finite dimensional simultaneously diagonalisable $\mathfrak{A}_3$-module one-dimensional—so each common eigenspace of $M$ is at most two-dimensional. This is a contradiction, thus $G_{23}v' = 0$.

□

## The General Case

This part is again parted into two parts, that is for $r = 1$ or $r = n$ (which are ›dual‹ to each other under the automorphism reverting the nodes of the Dynkin diagram) and $1 < r < n$ for which we shall rely on the results of the first two cases.

Set $R = \{1, \dots, n\} \setminus \{r\}$ and $R' = \{1, \dots, n\} \setminus \{r-1, r, r+1\}$.

DEFINITION 3.9 *Let $\eta \in k^*$ and $\lambda \in \Lambda$ and $M$ be a simultaneously diagonalisable $\mathfrak{A}$-module. Set*

$$M_{\eta,\lambda} = \Big\{ v \in M \mid G_{\beta_r^r} v = \eta q^{(-\alpha_r, \lambda)} v, K_{\alpha_i} v = q^{(\alpha_i, \lambda)} v \ (i \in R') \\ K_{\alpha_{r-1}}^2 K_{\alpha_r} = \eta^{-1} \zeta q^{(2\alpha_{r-1} + \alpha_r, \lambda)} v, K_{\alpha_r} K_{\alpha_{r+1}}^2 = \eta^{-1} \zeta q^{(\alpha_r + 2\alpha_{r+1})} v \Big\}.$$

*We call $M_{\eta,\lambda}$ the* weight space of weight $\lambda$ subject to $\eta$, *and $v \in M_{\eta,\lambda}$ a* weight vector of weight $\lambda$ subject to $\eta$. *As convention we will omit the ›subject to‹-part whenever it is clear subject to which $\eta$. In that case we write $M_\lambda$.*

REMARK 3.10 *The subalgebra $\mathfrak{A}^0$ is generated by $n$ elements, but the above weight spaces are parametrised by $n + 1$ parameters. The disadvantage is the lack of a general direct sum decomposition into weight spaces, however, in some special cases that can be fixed by fixing $\eta$, e.g. in the case that $M$ is generated by a single common eigenvector $v$ (well, this follows by equation (2.5)). This case is already included in the expression ›subject to‹. As we will see later every finite dimensional $U_q(\mathfrak{sl}_{n+1})$-module is semisimple as $\mathfrak{A}$-module, so the lack of a general decomposition into weight spaces is minor important.*

*The major benefit is the notation itself: We do not need a new functional—which would not be really compatible with notation induced from $U_q(\mathfrak{sl}_{n+1})$—and in the end we will examine $U_q(\mathfrak{sl}_{n+1})$-modules as $\mathfrak{A}$-modules.*

Let $\tilde{\Lambda} = \tilde{\Lambda}_{n,r}$ be the sublattice of $\Lambda$ generated by $\varpi_i, i \in R$. Let $\eta \in k$, $\eta \neq 0$ and $\tilde{\lambda} \in \tilde{\Lambda}$. For a tupel $(\eta, \tilde{\lambda})$ we define a (simultaneously diagonalisable) *Verma module* by

$$M(\eta, \tilde{\lambda}) = \mathfrak{A} \Big/ \Big( \mathfrak{A} G_{\beta_{r-1}^r} + \mathfrak{A} G_{\beta_r^{r+1}} + \sum_{i \in R} \mathfrak{A} E_{\alpha_i} + \mathfrak{A}(G_{\beta_r^r} - \eta) + \sum_{i = R} \mathfrak{A}\Big(K_{\alpha_i} - q^{(\alpha_i, \tilde{\lambda})}\Big) + \\ \mathfrak{A}\Big(K_{\alpha_{r-1}}^2 K_{\alpha_r} - \eta^{-1} \zeta q^{(2\alpha_{r-1}, \tilde{\lambda})}\Big) + \mathfrak{A}\Big(K_{\alpha_{r-1}}^2 K_{\alpha_r} - \eta^{-1} \zeta q^{(2\alpha_{r+1}, \tilde{\lambda})}\Big) \Big).$$

It is generated as an $\mathfrak{A}$-module by the coset of 1 which shall be denoted by $v_{\eta,\tilde{\lambda}}$. We have furthermore a direct sum decomposition of $\mathfrak{A}$ following from the ordered basis in (2.3): $\mathfrak{A} = U^-[w^-] \oplus C(w^+, \varphi, L)$. This implies that the map

$$U^-[w^-] \longrightarrow M(\eta, \tilde{\lambda}), \qquad u \mapsto uv_{\eta,\tilde{\lambda}} \tag{3.11}$$

is bijective and an isomorphism of $U^-[w^-]$-modules. The algebra $U^-[w^-]$ is a $P^-$-graded algebra, so the module $M(\eta, \tilde{\lambda})$ becomes via the map from above a graded $U^-[w^-]$-module with respect to $P^-$. With the formulas from equation (2.4) follows for $u \in U^-[w^-]_\mu$

$$\begin{aligned}
G_{\beta_r^r} uv_{\eta,\tilde{\lambda}} &= \eta q^{(-\alpha_r,\mu)} uv_{\eta,\tilde{\lambda}}, \\
K_{\alpha_{r-1}}^2 K_{\alpha_r} uv_{\eta,\tilde{\lambda}} &= \eta^{-1}\zeta q^{(2\alpha_{r-1}+\alpha_r,\mu+\tilde{\lambda})} uv_{\eta,\tilde{\lambda}}, \\
K_{\alpha_r} K_{\alpha_{r+1}}^2 uv_{\eta,\tilde{\lambda}} &= \eta^{-1}\zeta q^{(\alpha_r+2\alpha_{r-1},\mu+\tilde{\lambda})} uv_{\eta,\tilde{\lambda}}, \\
K_{\alpha_i} uv_{\eta,\tilde{\lambda}} &= q^{(\alpha_i,\mu+\tilde{\lambda})} uv_{\eta,\tilde{\lambda}} \qquad (i \in R').
\end{aligned}$$

Since $U^-[w^-] = \oplus_{\mu \in P^-} U^-[w^-]_\mu$, follows the decomposition of $M(\eta, \tilde{\lambda})$ into weight spaces, i.e.

$$M(\eta, \tilde{\lambda}) = \bigoplus_{\mu \in \Lambda} M(\eta, \tilde{\lambda})_\mu$$

In particular this module is simultaneously diagonalisable.

Definition 3.11 *Let $\lambda, \mu \in \Lambda$. $\mu$ is $P$-*smaller *than $\lambda$ if and only if $\lambda - \mu \in P^+$. The notation is $\mu <_P \lambda$*

This is a partial ordering on $\Lambda$. A reformulation of the statement above yields: The subspace $M(\eta, \tilde{\lambda})_\mu$ is non-trivial if and only if $\mu <_P \tilde{\lambda}$. In this language becomes the module $M(\eta, \tilde{\lambda})$ a highest weight module of weight $\tilde{\lambda}$. The parameter $\eta$ is minor important, it should be regarded as in index in $k^*$ indexing a family of very similar simultaneously diagonalisable Verma-modules of $\mathfrak{A}$. This view is also in align with Definition 3.9.

As in the case of quantum groups, each $M(\eta, \tilde{\lambda})_\mu$ is finite dimensional. In the classification process of finite dimensional, simple, simultaneously diagonalisable $\mathfrak{A}$-modules we use now the same route as for quantum enveloping algebras, like in [Jan96, p.72ff.], with the exception that we can take a shortcut later.

The module has the usual universal property: Let $M$ be a simultaneously diagonalisable $\mathfrak{A}$-module and $v \in M$ a vector such that

$$\begin{aligned}
G_{\beta_{r-1}^r} v = G_{\beta_r^{r+1}} v = E_{\alpha_i} v &= 0 \qquad (i \in R'), \\
G_{\beta_r^r} v &= \eta v, \\
K_{\alpha_{r-1}}^2 K_{\alpha_r} v &= \eta^{-1}\zeta q^{(2\alpha_{r-1},\tilde{\lambda})} v, \\
K_{\alpha_r} K_{\alpha_{r+1}}^2 v &= \eta^{-1}\zeta q^{(2\alpha_{r+1},\tilde{\lambda})} v, \\
K_{\alpha_i} v &= q^{(\alpha_i,\tilde{\lambda})} v \qquad (i \in R').
\end{aligned}$$

Then there is a unique $\mathfrak{A}$-module homomorphism $f\colon M(\eta,\tilde\lambda)\to M$ with $f(v_{\eta,\tilde\lambda})=v$.

Let $N\subset M(\eta,\tilde\lambda)$ be a submodule and define for $\mu\in\Lambda$ the weight space as $N_\mu = N\cap M(\eta,\tilde\lambda)_\mu$. Then $N$ becomes the direct sum of its weight spaces. There is, as in the classical case, a unique maximal submodule of $M(\eta,\tilde\lambda)$. Define $L(\eta,\tilde\lambda)$ as the factor module of $M(\eta,\tilde\lambda)$ by this unique maximal submodule.

From now on we are going to consider $\mathfrak{A}_{n,1}$.

LEMMA 3.12 *Let $M$ be a finite dimensional, simple, simultaneously diagonalisable $\mathfrak{A}_{n,1}$-module. Then there exist a $v\in M$, $v\neq 0$, a $\eta\in k^*$ and a dominant weight $\tilde\lambda\in\tilde\Lambda_{n,1}$ such that*

$$\begin{aligned} G_{\beta_1^2}v = E_{\alpha_i}v &= 0 && (3\le i\le n),\\ G_{\beta_1^1}v &= \eta v, &&\\ K_{\alpha_1}K_{\alpha_2}^2 v &= \eta^{-1}\zeta q^{(2\alpha_2,\tilde\lambda)}v, &&\\ K_{\alpha_i}v &= q^{(\alpha_i,\tilde\lambda)}v && (3\le i\le n),\end{aligned}$$

*and $F_{\alpha_i}^{(\alpha_i,\tilde\lambda)+1}v=0$ for $2\le i\le n$.*

*Proof.* The existence of a $v\in M$ such that $\mathfrak{A}^+$ operates as zero is clear from equation (2.5). From the embedding of $U_g(\mathfrak{sl}_{n-1})$ into $\mathfrak{A}_{n,1}$ the actions of $K_{\alpha_3},\dots,K_{\alpha_n}$ are determined, giving a dominant weight $\tilde\lambda'\in\mathbb{Z}\{\varpi_3,\dots,\varpi_n\}$. The embedding $\mathfrak{A}_2$ into $\mathfrak{A}_{n,1}$ gives a $\eta\in k^*$ and a $s\geq 0$ determining the action of $G_{\beta_1^1}$ and $K_{\alpha_1}K_{\alpha_2}^2$. Set $\tilde\lambda=s\varpi_2+\tilde\lambda'$.

From [Jan96, Lemma 5.4 b)] and Proposition 3.4 follows the claim about the $F_{\alpha_i}$. $\square$

Let $M$ be a finite dimensional, simple, simultaneously diagonalisable $\mathfrak{A}_{n,1}$-module. By the above Lemma 3.12 and the universal property follows that $M$ is isomorphic to $L(\eta,\tilde\lambda)$ for a $\eta\in k^*$ and a dominant weight $\tilde\lambda\in\tilde\Lambda_{n,1}$. The next theorem completes the classification of finite dimensional, simple, simultaneously diagonalisable $\mathfrak{A}_{n,1}$-modules.

THEOREM 3.13 *Let $\eta\in k^*$ and $\tilde\lambda\in\tilde\Lambda_{n,1}$ be a dominant weight. Then the $\mathfrak{A}_{n,1}$-module $L(\eta,\tilde\lambda)$ is finite dimensional.*

We want to argue as in the proof of Theorem 5.10 in [Jan96], so we like to have a module that is finite dimensional and has $L(\eta,\tilde\lambda)$ as a homomorphic image. A candidate is a module similar to that one in Proposition 5.9 in [Jan96]—that is as vector space isomorphic to that one in Proposition 5.9.

LEMMA 3.14 *Let $\tilde\lambda=\sum_{i=2}^n s_i\varpi\in\tilde\Lambda_{n,1}$ be a dominant weight and $\eta\in k^*$.*

1. *Set $\tilde\lambda'=\tilde\lambda-(s_2+1)(\alpha_2+\varpi_1)$ and $\eta'=\eta q^{-(s_2+1)}$. Then exists a homomorphism of $\mathfrak{A}_{n,1}$-module $f_{\alpha_2}\colon M(\eta',\tilde\lambda')\longrightarrow M(\eta,\tilde\lambda)$ such that $f_{\alpha_2}(v_{\eta',\tilde\lambda'})=F_{\alpha_2}^{s_2+1}v_{\eta,\tilde\lambda}$.*

2. *For all $3\le i\le n$ exists an $\mathfrak{A}_{n,1}$-module homomorphism $f_{\alpha_i}\colon M\big(\eta,\tilde\lambda-(s_i+1)\alpha_i\big)\longrightarrow M(\eta,\tilde\lambda)$ such that $f_{\alpha_i}(v_{\eta,\tilde\lambda-(s_i+1)\alpha_i})=F_{\alpha_i}^{s_i+1}v_{\eta,\tilde\lambda}$.*

*Proof.* The proof is very similar to that one of Lemma 5.6 in [Jan96].

1. $F_{\alpha_2}^{s_2+1} v_{\eta,\tilde{\lambda}}$ has weight $\tilde{\lambda} - (s_2+1)\alpha_2$ subject to $\kappa$. This is not in $\tilde{\Lambda}_{n,1}$. However, we find that

$$
\begin{aligned}
G_{\beta_1^1} F_{\alpha_2}^{s_2+1} v_{\eta,\tilde{\lambda}} &= \eta q^{-(s_2+1)} F_{\alpha_2}^{s_2+1} v_{\eta,\tilde{\lambda}} = \eta' F_{\alpha_2}^{r_2+1} v_{\eta,\tilde{\lambda}}, \\
K_{\alpha_1} K_{\alpha_2}^2 F_{\alpha_2}^{s_2+1} v_{\eta,\tilde{\lambda}} &= \eta^{-1} \zeta q^{-3(s_2+1)} q^{(2\alpha_2,\tilde{\lambda})} F_{\alpha_2}^{s_2+1} v_{\eta,\tilde{\lambda}} = (\eta')^{-1} q^{(2\alpha_2,\tilde{\lambda}')} F_{\alpha_2}^{s_2+1} v_{\eta,\tilde{\lambda}}, \\
K_{\alpha_i} F_{\alpha_2}^{s_2+1} v_{\eta,\tilde{\lambda}} &= q^{(\alpha_i,\tilde{\lambda})} F_{\alpha_2}^{s_2+1} v_{\eta,\tilde{\lambda}} = q^{(\alpha_i,\tilde{\lambda}')} F_{\alpha_2}^{s_2+1} v_{\eta,\tilde{\lambda}},
\end{aligned}
$$

so $F_{\alpha_2}^{s_2+1} v_{\eta,\tilde{\lambda}}$ has weight $\tilde{\lambda}'$ subject to $\eta'$. Since $E_{\alpha_i} F_{\alpha_2} = F_{\alpha_2} E_{\alpha_i}$ for each $3 \leq i \leq n$ holds $E_{\alpha_i} F_{\alpha_2}^{r_2+1} v_{\eta,\tilde{\lambda}} = 0$. From equation (3.1) follows

$$
\begin{aligned}
G_{\beta_1^2} F_{\alpha_2}^{s_2+1} v_{\eta,\tilde{\lambda}} &= \frac{1-q^{2(s_2+1)}}{1-q^2} \Big( q^{-s_2} F_{\alpha_2}^{s_2} G_{\beta^1} v_{\eta,\tilde{\lambda}} - q^{s_2} F_{\alpha_2}^{s_2} (K_{\alpha_1} K_{\alpha_2}^2)^{-1} v_{\eta,\tilde{\lambda}} \Big) \\
&= \frac{1-q^{2(s_2+1)}}{1-q^2} F_{\alpha_2}^{s_2} \Big( q^{-s_2}\eta - q^{s_2} \zeta \eta \zeta^{-1} q^{-2s_2} \Big) v_{\eta,\tilde{\lambda}} \\
&= 0.
\end{aligned}
$$

With the universal property of $M(\eta', \tilde{\lambda}')$ follows the claim.

2. $F_{\alpha_i}^{s_i+1} v_{\eta,\tilde{\lambda}}$ has weight $\tilde{\lambda} - (s_i+1)\alpha_i$ subject to $\kappa$. Since the weight is in $\tilde{\Lambda}_{n,1}$ and $G_{\beta_1^2} F_{\alpha_i} = F_{\alpha_i} G_{\beta_1^2}$ the procedure is identical with that given by Jantzen.

$\square$

*Proof of Theorem 3.13.* Let $f_{\alpha_2}, \dots, f_{\alpha_n}$ as in Lemma 3.14 and define

$$\tilde{L}(\eta, \tilde{\lambda}) = M(\eta, \tilde{\lambda}) / \Big( \sum_{i=2}^{n} \operatorname{im}(f_{\alpha_i}) \Big).$$

$L(\eta, \tilde{\lambda})$ is the homomorphic image of $\tilde{L}(\eta, \tilde{\lambda})$. In terms of the isomorphism from equation (3.11) the image of $f_{\alpha_i}$ is $U^-[w^-] F_{\alpha_i}^{s_i+1}$, and we find that there is an isomorphism of vector spaces between $\tilde{L}(\eta, \tilde{\lambda})$ and $U^-[w^-] / (\sum_{i=2}^{n} U^-[w^-] F_{\alpha_i}^{s_i+1})$.

The proof of Proposition 5.9 in [Jan96] shows that the there defined $U_q(\mathfrak{sl}_n)$-module $\tilde{L}(\tilde{\lambda})$ ($\tilde{\lambda}$ must of course be considered as a weight of $U_q(\mathfrak{sl}_n)$, this is possible by the definition of $\tilde{\Lambda}_{n,1}$.) is isomorphic to $U^-[w^-] / (\sum_{i=2}^{n} U^-[w^-] F_{\alpha_i}^{s_i+1})$ as a vector space. But since $\tilde{L}(\tilde{\lambda})$ is finite dimensional so is $\tilde{L}(\eta, \tilde{\lambda})$. $\square$

There is a natural projection from $\pi \colon \Lambda \to \tilde{\Lambda}_{n,r}$ given by $\pi(\sum_{i=1}^{n} s_i \varpi_i) = \sum_{i \in R}^{n} s_i \varpi_i$. In fact, this is the transition of weights from $U_q(\mathfrak{sl}_{n+1})$ to $U_q(\mathfrak{sl}_r) \otimes U_q(\mathfrak{sl}_{n-r+1})$ coming from the embedding of the latter in the first on the indices $i \in R$. Let $\lambda \in \Lambda$ be a weight of a $U_q(\mathfrak{sl}_{n+1})$-module, say $M$. Let $v \in M_\lambda$ be weight vector of weight $\lambda$. Consider now this module $M$ as $U_q(\mathfrak{sl}_r) \otimes U_q(\mathfrak{sl}_{n-r+1})$-module via the above embedding. Then $v$ has weight $\pi(\lambda)$. Let $u \in U_q(\mathfrak{sl}_r) \otimes U_q(\mathfrak{sl}_{n-r+1})$ be of weight $\pi(\mu)$, then $uv \in M_{\pi(\lambda+\mu)}$. This way we consider $\tilde{\Lambda}_{n,r}$ as the usual weight lattice of the algebra $U_q(\mathfrak{sl}_r) \otimes U_q(\mathfrak{sl}_{n-r+1})$. We shall use this identification for a transition of weight spaces from $\mathfrak{A}_{n,r}$-modules to ›corresponding‹

$U_q(\mathfrak{sl}_r) \otimes U_q(\mathfrak{sl}_{n-r+1})$-modules. In the next Lemmas and Corollaries we consider again the case $r = 1$.

Recall that the $U_q(\mathfrak{sl}_n)$-modules $L(\tilde{\lambda})$ and $\tilde{L}(\tilde{\lambda})$ are isomorphic (c.f. [Jan96, p. 6.26]). The proof above shows that there is a bijective linear map between $\tilde{L}(\eta, \tilde{\lambda})$ and $\tilde{L}(\tilde{\lambda})$ preserving weight spaces. This allows us to compute the dimension of $\tilde{L}(\eta, \tilde{\lambda})_\mu$ where $\mu$ is a weight subject to $\kappa$.

COROLLARY 3.15 *Let $\tilde{\lambda} \in \tilde{\Lambda}_{n,1}$ and $\mu \in \Lambda$ such that $\mu <_P \tilde{\lambda}$, and let $\eta \in k^*$. Then holds*

$$\dim \tilde{L}(\eta, \tilde{\lambda})_{\eta,\mu} = \dim L(\tilde{\lambda})_{\pi(\mu)}.$$

*Proof.* Let us perform the proof more explicitly than sketched above. By equation (3.11) and equation [Jan96, 5.5 (4)] there is an isomorphism of $U^-[w^-]$-modules $g\colon M(\eta, \tilde{\lambda}) \to M(\tilde{\lambda})$ with $g(uv_{\eta,\tilde{\lambda}}) = uv_{\tilde{\lambda}}$ where $u \in U^-[w^-]$.

Let us denote the highest weight vectors of $M(\eta, \tilde{\lambda})$ resp. $L(\tilde{\lambda})$ with $v_{\eta,\tilde{\lambda}}$ resp. $v_{\tilde{\lambda}}$, too. By equation (3.11) and equation [Jan96, 5.5 (4)] there are bijective linear maps $f\colon U^-[w^-]/(\sum_{i=2}^n F_{\alpha_i}^{r_i+1}) \to \tilde{L}(\eta, \tilde{\lambda})$ and $g\colon U^-[w^-]/(\sum_{i=2}^n F_{\alpha_i}^{r_i+1}) \to L(\tilde{\lambda})$ given by $f([u]) = [u]v_{\eta,\tilde{\lambda}}$ and $g([u]) = [u]v_{\tilde{\lambda}}$ where $[u]$ is the class of $u$.

By definition of $\mu <_P \tilde{\lambda}$ we have $\tilde{\lambda} - \mu \in P^+$. If $[u]$ has weight $-(\tilde{\lambda} - \mu)$[1] then $f([u]) = [u]v_{\eta,\tilde{\lambda}}$ has weight $\mu$ subject to $\kappa$ and $g([u]) = [u]v_{\tilde{\lambda}}$ has weight $\pi(\mu)$. We have $\dim M(\eta, \tilde{\lambda})_{\kappa,\mu} = \dim U^-[w^-]_{-(\tilde{\lambda}-\mu)} = \dim M(\tilde{\lambda})_{\pi(\mu)}$. □

For a better understanding of $L(\eta, \tilde{\lambda})$ it would be nice that $L(\eta, \tilde{\lambda})$ and $\tilde{L}(\eta, \tilde{\lambda})$ are isomorphic. This is indeed the case as the next Theorem shows. From the isomorphism we conclude also the universal property of $\tilde{L}(\eta, \tilde{\lambda})$ that comes from Lemma 3.12. The universal property for the latter is as follows: Let $M$ be a finite dimensional, simultaneously diagonalisable $\mathfrak{A}_{n,1}$-module, $v \in M$ a weight vector of weight $\tilde{\lambda} \in \tilde{\Lambda}_{n,1}$ (of course, $\tilde{\lambda}$ must be dominant) subject to $\eta$ with $\eta \in k^*$ such that $G_{\beta_1^2} v = E_{\alpha_3} v = \ldots = E_{\alpha_n} v = 0$. Then there is a unique $\mathfrak{A}_{n,1}$-module homomorphism from $f\colon \tilde{L}(\eta, \tilde{\lambda}) \to M$ such that $f(v_{\eta,\tilde{\lambda}}) = v$.

THEOREM 3.16 *Let $\eta \in k^*$ and let $\tilde{\lambda} \in \tilde{\Lambda}_{n,1}$ be a dominant weight. Then holds*

$$L(\eta, \tilde{\lambda}) \simeq \tilde{L}(\eta, \tilde{\lambda}).$$

*Proof.* By the universal property $\tilde{L}(\eta, \tilde{\lambda})$ it suffices to proof that $\tilde{L}(\eta, \tilde{\lambda})$ is simple. Let us consider $U_q(\mathfrak{sl}_n)$ as a subalgebra of $U_q(\mathfrak{sl}_{n+1})$ such that $\tilde{\Lambda}$ is the weight lattice of $U_q(\mathfrak{sl}_n)$. Write $\tilde{\lambda} = \sum_{i=2}^n s_i \varpi_i$ with $s_i \in \mathbb{N}_0$.

Consider the following infinite dimensional $U_q(\mathfrak{sl}_{n+1})$-module

$$M = U/\Big(\sum_{i=2}^n UF_{\alpha_i}^{s_i+1} + \sum_{i=1}^n UE_{\alpha_i} + U(K_{\alpha_1} - \eta^{-1}) + \sum_{i=2}^n U(K_{\alpha_i} - q^{(\alpha_i,\tilde{\lambda})})\Big)$$

Denote with $v_{\text{gen}}$ the class of the coset of 1. $v_{\text{gen}}$ is a weight vector of weight $\tilde{\lambda}$ subject to $\eta$. Denote with $N$ the subspace $U^-[w^-]v_{\text{gen}}$. $N$ is finite dimensional and as $\mathfrak{A}_{n,1}$-module

[1] As $U_q(\mathfrak{sl}_n)$-module. Then it is clear what shall be understood by a class of weight $\mu$.

isomorphic to $\tilde{L}(\eta, \tilde{\lambda})$ and a simple $U_q(\mathfrak{sl}_n)$-module with highest weight vector $v_{\text{gen}}$. Let $\mu \in \Lambda$, then

$$N_{\eta,\mu} = N_{\pi(\mu)}$$

and of course $N = \oplus_{\mu\in\Lambda} N_{\eta,\mu} = \oplus_{\mu\in\Lambda} N_{\pi(\mu)}$. Assume that $N$ is not a simple $\mathfrak{A}_{n,1}$-module. Then there is a proper $\mathfrak{A}_{n,1}$-submodule $\tilde{N}$. It decomposes into weight spaces, so

$$\tilde{N} = \bigoplus_{\mu\in\Lambda} \Big(\tilde{N} \cap N_{\eta,\mu}\Big) = \bigoplus_{\mu\in\Lambda} \Big(\tilde{N} \cap N_{\pi(\mu)}\Big).$$

Since $\tilde{N}$ is finite dimensional there exists a $\mu \in \Lambda$ with $\mu <_P \tilde{\lambda}$ and a $v \in \tilde{N} \cap N_{\eta,\mu}$ such that $G_{\beta_1^2} v = E_{\alpha_3} v = \ldots = E_{\alpha_n} v = 0$. We claim: Then $E_{\alpha_2} v = 0$, this would be a contradiction, since $v$ would span a simple $U_q(\mathfrak{sl}_n)$-submodule with highest weight $\pi(\mu)$ in the simple module $N$.

Since $v \in N = U^-[w^-]v_{\text{gen}}$, there exists a $u \in U^-[w^-]_{-(\tilde{\lambda}-\mu)}$ such that $uv_{\text{gen}} = v$. Recall that $G_{\beta_1^2} = K_{\alpha_1}^{-1} K_{\alpha_2}^{-1} \Big(E_{\alpha_1} E_{\alpha_2} - q^{-1} E_{\alpha_2} E_{\alpha_1} + (q - q^{-1})\zeta E_{\alpha_2}\Big)$. So if $(E_{\alpha_1} E_{\alpha_2} - q^{-1} E_{\alpha_2} E_{\alpha_1})v = 0$, then must be $E_{\alpha_2} v = 0$. Since $E_{\alpha_1} u = u E_{\alpha_1}$ we have $E_{\alpha_2} E_{\alpha_1} uv = E_{\alpha_2} u E_{\alpha_1} v = 0$. We have $E_{\alpha_2} u = u'$ with $u' \in U^0 U^-[w^-]_{-(\tilde{\lambda}-\mu)+\alpha_2}$, so there is a $u'' \in U^0 U^-[w^-]_{-(\tilde{\lambda}-\mu)+\alpha_2}$ such that $E_{\alpha_1} u' = u'' E_{\alpha_1}$, so follows as before that $E_{\alpha_1} E_{\alpha_2} uv = 0$, hence the contradiction. □

COROLLARY 3.17 *Let $\tilde{\lambda} \in \tilde{\Lambda}_{n,1}$ and $\mu \in \Lambda$ such that $\mu <_P \tilde{\lambda}$, and let $\eta \in k^*$. Then holds*

$$\dim L(\eta, \tilde{\lambda})_{\eta,\mu} = \dim L(\tilde{\lambda})_{\pi(\mu)}.$$

*In particular holds*

$$\dim L(\eta, \tilde{\lambda}) = \dim L(\tilde{\lambda}).$$

THEOREM 3.18 *Let $\tilde{\lambda} \in \tilde{\Lambda}_{n,r}$ be a dominant weight and $\eta \in k^*$. Then the module $L(\eta, \tilde{\lambda})$ is finite dimensional.*

*Proof.* For $r = 1$ and $r = n$ this is already done by Theorem 3.16. So let $1 < r < n$. We have embeddings $\mathfrak{A}_{r,r} \hookrightarrow \mathfrak{A}_{n,r}$ and $\mathfrak{A}_{n-r,1} \hookrightarrow \mathfrak{A}_{n,r}$. From the definition of $L(\eta, \tilde{\lambda})$ follows that there is a $v \in L(\eta, \tilde{\lambda})$ such that $\mathfrak{A}_{r,r}^+ v = \mathfrak{A}_{n-r,1}^+ v = 0$. This implies that $v$ spans a simple, finite dimensional, simultaneously diagonalisable $\mathfrak{A}_{r,r}$- resp.$\mathfrak{A}_{n-r,1}$-module, so there are $\tilde{\lambda}' \in \tilde{\Lambda}_{r,r}$, $\tilde{\lambda}'' \in \tilde{\Lambda}_{n-r,1}$ and a single $\eta \in k^*$ (since $G_{\beta_r^r}$ is in both subalgebras) such that $\mathfrak{A}_{r,r} v \simeq L(\eta, \tilde{\lambda}')$ and $\mathfrak{A}_{n-r,1} v \simeq L(\eta, \tilde{\lambda}'')$. Note that the generators of both commute with each other (some only up to a $q$-factor, which does no harm). This implies that $v$ spans a finite dimensional $\mathfrak{A}_{n,r}$ submodule. □

COROLLARY 3.19 *Let $\tilde{\lambda} \in \tilde{\Lambda}_{n,r}$, $\tilde{\lambda} = \sum_{i\in R} s_i \varpi_i$ and $\mu \in \Lambda$ such that $\mu <_P \tilde{\lambda}$ and let $\eta \in k^*$. Set $\lambda' = \sum_{i=1}^{r-1} s_i \varpi_i$ and $\lambda'' = \sum_{i=r+1}^{n} s_i \varpi_i$. Then hold*

$$\dim L(\eta, \tilde{\lambda})_{\eta,\mu} = \dim \Big(L(\lambda') \otimes L(\lambda'')\Big)_\mu,$$
$$\dim L(\eta, \tilde{\lambda}) = \dim L(\lambda') \cdot \dim L(\lambda'').$$

We shall use this Corollary to prove the next theorem. Additionally we like to know something more about the action of $G_{\beta_r^{r+1}}$ resp. $G_{\beta_{r-1}^r}$ on a finite dimensional, simple $U_q(\mathfrak{sl}_{n+1})$-module. The following Lemma implies besides Proposition 3.1.

LEMMA 3.20 *Let $\lambda \in \Lambda$ be a dominant weight and let $\mu$ be a weight of $L(\lambda)$. Let $v \in L(\lambda)_\mu$, $v \neq 0$ such that $E_{\alpha_{r-1}} v = E_{\alpha_{r+1}} v = 0$. Let $v_{diag}$ be its diagonalisation subject to $G_{\beta_r^r}$. Then $G_{\beta_{r-1}^r} v_{diag} = G_{\beta_r^{r+1}} v_{diag} = 0$*

*Proof.* Set $s = \max(k \mid E_{\alpha_r}^k v \neq 0)$. Set $a_0 = 1$ and for $1 \leq k \leq s$ via recursion $a_k = a_{k-1} \frac{-1}{\zeta(1-q^{2k})}$. Claim:

$$v_{\text{diag}} = \sum_{i=0}^{s} a_k E_{\alpha_r}^k v. \tag{3.12}$$

Since

$$G_{\beta_r^r} a_k E_{\alpha_r}^k v = a_k q^{-(\alpha_r,\mu)} \Big( \zeta q^{-(\alpha_r, k\alpha_r)} E_{\alpha_r}^k v + (1-\delta_{k,s}) q^{-(\alpha_r,(k+1)\alpha_r)} E_{\alpha_r}^{k+1} v \Big),$$

we have only to verify that

$$a_k \zeta q^{-2k} + a_{k-1} q^{-2k} = \zeta a_k$$

for $1 \leq k \leq s$ – which is indeed true. Concerning the action of $G_{\beta_r^{r+1}}$: Using the fact that $E_{\beta_r^{r+1}} E_{\alpha_r}^k = q^{-k} E_{\alpha_r}^k E_{\beta_r^{r+1}}$ follows for $0 \leq k \leq s$

$$G_{\beta_r^{r+1}} E_{\alpha_r}^k v = a_k q^{-(\alpha_r+\alpha_{r+1},\mu)} \Big( \zeta q^{-k-1} (q-q^{-1}) E_{\alpha_{r+1}} E_{\alpha_r}^k - q^{-2k-3} E_{\alpha_r}^{k-1} E_{\alpha_{r+1}} E_{\alpha_r} \Big) v,$$

so we have to check that

$$a_k \zeta (q-q^{-1}) q^{-k-1} E_{\alpha_{r+1}} E_{\alpha_r}^k v = a_{k-1} q^{-2(k-1)-3} E_{\alpha_1}^{k-1} E_{\alpha_{r+1}} E_{\alpha_r} v.$$

A short induction shows that $E_{\alpha_{r+1}} E_{\alpha_r}^k = q^k E_{\alpha_r}^k E_{\alpha_{r+1}} + q^{-k} \frac{1-q^{2k}}{1-q^{-2}} E_{\alpha_r}^{k-1} E_{\beta_r^{r+1}}$, so that the above condition becomes

$$-a_k \zeta (q-q^{-1}) q^{-k-1} q^{-k-1} \frac{1-q^{-2k}}{1-q^{-2}} = a_{k-1} q^{-2(k-1)-3}.$$

The left hand side computes with the recursion formula as

$$-a_k \zeta (q-q^{-1}) q^{-k-1} q^{-k-1} \frac{1-q^{-2k}}{1-q^{-2}} = a_{k-1} q^{-2(k-1)-3} \frac{\zeta(1-q^{2k})(1-q^{-2})}{\zeta(1-q^{2k})(1-q^{-2})} = a_{k-1} q^{-2(k-1)-3}.$$

The calculations are invariant under the mapping sending $E_{\alpha_{r+1}} \mapsto E_{\alpha_{r-1}}, K_{\alpha_{r+1}}^{\pm} \mapsto K_{\alpha_{r-1}}^{\pm}$ and fixing each other generator. This map sends $E_{\beta_r^{r+1}}$ to $E_{\alpha_{r+1}} E_{\alpha_r} - q^{-1} E_{\alpha_r} E_{\alpha_{r+1}}$ and $G_{\beta_r^{r+1}}$ to $G_{\beta_{r-1}^r}$ while $G_{\beta_r^r}$ is invariant. □

THEOREM 3.21 *Let $\lambda \in \Lambda$ be a dominant weight and let $L(\lambda)$ be the simple, finite dimensional $U_q(\mathfrak{sl}_{n+1})$-module of highest weight $\lambda$. Then $L(\lambda)$ is a semisimple, simultaneously diagonalisable $\mathfrak{A}_{n,r}$-module.*

*Proof.* We use the embedding of $U_q(\mathfrak{sl}_r) \otimes U_q(\mathfrak{sl}_{n-r+1})$ into $U_q(\mathfrak{sl}_{n+1})$ such that $\tilde{\Lambda}_{n,r}$ is the weight lattice of $U_q(\mathfrak{sl}_r) \otimes U_q(\mathfrak{sl}_{n-r+1})$. There is a unique decomposition of $L(\lambda)$ as $U_q(\mathfrak{sl}_r) \otimes U_q(\mathfrak{sl}_{n-r+1})$-module into simple modules, say $L(\lambda) = \oplus_{k=1}^t M_k$ with $M_k$ being simple $U_q(\mathfrak{sl}_r) \otimes U_q(\mathfrak{sl}_{n-r+1})$-modules. Let $v_k$ be the highest weight vector of $M_k$, then $v_k$ is a weight vector subject to $U_q(\mathfrak{sl}_{n+1})$—say $v_k$ has weight $\mu_k$. Let $v_k^d$ be its diagonalisation subject to $G_{\beta_r^r}$. By Lemma 3.20 follows $G_{\beta_{r-1}^r} v_k^d = G_{\beta_r^{r+1}} v_k^d = 0$. Therefore $v_k^d$ spans a simple, finite dimensional, simultaneously diagonalisable $\mathfrak{A}_{n,r}$-module that is isomorphic to $L\big(\zeta q^{-(\alpha_r,\mu_k)}, \pi(\mu_k)\big)$. Denote this module by $M_k^d$. From Corollary 3.17 follows $\dim M_k^d = \dim M_k$ and hence

$$L(\lambda) = \bigoplus_{k=1}^{t} M_k^d,$$

i.e. a decomposition of $L(\lambda)$ as $\mathfrak{A}_{n,r}$-module into simple, simultaneously diagonalisable $\mathfrak{A}_{n,r}$-modules. □

Theorem 3.21 together with Krämers 1. Behauptung (1) in [Krä79] ensures that whenever $r \neq n - r + 1$ part 1 and 2 of Definition 1.2 are fulfilled: For the with multiplicity at most one appearing trivial representation of $U_q(\mathfrak{sl}_r) \otimes U_q(\mathfrak{sl}_{n-r+1})$ there is one and only one one dimensional $\mathfrak{A}_{n,r}$-module, though at most one trivial $\mathfrak{A}_{n,r}$-module. Note that not every one dimensional $\mathfrak{A}_{n,r}$-module is necessary trivial. This is especially clear by Proposition 3.4. For the case $r = n - r + 1$ we give an example of a non-spherical module.

In the remainder of this chapter we want partially examine the homogeneous space of $\mathfrak{A}_{n,r}$. The first question arising asks if there is a (sufficient amount of) invariant vector(s). The answer is ›yes‹ as we shall see below. With this answer in mind we are forced to ask what are the invariant vectors or at least, which simple $U_q(\mathfrak{sl}_{n+1})$-modules possess invariant vectors. The answer is not completely satisfying.

Let $\lambda \in \Lambda$ be a dominant weight and let $v \in L(\lambda)$ be an $\mathfrak{A}_{n,r}$-invariant vector. Write $v = \sum_{\mu \leq \lambda} v_\mu$ as a sum of weight vectors. Since $\epsilon(G_{\beta_r^r}) = \zeta$ and $\epsilon(K_i) = 1$ follows with Proposition 3.1 that $v = \sum_{k \leq 0} v_{k\alpha_r}$ with $v_0 \neq 0$. This means that the diagonalisation of a $U_q(\mathfrak{sl}_r) \otimes U_q(\mathfrak{sl}_{n-r+1})$-invariant vector $w \in L(\lambda)$ is $\mathfrak{A}_{r,n}^{(\zeta)}$-invariant if and only if $w$ has weight 0, i.e. $w \in L(\lambda)_0$.

Proposition 3.22 *Let $r \neq n - r + 1$. Then $\mathfrak{A}_{n,r}$ is a spherical right-coideal subalgebra. Let $\lambda \in \Lambda$ and set $s = \min(r, n-r+1)$. The $U_q(\mathfrak{sl}_{n+1})$-module $L(\lambda)$ has an $\mathfrak{A}_{n,r}$-invariant vector if*

$$\lambda \in \bigoplus_{k=1}^{s} (\varpi_k + \varpi_{n+1-k})\mathbb{N}_0.$$

*Proof.* This follows directly from Krämers Tabelle 1 in [Krä79], lines no. 2 and 4 since $S\big(U(r) \times U(n-r+1)\big)$ contains all diagonal matrices of $SU(n+1)$, i.e. a full torus, and thus the invariant vectors all have weight 0. All these vectors are $SU(r) \times SU(n-r+1)$-invariant. □

Let $r \neq n-r+1$, then are by Tabelle 1 in [Krä79] $L(\varpi_r)$ and $L(\varpi_{n-r+1})$ modules with an invariant vector. For the first one it is clear that the highest weight vector is $U_q(\mathfrak{sl}_r) \otimes U_q(\mathfrak{sl}_{n-r+1})$-invariant while $G_{\beta_r^r} v_{\varpi_r} = \zeta q^{-1} v_{\varpi_r}$—though not invariant under $\mathfrak{A}_{n,r}$. But since $L(\varpi_{n-r+1})$ is dual to $L(\varpi_r)$ we find that the lowest weight vector is $U_q(\mathfrak{sl}_r) \otimes U_q(\mathfrak{sl}_{n-r+1})$—but not $\mathfrak{A}_{n,r}$-invariant.

In the case $r = n-r+1$ the right-coideal subalgebra $\mathfrak{A}_{n,r}$ is not spherical as $U_q(\mathfrak{sl}_r) \otimes U_q(\mathfrak{sl}_{n-r+1})$ is not spherical; the latter his is a well known fact, c.f. [Krä79, 1. Behauptung or Tabelle 1 ]. Any $U_q(\mathfrak{sl}_r) \otimes U_q(\mathfrak{sl}_{n-r+1})$-invariant vector has clearly weight $k\varpi_r, k \in \mathbb{Z}$—which is a weight of the $U_q(\mathfrak{sl}_{n+1})$-module $L(\varpi_1 + \varpi_n)$ solemnly for $k = 0$.

In case $n = 2$ we are able to proof a bit more and describe the homogeneous space in the next chapter—this is due to the well-known structure of $U_q(\mathfrak{sl}_3)$ (or, to be more precise, of $\mathfrak{sl}_3$).

# Chapter 4

# Homogeneous Space of $\mathfrak{A}_2$

In this chapter we examine the homogeneous spaces attached to $\mathfrak{A}_2$. We compute the invariant vector, the homogeneous space and show that the space coincides with the homogeneous space of Noumi and Dijkhuizen in [DN98]—for which we shall use the quantum sphere introduced by Podleś, c.f. [Pod87], [KV92], too—in the sense that they have the same algebraic structure. Though, there are some differences: First, the space of Nuomi and Dijkhuizen is equipped with an induced $*$-structure from $\mathcal{A}_q$. Secondly, the space of the two authors is embedded ›symmetrically‹ which is due to the fact that whenever their $\mathfrak{k}^\sigma$-invariant vector of a simple $U_q(\mathfrak{sl}_{n+1})$-simple has a component of weight—say $\mu$—,then also of weight $-\mu$.

Let us start with the description of the invariant vector and the elements generating the homogeneous space of $\mathfrak{A}_2$ in $\mathcal{A}_q$ first.

## Homogeneous Space

Let $\lambda = k_1\varpi_1 + k_2\varpi_2$ be a dominant weight and $v_\lambda$ be the highest weight vector of $L(\lambda)$. Then there is

$$\begin{aligned} K_1 v_\lambda &= q^{k_1} v_\lambda, \\ K_2 v_\lambda &= q^{k_2} v_\lambda \end{aligned}$$

and let on the other hand $m_1, m_2 \geq 0$ be integers and consider now a weight vector $v \in L(\lambda)$ of weight $\mu - (m_1\alpha_1 + m_2\alpha)$. Then there is

$$\begin{aligned} K_1 v = v &\iff k_1 - 2m_1 + m_2 = 0, \\ K_2 v = \zeta v &\iff k_2 + m_1 - 2m_2 = 0. \end{aligned}$$

Reading the equations on the right hand side as conditions on $m_1$ and $m_2$ yields

$$m_1 = \frac{2k_1 + k_2}{3}$$

and

$$m_2 = \frac{k_1 + 2k_2}{3}.$$

In particular, to achieve integrity of $m_1$ and $m_2$, $k_1$ and $k_2$ have to lie in the same residue class mod 3.

The next proposition is known by the end of the last chapter. Nevertheless we give another proof which does work in this special case only—it does rely on dimension comparisons.

PROPOSITION 4.1 *Let $k \in \mathbb{N}$ and $\lambda = k(\varpi_1 + \varpi_2)$, then the trivial representation occurs precisely once in the decomposition of the $U_q(\mathfrak{sl}_3)$-module $L(\lambda)$ as an $\mathfrak{A}_2$-module.*

*Proof.* It is enough to prove that the trivial representation of $U_q(\mathfrak{sl}_2)$ is a subspace of $L(\lambda)_0$ if we embed $U_q(\mathfrak{sl}_2)$ on the first index in $U_q(\mathfrak{sl}_3)$. This comes again from the proof of Theorem 3.21. To show this it suffices to show that $\dim L(\lambda)_0 > \dim L(\lambda)_{\alpha_2}$ since there is exactly one $U_q(\mathfrak{sl}_2)$-invariant vector. This can also be deduced by Pieri's formula ([FH96, Ex. 6.12]. There is a bijection (c.f. [Jan96, Lemma 5.18])

$$U_q(\mathfrak{sl}_3)^-_{-\mu} \longrightarrow L(\lambda)_{\lambda-\mu}, \quad u \longmapsto uv_\lambda$$

for $0 \leq \mu \leq k(\alpha_1 + \alpha_2)$. Hence we compare the dimensions of $U^-_{k(\alpha_1+\alpha_2)}$ and $U^-_{(k-1)\alpha_1+k\alpha_2}$. By [Jan96, Section 8.24] the dimensions og these spaces are given by the Kostant partition function, to be denoted with $\mathcal{P}$, which gives the number of possibilities to write a weight as linear combination of positive roots with non-negative integral coefficients. From this definition it is clear that $\mathcal{P}(k(\alpha_1 + \alpha_2)) > \mathcal{P}((k-1)\alpha_1 + k\alpha_2)$ and accordingly the dimensions. $\square$

We shall use weight diagrams for the proof of the next proposition. They were used by Antoine and Speiser in [AS64] to prove the Weyl character formula for an arbitrary simple Lie group. Drawing a weight diagram for a two dimensional root system in which one can read of the dimensions of the weight spaces is rather easy, but for higher dimensions more or less not possible in general (which is in terms of the presentations of the Weyl character formula reasonable). In the case of the root system $A_2$ we use the notation and results of [FH96, §13.2]. Let us review the notation and facts of and about the weight diagrams of $A_2$.

Let $k_1\varpi_1 + k_2\varpi_2$ be a dominant weight. We may assume without loss of generality that $k_1 \leq k_2$. For each weight $\mu \in \Lambda$ define $m_\mu$ to be its multiplicity in the module $L(k_1\varpi_1 + k_2\varpi_2)$. If $m_\mu = 0$ for a weight do nothing, for $m_\mu = 1$ draw a point in the weight lattice at point $\mu$, for $m_\mu \geq 2$ draw $m_\mu - 1$ circles around the point. Of course the highest weight vector is represented by a single point. In the case that $k_1 > 0$ all one-dimensional weights form a hexagon with two length of edges, having $k_2 + 1$ points on the longer one and $k_1 + 1$ on the shorter one. The two dimensional weight spaces form a hexagon or triangle in this hexagon - each edge having one point less in length.

In general: There will be $k_1$ hexagons, denoted by $H_0, \ldots, H_{k_1}$ and $s = [(k_2 - k_1)/3] + 1$ triangles. If the corresponding point of a weight $\mu$ is on the hexagon $H_i$, then it has the multiplicity $m_\mu = i + 1$. The multiplicities on the triangles are all the same and given by $k_1$. Below is a diagram for the highest weight module of weight $2\varpi_1 + 2\varpi_5$.

As one sees immediately the corner of the hexagons in the fundamental chamber differ from the highest weight by a multiple of $\alpha_1 + \alpha_2$. Of course this is valid for all weight diagrams of type $A_2$.

An slightly bit technically but easy to proof result for $U_q(\mathfrak{sl}_2)$ is as follows: Let $M$ be a finite dimensional $U_q(\mathfrak{sl}_2)$-module and let $\mu$ be a weight of $M$. If $\dim M_\mu = \dim M_{\mu+\alpha}$, then $E(M_\mu) = M_{\mu+\alpha}$.

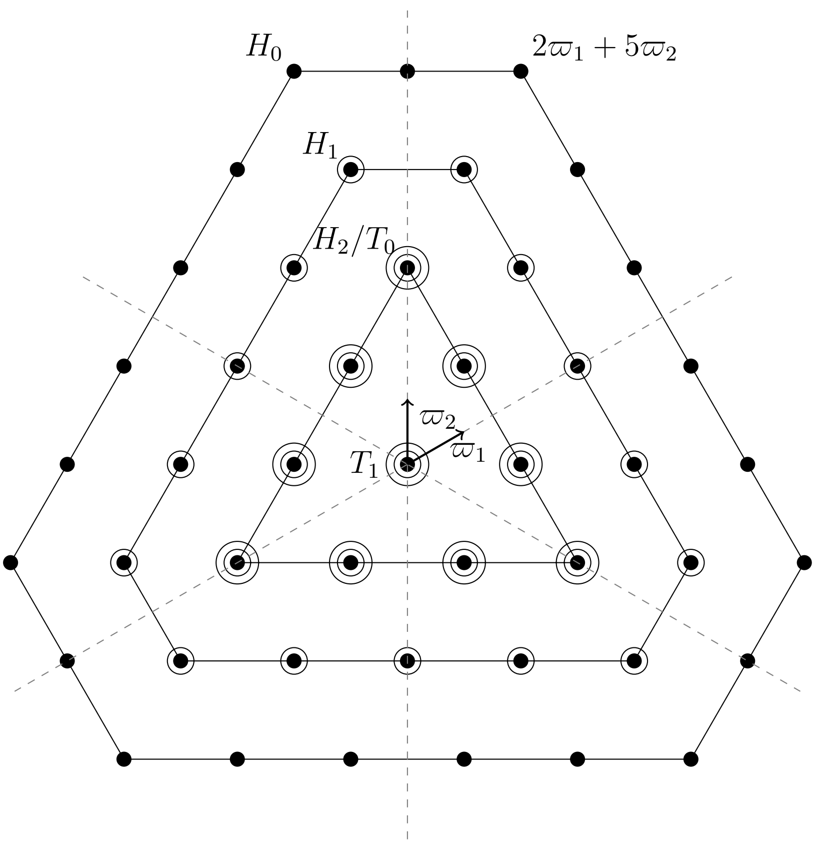


Figure 1: Weight diagram for weight $2\varpi_1 + 2\varpi_5$

PROPOSITION 4.2 *Let $k_1 \neq k_2$ be non-negative integers that are congruent* $\mod 3$ *and $\lambda = k_1\varpi_1 + k_2\varpi_2$. Then there is no trivial representation in the decomposition of the $U_q(\mathfrak{sl}_3)$-module $L(\lambda)$ as an $\mathfrak{A}_2$-module.*

*Proof.* With the theory of weight diagrams we have $\dim L(\lambda)_0 = \dim L(\lambda)_{\alpha_2}$ since there are at least two triangles. We embed $U_q(\mathfrak{sl}_2)$ into $U_q(\mathfrak{sl}_3)$ on the first index, it follows that $E_1 v \neq 0$ for all nonzero $v \in L(\lambda)_0$, so no $v \in L(\lambda)$ spans a trivial $\mathfrak{A}_2$-module. □

In the next step we want to compute the invariant element in $V^* \otimes V$ where $V$ is the vector representation of $U_q(\mathfrak{sl}_3)$. Denote with $v_1$ the highest weight vector of $V$ and set $v_2 = F_1 v_1$ and $v_3 = F_2 F_1 v_1$. This is the standard notation, it is used by Wen in [APW91, Appendix] as in [DN98]. The $U_q(\mathfrak{sl}_3)$-module $V^* \otimes V$ decomposes as

$$V^* \otimes V \simeq V(\varpi_1 + \varpi_2) \oplus V(0),$$

where the first one has $v_i^* \otimes v_j$ for $1 \leq i \neq j \leq 3$ together with $v_i^* \otimes v_i - v_{i+1}^* \otimes v_{i+1}$ for $i = 1, 2$ as basis, the latter one $v_1^* \otimes v_1 + q^{-2} v_2^* \otimes v_2 + q^{-4} v_3^* \otimes v_3$ as basis. Consider

$$v_1^* \otimes v_1 + q^{-2} v_2^* \otimes v_2 - (1 + q^{-2}) v_3^* \otimes v_3.$$

Since it is an element in $L(\varpi_1+\varpi_2)_0$ and

$$\Delta(E_1)\Big(v_1^*\otimes v_1+q^{-2}v_2^*\otimes v_2-(1+q^{-2})v_3^*\otimes v_3\Big)=0$$

follows that it is the $U_q(\mathfrak{sl}_2)$-invariant vector. With Lemma 3.20 and equation (3.12) we compute that

$$v_1^*\otimes v_1+q^{-2}v_2^*\otimes v_2-(1+q^{-2})v_3^*\otimes v_3+\frac{q+q^{-1}+q^{-3}}{\zeta(1-q^2)}v_3^*\otimes v_2 \tag{4.1}$$

is the $\mathfrak{A}_2$-invariant in $L(\varpi_1+\varpi_2)\subset V^*\otimes V$. Substracting $v_1^*\otimes v_1+q^{-2}v_2^*\otimes v_2+q^{-4}v_3^*\otimes v_3$ and stretching with $\frac{\zeta(1-q^2)}{q+q^{-1}+q^{-3}}$ yields

$$\zeta(q-q^{-1})v_3^*\otimes v_3+v_3^*\otimes v_2.$$

The subset

$$\mathcal{A}_q^{\mathfrak{A}_2}=\{\,b\in\mathcal{A}_q\mid ub=\epsilon(u)b\text{ for all }u\in\mathfrak{A}_2\}$$

is the set of left $\mathfrak{A}_2$ invariant elements. In terms of the Peter-Weyl decomposition (1.2) it decomposes by Propositions 4.1 and 4.2 as

$$\mathcal{A}_q^{\mathfrak{A}_2}=\bigoplus_{k\geq 0}L\Big(k(\varpi_1+\varpi_2)\Big)^*. \tag{4.2}$$

The following theorem gives a description of $\mathcal{A}_q^{\mathfrak{A}_2}$ in terms of generators.

THEOREM 4.3 *The subspace spanned by the elements*

$$x_{ij}=\zeta(q-q^{-1})t_{i3}^*t_{j3}+t_{i3}^*t_{j2}\quad(1\leq i,j\leq 3)$$

*is left $\mathfrak{A}_2$ and right $U_q(\mathfrak{sl}_3)$ invariant. The linear map*

$$\begin{aligned}V^*\otimes V&\longrightarrow\mathcal{A}_q\\ v_i^*\otimes v_j&\longmapsto q^{-2(3-i)}x_{ij}\end{aligned}$$

*is an injective right $U_q(\mathfrak{sl}_3)$ module homomorphism. The $x_{ij}$ generate the algebra $\mathcal{A}_q^{\mathfrak{A}_2}$.*

*Proof.* The proof is similar to the proof given in [DN98, Prop. 3.11]. Therefore we show, that the given map is injective. This is indeed true, we have $\sum_k x_{kk}=\zeta(q-q^{-1})\cdot 1$ by [NYM93, Corollary to Proposition 1.1] which means that the trivial representation occurs with non-zero multiplicity, and we have $x_{31}=t_{33}^*\Big(\zeta(q-q^{-1})t_{13}-t_{12}\Big)\neq 0$—which is a vector of highest weight—since $\mathcal{A}_q$ does not have any zero divisors. □

## Construction of $\mathcal{A}_q^{\mathfrak{A}_2}$ via Quantum Spheres

In [DN98] Dijkhuizen and Noumi constructed their subalgebra of left $\mathfrak{k}^\sigma$ invariant elements in $\mathcal{A}_q$ via ›deformed‹ quantum spheres. With some adjustments it is possible to construct their subalgebra of left $\mathfrak{k}^\sigma$ invariants in $\mathcal{A}_q$ with the help of deformed quantum spheres. The method does also work for $\mathcal{A}_q^{\mathfrak{A}_2}$. In the rest of the section there is short route of the construction given using the notation in said publication—taking here care of some differences. They are due to the fact that we consider a left instead of right action. For proofs see the original paper of Dijkhuizen and Noumi.

Definition 4.4 *Let $n \in \mathbb{N}$ and let $\mathcal{A}_q(\tilde{S}) = \mathcal{A}_q(\tilde{S}^{2n-1})$ be the algebra generated by $z_i, w_i$, $1 \le i \le n$ and $c, d$ subject to the relations*

$$
\begin{aligned}
&z_i z_j = q z_j z_i, \quad (1 \le i < j \le n),\\
&w_i w_j = q^{-1} w_j w_i \quad (1 \le i < j \le n),\\
&w_i z_j = q z_j w_i \quad (1 \le i \ne j \le n),\\
&z_j w_j = w_j z_j + (1-q^{-2}) \sum_{k<j} w_k z_k - (1-q^{-2}) d \quad (1 \le j \le n),\\
&\sum_{k=1}^{n} w_k z_k = c + d.
\end{aligned}
$$

With these relations the following equations are holds for all $1 \le j \le n$:

$$
\begin{aligned}
&c z_j = z_j c,\ c w_j = w_j c,\ cd = dc,\\
&d z_j = q^2 z_j d,\ d w_j = q^{-2} w_j d,\\
&w_j z_j = z_j w_j - (1-q^{-2}) \sum_{k=1}^{j-1} q^{-2(j-k-1)} z_k w_k + (1-q^{-2}) q^{-2(j-1)} d,\\
&\sum_{k=1}^{n} q^{-2(n-k)} z_k w_k = c + q^{-2n} d.
\end{aligned}
$$

There is also a unique $*$-structure on the algebra $\mathcal{A}_q(\tilde{S})$ such that

$$z_j^* = w_j,\ c^* = c,\ d^* = d.$$

We set $\mathcal{C} = \mathbb{C}[\alpha, \beta, \gamma, \delta]$ and $\theta \colon \mathcal{C} \to \mathcal{C}$ with

$$\theta(\alpha) = \alpha,\ \theta(\beta) = q\beta,\ \theta(\gamma) = q\gamma,\ \theta(\delta) = \delta.$$

We set $\mathcal{C}[\theta^{\pm 1}]$ as the subalgebra of $\mathrm{End}_{\mathbb{C}} \mathcal{C}$ generated by the left multiplication by $\alpha, \beta\, \gamma, \delta, \theta, \theta^{-1}$. This algebra is isomorphic to $\mathbb{C}[\theta^{\pm 1}] \otimes \mathbb{C}[\alpha, \beta, \gamma, \delta]$ with the multiplication on the tensor product given by $\theta P = \theta(P)\theta$ where $P$ is a polynomial in $\mathcal{C}$.

We define the maps

$$
\begin{aligned}
R_1 &\colon \mathcal{A}_q(\tilde{S}) \longrightarrow \mathcal{A}_q(\tilde{S}) \otimes \mathcal{A}(U(1)),\\
L_1 &\colon \mathcal{A}_q(\tilde{S}) \longrightarrow \mathcal{A}_q \otimes \mathcal{A}_q(\tilde{S}),
\end{aligned}
$$

$$R_2\colon \mathcal{A}_q \otimes \mathcal{C}[\theta^{\pm 1}] \longrightarrow \mathcal{A}_q \otimes \mathcal{C}[\theta^{\pm 1}] \otimes \mathcal{A}(U(1)),$$
$$L_2\colon \mathcal{A}_q \otimes \mathcal{C}[\theta^{\pm 1}] \longrightarrow \mathcal{A}_q \otimes \mathcal{A}_q \otimes \mathcal{C}[\theta^{\pm 1}]$$

by

$$R_1(z_j) = z_j \otimes z,\ R_1(w_j) = w_j \otimes z^{-1},\ R_1(c) = c \otimes 1,\ R_1(d) = d \otimes 1,$$
$$L_1(z_j) = \sum_{i=1}^{n} t_{ji} \otimes z_i,\ L_1(w_j) = \sum_{i=1}^{n} S(t_{ij}) \otimes w_i,\ L_1(c) = 1 \otimes c,\ L_1(d) = 1 \otimes d,$$
$$R_2(\theta) = \theta \otimes z^{-1}, R_2(g) = g \otimes 1,\ \text{for } g \text{ generator},$$
$$L_2 = \Delta \otimes \mathrm{id}\,.$$

These morphism turn $\mathcal{A}_q(\tilde{S})$ and $\mathcal{A}_q \otimes \mathcal{C}[\theta^{\pm 1}]$ into two-sided $\big(\mathcal{A}_q, \mathcal{A}(U(1))\big)$-comodule algebras. In the rest of the section we choose $n$ to be 3, since the subalgebra $\mathfrak{A}_2$ only lives in $U_q(\mathfrak{sl}_3)$.

THEOREM 4.5 *There are unique algebra homomorphisms $\Psi, \Phi\colon \mathcal{A}_q(\tilde{S}) \to \mathcal{A}_q \otimes \mathcal{C}[\theta^{\pm 1}]$ such that for $1 \leq j \leq 3$ we have*

$$\Psi(z_j) = \theta^{-1}(\gamma t_{j2} + \delta t_{j3}),$$
$$\Phi(z_j) = \theta^{-1}(\gamma t_{j1} + \delta t_{j3}),$$
$$\Psi(w_j) = \alpha t^*_{j3}\theta,$$
$$\Phi(w_j) = (-\beta t^*_{j1} + \alpha t^*_{j3})\theta,$$
$$\Psi(c) = \Phi(c) = \alpha\delta,$$
$$\Phi(d) = 0,$$
$$\Psi(d) = -\beta\gamma.$$

*The mappings $\Psi, \Phi$ are injective and two-sided $\big(\mathcal{A}_q, \mathcal{A}(U(1))\big)$-comodule homomorphmisms. $\Phi$ is also a $*$-homomorphism.*

COROLLARY 4.6 *The isotypical decomposition of $\mathcal{A}_q(\tilde{S})$ with respect to the $\mathcal{A}_q$ coaction is*

$$\mathcal{A}_q(\tilde{S}) \cong \bigoplus_{l,m \geq 0} V(l\varpi_1 + m\varpi_2) \otimes \mathbb{C}[c, d],$$

*where $V(l\omega_1 + m\omega_2)$ is the $\mathcal{A}_q$-subcomodule of $\mathcal{A}_q(\tilde{S})$ generated by the highest weight vector $w_3^m z_1^l$.*

We define

$$\mathcal{A}_q(\widetilde{\mathbb{CP}}^2) = \{a \in \mathcal{A}_q(\tilde{S}^5) \mid R(a) = a \otimes 1\}.$$

This algebra is generated by $w_i z_j$, $1 \leq i, j \leq 3$ and $c, d$. As already in $\mathcal{A}_q(\tilde{S})$ in the above algebra $c, d$ are central. We can specialise $c, d$ to elements in $\mathbb{C}$ with not both chosen as zero. We denote this algebra by $\mathcal{A}_q\big(\widetilde{\mathbb{CP}}^2(c, d)\big)$, it has the following decomposition subject to the left $\mathcal{A}_q$-comodule structure

$$\mathcal{A}_q\big(\widetilde{\mathbb{CP}}^2(c, d)\big) = \bigoplus_{l \geq 0} V\big(l(\varpi_1 + \varpi_2)\big).$$

Set $\mathcal{A}_q^{\mathfrak{k}^\sigma} = \{\, b \in \mathcal{A}_q \mid ub = \epsilon(u)b$ for all $u \in \mathfrak{k}^\sigma\}$. This algebra is generated by the elements

$$y_{ij} = qdt_{i1}^* t_{j1} + q^{-1}ct_{i3}^* t_{j3} + \sqrt{cd}t_{i3}^* t_{j1} + \sqrt{cd}t_{i1}^* t_{j3} \quad (1 \le i, j \le 3)$$

—so it is the homogeneous space in [DN98].

Theorem 4.7 *Let $c_0 > 0$ be a real number and $\zeta = c_0 \frac{1}{q-q^{-1}}$. Then there are unique algebra homomorphisms given by*

$$\mathcal{A}_q(\widetilde{\mathbb{CP}}^2) \longrightarrow \mathcal{A}_q^{\mathfrak{A}_2},\ w_i z_j \mapsto x_{ij},\ c \mapsto c_0, d \mapsto 0,$$
$$\mathcal{A}_q(\widetilde{\mathbb{CP}}^2) \longrightarrow \mathcal{A}_q^{\mathfrak{k}^{c_0,0}},\ w_i z_j \mapsto y_{ij},\ c \mapsto c_0,\ d \mapsto 0.$$

*These two mappings are $\mathcal{A}_q$-module homomorphisms and induce isomorphisms from $\mathcal{A}_q(\widetilde{\mathbb{CP}}^{n-1}(c_0, 0))$ to $\mathcal{A}_q^{\mathfrak{k}^{c_0,0}}$ resp. $\mathcal{A}_q^{\mathfrak{A}_2}$. The first homomorphism is a $*$-algebra homomorphism. In particular we have an isomorphism between $\mathcal{A}_q^{\mathfrak{A}_2}$ and $\mathcal{A}_q^{\mathfrak{k}^{c_0,0}}$.*

*Proof.* We define algebra homomorphisms $\phi, \psi\colon \mathbb{C}[\alpha, \beta, \gamma, \delta] \to \mathbb{C}$ by

$$\begin{array}{llll} \psi(\alpha) = 1, & \psi(\beta) = 0, & \psi(\gamma) = 1, & \psi(\delta) = c_0, \\ \phi(\alpha) = 0, & \psi(\beta) = 0, & \psi(\gamma) = 0, & \psi(\delta) = c_0, \end{array}$$

then the maps $(\mathrm{id} \otimes \psi) \circ \Psi$ and $(\mathrm{id} \otimes \phi) \circ \Phi$ are well-defined maps, since $\Psi\Big(\mathcal{A}_q(\widetilde{\mathbb{CP}}^2)\Big)$ and $\Phi\Big(\mathcal{A}_q(\widetilde{\mathbb{CP}}^2)\Big)$ are contained in $\mathcal{A}_q \otimes \mathbb{C}[\alpha, \beta, \gamma, \delta]$. □

Remark 4.8 *In the case $r = 1$, that is $\mathfrak{A}_{2,1}^{(\zeta)}$, the corresponding homogeneous space is isomorphic to $\mathcal{A}_q^{\mathfrak{k}^{d_0,0}}$ for suitable pairs $d_0 > 0$ and $\zeta \in k^*$, namely $\zeta = d_0 \frac{1}{q-q^{-1}}$.*

## The Case $\zeta = 0$

If the parameter $\zeta = 0$, then $G_2$ is obviously nilpotent. In particular, we lose the semisimplicity for at least all finite dimensional simple $U_q(\mathfrak{sl}_3)$ modules. However, we can still show that the space of $\mathfrak{A}_2^{(0)}$-invariant elements is one-dimensional. To perform this we lift the case $\zeta = 0$ back to the case $\zeta \neq 0$.

Similarly as in the case for $\zeta \neq 0$ we get for the case $\zeta = 0$ the following necessary condition: Let $\mu = k_1\varpi_1 + k_2\varpi_2$ be a dominant weight, then we have

$$K_{12} v_\mu = q^{2k_1 + 2k_1} v_\mu$$

and let $m_1, m_2 \ge 0$ be integers and choose $v \in L(\mu)$ with weight $\mu - (m_1\alpha_1 + m_2\alpha_2)$ that is $\mathfrak{A}_2^{(0)}$-invariant, then we have the condition

$$K_{12} v = v \Longleftrightarrow -3m_1 + 2k_1 + k_2 = 0,$$

which means that we have $m_1 = \frac{2k_1+k_2}{3}$ and no condition on $m_2$. This requires $k_1$ and $k_2$ lying in same residue class $\mod 3$ and we know, that an invariant vector $v$ must have weight $0 + t\alpha_2$ for a suitable $t$.

For given $\zeta \in k$ we define

$$G_2^{(\zeta)} = K_2^{-1}(E_2 + \zeta \cdot 1),$$
$$G_{12}^{(\zeta)} = K_1^{-1}K_2^{-1}(E_2E_1 - q^{-1}E_1E_2 + (q - q^{-1})\zeta E_1).$$

PROPOSITION 4.9 *Let $k \in \mathbb{N}_0$ and $\mu = k(\varpi_1 + \varpi_2)$, then the subspace of $\mathfrak{A}_2^{(0)}$-invariant vectors in the $U_q(\mathfrak{sl}_3)$-module $L(\mu)$ is one-dimensional.*

*Proof.* The one-dimensional weight space $L(\mu)_{0+k\alpha_2} = L(\mu)_{\mu-k\alpha_1}$ is invariant under $\mathfrak{A}_2^{(0)}$.

Let $0 \leq t < k$. Assume there is a vector $w \in L(\mu)$ with weight $t\alpha_2$ such that $G_2^{(0)}w = 0$ an $F_1w = 0$. Then we have $G_2^{(1)}w = q^{-2t}\zeta w$, i.e. $w$ is an eigenvector of $G_2^1$ and $K_{12}$. Moreover, $w$ spans an $\mathfrak{A}_2^{(1)}$-module of dimension $t + 1$. Since $w$ is an weight vector of $U_q(\mathfrak{sl}_3)$, follows with $U_q(\mathfrak{sl}_2)$-theory that $(G_{12}^{(1)})^t w$ is a weight vector of weight $t\alpha_1 + t\alpha_2$. Therefore we have that $G_2^{(1)}(G_{12}^{(1)})^t w = q^{-t}\zeta(G_{12}^{(1)})^t w$, and it follows that $G_2^{(0)}(G_{12}^{(1)})^t w = 0$, in particular we have $E_2(G_{12}^{(1)})^t w = 0$. On the other hand we have $(G_{12}^{(1)})^{t+1}w = 0$ from which follows that $E_1(G_{12}^{(1)})^t w = 0$. This means $(G_{12}^{(1)})^t w$ is a highest weight vector in $L(\mu)$, which is contradiction. □

PROPOSITION 4.10 *Let $k_1 \neq k_2$ be non-negative integers that are in the same residue class* $\mod 3$ *and set $\mu = k_1\varpi_1 + k_2\varpi_2$. Then there is no non-trivial $\mathfrak{A}_2^{(0)}$-invariant vector in the $U_q(\mathfrak{sl}_3)$-module $L(\mu)$.*

*Proof.* Set $m_2 = \frac{2k_1+k_2}{3}$ and choose $-m_2 \leq t \leq m_2$. We distinguish two cases.

Assume $k_1 < k_2$ and set $s = \frac{k_2-k_1}{3}$. There are two cases:

1. $t \leq s - 1$: For any vector $w \neq 0$ of weight $t\alpha_2$ we have $E_2w \neq 0$ and thus $G_2w \neq \epsilon(G_2)w = 0$.

2. $t \geq s$: Assume there is a $w \in L(\mu)_{t\alpha_2}$, $w \neq 0$ such that $\mathfrak{A}_2^{(0)}w = 0$. By Proposition 3.4 spans $w$ a $t+1$-dimensional $\mathfrak{A}_2^{(1)}$-module. Since $(E_2E_1 - E_1E_2)E_1^t = q^t(E_2E_1 - E_1E_2)$ and $G_{12}^{(0)}w = 0$ follows

   $$(G_{12}^{(1)})^t w \in L(\mu)_{t(\alpha_1+\alpha_2)}.$$

   In particular this is an eigenvector of $G_2^{(1)}$. But since $\mu$ is not a multiple of $\alpha_1 + \alpha_2$ and $t(\alpha_1 + \alpha_2)$ is weight of $H_{m_2-t}$ follows $t(\alpha_1 + \alpha_2)$ is not a corner of $H_{m_2-t}$ and so $E_2(G_{12}^{(1)})^t w \neq 0$ which contradicts to $(G_{12}^{(1)})^t w$ being an eigenvector of $G_2^{(1)}$.

Assume that $k_1 > k_2$ and set $s = \frac{k_1-k_2}{3}$. There are again two cases:

1. $t \leq s - 1$: This is identically to the first case above.

2. $t \geq s$: The weight space $L(\mu)_{t\alpha_2}$ is not a corner of $H_{m_2-t}$, since otherwise $s_{\alpha_1}(t\alpha_2) = t(\alpha_1+\alpha_2)$ would be a corner of $H_{m_2-t}$ which is not true, c.f. second case above. Thus $\dim L(\mu)_{t\alpha_2} = \dim L(\mu)_{t\alpha_2-\alpha_1}$ and for any $w \neq 0$ of weight $t\alpha_2$ holds $F_1 w \neq 0$.

$\square$

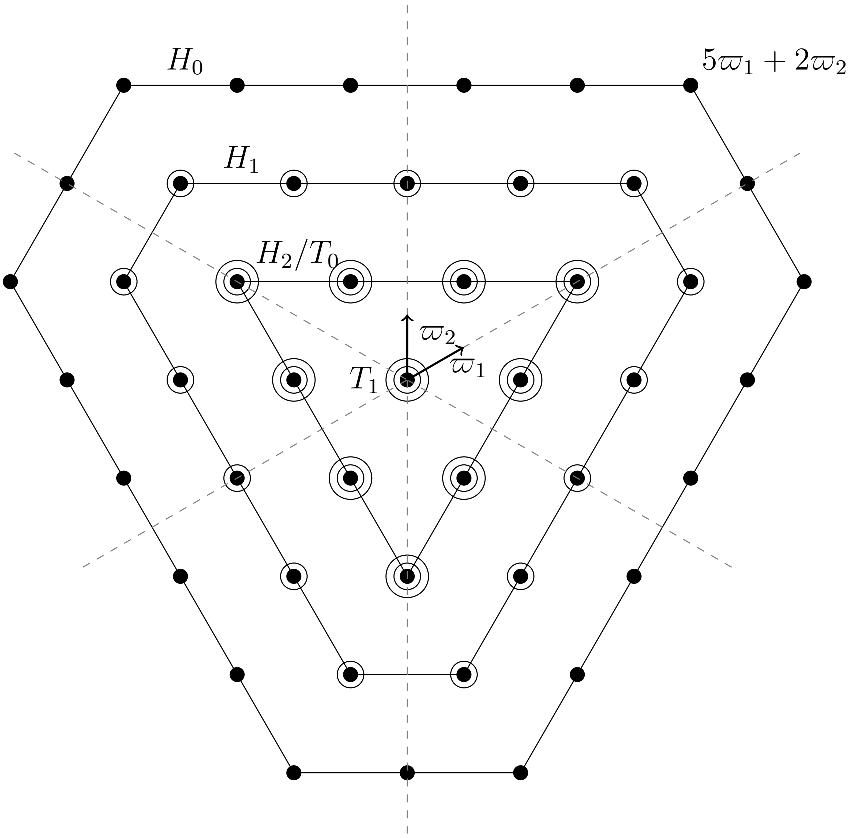


Figure 2: Weight diagram of weight $5\varpi_1 + 2\varpi_2$

The space of invariant elements in $V^* \otimes V$ is spanned by $v_3^* \otimes v_2$ and $q^4 v_1^* \otimes v_1 + q^2 v_2^* \otimes v_2 + v_3^* \otimes v_3$.

THEOREM 4.11 *The subspace spanned by the elements*

$$x_{ij} = \delta_{ij} 1 + t_{i3}^* t_{j2} \quad (1 \leq i, j \leq 3),$$

*is left* $\mathfrak{A}_2^{(0)}$ *and right* $U_q(\mathfrak{sl}_3)$ *invariant. The linear map*

$$\begin{aligned} V^* \otimes V &\longrightarrow \mathcal{A}_q \\ v_i^* \otimes v_j &\longmapsto x_{ij} \end{aligned}$$

*is an injective right* $U_q(\mathfrak{sl}_3)$ *module homomorphism. The* $x_{ij}$ *generate the algebra* $\mathcal{A}_q^{\mathfrak{A}_2}$.

It is not possible to construct a map as in the previous section from $\mathcal{A}_q(\tilde{S})$ to $\mathcal{A}_q^{\mathfrak{A}_2}$. The reason is very simple: There is no factorisation for the elements $1+t_{k3}^* t_{k2}$, $1 \le k \le 3$. However, it is clear that the algebra $\mathcal{A}_q^{\mathfrak{A}_2}$ is generated by $t_{i3}^* t_{j2}$, $1 \le i, j \le 3$, and 1. Clearly the first nine elements can be factorised and we can define an injective map $\Psi' \colon \mathcal{A}_q(\tilde{S}) \longrightarrow \mathcal{A}_q^{\mathfrak{A}_2}$ given by

$$\Psi'(z_j) = t_{j2}, \quad \Psi'(w_j) = t_{j3}^*, \quad \Psi'(c) = \Psi'(d) = 0,$$

and we get an injective map from $\mathcal{A}_q(\widetilde{\mathbb{CP}}^{n-1}(0,0))$ to $\mathcal{A}_q^{\mathfrak{A}_2}$, where the image is generated by the $t_{i3}^* t_{i2}$. If we add a 1 to $\mathcal{A}_q(\tilde{S})$ and hence to $\mathcal{A}_q(\widetilde{\mathbb{CP}}^{n-1}(c_0,0))$ and define $\Psi'(1) = 1$, then we get an isomorphism.

# Chapter 5

# Prospects

## The Case $B_2$

We shall examine here briefly and elementary an analogous family like in Chapter 3 and 4. The proofs will be more or less the same as in Chapter 3 concerning the representation theory and are therefore mostly omitted.

Let $\alpha_1$ and $\alpha_2$ be the simple roots of type $B_2$ with $\alpha_1$ being the long root. Set $w^+ = s_{\alpha_2} s_{\alpha_1}$ and $w^- = s_{\alpha_1}$. Set $\beta = s_{\alpha_2}(\alpha_1) = \alpha_1 + 2\alpha_2$,

$$
\begin{aligned}
&E_2^{(2)} = \frac{E_2^2}{[2]} \quad \text{and} \\
&E_\beta = T_{\alpha_2}(E_1) = E_2^{(2)} E_1 - q^{-1} E_2 E_1 E_2 + q^{-2} E_1 E_2^{(2)}.
\end{aligned}
$$

We have

$$
\begin{aligned}
&\Delta(E_\beta) \\
&\quad = E_\beta \otimes 1 + K_1 K_2^2 \otimes E_\beta \\
&\qquad + (qE_2K_2E_1 - q^{-1}E_2E_1K_2 - q^{-1}K_2E_1E_2 + q^{-1}E_1E_2K_2) \otimes E_2 \\
&\qquad + (K_2^2E_1 - q^{-1}q^2K_2E_1K_2 + q^{-2}E_1K_2^2) \otimes E_2^{(2)} \\
&\qquad + (E_2^{(2)}K_1 - q^{-1}E_2K_1E_2 + q^{-2}K_1E_2^{(2)}) \otimes E_2 \\
&\qquad + (qE_2K_2K_1 - q^{-1}K_2K_1E_2) \otimes E_2E_1 \\
&\qquad + (-q^{-1}E_2K_1K_2 + q^{-1}K_1E_2K_1) \otimes E_1E_2 \\
&\quad = E_\beta \otimes 1 + K_1K_1^2 \otimes E_\beta + q^{-1}(1-q^{-2})^2 E_2^2 K_1 \otimes E_1 \\
&\qquad + (q-q^{-1})E_2K_1K_2 \otimes E_2E_1 - q^{-2}(q-q^{-1})E_2K_1K_2 \otimes E_1E_2
\end{aligned}
$$

and with equation (5) of [Jan96, p. 8.17] follows

$$
\begin{aligned}
&\quad = E_\beta \otimes 1 + K_1K_1^2 \otimes E_\beta + \\
&\qquad + (q-q^{-1})E_2K_1K_2 \otimes T_{\alpha_2}T_{\alpha_1}(E_2) + q^{-1}(1-q^{-2})^2 E_2^2 K_1 \otimes E_1,
\end{aligned}
$$

so we compute

$$
\begin{aligned}
\Delta \circ \psi(E_2) &= q^{-1} \cdot 1 \otimes E_2K_2^{-1} + q^{-1}E_2K_2^{-1} \otimes K_2^{-1} \\
\Delta \circ \psi(E_\beta) &= q^{-2}E_\beta K_1^{-1}K_2^{-2} \otimes K_2^{-2} + q^{-2} \cdot 1 \otimes E_\beta K_1^{-1}K_2^{-2} \\
&\quad + q^{-2}(q-q^{-1})E_2K_2^{-1} \otimes T_{\alpha_2}T_{\alpha_1}(E_2)K_1^{-1}K_2^{-2} \\
&\quad + q^{-5}(q-q^{-1})^2 E_2^2 K_2^{-2} \otimes E_1K_1^{-1}K_2^{-2}.
\end{aligned}
$$

Let $\zeta \in k^*$ and set $\varphi(E_2) = q\zeta$. We apply the contraction on the first tensor with $q^{-1}(\varphi\circ\psi)$ and obtain

$$\begin{aligned}
&K_2^{-1}(E_2 + \zeta \cdot 1),\\
&K_1^{-1}K_2^{-2}(qE_\beta + q(q-q^{-1})\zeta T_{\alpha_2}T_{\alpha_1}(E_2) + (q-q^{-1})^2\zeta E_1).
\end{aligned}$$

Denote with $G_2$ as the first and with $G_{12}$ the latter stretched by the factor $q^{-1}\frac{q+q^{-1}}{q-q^{-1}}$. Define $\mathfrak{B}_2$ as the subalgebra of $U_q(\mathfrak{o}_5)$ generated by

$$\begin{aligned}
&G_2,\\
&G_{12},\\
&F_1,\\
&K_2^2K_2^2, K_1^{-2}K_2^{-2}.
\end{aligned}$$

The following relations are holding

$$\begin{aligned}
G_2K_2^2K_2^2 &= K_2^2K_2^2G_2,\\
G_2F_1 &= q^{-2}F_1G_2,\\
G_2G_{12} &= q^2G_{12}G_2,\\
K_2^2K_2^2F_1 &= q^{-4}F_1K_2^2K_2^2,\\
K_2^2K_2^2G_{12} &= q^4F_1K_2^2K_2^2,\\
G_{12}F_1 &= F_1G_{12} + G_2^2 - \zeta^2K_1^{-2}K_2^{-2},
\end{aligned}$$

where the last relation generalises for $s \in \mathbb{N}$

$$G_{12}F_1^s = F_1^sG_{12} + \frac{1-q^{4s}}{1-q^4}F_1^{s-1}\Big(q^{4(s-1)}G_2^2 - \zeta^2K_1^{-2}K_2^{-2}\Big).$$

By Theorem [HK11, p. 2.17] and [HK11, p. 3.18] is $\mathfrak{B}_2$ a right coideal subalgebra. It is possible to choose $K_1K_2$ and its inverse instead of $K_2^2K_2^2$. In terms of the spherical property of $\mathfrak{B}_2$ this has no effect, but the representation theory differs in general slightly. We shall give a remark later.

For each pair $\eta, \kappa \in k$ with $\eta, \kappa \neq 0$ exists an infinite dimensional $\mathfrak{B}_2$-module $M(\eta,\kappa)$ with a basis $m_0, m_1, m_2, \ldots$ such that for all $i \geq 0$

$$\begin{aligned}
G_2m_i &= \eta q^{-2i}m_i,\\
K_2^2K_2^2m_i &= \kappa q^{-4i}m_i,\\
F_1m_i &= m_{i+1},\\
G_{12}m_i &= \begin{cases} 0, & \text{if } i = 0,\\ \dfrac{1-q^{4i}}{1-q^4}\Big(q^{4(i-1)}\eta^2 - \zeta^2\kappa^{-1}\Big)m_{i-1}, & \text{if } i \geq 1.\end{cases}
\end{aligned}$$

PROPOSITION 5.1 *The module $M(\eta,\kappa)$ contains exactly one proper submodule if and only if $\eta^2\kappa = \zeta^2q^{4(s-1)}$ holds for an integer $s \geq 1$. Otherwise it is simple.*

Proposition 5.2 *For each pair $\eta \in k^*$ and $s \in \mathbb{N}_0$ exists a simple simultaneously diagonalisable $\mathfrak{B}_2$-module of dimension $s+1$ and a basis $m_0, \dots, m_s$ such that*

$$\begin{aligned} G_2 m_i &= \eta q^{-2i} m_i, \\ K_2^2 K_2^2 m_i &= \eta^{-2} \zeta^2 q^{4(s-i)} m_i, \\ F_1 m_i &= \begin{cases} m_{i+1}, & \text{if } i < s, \\ 0, & \text{if } s = 0, \end{cases} \\ G_{12} m_i &= \begin{cases} 0, & \text{if } i = 0, \\ \dfrac{1-q^{4i}}{1-q^4}\Big(q^{4(i-1)}\eta^2 - q^{4s}\eta^2\Big) m_{i-1}, & \text{if } i \geq 1. \end{cases} \end{aligned}$$

*Denote this module by $L(\eta, s)$. Each simple simultaneously diagonalisable $\mathfrak{B}_2$-module of dimension $s+1$ is isomorphic to a $L(\eta, s)$ for a suitable $\eta$.*

Remark 5.3 *Having an $\eta^2$ in the condition of Proposition 5.1 yields in Proposition 5.2 in principle a sign. But by choosing $\eta$ this is already encoded in the act of choosing. Determining instead $\kappa$ for the element $K_2^2 K_2^2$ first would give a sign.*

Remark 5.4 *Taking $K_1 K_2$ and its inverse as generating elements of $\mathfrak{B}_2$ gives in Proposition 5.1 the condition $\eta^2 \kappa^2 = \zeta^2 q^{4(s-1)}$. Consequently we get then in Proposition 5.2 an additional sign for the eigenvalue of $K_1 K_2$ on $m_0$. But this is the only difference.*

Proposition 5.5 *Each finite dimensional simultaneously diagonalisable $\mathfrak{B}_2$-module is semisimple.*

Remark 5.6 *Applying the proof of Proposition 3.5 produces in the calculations of case 2*

$$s + s' = 0$$

*instead of $s + s' = -2$. This does not give a contradiction whenever $s = s' = 0$. But in that case we have necessarily $F_1 v = 0$, since $M$ is two-dimensional, and so does $v$ span a one-dimensional submodule as does $w$. Their direct sum must have a disjoint decomposition into two common eigenspaces.*

Let $L(k_1\varpi_1 + k_2\varpi_2)$ be a simple, finite dimensional $U_q(\mathfrak{o}_5)$-module. Let $v \in L$ be a simultaneously diagonalised, non-zero vector such that $G_{12} v = 0$. Denote with $w$ the component of highest weight of $v$ with respect to $U_q(\mathfrak{o}_5)$. By the presentation of $G_2$ and $G_{12}$ follows immediately that $E_1 w = 0$ and that $w$ generates a simple $U_q(\mathfrak{sl}_2)$-submodule of the same dimension as the $\mathfrak{B}_2$-submodule generated by $v$. From this follows a one-to-one correspondence of decompositions of $L(k_1\varpi_1 + k_2\varpi_2)$.

Requiring from $v \in L(k_1\varpi_1 + k_2\varpi_2)$ to be a $\mathfrak{B}_2$-invariant vector implies that its component of highest weight has weight 0. As necessary condition on $k_1$ and $k_2$ this produces $k_2 \in 2\mathbb{N}_0$, so $L(k_1\varpi_1 + k_2\varpi)$ must be a spin representation. This is also a sufficient condition. For showing

$$\dim L(k_1\varpi_1 + 2k_2\varpi_2)_0 - \dim L(k_1\varpi_1 + 2k_2\varpi_2)_{\alpha_1} = 1$$

for all $k_1, k_2 \in \mathbb{N}_0$, we are going to use the two recursion formula (9) computed by Fernández-Núñez and García-Fuertes in [FNGF14]. For an easier computation we compute

$$\dim L(k_1\varpi_1 + 2k_2\varpi_2)_0 - \dim L(k_1\varpi_1 + 2k_2\varpi_2)_{2\varpi_2}$$

which is the same number as above since $s_{\alpha_2}(\alpha_1) = 2\varpi_2$. We perform a double induction on $k_1$ and $k_2$. Define $y_{k_1,k_2}(0,0)$ resp. $y_{k_1,k_2}(0,2)$ as in at (9) in [FNGF14] but for $B_2$ instead. Note that $y_{k_1,k_2}(0,0) = y_{k_1,k_2}(0,2)$.

Let $k_1 = 0$. By equation (9) in [FNGF14] and the fact that $\dim L(0)_0 = 1$ and $\dim L(0)_{2\varpi_2} = 0$ follows via induction on $k_2$ immediately

$$\begin{aligned}\dim L(2k_2\varpi_2)_0 &= k_2 + 1,\\ \dim L(2k_2\varpi_2)_{2\varpi_2} &= k_2.\end{aligned}$$

Let $k_1 \geq 1$. We distinguish two cases in which both we apply equation (9):

1. $k_2 = 0$. Then

$$\begin{aligned}&\dim L(k_1\varpi_1)_0 - \dim L(k_1\varpi_1)_{2\varpi_2}\\ &= \dim L\Big((k_1-1)\varpi_1\Big)_0 + y_{k_1,0}(0,0) - \dim L\Big((k_1-1)\varpi_1\Big)_{2\varpi_2} - y_{k_1,0}(0,2)\\ &= 1\end{aligned}$$

   by induction.

2. $k_2 \geq 1$. Then

$$\begin{aligned}&\dim L(k_1\varpi_1 + 2k_2\varpi_2)_0 - \dim L(k_1\varpi_1 + 2k_2\varpi_2)_{2\varpi_2}\\ &= \dim L\Big(k_1\varpi_1 + 2(k_2-1)\varpi_2\Big)_0 + \dim L\Big((k_1-1)\varpi_1 + 2k_2\varpi_2\Big)_0\\ &- \dim L\Big((k_1-1)\varpi_1 + 2(k_2-1)\varpi_2\Big)_0 + y_{k_1,2k_2}(0,0)\\ &- \dim L\Big(k_1\varpi_1 + 2(k_2-1)\varpi_2\Big)_{2\varpi_2} - \dim L\Big((k_1-1)\varpi_1 + 2k_2\varpi_2\Big)_{2\varpi_2}\\ &+ \dim L\Big((k_1-1)\varpi_1 + 2(k_2-1)\varpi_2\Big)_{2\varpi_2} - y_{k_1,2k_2}(0,2)\\ &= 1.\end{aligned}$$

□

Theorem 5.7 *The right coideal subalgebra $\mathfrak{B}_2$ is spherical in $U_q(\mathfrak{o}_5)$.*

## Generalisation for Other Types

As the calculus for the quantum enveloping algebra of type $B_2$ shows it is possible to apply the method of deformation done in this work on other types than $A_n$. In principle there are two possibilities: Either performing case-by-case calculus or try to find a suitable theory for all cases.

Krämers [Krä79, Tabelle 1] and the theory of riemannian symmetric spaces suggest some kind of grouping at least concerning the property of being spherical (or not spherical).

One part is nevertheless the treatment of small cases like $\mathfrak{A}_2$ and $\mathfrak{B}_2$ since the rest of the right coideal subalgebra is a covered by the theory of quantum enveloping algebras, c.f. ›The General Case‹ in Chapter 3. Since the theories for $\mathfrak{A}_2$ and $\mathfrak{B}_2$ are very similar it is natural to ask for a unifying theory of these two cases. The treated algebra $\mathfrak{A}_{3,2}$ plays a minor role since its representations behave more less like that $\mathfrak{A}_2 \otimes \mathfrak{A}_2$—but sharing one element (and producing some $q$-factors).

## Generalisation of $\mathfrak{A}_2$ and $\mathfrak{B}_2$

As pointed out in the above treatment of case $B_2$ the differences to $A_2$ are quite small. The main difference appears with the relations between $G_{12}$ and $F_1$, where $B_2$ produces some squares. The $q$-factor appearing in $A_2$ seems to the author minor important since it does not change the representation theory. But, as done in this work, the square can neglected by replacing it. A similar calculus in type $G_2$ with $\alpha_1$ long and $\alpha_2$ short gives indeed a third power of $G_2$ but no power of an element of $U^0$, explicitly one computes

$$G_{12}F_1 = q^{-3}F_1G_{12} + G_2^3 - \zeta^3 K_1^{-2}K_2^{-3}$$

as relation.

Interchanging long and short root yields power-free relations among $G_{12}$ and $F_1$ independently of the length of the long root. This is reasonable, since in this case the element element leading to $G_{12}$ looks like $E_1E_2 - q^{-k}E_2E_1$ for suitable $k \in \mathbb{N}$ and is up to the $q$-factor the same as in case $A_2$.

A generalisation could be as follows: Let $k, l, m, n \in \mathbb{Z}$ be integers with $k, l \neq 0$. Define $\mathscr{A}$ as the unital algebra generated over $k$ by the elements $F, G, H_1, H_2^{\pm}$ subject to the relations

$$\begin{aligned}
H_1H_2 &= H_2H_1,\\
H_1F &= q^kFH_1,\\
H_1G &= q^{-k}GH_1,\\
H_2F &= q^lFH_2,\\
H_2G &= q^{-l}GH_2,\\
GF &= q^{-m}FG + H_1^n - H_2.
\end{aligned}$$

The parameter $\zeta$ is left out since it is not important for the relations (it does only appear as a stretching factor of $H_2$).

Conjecture 5.8 *The theory of dimensional $\mathscr{A}$-modules on which $H_1$ and $H_2$ are diagonalisable and $H_1$ has no eigenvalue equals to 0 is analogue to the theory of $\mathfrak{A}_2$ resp. $\mathfrak{B}_2$.*

Another question concerning the representation theory and concerns Definition 3.2 is whether $H_1$ is invertible or diagonalisable on a finite dimensional $\mathscr{A}$-module.

CONJECTURE 5.9 *Each finite dimensional $\mathscr{A}$-module is simultaneously diagonalisable by $H_1$ and $H_2$ and $H_1$ is an epimorphism.*

## Other Conjectures

The discussion around Proposition 3.22 suggests

CONJECTURE 5.10 *Under the assumptions of Proposition 3.22 holds: The $U_q(\mathfrak{sl}_{n+1})$-module $L(\lambda)$ has an $\mathfrak{A}_{n,r}$-invariant vector* if and only if

$$\lambda \in \bigoplus_{k=1}^{s} (\varpi_k + \varpi_{n+1-k})\mathbb{N}_0.$$

In Chapter 4 was shown for the case $n = 2$ that the homogeneous space of Dijkhuizen-Noumi is as algebra isomorphic to the homogeneous space of this work. In assuming the correctness of the above conjecture it is possible the extend the calculations of the case $n = 2$ to the general case.

Taking a look at the invariant $\mathfrak{A}_2$-vector of $L(\varpi_1 + \varpi_2)$ in equation (4.1) we see immediately that it has a non-zero component of maximal weight in direction $\alpha_2$. We shall understand ›maximal weight in direction of $\alpha_r$‹ as the weight with the property: There is a $k \in \mathbb{N}$ such that $k\alpha_r$ is a weight of the simple module but $(k+1)\alpha_r$ is not.

CONJECTURE 5.11 *Let $L$ be a simple $U_q(\mathfrak{sl}_{n+1})$-module and let $v \in L$ be an $\mathfrak{A}_{n,r}$-invariant vector. Then $v$ has non-zero component of maximal weight in direction $\alpha_r$.*

In the proof of Proposition 4.9 we gave the invariant vector for the case $n = 2$ and $\zeta = 0$. This was the main motivation for the above conjecture since it is clear: Let $v = \sum_{k=0}^{l} v_k$ with $v_k \in L_{k\alpha_r}$ and $v_l \neq 0$ be $\mathfrak{A}_{n,r}^{(\zeta)}$-invariant for a $\zeta \neq 0$, then $v_l$ is clearly $\mathfrak{A}_{n,r}^{(0)}$-invariant. In terms of Proposition 4.9 and Proposition 4.10 it is reasonable to formulate

CONJECTURE 5.12 *Let $L$ be a simple $U_q(\mathfrak{sl}_{n+1})$-module. A vector $w \in L$ is $\mathfrak{A}_{n,r}^{(0)}$-invariant if and only if there is an $\mathfrak{A}_{n,r}$-invariant $v = \sum_{k=0}^{l} v_k$ with $v_k \in L_{k\alpha_r}$ and $v_l = w$.*

These two conjectures should extend to other cases constructed via the method applied in this work.

## Programme

As pointed out in the introduction the method for proving the spherical property relies on Krämers work. Unlike the works of Noumi-Sugitani, Dijkhuizen-Noumi or Letzter it does not really use the ›freedom‹ of a right coideal subalgebra that was explained by Noumi-Sugitani.

Additionally there is another property of the right coideal subalgebras of Noumi-Dijkhuizen and Letzter ([Let03]) that is that their invariant vector has a non-zero highest weight component, i.e. fixing a one-dimensional weight space plus a ›tail‹. This is completely contrary to the here presented concept.

Nevertheless in point of view of Conjectures 5.11 and 5.12 it seems to be possible to fix an arbitrary one-dimensional weight space or least one whose weight is conjugated to the highest weight under the weyl group.

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# Appendix

## German Abstract / Deutsche Zusammenfassung

Ende der 1980er Jahre konstruierten unabhängig voneinander Jimbo ([Jim86]) und Drinfel'd ([Dri86]) Quantengruppen. Diese Quantengruppen spielen inzwischen eine wichtige Rolle in der Lietheorie als auch in der Theorie der Hopfalgebren. Bereits kurz nach ihrer Einführung erschienen Verallgemeinerungen, so gab Woronowicz in 1987 eine Definition für kompakte Quantengruppen bzw. kompakte Matrixpseudogruppen ([Wor87]). Mit Hilfe einer $*$-Struktur konnte er auf seinen kompakten Quantengruppen einen Haarzustand konstruieren, welcher dem klassischen Haarmaß sehr ähnlich ist, vor allem aber zentrale Eigenschaften des Haarmaßes besitzt. Podleś nutzte diesen Zustand um einen homogenen Raum für $S_qU(2)$ zu konstruieren ([Pod87]).

Im Jahr 1995 präsentierten Noumi und Sugitani eine neue Methode um quantensymmetrische Räume zu definieren ([NS95]): Anstatt einen homogenen Raum als Invariantenraum einer Hopfunteralgebra zu betrachten, schlugen sie einen Wechsel zu Invariantenräumen von Koidealunteralgebren vor. Um solche Koideale zu finden, nutzten sie Lösungen einer Spiegelungsgleichung; sie konstruierten Koideale für alle Riemannschen symmetrischen Paare außer vom Typ AIII. Später folgte eine Publikation von Dijkhuizen und Noumi ([DN98]), in welcher sie auch Koideale für den Typ AIII konstruierten, ferner erschienen von Letzter ebenfalls ein Beitrag zu Rechtskoidealunteralgebren ([Let97]). Letzter publizierte einen Ansatz, um alle Typen Riemannscher symmetrischer Räume ohne Fallunterscheidung abzudecken ([Let99]). Weitere Arbeiten in diese Richtung erschienen mit [Let02], [Let03], [KL08] und [Kol08].

Kharchenko startete ein Programm zur Klassifikation vom homogenen Rechtskoidealunteralgebren. Solche Rechtskoidealunteralgebren haben die Eigenschaft, dass sie das Erzeugnis aller gruppenartigen Elemente von $U_q(\mathfrak{g})$ enthalten, also $U^0$. Mit [Kha11] absolvierte er die Klassifikation dieser Rechtskoidealunteralgebren für $U_q^+(\mathfrak{so}_{2n+})$ und zusammen mit Sagahon für $U_q(\mathfrak{sl}_{n+1})$ ([Kha08]). Das Klassifikationsprojekt wurde dann von Heckenberger und Kolb für alle Typen abgeschlossen ([HK12]). Wie auch Heckenberger und Schneider ([HS13, Theorem 7.13]) stellten sie die homogenen Rechtkoidealunteralgebren in Beziehung zu der Weylgruppe der sie enthaltenden universellen Einhüllenden. Durch die Verknüpfung können homogene Rechtskoidealunteralgebren vermittels PBW-Elementen angegeben werden. In einer weiteren Arbeit [HK11] behandelten Heckenbeger und Kolb diejenigen Rechtskoidealunteralgebren des Borelteils, deren Schnitt mit $U^0$ eine Hopfalgebra liefert. Wieder konnten sie die Rechtskoidealunteralgebren mit der Weylgruppe verknüpfen. Zusätzlich zur Beschreibung mit PBW-Elementen tauchten Charaktere, die zu einer Deformation von $U^+[w]$ führen. Sie wurden ebenfalls klassifiziert.

Aufbauend auf den beiden Arbeiten solle in dieser Arbeit eine weitere Familie von Rechtskoidealunteralgebren vom Typ $A_n$ vorgestellt und behandelt werden. Während die Arbeiten von Dijkhuizen, Letzter und Noumi ihren Schwerpunkt auf symmetrische

Räume und $*$-Struktueren legen, sollen diese hier nicht beachtet werden - denn homogene Räume sind auch mit keiner der beiden Eigenschaften durchaus interessant. Der hier verfolgte Ansatz ist rein algebraisch, und motiviert durch eine Arbeit Müller und Schneider ([MS99]) sollen hier reduktive Rechtskoidealunteralgebren betrachtet werden, denn diese verfügen über günstige algebraische Eigenschaften. Wegen dieser Anforderung sind die Rechtskoidealunteralgebren in der vorliegenden Arbeit groß in dem Sinne, dass sie eine große Hopfalgebra unterhalten und als zu $U_q(\mathfrak{sl}_r) \otimes U_q(\mathfrak{sl}_{n-r+1})$ verwandt gelten dürfen. Aus letzterem resultiert: Weil die Erzeuger einer quantisierten universellen Einhüllenden in Verbindung mit dem Wurzelsystem stehen, gibt es eine ›ungenutzte‹ Wurzel: $\alpha_r$.

Wir bezeichnen mit $\mathfrak{A}_{n,r} \subset U_q(\mathfrak{sl}_{n+1})$ die Rechtskoidealunteralgebra, und sei $\Lambda$ das Gewichtsgitter von $U_q(\mathfrak{sl}_{n+1})$. Ein Modul $U_q(\mathfrak{sl}_{n+1})$-Modul ist sphärisch, wenn die Fixpunktmenge von $\mathfrak{A}_{n,r}$ höchstens eindimensional ist. Theorem 3.21 und Proposition 3.22 ergeben zusammen:

*Sei $\lambda \in$ Lambda ein dominantes Gewicht und sei $L(\lambda)$ der einfache $U_q(\mathfrak{sl}_{n+1})$-Modul zum Gewicht $\lambda$. Dann ist $L(\lambda)$ ein halbeinfacher $\mathfrak{A}_{n,r}$-Modul. Ferner gilt: Ist $L(\lambda)$ ein sphärischer $U_q(\mathfrak{sl}_r) \otimes U_q(\mathfrak{sl}_{n-r+})$-Modul, so ist er auch ein sphärischer $\mathfrak{A}_{n,r}$-Modul.*

Krämer zeigte in seiner Arbeit [Krä79], dass $U_q(\mathfrak{sl}_r) \otimes U_q(\mathfrak{sl}_{n-r+1})$ genau dann sphärisch ist, wenn $r \neq n-r+1$. (Um genau zu sein: Er zeigte die entsprechende Aussagen für die spezielle unitäre Gruppe, deren Darstellungstheorie aber identisch mit der von $U_q(\mathfrak{sl}_{n+1})$ ist.)

Die Algebra $\mathfrak{A}_{2,2}$ hat zudem eine gute Darstellungstheorie, wie Propositionen 3.4 und 3.5 zeigen. Sie ist vergleichbar mit der Darstellungstheorie von $U_q(\mathfrak{sl}_2)$. Ihre Erzeuger sind $G_2 = K_2^{-1}(E_2 + \zeta \cdot 1)$, $G_{12} = K_1^{-1}K_2^{-1}(E_2E_1 - q^{-1}E_1E_2 + \zeta(q-q^{-1})E_1)$, $(K_1^2K_2)^{\pm 1}$ and $F_1$, wobei $\zeta$ ein Skalar ungleich 0 aus dem Grundkörper $k$ ist.

*Für jedes Paar $s \in \mathbb{N}_0$ und $\kappa \in k, \kappa \neq 0$ gibt es genau einen einfachen $\mathfrak{A}_{2,2}$-Modul der Dimension $s+1$, so dass $G_2$ und $K_1^2K_2$ simultan diagonalisierbar sind. Jeder endlich dimensionale, einfache $\mathfrak{A}_{2,2}$-Modul mit der letztgenannten Eigenschaft von $G_2$ und $K_1^2K_2$ ist isomorph zu einem der oben aufgeführten einfachen Moduln. Ferner ist jeder endlich dimensionale $\mathfrak{A}_{2,2}$-Modul mit besagter Eigenschaft halbeinfach.*

Im ersten Kapitel werden die Notation und wichtige Eigenschaften über quantisierte universelle Einhüllende sowie deren duale Hopfalgebra, den quantisierten Koordinatenring, vorgestellt.

Im zweiten Kapitel werden zu Beginn die für die Konstruktion wichtigen Sätze aus [HK11] zitiert. Anschließend wird die Deformation durchgeführt, hierzu wird ein geeignetes Wort – das längste Wort Unterdiagramms $A_{r-1} \times A_{n-r}$ von $A_n$ – gewählt. Die Deformation über den Charakter liefert dann das Element $G_{\beta_r^r} = K_{\alpha_r}^{-1}(E_{\alpha_r} + \zeta \cdot 1)$, wobei $\zeta$ ein invertierbarer Skalar ist, welches die Rolle des Elements $K_{\alpha_r}$ übernimmt, aber nicht in $U^0$ liegt. Dieses ›schiefe‹ Element erfordert eine Deformation bzgl. der Wurzeln $\alpha_{r-1}$ und $\alpha_{r+1}$, welche zu Elementen $G_{\beta_{r-1}^r}$ bzw $G_{\beta_r^{r+1}}$ führt. Nach hinzufügen von Elemente aus dem negativen Borelteil ergibt sich die Relation

$$G_{\beta_{r-1}^r} F_{\alpha_{r-1}} = qF_{\alpha_{r-1}} G_{\beta_{r-1}^r} + G_{\beta_r^r} - \zeta K_{\alpha_{r-1}}^{-2} K_{\alpha_r}^{-1},$$

sowie ihr Korrespondat für $r+1$ statt $r-1$. Das Element $G_{\beta_{r-1}^r}$ sollte als eine Deformation des Elements $E_{\alpha_r}$ betrachtet werden, obgleich von dem Element $E_{\alpha_r}E_{\alpha_{r-1}} - q^{-1}E_{\alpha_{r-1}}E_{\alpha_r}$

herrührt. Wie bereits oben angedeutet wird die resultierende Rechtskoidealunteralgebra mit $\mathfrak{A}_{n,r}$ bezeichnet. Nach Ausführung der Deformation werden noch einige algebraische Eigenschaften abgeleitet, wie etwa die Existenz einer PBW-Basis und eine Graduierung.

Im dritten Kapitel wird die Darstellungstheorie untersucht - mit der Annahme, dass $G_{\beta_r^r}$ diagonalisierbar ist. Dies stellt insofern keine Einschränkung dar, da $G_{\beta_r^r}$ auf jedem endlich dimensionalen $U_q(\mathfrak{sl}_{n+1})$-Modul diagonalisierbar ist (Proposition 3.1). Die Halbeinfachheit wird zunächst für die Algebren $\mathfrak{A}_{2,2}$ und $\mathfrak{A}_{3,2}$ gezeigt. Die angewandte Methodik ist vergleichbar zu der für $U_q(\mathfrak{sl}_2)$ bzw. $U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$. Im allgemeinen Fall resultiert dann die Halbeinfachheit aus der von $U_q(\mathfrak{sl}_r) \otimes U_q(\mathfrak{sl}_{n-r+1})$. Dies gilt ebenso wie für den Nachweis, dass $\mathfrak{A}_{n,r}$ sphärisch ist.

Das vierte Kapitel behandelt den homogenen Raum von $\mathfrak{A}_{2,2}$. In diesem Fall kann der invariante Vektor leicht bestimmt werden. Es wird gezeigt, dass der homogene Raum isomorph zu demjenigen von Dijkhuizen und Noumi ist. Allerdings verfügt hiesieger über keine $*$-Struktur. Der Nachweis erfolgt vermittels Quantensphären. Wie nämlich auch Dijkhuizen und Noumi einen Isomorphismus zu einem quantenprojektiven Raum zeigen konnten, ist dies mit der gleichen Methode auch für den hier präsentierten homogenen Raum möglich. Im letzten Abschnitt des Kapitels wird für $n = 2$ der Fall $\zeta = 0$ betrachtet: Dann verliert $\mathfrak{A}_{2,2}$ sofort ihre Halbeinfachheit und $G_{\beta_2^2}$ wird nilpotent. Mit Hilfe von Gewichtsdiagrammen kann dennoch gezeigt werden, dass der Raum der Invarianten höchsten eindimensional ist.

Das letzte Kapitel beginnt mit einer Untersuchung für den Typ $B_2$. Die Ergebnisse sind nicht wesentlich verschieden wie für Typ $A_2$ und entsprechend werden einige Verallgemeinerung der zugehörigen Algebren vorgestellt. Abschließend sind noch einige Vermutungen hinsichtlich aufgetretener Fragen formuliert.