Dissertation

Adaptive wavelet methods for a class of stochastic partial differential equations

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2015

to Katarina

Adaptive Wavelet Methoden für eine Klasse von stochastischen partiellen Differentialgleichungen

Dissertation

zur
Erlangung des akademischen Grades
Doktor der Naturwissenschaften

dem
Fachbereich Mathematik und Informatik
der Philipps-Universität Marburg
(Hochschulkennziffer: 1180)

in
englischer Sprache vorgelegt
von

Dipl.-Math. Stefan Kinzel
geb. in
Lutherstadt Wittenberg

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Einreichungstermin: 27. Mai 2015
Prüfungstermin: 02. November 2015

Erscheinungsort: Marburg
Erscheinungsjahr: 2016

Zusammenfassung

Eine abstrakte Interpretation der Rothe Methode zur Diskretisierung von Evolutionsgleichungen wird hergeleitet. Die Fehlerfortpflanzung wird untersucht und Bedingungen an die Toleranzen werden bewiesen, welche die Konvergenz im Falle von approximativen Operatorauswertungen sicher stellen. Zur Untermauerung der abstrakten Analysis wird das linear implizite Eulerschema mit uniformer Zeitdiskretisierung auf eine Klasse von semi-linearen parabolischen stochastischen partiellen Differentialgleichungen angewendet. Unter Verwendung der Existenz von optimalen adaptiven Methoden für die elliptischen Teilprobleme werden hinreichende Bedingungen gezeigt, welche die Konvergenz mit zugehörigen Konvergenzordnungen auch im Fall von approximativen Operatorauswertungen sichern. Obere Komplexitätsschranken werden im deterministischen Fall bewiesen.

Die stochastische Poissongleichung mit zufälligen rechten Seiten dient als Modellgleichung für die elliptischen Teilprobleme. Die zufälligen rechten Seiten werden, basierend auf Waveletentwicklungen, eingeführt anhand eines stochastischen Modells, welches, wie gezeigt wird, eine explizite Regularitätskontrolle deren Realisierungen bietet und dünn besetzte Entwicklungen induzieren kann. Für diese Klasse von Gleichungen werden obere Fehlerschranken der besten N-term Waveletapproximation auf verschiedenen beschränkten Gebieten bewiesen. Sie zeigen, dass die Verwendung von nichtlinearen (adaptiven) Methoden gegenüber uniformen linearen Methoden gerechtfertigt ist, insbesondere bei dünn besetzten Entwicklungen auf zwei oder drei dimensionalen Lipschitzgebieten.

Die Klasse von zufälligen Funktionen, welche aus dem stochastischen Modell abgeleitet werden kann, ist an sich interessant, da sie dünn besetzte Varianten von allgemeinen Gauß'schen zufälligen Funktionen liefert. In verschiedenen Glattheitsräumen wird die Regularität der zufälligen Funktionen analysiert, ebenso werden lineare und nichtlineare Approximationsergebnisse bewiesen, welche deren Anwendbarkeit in numerischen Experimenten verdeutlicht.

Preface

I would like to express my gratitude to everyone who has supported me in regards to this dissertation. First and foremost, I am deeply grateful for my research advisor Prof. Stephan Dahlke. Stephan, you have been a tremendous supporter of my work in all aspects and I am so grateful that you have taken me on as a doctoral candidate. In particular your demand for a broad and precise mathematical background together with your professional insights have been the foundation of my mathematical research. Additionally, you have assembled a great team of colleagues and an excellent work environment, which have helped to develop my work.

Furthermore, I deeply appreciate the support from the Deutsche Forschungsgemeinschaft and the priority program DFG-SPP 1324[1]. I would like to especially thank and recognize the members of our research group 'Adaptive Wavelet Methods for SPDEs', Prof. Klaus Ritter, Prof. René L. Schilling, Jun.-Prof. Felix Lindner, and Nicolas Döhring. Additionally, I would like to thank Prof. Thorsten Raasch, Petru A. Cioica, and Ulrich Friedrich, who also worked side-by-side with me as co-authors on our publications. It has been a tremendous honor to work with all of you.

I would like to distinctly recognize the Mathematics and Computer Science Department at the Philipps-Universität Marburg for its great research facilities and extremely supportive faculty and staff. Moreover, I would like to express the deepest appreciation to my fellow colleagues in the Numerics and Optimization Workgroup. It has been a great pleasure working alongside all of you for the last several years.

A special thanks goes to my family and friends. Your unyielding support for me sustained me thus far. My utmost gratitude and appreciation is with my beloved wife. Words cannot express how grateful I am for all of the sacrifices that you have made on my behalf. To you, Katarina, I dedicate this dissertation.

Stefan Kinzel

[1]The author has been financially supported by the Deutsche Forschungsgemeinschaft (Grant DA 360/13-1, 13-2), [44].

Contents

Chapter 1

Introduction

The numerical treatment of stochastic partial differential equations (SPDEs) is a recent and active area of research. It combines the fields of numerics of partial differential equations with stochastic analysis. Evolution equations of the parabolic type, for instance, describe diffusion processes that are very often used for the mathematical modeling of economical, biological, chemical, and physical processes. The inclusion of a stochastic driving process allows to incorporate random distortions or noise into the model. In a growing number of applications, e.g., in computational finance, epidemiology, population genetics, and many more, this has become important in order to, e.g., account for uncertainties and therefore improve the accuracy in the predictions of the model.

In general, partial differential equations are not solved in a direct fashion due to complexity reasons. Instead, a discretization scheme is applied to the equation and iterative numerical schemes are employed to obtain an approximation to the solution up to a prescribed tolerance. Also approximations to explicitly given objects of the equations are required, due to storage constraints. Considering evolution equations, aside of simultaneous space-time numerical approximation schemes, there are two principally different discretization approaches: the *vertical method of lines* and the *horizontal method of lines*. The former starts with an approximation first in space, and then proceeds in time, while the latter starts with a discretization first in time, and then in space; it is also known as *Rothe's method*.

Very often, the vertical method of lines is preferred, since after the discretization in space is performed just finite dimensional ordinary stochastic differential equations (SDE) in time direction have to be solved, for which there are many approaches available. However, there are also certain drawbacks; in many applications the utilization of adaptive strategies allows to increase efficiency, but in the context of the vertical method of lines the combination with spatial adaptivity is at least not straightforward. In contrast, the use of adaptive methods somewhat suggests itself when investigating the horizontal method of lines. Namely, using Rothe's method, the parabolic equation can be interpreted as an abstract Cauchy problem, i.e., as a SDE in some suitable function space. Then, in time direction, one can use one of the SDE-solver with step size control. Note that any such solver must be based on an implicit discretization scheme due to stability reasons, since the equation under consideration is usually stiff. On this account, in each time step, a system of elliptic equations with random functions as right-hand sides has to be solved. To this end, adaptive numerical schemes that are well-established for elliptic deterministic equations, can be used, e.g., adaptive finite element or wavelet methods.

The motivation of this dissertation is based exactly on this line of thought and its results are organized as follows. Chapter 2 provides the setting and theoretical foundation for the subject of this dissertation. In Chapter 3 we introduce and investigate a new class of random functions for the numerical modeling of stochastic equations. In Chapter 4 we consider the stationary case and employ this class of random functions as right-hand sides to the Poisson equation, which serves as model problem for the elliptic subproblems. Finally, in Chapter 5 we investigate the error propagation and analyze the convergence of spatially adaptive Rothe methods for deterministic and stochastic evolution equations of the parabolic type.

In this introduction, following the summary, we give an overview of related research results that are within the scope of the subject matter in Section 1.2. In the subsequent Sections 1.3, 1.4, and 1.5, we state the introductions to the individual chapters including the main results.

1.1 Summary

An abstract interpretation of Rothe's method for the discretization of evolution equations is derived. The error propagation is analyzed and condition on the tolerances are proven, which ensure convergence in the case of inexact operator evaluations. Substantiating the abstract analysis, the linearly implicit Euler scheme on a uniform time discretization is applied to a class of semi-linear parabolic stochastic partial differential equations. Using the existence of asymptotically optimal adaptive solver for the elliptic subproblems, sufficient conditions for convergence with corresponding convergence orders also in the case of inexact operator evaluations are shown. Upper complexity bounds are proven in the deterministic case.

The stochastic Poisson equation with random right hand sides is used as model equation for the elliptic subproblems. The random right hand sides are introduced based on wavelet decompositions and a stochastic model that, as is shown, provides an explicit regularity control of their realizations and induces sparsity of the wavelet coefficients. For this class of equations, upper error bounds for best N-term wavelet approximation on different bounded domains are proven. They show that the use of nonlinear (adaptive) methods over uniform linear methods is justified whenever sparsity is present, which in particularly holds true on Lipschitz domains of two or three dimensions.

By providing sparse variants of general Gaussian random functions, the class of random functions derived from the stochastic model is interesting on its own. The regularity of the random functions is analyzed in certain smoothness spaces, as well as linear and nonlinear approximation results are proven, which clarify their applicability for numerical experiments.

1.2 Overview of related research results

We give an overview of research results that are related to the scope of the subject matter. Additional research results, which are more specifically related to the results of this dissertation, are given in the subsequent introductions.

When it comes to numerical approximations of the objects of interest, e.g., the solutions of partial differential equations, then the approximation order that can be achieved usually depends on the membership of these objects in specific scales of smoothness spaces. For approximation schemes based on wavelets, it is well-known that the approximation order of linear uniform wavelet algorithms depends on the Sobolev smoothness of the underlying object, whereas the approximation order of nonlinear algorithms such as best N-term wavelet approximation in L_2 depends on the regularity in the specific scale

$$B^s_\tau(L_\tau), \quad \text{where} \quad \frac{1}{\tau} = \frac{s}{d} + \frac{1}{2}, \quad s > 0, \tag{1.1}$$

of Besov spaces. We refer to Dahlke et al. [45], DeVore [65], DeVore et al. [66], and the references therein for further information. These relationships are a consequence of the fact that wavelets are able to characterize smoothness spaces such as Besov and Sobolev spaces, respectively, i.e., the corresponding smoothness norms are equivalent to weighted sequence norms of wavelet decomposition coefficients, see, e.g., DeVore et al. [66], Frazier, Jawerth [80], Meyer [129], Runst, Sickel [143], Triebel [162] for details. Furthermore, this connection often motivates to analyze the Besov regularity of solutions of various problems, see, e.g., Cioica et al. [26], Dahlke [41, 42, 43], Dahlke, DeVore [47], Dahlke, Sickel [55, 56], Dahlke, Weimar [57], Eckhardt [73], Hansen [99], Hansen, Sickel [100].

Adaptive wavelet methods for deterministic elliptic and parabolic partial differential equations have been studied intensively in recent years, see, e.g., Cohen et al. [29, 30, 31], Dahlke et al. [46, 49, 50, 51], Gantumur et al. [81], Kappei [106], Lellek [122], Raasch [138], Schwab, Stevenson [146, 147], Stevenson [150, 152], Stevenson, Werner [153, 154], Werner [174]. Usually, best N-term wavelet approximation is used as a benchmark for adaptive wavelet schemes, since it is an almost optimal approximation scheme, see Dahlke et al. [53, 54].

Also motivated by above observations, the relations of stochastic analysis and the theory of function spaces has become a field of increasing interest. For instance, approximations and the regularity of the solutions to SDEs and SPDEs in several function spaces has been studied in, e.g., Cioica [20], Cioica et al. [27], Jentzen, Kloeden [102], Jentzen, Röckner [103], Kim [109, 110], Kruse, Larsson [116], Krylov [117], Lindner [124], van Neerven et al. [164, 165]. We also we refer to Kovács et al. [112, 113, 114], Walsh [171] for convergence results based on finite element discretization applied to SPDEs. For stochastic ordinary differential equations nonlinear approximation of the solution process is studied in, e.g., Creutzig et al. [38], Slassi [149]. Note that these references are indicative only.

1.3 A class of random functions

In Chapter 3 we analyze the regularity of a class of random functions in certain smoothness spaces and state linear and nonlinear approximation results. The random functions are defined in terms of wavelet decompositions according to a stochastic model that provides an explicit regularity control of their realizations and, in particular, induces sparsity of the wavelet coefficients. We expect this stochastic model to be an interesting tool to generate test functions in numerical experiments.

Some effort has been spent to create random functions whose realizations possess, almost everywhere, a prescribed regularity in Besov or Sobolev spaces, see, e.g., Abramovich et al. [1], Bochkina [12, 13], Cohen, d'Ales [32], Cohen et al. [34], Creutzig et al. [38], Kon, Plaskota [111]. Often, one major tool has been the wavelet characterization of smoothness spaces. Based on a fixed wavelet basis, random coefficients have been designed which, by means of the norm equivalences, guarantee the desired regularity. In, e.g., Abramovich et al. [1], Bochkina [12], in the context of Bayesian non-parametric regression, the random wavelet coefficients $w_{j,k}$ have been modeled as an independent mixture of Bernoulli distributions Y_j and standard normal distributions Z_j:

$$w_{j,k} \sim (1-\pi_j)Y_j + \pi_j \tau_j Z_j,$$

where $\pi_j \in [0,1]$ and $\tau_j > 0$. In particular, $\tau_j^2 = 2^{-\alpha j} C_1$ and $\pi_j = \min\{1, 2^{-\beta j} C_2\}$ have been studied and it has been investigated how the parameters $\alpha, \beta \geq 0$ have to be tuned to yield a certain prescribed Besov smoothness. However, only the one-dimensional setting $d = 1$, smoothness parameters $s > 0$, and integrability parameters $p, q > 1$ have been analyzed. The paper Bochkina [13] considers more general parametrizations and also focuses on Bayesian non-parametric wavelet regression. While nonlinear approximation methods are extensively studied in the deterministic case, see DeVore [65] and the references therein for details and a survey, far less is known for random functions. For the latter, we refer to Cohen, d'Ales [32], Cohen et al. [34], where wavelet methods are analyzed, and to Creutzig et al. [38], Kon, Plaskota [111], where free knot splines are used. Again, in these papers only the one-dimensional case is studied.

In Chapter 3, in particular in Section 3.1, we define and analyze random functions by generalizing the stochastic model as introduced in Abramovich et al. [1], Bochkina [12] and study different approximations. First, let us summarize the setting. The random function (stochastic field) X is defined in terms of a stochastic wavelet decomposition

$$X = \sum_{j \geq 0} \sum_{k \in \nabla_j} Y_{j,k} Z'_{j,k} \psi_{j,k}. \tag{1.2}$$

Here $\{\psi_{j,k} : j \geq 0,\, k \in \nabla_j\}$ is a wavelet Riesz basis for $L_2(\mathcal{O})$, where $\mathcal{O} \subset \mathbb{R}^d$ is a bounded domain and j denotes the scale parameter. Furthermore, ∇_j is a finite set with, in order of magnitude, 2^{jd} elements, and $Y_{j,k}$ and $Z'_{j,k}$ are independent random variables. In a slightly simplified version of the stochastic model, $Y_{j,k}$ is Bernoulli distributed with parameter $2^{-\beta jd}$ and $Z'_{j,k}$ is normally distributed with mean zero and variance $2^{-\alpha jd}$, where $\beta \in [0,1]$ and $\alpha + \beta > 1$. Note that the sparsity of the decomposition (1.2) depends monotonically on β. For $\beta = 0$, i.e., with no sparsity present, (1.2) is the Karhunen-Loève decomposition of a Gaussian random function X if the wavelets form an orthonormal basis of $L_2(\mathcal{O})$. Additionally, we discuss stochastic fields X with realizations in Besov spaces with negative smoothness, i.e., we allow $\alpha \in \mathbb{R}$. Such stochastic fields are in particular natural to consider for the modeling of stochastic Poisson equations with random right-hand sides, where the Laplacian is a bounded operator from H_0^1 onto H^{-1}, the normed dual of H_0^1, cf. Chapter 4.

Let us now point to the main results of Section 3.1. The random function X takes values in the (quasi-) Besov space $B^s_q(L_p(\mathcal{O}))$, $0 < p, q < \infty$, and $p, q > 1$ for $s < 0$, with probability one if and only if

$$s < d \cdot \left(\frac{\alpha - 1}{2} + \frac{\beta}{p} \right),$$

see Theorem 3.10. In Abramovich et al. [1], Bochkina [12] the result was stated for $d = 1$, $s > 0$, and $p, q \geq 1$. In particular, the smoothness of X along the scale of Sobolev spaces $H^s(\mathcal{O}) = B^s_2(L_2(\mathcal{O}))$ is determined by $\alpha + \beta$, and for $\beta > 0$ with decreasing $p \in (0, 2]$ the smoothness can get arbitrarily large.

We study different approximations $\widehat{X}$ of X with respect to the norm in $L_2(\mathcal{O})$, where we always consider the average error

$$(\mathrm{E}[\|X - \widehat{X}\|^2_{L_2(\mathcal{O})}])^{1/2}$$

for any approximation $\widehat{X}$. Let $\beta \in [0, 1]$ and $\alpha + \beta > 1$. For the optimal linear uniform approximation, i.e., for the approximation from an optimally chosen N-dimensional subspace of $L_2(\mathcal{O})$, the corresponding error rate is asymptotically equivalent to $N^{-\varrho}$ with

$$\varrho = \frac{\alpha + \beta - 1}{2},$$

see Theorem 3.15. In contrast, for the best average N-term wavelet approximation we only require that the average number of non-zero wavelet coefficients is at most N. In this case the corresponding errors exhibit asymptotically at most the rate $N^{-\varrho}$ with

$$\varrho = \frac{\alpha + \beta - 1}{2(1 - \beta)}$$

and $\beta < 1$, see Theorem 3.17. The best average N-term wavelet approximation is superior to optimal linear uniform approximation if $\beta > 0$. The simulation of the respective average N-term wavelet approximation is possible at an average computational cost of order N, which is crucial in computational practice, see Remark 3.18.

Furthermore, we extend our findings and study different approximations $\widehat{X}$ of X with respect to the norms of the Besov spaces $B^\nu_p(L_p(\mathcal{O}))$, where $\nu \in \mathbb{R}$ and $1 < p < \infty$. Analogously to above, we consider the average error

$$(\mathrm{E}[\|X - \widehat{X}\|^p_{B^\nu_p(L_p(\mathcal{O}))}])^{1/p}$$

for any approximation $\widehat{X}$. Let $\beta \in [0, 1)$ and $\alpha \in \mathbb{R}$. For the optimal linear uniform approximation, the corresponding errors exhibit asymptotically at most the rate $N^{-\varrho}$ with

$$\varrho = \frac{\alpha - 1}{2} + \frac{\beta}{p} - \frac{\nu}{d},$$

see Theorem 3.20. In contrast, for best average N-term wavelet approximation with respect to $B^\nu_p(L_p(\mathcal{O}))$, the corresponding errors exhibit asymptotically at most the rate $N^{-\varrho}$ with

$$\varrho = \frac{1}{1 - \beta} \left(\frac{\alpha - 1}{2} + \frac{\beta}{p} - \frac{\nu}{d} \right)$$

and $\beta < 1$, see Theorem 3.23. Again, the best average N-term wavelet approximation is superior to optimal linear uniform approximation if $\beta > 0$.

Moreover, with respect to the norm in $H^\mu(\mathcal{O})$ we also obtain lower error bounds, see Theorem 3.22 for optimal linear uniform approximation, and Theorem 3.25 for best average N-term wavelet approximation.

In Section 3.2, we discuss additional important classes of smoothness spaces, specifically the anisotropic Sobolev and Besov spaces. Once again, to design random functions in these spaces is of independent interest, but nevertheless, we are convinced that there are a lot of possible applications. As an example, let us mention certain elliptic equations with random coefficients as they occur, e.g., in the modeling of groundwater flow problems Ernst et al. [78], Ernst, Ullmann [79], Ullmann et al. [163]. Usually, the random coefficients are modeled by highly isotropic lognormal distributions. However, due to certain anisotropic features that might show up in the physical environment, it could be more appropriate to use stochastic models that reflect these kinds of anisotropies. Therefore, we derive stochastic fields with prescribed smoothness in anisotropic smoothness spaces. The major tool is again the wavelet characterization of these spaces as derived, e.g., in Garrigós et al. [82], Garrigós, Tabacco [83].

Finally, in Section 3.3, we construct new bases of stochastic tensor wavelets. To our best knowledge, these kinds of stochastic fields have not been considered before. Tensor wavelets are in a certain sense the wavelet version of the sparse grid approach, see, e.g., Bungartz, Griebel [16] for a detailed discussion on sparse grids. They are very important for the following reason: Similar to sparse grids, (adaptive) approximation schemes based on these wavelets can give rise to dimension-independent convergence rates, see Schwab, Stevenson [146]. In this sense, tensor wavelets provide a way to break the famous curse of dimensionality. The spaces that can be characterized by tensor wavelets are generalized dominated mixed smoothness spaces, see Section 2.2 for details. Therefore, in Section 3.3, we derive stochastic fields with prescribed regularity in these spaces.

1.4 Application to the stochastic Poisson equation

In Chapter 4 we consider the stochastic Poisson equation on bounded domains, where the right-hand side is a random function which is given by the stochastic model that is analyzed in Section 3.1. In order to obtain approximations to the realizations of the solution, we employ asymptotically optimal adaptive wavelet algorithms as they asymptotically realize the approximation rate of best N-term wavelet approximation. Since the related convergence analysis of these adaptive wavelet algorithms relies on the energy norm, which is equivalent to the norm in H^1, we analyze best N-term wavelet approximation in H^1 for the considered class of stochastic Poisson equations. Moreover, the asymptotic results are matched by numerical experiments.

Solving stochastic evolution equations by application of Rothe's method, i.e., the evolution equation is first discretized in time, and then in space, due to stability reasons, one has to use an implicit time discretization scheme. This leads to elliptic boundary value problems with random right-hand sides that need to be solved in every time step. Therefore, a particular but nevertheless very important model problem is the Poisson equation

$$\begin{aligned} -\Delta U &= X \qquad \text{in} \quad \mathcal{O}, \\ U &= 0 \qquad \text{on} \quad \partial\mathcal{O}, \end{aligned} \tag{1.3}$$

with random right-hand side X and $\mathcal{O} \subset \mathbb{R}^d$ a bounded (Lipschitz) domain. Different numerical problems have been studied for Poisson equations, or more generally, for elliptic equations with a random right-hand side and/or a random diffusion coefficient. The computational task is to approximate either the realizations of the solution or at least their moments, and different techniques like stochastic finite element methods, sparse grids, or polynomial chaos decompositions are employed. A, by no means complete, list of papers includes Babuška et al. [7], Cohen et al. [35], Ernst et al. [78], Nobile et al. [131], Ritter, Wasilkowski [140], Todor, Schwab [156], Wan, Karniadakis [172], Xiu, Karniadakis [176]. Stochastic differential equations, in general, yield implicitly given random functions, which holds true in particular for U in (1.3). For stochastic ordinary differential equations nonlinear approximation of the solution process is studied in Creutzig et al. [38], Slassi [149].

Stochastic elliptic equations of the form (1.3) also arise in, e.g., Breckner, Grecksch [15], Cox, van Neerven [37], Debussche, Printems [64], Grecksch, Tudor [84], Gyöngy, Nualart [93], Printems [137] as sub-problems of stochastic evolution equations that are discretized by means of Rothe's method, cf. Section 1.5.

In Section 4.1, we analyze best N-term wavelet approximation for the Poisson equation (1.3) with right-hand sides X which are based on the stochastic model considered in Section 3.1. The solution U of the Poisson equation is approximated with respect to the norm in $H^1(\mathcal{O})$ and we consider the average error $(\mathrm{E}\|U - \widehat{U}\|^2_{H^1(\mathcal{O})})^{1/2}$ for any approximation $\widehat{U}$. Here, the space $H^1(\mathcal{O})$ is the natural choice, since its norm is equivalent to the energy norm and the convergence analysis of adaptive wavelet algorithms relies on this norm. We study the N-term wavelet approximation under different assumptions on the domain $\mathcal{O}$, and we establish upper bounds of the form $N^{-(\varrho-\varepsilon)}$, which hold for every $\varepsilon > 0$. For any bounded Lipschitz domain $\mathcal{O}$ in dimension $d = 2$ or 3 we obtain

$$\varrho = \min\left\{\frac{1}{2(d-1)}, \frac{\alpha+\beta-1}{6} + \frac{2}{3d}\right\},$$

see Theorem 4.1. Regardless of the smoothness of X we have $\varrho \leq 1/(2(d-1))$, e.g., due to possible singularities of U at the boundary of $\mathcal{O}$. On the other hand, uniform approximation schemes can only achieve the order $N^{-1/(2d)}$ on general Lipschitz domains $\mathcal{O}$, and we always have $\varrho > 1/(2d)$. For more specific domains we fully benefit from the smoothness of the right-hand side. First,

$$\varrho = \frac{\alpha+\beta}{2}$$

if $\mathcal{O}$ is a simply connected polygonal domain in $\mathbb{R}^2$, see Theorem 4.5, and

$$\varrho = \frac{1}{1-\beta}\left(\frac{\alpha-1}{2} + \beta\right) + \frac{1}{d}$$

for bounded C^∞-domains $\mathcal{O} \subset \mathbb{R}^d$, see Theorem 4.6.

These rates for the best N-term wavelet approximation of U are actually achieved by suitable adaptive wavelet algorithms, which have been developed for deterministic elliptic PDEs. Those algorithms converge for a large class of operators, including operators of negative order, and they are asymptotically optimal in the sense that they realize the optimal order of convergence, while staying efficient, i.e., the computational cost is proportional to the number N, see Cohen et al. [29, 30], Dahlke et al. [45], Gantumur et al. [81]. Moreover, the algorithmic approach can be extended to wavelet frames, i.e., to redundant wavelet systems, which are much easier to construct than wavelet bases on general domains, see Dahlke et al. [50], Stevenson [150].

Numerical experiments are presented in Section 4.2 to complement the asymptotic error analysis. We determine empirical rates of convergence for adaptive and uniform approximation of the solution U to the Poisson equation (1.3) in dimension $d = 1$. It turns out that the empirical rates fit very well to the asymptotic results, and we observe superiority of the adaptive scheme already for moderate accuracies.

1.5 On the convergence of the inexact linearly implicit Euler scheme

In Chapter 5 we investigate the error propagation and analyze the convergence of Rothe's method for evolution equations of the parabolic type with focus on linearly implicit one-step methods. We use uniform discretizations in time and non-uniform (adaptive) discretizations in space. The space discretization methods are assumed to converge up to a given tolerance ε when applied to the resulting elliptic subproblems. Typical examples are adaptive finite element or wavelet methods. We investigate how the tolerances ε in each time step have to be tuned so that the overall scheme converges with the same order as in the case of exact evaluations of the elliptic subproblems.

As mentioned above, usually the exact solution to a partial differential equation cannot be computed explicitly. In those cases a numerical scheme for the constructive approximation of the solution is required. For the vertical method of lines we refer to Hanke-Bourgeois [97], Johnson [105], Thomée [155], as well as to Gyöngy [90], Gyöngy, Krylov [91], Gyöngy, Millet [92], Hall [96] for detailed information. Our method of choice is Rothe's method, or the horizontal method of lines as it starts with a discretization first in time, and then in space. It has also been studied in, e.g., Breckner, Grecksch [15], Grecksch, Tudor [84] in the stochastic setting and in, e.g. Lang [121], Lubich, Ostermann [126] in the deterministic setting. With this approach the parabolic equation is interpreted as an abstract Cauchy problem, i.e., as an ordinary deterministic or stochastic differential equation in a suitable function space. Then, in time direction, one can apply an ODE/SDE-solver. Since the equation under consideration is usually stiff this solver must be based on an implicit discretization scheme. Linearly-implicit one-step methods are of primary interest because their realization only requires to solve a system of linear elliptic stage equations per time step. To solve the elliptic stage equations, well-established adaptive numerical schemes based, e.g., on wavelets or finite elements, can be used. We refer to Cohen et al. [29, 30], Dahlke et al. [51] for suitable wavelet methods, and to Babuška [5], Babuška, Rheinboldt [6], Bank, Weiser [8], Bornemann et al. [14], Eriksson [74], Eriksson, Johnson [75, 76], Eriksson et al. [77], Hansbo, Johnson [98], Verfürth [166, 167] for the finite element case. Note that all references are indicative and are by no means complete.

To the best of our knowledge, the most far reaching results concerning a rigorous convergence analysis of Rothe's method have been obtained in Lang [121], where finite element discretization in space is used. In the stochastic setting Rothe's method with unperturbed or *exact* evaluation of the elliptic subproblems, has been considered in Breckner, Grecksch [15], Grecksch, Tudor [84], and explicit convergence rates have been established in Cox, van Neerven [37], Gyöngy, Nualart [93], Printems [137]. First results concerning the combination with adaptive space discretization methods based on wavelets have been shown in Kovács et al. [115], where additive noise is considered, a splitting method is applied, and adaptivity is only used for the deterministic part of the equation. At this point, let us remark that the use of spatially adaptive schemes is useful especially for stochastic equations, where singularities appear naturally near the boundary due to the irregular behavior of the noise, cf. Cioica et al. [25] and the references therein.

In Section 5.1, we start with the observation that at an abstract level, Rothe's method can be reformulated as the consecutive application of two types of operators, the inverse of a (linear) elliptic differential operator $L_{\tau,i}^{-1}$ and certain (nonlinear) evaluation operators $R_{\tau,k,i}$, i.e.,

$$L_{\tau,i}^{-1} : \mathcal{G} \to \mathcal{H} \quad \text{and} \quad R_{\tau,k,i} : \underbrace{\mathcal{H} \times \cdots \times \mathcal{H}}_{i} \to \mathcal{G},$$

where $\mathcal{H}$ and $\mathcal{G}$ are suitable Hilbert spaces, τ denotes the time step size, k denotes the current time step, and $i = 1, \dots, S$ denotes the current stage. Then, an abstract S-stage scheme to compute an approximation $(\widetilde{u}_k)_k$ to a mapping $u : [0,T] \to \mathcal{H}$ can be defined as

$$\begin{aligned} \widetilde{u}_{k+1} &:= \sum_{i=1}^{S} \widetilde{w}_{k,i}, \quad \widetilde{u}_0 := u(0), \\ \widetilde{w}_{k,i} &:= \big[L_{\tau,i}^{-1} R_{\tau,k,i}(\widetilde{u}_k, \widetilde{w}_{k,1}, \dots, \widetilde{w}_{k,i-1})\big]_{\varepsilon_{k,i}}, \quad i = 1, \dots, S. \end{aligned}$$

Here, $[\,\cdot\,]_\varepsilon$ stands for any numerical scheme that, for any prescribed tolerance ε, yields an approximation of the evaluation of both operators, which is necessary since the mapping u is understood to be the solution of a partial differential equation and therefore the inverses $L_{\tau,i}^{-1}$ are not given explicitly in most cases. In the presence of spatial discretization errors, we investigate how the tolerances $\varepsilon_{k,i}$ in each time step must be tuned in order to preserve the asymptotic temporal convergence order δ of the time stepping. We derive sufficient conditions for convergence in the case of perturbed or *inexact* operator evaluations and obtain conditions on the tolerances which guarantee the overall convergence with corresponding convergence order δ, see Theorems 5.21 and 5.26, i.e., by choosing

$$0 < \varepsilon_{k,i} \le \frac{1}{S}\, C_a\, \tau^{1+\delta},$$

we get

$$\big\| u(T) - \widetilde{u}_K \big\|_{\mathcal{H}} \le C_b\, \tau^{\delta}$$

with specified constants C_a and C_b. Moreover, we derive abstract complexity estimates. Furthermore, in a Gel'fand triple setting (V, U, V^*), we consider $u : (0,T] \to V$ to be a solution to initial value problems of the form

$$u'(t) = F(t, u(t)), \quad u(0) = u_0, \quad t \in [0, T],$$

where $F : [0, T] \times V \to V^*$ is a nonlinear right-hand side and $u_0 \in V$ is some initial value. We substantiate our analysis and show that any linearly-implicit 1-step S-stage scheme of W-type can be written as abstract Rothe methods. See Observation 5.33 for the case $\mathcal{H} = V$ and $\mathcal{G} = V^*$ and Observation 5.41 for the case $\mathcal{H} = \mathcal{G} = U$. By combining our analysis with the convergence results for the unperturbed schemes, which, e.g., are outlined in Lubich, Ostermann [126], we are therefore able to provide rigorous convergence proofs for spatially adaptive versions of W-methods. In the examples, special emphasis is placed on the semi-linear case

$$u'(t) = Au(t) + f(t, u(t)), \quad u(0) = u_0, \quad t \in [0, T],$$

where in practical applications usually A is a differential operator and f a linear or nonlinear drift term.

In Section 5.2, we show that also semi-linear parabolic SPDEs can be treated, if the linearly-implicit Euler scheme is the method of choice. We consider a separable real Hilbert space U and the U-valued SDE

$$\mathrm{d}u(t) = Au(t)\mathrm{d}t + F(u(t))\mathrm{d}t + B(u(t))\mathrm{d}W(t), \quad u(0) = u_0, \quad t \in [0, T], \tag{1.4}$$

driven by a Q-Wiener process W over the sequence space ℓ_2 with respect to a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$ on a complete probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Here, $\mathrm{d}u(t)$ denotes the stochastic differential of Itō type with respect to time $t \in [0, T]$. Furthermore,

$$A : D(A) \subset U \to U$$

is a densely defined, strictly negative definite, self-adjoint, linear operator such that zero belongs to the resolvent set, and A^{-1} is compact on U. The drift term

$$F : D((-A)^{\varrho}) \to D((-A)^{\varrho-\varrho_F})$$

and the diffusion term

$$B : D((-A)^{\varrho}) \to \mathcal{L}(\ell_2, D((-A)^{\varrho-\varrho_B}))$$

are Lipschitz continuous maps for suitable constants ϱ, ϱ_F, and ϱ_B. To put the focus on the stochastic forcing term, we sometimes restrict F to the linear case where it is independent of u, i.e., $F(u) \equiv F$ or even set $F \equiv 0$. In practical applications usually $B(u(t))\mathrm{d}W(t)$ describes additive or multiplicative noise. This setting is based on the one considered in Printems [137] where the convergence of semi-discretizations in time is investigated. However, we allow the spatial regularity of the whole setting to be 'shifted' in terms of the additional parameter ϱ.

We start with a detailed description of the considered class of semi-linear parabolic SPDEs and show the existence of a unique mild solution, see Proposition 5.52. We call a mild solution to Eq. (1.4) a predictable process $u : \Omega \times [0, T] \to D((-A)^{\varrho})$ with

$$\sup_{t\in[0,T]} \mathrm{E}[\|u(t)\|^2_{D((-A)^{\varrho})}] < \infty$$

and such that for every $t \in [0,T]$ the equality

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}F(u(s))\mathrm{d}s + \int_0^t e^{(t-s)A}B(u(s))\mathrm{d}W(s) \tag{1.5}$$

P-almost surely holds in $D((-A)^{\varrho})$. The first integral in (1.5) is a $D((-A)^{\varrho})$-valued Bochner integral for P-almost every $\omega \in \Omega$ and the second integral is a $D((-A)^{\varrho})$-valued stochastic integral.

Furthermore, in Observation 5.57 we show that the stochastic analogue of the linearly-implicit Euler scheme fits into the abstract setting of Section 5.1. Based on an error estimate for the inexact scheme, see Proposition 5.65, we derive sufficient conditions for convergence with corresponding convergence order in the case of inexact operator evaluations in Theorem 5.63.

Our analysis so far holds for any spatially adaptive numerical scheme that provides an approximation of the unknown solution up to any prescribed tolerance. In the finite element setting, such strategies have been derived in, e.g., Binev et al. [10], Dörfler [72], Stevenson [151]. However, we are in particular interested in spatially adaptive schemes based on *wavelets* due to the strong analytical properties of wavelets, which can be used to design adaptive numerical schemes that are guaranteed to converge with optimal order, i.e., with the same order as best N-term wavelet approximation, see, e.g., Cohen et al. [29, 30], Gantumur et al. [81]. These relationships pave a way to rigorous complexity estimates in the wavelet case. Indeed, as pointed out above, the convergence order of best N-term wavelet approximation depends on the smoothness of the object one wants to approximate in specific scales of Besov spaces, cf. (1.1). So, the overall complexity can be determined by combining our abstract analysis with estimates for the Besov smoothness of the solutions to the elliptic subproblems in each time step. In Section 5.3, we therefore present a detailed analysis of Rothe's method, where adaptive wavelet discretizations are applied to the elliptic subproblems. In the first part, see Theorem 5.71, we concentrate on the case, where the solutions of the stage equations are approximated by using best N-term wavelet approximation and, in the second part, see Theorem 5.73, we consider an implementable and asymptotically optimal numerical wavelet solver for the stage equations. In particular for the linearly-implicit Euler scheme applied to the classical heat equation, we determine upper bounds for the overall number of degrees of freedom that are needed to approximate the solution up to a prescribed tolerance. See Theorem 5.78 where we assume that best N-term wavelet approximation for the spatial approximation of the stage equations is applied, and see Theorem 5.79 where an implementable and asymptotically optimal numerical wavelet solver is employed.

Chapter 2

Preliminaries

We provide the mathematical foundation on which the subject and results of this dissertation are based upon.

In Section 2.1 we begin with a few words about the structure of the motivating equations as well as the existence of solutions. In Section 2.2 we state the definitions and important results concerning the considered function spaces. Sections 2.3 and 2.4 provide an overview of wavelets and the applied numerical approximation methods which are based on wavelets. A nomenclature of frequently used notations as well as the index can be found at the end of this dissertation.

2.1 Stochastic partial differential equations

A few words about the structure of the motivating equations as well as the existence of solutions are in order.

We apply our analysis to semi-linear second order stochastic partial differential equations of the parabolic type on a bounded (Lipschitz) domain $\mathcal{O} \subset \mathbb{R}^d$ over a finite time horizon $[0, T]$ which are driven by an infinite-dimensional Wiener process W. We formulate such equations as an abstract evolution problem in an infinite-dimensional state space U:

$$\begin{aligned} \mathrm{d}X(t) &= AX(t)\mathrm{d}t + F(X(t))\mathrm{d}t + B(X(t))\mathrm{d}W(t), \quad t \in (0, T], \\ X(0) &= X_0 \in U. \end{aligned} \tag{2.1}$$

We in particular investigate the case, where A is an unbounded linear operator, like the Laplace operator with Dirichlet boundary conditions, and where the drift term F as well as the diffusion term B are globally Lipschitz continuous mappings. The case where B is independent of X, $B(X) \equiv B$, refers to *additive noise* $B\mathrm{d}W$, while $B(X)\mathrm{d}W$ is called *multiplicative noise*.

Several approaches to solve the problem (2.1) are frequently studied in the literature. The *analytically strong formulation* of a solution to (2.1) is given by a $D(A)$-valued predictable process X which satisfies

$$X(t) = X_0 + \int_0^t \Big(AX(s) + F(X(s)) \Big)\mathrm{d}s + \int_0^t B(X(s))\mathrm{d}W(s), \quad \text{P-a.s.},$$

for each $t \in [0,T]$. The *analytically weak formulation*of a solution to (2.1) is given by an U-valued predictable process X which satisfies

$$\langle X(t),\eta\rangle = \langle X_0,\eta\rangle + \int_0^t \langle X(s), A^*\eta\rangle + \langle F(X(s)),\eta\rangle \mathrm{d}s + \int_0^t \langle \eta, B(X(s))\mathrm{d}W(s)\rangle, \text{ P-a.s.},$$

for all $\eta \in \mathcal{D}(A^*)$ and each $t \in [0,T]$. The *mild formulation* is based on the *semi-group approach* of Da Prato, Zabczyk [40]. It is given by an U-valued predictable process X which satisfies

$$X(t) = e^{At}X_0 + \int_0^t e^{A(t-s)}F(X(s))\mathrm{d}s + \int_0^t e^{A(t-s)}B(X(s))\mathrm{d}W(s), \quad \text{P-a.s.},$$

for each $t \in [0,T]$. Here, $\{e^{tA}\}_{t\geq 0}$ is the semi-group generated by A. Of course, the above formulations only make sense if, in particular, the appearing integrals are well-defined. We refer to Appendix A.2 for the definition of the Bochner integral and Appendix A.5 for details on the stochastic integral.

Under various conditions on (2.1) existence and uniqueness results for these formulations have been obtained in the literature. For instance, existence proofs of a unique solution in the mild formulation are given, e.g., in Da Prato, Zabczyk [40, Theorem 5.4] and Jentzen, Kloeden [102, Theorem 5.1], while Prévôt, Röckner [135] and Rozovskii [141] consider a more general *variational formulation*. Its relation to above formulations as well as the corresponding uniqueness and existence results can be found, e.g., in Prévôt, Röckner [135, Appendix F]. We also refer to Chow [19], Grecksch, Tudor [84], Hairer [95], Krylov [118], Krylov, Rozovskii [119], Walsh [170] for further information.

2.2 Smoothness and function spaces

We state the definitions and important properties of the considered smoothness spaces, which are mainly of Sobolev and Besov type.

In this dissertation, the following smoothness scales are of particular importance. Let $f : \mathcal{O} \to \mathbb{R}$ be a measurable function, where $\mathcal{O} \subseteq \mathbb{R}^d$ is a *domain*, i.e., an open and connected set. The order of differentiability of a continuous function f is the maximal number $m \in \mathbb{N}_0$ such that all partial derivatives $\partial^\alpha f$, $\alpha \in \mathbb{N}_0^d$, $|\alpha| := \alpha_1 + \cdots + \alpha_d \leq m$, are bounded and continuous. The space of all such functions is denoted by $\mathcal{C}^m(\mathcal{O})$ and it can be shown it is a Banach space with respect to the norm

$$\|f\|_{\mathcal{C}^m(\mathcal{O})} := \max_{|\alpha|\leq m} \sup_{x\in\mathcal{O}} |\partial^\alpha f(x)|,$$

cf., e.g., Adams, Fournier [2, §1.26ff]. Furthermore, $\mathcal{C}^\infty(\mathcal{O}) := \bigcap_{m=0}^\infty \mathcal{C}^m(\mathcal{O})$ denotes the space of infinitely often continuously differentiable functions and $\mathcal{C}^0(\mathcal{O}) =: \mathcal{C}(\mathcal{O})$. The compactly supported functions in $\mathcal{C}(\mathcal{O})$ and $\mathcal{C}^\infty(\mathcal{O})$ are denoted by $\mathcal{C}_0(\mathcal{O})$ and $\mathcal{C}_0^\infty(\mathcal{O})$, respectively. A function f is by definition compactly supported if $\operatorname{supp}(f) := \operatorname{clos}\{x \in \mathcal{O} : f(x) \neq 0\}$ is a compact set in $\mathcal{O}$.

Smoothness scales of Sobolev type are taking the integrability of a function into account. For each $1 \leq p < \infty$ one can define the space $L_p(\mathcal{O})$ of p-integrable functions, cf. Appendix A.2, as the completion of the space $\mathcal{C}_0(\mathcal{O})$ with respect to the norm

$$\|f\|_{L_p(\mathcal{O})} := \left(\int_{\mathcal{O}} |f(x)|^p \,\mathrm{d}x\right)^{1/p},$$

where $\mathrm{d}x$ denotes the Lebesgue measure, see, e.g., Adams, Fournier [2, Theorem 2.19]. Then, one can also compare smoothness by asking for the maximal number $m \in \mathbb{N}_0$ such that all generalized derivatives $D^\alpha f$ up to the order m are p-integrable. The definition of generalized derivative is given in Appendix A.3. The space consisting of all functions $f \in L_p(\mathcal{O})$ for which $\sum_{|\alpha|\leq m} \|D^\alpha f(x)\|_{L_p(\mathcal{O})} < \infty$ is known as the *classical Sobolev space* $W_p^m(\mathcal{O})$. It is also a complete space with respect to the norm

$$\|f\|_{W_p^m(\mathcal{O})} := |f|_{W_p^m(\mathcal{O})} + \|f\|_{L_p(\mathcal{O})},$$

where

$$|f|_{W_p^m(\mathcal{O})} := \sum_{|\alpha|=m} \|D^\alpha f(x)\|_{L_p(\mathcal{O})}$$

defines the Sobolev semi-norm. We refer to Adams, Fournier [2] for details.

Smoothness scales of Besov type can be defined based on the behavior of the modulus of smoothness. The *k-th order L_p-modulus of smoothness*, $k \in \mathbb{N}$, of a function f is defined as

$$\omega^k(t, f)_p := \sup_{|h_1+\ldots+h_d|<t} \|\Delta_h^k f\|_{L_p(\mathcal{O})}, \quad t > 0,$$

where

$$\Delta_h^k f(x) := \prod_{i=0}^{k} \mathbb{1}_{\mathcal{O}}(x + ih) \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} f(x + jh), \quad x \in \mathbb{R}^d,$$

denotes its *k-th difference* with step size $h \in \mathbb{R}^d$. In this setting, one can compare the smoothness of a function f depending on how fast $\omega^k(f, t)_p$ goes to zero as $t \to 0$. For example, Besov spaces of smoothness $s > 0$ contain those functions for which

$$\left(2^{js}\omega^k(f, 2^{-j})_p\right)_{j\geq 0} \in \ell_q,$$

i.e., $\omega^k(f, t)_p$ goes to zero like $O(t^s)$ as $t \to 0$, see, e.g., Cohen [28, Chapter 3.2].

2.2.1 Besov and Sobolev spaces

Let

$$0 < p, q \leq \infty \quad \text{and} \quad d(1/p - 1)_+ < s < \infty,$$

as well as $k > s$, where $d(1/p-1)_+ := \min\{0,\, d(1/p-1)\}$. The *Besov space* $B_q^s(L_p(\mathcal{O}))$ is defined as the set of all functions $f \in L_p(\mathcal{O})$ such that the term

$$|f|_{B_q^s(L_p(\mathcal{O}))} := \begin{cases} \left(\displaystyle\int_0^\infty \left(t^{-s}\,\omega^k(t, f)_p\right)^q \frac{\mathrm{d}t}{t}\right)^{1/q} & : 0 < q < \infty, \\ \displaystyle\sup_{t>0} t^{-s}\,\omega^k(t, f)_p & : q = \infty, \end{cases}$$

is finite. A quasi-norm on $B_q^s(L_p(\mathcal{O}))$ is given by

$$\|f\|_{B_q^s(L_p(\mathcal{O}))} := |f|_{B_q^s(L_p(\mathcal{O}))} + \|f\|_{L_p(\mathcal{O})}, \tag{2.2}$$

which is independent of the order $k > s$ in the sense of equivalent quasi-norms, see, e.g., DeVore, Sharpley [68].

Aside of the inner description above, Besov spaces $\bar{B}^s_q(L_p(\mathbb{R}^d))$ for $0 < p, q \leq \infty$ and $s \in \mathbb{R}$ can also be defined as the set of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that the quasi-norm

$$\|f\|_\varphi := \begin{cases} \left(\sum_{j=0}^{\infty} \left(2^{js} \, \|\mathfrak{F}^{-1}(\varphi_j \mathfrak{F} f)\|_{L_p(\mathbb{R}^d)} \right)^q \right)^{1/q} & : 0 < q < \infty, \\ \sup_{j \geq 0} 2^{js} \, \|\mathfrak{F}^{-1}(\varphi_j \mathfrak{F} f)\|_{L_p(\mathbb{R}^d)} & : q = \infty, \end{cases} \tag{2.3}$$

is finite. Here, we require a fixed test function $\varphi \in \mathcal{D}(\mathbb{R}^d)$ with $\varphi \geq 0$ and $\varphi(0) \neq 0$, since then every $f \in \mathcal{S}'(\mathbb{R}^d)$ can be decomposed as

$$f = \sum_{j=0}^{\infty} \mathfrak{F}^{-1}(\varphi_j \mathfrak{F} f),$$

where $\varphi_0 := \varphi$ and $\varphi_j(x) := \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$, $j \in \mathbb{N}$. Moreover, $\|\cdot\|_\varphi$ is independent of the choice of φ in the sense of equivalent quasi-norms, in particular φ can be chosen such that $\{\varphi_j\}_j$ forms a resolution of unity, see, e.g., Triebel [158, Sections 2.3.1, 2.3.2] for details. Based on this approach, spaces on domains are usually defined by restriction, e.g.,

$$\bar{B}^s_q(L_p(\mathcal{O})) := \left\{ f \in \mathcal{D}'(\mathcal{O}) \, : \, \exists g \in \bar{B}^s_q(L_p(\mathbb{R}^d)) \text{ with } g|_{\mathcal{O}} = f \right\}$$

together with the quasi-norm $\|f\|_{\varphi,\mathcal{O}} := \inf\{\|g\|_\varphi \, : \, g \in \bar{B}^s_q(L_p(\mathbb{R}^d)),\, g|_{\mathcal{O}} = f\}$. The definitions of test functions, $\mathcal{D}'(\mathcal{O})$, $\mathcal{S}'(\mathbb{R}^d)$, and of the Fourier transform are given in Appendix A.3.

As it turns out, for $0 < p, q < \infty$ and $s > d(1/p - 1)_+$ both definitions coincide on $\mathbb{R}^d$ and, by employing bounded extension operators which exist for Lipschitz domains and Besov spaces, see Rychkov [144], the two approaches also yield the same space for bounded Lipschitz domains, see also Cohen [28, Remark 3.9.1] and the references therein. We refer to Triebel [158, Theorem 2.5.12] for a proof on the whole space and Dispa [70, Theorem 3.18] for a proof on bounded Lipschitz domains. A domain $\mathcal{O} \subset \mathbb{R}^d$ is by definition a *Lipschitz domain*, cf. Triebel [162, Definition 1.103], if each point on the boundary $\partial\mathcal{O}$ has a neighborhood whose intersection with the boundary (after relabeling and reorienting the coordinate axes if necessary) is the graph of a Lipschitz function.

Remark 2.1. A proof of the completeness of $B^s_q(L_p(\mathbb{R}^d))$ can be found in Triebel [158, Theorem 2.3.3(i)] and in Triebel [158, Proposition 3.2.3(i)] for $B^s_q(L_p(\mathcal{O}))$ on any domain $\mathcal{O} \subset \mathbb{R}^d$, since the completeness part of the proof only requires $\mathcal{O}$ to be a domain.

Remark 2.2. For general information on Besov spaces, we also refer to the monographs Meyer [129], Nikol'skij [130], Peetre [134], Runst, Sickel [143].

The Besov spaces $B^s_p(L_p(\mathcal{O}))$, $1 < p < \infty$, $s \in \mathbb{R}_+ \setminus \mathbb{N}$, coincide on certain domains with the following fractional Sobolev spaces $W^s_p(\mathcal{O})$, which extend the classical Sobolev spaces $W^m_p(\mathcal{O})$, $m \in \mathbb{N}_0$, to the whole smoothness scale $s \geq 0$: Let $s := m + \sigma$ with

$\sigma \in (0,1)$ and $1 \le p < \infty$. The *Sobolev(-Slobodeckij)* space $W_p^s(\mathcal{O})$ is defined as the set of all functions $f \in L_p(\mathcal{O})$ such that the term

$$|f|^p_{W_p^s(\mathcal{O})} := \sum_{|\alpha|=m} \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{|D^\alpha f(x) - D^\alpha f(y)|^p}{|x-y|^{d+\sigma p}} \,\mathrm{d}x\mathrm{d}y$$

is finite. A norm on $W_p^s(\mathcal{O})$ is given by

$$\|f\|_{W_p^s(\mathcal{O})} := \left(\|f\|^p_{W_p^m(\mathcal{O})} + |f|^p_{W_p^s(\mathcal{O})} \right)^{1/p}. \tag{2.4}$$

The equivalence of (2.2) and (2.4), where $1 < p < \infty$ and $s \in \mathbb{R}_+ \setminus \mathbb{N}$, has been shown in, e.g., Triebel [159, Section 2.5.1 & Remark 4.4.2/2] for $\mathcal{O} = \mathbb{R}^d$ and for bounded Lipschitz domains. Note that the classical Sobolev spaces $W_p^m(\mathcal{O})$, $m \in \mathbb{N}_0$, with $p \neq 2$ are not Besov spaces, cf. DeVore [65, Section 4.5].

Remark 2.3. A proof that $(W_p^s(\mathcal{O}), \|\cdot\|_{W_p^s(\mathcal{O})})$ is a Banach space and a Hilbert space for $p = 2$, can be found in, e.g., Dobrowolski [71, Section 6.10].

Remark 2.4. Sobolev spaces of fractional smoothness can also be introduced by means of the Fourier transform. For example, the Sobolev space $H^{s,p}(\mathbb{R}^d)$, $1 \le p < \infty$, $s \in \mathbb{R}$, of *Bessel potentials* is defined as

$$H^{s,p}(\mathbb{R}^d) := \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \,:\, \|f\|_{H^{s,p}(\mathbb{R}^d)} := \|\mathfrak{F}^{-1}(1+|\cdot|^2)^{s/2}\,\mathfrak{F}f\|_{L_p(\mathbb{R}^d)} < \infty \right\},$$

where on domains $\mathcal{O} \subseteq \mathbb{R}^d$ it is defined by restriction, i.e.,

$$\|f\|_{H^{s,p}(\mathcal{O})} := \inf\{\|g\|_{H^{s,p}(\mathbb{R}^d)} \,:\, g \in H^{s,p}(\mathbb{R}^d),\, g|_{\mathcal{O}} = f\},$$

cf. Adams, Fournier [2, §7.63]. For $\mathcal{O} = \mathbb{R}^d$ or a Lipschitz domain, it has been shown that $H^{s,2}(\mathcal{O})$ coincides with $W_2^s(\mathcal{O})$, $s > 0$, and that $H^{m,p}(\mathcal{O})$, $m \in \mathbb{N}$, coincides with $W_p^m(\mathcal{O})$. However, $H^{s,p}(\mathcal{O})$ and $W_p^s(\mathcal{O})$ differ for $p \neq 2$ and $s \in \mathbb{R}_+ \setminus \mathbb{N}$, yet still $B_1^s(L_p(\mathcal{O})) \subset H^{s,p}(\mathcal{O}) \subset B_\infty^s(L_p(\mathcal{O}))$. We refer to Triebel [158, Remark 2.2.2/3] and Triebel [160] for details.

In our approach, Besov and Sobolev spaces with negative smoothness are defined by duality. Therefore, let

$$1 < p, p', q, q' < \infty, \quad \frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1, \quad \text{and} \quad s > 0.$$

We define

$$B_q^{-s}(L_p(\mathcal{O})) := \left(B_{q'}^s(L_{p'}(\mathcal{O})) \right)^* = \mathcal{L}\left(B_{q'}^s(L_{p'}(\mathcal{O})), \mathbb{R} \right), \tag{2.5}$$

which is equipped with the canonical dual norm

$$\|f\|_{B_q^{-s}(L_p(\mathcal{O}))} := \sup\left\{ |fg| \,:\, g \in B_{q'}^s(L_{p'}(\mathcal{O})),\, \|g\|_{B_{q'}^s(L_{p'}(\mathcal{O}))} = 1 \right\},$$

cf. (A.1) in Appendix A.1.

Since $\mathcal{D}(\mathcal{O})$ is in general not dense in $W_p^s(\mathcal{O})$, the dual space of $W_p^s(\mathcal{O})$ can in those cases not be identified with a space of distributions, cf. Grisvard [88, Section 1.3.2]. Instead, the dual space of the closure of $\mathcal{D}(\mathcal{O})$ with respect to $\|\cdot\|_{W_p^s(\mathcal{O})}$, which is denoted by $\mathring{W}_p^s(\mathcal{O})$, is used to extend the Sobolev spaces to negative smoothness, i.e.,

$$W_p^{-s}(\mathcal{O}) := \left(\mathring{W}_{p'}^s(\mathcal{O}) \right)^* = \mathcal{L}\left(\mathring{W}_{p'}^s(\mathcal{O}), \mathbb{R} \right), \quad s > 0, \tag{2.6}$$

which is equipped with the canonical dual norm.

Remark 2.5. We have that $(\mathring{W}_p^s(\mathcal{O}), \|\cdot\|_{W_p^s(\mathcal{O})})$ is a Banach space, since $\mathring{W}_p^s(\mathcal{O})$ is a closed subspace of $W_p^s(\mathcal{O})$.

In the Hilbert space case, $p=2$, it is customary to use the abbreviations

$$H^s(\mathcal{O}) := W_2^s(\mathcal{O}) \quad \text{and} \quad H_0^s(\mathcal{O}) := \mathring{W}_2^s(\mathcal{O}).$$

A direct consequence of the characterization of Besov spaces by means of the Fourier transform, i.e., using (2.3), is that $H^s(\mathbb{R}^d)$ and $B_2^s(L_2(\mathbb{R}^d))$ coincide for all $s \geq 0$ in the sense of equivalent norms, see, e.g., Triebel [159, Theorems 2.3.2(d), 2.3.3(b)]. We refer to Triebel [159, Proposition 4.2.4, Theorem 4.6.1(b)] for a proof that

$$H^s(\mathcal{O}) = B_2^s(L_2(\mathcal{O})), \quad s \geq 0, \tag{2.7}$$

also holds in the sense of equivalent norms for domains $\mathcal{O} \subset \mathbb{R}^d$ of cone-type and in particular for bounded Lipschitz domains, since they are of cone-type, cf. Adams, Fournier [2, §4.11].

Here, we frequently apply the following embedding results of Besov spaces, where $A_1 \hookrightarrow A_2$ denotes that A_1 is continuously embedded in A_2, i.e., there exists a constant $c > 0$ such that for all $a \in A_1$ we have $\|a\|_{A_2} \leq c\|a\|_{A_1}$.

Theorem 2.6. *Let $\mathcal{O} \subseteq \mathbb{R}^d$ be a domain and $\varepsilon > 0$.*

(i) *Let $0 < p \leq \infty$, $0 < q_1 \leq q_2 \leq \infty$, and $s \in \mathbb{R}$. Then*

$$B_{q_1}^s(L_p(\mathcal{O})) \hookrightarrow B_{q_2}^s(L_p(\mathcal{O})).$$

(ii) *Let $0 < p, q_1, q_2 \leq \infty$ and $s \in \mathbb{R}$. Then*

$$B_{q_2}^{s+\varepsilon}(L_p(\mathcal{O})) \hookrightarrow B_{q_1}^s(L_p(\mathcal{O})).$$

(iii) *Let $\mathcal{O}$ be bounded, $0 < p_1 \leq p_2 \leq \infty$, $0 < q \leq \infty$, and $s \in \mathbb{R}$. Then*

$$B_q^s(L_{p_2}(\mathcal{O})) \hookrightarrow B_q^s(L_{p_1}(\mathcal{O})).$$

(iv) *Let $\mathcal{O}$ be bounded, $0 < p_1 \leq p_2 < \infty$, and $s \in \mathbb{R}$. Then*

$$B_{p_2}^{s+\varepsilon}(L_{p_2}(\mathcal{O})) \hookrightarrow B_{p_1}^s(L_{p_1}(\mathcal{O})).$$

(v) *Let $\mathcal{O}$ be bounded, $0 < p_1 < p_2 \leq \infty$, $0 < q_1 \leq q_2 \leq \infty$, and $s_1, s_2 \in \mathbb{R}$. Then*

$$B_{q_1}^{s_1}(L_{p_1}(\mathcal{O})) \hookrightarrow B_{q_2}^{s_2}(L_{p_2}(\mathcal{O})) \quad \textit{if} \quad \frac{1}{p_1} = \frac{s_1 - s_2}{d} + \frac{1}{p_2}.$$

(vi) *Let $\mathcal{O}$ be a bounded Lipschitz domain or $\mathcal{O} = \mathbb{R}^d$. Let $1 < p < \infty$, $s > 0$, and*

$$\frac{1}{\tau} = \frac{s}{d} + \frac{1}{p}.$$

Then

$$B_\tau^s(L_\tau(\mathcal{O})) \hookrightarrow L_p(\mathcal{O}).$$

(vii) *Let $\mathcal{O}$ be a bounded Lipschitz domain or $\mathcal{O} = \mathbb{R}^d$. Let $0 < p, q \leq \infty$ and $s > d(1/p - 1)_+$. Then*

$$B_q^s(L_p(\mathcal{O})) \hookrightarrow L_u(\mathcal{O}) \quad \textit{for some } u > 1.$$

Proof. The embeddings **(i)** and **(ii)** follow from the monotonicity of the ℓ_q spaces and the arguments given in TRIEBEL [158, Proposition 2.3.2/2]. An application of Hölders inequality yields **(iii)** and a combination of **(ii)** and **(iii)** yields **(iv)**. For the embedding **(v)** we refer to TRIEBEL [162, Section 1.11] and the references therein. The embedding **(vi)** is given in TRIEBEL [162, Theorem 1.73] for the case $\mathcal{O} = \mathbb{R}^d$. On bounded domains, **(vi)**, see, e.g., TRIEBEL [162, Section 1.11.5], is shown by using a bounded extension operator, see e.g., RYCHKOV [144] for bounded Lipschitz domains. The embedding **(vii)** is given in TRIEBEL [162, Proposition 4.6]. □

Remark 2.7. The embedding relations **(i)** and **(ii)** of Theorem 2.6 for $B^s_q(L_p(\mathcal{O}))$ show that s can be understood as primary smoothness parameter, while q is considered as fine tuning parameter.

Moreover, the following characterizations of Besov spaces in terms of interpolation spaces hold. See Appendix A.1.4 for the definition of the considered interpolation spaces. Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded Lipschitz domain or $\mathcal{O} = \mathbb{R}^d$. Let $1 \le p \le \infty$ and $r \in \mathbb{N}$. We have

$$(L_p(\mathcal{O}), W^r_p(\mathcal{O}))_{\theta,q} = B^{\theta r}_q(L_p(\mathcal{O})), \quad \theta \in (0,1),\ 0 < q \le \infty,$$

in the sense of equivalent norms. For $s_1 < s_2$, and $0 < q_1, q_2 \le \infty$ we have

$$(B^{s_1}_{q_1}(L_p(\mathcal{O})), B^{s_2}_{q_2}(L_p(\mathcal{O})))_{\theta,q} = B^{(1-\theta)s_1+\theta s_2}_q(L_p(\mathcal{O})), \quad \theta \in (0,1),\ 0 < q \le \infty,$$

in the sense of equivalent norms, and

$$(L_p(\mathcal{O}), B^{s_2}_{q_2}(L_p(\mathcal{O})))_{\theta,q} = B^{\theta s_2}_q(L_p(\mathcal{O})), \quad \theta \in (0,1),\ 0 < q \le \infty,$$

in the sense of equivalent norms. We refer to DEVORE [65, Section 4.6] and BERGH, LÖFSTRÖM [9] for details.

A type diagram for function spaces is given in Figure 2.1, which in this context is often referred to as DeVore-Triebel diagram: It illustrates function spaces of "s degrees of smoothness in L_p" in a coordinate system with respect to the parameters. On the x-axis, the inverse $1/p$ of the integrability parameter is plotted, while the smoothness parameter s is plotted on the y-axis. This way, the x-axis represents the spaces with smoothness zero and the dashed line $1/2 \mapsto s$ represents the Sobolev spaces $H^s(\mathcal{O})$ with smoothness s. The shaded area represents the defined range of parameter pairs $(1/\bar{p}, \bar{s})$ of the Besov spaces $B^{\bar{s}}_q(L_{\bar{p}}(\mathcal{O}))$, where the fine tuning parameter q is usually omitted. The arrows indicate the directions of the principal embeddings on bounded Lipschitz domains, cf. Theorem 2.6. In particular, the rays with slope d indicate the so-called Sobolev embedding lines: While keeping $1/p - s/d$ fixed, one can enlarge the function space by trading smoothness for integrability.

Remark 2.8. For our analysis, we are going to employ wavelet multiscale characterizations of the considered Besov spaces, cf. Section 2.3 below. Therefore, we do not consider Besov spaces for the parameters $0 < p < 1$ and $s < d(1/p - 1)$, i.e., which are outside of the shaded area in Figure 2.1, since it is not clear whether they allow a wavelet characterization of the type we require for our analysis. We refer to COHEN [28, Remark 3.7.4] for details.

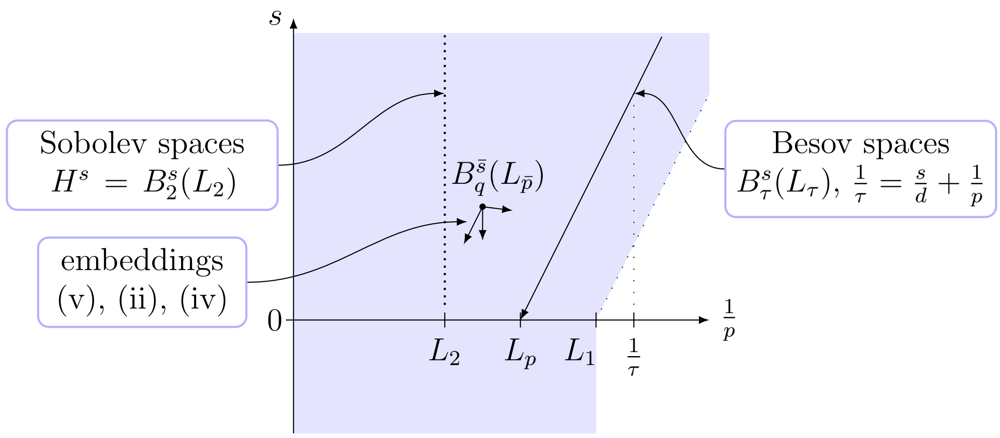


Figure 2.1: DeVore-Triebel diagram: any function space of "s degrees of smoothness in L_p" is represented by the point $(1/p, s)$

2.2.2 Anisotropic Besov spaces

Let us now consider the anisotropic setting. First, we fix an *anisotropy*

$$\boldsymbol{a} = (a_1, \ldots, a_d) \in \mathbb{R}^d_+, \quad \text{with} \quad \sum_{i=1}^{d} \frac{1}{a_i} = d. \tag{2.8}$$

Let $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d\}$ denote the canonical basis of $\mathbb{R}^d$. For a function $f : \mathbb{R}^d \to \mathbb{R}$ let

$$\Delta^{\boldsymbol{k}}_{\boldsymbol{h}} f(x) := (\Delta^{k_1}_{h_1 \boldsymbol{e}_1} \circ \ldots \circ \Delta^{k_d}_{h_d \boldsymbol{e}_d}) f(x), \quad x \in \mathbb{R}^d,$$

be the *mixed difference* of order $\boldsymbol{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$ and step $\boldsymbol{h} = (h_1, \ldots, h_d) \in \mathbb{R}^d$. For $p \in (0, \infty)$ the *mixed modulus of smoothness* with respect to $\boldsymbol{a}$ is defined by

$$\omega^{\boldsymbol{k}}_{\boldsymbol{a}}(t, f)_p := \sup_{|\boldsymbol{h}|_{\boldsymbol{a}} < t} \|\Delta^{\boldsymbol{k}}_{\boldsymbol{h}} f\|_{L_p(\mathcal{O})}, \quad t > 0,$$

where

$$|\boldsymbol{h}|_{\boldsymbol{a}} := \sum_{j=1}^{d} |h_j|^{a_j}, \quad h \in \mathbb{R}^d,$$

is the *anisotropic pseudo-distance* of the step $\boldsymbol{h}$ related to the anisotropy $\boldsymbol{a}$.

Now, let

$$0 < p, q < \infty \quad \text{and} \quad d(1/p - 1)_+ < s < \infty,$$

as well as $\mathbb{N} \ni K > \max\{s_1, ..., s_d\}$ with $s_i := s a_i$, $i = 1, ..., d$. The *anisotropic Besov space* $B^{s,\mathbf{a}}_q(L_p(\mathbb{R}^d))$ is the set of all functions $f \in L_p(\mathbb{R}^d)$ such that the term

$$|f|_{B^{s,\mathbf{a}}_q(L_p(\mathbb{R}^d))} := \sum_{|\boldsymbol{k}|=K} \left(\int_0^\infty \left(t^{-s}\, \omega^{\boldsymbol{k}}_{\boldsymbol{a}}(t, f)_p\right)^q \frac{\mathrm{d}t}{t} \right)^{1/q}$$

is finite. A quasi-norm on $B^{s,\mathbf{a}}_q(L_p(\mathbb{R}^d))$ is given by

$$\|f\|_{B^{s,\mathbf{a}}_q(L_p(\mathbb{R}^d))} := |f|_{B^{s,\mathbf{a}}_q(L_p(\mathbb{R}^d))} + \|f\|_{L_p(\mathbb{R}^d)},$$

which is independent of the choice of K in the sense of equivalent quasi-norms, cf. Triebel [162, Theorem 5.8]. Observe that, if $\boldsymbol{a} = \mathbf{1}$ we are in the isotropic case.

In this setting, anisotropic Besov spaces on domains $\mathcal{O} \subset \mathbb{R}^d$ are defined by restriction, i.e.,

$$B_q^{s,\mathbf{a}}(L_p(\mathcal{O})) := \left\{ f \in L_p(\mathcal{O}) \;:\; \exists g \in B_q^{s,\mathbf{a}}(L_p(\mathbb{R}^d)),\, g|_{\mathcal{O}} = f \right\},$$

together with the quasi-norm

$$\|f\|_{B_q^{s,\mathbf{a}}(L_p(\mathcal{O}))} := \inf\{\|g\|_{B_q^{s,\mathbf{a}}(L_p(\mathbb{R}^d))} \;:\; g \in B_q^{s,\mathbf{a}}(L_p(\mathbb{R}^d)),\, g|_{\mathcal{O}} = f\}.$$

Remark 2.9. The above definition of anisotropic Besov spaces is equivalent to the definitions used in Garrigós et al. [82, 83], see [82, Proposition 2.2].

Remark 2.10. The space $B_2^{s,\mathbf{a}}(L_2(\mathbb{R}^d))$ coincides with the anisotropic Sobolev space of Bessel potentials

$$H^{s,2,\mathbf{a}}(\mathbb{R}^d) := \{f \;:\; \mathfrak{F}^{-1}(1 + |\xi_i|^2)^{sa_i/2}\, \mathfrak{F}f \in L_2(\mathbb{R}^d),\ i = 1, \ldots, d\}.$$

In the case $\mathbf{a} = (sa_1, \ldots, sa_d) \in \mathbb{N}^d$, the space $B_2^{s,\mathbf{a}}(L_2(\mathbb{R}^d))$ coincides with the classical anisotropic Sobolev space

$$W_2^{s,\mathbf{a}}(\mathbb{R}^d) := \left\{ f \in L_2(\mathbb{R}^d) \;:\; \sum_{i=1}^{d} \left\| \frac{\partial^{sa_i} f}{\partial x_i^{sa_i}} \right\|_{L_2(\mathbb{R}^d)} < \infty \right\}.$$

Aside of Garrigós et al. [82, 83], we also refer to Triebel [162, Chapter 5] for details on anisotropic Besov spaces.

2.2.3 Tensor spaces of generalized dominating mixed derivatives

Let the domain $\mathcal{O} \subset \mathbb{R}^d$ be an n-fold product of component domains $\mathcal{O}_m \subset \mathbb{R}^{d_m}$, $m = 1, ..., n$, $n \geq 2$, with $\sum_{m=1}^{n} d_m = d$. On the component domains, let $\bar{H}^s(\mathcal{O}_m)$ be either the Sobolev space $H^s(\mathcal{O}_m)$, or a closed subspace of it, in which boundary conditions are incorporated, e.g., $H_0^s(\mathcal{O}_m)$, cf. Section 2.2.1.

Let $\boldsymbol{t} = (t_1, \ldots, t_n) \in [0, \infty)^n$ and $\ell \in [0, \infty)$. The tensor space $H^{\boldsymbol{t},\ell}(\mathcal{O})$ of generalized dominating mixed derivatives is defined as

$$H^{\boldsymbol{t},\ell}(\mathcal{O}) := \bigcap_{i=1}^{n} \bigotimes_{m=1}^{n} \bar{H}^{t_m + \delta_{m,i}\ell}(\mathcal{O}_m),$$

where $\delta_{m,i}$ is the Kronecker delta. That is, $H^{\boldsymbol{t},\ell}(\mathcal{O}) \subset L_2(\mathcal{O})$ is the set of all functions $f = f_1 \otimes \cdots \otimes f_n$ for which

$$\|f\|_{H^{\boldsymbol{t},\ell}(\mathcal{O})} := \sum_{i=1}^{n} \prod_{m=1}^{n} \|f_m\|_{H^s(\mathcal{O}_m)}, \quad \text{with} \quad s = t_m + \delta_{m,i}\ell,$$

is finite.

Remark 2.11. The spaces $H^{\boldsymbol{t},\ell}(\mathcal{O})$ are generalizations of spaces with dominating mixed derivatives $H^{\boldsymbol{t},0}(\mathcal{O})$ as introduced in Lizorkin, Nikol'skij [125], see also Griebel, Knapek [85] and Schwab, Stevenson [146]. Also note that, since the Lebesgue measure is a product measure, $H^{\mathbf{0},\ell}(\mathcal{O})$ is isomorphic to the standard Sobolev space $H^\ell(\mathcal{O})$ on bounded domains.

2.3 The wavelet setting

In general, we employ biorthogonal wavelet bases on a domain $\mathcal{O} \subseteq \mathbb{R}^d$ that characterize certain function spaces by the decay properties of the coefficients in their wavelet decompositions. The considered function spaces are defined in Section 2.2. For our analysis, we have wavelets in mind which are constructed by means of a multiresolution analysis and in the multivariate case they are given as tensor products of univariate wavelet bases.

2.3.1 Wavelet multiscale decomposition and the characterization of Besov spaces

We start our exposition by explaining how biorthogonal wavelet bases are constructed from a multiresolution analysis for $L_2(\mathbb{R}^d)$.

Let $\varphi : \mathbb{R} \to \mathbb{R}$ and $\widetilde{\varphi} : \mathbb{R} \to \mathbb{R}$ be two compactly supported *refinable* functions, i.e.,

$$\varphi = \sum_{k\in\mathbb{Z}} h_k \varphi(2\cdot -k) \quad \text{and} \quad \widetilde{\varphi} = \sum_{k\in\mathbb{Z}} \widetilde{h}_k \widetilde{\varphi}(2\cdot -k)$$

with a finite number of non-zero coefficients $(h_k)_{k\in\mathbb{Z}}$ and $(\widetilde{h}_k)_{k\in\mathbb{Z}}$. Furthermore, let φ and $\widetilde{\varphi}$ be *dual* to each other, that is

$$\int_{\mathbb{R}} \varphi(x-k)\widetilde{\varphi}(x-l)\,\mathrm{d}x = \delta_{k,l}, \quad k,l \in \mathbb{Z},$$

where $\delta_{k,l}$ denotes the Kronecker delta. This way, the multivariate functions

$$\phi(x) := \varphi(x_1)\cdots\varphi(x_d) \quad \text{and} \quad \widetilde{\phi}(x) := \widetilde{\varphi}(x_1)\cdots\widetilde{\varphi}(x_d), \quad x = (x_1,\ldots,x_d) \in \mathbb{R}^d,$$

are also dual to each other, i.e.,

$$\int_{\mathbb{R}^d} \phi(x-k)\widetilde{\phi}(x-l)\,\mathrm{d}x = \delta_{k,l}, \quad k,l \in \mathbb{Z}^d.$$

From the existence of a compactly supported dual function $\widetilde{\phi}$, we have that the set $\{\phi(\,\cdot - k) \,:\, k \in \mathbb{Z}^d\}$ is a Riesz basis for the space

$$V_0 := \operatorname{clos}_{L_2(\mathbb{R}^d)} \left(\operatorname{span}\{\phi(\,\cdot - k) \,:\, k \in \mathbb{Z}^d\}\right),$$

cf. Dahlke et al. [45]. The definition of a Riesz basis is given in Appendix A.1.1. Observe that each dilated space

$$V_j := \{g(2^j\,\cdot) \,:\, g \in V_0\},$$

for a fixed $j \in \mathbb{Z}$, is spanned by the functions $\phi_{j,k}$, $k \in \mathbb{Z}^d$, where

$$\eta_{j,k} := 2^{jd/2}\eta(2^j\cdot -k), \quad j \in \mathbb{Z},\ k \in \mathbb{Z}^d,$$

are the scaled and shifted dilates of $\eta \in L_2(\mathbb{R}^d)$ such that $\|\eta_{j,k}\|_{L_2(\mathbb{R}^d)} = \|\eta\|_{L_2(\mathbb{R}^d)}$ for all $j \in \mathbb{Z}$, $k \in \mathbb{Z}^d$. The elements $g \in V_j$ can therefore be decomposed as

$$g = \sum_{k\in\mathbb{Z}^d} \langle g, \widetilde{\phi}_{j,k}\rangle \phi_{j,k},$$

where $\langle g_1, g_2\rangle := \int_{\mathbb{R}^d} g_1(x)g_2(x)\,\mathrm{d}x$ denotes the inner product of $L_2(\mathbb{R}^d)$. Furthermore, $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L_2(\mathbb{R}^d)$, since ϕ is refinable and thus

$$V_j \subset V_{j+1}, \quad j \in \mathbb{Z}, \tag{2.9}$$

see DE BOR ET AL. [63]. Going back to MALLAT [127], such a nested sequence $(V_j)_{j\in\mathbb{Z}}$ is called *multiresolution analysis*. By (2.9), we obtain that

$$\{\phi_{j,k} \,:\, k \in \mathbb{Z}^d\}$$

forms a Riesz basis for V_j: Any $g_j \in V_j$ can be written as $g_j = 2^{jd/2}g_0(2^j\,\cdot\,)$ with $g_0 = \sum_{k\in\mathbb{Z}^d} c_k\phi(\,\cdot - k) \in V_0$, i.e., $g_j = \sum_{k\in\mathbb{Z}^d} c_k\,\phi_{j,k}$, and since $\|g_j\|_{L_2} = \|g_0\|_{L_2}$ the constants in the Riesz basis property (A.3) are independent of j. In this setting, the projectors

$$\begin{aligned} P_j : L_2(\mathbb{R}^d) &\to V_j \\ f &\mapsto \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\phi}_{j,k}\rangle \phi_{j,k}, \quad j \in \mathbb{Z}, \end{aligned} \tag{2.10}$$

are uniformly bounded on $L_2(\mathbb{R}^d)$ with

$$\lim_{j\to\infty} \|f - P_j f\|_{L_2(\mathbb{R}^d)} = 0, \quad f \in L_2(\mathbb{R}^d),$$

see COHEN [28, Sections 2.3, 2.12] for details.

Remark 2.12. In general, a dual $\widetilde{\varphi}$ is not uniquely determined for a given compactly supported refinable function φ. However, under the assumptions on $\widetilde{\varphi}$ being refinable and compactly supported, the dual spaces $\widetilde{V}_j$, spanned by $\widetilde{\phi}_{j,k}$, $k \in \mathbb{Z}^d$, also form a multiresolution analysis, see COHEN [28, Section 2.2].

A wavelet multiscale decomposition allows for a successive update of a decomposition in V_j to obtain a decomposition in V_{j+1}. Therefore, one considers the operators

$$Q_j := P_{j+1} - P_j, \quad j \in \mathbb{Z},$$

which are also uniformly bounded on $L_2(\mathbb{R}^d)$. Observe that Q_j maps $L_2(\mathbb{R}^d)$ onto some complement space W_j of V_j in V_{j+1}, which represents the *detail* that needs to be added to V_j to obtain V_{j+1}. These, so called *wavelet spaces*, have the following properties:

$$W_j = \{g(2^j\,\cdot\,) \,:\, g \in W_0\}, \quad j \in \mathbb{Z},$$

and

$$W_0 = \operatorname{clos}_{L_2(\mathbb{R}^d)}\left(\operatorname{span}\{\psi_e(\,\cdot - k) \,:\, e = 1, \dots, 2^d - 1,\, k \in \mathbb{Z}^d\}\right),$$

for suitable $2^d - 1$ functions ψ_e. Moreover, there exist $2^d - 1$ dual functions $\widetilde{\psi}_e$ such that

$$\langle \psi_e(\cdot - k), \widetilde{\psi}_{e'}(\cdot - k')\rangle = \delta_{e,e'}\delta_{k,k'}, \quad e, e' = 1, \dots, 2^d - 1,\ k, k' \in \mathbb{Z}^d.$$

For instance, one may choose

$$\psi_e(x) := \psi^{e_1}(x_1)\cdots\psi^{e_d}(x_d), \quad (e_1, \dots, e_d) \in \{0,1\}^d \setminus \{0\},\ x = (x_1, \dots, x_d) \in \mathbb{R}^d,$$

with

$$\psi^0 := \varphi \quad \text{and} \quad \psi^1 := \sum_{k\in\mathbb{Z}} (-1)^k\,\widetilde{h}_{1-k}\,\varphi(2\,\cdot - k),$$

and $\widetilde{\psi}_e$ analogously with

$$\widetilde{\psi}^0 := \widetilde{\varphi} \quad \text{and} \quad \widetilde{\psi}^1 := \sum_{k\in\mathbb{Z}} (-1)^k \, h_{1-k} \, \widetilde{\varphi}(2 \cdot -k),$$

see Cohen [28, Sections 2.6, 2.9]. Under the assumption that $\varphi, \widetilde{\varphi} \in H^{\varepsilon}(\mathbb{R}^d)$ for some $\varepsilon > 0$, it is possible to obtain a pair of biorthogonal *wavelet Riesz bases*

$$\Psi := \{\phi_{0,k} \,:\, k \in \mathbb{Z}^d\} \cup \{\psi_{e,j,k} \,:\, e = 1, \ldots, 2^d - 1, \; j \geq 0, \; k \in \mathbb{Z}^d\}$$

and

$$\widetilde{\Psi} := \{\widetilde{\phi}_{0,k} \,:\, k \in \mathbb{Z}^d\} \cup \{\widetilde{\psi}_{e,j,k} \,:\, e = 1, \ldots, 2^d - 1, \; j \geq 0, \; k \in \mathbb{Z}^d\}$$

for $L_2(\mathbb{R}^d)$, see Cohen [28, Section 3.8]. Note that the basis functions are assumed to be L_2-normalized. Every $f \in L_2(\mathbb{R}^d)$ can therefore be written in form of the *wavelet multiscale decomposition*

$$f = \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\phi}_{0,k} \rangle \phi_{0,k} + \sum_{e=1}^{2^d-1} \sum_{j\geq 0} \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\psi}_{e,j,k} \rangle \psi_{e,j,k}.$$

As we see, wavelet indices encode several types of information: the *scale* or *level* j, the *spatial location*, e.g., $2^{-j}k$ for $d = 1$, and also the *type* e of the wavelet. In asymptotic analysis, an explicit dependence on the type of the wavelets can be ignored, whenever it only produces additional constants. Therefore, we are also using the following notation: We set $\psi_{0,0,k} := \phi_{0,k}$, $\widetilde{\psi}_{0,0,k} := \widetilde{\phi}_{0,k}$, and $\nabla := \cup_{j\geq 0} \nabla_j$, where ∇_j denotes the set of all indices on level j encoding type and spatial location. Furthermore, we abbreviate the wavelet bases by

$$\Psi = \{\psi_\lambda \,:\, \lambda \in \nabla\} \quad \text{and} \quad \widetilde{\Psi} = \{\widetilde{\psi}_\lambda \,:\, \lambda \in \nabla\}$$

with $\lambda = (j, k)$, $|\lambda| = j$, and $\nabla = \{(j, k) \,:\, j \geq 0, \; k \in \nabla_j\}$. This way, we can write

$$f = \sum_{j\geq 0} \sum_{k\in\nabla_j} \langle f, \widetilde{\psi}_{j,k} \rangle \psi_{j,k} = \sum_{\lambda\in\nabla} \langle f, \widetilde{\psi}_\lambda \rangle \psi_\lambda, \tag{2.11}$$

as well as

$$\|f\|^2_{L_2(\mathbb{R}^d)} \asymp \sum_{\lambda\in\nabla} |\langle f, \widetilde{\psi}_\lambda \rangle|^2 \asymp \sum_{\lambda\in\nabla} |\langle f, \psi_\lambda \rangle|^2 \tag{2.12}$$

for the Riesz bases properties, cf. (A.3), where constants are independent of λ.

Remark 2.13. For details on the construction of wavelet bases in the biorthogonal setting we refer to Cohen [28], Cohen et al. [33], Daubechies [62], Lemarié-Rieusset [123], Mallat [128], Meyer [129]. In particular, for the adaption of compactly supported wavelet bases to bounded domains $\mathcal{O} \subset \mathbb{R}^d$ and boundary conditions we refer to Cohen [28, Section 2.12]. Note that this might require to adjust the starting level to $j \geq j_0$, $j_0 \in \mathbb{Z}$, in (2.11). Here, we just like to point out that for a wide class of bounded domains, e.g., coordinatewise Lipschitz domains, compactly supported wavelet bases can be constructed such that the index set ∇_j is finite, see Dahlke et al. [45, Section 3.5].

In order to characterize Besov spaces $B_q^s(L_p(\mathbb{R}^d))$ by means of wavelet multiscale decompositions both wavelet bases Ψ, $\widetilde{\Psi}$ need to satisfy certain locality, smoothness, and vanishing moment conditions. For instance, let $r \in \mathbb{N}$, $M > 0$, and suppose that for all $e = 1, \ldots, 2^d - 1$, we have

$$\operatorname{supp}\phi,\ \operatorname{supp}\psi_e \subset [-M, M]^d, \tag{2.13}$$

$$\phi,\ \psi_e \in C^r(\mathbb{R}^d), \tag{2.14}$$

$$\int x^\alpha\, \psi_e(x)\,\mathrm{d}x = 0 \quad \text{for all } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| = \textstyle\sum_{i=1}^d \alpha_i \le r, \tag{2.15}$$

as well as (2.13), (2.14), and (2.15) with ϕ and ψ_e replaced by $\widetilde{\phi}$ and $\widetilde{\psi}_e$ for possibly different parameters $\widetilde{r} \in \mathbb{N}$ and $\widetilde{M} > 0$. To shorten the notation, we set

$$\psi_{e,j,k,p} := 2^{jd(1/p-1/2)}\psi_{e,j,k} \quad \text{and} \quad \widetilde{\psi}_{e,j,k,p'} := 2^{jd(1/p'-1/2)}\widetilde{\psi}_{e,j,k},$$

for the L_p-normalized wavelets and the corresponding duals with $p' := p/(p-1)$ if $p \in (0,\infty)$, $p \neq 1$, and $p' := \infty$, $1/p' := 0$ if $p = 1$.

Theorem 2.14. *Let $0 < p, q < \infty$ and $s > d(1/p - 1)_+$. Choose $r \in \mathbb{N}$ such that $r > s$ and construct a dual pair of wavelet Riesz bases with the properties* (2.13), (2.14), *and* (2.15). *Then a locally integrable function $f : \mathbb{R}^d \to \mathbb{R}$ is in the Besov space $B_q^s(L_p(\mathbb{R}^d))$, if and only if*

$$f = \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\phi}_{0,k}\rangle\, \phi_{0,k} + \sum_{e=1}^{2^d-1} \sum_{j\ge0} \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\psi}_{e,j,k,p'}\rangle\, \psi_{e,j,k,p} \tag{2.16}$$

(convergence in $\mathcal{D}'(\mathbb{R}^d)$) and

$$\left(\sum_{k\in\mathbb{Z}^d} |\langle f, \widetilde{\phi}_{0,k}\rangle|^p\right)^{1/p} + \left(\sum_{e=1}^{2^d-1} \sum_{j\ge0} 2^{jsq} \left(\sum_{k\in\mathbb{Z}^d} |\langle f, \widetilde{\psi}_{e,j,k,p'}\rangle|^p\right)^{q/p}\right)^{1/q} < \infty. \tag{2.17}$$

Moreover, (2.17) *is an equivalent (quasi-)norm for $B_q^s(L_p(\mathbb{R}^d))$.*

A proof of this theorem for $p \ge 1$ can be found in Meyer [129, Chapter 6, §10]. For the general case see, e.g., Kyriazis [120] or Cohen [28, Theorem 3.7.7]. Moreover, the adaption of Theorem 2.14 to bounded (Lipschitz) domains and to boundary conditions are discussed in Cohen [28, Sections 3.9, 3.10]. The characterization of the dual spaces (2.5), i.e., the case of negative smoothness, relies only on the ability to characterize the primal space by the dual decomposition. Therefore, it is also possible to obtain the norm equivalence

$$\|\cdot\|_{B_q^{-s}(L_p(\mathbb{R}^d))} \asymp \left(\sum_{k\in\mathbb{Z}^d} |\langle\,\cdot\,, \phi_{0,k}\rangle|^p\right)^{1/p} + \left(\sum_{e=1}^{2^d-1} \sum_{j\ge0} 2^{-jsq} \left(\sum_{k\in\mathbb{Z}^d} |\langle\,\cdot\,, \psi_{e,j,k,p'}\rangle|^p\right)^{q/p}\right)^{1/q}$$

for $1 < p, q < \infty$ and $0 < s < \widetilde{r}$, see Cohen [28, Theorem 3.8.1].

Remark 2.15. In particular, the special case

$$\|\cdot\|^2_{H^s(\mathbb{R}^d)} \asymp \sum_{\lambda\in\nabla} 2^{2s|\lambda|} |\langle\,\cdot\,, \widetilde{\psi}_\lambda\rangle|^2$$

extends the stability properties (2.12) to the scale of Sobolev spaces within $-\widetilde{r} < s < r$.

Remark 2.16. We also refer to DeVore et al. [66], Frazier, Jawerth [80], Meyer [129], Runst, Sickel [143], Triebel [162] for further information on the fact that wavelets are able to characterize smoothness spaces such as Besov and Sobolev spaces.

Remark 2.17. In Cohen et al. [33] it has been shown that compactly supported dual pairs φ, $\widetilde{\varphi}$ exist with order of polynomial reproduction $n, \widetilde{n} \in \mathbb{N}$, respectively, as long as $n + \widetilde{n}$ is even. Therefore, the biorthogonal setting allows to construct wavelets with an arbitrarily high number of vanishing moments, see Dahlke et al. [45].

2.3.2 Linear and nonlinear approximation

The goal of approximation is to describe the elements of a possibly infinitely dimensional (quasi-)normed space by elements of finite dimensional normed spaces or a finite number of building blocks, cf. DeVore [65]. Let $(X, \|\cdot\|_X)$ be the (quasi-)normed space of elements we wish to approximate and let $X_n \subset X$, $n \in \mathbb{N}_0$, be closed subspaces, called *approximation spaces*, from which the approximating elements are derived. The *approximation error* of an element $f \in X$ that is going to arise is, e.g., given by

$$E_n(f, (X_n))_X := \operatorname{dist}(f, X_n)_X := \inf_{g \in X_n} \|f - g\|_X, \tag{2.18}$$

or some similar term. Now, the task is to design approximation procedures $f \mapsto f_n$, $f_n \in X_n$, that avoid to solve the minimization problem (2.18) directly, which in general can be quite demanding. Preferably, such approximation procedures should still be near-optimal or *asymptotically optimal*, i.e.,

$$\|f - f_n\|_X \preceq E_n(f, (X_n))_X, \quad f \in X, \tag{2.19}$$

with a constant independent of f and n. In practical applications, the performance of any numerical approximation scheme is not just determined by (2.19), but also by its *computational cost*. A numerical approximation scheme is *efficient*, if its computational cost is proportional to the number n, and it is asymptotically optimal if it realizes (2.19) while staying efficient.

Depending on the properties of the employed approximation spaces X_n, there are two distinct types of approximation procedures, cf. DeVore [65]. In *linear approximation* one is interested in the approximation properties of finite dimensional linear spaces X_n and the number n of parameters that describe the approximation is usually the dimension of X_n or closely related to it. In *nonlinear approximation* the X_n are usually nonlinear spaces and the number n relates to the maximal number of free parameters or building blocks, like the number of discretization knots or the number of elements in a decomposition of f_n. In the latter case, n is also referred to as the number of *degrees of freedom* for the approximation.

In either case, the spaces X_n, which are allowed so far, are way too general for the scope of this exposition. Therefore, following DeVore [65, Section 4.1], we only consider spaces X_n that satisfy the properties:

(A1) $X_0 := \{0\}$, $X_n \subset X_{n+1}$,

(A2) each $f \in X$ has a best approximation from X_n, (A3) $aX_n = X_n$, $a \in \mathbb{R} \setminus \{0\}$,

(A4) $X_n + X_n \subseteq X_{cn}$ for some constant $c \in \mathbb{N}$,

(A5) $\lim_{n\to\infty} E_n(f,(X_n))_X = 0$ for all $f \in X$.

While items (A1) and (A2) may be modified to suit a different setting, items (A3)–(A5) are characteristic to the considered approximation theory.

Given suitable spaces X_n, the elements of X can be described by the *approximation rate*, i.e., the rate at which the approximation error decreases. For every $\alpha > 0$ and $0 < q \leq \infty$, the *approximation class* $\mathcal{A}^\alpha_q(X,(X_n))$ consist of all the elements $f \in X$ such that

$$|f|_{\mathcal{A}^\alpha_q(X,(X_n))} := \begin{cases} \left(\sum_{n=1}^{\infty} \frac{1}{n} \Big(n^\alpha E_n(f,(X_n))_X \Big)^q \right)^{1/q} & : 0 < q < \infty, \\ \sup_{n \geq 1} n^\alpha E_n(f,(X_n))_X & : q = \infty, \end{cases}$$

is finite. We obtain a (quasi-)norm on this class by $\|f\|_{\mathcal{A}^\alpha_q(X,(X_n))} := |f|_{\mathcal{A}^\alpha_q(X,(X_n))} + \|f\|_X$ and we have the set relation

$$\mathcal{A}^\alpha_{q_1}(X,(X_n)) \subseteq \mathcal{A}^\alpha_{q_2}(X,(X_n)) \quad \text{for} \quad 0 < q_1 \leq q_2 \leq \infty.$$

Since $\mathcal{A}^\alpha_\infty(X,(X_n))$ contains exactly the $f \in X$ for which $E_n(f,(X_n))_X = O(n^{-\alpha})$, all elements in $\mathcal{A}^\alpha_q(X,(X_n))$ exhibit *approximation order* $\alpha > 0$, i.e.,

$$E_n(f,(X_n))_X \preceq n^{-\alpha}, \quad f \in \mathcal{A}^\alpha_q(X,(X_n)),$$

where $0 < q \leq \infty$ is some fine tuning parameter. Note that, since $(E_n(f,(X_n))_X)_{n\in\mathbb{N}}$ is a monotone sequence due to (A1) and (A5), we also have the norm equivalence

$$|\cdot|_{\mathcal{A}^\alpha_q(X,(X_{2^j}))} \asymp \begin{cases} \left(\sum_{j=0}^{\infty} \Big(2^{j\alpha} E_{2^j}(f,(X_{2^j}))_X \Big)^q \right)^{1/q} & : 0 < q < \infty, \\ \sup_{j \geq 0} 2^{j\alpha} E_{2^j}(f,(X_{2^j}))_X & : q = \infty. \end{cases}$$

The aim is now to actually characterize $\mathcal{A}^\alpha_q(X,(X_n))$ by some analytic conditions that are known to us, e.g., in terms of function spaces and smoothness scales. This way, the approximation rates that can be achieved for a given sequence $(X_n)_{n\in\mathbb{N}}$ are determined by the related smoothness scales. A main tool in this direction is to employ *direct* (or *Jackson type*) and *inverse* (or *Bernstein type*) inequalities to obtain a characterization in terms of interpolation spaces. See Appendix A.1.4 for their definition. Given a number $r > 0$ and a (possibly quasi-)semi-normed linear space $(Y, |\cdot|_Y)$ which is continuously embedded in X, i.e., $Y \hookrightarrow X$ and $\|\cdot\|_X \preceq |\cdot|_Y$, suppose that for all $n \in \mathbb{N}$ the following two inequalities hold

Jackson inequality: $\quad E_n(f,(X_n))_X \preceq n^{-r}\,|f|_Y, \quad f \in Y, \quad$ and

Bernstein inequality: $\quad |g|_Y \preceq n^r\,\|g\|_X, \quad g \in X_n.$

Then for all $0 < q \leq \infty$ the following relation between approximation classes and interpolation spaces can be shown

$$\mathcal{A}_q^\gamma(X,(X_n)) = (X,Y)_{\gamma/r,q}, \quad 0 < \gamma < r, \tag{2.20}$$

in the sense of equivalent norms, see DeVore [65, Theorem 1]. We also refer to Cohen [28, Theorems 3.5.2, 4.2.1]. By this approach, in order to characterize $\mathcal{A}_q^\alpha(X,(X_n))$ for certain X_n, one has to find a suitable space Y for which the above Jackson and Bernstein inequalities hold, as well as a characterization of the interpolation spaces in (2.20) in terms of function spaces and smoothness scales.

In the context of linear approximation, we are interested in the characterization of the approximation classes $\mathcal{A}_q^\alpha(X,(X_n))$ related to multiresolution spaces $(V_j)_{j\geq 0}$. That is, the approximation spaces X_n are dilates of a shift-invariant space which is generated by a compactly supported function, where the underlying discretization steps or mesh size $h = n^{-1/d}$ tends *uniformly* to zero. For instance,

$$X_n = V_j, \quad j \geq 0, \quad \text{with} \quad n = \dim(V_j) \asymp 2^{jd}$$

for the multiresolution spaces described in Section 2.3.1.

The proofs of the following norm equivalences corresponding to linear approximation can be found in Cohen [28, Theorem 3.6.1, Corollary 3.6.1].

Theorem 2.18. *Let $1 \leq p, q \leq \infty$. Let $\varphi \in L_p(\mathbb{R}^d)$ and $\widetilde{\varphi} \in L_{p'}(\mathbb{R}^d)$ with $p' := p/(p-1)$ if $p \in (1,\infty)$ and $p' := \infty$, $1/p' := 0$ if $p = 1$. Suppose $\varphi \in B_{q_1}^{s_1}(L_p(\mathbb{R}^d))$ for some $s_1 > 0$ and $1 \leq q_1 \leq \infty$. Then*

$$\mathcal{A}_q^s(L_p(\mathbb{R}^d),(V_j)) = B_q^s(L_p(\mathbb{R}^d)), \quad 0 < s < \min\{n, s_1\},$$

in the sense of equivalent norms, where $n-1$ is the order of polynomial reproduction in the V_j spaces. Furthermore, let $1 \leq q_2 \leq \infty$, then

$$\mathcal{A}_q^{s-\nu}(B_{q_2}^\nu(L_p(\mathbb{R}^d)),(V_j)) = B_q^s(L_p(\mathbb{R}^d)), \quad 0 < \nu < s < \min\{n, s_1\},$$

in the sense of equivalent norms.

Remark 2.19. For the approximation properties of the multiresolution spaces V_j considered in Section 2.3.1, which are traced back to the smoothness of φ and polynomial reproduction properties, we refer to Cohen [28, Sections 2.7, 2.8]. In particular, the projectors $P_j f$, $j \in \mathbb{Z}$, defined in (2.10), are uniformly stable approximations to f in V_j providing an optimal error estimate since $\|f - P_j f\|_{L_2(\mathbb{R}^d)} \asymp \inf_{g\in V_j} \|f-g\|_{L_2(\mathbb{R}^d)}$.

Remark 2.20. The adaption of Theorem 2.18, and of Theorem 2.21, to bounded domains and to boundary conditions is discussed in Cohen [28, Sections 3.9, 3.10].

Given a basis decomposition of a function, e.g., (2.11), one can ask for the approximation properties of approximations which are only allowed to have up to $N \in \mathbb{N}_0$ terms in their decomposition. Therefore, in the context of wavelet decompositions let

$$S_N := \left\{ \sum_{\lambda\in\Lambda\subset\nabla} c_\lambda \psi_\lambda \; : \; |\Lambda| \leq N \right\}, \quad N \in \mathbb{N}_0,$$

be the corresponding N-term approximation spaces, i.e., $X_n = S_N$. Note that $(S_N)_{N\in\mathbb{N}}$ is a sequence of nonlinear approximation spaces, since in particular $S_{N_1} + S_{N_2} = S_{N_1+N_2}$.

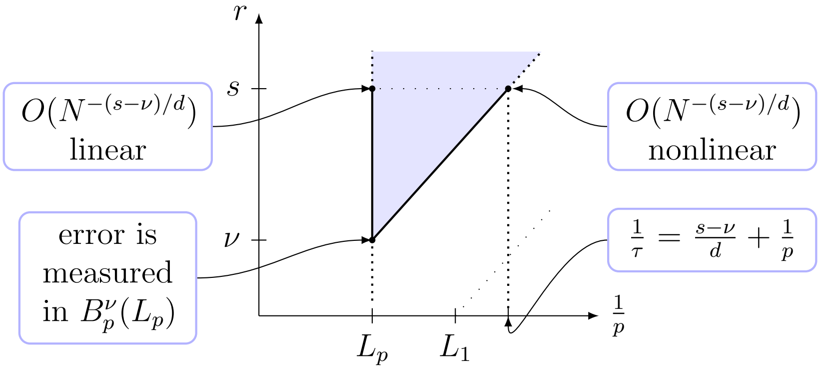


Figure 2.2: Linear and nonlinear wavelet approximation

One option to obtain a N-term approximation f_N to $f = \sum_{\lambda\in\nabla} c_\lambda\psi_\lambda$ is *thresholding*, i.e., the choice

$$f_N := \sum_{\lambda\in\Lambda_N(f,X)} c_\lambda\psi_\lambda,$$

where $\Lambda_N(f,X) \subset \nabla$ is a set containing N indices referring to contributions $\|c_\lambda\psi_\lambda\|_X$ that are largest in the decomposition of f. A remarkable property of wavelet bases is that f_N is an asymptotically optimal approximation to f in S_N, i.e.,

$$\|f - f_N\|_X \preceq E_N(f,(S_N))_X,$$

not just for $X = L_2(\mathbb{R}^d)$, but also for certain Besov spaces and $L_p(\mathbb{R}^d)$, $1 < p < \infty$, cf. Cohen [28, Section 4.1]. The proofs of the following norm equivalences for *best N-term wavelet approximation* can be found in Cohen [28, Theorems 4.2.2, 4.3.3].

Theorem 2.21. *Let $0 < p < \infty$. Suppose $B^s_\tau(L_\tau(\mathbb{R}^d))$ with*

$$\frac{1}{\tau} = \frac{s-\nu}{d} + \frac{1}{p} \tag{2.21}$$

admits a wavelet characterization of the type (2.17) *within $\nu \le s \le s_1$ for some $s_1 > 0$, $s_1 > \nu$. Then*

$$\mathcal{A}^{s-\nu}_\tau(B^\nu_p(L_p(\mathbb{R}^d)),(S_N)) = B^s_\tau(L_\tau(\mathbb{R}^d))$$

in the sense of equivalent (quasi-)norms. Furthermore, for $1 < p < \infty$ and $\nu = 0$, we have

$$\mathcal{A}^s_\tau(L_p(\mathbb{R}^d),(S_N)) = B^s_\tau(L_\tau(\mathbb{R}^d))$$

in the sense of equivalent (quasi-)norms.

Figure 2.2 illustrates linear approximation and N-term wavelet approximation in a DeVore-Triebel diagram. The vertical line, representing Besov spaces $B^{s_0}_q(L_p(\mathbb{R}^d))$ within $\nu \le s_0 \le s$, refers to linear approximation in $B^\nu_{q_2}(L_p(\mathbb{R}^d))$ due to Theorem 2.18. Therefore, $1/p \mapsto s$ is called *linear approximation line*. The line with slope d is generally called *nonlinear approximation line*. It represents the Besov spaces $B^s_\tau(L_\tau(\mathbb{R}^d))$, in the scale (2.21) and, due to Theorem 2.21, it refers to best N-term wavelet approximation in $B^\nu_p(L_p(\mathbb{R}^d))$. The shaded area indicates that embeddings into $B^\nu_p(L_p(\mathbb{R}^d))$ hold, cf. Theorem 2.6. Note that for bounded domains of the type considered below, Section 2.3.3, the situation is the same.

Remark 2.22. Theorem 2.18 essentially states that the approximation order which can be achieved by uniform linear schemes depends on the regularity of the object under consideration in the same scale of smoothness spaces, while according to Theorem 2.21 the approximation order which can be achieved by nonlinear approximation depends on the regularity in the scale (2.21) of Besov spaces.

Remark 2.23. On account that thresholding yields an asymptotically optimal approximation in S_N, the scale (2.21) relates to the *sparsity* of the underlying wavelet decomposition, i.e., the $B^\nu_p(L_p(\mathbb{R}^d))$ norm is concentrated on a small number of wavelet coefficients. If a function has isolated singularities it usually has a sparse wavelet decomposition, due to the polynomial reproduction properties of the multiscale spaces. Therefore, also its smoothness in the scale (2.21) is usually higher than on $1/p \mapsto s$, cf. Cohen [28, Chapter 4].

2.3.3 Assumptions on the underlying wavelet basis

We assume that the domain $\mathcal{O} \subseteq \mathbb{R}^d$ under consideration enables us to construct compactly supported dual wavelet bases $\Psi = \{\psi_\lambda : \lambda \in \nabla\}$ and $\widetilde{\Psi} = \{\widetilde{\psi}_\lambda : \lambda \in \nabla\}$, where

$$\lambda = (j,k), \quad \nabla = \{(j,k) : j \geq 0,\ k \in \nabla_j\}, \quad \text{and} \quad |\lambda| = j,$$

for $L_2(\mathcal{O})$ with the following properties:

(W1) Ψ is a *Riesz basis* for $L_2(\mathcal{O})$.

(W2) The wavelets are *local* in the sense that

$$\operatorname{diam}(\operatorname{supp}\psi_\lambda) \asymp 2^{-|\lambda|}, \quad \psi_\lambda \in \Psi.$$

(W3) The *cardinalities* of the index sets ∇_j satisfy

$$\#\nabla_j \asymp 2^{jd}, \quad j \in \mathbb{N}_0.$$

(W4) The wavelets satisfy the *cancellation property*

$$|\langle v, \psi_\lambda\rangle_{L_2(\mathcal{O})}| \preceq 2^{-|\lambda|(\frac{d}{2}+\bar{m})}|v|_{W^{\bar{m}}(L_\infty(\operatorname{supp}\psi_\lambda))}, \quad \psi_\lambda \in \Psi,\ v \in L_2(\mathcal{O}),$$

for $|\lambda| > 0$ with some parameter $\bar{m} = \bar{m}(\Psi) \in \mathbb{N}$.

(W5) The *dual wavelet basis* $\widetilde{\Psi} = \{\widetilde{\psi}_\lambda : \lambda \in \nabla\}$ to Ψ also fulfills (W1)–(W4) and is *biorthogonal* to Ψ, i.e.,

$$\langle \psi_\lambda, \widetilde{\psi}_{\lambda'}\rangle_{L_2(\mathcal{O})} = \delta_{\lambda,\lambda'}, \quad \psi_\lambda \in \Psi,\ \widetilde{\psi}_\lambda \in \widetilde{\Psi}.$$

(W6) The biorthogonal wavelet bases Ψ, $\widetilde{\Psi}$ induce a *characterization* of the Besov spaces $B^s_q(L_p(\mathcal{O}))$ — within $-s_2 < s < s_1$, where $s_1, s_2 > 0$ are bounds which are determined by the smoothness and the approximation properties of Ψ and $\widetilde{\Psi}$ — of the form

$$\|\cdot\|_{B^s_q(L_p(\mathcal{O}))} \asymp \left(\sum_{j=0}^{\infty} 2^{jq(s+d(\frac{1}{2}-\frac{1}{p}))}\left(\sum_{k\in\nabla_j} |\langle\,\cdot\,, \widetilde{\psi}_{j,k}\rangle_{L_2(\mathcal{O})}|^p\right)^{q/p}\right)^{1/q}$$

for $d(1/p-1)_+ < s < s_1$. For $-s_2 < s < 0$ and $1 < p, q < \infty$ it is of the form

$$\|\cdot\|_{B^s_q(L_p(\mathcal{O}))} \asymp \left(\sum_{j=0}^{\infty} 2^{jq(s+d(\frac{1}{2}-\frac{1}{p}))} \left(\sum_{k \in \nabla_j} |\langle \cdot , \psi_{j,k} \rangle_{L_2(\mathcal{O})}|^p \right)^{q/p} \right)^{1/q}.$$

Remark 2.24. Suitable constructions of wavelets on domains satisfying (W1)–(W6) can be found in, e.g., Canuto et al. [17, 18], Cohen et al. [33], Dahmen, Schneider [58, 59, 60], and Primbs [136]. In general, we refer to Cohen [28] for a detailed discussion.

Remark 2.25. Properties (W1) and (W2) imply numerical stability in a sense that a change of a coefficient in a decomposition only induces a local change of the same order in the underlying function. The finiteness of the index sets on each level in (W3) is a beneficial consequence of (W2) and the form is chosen according to the scaling of the underlying multiresolution analysis. Property (W4) is a Jackson type inequality and induces the approximation capability of the wavelets. Since the construction of orthogonal wavelet bases is not necessary in our context, we only assume the more general property (W5). Lastly, (W6) requires a Bernstein type inequality to hold and therefore complements the set of properties required to characterize approximation spaces by smoothness.

Remark 2.26. In the previous section, Theorem 2.21 follows by proving Jackson- and Bernstein type inequalities together with a characterization of the corresponding interpolation spaces in terms of Besov spaces. Suitable Jackson and Bernstein inequalities can be shown by exploiting the existence of wavelet bases that provide equivalent weighted sequence norms for the Besov spaces in terms of wavelet coefficients. In our analysis below and under the assumptions (W1)–(W6), we are in particular going to use that $f \in B^s_\tau(L_\tau(\mathcal{O}))$ in the scale (2.21) implies the validity of the Jackson inequality

$$E_N(f, (S_N))_{B^\nu_p(L_p(\mathcal{O}))} \preceq \|f\|_{B^s_\tau(L_\tau(\mathcal{O}))} \, N^{-(s-\nu)/d} \tag{2.22}$$

with a constant that does not depend on f or N. Error estimates of the form (2.22) in Sobolev spaces trace back to DeVore et al. [66]. In particular for approximation in $H^\nu(\mathcal{O})$, the estimate (2.22) with $p = 2$ can be derived from DeVore [65, Section 7.7]. In the context of elliptic problems we also refer to Dahlke et al. [45, 53], Hackbusch [94] for further information. Results of this type also hold for negative ν, see, e.g., Dahlke et al. [54] and the discussion in Cohen [28, Section 3.8].

Remark 2.27. In the context of elliptic boundary value problems, cf. Chapter 4, Sobolev and Besov spaces involving boundary conditions come into play. The most prominent example is the Sobolev space $H^1_0(\mathcal{O})$ which is used to describe Dirichlet boundary conditions for second order elliptic differential operators. In this case, the dual space is slightly differently defined as $H^{-1}(\mathcal{O}) := (H^1_0(\mathcal{O}))^*$, cf. (2.6). In many cases, it is possible to find a boundary adapted wavelet basis that characterizes $H^1_0(\mathcal{O})$ in the sense of (W6), while the dual basis gives rise to similar norm equivalences for the dual spaces, see, e.g., Dahmen, Schneider [60, Theorem 3.4.3]. Moreover, these biorthogonal wavelet bases very often also exist for much more general boundary conditions. Again, we refer to Dahmen, Schneider [60] for further information. Once such a wavelet basis is available, the analysis that depends on (W6) can also be generalized to this case.

2.3.4 The anisotropic wavelet setting

In order to characterize a given anisotropic Besov space $B_q^{s,\mathbf{a}}(L_p(\mathbb{R}^d))$, cf. Section 2.2.2, the anisotropy has to be built into the wavelet basis. Therefore, we employ suitable $\mathbf{M}$-scaling functions $\phi : \mathbb{R}^d \to \mathbb{R}$, which satisfy

$$\phi(\,\cdot\,) = |\det(\mathbf{M})|^{1/2} \sum_{k\in\mathbb{Z}^d} h_k\, \phi(\mathbf{M}\,\cdot\, - k) \tag{2.23}$$

with a finite number of non-zero coefficients $h_k \in \mathbb{R}$. Here, $\mathbf{M}$ is an anisotropic integer scaling matrix of the form

$$\mathbf{M} := \operatorname{diag}(\lambda^{1/a_1}, \ldots, \lambda^{1/a_d}), \quad \text{for some} \quad \lambda > 1, \tag{2.24}$$

where $\boldsymbol{a} = (a_1, \ldots, a_d) \in \mathbb{R}_+^d$ is the given anisotropy, cf. (2.8). Since $\sum_{i=1}^d \frac{1}{a_i} = d$, we have

$$\lambda^d = |\det(\mathbf{M})| =: m.$$

We assume to have an $\mathbf{M}$-scaling function ϕ at hand, which satisfies the following properties:

(M1) $\phi \in H^s(\mathbb{R}^d)$ for $s > d/2$,

(M2) ϕ is compactly supported and $\int_{\mathbb{R}^d} \phi(x)\,\mathrm{d}x = 1$,

(M3) ϕ is a refinable function in the sense of (2.23),

(M4) $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}^d}$ is a Riesz basis of the space it spans,

(M5) $\phi \in B_q^{s_0,\mathbf{a}}(L_p(\mathbb{R})) \cap H^{L,\mathbf{a}}(\mathbb{R}^d)$ for some $s_0 > 0$ and $\mathbb{N} \ni L > d/2$.

(M6) There exists an $\mathbf{M}$-scaling function $\widetilde{\phi}$, which also satisfies (M1)–(M5) with potentially different constants, that is biorthogonal to ϕ, i.e., $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}^d}$ and $\{\widetilde{\phi}(\cdot - k)\}_{k\in\mathbb{Z}^d}$ satisfy

$$\int_{\mathbb{R}^d} \phi(x)\widetilde{\phi}(x-k)\,\mathrm{d}x = \delta_{0,k} \quad \text{for all } k \in \mathbb{Z}^d.$$

Remark 2.28. The existence of nontrivial scaling functions satisfying (M1)–(M6) is of course not obvious, nevertheless a lot of examples exist. We refer to Garrigós, Tabacco [83], Hochmuth [101] for a detailed discussion. Moreover, matrices of the form (2.24) are the only ones compatible with the anisotropy for the type of multiscale decomposition we wish to apply. For instance, it is necessary that $\mathbf{M}$ is integer-valued in order for (M2) to hold, see Auscher [4]. Again, we refer to Garrigós, Tabacco [83, Section 3.1-3.3] for a detailed discussion.

Remark 2.29. By the Sobolev embedding theorem (M1) and (M2) imply ϕ and $\widetilde{\phi}$ to be continuous and that they are contained in $L_p(\mathbb{R}^d)$ for all $1 \le p \le \infty$. Therefore, with $1 \le p < \infty$ and $1/p + 1/p' = 1$, where $p' = \infty$, if $p = 1$, we have, in particular, $\phi \in L_p(\mathbb{R}^d)$ and $\widetilde{\phi} \in L_{p'}(\mathbb{R}^d)$. Assumption (M3) is necessary for ϕ to even qualify as generator function for a multiresolution analysis, while (M4) ensures that a wavelet basis constructed with these scaling functions really characterizes anisotropic Besov spaces, see Cohen et al. [36, Theorem 2.1, Proposition 2.1] and Garrigós, Tabacco [83, Theorem 1.2, Proposition 3.3]. The anisotropic smoothness of ϕ and $\widetilde{\phi}$ in (M5) directly affects the range of smoothness that can be characterized later, cf. Theorem 2.31 below. Lastly, to only consider orthogonal wavelet bases turns out to be a too strong restriction, therefore the more general case (M6) leading to biorthogonal wavelet bases is considered.

Let ϕ be an $\mathbf{M}$-scaling function satisfying (M1)–(M6), and let $1 \le p < \infty$ and $1/p + 1/p' = 1$, where $p' = \infty$, if $p = 1$. We set

$$\langle f_1, f_2 \rangle := \int_{\mathbb{R}^d} f_1(x) f_2(x) \,\mathrm{d}x, \quad f_1 \in L_p(\mathbb{R}^d),\ f_2 \in L_{p'}(\mathbb{R}^d).$$

Since, in particular, $\phi, \widetilde{\phi} \in L_c^\infty(\mathbb{R}^d)$ they generate multiresolution analyses $(V_j^{(p)})_{j\in\mathbb{Z}}$ and $(\widetilde{V}_j^{(p')})_{j\in\mathbb{Z}}$ by

$$V_j^{(p)} := \operatorname*{clos\,span}_{L_p(\mathbb{R}^d)} \left\{ \phi_{j,k}^{(p)} := |\det(\mathbf{M})|^{j/p} \phi(\mathbf{M}^j \cdot - k) \ :\ k \in \mathbb{Z}^d \right\}, \quad j \in \mathbb{Z},$$

and

$$\widetilde{V}_j^{(p')} := \operatorname*{clos\,span}_{L_{p'}(\mathbb{R}^d)} \left\{ \widetilde{\phi}_{j,k}^{(p')} := |\det(\mathbf{M})|^{j/p'} \widetilde{\phi}(\mathbf{M}^j \cdot - k) \ :\ k \in \mathbb{Z}^d \right\}, \quad j \in \mathbb{Z},$$

see Garrigós, Tabacco [83, Section 3.1]. Moreover, $(V_j^{(p)})_{j\in\mathbb{Z}}$ and $(\widetilde{V}_j^{(p')})_{j\in\mathbb{Z}}$ are biorthogonal, that is

$$\langle \phi_{j,k}^{(p)}, \widetilde{\phi}_{j,k'}^{(p')} \rangle = \delta_{k,k'}, \quad j \in \mathbb{Z}.$$

Remark 2.30. Here, a multiresolution analysis $(V_j^{(q)})_{j\in\mathbb{Z}}$, $1 \le q \le \infty$, is a sequence of closed linear subspaces of $L_q(\mathbb{R}^d)$ with the following properties: The spaces are nested, i.e., $V_j^{(q)} \subset V_{j+1}^{(q)}$ for all $j \in \mathbb{Z}$. We have $\bigcup_{j\in\mathbb{Z}} V_j^{(q)}$ is dense in $L_q(\mathbb{R}^d)$ and $\bigcap_{j\in\mathbb{Z}} V_j^{(q)} = \{0\}$. There exists a function $\phi \in V_0^{(q)}$ such that $\{\phi(\,\cdot - k)\}_{k\in\mathbb{Z}^d}$ is a q-stable basis of $V_0^{(q)}$, i.e., $\operatorname{span}\{\phi(\,\cdot\, - k) \,:\, k \in \mathbb{Z}^d\} = V_0^{(q)}$ and

$$\Big\| \sum_{k\in\mathbb{Z}^d} a_k\, \phi(\,\cdot\, - k) \Big\|_{L_q(\mathbb{R}^d)} \asymp \Big(\sum_{k\in\mathbb{Z}^d} |a_k|^q \Big)^{1/q}.$$

Finally, we have $f(\,\cdot\,) \in V_j^{(q)}$ if and only if $f(\mathbf{M}\,\cdot\,) \in V_{j+1}^{(q)}$ for all $j \in \mathbb{Z}$. Note, in the case $q = \infty$ the space $L_\infty(\mathbb{R}^d)$ is replaced by $C_0(\mathbb{R}^d)$, the space of continuous functions with compact support, and $\ell_\infty(\mathbb{Z}^d)$ is replaced by $c_0(\mathbb{Z}^d)$, the space of sequences converging to zero.

Wavelets come into play as basis functions for the complement spaces $W_j^{(p)}$ and $\widetilde{W}_j^{(p')}$ to the multiresolution analyses, which are defined by

$$W_j^{(p)} := \Big\{ f \in V_{j+1}^{(p)} \ :\ \langle f, \widetilde{f} \rangle = 0 \ \text{ for all } \widetilde{f} \in \widetilde{V}_j^{(p')} \Big\}, \quad j \in \mathbb{Z},$$

and $\widetilde{W}_j^{(p')}$ is defined analogously. In this way, one obtains a decomposition of the form

$$L_p(\mathbb{R}^d) = V_0^{(p)} \oplus \left(\bigoplus_{j=0}^{\infty} W_j^{(p)} \right), \quad 1 \le p < \infty.$$

Now, the remarkable result is, that there exist families of compactly supported functions $\{\psi_1, \dots, \psi_{m-1}\}$ and $\{\widetilde{\psi}_1, \dots, \widetilde{\psi}_{m-1}\}$, $m = |\det(\mathbf{M})|$, such that

$$\left\{\psi_{e,j,k}^{(p)}\right\}_{e,j,k} := \left\{\psi_e(\mathbf{M}^j \cdot - k) \,:\, e = 1, ..., m-1,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d\right\}, \tag{2.25}$$

and analogously for $\{\widetilde{\psi}_{e,j,k}^{(p')}\}_{e,j,k}$, are uniformly p, p'-stable bases of $W_j^{(p)}$ and $\widetilde{W}_j^{(p')}$, respectively, see Garrigós, Tabacco [83, Section 3.2] for details. Accordingly, a pair of dual anisotropic wavelet bases for $L_p(\mathbb{R}^d)$, $1 \le p < \infty$, are given by

$$\left\{\phi_{0,k}^{(p)} \,:\, k \in \mathbb{Z}^d\right\} \cup \left\{\psi_{e,j,k}^{(p)} \,:\, e = 1, \dots, m-1,\ j \in \mathbb{N}_0,\ k \in \mathbb{Z}^d\right\}$$

and

$$\left\{\widetilde{\phi}_{0,k}^{(p')} \,:\, k \in \mathbb{Z}^d\right\} \cup \left\{\widetilde{\psi}_{e,j,k}^{(p')} \,:\, e = 1, \dots, m-1,\ j \in \mathbb{N}_0,\ k \in \mathbb{Z}^d\right\},$$

which are biorthogonal, that is $\langle \psi_{e,j,k}^{(p)}, \widetilde{\psi}_{e,j',k'}^{(p')} \rangle = \delta_{k,k'}\delta_{j,j'}$. In particular, we obtain the anisotropic multiscale wavelet decomposition

$$f = \sum_{k \in \mathbb{Z}^d} \langle f, \widetilde{\phi}_{0,k}^{(p')} \rangle\, \phi_{0,k}^{(p)} + \sum_{j=0}^{\infty} \sum_{e=1}^{m-1} \sum_{k \in \mathbb{Z}^d} \langle f, \widetilde{\psi}_{e,j,k}^{(p')} \rangle\, \psi_{e,j,k}^{(p)}, \quad f \in L_p(\mathbb{R}^d).$$

The following theorem states the wavelet characterization of the anisotropic Besov space $B_q^{s,\mathbf{a}}(L_p(\mathbb{R}^d))$. Part 1 is cited from Garrigós, Tabacco [83, Theorem 5.3] and considers the case $1 \le p, q < \infty$, while Part 2 considers the quasi-Banach case and is taken from Garrigós et al. [82, Theorem 6.2]. Since the biorthogonal setting described above is restricted to L_p-spaces with $p \ge 1$, only anisotropic quasi-Banach spaces which are embedded in some L_p-space with $p \ge 1$ can be characterized in this way. In Garrigós et al. [82], it has been shown that $B_\tau^{s,\mathbf{a}}(L_\tau(\mathbb{R}^d)) \hookrightarrow L_p(\mathbb{R}^d)$ for every finite p such that $\tau \le p \le p(s, \tau)$, where

$$p(s,\tau) := \begin{cases} (1/\tau - s/d)^{-1} & : s < d/\tau, \\ \infty & : \text{otherwise}. \end{cases} \tag{2.26}$$

This explains the restrictions in Part 2.

Theorem 2.31. Part 1 [83, Theorem 5.3]. *Suppose (M1)–(M6) are satisfied and let* $1 \le p, q < \infty$. *If* $0 < s < \min\{s_0,\ L/a_1, \dots, L/a_d\}$, *then*

$$B_q^{s,\mathbf{a}}(L_p(\mathbb{R}^d)) = \left\{ f \in L_p(\mathbb{R}^d) \,:\, \sum_{k \in \mathbb{Z}^d} |\langle f, \widetilde{\phi}_{0,k}^{(p')} \rangle|^p + \sum_{j=0}^{\infty} m^{\frac{sq}{d}j} \left(\sum_{e=1}^{m-1} \sum_{k \in \mathbb{Z}^d} |\langle f, \widetilde{\psi}_{e,j,k}^{(p')} \rangle|^p \right)^{q/p} < \infty \right\}.$$

Moreover, the following norm equivalence holds:

$$\|\cdot\|_{B_q^{s,\mathbf{a}}(L_p(\mathbb{R}^d))} \asymp \left(\sum_{k \in \mathbb{Z}^d} |\langle \cdot\,, \widetilde{\phi}_{0,k}^{(p')} \rangle|^p \right)^{1/p} + \left(\sum_{j=0}^{\infty} m^{\frac{sq}{d}j} \left(\sum_{e=1}^{m-1} \sum_{k \in \mathbb{Z}^d} |\langle \cdot\,, \widetilde{\psi}_{e,j,k}^{(p')} \rangle|^p \right)^{q/p} \right)^{1/q}. \tag{2.27}$$

Part 2 [82]. *Suppose (M1)–(M6) are satisfied. Let* $0 < \tau < 1$, $d(1/\tau - 1)_+ < s < \min\{s_0, L/a_1, \dots, L/a_d\}$, *and* $p(s,\tau)$ *be defined by* (2.26). *Then, for any* $1 < p < \infty$ *with* $p \le p(s,\tau)$ *we have* $B^{s,\mathbf{a}}_\tau(L_\tau(\mathbb{R}^d)) \hookrightarrow L_p(\mathbb{R}^d)$ *and*

$$B^{s,\mathbf{a}}_\tau(L_\tau(\mathbb{R}^d)) = \left\{ f \in L_p(\mathbb{R}^d) \ : \ \sum_{k\in\mathbb{Z}^d} |\langle f, \widetilde{\phi}^{(p')}_{0,k}\rangle|^\tau + \sum_{j=0}^\infty m^{\frac{s\tau}{d}j} \sum_{e=1}^{m-1} \sum_{k\in\mathbb{Z}^d} |\langle f, \widetilde{\psi}^{(p')}_{e,j,k}\rangle|^\tau < \infty \right\}.$$

Moreover, the following quasi-norm equivalence holds:

$$\| \cdot \|_{B^{s,\mathbf{a}}_\tau(L_\tau(\mathbb{R}^d))} \asymp \left(\sum_{k\in\mathbb{Z}^d} |\langle \cdot\,, \widetilde{\phi}^{(p')}_{0,k}\rangle|^p \right)^{1/\tau} + \left(\sum_{j=0}^\infty m^{\frac{s\tau}{d}j} \sum_{e=1}^{m-1} \sum_{k\in\mathbb{Z}^d} |\langle \cdot\,, \widetilde{\psi}^{(p')}_{e,j,k}\rangle|^\tau \right)^{1/\tau}. \quad (2.28)$$

Remark 2.32. In TRIEBEL [161] a wavelet characterization for the anisotropic Besov spaces in the general case $0 < p, q < \infty$, $s \in \mathbb{R}$ is discussed, where orthogonal wavelet bases, e.g., Daubechies type wavelets, cf. DAUBECHIES [62], are employed. The (more general) biorthogonal case is not considered.

In Chapter 3 it is our goal to construct random functions on bounded (Lipschitz) domains $\mathcal{O} \subset \mathbb{R}^d$ with the aid of wavelet characterizations of Besov spaces. The analysis in Section 3.1 for the isotropic case is particularly designed for bounded domains and the cardinality of ∇_j, cf. (W3), is going to play a central role. In Section 3.2 we aim to construct random functions on bounded domains taking values in anisotropic Besov spaces with the aid of the wavelet characterization as outlined in Theorem 2.31. However, here we are facing a nontrivial problem. As it is, Theorem 2.31 is concerned with function spaces on the whole Euclidean d-plane, and a generalization to bounded domains is at least not obvious, since this would require the construction of specific boundary-adapted anisotropic wavelet bases on domains. To our best knowledge no result in this direction has been reported so far. Still, in order to obtain an analogous assumption to (W3) for the anisotropic case, we proceed in the following way:

Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded Lipschitz domain, then there exists a cube $\square \supset \mathcal{O}$ such that

$$\phi^{(p)}_{0,k} \cap \mathcal{O} \neq \emptyset \quad \text{and} \quad \psi^{(p)}_{e,j,k} \cap \mathcal{O} \neq \emptyset$$

implies $\operatorname{supp} \phi^{(p)}_{0,k} \subset\subset \square$ and $\operatorname{supp} \psi^{(p)}_{e,j,k} \subset\subset \square$. Based on this cube $\square$ we define the sets

$$\begin{aligned} \nabla_0 &:= \{k \in \mathbb{Z}^d \ : \ \operatorname{supp} \phi^{(p)}_{0,k} \subset\subset \square\}, \\ \nabla_{e,j} &:= \{k \in \mathbf{M}^{-j}\mathbb{Z}^d \ : \ \operatorname{supp} \psi_{e,j,k} \subset\subset \square\} \end{aligned} \quad (2.29)$$

for $e = 1, \dots, m-1$ and $j \in \mathbb{N}_0$.

(M7) We assume to have compactly-supported wavelets at hand such that ∇_0 is a finite set and the sets $\nabla_{e,j}$ in (2.29) fulfill

$$\#\nabla_{e,j} \asymp m^j = |\det(\mathbf{M})|^j.$$

Remark 2.33. Assumption (M7) on the cardinality of $\nabla_{e,j}$ slightly varies from (W3) for the isotropic setting. It is motivated by the following observation. If we start with a tensor wavelet construction on the whole Euclidean plane, then, since $\mathbf{M}$ is a diagonal matrix, the relevant grid points can be computed coordinate-wise and are of the order λ^{j/a_i}, $i = 1, \dots, d$. By observing $\prod_{i=1}^d \lambda^{j/a_i} = |\det(\mathbf{M})|^j$ (M7) is satisfied in this case.

2.3.5 The tensor wavelet characterization

In the spirit of the previous sections, we employ a wavelet characterization of the space under consideration. Here, note that, the domain $\mathcal{O} \subset \mathbb{R}^d$, $d > 1$ is assumed to be an n-fold product of component domains $\mathcal{O}_m \subset \mathbb{R}^{d_m}$, $m = 1, \ldots, n$, $n \geq 2$, with $\sum_{m=1}^n d_m = d$.

(T1) We assume that all domains $\mathcal{O}_m$, $m = 1, \ldots, n$, allow the construction of a wavelet basis $(\psi^{(m)}_{\lambda_m})_{\lambda_m \in \Lambda_m}$, which is sufficiently smooth and has sufficiently many vanishing moments, such that for all $\ell' \in [t_m, t_m + \ell]$ the scaled wavelets

$$\{2^{-|\lambda_m|\ell'} \psi^{(m)}_{\lambda_m} \, : \, \lambda_m \in \Lambda_m\}$$

are Riesz bases for $H^{\ell'}(\mathcal{O}_m)$.

Similar to the previous sections, the wavelet indices $\lambda_m \in \Lambda_m$, $m = 1, \ldots, n$, are of the form $\lambda_m = (j_m, k_m)$, where $|\lambda_m| = j_m \in \mathbb{N}_0$ is the scale of the wavelet and $k_m \in \nabla_{j_m}$ encodes the shift and type of the wavelet. The set ∇_{j_m} is finite, if the wavelets are compactly supported. Note that, in case of the domain being an n-fold with $n < d$ the respective k_m are then vectors of dimension $d_m > 1$. For the remaining part, we impose the following assumption which is the natural generalization of (W3):

(T2) We assume to have suitable compactly supported wavelets at hand for which

$$\#\nabla_{j_m} \asymp 2^{j_m d_m}.$$

A *tensor wavelet basis* on the domain $\mathcal{O}$ is defined as the collection of all functions $(\psi_{\boldsymbol{\lambda}})_{\boldsymbol{\lambda} \in \boldsymbol{\Lambda}}$ of the form

$$\psi_{\boldsymbol{\lambda}} := \bigotimes_{m=1}^{n} \psi^{(m)}_{\lambda_m}$$

with $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_n)$, $|\boldsymbol{\lambda}| := (|\lambda_1|, \ldots, |\lambda_n|)$, and $\boldsymbol{\Lambda} := \prod_{m=1}^n \Lambda_m$.

Remark 2.34. In comparison to standard isotropic wavelets, tensor wavelets are in a certain sense the wavelet version of the sparse grid approach, see, e.g., Bungartz, Griebel [16] for a detailed discussion on sparse grids. The application of (adaptive) tensor product approximation schemes gives rise to convergence rates that only depend on the component domains. We refer to, e.g., Schwab, Stevenson [146] and the references therein for further information. In particular, if all the component domains are one-dimensional one obtains dimensional independent convergence rates.

The wavelet characterization for $H^{\boldsymbol{t},\ell}(\mathcal{O})$ is now given by the following statement, which has been shown in Griebel, Knapek [85], Griebel, Oswald [86].

Theorem 2.35 [85, 86]**.** *Suppose (T1) is satisfied. Then*

$$\left\{2^{-\boldsymbol{t}|\boldsymbol{\lambda}| - \ell \| |\boldsymbol{\lambda}| \|_\infty} \psi_{\boldsymbol{\lambda}} \, : \, \boldsymbol{\lambda} \in \boldsymbol{\Lambda}\right\}$$

is a Riesz basis for $H^{\boldsymbol{t},\ell}(\mathcal{O})$. In particular, for every $f = \sum_{\boldsymbol{\lambda} \in \boldsymbol{\Lambda}} a_{\boldsymbol{\lambda}}(f)\, \psi_{\boldsymbol{\lambda}}$ the following two statements are equivalent:

i) $f \in H^{\boldsymbol{t},\ell}(\mathcal{O})$, *ii)* $\sum_{\boldsymbol{\lambda}\in\boldsymbol{\Lambda}} |a_{\boldsymbol{\lambda}}(f)|^2\, 2^{2\boldsymbol{t}|\boldsymbol{\lambda}|+2\ell\|\boldsymbol{\lambda}\|_\infty} < \infty.$

Remark 2.36. The proof of Theorem 2.35 requires, for every coordinate direction, a set of sufficiently smooth wavelets $\{\psi^{(m)}_{\lambda_m} : \lambda_m \in \Lambda_m\}$ such that for all ℓ' within a region including the range $[t_m, t_m + \ell]$, the scaled wavelets $\{2^{-|\lambda_m|\ell'}\psi^{(m)}_{\lambda_m} : \lambda_m \in \Lambda_m\}$ are Riesz bases for the spaces $H^{\ell'}(\mathcal{O}_m)$. Suitable (compactly supported) wavelet bases satisfying (T1) and (T2) can, e.g., be found in [17, 33, 58–60, 136], cf. Remark 2.24.

2.4 Adaptive wavelet methods for operator equations

We briefly review the class of adaptive wavelet methods that we have in mind for the spatial numerical discretization in our approach of Rothe's method.

2.4.1 Operator equations in wavelet coordinates

We begin with an explanation of how operator equations are discretized with respect to a given wavelet basis at hand of an elliptic operator equation

$$AU = X, \tag{2.30}$$

where A is a boundedly invertible operator from some Hilbert space V into its normed dual V^*, i.e.,

$$\|Av\|_{V^*} \asymp \|v\|_V, \quad v \in V.$$

Consequently, we are in a Gel'fand triple setting (V, U, V^*), cf. Appendix A.1.3, and consider the case where

$$a(v, w) = \langle -Av, w\rangle_{V^*\times V}$$

is a symmetric, bounded, and elliptic bilinear form on V in the sense of (A.6). In this setting the bilinear form induces a norm on V, the *energy norm*, by

$$\|\cdot\|_a := a(\cdot,\cdot)^{1/2}. \tag{2.31}$$

It is equivalent to the Sobolev norm, i.e.,

$$c_{\text{energy}}\|\cdot\|_{H^\nu(\mathcal{O})} \le \|\cdot\|_a \le C_{\text{energy}}\|\cdot\|_{H^\nu(\mathcal{O})}. \tag{2.32}$$

In case of boundary conditions, i.e., $V = H^1_0(\mathcal{O})$, and whenever we have a wavelet Riesz basis

$$\Psi = \{\psi_\lambda : \lambda \in \nabla\}$$

of $L_2(\mathcal{O})$ fulfilling certain assumptions, e.g., (W2)–(W6) as given in Section 2.3.3, the equation (2.30) has an equivalent reformulation

$$\mathbf{AU} = \mathbf{X}, \tag{2.33}$$

in wavelet coordinates. That is, we define $\mathbf{A} = (\mathbf{A}_{\lambda,\mu})_{\lambda,\mu}$ and $\mathbf{X} = (\mathbf{X}_\lambda)_\lambda$ for all $\mu, \lambda \in \nabla$ by

$$\mathbf{A}_{\lambda,\mu} := 2^{-(|\mu|+|\lambda|)} a(\psi_\mu, \psi_\lambda), \quad \psi_\mu, \psi_\lambda \in \Psi,$$

and

$$\mathbf{X}_\lambda := 2^{-|\lambda|}\langle X, \psi_\lambda\rangle_{H^{-1}(\mathcal{O})\times H_0^1(\mathcal{O})}, \quad \psi_\lambda \in \Psi.$$

Here, equivalent reformulation means the following: For each solution $\mathbf{U} \in \ell_2(\nabla)$ to (2.33), the associated wavelet decomposition

$$U = \sum_{\lambda\in\nabla} 2^{-|\lambda|}\,\mathbf{U}_\lambda \psi_\lambda \in V,$$

solves (2.30). Conversely, the unique decomposition coefficients

$$\mathbf{U}_\lambda = 2^{|\lambda|}\langle \widetilde{\psi}_\lambda, U\rangle_{H^{-1}(\mathcal{O})\times H_0^1(\mathcal{O})},$$

of a solution $U \in V$ to (2.30) satisfy (2.33).

Remark 2.37. In Chapter 4, we analyze elliptic operator equations with random right hand sides defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. In this case, we consider

$$AU(\omega) = X(\omega), \quad \omega \in \Omega,$$

and have in mind the ω-wise application of well-established adaptive numerical wavelet schemes, as, e.g., published in Cohen et al. [29, 30], Dahlke et al. [51].

2.4.2 Adaptive wavelet frame methods

On, e.g., polygonal domains in $\mathbb{R}^2$, wavelet frames are much easier to construct than bases, see Dahlke et al. [50], Stevenson [150]. The definition of a frame, which is a possibly redundant system allowing for stable analysis and synthesis operations, is given in Appendix A.1.1. Fortunately, an equivalence between (2.30) and its discrete formulation (2.33) still holds if the underlying wavelet Riesz basis is replaced by a wavelet frame. In the case of frames, each solution $\mathbf{U} \in \ell_2(\nabla)$ to (2.33) still expands into a solution $U \in V$ to (2.30). However, uniqueness of the decomposition coefficients $\mathbf{U}$ of U can only be expected to hold within the range of $\mathbf{A}$, being a proper subset of $\ell_2(\nabla)$. Note that the chosen wavelet frame is still required to fulfill assumptions as, e.g., (W2)–(W6) of Section 2.3.3, i.e., removing only (W1) completely, see Dahlke et al. [49] for details.

Therefore, we can employ overlapping domain decomposition methods and a class of adaptive wavelet methods that also work with frames. Let $M \in \mathbb{N}$ be a finite number and $\Psi = \bigcup_{i=1}^M \Psi_i$ be a finite union of wavelet bases Ψ_i for $L_2(\mathcal{O}_i)$, subordinate to an overlapping partition of the domain $\mathcal{O}$ into patches $\mathcal{O}_i$, $i = 1, \ldots, M$. The properties (W2)–(W6), given in Section 2.3.3, are assumed to hold for each basis Ψ_i in a patchwise sense. In that case, the main diagonal blocks $\mathbf{A}_{i,i}$ of $\mathbf{A}$ are invertible, so that the following abstract Gauss-Seidel iteration

$$\mathbf{U}^{(n+1)} = \mathbf{U}^{(n)} + \mathbf{R}(\mathbf{X} - \mathbf{A}\mathbf{U}^{(n)}), \quad n = 0, 1, \ldots \tag{2.34}$$

with the preconditioner

$$\mathbf{R} = (\mathbf{A}_{j,j})^{-1}_{j\le i}$$

makes sense. The convergence properties of the iteration (2.34) and, in particular, of fully adaptive variants involving inexact operator evaluations and inexact applications of the preconditioner $\mathbf{R}$ have been analyzed in Stevenson, Werner [154] and Werner [174]. In order to turn the abstract iteration (2.34) into an implementable scheme, all infinite-dimensional quantities have to be replaced by computable ones. Such realizations involve the inexact evaluation of the right-hand side $\mathbf{X}$ and of the various biinfinite matrix-vector products $\mathbf{A}_{i,j}\mathbf{v}$, both enabled by the compression properties of the wavelet system Ψ. We refer to Cohen [28], Cohen et al. [29], Stevenson [150] for details and properties of the corresponding numerical subroutines.

A full convergence and cost analysis of the resulting wavelet frame domain decomposition algorithm is available in case that a particular set of quadrature rules is used in the overlapping regions of the domain decomposition, see Stevenson, Werner [153], and under the assumption that the local subproblems $\mathbf{A}_{i,i}\mathbf{v} = \mathbf{g}$ are solved with a suitable adaptive wavelet scheme, e.g., the wavelet-Galerkin methods from Cohen et al. [29] and Gantumur et al. [81]. We postpone their discussion to Section 5.3.3 below. By the properties of the aforementioned numerical subroutines and the findings of Stevenson, Werner [153, 154], an implementable numerical routine

$$\mathbf{SOLVE}[\varepsilon] \to \mathbf{U}_\varepsilon$$

with the following key properties exists:

- For each $\varepsilon > 0$, **SOLVE** outputs a finitely supported sequence $\mathbf{U}_\varepsilon$ with guaranteed accuracy
$$\|\mathbf{U} - \mathbf{U}_\varepsilon\|_{\ell_2(\nabla)} \leq \varepsilon \quad (\textit{convergence}).$$
- Whenever the best N-term wavelet approximation of U with respect to Ψ converges at a rate $s > 0$ in $H^1(\mathcal{O})$, i.e.,
$$U \in \mathcal{A}^s_\infty(H^1(\mathcal{O}), (S_N)),$$
then the outputs $\mathbf{U}_\varepsilon$ realize the same work/accuracy ratio, as $\varepsilon \to 0$, i.e.,
$$\#\,\mathrm{supp}\,\mathbf{U}_\varepsilon \preceq \varepsilon^{-1/s}\|U\|^{1/s}_{\mathcal{A}^s_\infty(H^1(\mathcal{O}),(S_N))} \quad (\textit{convergence rates}).$$
- The associated computational cost asymptotically scales in the same way,
$$\#\,\mathrm{flops}_\varepsilon \preceq \varepsilon^{-1/s}\|U\|^{1/s}_{\mathcal{A}^s_\infty(H^1(\mathcal{O}),(S_N))} \quad (\textit{linear cost}).$$

Generalizations of these properties towards the average case setting are straightforward. The results given in Section 4.1 provide upper bounds for $\|U\|_{\mathcal{A}^s_\infty(H^1(\mathcal{O}),(S_N))}$ for certain values of s in terms of suitable Besov norms of the right-hand side X, and the latter norms may be chosen to have arbitrarily high moments, cf. Section 3.1.

Remark 2.38. In Section 4.2 we apply domain decomposition methods based on wavelet frames to solve the Poisson equation with random right-hand side and zero-Dirichlet boundary conditions on the L-shaped domain.

Chapter 3

A class of random functions

We analyze the regularity of a class of random functions in certain smoothness spaces and state linear and nonlinear approximation results. The random functions are defined in terms of wavelet decompositions according to a stochastic model that provides an explicit regularity control of their realizations and, in particular, induces sparsity of the wavelet coefficients. Therefore, we expect this stochastic model to be an interesting tool to generate test functions in numerical experiments. See Section 1.3 for the complete introduction.

In Section 3.1 we analyze the Besov regularity of such random functions and state error bounds for linear and nonlinear approximations. In Sections 3.2 and 3.3 we extend the regularity results to anisotropic Besov spaces and tensor wavelet decompositions.

The results of this chapter have been partly worked out by the author and collaborators in [24] and [48].

3.1 A class of random functions in Besov spaces

We analyze the regularity of a class of random functions in $B^s_q(L_p(\mathcal{O}))$ for the range of parameters $0 < p, q < \infty$ and $s > d(1/p - 1)_+$, as well as $1 < p, q < \infty$ and $s < 0$, see Theorem 3.10, on bounded domains $\mathcal{O} \subset \mathbb{R}^d$, $d \geq 1$, which allow biorthogonal wavelet bases Ψ, $\widetilde{\Psi}$ that can characterize the Besov space. See Section 2.2.1 for the definition of $B^s_q(L_p(\mathcal{O}))$ and Section 2.3.3 for the assumptions (W1)–(W6) on the wavelets. Furthermore, we study different approximations with respect to appropriate norms and state error bounds, where we always consider the average error $(\mathrm{E}[\|X - \widehat{X}\|^2])^{1/2}$ for any approximation $\widehat{X}$ of a random function X. In particular, the construction of the average nonlinear approximation of these random functions can be done at an average computational cost of linear order, which is crucial in computational practice. Moreover, we complement the discussion of the stochastic model with illustrating realizations and moments of Besov norms of X.

3.1.1 The stochastic model

Let

$$\alpha, \gamma \in \mathbb{R}, \quad \beta \in [0, 1],$$
$$\rho_j := \min\left\{1,\, C_1 2^{-\beta j d}\right\}, \quad \text{and} \quad \sigma_j^2 := \begin{cases} C_2 j^{\gamma d} 2^{-\alpha j d} & : j > j_0, \\ 1 & : j = 0, \end{cases} \tag{3.1}$$

where $j_0 \in \mathbb{N}_0$ and $C_1, C_2 > 0$. The stochastic model is based on a family of independent random variables $(Y_{j,k},\ Z_{j,k})$ for $j \geq j_0$ and $k \in \nabla_j$ on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$: The $Y_{j,k}$ are Bernoulli distributed with parameter ρ_j and the $Z_{j,k}$ are standard normally distributed, i.e.,

$$Z_{j,k} \sim \mathcal{N}(0,1), \quad \text{and} \quad \mathrm{P}(Y_{j,k} = 1) = 1 - \mathrm{P}(Y_{j,k} = 0) = \rho_j. \tag{3.2}$$

Given biorthogonal wavelet bases Ψ, $\widetilde{\Psi}$ which satisfy the assumptions (W1)–(W6), we define the random functions

$$X := \sum_{j=j_0}^{\infty} \sum_{k \in \nabla_j} \sigma_j\, Y_{j,k}\, Z_{j,k}\, \psi^*_{j,k}, \tag{3.3}$$

where $\{\psi^*_{j,k}\}_{j \geq j_0, k \in \nabla_j}$ is either the basis $\Psi = \{\psi_{j,k}\}_{j,k}$ or $\widetilde{\Psi} = \{\widetilde{\psi}_{j,k}\}_{j,k}$, respectively. Let us emphasize that only within a range $-s_2 < s < s_1$, which depends on the employed wavelet bases, Besov spaces may be characterized by the decay properties of the wavelet coefficients. However, since it is possible to construct wavelet bases for any desired finite range, we are not going to explicitly mention this range further on. The parameters j_0, C_1, and C_2 allow us to scale the stochastic model to the employed wavelet basis, which come in handy for the numerical experiments in Section 3.1.4.

Remark 3.1. Using (W3), i.e., $\#\nabla_j \asymp 2^{jd}$, we have

$$\mathrm{E}\left[\sum_{j=j_0}^{\infty} \sum_{k \in \nabla_j} \sigma_j^2 Y_{j,k}^2 Z_{j,k}^2\right] = \sum_{j=j_0}^{\infty} \#\nabla_j \sigma_j^2 \rho_j \asymp \sum_{j=j_0}^{\infty} j^{\gamma d} 2^{-(\alpha+\beta-1)jd} < \infty,$$

if $\alpha + \beta > 1$ or $\alpha + \beta \geq 1$ and $\gamma d < -1$, and together with (W1), i.e., the Riesz basis property, we can conclude that (3.3) converges P-almost surely in $L_2(\mathcal{O})$. In this case we have $\mathrm{E}[\langle \xi, X \rangle_{L_2(\mathcal{O})}] = 0$, $\xi \in L_2(\mathcal{O})$, i.e., X is a *mean zero* random function. Moreover,

$$\mathrm{E}\left[\langle \xi, X \rangle_{L_2(\mathcal{O})} \langle \zeta, X \rangle_{L_2(\mathcal{O})}\right] = \sum_{j=j_0}^{\infty} \sigma_j^2 \rho_j \sum_{k \in \nabla_j} \langle \xi, \psi_{j,k} \rangle_{L_2(\mathcal{O})} \langle \zeta, \psi_{j,k} \rangle_{L_2(\mathcal{O})}, \quad \xi, \zeta \in L_2(\mathcal{O}).$$

Using the dual basis we obtain

$$Q\xi = \sum_{j=j_0}^{\infty} \sigma_j^2 \rho_j \sum_{k \in \nabla_j} \langle \xi, \widetilde{\psi}_{j,k} \rangle_{L_2(\mathcal{O})} \psi_{j,k}, \quad \xi \in L_2(\mathcal{O}),$$

for the *covariance operator* Q associated with X.

Remark 3.2. Let $s \in \mathbb{R}$ and suppose $X \in H^s(\mathcal{O})$ P-a.s. Since $H^s(\mathcal{O})$ is a Hilbert space and $\{2^{-js}\psi_{j,k} : j \geq j_0,\ k \in \nabla_j\}$ is a Riesz basis for $H^s(\mathcal{O})$, we know that there exists an orthonormal basis $\{e_{j,k}\}_{j,k}$ of $H^s(\mathcal{O})$ with $2^{-js}\psi_{j,k} = \Phi_{\mathrm{blb}} e_{j,k}$ for a bounded linear bijection $\Phi_{\mathrm{blb}} : H^s(\mathcal{O}) \to H^s(\mathcal{O})$, using Gram-Schmidt. Therefore, we have $\mathrm{E}[\langle \xi, \Phi_{\mathrm{blb}}^{-1} X \rangle_{H^s(\mathcal{O})}] = 0$, $\xi \in H^s(\mathcal{O})$, and since

$$\mathrm{E}\left[\langle \xi, \Phi_{\mathrm{blb}}^{-1} X \rangle_{H^s(\mathcal{O})} \langle \zeta, \Phi_{\mathrm{blb}}^{-1} X \rangle_{H^s(\mathcal{O})}\right] = \sum_{j=j_0}^{\infty} 2^{2js} \sigma_j^2 \rho_j \sum_{k \in \nabla_j} \langle \xi, e_{j,k} \rangle_{H^s(\mathcal{O})} \langle \zeta, e_{j,k} \rangle_{H^s(\mathcal{O})},$$

$\xi, \zeta \in H^s(\mathcal{O})$, we obtain for the *covariance operator* $\widetilde{Q}$ associated with $\Phi_{\text{blb}}^{-1}X$

$$\widetilde{Q}\xi = \sum_{j=j_0}^{\infty} 2^{2js}\sigma_j^2\rho_j \sum_{k\in\nabla_j} \langle \xi, e_{j,k}\rangle_{H^s(\mathcal{O})} e_{j,k}, \quad \xi \in H^s(\mathcal{O}).$$

Remark 3.3. Observe that in this stochastic model the parameter α determines the exponential growth/decay of the wavelet coefficients on increasing wavelet scales, whereas the influence of γ is polynomial. The sparsity of the decomposition (3.3) depends monotonically on the parameter β. For $\beta = 0$, i.e., with no sparsity present, (3.3) coincides with the Karhunen-Loève decomposition of the (in this case) Gaussian random function X provided the wavelets form an orthonormal basis of $L_2(\mathcal{O})$.

Remark 3.4. Essentially, this stochastic model was introduced and analyzed in the context of Bayesian non-parametric regression in Abramovich et al. [1] and generalized in Bochkina [12] in the case $\mathcal{O} = [0, 1]$ for Besov parameters $p, q \geq 1$ and $s > 0$.

We continue by stating necessary properties. The proofs can be found in the Appendix B. We set

$$S_{j,p} := \sum_{k\in\nabla_j} Y_{j,k}|Z_{j,k}|^p, \quad j \geq j_0,\ 0 < p < \infty \tag{3.4}$$

for an independent family of random variables $(Y_{j,k}, Z_{j,k})_{j\geq j_0, k\in\nabla_j}$ as defined by (3.2). Note, $(S_{j,p})_{j\geq j_0}$ forms an independent sequence for every fixed $0 < p < \infty$. Also, with ν_p denoting the p-th absolute moment of the standard normal distribution, i.e., $\nu_p := 2^{p/2}\,\Gamma((p+1)/2)/\pi^{1/2}$, we have

$$\mathrm{E}[S_{j,p}] = \#\nabla_j\rho_j\nu_p. \tag{3.5}$$

Lemma 3.5. *Let $(X_i)_{i\in\mathbb{N}}$ be a family of independent, non-negative random variables. Then $\sum_{i=1}^{\infty} X_i < \infty$, P-a.s., if and only if $\sum_{i=1}^{\infty} \mathrm{E}\big[\frac{X_i}{1+X_i}\big] < \infty$.*

Proof. See Appendix B.1. □

Lemma 3.6. *Let $n \in \mathbb{N}, p \in [0, 1]$, and $X_{n,p} \sim \mathrm{Bin}(n, p)$. For all n there exists a constant $c = c(n) > 0$ such that for all $r > 0$ and p*

$$\mathrm{E}[X_{n,p}^r] \leq c\,(1 + (np)^r).$$

Proof. See Appendix B.2. □

Lemma 3.7. *Let $\beta \in [0, 1)$. Then*

$$\lim_{j\to\infty} \frac{S_{j,p}}{\#\nabla_j\rho_j} = \nu_p$$

holds with probability one. Further, for every $r > 0$

$$\sup_{j\geq j_0} \frac{\mathrm{E}\,[S_{j,p}^r]}{(\#\nabla_j\rho_j)^r} < \infty.$$

Proof. See Appendix B.3. □

Lemma 3.8. *Let $\beta = 1$ and*

$$\lim_{j\to\infty} \#\nabla_j \, 2^{-jd} = C_0 \quad \textit{for some} \quad C_0 > 0. \tag{3.6}$$

Let μ_p denote the distribution of $|Z_{j,k}|^p$, and let S_p be a compound Poisson distributed random variable with intensity measure $C_0\mu_p$. Then $(S_{j,p})_{j\geq j_0}$ converges in distribution to S_p, and for every $r > 0$

$$\sup_{j\geq j_0} \mathrm{E}\,[S_{j,p}^r] < \infty. \tag{3.7}$$

Proof. See Appendix B.4. □

Remark 3.9. Note that we only assumed (W3), i.e., $\#\nabla_j \asymp 2^{jd}$, instead of (3.6) so far. For $\beta = 1$ the upper bound (3.7) remains valid in the general case. In all known constructions of wavelet bases on bounded domains, see, e.g., [17, 33, 58–60, 136], as are stated in Remark 2.24, and also for wavelet frames, see, e.g., Dahlke et al. [50], Stevenson [150], the number $\#\nabla_j$ of wavelets per level $j > 0$ is a constant multiple of 2^{jd}. For those kinds of bases, (3.6) clearly holds.

3.1.2 Regularity theorem

The following theorem states the conditions on the parameters α, β, γ in (3.1) of the stochastic model which guarantee that a random function X, defined by (3.3), almost surely has a certain smoothness in $B^s_q(L_p(\mathcal{O}))$.

Theorem 3.10. *Let Ψ, $\widetilde{\Psi}$ be dual wavelet bases for which (W1)–(W6) holds. Let X be a random function, defined by* (3.3) *with respect to Ψ in the case $s > d(1/p-1)_+$, $0 < p, q < \infty$, and with respect to the dual basis $\widetilde{\Psi}$ in the case $s < 0$, $1 < p, q < \infty$. Then X is* P*-almost surely contained in $B^s_q(L_p(\mathcal{O}))$, if and only if*

$$s < d\left(\frac{\alpha-1}{2} + \frac{\beta}{p}\right) \tag{3.8}$$

or

$$s \leq d\left(\frac{\alpha-1}{2} + \frac{\beta}{p}\right) \quad \textit{and} \quad q\gamma d < -2. \tag{3.9}$$

In both cases

$$\mathrm{E}\left[\|X\|^q_{B^s_q(L_p(\mathcal{O}))}\right] < \infty. \tag{3.10}$$

Proof. Using the wavelet characterization (W6) and $S_{j,p}$, which is defined in (3.4), we have $X \in B^s_q(L_p(\mathcal{O}))$ P-almost surely if and only if

$$\|X\|^q_{B^s_q(L_p(\mathcal{O}))} \asymp \sum_{j=j_0}^{\infty} 2^{j(s+d(1/2-1/p))q}\, \sigma_j^q\, S_{j,p}^{q/p} < \infty, \quad \text{P-a.s.}$$

Thus, using the abbreviation $a_j := 2^{j(s+d(1/2-1/p))q}\,\sigma_j^q$, we have to show when

$$\sum_{j=j_0}^{\infty} a_j\, S_{j,p}^{q/p} < \infty, \quad \text{P-a.s.} \tag{3.11}$$

It is enough to show that (3.11) is equivalent to

$$\sum_{j=j_0}^{\infty} a_j \left(\#\nabla_j \rho_j\right)^{q/p} < \infty, \tag{3.12}$$

because inserting (W3), i.e., $\#\nabla_j \asymp 2^{jd}$, and (3.1) into (3.12) yields

$$\sum_{j=j_0}^{\infty} a_j \left(\#\nabla_j \rho_j\right)^{q/p} \asymp \sum_{j=j_0}^{\infty} j^{q\gamma d/2}\, 2^{qjd(s/d-(\alpha-1)/2-\beta/p)},$$

and we see that (3.12) holds if and only if the conditions (3.8) or (3.9) are satisfied.

We continue to show the equivalence of (3.11) and (3.12). In the case $0 \le \beta < 1$ it follows from Lemma 3.7. In the case $\beta = 1$ observe that (3.12) with (W3) reduces to

$$\sum_{j=j_0}^{\infty} a_j < \infty, \tag{3.13}$$

while (3.11) is, due to Lemma 3.5, equivalent to

$$\sum_{j=j_0}^{\infty} \mathrm{E}\left[\frac{a_j\, S_{j,p}^{q/p}}{1 + a_j\, S_{j,p}^{q/p}}\right] < \infty. \tag{3.14}$$

The equivalence of (3.13) and (3.14) is shown in two parts. To show that (3.13) implies (3.14), we use Lemma 3.8 to conclude

$$\sum_{j=j_0}^{\infty} \mathrm{E}\left[\frac{a_j\, S_{j,p}^{q/p}}{1 + a_j\, S_{j,p}^{q/p}}\right] \le \sum_{j=j_0}^{\infty} a_j\, \mathrm{E}[S_{j,p}^{q/p}] < \infty, \quad \text{if} \quad \sum_{j=j_0}^{\infty} a_j < \infty.$$

The second part, i.e., (3.14) implies (3.13), is shown by contradiction. We assume (3.14) and $\sum_{j=j_0}^{\infty} a_j = \infty$ to hold. Now, by (3.6) we obtain that $c_p := \inf_{j \ge 0} \mathrm{P}(S_{j,p} \ge 1) > 0$, and, using the specific form of a_j, we can conclude

$$\sum_{j=j_0}^{\infty} \mathrm{E}\left[\frac{a_j\, S_{j,p}^{q/p}}{1 + a_j\, S_{j,p}^{q/p}}\right] \ge c_p \sum_{j=j_0}^{\infty} \frac{a_j}{1 + a_j} = \infty,$$

which contradicts the assumption (3.14). All together, the equivalence of (3.11) and (3.12) is proven.

It remains to show (3.10). We use the wavelet characterization (W6), Lemma 3.7, Lemma 3.8, and (3.12) to derive

$$\mathrm{E}\left[\|X\|^q_{B^s_q(L_p(\mathcal{O}))}\right] \preceq \sum_{j=j_0}^{\infty} a_j \mathrm{E}[S_{j,p}^{q/p}] \preceq \sum_{j=j_0}^{\infty} a_j (\#\nabla_j \rho_j)^{q/p} < \infty. \qquad \square$$

Remark 3.11. In the case $X \in H^s(\mathcal{O})$, $s \in \mathbb{R}$, one can compute the moment in (3.10) explicitly, i.e.,

$$\mathrm{E}\left[\|X\|^2_{H^s(\mathcal{O})}\right] \asymp \mathrm{E}\left[\sum_{j=j_0}^{\infty} 2^{2js}\, \sigma_j^2\, S_{j,2}\right] \asymp \sum_{j=j_0}^{\infty} 2^{2js}\, \sigma_j^2\, \mathrm{E}\left[S_{j,2}\right]$$

$$\asymp \sum_{j=j_0}^{\infty} 2^{2js}\, \sigma_j^2 \,\#\nabla_j\, 2^{-jd\beta} \asymp \sum_{j=j_0}^{\infty} 2^{-jd(\alpha+\beta-1-2s/d)}\, j^{\gamma d},$$

which is finite if and only if $\alpha+\beta > 1+2s/d$ or $\alpha+\beta \geq 1+2s/d$ and $\gamma d < -1$.

As a special case of Theorem 3.10 we emphasize the regularity of X in $B^s_\tau(L_\tau(\mathcal{O}))$, where

$$\frac{1}{\tau} = \frac{s-\nu}{d} + \frac{1}{p}, \quad 1 < p < \infty, \quad \text{and} \quad \nu < s, \tag{3.15}$$

since this scale is related with nonlinear wavelet approximation in $B^\nu_p(L_p(\mathcal{O}))$, cf. (2.21) in Section 2.3.2. Furthermore, in the next section and in Section 4.1 average errors are defined by second moments, and therefore we also consider the second moments of the norm in $B^s_\tau(L_\tau(\mathcal{O}))$.

Corollary 3.12. *Let $\beta \in [0,1)$, $1 < p < \infty$, and $-d/p < \nu < d((\alpha-1)/2 + \beta/p)$. Then*

$$X \in B^s_\tau(L_\tau(\mathcal{O})) \quad \textit{for all} \quad s < s^*$$

in the scale (3.15) *holds with probability one, where*

$$s^* := \frac{d}{1-\beta}\left(\frac{\alpha-1}{2} + \frac{\beta}{p}\right) - \frac{\beta\nu}{1-\beta}. \tag{3.16}$$

Moreover, we have

$$\mathrm{E}\|X\|^2_{B^s_\tau(L_\tau(\mathcal{O}))} < \infty. \tag{3.17}$$

Proof. Using Theorem 3.10 we have to show that

$$s^* = d\left(\frac{\alpha-1}{2} + \frac{\beta}{\tau^*}\right). \tag{3.18}$$

Inserting (3.15) with $s = s^*$ and $\tau = \tau^*$ into (3.16) yields (3.18). To show (3.17), observe that actually $X \in B^{s+\delta}_2(L_\tau(\mathcal{O}))$ holds with probability one if $\delta > 0$ is sufficiently small, and $X \in B^{s+\delta}_2(L_\tau(\mathcal{O}))$ is continuously embedded in $X \in B^s_\tau(L_\tau(\mathcal{O}))$, cf. Theorem 2.6 (i), (ii). The moment bound (3.10) therefore implies

$$\mathrm{E}\left[\|X\|^2_{B^s_\tau(L_\tau(\mathcal{O}))}\right] \preceq \mathrm{E}\left[\|X\|^2_{B^{s+\delta}_2(L_\tau(\mathcal{O}))}\right] < \infty,$$

which completes the proof. □

Remark 3.13. Corollary 3.12 implies that by choosing β closer to one, an arbitrary high regularity in the nonlinear approximation scale (3.15) can be achieved, provided that the underlying wavelet basis is sufficiently smooth. This is obviously *not* possible in the L_2-Sobolev scale, see Theorem 3.10 with $p = q = 2$.

Remark 3.14. The lower bound on ν in Corollary 3.12 is caused by the fact that we must ensure the fundamental condition $s > d(1/\tau - 1)$, which guarantees the existence of a wavelet characterization, cf. Remark 2.8. See Figure 3.1 for an illustration.

3.1.3 Linear and nonlinear approximation results

We state error bounds to linear uniform and nonlinear approximations of X, first with respect to the L_2-norm and, subsequently, we extend our studies and state error bound with respect to Besov norms.

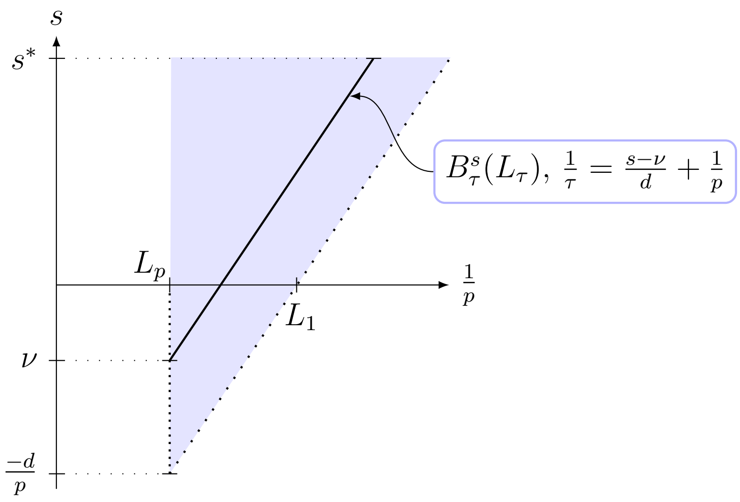


Figure 3.1: Setting for Corollary 3.12 illustrated in a DeVore-Triebel diagram

Error bounds with respect to the L_2-norm

Here, we have to consider

$$\alpha + \beta > 1, \quad \text{or} \quad \alpha + \beta \geq 1 \text{ and } \gamma d < -1, \quad \alpha, \gamma \in \mathbb{R},\ \beta \in [0,1], \tag{3.19}$$

in order to ensure $X \in L_2(\mathcal{O})$ P-a.s., cf. Remark 3.1.

For linear approximation one considers the best approximation from linear subspaces of dimension at most N, which is given by the orthogonal projection on these subspaces, cf. Section 2.3.2. The corresponding *linear approximation error* of X with respect to $L_2(\mathcal{O})$ is given by

$$e_N^{\mathrm{lin}}(X) := \inf \left(\mathrm{E}\,[\|X - \widehat{X}\|^2_{L_2(\mathcal{O})}] \right)^{1/2}$$

with the infimum taken over all measurable mappings $\widehat{X} : \Omega \to L_2(\mathcal{O})$ such that

$$\dim(\mathrm{span}(\widehat{X}(\Omega))) \leq N.$$

Theorem 3.15. *Let α, β, γ in* (3.19) *be fixed and let X be defined by* (3.3)*. The linear approximation error with respect to $L_2(\mathcal{O})$ satisfies*

$$e_N^{\mathrm{lin}}(X) \asymp (\log_2 N)^{\frac{\gamma d}{2}}\, N^{-\frac{\alpha+\beta-1}{2}}.$$

Proof. To prove the upper bound, we truncate the decomposition (3.3) of X at some level $j_1 \geq j_0$ and we obtain a uniform linear approximation

$$\widehat{X}_{j_1} := \sum_{j=j_0}^{j_1} \sum_{k \in \nabla_j} \sigma_j Y_{j,k} Z_{j,k}\, \psi_{j,k}, \tag{3.20}$$

which satisfies

$$\mathrm{E}\left[\|X - \widehat{X}_{j_1}\|^2_{L_2(\mathcal{O})}\right] \asymp \sum_{j=j_1+1}^{\infty} \#\nabla_j \sigma_j^2 \rho_j \asymp \sum_{j=j_1+1}^{\infty} j^{\gamma d} 2^{-(\alpha+\beta-1)jd} \asymp j_1^{\gamma d} 2^{-(\alpha+\beta-1)j_1 d}. \tag{3.21}$$

Since $\dim(\mathrm{span}(\widehat{X}_{j_1}(\Omega))) \asymp \sum_{j=j_0}^{j_1} \#\nabla_j \asymp 2^{j_1 d}$, we get the upper bound as claimed.

To prove the lower bound we use the fact that $\psi_{j,k} = \Phi_{\mathrm{blb}} e_{j,k}$ for an orthonormal basis $(e_{j,k})_{j,k}$ in $L_2(\mathcal{O})$ and a bounded linear bijection $\Phi_{\mathrm{blb}} : L_2(\mathcal{O}) \to L_2(\mathcal{O})$, cf. Remark 3.2. This implies

$$e_N^{\mathrm{lin}}(X) \asymp e_N^{\mathrm{lin}}(\Phi_{\mathrm{blb}}^{-1} X).$$

Furthermore, $e_N^{\mathrm{lin}}(\Phi_{\mathrm{blb}}^{-1} X)$ depends on $\Phi_{\mathrm{blb}}^{-1} X$ only via its covariance operator $\widetilde{Q}$ which, as we also know from Remark 3.2, is given by

$$\widetilde{Q}\xi = \sum_{j=j_0}^{\infty} \sigma_j^2 \rho_j \sum_{k \in \nabla_j} \langle \xi, e_{j,k} \rangle_{L_2(\mathcal{O})} \, e_{j,k}. \tag{3.22}$$

Consequently, the functions $e_{j,k}$ form an orthonormal basis of eigenfunctions of $\widetilde{Q}$ with associated eigenvalues $\sigma_j^2 \rho_j$. Due to a theorem by Micchelli and Wahba, see, e.g., Ritter [139, Proposition III.24], we can conclude

$$e_N^{\mathrm{lin}}(\Phi_{\mathrm{blb}}^{-1} X) = \left(\sum_{j=j_1+1}^{\infty} \#\nabla_j \sigma_j^2 \rho_j \right)^{1/2} \quad \text{with} \quad N = \sum_{j=j_0}^{j_1} \#\nabla_j. \tag{3.23}$$

Inserting (W3), i.e., $\#\nabla_j \asymp 2^{jd}$, and (3.1) into (3.23) implies the asserted lower bound. □

The best N-term wavelet approximation imposes a restriction only on

$$\eta(g) := \# \left\{ \lambda \, : \, \lambda \in \nabla, \ g = \sum_{\lambda \in \nabla} c_\lambda \, \psi_\lambda, \ c_\lambda \neq 0 \right\}, \tag{3.24}$$

the number of non-zero wavelet coefficients of g. Therefore, the corresponding *error of best N-term wavelet approximation* for X with respect to $L_2(\mathcal{O})$ is given by

$$e_N^{\mathrm{best}}(X) := \inf \left(\mathrm{E} \, [\| X - \widehat{X} \|_{L_2(\mathcal{O})}^2] \right)^{1/2}$$

with the infimum taken over all measurable mappings $\widehat{X} : \Omega \to L_2(\mathcal{O})$ such that

$$\eta(\widehat{X}(\omega)) \leq N \quad \text{P-a.s.}$$

For deterministic functions x on $\mathcal{O}$ the error of best N-term wavelet approximation with respect to the L_2-norm is defined by

$$e_N^{\mathrm{det}}(x) := \inf \left\{ \| x - \widehat{x} \|_{L_2(\mathcal{O})} \, : \, \widehat{x} \in L_2(\mathcal{O}), \ \eta(\widehat{x}) \leq N \right\}, \tag{3.25}$$

cf. Section 2.3.2. Clearly, we have

$$e_N^{\mathrm{best}}(X) = \left(\mathrm{E} \, [e_N^{\mathrm{det}}(X)^2] \right)^{1/2}.$$

Theorem 3.16. *Let α, β, γ in* (3.19) *be fixed and let X be defined by* (3.3)*. For every $\varepsilon > 0$, the error of best N-term wavelet approximation with respect to $L_2(\mathcal{O})$ satisfies*

$$e_N^{\mathrm{best}}(X) \preceq \begin{cases} N^{-1/\varepsilon} & : \textit{if } \beta = 1, \\ N^{-\frac{\alpha+\beta-1}{2(1-\beta)}+\varepsilon} & : \textit{otherwise.} \end{cases}$$

Proof. The case $\beta = 1$ is a direct consequence of the definition of X. For $\beta < 1$, let s and τ satisfy (3.15) with $\nu = 0$ and $p = 2$, i.e., $1/\tau = s/d + 1/2$. By Remark 2.26 in Section 2.3.3 we have that $x \in B^s_\tau(L_\tau(\mathcal{O}))$ implies $e_N^{\mathrm{det}}(x) \preceq \|x\|_{B^s_\tau(L_\tau(\mathcal{O}))}\, N^{-s/d}$ and therefore it remains to apply Corollary 3.12. $\square$

For random functions it is also reasonable to impose a constraint on the average number of non-zero wavelet coefficients only, and to study the *error of best average N-term wavelet approximation*

$$e_N^{\mathrm{avg}}(X) := \inf \left(\mathrm{E}\,[\|X - \widehat{X}\|^2_{L_2(\mathcal{O})}] \right)^{1/2}$$

with the infimum taken over all measurable mappings $\widehat{X} : \Omega \to L_2(\mathcal{O})$ such that

$$\mathrm{E}\,[\eta(\widehat{X})] \leq N.$$

Theorem 3.17. *Let α, β, γ in* (3.19) *be fixed and let X be defined by* (3.3)*. The error of best average N-term wavelet approximation with respect to $L_2(\mathcal{O})$ satisfies*

$$e_N^{\mathrm{avg}}(X) \preceq \begin{cases} N^{\frac{\gamma d}{2}}\, 2^{-\frac{\alpha d N}{2}} & : \textit{if } \beta = 1, \\ (\log_2 N)^{\frac{\gamma d}{2}}\, N^{-\frac{\alpha+\beta-1}{2(1-\beta)}} & : \textit{otherwise.} \end{cases}$$

Proof. Let $N_{j_1} := \mathrm{E}[\eta(\widehat{X}_{j_1})]$ for $\widehat{X}_{j_1}$ as in (3.20). Clearly

$$N_{j_1} = \sum_{j=j_0}^{j_1} \#\nabla_j \rho_j \asymp \sum_{j=j_0}^{j_1} 2^{(1-\beta)jd} \asymp \begin{cases} j_1 & : \text{if } \beta = 1 \\ 2^{(1-\beta)j_1 d} & : \text{otherwise.} \end{cases}$$

In particular, $2^{j_1 d} \asymp N_{j_1}^{1/(1-\beta)}$ if $\beta \in [0,1)$. Now, observe the error bound (3.21). $\square$

The asymptotic behavior of the linear approximation error $e_N^{\mathrm{lin}}(X)$ is determined by the *decay of the eigenvalues* $\sigma_j^2 \rho_j$ of the covariance operator $\widetilde{Q}$, see (3.22), i.e., it is essentially determined by the sum $\alpha + \beta$. According to Theorem 3.10, the sum $\alpha + \beta$ also determines the regularity of X in the scale of Sobolev spaces $H^s(\mathcal{O})$.

For $\beta \in (0,1]$ nonlinear approximation is *superior* to linear approximation. More precisely, the following holds true. By definition, $e_N^{\mathrm{avg}}(X) \leq e_N^{\mathrm{best}}(X)$, and for $\beta > 0$ the convergence of $e_N^{\mathrm{best}}(X)$ to zero is faster than that of $e_N^{\mathrm{lin}}(X)$. For $\beta \in (0,1)$ the upper bounds for $e_N^{\mathrm{avg}}(X)$ and $e_N^{\mathrm{best}}(X)$ slightly differ, and any dependence of $e_N^{\mathrm{best}}(X)$ on the parameter γ is swallowed by the term N^ε in the upper bound. For linear and best average N-term approximation we have

$$e^{\mathrm{avg}}_{N^{1-\beta}}(X) \preceq e_N^{\mathrm{lin}}(X) \text{ if } \beta \in (0,1) \quad \text{and} \quad e^{\mathrm{avg}}_{c \log_2 N}(X) \preceq e_N^{\mathrm{lin}}(X) \text{ if } \beta = 1$$

with a suitably chosen constant $c > 0$.

Remark 3.18. We stress that for $\beta \in (0,1]$ the *simulation* of the approximation $\widehat{X}_{j_1}$, which achieves the upper bound in Theorem 3.17, is possible at an *average computational cost* of order N_{j_1}. Let us briefly sketch the method of simulation. Set $n_j := \#\nabla_j$. For each level j we first simulate a binomial distribution with parameters n_j and ρ_j, which is possible at an average cost of order at most $n_j \rho_j$. Conditional on a realization $L(\omega)$ of this step, the locations of the non-zero coefficients on level j are uniformly distributed on the set of all subsets of $\{0,\dots,n_j\}$ of cardinality $L(\omega)$. Thus, in the second step, we employ acceptance-rejection to collect the elements of such a random subset sequentially. If $L(\omega) \le n_j/2$, then all acceptance probabilities are at least $1/2$, and otherwise we switch to complements to obtain the same bound for the acceptance probability. In this way, the average cost of the second step is of order $n_j\rho_j$, too. In the last step we simulate the values of the non-zero coefficients. In total, the average computational cost for each level j is of order $n_j\rho_j$.

Remark 3.19. For Theorems 3.15 and 3.17 we only need the Riesz basis property (W1) and the property (W3), i.e., $\#\nabla_j \asymp 2^{jd}$, of the basis Ψ, and (W3) enters only via the asymptotic behavior of the parameters ρ_j and σ_j. After a lexicographic reordering of the indices (j,k) the two assumptions essentially amount to

$$X = \sum_{n=1}^{\infty} \sigma_n Y_n Z_n \psi_n$$

with any Riesz basis $\{\psi_n\}_{n\in\mathbb{N}}$ for $L_2(\mathcal{O})$, and $\sigma_n \asymp (\log_2 n)^{\gamma d/2} n^{-\alpha/2}$ as well as independent random variables Y_n and Z_n, where Z_n is $\mathcal{N}(0,1)$-distributed and Y_n is Bernoulli distributed with parameter $\rho_n \asymp n^{-\beta}$. Therefore, Theorems 3.15 and 3.17 remain valid beyond the wavelet setting. For instance, let $\rho_n = 1$, which corresponds to $\beta = 0$. Classical examples for Gaussian random functions on $\mathcal{O} = [0,1]^d$ are the *Brownian sheet*, which corresponds to $\alpha = 2$ and $\gamma = 2(d-1)/d$, and *Lévy's Brownian motion*, which corresponds to $\alpha = (d+1)/d$ and $\gamma = 0$. Theorem 3.15 is due to PAPAGEORGIOU, WASILKOWSKI [132], WOŹNIAKOWSKI [175] for the Brownian sheet and due to WASILKOWSKI [173] for Lévy's Brownian motion. See RITTER [139, Chapter VI] for further results and references on approximation of Gaussian random functions. Therefore, for $\beta > 0$ our stochastic model provides sparse variants of general Gaussian random function.

Error bounds with respect to Besov norms

We extend above findings and state error bounds for linear and nonlinear approximation schemes for the considered random functions with respect to the norms of the Besov spaces $B^\nu_p(L_p(\mathcal{O}))$ with $\nu \in \mathbb{R}$ and $1 < p < \infty$.

We define the *linear approximation error* of X with respect to $B^\nu_p(L_p(\mathcal{O}))$ by

$$e^{\mathrm{lin}}_{N,p,\nu}(X) := \inf \left(\mathrm{E}[\|X - \widehat{X}\|^p_{B^\nu_p(L_p(\mathcal{O}))}] \right)^{1/p}$$

with the infimum taken over all measurable mappings $\widehat{X}$ such that

$$\dim(\mathrm{span}(\widehat{X}(\mathcal{O}))) \le N.$$

Theorem 3.20. *Let $\beta \in [0,1)$, $m > 0$. For a fixed approximation space $B^\nu_p(L_p(\mathcal{O}))$, $\nu \in \mathbb{R}$, $1 < p < \infty$, let X be given by* (3.3) *with*

$$\nu + m < d((\alpha-1)/2 + \beta/p) =: \nu + m^*, \tag{3.26}$$

i.e., $X \in B^{\nu+m}_p(L_p(\mathcal{O}))$ for all $m < m^*$. The linear approximation error with respect to $B^\nu_p(L_p(\mathcal{O}))$ satisfies

$$e^{\mathrm{lin}}_{N,p,\nu}(X) \preceq (\log_2 N)^{\frac{\gamma d}{2}} N^{-(\frac{\alpha-1}{2} + \frac{\beta}{p} - \frac{\nu}{d})}. \tag{3.27}$$

Proof. Again, as a specific linear approximation, we consider a uniform approximation of the form

$$\widehat{X}_{j_1} = \sum_{j=j_0}^{j_1} \sum_{k \in \nabla_j} \sigma_j Y_{j,k} Z_{j,k} \psi_{j,k}$$

for some $j_1 \geq j_0$, where in particular $N \asymp 2^{j_1 d}$. With $S_{j,p}$ as defined in (3.4) and with (3.5), we obtain

$$\begin{aligned}
\mathrm{E}[\|X - \widehat{X}_{j_1}\|^p_{B^\nu_p(L_p(\mathcal{O}))}] &\asymp \mathrm{E}\left[\sum_{j=j_1+1}^{\infty} 2^{jp(\nu+d(\frac{1}{2}-\frac{1}{p}))} \sigma_j^p S_{j,p}\right] \\
&= \mathrm{E}\left[\sum_{j=j_1+1}^{\infty} 2^{jp\left((\nu+m^*)+d(\frac{1}{2}-\frac{1}{p})\right)} 2^{-jpm^*} \sigma_j^p S_{j,p}\right] \\
&= \sum_{j=j_1+1}^{\infty} 2^{jpd\left((\nu+m^*)+d(\frac{1}{2}-\frac{1}{p})\right)} 2^{-jpm^*} \sigma_j^p \# \nabla_j \rho_j.
\end{aligned}$$

Inserting (W3), i.e., $\#\nabla_j \asymp 2^{jd}$, (3.1), and (3.26) we get

$$\begin{aligned}
\mathrm{E}[\|X - \widehat{X}_{j_1}\|^p_{B^\nu_p(L_p(\mathcal{O}))}] &\asymp \sum_{j=j_1+1}^{\infty} 2^{jpd\left(\frac{\alpha-1}{2}+\frac{\beta}{p}+\frac{1}{2}-\frac{1}{p}\right)} 2^{-jpm^*} j^{\frac{\gamma dp}{2}} 2^{-\frac{\alpha j dp}{2}} 2^{jd} 2^{-\beta jd} \\
&= \sum_{j=j_1+1}^{\infty} j^{\frac{\gamma dp}{2}} 2^{-jpm^*} \\
&\asymp j_1^{\frac{\gamma dp}{2}} 2^{-j_1 pm^*} \\
&\asymp (\log_2 N)^{\frac{\gamma dp}{2}} N^{-p\left(\frac{\alpha-1}{2}+\frac{\beta}{p}-\frac{\nu}{d}\right)},
\end{aligned}$$

which yields (3.27). □

Remark 3.21. In the setting of Theorem 3.20, with a slightly coarser error estimation, for all $m < m^*$ we get

$$\begin{aligned}
\mathrm{E}[\|X - \widehat{X}_{j_1}\|^p_{B^\nu_p(L_p(\mathcal{O}))}] &\preceq 2^{-j_1 pm}\, \mathrm{E}\left[\sum_{j=j_0}^{\infty} 2^{jp\left((\nu+m)+d(\frac{1}{2}-\frac{1}{p})\right)} \sigma_j^p S_{j,p}\right] \\
&\preceq N^{-pm/d}\, \mathrm{E}\left[\|X\|^p_{B^{\nu+m}_p(L_p(\mathcal{O}))}\right].
\end{aligned}$$

Since we have $\mathrm{E}[\|X\|^p_{B^{\nu+m}_p(L_p(\mathcal{O}))}] < \infty$, $m < m^*$, by (3.10), we derive that the linear approximation error satisfies

$$e^{\mathrm{lin}}_{N,p,\nu}(X) \preceq N^{-m/d} \left(\mathrm{E}[\|X\|^p_{B^{\nu+m}_p(L_p(\mathcal{O}))}]\right)^{1/p}. \tag{3.28}$$

From (3.28), we observe that, similar to the well-known deterministic setting, see Section 2.3.2, the approximation order which can be achieved by uniform linear schemes depends on the regularity of the object under consideration in the same scale of smoothness spaces.

The following theorem is a generalization of Theorem 3.15. It states the error bounds for linear wavelet approximation with respect to H^ν.

Theorem 3.22. *Let $\beta \in [0,1)$ and $m > 0$. For a fixed approximation space $H^\nu(\mathcal{O})$, $\nu \in \mathbb{R}$, let X be given by (3.3) with $\nu + m < d(\alpha - 1 + \beta)/2 =: \nu + m^*$, that is, $X \in H^{\nu+m}(\mathcal{O})$ for all $m < m^*$. The linear approximation error with respect to $H^\nu(\mathcal{O})$ satisfies*

$$e_{N,2,\nu}^{\mathrm{lin}}(X) \asymp (\log_2 N)^{\frac{\gamma d}{2}} N^{-\left(\frac{\alpha-1+\beta}{2} - \frac{\nu}{d}\right)}.$$

Proof. The upper bound is proven in Theorem 3.20, where $p = 2$. The lower bound is analogously shown as in the proof of Theorem 3.15. Given $\{e_{j,k}\}_{j,k}$ and Φ_{blb} as in Remark 3.2, we know that

$$e_{N,2,\nu}^{\mathrm{lin}}(X) \asymp e_{N,2,\nu}^{\mathrm{lin}}(\Phi_{\mathrm{blb}}^{-1} X).$$

We also know from Remark 3.2 that the covariance operator $\widetilde{Q}$ of $\Phi_{\mathrm{blb}}^{-1} X$ is given by

$$\widetilde{Q}\xi = \sum_{j=j_0}^{\infty} 2^{2j\nu} \sigma_j^2 \rho_j \sum_{k \in \nabla_j} \langle \xi, e_{j,k} \rangle_{H^\nu(\mathcal{O})} e_{j,k},$$

which means, that the functions $e_{j,k}$ form an orthonormal basis of eigenfunctions of $\widetilde{Q}$ with associated eigenvalues $2^{2j\nu}\sigma_j^2\rho_j$. Using methods, e.g. shown in Ritter [139, Chapter III], we get

$$e_{N,2,\nu}^{\mathrm{lin}}(\Phi_{\mathrm{blb}}^{-1} X) = \left(\sum_{j=j_1+1}^{\infty} \#\nabla_j 2^{2j\nu} \sigma_j^2 \rho_j \right)^{1/2} \quad \text{with} \quad N = \sum_{j=j_0}^{j_1} \#\nabla_j. \tag{3.29}$$

Inserting (W3), i.e., $\#\nabla_j \asymp 2^{jd}$, and (3.1) into (3.29) yields the claim. □

We define the *average nonlinear approximation error* of $X : \Omega \to B_\tau^s(L_\tau(\mathcal{O}))$ with respect to $B_p^\nu(L_p(\mathcal{O}))$ in the scale (3.15), i.e.,

$$\frac{1}{\tau} = \frac{s-\nu}{d} + \frac{1}{p}, \quad 1 < p < \infty, \quad \text{and} \quad \nu < s,$$

cf. Corollary 3.12, by

$$e_{N,p,\nu}^{\mathrm{avg}}(X) := \inf \left(\mathrm{E}[\|X - \widehat{X}\|_{B_p^\nu(L_p(\mathcal{O}))}^p] \right)^{1/p}$$

with the infimum taken over all measurable mappings $\widehat{X}$ such that $\mathrm{E}[\eta(\widehat{X})] \le N$. Again, $\eta(g)$ denotes the number of nonzero wavelet coefficients of g, see (3.24).

Theorem 3.23. *Let $\beta \in [0,1)$. For a fixed approximation space $B_p^\nu(L_p(\mathcal{O}))$, $\nu \in \mathbb{R}$, $1 < p < \infty$, let X be given by (3.3) with $-d/p \le \nu < d((\alpha-1)/2 + \beta/p)$, that is, $X \in B_\tau^s(L_\tau(\mathcal{O}))$ in the scale (3.15) for all $s < s^*$, where s^* is given by (3.16). Then the average nonlinear approximation error with respect to $B_p^\nu(L_p(\mathcal{O}))$ satisfies*

$$e_{N,p,\nu}^{\mathrm{avg}}(X) \preceq (\log_2 N)^{\frac{\gamma d}{2}} N^{-\frac{1}{1-\beta}\left(\frac{\alpha-1}{2} + \frac{\beta}{p} - \frac{\nu}{d}\right)}. \tag{3.30}$$

Proof. As a specific nonlinear approximation of X we consider

$$\widehat{X}_{j_1,N} := \sum_{j=j_0}^{j_1} \sum_{k\in\nabla_j} \sigma_j Y_{j,k} Z_{j,k} \psi_{j,k}$$

for some $j_1 \geq j_0$, where only the non-zero coefficients $N := \mathrm{E}[\eta(\widehat{X}_{j_1,N})]$ are retained. We have

$$N = \sum_{j=j_0}^{j_1} \#\nabla_j \rho_j \asymp 2^{(1-\beta)j_1 d}.$$

With $S_{j,p}$ being defined in (3.4) and with (3.5), we use (3.15), where $s = s^*$ and $\tau = \tau^*$, to obtain

$$\begin{aligned}
\mathrm{E}[\|X - \widehat{X}_{j_1,N}\|^p_{B^\nu_p(L_p(\mathcal{O}))}] &\asymp \mathrm{E}\left[\sum_{j=j_1+1}^{\infty} 2^{jp\left(\nu+d(\frac{1}{2}-\frac{1}{p})\right)} \sigma_j^p S_{j,p}\right] \\
&= \mathrm{E}\left[\sum_{j=j_1+1}^{\infty} 2^{jp\left(s^*+d(\frac{1}{2}-\frac{1}{\tau^*})\right)} \sigma_j^p S_{j,p}\right] \\
&= \sum_{j=j_1+1}^{\infty} 2^{jp\left(s^*+d(\frac{1}{2}-\frac{1}{\tau^*})\right)} \sigma_j^p \#\nabla_j \rho_j.
\end{aligned}$$

Inserting (W3), i.e., $\#\nabla_j \asymp 2^{jd}$, (3.1), (3.18), and (3.15), where $s = s^*$ and $\tau = \tau^*$, we get

$$\begin{aligned}
\mathrm{E}[\|X - \widehat{X}_{j_1,N}\|^p_{B^\nu_p(L_p(\mathcal{O}))}] &\asymp \sum_{j=j_1+1}^{\infty} 2^{jpd\left(\frac{\alpha-1}{2}+\frac{\beta}{\tau^*}+\frac{1}{2}-\frac{1}{\tau^*}\right)} j^{\frac{\gamma dp}{2}} 2^{-\frac{\alpha j dp}{2}} 2^{jd} 2^{-\beta jd} \\
&= \sum_{j=j_1+1}^{\infty} j^{\frac{\gamma dp}{2}} 2^{-jp(1-\beta)(s^*-\nu)} \\
&\asymp j_1^{\frac{\gamma dp}{2}} 2^{-j_1 p(1-\beta)(s^*-\nu)} \\
&\asymp (\log_2 N)^{\frac{\gamma dp}{2}} N^{-\frac{p}{1-\beta}\left(\frac{\alpha-1}{2}+\frac{\beta}{p}-\frac{\nu}{d}\right)},
\end{aligned}$$

which yields (3.30). □

An analogous statement to Remark 3.21 also holds for the average nonlinear approximation error.

Remark 3.24. Let $\varepsilon > 0$ and $s := s^* - \varepsilon$ with s^* being defined in (3.16). In the setting of Theorem 3.23, with a slightly coarser error estimation, we get

$$\begin{aligned}
\mathrm{E}[\|X - \widehat{X}_{j_1,N}\|^p_{B^\nu_p(L_p(\mathcal{O}))}] &\asymp \mathrm{E}\left[\sum_{j=j_1+1}^{\infty} 2^{j(p-\tau)\left(s+d(\frac{1}{2}-\frac{1}{\tau})\right)} \sigma_j^{p-\tau}\, 2^{j\tau\left(s+d(\frac{1}{2}-\frac{1}{\tau})\right)} \sigma_j^\tau S_{j,p}\right] \\
&\asymp \mathrm{E}\left[\sum_{j=j_1+1}^{\infty} 2^{j(p-\tau)\left(s+d(\frac{1}{2}-\frac{1}{\tau}-\frac{\alpha}{2})\right)} j^{\frac{\gamma d}{2}(p-\tau)}\, 2^{j\tau\left(s+d(\frac{1}{2}-\frac{1}{\tau})\right)} \sigma_j^\tau S_{j,p}\right]
\end{aligned}$$

$$\preceq \mathrm{E}\left[\sum_{j=j_1+1}^{\infty} 2^{j(p-\tau)\left(s+d(\frac{1}{2}-\frac{1}{\tau}-\frac{\alpha}{2})\right)+\delta j}\, 2^{j\tau\left(s+d(\frac{1}{2}-\frac{1}{\tau})\right)}\, \sigma_j^\tau S_{j,p}\right],$$

for any $\delta > 0$. Inserting $s = s^* - \varepsilon$ and $\delta := p(s-\nu)(1-\beta)(\varepsilon\tau)/d$, as well as using (3.18), (3.15), and also (3.15) with $s = s^*$ and $\tau = \tau^*$, which yields $1/\tau^* = 1/\tau + \varepsilon/d$, we get

$$\begin{aligned}
\mathrm{E}[\|X - \widehat{X}_{j_1,N}\|^p_{B^\nu_p(L_p(\mathcal{O}))}] &\preceq \mathrm{E}\left[\sum_{j=j_1+1}^{\infty} 2^{j(p-\tau)\left(s^*-\varepsilon+d(\frac{1}{2}-\frac{1}{\tau}-\frac{\alpha}{2})\right)+\delta j}\, 2^{j\tau\left(s+d(\frac{1}{2}-\frac{1}{\tau})\right)}\, \sigma_j^\tau S_{j,p}\right]\\
&\asymp \mathrm{E}\left[\sum_{j=j_1+1}^{\infty} 2^{j\left((p-\tau)d(\beta-1)\left(\frac{1}{\tau}+\frac{\varepsilon}{d}\right)+\delta\right)}\, 2^{j\tau\left(s+d(\frac{1}{2}-\frac{1}{\tau})\right)}\, \sigma_j^\tau S_{j,p}\right]\\
&\asymp \mathrm{E}\left[\sum_{j=j_1+1}^{\infty} 2^{j\left(p(s-\nu)\tau(\beta-1)\left(\frac{1}{\tau}+\frac{\varepsilon}{d}\right)+\delta\right)}\, 2^{j\tau\left(s+d(\frac{1}{2}-\frac{1}{\tau})\right)}\, \sigma_j^\tau S_{j,p}\right]\\
&\asymp \mathrm{E}\left[\sum_{j=j_1+1}^{\infty} 2^{-jp(1-\beta)(s-\nu)}\, 2^{j\tau\left(s+d(\frac{1}{2}-\frac{1}{\tau})\right)}\, \sigma_j^\tau S_{j,p}\right]\\
&\asymp 2^{-j_1p(1-\beta)(s-\nu)}\,\mathrm{E}\left[\sum_{j=j+1}^{\infty} 2^{j\tau\left(s+d(\frac{1}{2}-\frac{1}{\tau})\right)}\, \sigma_j^\tau S_{j,p}\right]\\
&\preceq 2^{-j_1p(1-\beta)(s-\nu)}\,\mathrm{E}\left[\sum_{j=0}^{\infty} 2^{j\tau\left(s+d(\frac{1}{2}-\frac{1}{\tau})\right)}\, \sigma_j^\tau S_{j,p}\right]\\
&\preceq N^{-p\frac{s-\nu}{d}}\,\mathrm{E}\left[\|X\|^\tau_{B^s_\tau(L_\tau(\mathcal{O}))}\right]
\end{aligned}$$

with $\mathrm{E}[S_{j,p}] = \#\nabla_j \rho_j \nu_p = \#\nabla_j \rho_j \nu_\tau \frac{\nu_p}{\nu_\tau} = \mathrm{E}[S_{j,\tau}]\frac{\nu_p}{\nu_\tau}$. Since we have $\mathrm{E}[\|X\|^\tau_{B^s_\tau(L_\tau(\mathcal{O}))}] < \infty$ for $s < s^*$, by (3.10), we see that the average nonlinear approximation error satisfies

$$e^{\mathrm{avg}}_{N,p,\nu}(X) \preceq N^{-\frac{s-\nu}{d}} \left(\mathrm{E}[\|X\|^\tau_{B^s_\tau(L_\tau(\mathcal{O}))}]\right)^{1/p}. \tag{3.31}$$

From (3.31) we observe that, similar to the deterministic setting, the approximation order which can be achieved by nonlinear approximation does not depend on the regularity in the same scale of smoothness spaces of the object under consideration, but on the regularity in the corresponding scale (3.15) of Besov spaces.

For the case $p = 2$, i.e., for nonlinear wavelet approximation with respect to H^ν, also a lower bound for the average nonlinear approximation error can be derived.

Theorem 3.25. *Let $\beta \in [0,1)$. For a fixed approximation space $H^\nu(\mathcal{O})$, $\nu \in \mathbb{R}$, let X be given by* (3.3) *with $-d/2 \le \nu < d(\alpha - 1 + \beta)/2$, i.e., $X \in B^s_\tau(L_\tau(\mathcal{O}))$ in the scale* (3.15) *for all $s < s^*$, where s^* is given by* (3.16) *with $p = 2$. Then the average nonlinear approximation error in $H^\nu(\mathcal{O})$ satisfies*

$$e^{\mathrm{avg}}_{N,2,\nu}(X) \succeq (\log_2 N)^{\frac{\gamma d}{2}} N^{-\frac{1}{1-\beta}\left(\frac{\alpha-1+\beta}{2}-\frac{\nu}{d}\right)}. \tag{3.32}$$

Proof. Let X be defined by (3.3). For every level j, we define the number of scaled coefficients of X larger than a threshold $\delta_j > 0$ as

$$M(j,\delta_j) := \#\left\{2^{j\nu}\sigma_j Y_{j,k}|Z_{j,k}| \,:\, 2^{j\nu}\sigma_j Y_{j,k}|Z_{j,k}| > \delta_j,\ k \in \nabla_j\right\}. \tag{3.33}$$

We set $Y_{j,\beta} := \sum_{k\in\nabla_j} Y_{j,k}$ and obtain $Y_{j,\beta} \sim \mathrm{Bin}(2^{jd}, 2^{-\beta jd})$. Since the $(Y_{j,k})_{j,k}$ are discrete and $(Z_{j,l})_{j,l}$ are identically distributed we can compute

$$\begin{aligned}
\mathrm{E}\left[M(j,\delta_j)\right] &= \sum_{l=0}^{2^{jd}} \mathrm{E}\left[M(j,\delta_j)\,\Big|\sum_{k\in\nabla_j} Y_{j,k} = l\right] \mathrm{P}\left(\sum_{k\in\nabla_j} Y_{j,k} = l\right) \\
&= \sum_{l=0}^{2^{jd}} l\,\mathrm{P}\left(2^{j\nu}\sigma_j |Z_{j,l}| > \delta_j\right) \mathrm{P}\left(Y_{j,\beta} = l\right) \\
&= \mathrm{E}[Y_{j,\beta}]\,\mathrm{P}(2^{j\nu}\sigma_j|Z_{j,k}| > \delta_j) \\
&= 2^{jd(1-\beta)}\, 2\left(1 - \Phi_{\mathrm{cdf}}\left(\frac{\delta_j}{2^{j\nu}\sigma_j}\right)\right),
\end{aligned}$$

where Φ_{cdf} denotes the cumulative distribution function of the standard normal distribution. Now, we choose

$$\delta_j := 2^{j\nu}\sigma_j \tag{3.34}$$

and we obtain $\mathrm{E}[M(j, 2^{j\nu}\sigma_j)] = c_1\, 2^{jd(1-\beta)}$ with $c_1 := 2(1-\Phi_{\mathrm{cdf}}(1))$. For a given $N \in \mathbb{N}_0$ we set $j_1 := \min\{j \,:\, N \le 2^{\frac{jd}{2}}\}$ and determine a level j_2, such that

$$\mathrm{E}\left[M(j_2, 2^{j_2\nu}\sigma_{j_2})\right] \ge c_1 N^2. \tag{3.35}$$

This holds for

$$j_2 = \left\lceil \frac{j_1}{1-\beta} \right\rceil. \tag{3.36}$$

Up to this point we have shown that, for X and any given $N \in \mathbb{N}_0$, we can find a level j_2, which contains on average at least $c_1 N^2$ coefficients, that are larger than δ_{j_2}.

Let

$$\widehat{X}_N := \sum_{j=j_0}^{\infty} \sum_{k\in\widehat{\nabla}_j} c_{j,k}\psi_{j,k} := \sum_{\lambda\in\widehat{\nabla}} c_\lambda \psi_\lambda$$

with $\mathrm{E}[\#\widehat{\nabla}] = \mathrm{E}[\eta(\widehat{X}_N)] \le N$ be any approximation of

$$X = \sum_{j=j_0}^{\infty} \sum_{k\in\nabla_j} \sigma_j Y_{j,k} Z_{j,k} \psi_{j,k} := \sum_{\lambda\in\nabla} d_\lambda \psi_\lambda.$$

We set $|\lambda| := j$. Then, by using the norm equivalence from (W6), we obtain

$$\begin{aligned}
\mathrm{E}[\|X - \widehat{X}_N\|^2_{H^\nu(\mathcal{O})}] &= \mathrm{E}\left[\left\| \sum_{\lambda\in\nabla} d_\lambda\psi_\lambda - \sum_{\lambda\in\widehat{\nabla}} c_\lambda\psi_\lambda \right\|^2_{H^\nu(\mathcal{O})}\right] \\
&= \mathrm{E}\left[\left\| \sum_{\lambda\in\nabla\setminus\widehat{\nabla}} d_\lambda\psi_\lambda + \sum_{\lambda\in\widehat{\nabla}} (d_\lambda - c_\lambda)\psi_\lambda \right\|^2_{H^\nu(\mathcal{O})}\right] \\
&\asymp \mathrm{E}\left[\sum_{\lambda\in\nabla\setminus\widehat{\nabla}} 2^{2|\lambda|\nu}|d_\lambda|^2 + \sum_{\lambda\in\widehat{\nabla}} 2^{2|\lambda|\nu}|d_\lambda - c_\lambda|^2 \right].
\end{aligned}$$

If we omit the second sum and by (3.33) and (3.34), we get

$$
\begin{aligned}
\mathrm{E}[\|X - \widehat{X}_N\|^2_{H^\nu(\mathcal{O})}] &\succeq \mathrm{E}\left[\sum_{\lambda \in \nabla \setminus \widehat{\nabla}} 2^{2|\lambda|\nu} |d_\lambda|^2\right] \\
&\succeq \mathrm{E}\left[\sum_{\lambda \in \nabla_{j_2} \setminus \widehat{\nabla}} 2^{2|\lambda|\nu} |d_\lambda|^2\right] \\
&\geq \mathrm{E}\left[\sum_{k \in \nabla_{j_2} \setminus \widehat{\nabla}} 2^{2j_2\nu} |d_{j_2,k}|^2\right] \\
&\geq \mathrm{E}\left[\#\left\{k \in \nabla_{j_2} \setminus \widehat{\nabla} \,:\, 2^{2j_2\nu}|d_{j_2,k}|^2 > \delta^2_{j_2}\right\}\right] \cdot \delta^2_{j_2} \\
&\geq \mathrm{E}\left[M(j_2, 2^{j_2\nu}\sigma_{j_2}) - \#\widehat{\nabla}\right] \cdot 2^{2j_2\nu}\sigma^2_{j_2} \\
&= \left(\mathrm{E}[M(j_2, 2^{j_2\nu}\sigma_{j_2})] - \mathrm{E}[\#\widehat{\nabla}]\right) \cdot 2^{2j_2\nu}\sigma^2_{j_2},
\end{aligned}
$$

so that, by inserting (3.1), (3.35), (3.36), $\mathrm{E}[\#\widehat{\nabla}] \leq N \leq 2^{\frac{j_1 d}{2}}$, we can conclude

$$
\begin{aligned}
\mathrm{E}[\|X - \widehat{X}_N\|^2_{H^\nu(\mathcal{O})}] &\succeq \left(2^{j_1 d} - 2^{\frac{j_1 d}{2}}\right) 2^{2j_2\nu} j_2^{\gamma d} 2^{-\alpha j_2 d} \\
&\succeq j_1^{\gamma d} 2^{j_1 d + \frac{2j_1\nu}{1-\beta} - \frac{\alpha j_1 d}{1-\beta}} \\
&\asymp (\log_2 N)^{\gamma d} N^{-\frac{1}{1-\beta}(\alpha - 1 + \beta - \frac{2\nu}{d})},
\end{aligned}
$$

which yields (3.32). □

Remark 3.26. For the proof of Theorem 3.25 it is essential to be able to compute the expected value of $M(j, \delta_j)$, i.e., the average number of coefficients on level j which are larger than the threshold δ_j. This random variable can be derived solely due to the structure of X. Since the threshold $\delta_j = 2^{j\nu} j^{\gamma d/2} 2^{-\alpha j d/2}$ decays with increasing level j, cf. (3.34), the growth of $\mathrm{E}[M(j, \delta_j)]$ is in compliance with Theorem 3.10.

Remark 3.27. Observe that the upper bound in Theorem 3.23 for $p = 2$ coincides with the lower bound in Theorem 3.25.

3.1.4 Realizations and moments of Besov norms of X

We illustrate the impact of the parameters α, γ, and $\beta \in [0, 1)$ on individual realizations of X, defined in (3.3), as well as on the sparsity and decay of its wavelet coefficients in the case $\mathcal{O} = (0, 1)$. The parameter α influences decay, while β induces sparsity patterns in the wavelet coefficients. By providing a polynomial scaling, the parameter γ allows to emphasize or convey any singularities within finitely many coefficients.

The numerical experiments were performed by using a stable biorthogonal spline wavelet basis as constructed in Primbs [136]. The primal wavelets consist of cardinal splines of order $m = 3$, i.e., they are piecewise quadratic polynomials, and the condition (W4) is satisfied with $\widetilde{m} = 5$. The wavelet basis satisfies (W6) along the nonlinear approximation line with $s_1 = 3$, while $s_1 = 2.5$ along the linear approximation line, see Section 2.3.2. Moreover, $j_0 = 2$ and $\#\nabla_{j_0} = 10$, while $\#\nabla_j = 2^j$ for $j > j_0$. In parameters (3.1) of the stochastic model for X we set

$$C_1 = 2^{\beta j_0}, \quad C_2 = 2^{\alpha j_0},$$

which means that sparsity is only induced at levels $j > j_0$ and the coefficients at level j_0 are standard normally distributed. This ensures that we keep the entire polynomial part of X.

In any simulation, only a finite number of coefficients can be handled. Therefore, we truncate the wavelet decomposition in a suitable way, i.e.,

$$\widehat{X}_{j_1} := \sum_{j=j_0}^{j_1} \sum_{k \in \nabla_j} \sigma_j Y_{j,k} Z_{j,k}\, \psi_{j,k}, \tag{3.37}$$

is the truncation of X at level j_1, cf. (3.20).

Theorems 3.17 and 3.23 provide error bounds for the approximation of X by $\widehat{X}_{j_1}$ in terms of the expected number of non-zero coefficients. An efficient way of simulating $\widehat{X}_{j_1}$ is presented in Remark 3.18. Specifically, we choose

$$\alpha \in \{2.0, 1.8, 1.5, 1.2\} \quad \text{and} \quad \beta = 2 - \alpha,$$

which is motivated as follows. At first, $\alpha = 2$ and $\beta = 0$ corresponds to the smoothness of a Brownian motion, see Remark 3.19, and secondly, according to Theorems 3.15 and 3.17 for our choice of α and β the order of linear approximation is kept constant while the order of best average N-term approximation increases with β. The two underlying scales of Besov spaces $B^s_\tau(L_\tau(\mathcal{O}))$ are the linear approximation scale, where $\tau = 2$, and the nonlinear approximation scale, where $1/\tau = s + 1/2$, cf. (3.15) with $\nu = 0$ and $p = 2$ for L_2-approximation.

We set

$$V_{j_1} := \max_{j \le j_1} \max_{k \in \nabla_j} \sigma_j Y_{j,k} |Z_{j,k}|$$

in order to normalize the absolute values of the coefficients. The left column in Figure 3.2 shows realizations of the normalized absolute values $\sigma_j Y_{j,k}|Z_{j,k}|/V_{j_1}$ of all coefficients up to level

$$j_1 = 12.$$

It exhibits that the parameter β induces sparsity patterns, for larger values of β more coefficients are zero and the wavelet decomposition of X is sparser. Figure 3.2 also illustrates the corresponding sample functions. We observe that for $\beta = 0$ the sample function is irregular everywhere, and by increasing β the irregularities become more and more isolated. This does not affect the L_2-Sobolev smoothness, while on the other hand it is well known that piecewise smooth functions with isolated singularities have a higher Besov smoothness on the nonlinear approximation scale, cf. Remark 2.23. According to Theorem 3.10 and Corollary 3.12, X belongs to a space on the linear approximation scale with probability one if and only if

$$s < \frac{1}{2}, \tag{3.38}$$

while X belongs to a space on the nonlinear approximation scale with probability one if and only if

$$s < \frac{1}{2(1-\beta)}. \tag{3.39}$$

Hence, these upper bounds reflect the orders of convergence for linear and best average N-term approximation, respectively.

Moments of Besov norms or of equivalent norms on sequence spaces appear in the constants in the error bounds that are derived in this section, see, e.g., Remarks 3.21 and 3.24 where this has been worked out. Here, we consider $\mathcal{O} = (0,1)$, and the moments of X along the linear and nonlinear approximation scale. We set

$$b_j(s,\tau) := 2^{j(s+(\frac{1}{2}-\frac{1}{\tau}))\tau}.$$

By (W6) we get

$$\|X\|^\tau_{B^s_\tau(L_\tau(\mathcal{O}))} \asymp \sum_{j=j_0}^{\infty} b_j(s,\tau)\,\sigma_j^\tau \sum_{k\in\nabla_j} Y_{j,k}|Z_{j,k}|^\tau.$$

Denoting with ν_τ the absolute moment of order τ of the standard normal distribution, i.e., $\nu_\tau := 2^{\tau/2}\,\Gamma((\tau+1)/2)/\pi^{1/2}$, we obtain $\mathrm{E}[\|X\|^\tau_{B^s_\tau(L_\tau(\mathcal{O}))}] \asymp M(s,\tau)$ with

$$M(s,\tau) = \nu_\tau \sum_{j=j_0}^{\infty} b_j(s,\tau)\,\#\nabla_j \sigma_j^\tau \rho_j.$$

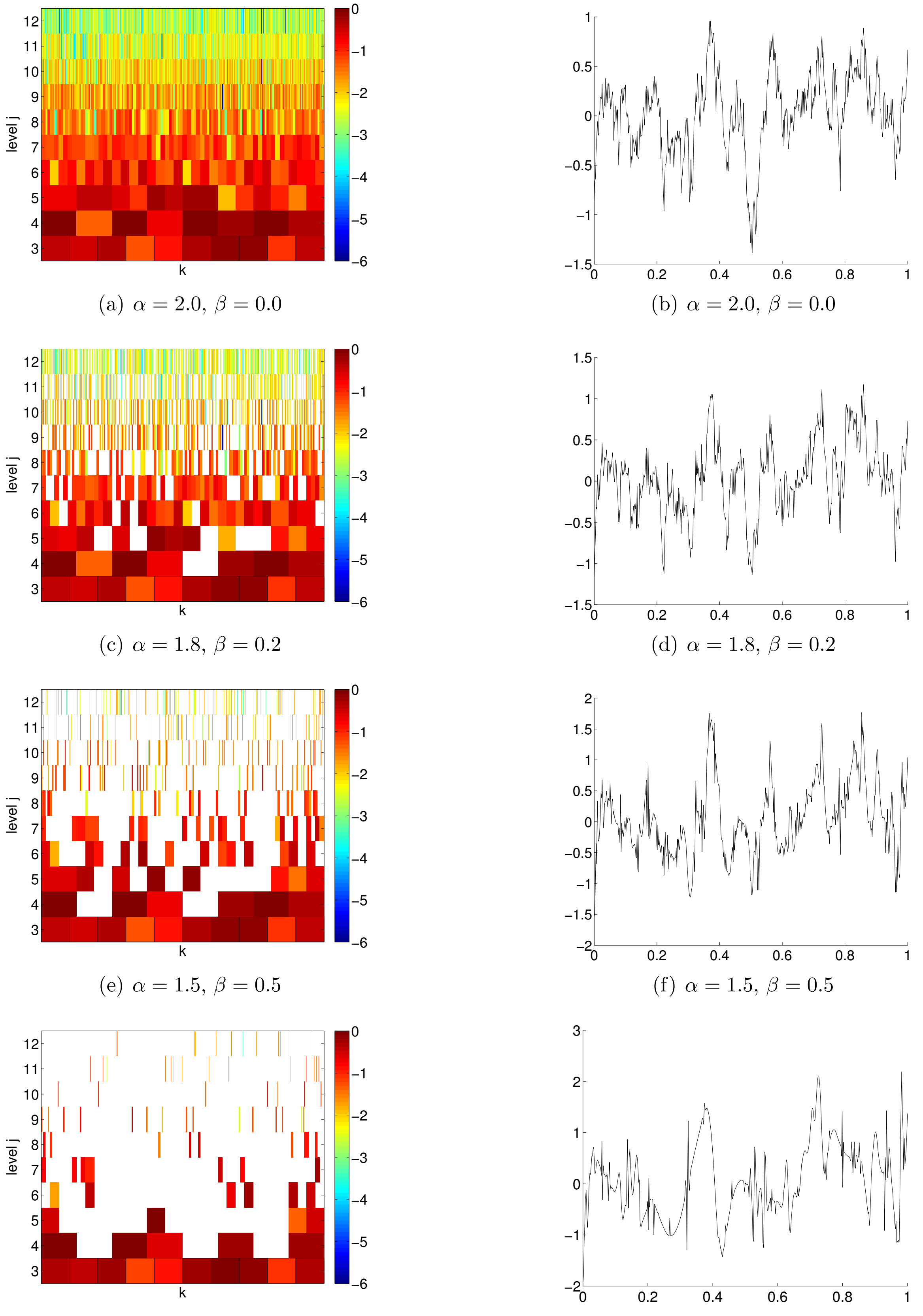
Figure 3.5 contains the graphs of M along the linear and nonlinear approximation scales, i.e., $s \mapsto M(s,2)$ and $s \mapsto M(s,1/(s+1/2))$, for the selected values of α and β, while $\gamma = 0$. Note that the upper bounds (3.38) and (3.39), respectively, also provide the location of the singularities of M along the two scales.

The effect of truncation in Figure 3.5 (a) at level $j_1 = 20$, (b) at level $j_1 = 78$ and finite sample size is illustrated in Figure 3.5, as well, by presenting sample means of the right-hand side in

$$\|\widehat{X}_{j_1}\|^\tau_{B^s_\tau(L_\tau(\mathcal{O}))} \asymp \sum_{j=j_0}^{j_1} b_j(s,\tau)\cdot\sigma_j^\tau \sum_{k\in\nabla_j} Y_{j,k}|Z_{j,k}|^\tau.$$

Specifically, for each scale and each choice of α and β we consider 4 different values of s and use 1000 independent samples. Moreover, for each choice of the parameters α and β, the truncation level j_1 is chosen according to Table 3.1 so that the expected number $\sum_{j=j_0}^{j_1} \#\nabla_j\,\rho_j$ of non-zero coefficients is approximately 10^6 in all cases. We observe the strongest impact of truncation for the nonlinear approximation scales and small values of β. We add that confidence intervals for the level 0.95 are of length less than 10 in all cases.

Likewise we proceed for $(\mathrm{E}[\|X\|^2_{B^s_\tau(L_\tau(\mathcal{O}))}])^{1/2}$ along the linear and nonlinear approximation scales. Figure 3.6 shows the results after truncation (a) at level $j_1 = 20$, (b) at level $j_1 = 78$, sampling, and applying the norm equivalence. It is worth noting that the sampled Besov norms are smaller than the sampled L_2-Sobolev norms. We add that confidence intervals for the level 0.95 are of length less than one percent of the estimate in all cases.



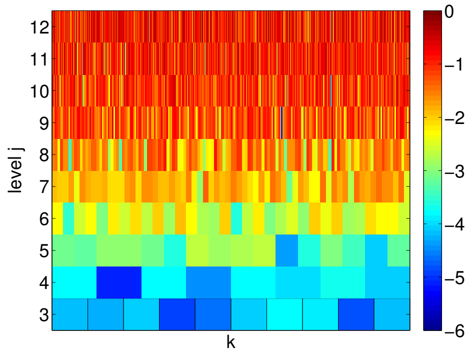
(a) $\alpha = 2.0$, $\beta = 0.0$ (b) $\alpha = 2.0$, $\beta = 0.0$

(c) $\alpha = 1.8$, $\beta = 0.2$ (d) $\alpha = 1.8$, $\beta = 0.2$

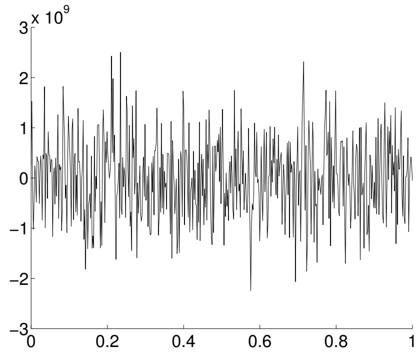
(e) $\alpha = 1.5$, $\beta = 0.5$ (f) $\alpha = 1.5$, $\beta = 0.5$

(g) $\alpha = 1.2$, $\beta = 0.8$ (h) $\alpha = 1.2$, $\beta = 0.8$

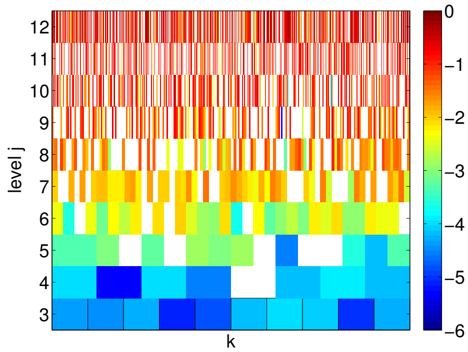
Figure 3.2: $\widehat{X}_{j_1}(\omega)$, $\gamma = 0$. Left: absolute values of normalized coefficients. Right: respective sample function



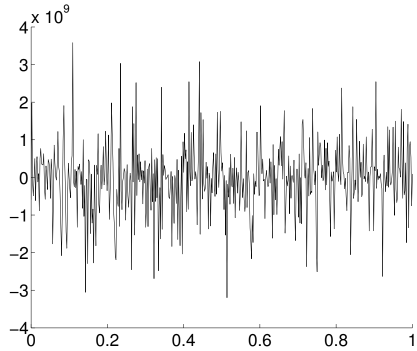
(a) $\alpha = 2.0,\ \beta = 0.0,\ \gamma = 10.0$



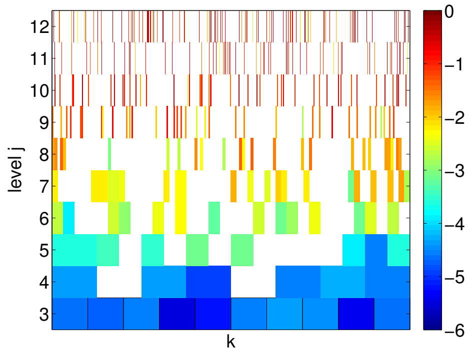
(b) $\alpha = 2.0,\ \beta = 0.0,\ \gamma = 10.0$



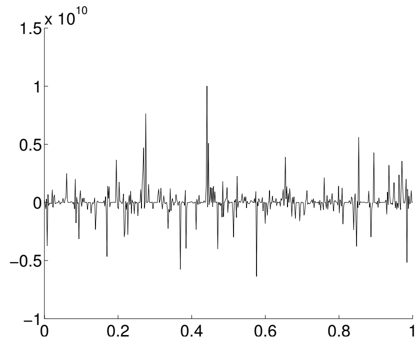
(c) $\alpha = 1.8,\ \beta = 0.2,\ \gamma = 10.0$



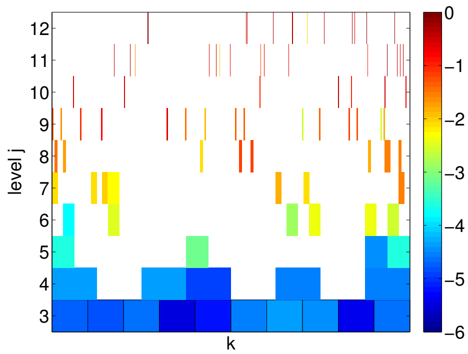
(d) $\alpha = 1.8,\ \beta = 0.2,\ \gamma = 10.0$



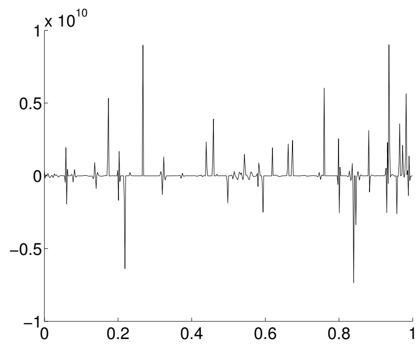
(e) $\alpha = 1.5,\ \beta = 0.5,\ \gamma = 10.0$



(f) $\alpha = 1.5,\ \beta = 0.5,\ \gamma = 10.0$

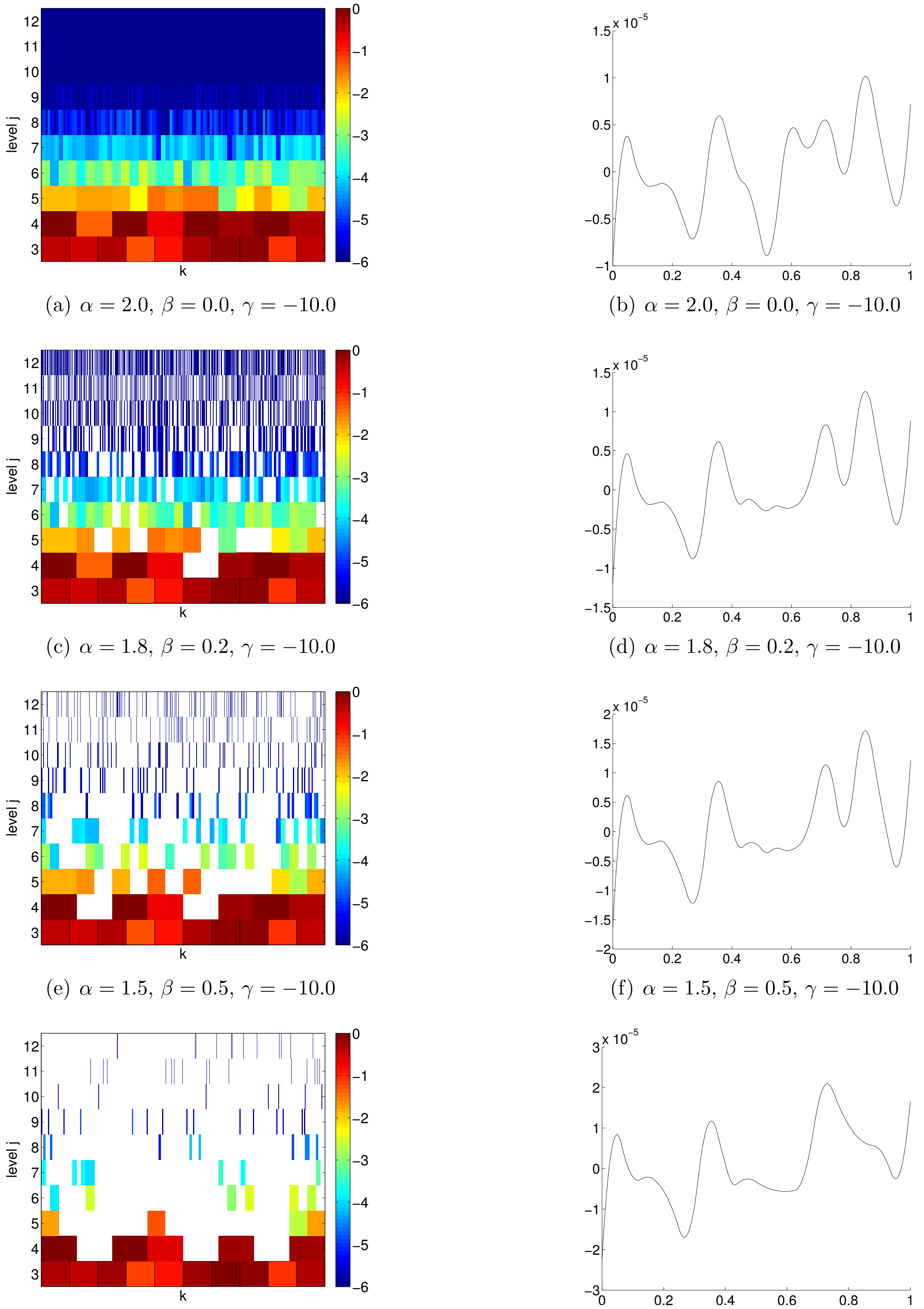


(g) $\alpha = 1.2,\ \beta = 0.8,\ \gamma = 10.0$



(h) $\alpha = 1.2,\ \beta = 0.8,\ \gamma = 10.0$

Figure 3.3: $\widehat{X}_{j_1}(\omega)$. Left: absolute values of normalized coefficients. Right: respective sample function.



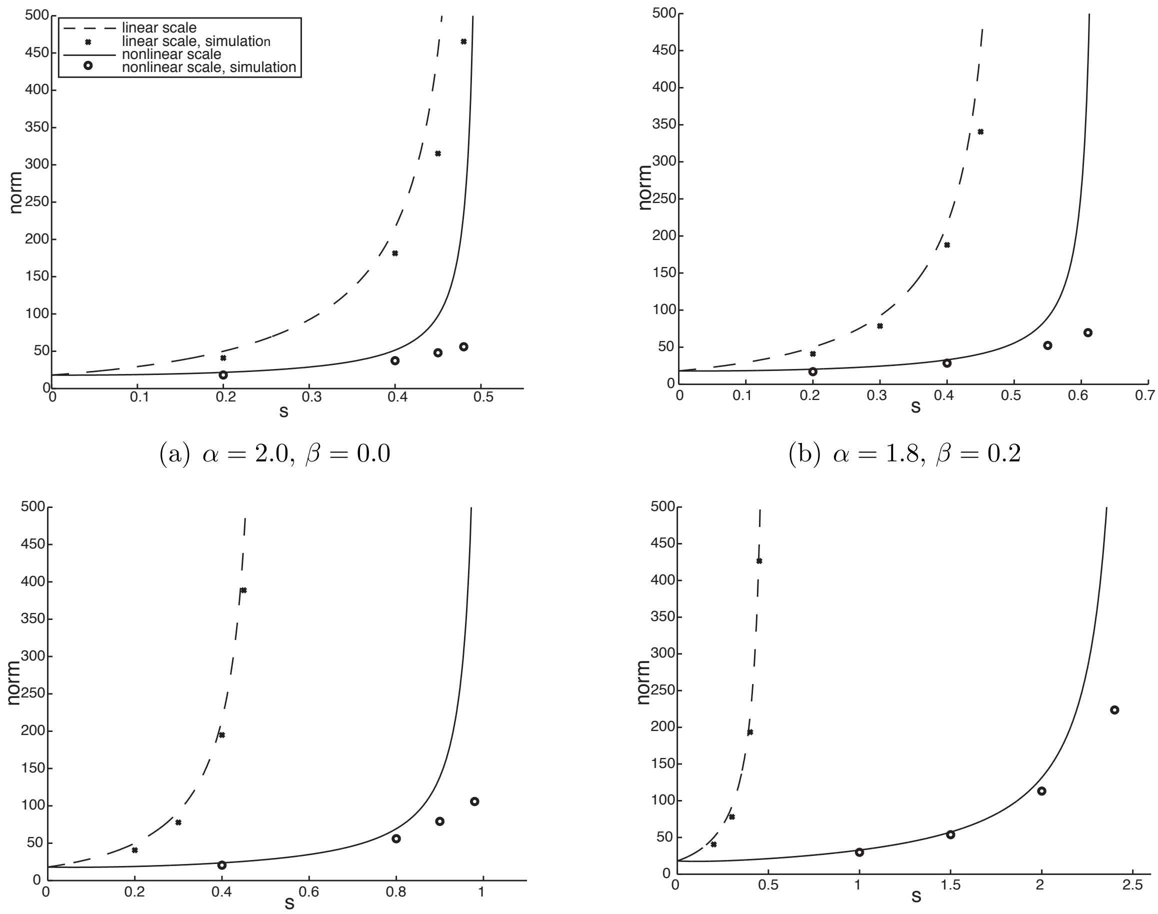
(a) $\alpha = 2.0$, $\beta = 0.0$, $\gamma = -10.0$ (b) $\alpha = 2.0$, $\beta = 0.0$, $\gamma = -10.0$

(c) $\alpha = 1.8$, $\beta = 0.2$, $\gamma = -10.0$ (d) $\alpha = 1.8$, $\beta = 0.2$, $\gamma = -10.0$

(e) $\alpha = 1.5$, $\beta = 0.5$, $\gamma = -10.0$ (f) $\alpha = 1.5$, $\beta = 0.5$, $\gamma = -10.0$

(g) $\alpha = 1.2$, $\beta = 0.8$, $\gamma = -10.0$ (h) $\alpha = 1.2$, $\beta = 0.8$, $\gamma = -10.0$

Figure 3.4: $\widehat{X}_{j_1}(\omega)$. Left: absolute values of normalized coefficients. Right: respective sample function



(a) $\alpha = 2.0$, $\beta = 0.0$

(b) $\alpha = 1.8$, $\beta = 0.2$

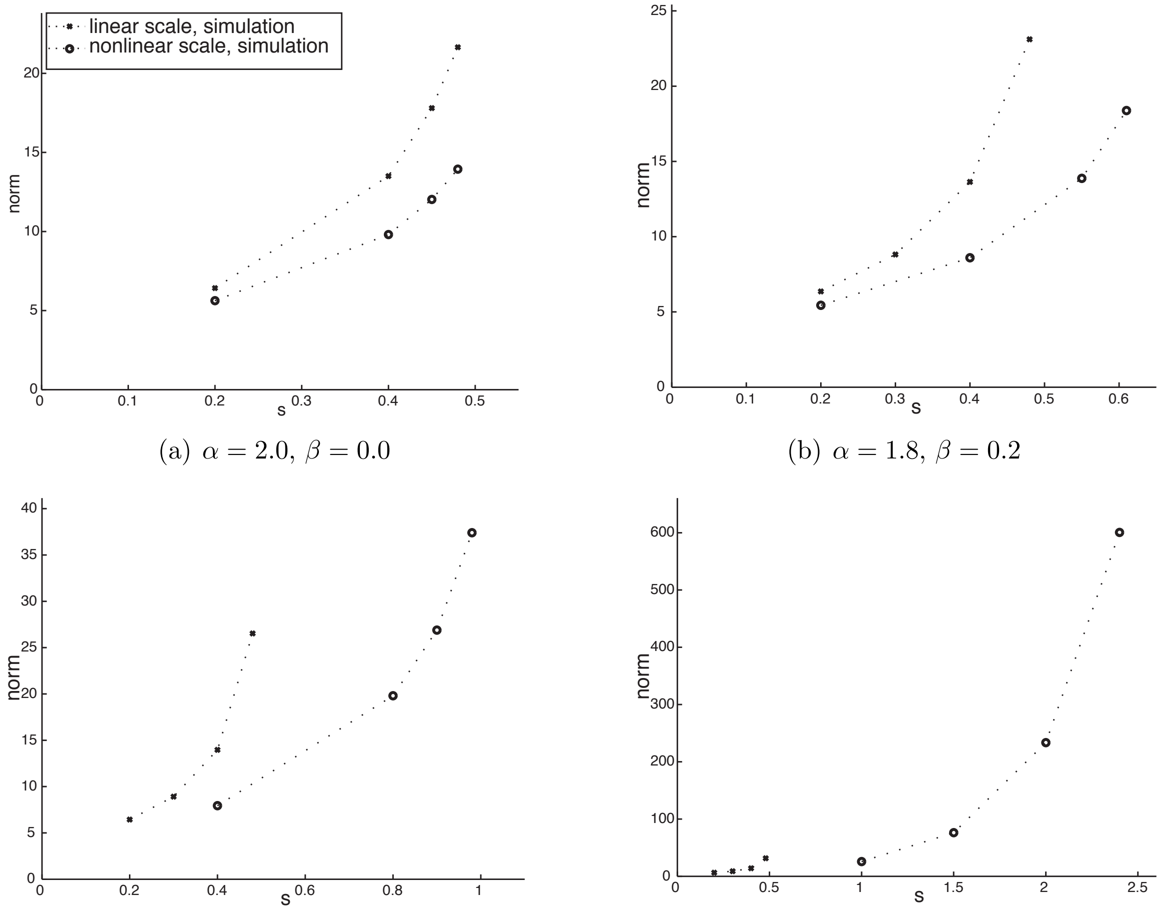
(c) $\alpha = 1.5$, $\beta = 0.5$

(d) $\alpha = 1.2$, $\beta = 0.8$

Figure 3.5: $\mathrm{E}[\|X\|^{\tau}_{B^s_\tau(L_\tau(\mathcal{O}))}]$ along the linear and nonlinear approximation scales

α	2.0	1.8	1.5	1.2
j_1	20	24	36	78

Table 3.1: Truncation levels in Figures 3.5 and 3.6



(a) $\alpha = 2.0$, $\beta = 0.0$

(b) $\alpha = 1.8$, $\beta = 0.2$

(c) $\alpha = 1.5$, $\beta = 0.5$

(d) $\alpha = 1.2$, $\beta = 0.8$

Figure 3.6: $(\mathrm{E}[\|X\|^2_{B^s_\tau(L_\tau(\mathcal{O}))}])^{1/2}$ along the linear and nonlinear approximation scales

3.2 A class of random functions in anisotropic Besov spaces

We analyze the regularity of a class of random functions in anisotropic Besov spaces. We introduce the random functions based on a similar stochastic model as considered in the previous section and we derive conditions, see Theorem 3.28, under which such a random function almost surely has a certain smoothness in $B_q^{s,\mathbf{a}}(L_p(\mathcal{O}))$, where $0 < p,q < \infty$, $s > d(1/p-1)_+$, and $\boldsymbol{a} = (a_1, \ldots, a_d) \in \mathbb{R}_+^d$ is a given anisotropy. See Section 2.2.2 for the definition of anisotropic Besov spaces $B_q^{s,\mathbf{a}}(L_p(\mathcal{O}))$ and Section 2.3.4 for the construction of an anisotropic wavelet basis. In particular, the employed wavelet characterization by Garrigós et al. [82, 83] is stated in Theorem 2.31.

3.2.1 The stochastic model in the anisotropic case

Let Ψ and $\widetilde{\Psi}$ be a dual pair of anisotropic wavelet bases, such that the smoothness range that can be characterized is large enough for all further considerations. Furthermore, we assume Ψ and $\widetilde{\Psi}$ to satisfy (M7), i.e., $\#\nabla_{e,j} \asymp m^j$, which is based on our approach to treat bounded domains, cf. (2.29) in Section 2.3.4.

In the anisotropic case, the random functions are defined by

$$X := \sum_{k\in\nabla_0} \sigma_0\, Y_{0,k}\, Z_{0,k}\, \phi_{0,k}^{(p)} + \sum_{j=0}^{\infty}\sum_{e=1}^{m-1}\sum_{k\in\nabla_{e,j}} \sigma_j\, Y_{e,j,k}\, Z_{e,j,k}\, \psi_{e,j,k}^{(p)}, \tag{3.40}$$

where $(Y_{e,j,k}, Z_{e,j,k})_{e,j,k}$, $e = 1, ..., m-1$, $j \in \mathbb{N}_0$, $k \in \nabla_{e,j}$, is an independent family of random variables on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. As before, the stochastic model depends on

$$\alpha, \gamma \in \mathbb{R}, \quad \beta \in [0,1]. \tag{3.41}$$

The variables $Y_{e,j,k}$ are Bernoulli distributed with parameter

$$\rho_j := 2^{-\beta j d}, \quad \text{where} \quad \mathrm{P}(Y_{e,j,k} = 1) = \rho_j \quad \text{and} \quad \mathrm{P}(Y_{e,j,k} = 0) = 1 - \rho_j.$$

The variables $Z_{e,j,k}$ are $\mathcal{N}(0,1)$-distributed, and we set

$$\sigma_j^2 := j^{\gamma d} 2^{-\alpha j d}, \quad j \in \mathbb{N}, \quad \text{and} \quad \sigma_0 := 1.$$

3.2.2 Regularity theorem in the anisotropic case

We state conditions on the parameters (3.41) under which a random function X almost surely belongs to a given anisotropic Besov space $B_q^{s,\mathbf{a}}(L_p(\mathcal{O}))$.

Theorem 3.28. *Let the assumptions of Theorem 2.31 and (M7) be satisfied, i.e., in particular the wavelet characterizations* (2.27) *and* (2.28) *hold. Let X be a random function as defined in* (3.40). *Then $X|_{\mathcal{O}}$ is* P*-almost surely contained in $B_q^{s,\mathbf{a}}(L_p(\mathcal{O}))$, for $s > d(1/p-1)_+$ and either $1 \le p,q < \infty$ or $0 < p = q < 1$, if*

$$s < \frac{d^2}{\log_2 m}\left(\frac{\alpha}{2} + \frac{\beta}{p}\right) - \frac{d}{p} \tag{3.42}$$

or

$$s \leq \frac{d^2}{\log_2 m}\left(\frac{\alpha}{2}+\frac{\beta}{p}\right)-\frac{d}{p} \quad \textit{and} \quad q\gamma d < -2. \tag{3.43}$$

In both cases

$$\mathrm{E}[\|X|_{\mathcal{O}}\|^q_{B^{s,\mathbf{a}}_q(L_p(\mathcal{O}))}] < \infty. \tag{3.44}$$

Proof. Since

$$\|f\|_{B^{s,\mathbf{a}}_q(L_p(\mathcal{O}))} := \inf\left\{\|g\|_{B^{s,\mathbf{a}}_q(L_p(\mathbb{R}^d))} \;:\; g \in B^{s,\mathbf{a}}_q(L_p(\mathbb{R}^d)),\; g|_{\mathcal{O}} = f\right\},$$

it is sufficient to show that X as defined by (3.40) is P-a.s. contained in $B^{s,\mathbf{a}}_q(L_p(\mathbb{R}^d))$.

Using Theorem 2.31 and that the support of X is contained in the cube $\square$ by (2.29), we have $X \in B^{s,\mathbf{a}}_q(L_p(\mathbb{R}^d))$ P-a.s. if and only if

$$\left(\sum_{k\in\nabla_0} |\langle X, \widetilde{\phi}^{(p')}_{0,k}\rangle|^p\right)^{1/p} + \sum_{j=0}^{\infty} m^{\frac{sq}{d}j}\left(\sum_{e=1}^{m-1}\sum_{k\in\nabla_{e,j}} |\langle X, \widetilde{\psi}^{(p')}_{e,j,k}\rangle|^p\right)^{q/p} < \infty \quad \text{P-a.s.}$$

Observe that the first sum in this formula is finite, since by (M7) the set ∇_0 is assumed to be finite. Therefore we are left to show that the second sum is also finite. Inserting (3.40) and using the abbreviation

$$S_{e,j,p} := \sum_{k\in\nabla_{e,j}} Y_{e,j,k}|Z_{e,j,k}|^p,$$

yields

$$\sum_{j=0}^{\infty} m^{\frac{sq}{d}j}\left(\sum_{e=1}^{m-1}\sum_{k\in\mathbb{Z}^d} |\langle X, \widetilde{\psi}^{(p')}_{e,j,k}\rangle|^p\right)^{q/p} \asymp \sum_{j=0}^{\infty} m^{\frac{sq}{d}j}\sigma_j^q\left(\sum_{e=1}^{m-1} S_{e,j,p}\right)^{q/p}.$$

So, with $a_j := m^{\frac{sq}{d}j}\sigma_j^q$, we have to show when

$$\sum_{j=0}^{\infty} a_j\left(\sum_{e=1}^{m-1} S_{e,j,p}\right)^{q/p} < \infty, \quad \text{P-a.s.} \tag{3.45}$$

Analogously to the corresponding steps in the proof of Theorem 3.10 it can be shown that (3.45) is equivalent to

$$\sum_{j=0}^{\infty} a_j\left(\sum_{e=1}^{m-1} \#\nabla_{e,j}\rho_j\right)^{q/p} < \infty. \tag{3.46}$$

Inserting (M7) and the stochastic model into (3.46) yields

$$\begin{aligned}\sum_{j=0}^{\infty} a_j\left(\sum_{e=1}^{m-1} \#\nabla_{e,j}\rho_j\right)^{q/p} &\asymp \sum_{j=0}^{\infty} a_j\ \left((m-1)m^j\rho_j\right)^{q/p}\\ &\asymp \sum_{j=0}^{\infty}(m-1)^{q/p}\, j^{\frac{\gamma q d}{2}}\, 2^{jqR},\end{aligned}$$

with $R := (\frac{s}{d}\log_2 m - \frac{\alpha d}{2} + \frac{1}{p}\log_2 m - \frac{\beta d}{p})$. Therefore, (3.46) holds if and only if the conditions (3.42) or (3.43) are satisfied.

It remains to prove (3.44). Note, since (3.42) or (3.43) are satisfied, we have that (3.46) holds. Using the norm equivalences (2.27) and (2.28), as well as Lemma 3.7 and Lemma 3.8, with $\nabla_j := \bigcup_{e=1}^{m-1} \nabla_{e,j}$ and $S_{j,p} := \sum_{e=1}^{m-1} S_{e,j,p}$ instead of (3.4), we can conclude

$$\mathrm{E}[\|X\|^q_{B^{s,\mathbf{a}}_q(L_p(\mathbb{R}^d))}] \preceq \sum_{j=0}^{\infty} a_j \mathrm{E}\left[\left(\sum_{e=1}^{m-1} S_{e,j,p}\right)^{q/p}\right] \preceq \sum_{j=0}^{\infty} a_j \left(\sum_{e=1}^{m-1} \#\nabla_{e,j}\rho_j\right)^{q/p} < \infty. \quad \square$$

Remark 3.29. Due to our approach (2.29) to handle bounded domains, we are only able to state sufficient conditions in Theorem 3.28.

Remark 3.30. For $p = 2$ and in the isotropic case, i.e., $\boldsymbol{a} = \mathbf{1}$, $\lambda = 2$, and $\log_2 m = d$, the statements (3.42) and (3.43) of Theorem 3.28 coincide with (3.8) and (3.9) of Theorem 3.10. For $p \neq 2$ and in the isotropic case, note that the wavelets in (2.25) are normalized in $L_p(\mathbb{R}^d)$ and a renormalization of these wavelets to $L_2(\mathbb{R}^d)$ produces the additional factor $m^{d(1/2-1/p)}$. Therefore, the statements (3.42) and (3.43) coincide with (3.8) and (3.9) also in this case.

3.3 A class of random tensor wavelet decompositions

We study a class of random tensor wavelet decompositions that we also introduce based on the stochastic model of the previous sections. We derive conditions, see Theorem 3.31, under which such a random tensor decomposition almost surely has a certain smoothness in the tensor space $H^{\boldsymbol{t},\ell}(\mathcal{O})$, $\boldsymbol{t} \in [0,\infty)^n$, $\ell \in [0,\infty)$, of generalized dominating mixed derivatives. See Section 2.2.3 for the definition of $H^{\boldsymbol{t},\ell}(\mathcal{O})$ and Section 2.3.5 for the tensor wavelet setting. In particular, the employed wavelet characterization by Griebel et al. [85, 86] is stated in Theorem 2.35.

3.3.1 The stochastic model for random tensor decompositions

The aim is to derive random tensor decompositions of the form

$$X := \sum_{\boldsymbol{\lambda}\in\boldsymbol{\Lambda}} a_{\boldsymbol{\lambda}}(X) \bigotimes_{m=1}^{n} \psi^{(m)}_{\lambda_m}, \quad \text{where} \quad a_{\boldsymbol{\lambda}}(X) = \prod_{\lambda_m\in\Lambda_m} a_{\lambda_m}(X), \tag{3.47}$$

which have a prescribed smoothness in $H^{\boldsymbol{t},\ell}(\mathcal{O})$. Here, the sequence $(a_{\lambda_m}(X))_{\lambda_m\in\Lambda_m}$ of random variables $a_{\lambda_m} : \Omega \to \mathbb{R}$ is based on a stochastic model, which is similar to the stochastic models of the previous sections. Given $\alpha_m, \gamma_m \in \mathbb{R}$, and $\beta_m \in [0,1]$, $m = 1,\ldots,n$, and a probability space $(\Omega, \mathcal{F}, \mathrm{P})$, we set

$$a_{\lambda_m} := a_{j_m,k_m} := \sigma_{j_m} Y^{(m)}_{j_m,k_m} Z^{(m)}_{j_m,k_m}, \qquad j_m \in \mathbb{N}_0,\ k_m \in \nabla_{j_m},\ m = 1,\ldots,n, \tag{3.48}$$

where $Z^{(m)}_{j_m,k_m} \sim \mathcal{N}(0,1)$ are standard-normally distributed,

$$\sigma^2_{j_m} := j_m^{\gamma_m d_m} 2^{-\alpha_m j_m d_m}, \quad \sigma_{j_m} := 1 \text{ for } j_m = 0, \tag{3.49}$$

and $Y^{(m)}_{j_m,k_m}$ are Bernoulli distributed random variables with parameter

$$\rho_{j_m} := 2^{-\beta_m j_m d_m} \quad \text{and} \quad \mathrm{P}(Y^{(m)}_{j_m,k_m} = 1) = 1 - \mathrm{P}(Y^{(m)}_{j_m,k_m} = 0) = \rho_{j_m}. \tag{3.50}$$

Also, we assume the family of random variables $(Y^{(m)}_{j_m,k_m}, Z^{(m)}_{j_m,k_m})_{m,j_m,k_m}$ to be independent.

3.3.2 Regularity theorem in the tensor case

We state conditions on the parameters $\alpha_m, \beta_m, \gamma_m$, $m = 1, \dots, n$, such that the decomposition (3.47) with (3.48) is almost surely contained in a given tensor space $H^{\boldsymbol{t},\ell}(\mathcal{O})$.

Theorem 3.31. *Let the assumptions of Theorem 2.35 and (T2) be satisfied. Let X be a random tensor decomposition of the form* (3.47) *with* (3.48)*. Then*

i) X is contained in $H^{\boldsymbol{t},\ell}(\mathcal{O})$, $\boldsymbol{t} = (t_1, \dots, t_n)$, P*-almost surely if and only if*

$$t_m + \ell < d_m \left(\frac{\alpha_m + \beta_m - 1}{2} \right), \quad \gamma_m = 0, \quad m = 1, \dots, n. \tag{3.51}$$

ii) X is contained in $H^{\boldsymbol{t},\ell}(\mathcal{O})$, $\boldsymbol{t} = (t_1, \dots, t_n)$, P*-almost surely if*

$$t_m + \ell \le d_m \left(\frac{\alpha_m + \beta_m - 1}{2} \right), \quad \gamma_m d_m < -1, \quad m = 1, \dots, n. \tag{3.52}$$

In both cases

$$\mathrm{E}[\|X\|^2_{H^{\boldsymbol{t},\ell}(\mathcal{O})}] < \infty. \tag{3.53}$$

Proof. In order to prove (3.51) and (3.52), according to Theorem 2.35, we check under which conditions on α_m, β_m, and γ_m, $m = 1, \dots, n$, we have

$$\sum_{\boldsymbol{\lambda} \in \boldsymbol{\Lambda}} |a_{\boldsymbol{\lambda}}(X)|^2 \, 2^{2\boldsymbol{t}|\boldsymbol{\lambda}| + 2\ell \||\boldsymbol{\lambda}|\|_\infty} < \infty, \quad \text{P-a.s.} \tag{3.54}$$

Step 1. Inserting (3.48) into (3.54) and using the abbreviations

$$S_{j_m,2} := \sum_{k_m \in \nabla_{j_m}} Y^{(m)}_{j_m,k_m} |Z^{(m)}_{j_m,k_m}|^2, \quad m = 1, \dots, n, \tag{3.55}$$

as well as $\bar{a}_{\boldsymbol{j}} := 2^{2\boldsymbol{t}\boldsymbol{j} + 2\ell\|\boldsymbol{j}\|_\infty} \sigma^2_{j_1} \cdots \sigma^2_{j_n}$, where $\boldsymbol{j} = (j_1, \dots, j_n)$, yields that we have to check when

$$\sum_{\boldsymbol{j} \in \mathbb{N}_0^n} \bar{a}_{\boldsymbol{j}} \prod_{m=1}^n S_{j_m,2} < \infty, \quad \text{P-a.s.} \tag{3.56}$$

Applying Lemma 3.7 and Lemma 3.8, we obtain that (3.56) is equivalent to

$$\sum_{\boldsymbol{j} \in \mathbb{N}_0^n} \bar{a}_{\boldsymbol{j}} \prod_{m=1}^n \#\nabla_{j_m} \rho_{j_m} < \infty, \tag{3.57}$$

which is shown analogously to the corresponding steps in the proof of Theorem 3.10.

Step 2. We first prove the claim for the case $n = 2$, which illustrates the key ideas of the proof. Moreover, we restrict this case to $\gamma_1 = \gamma_2 = 0$ in (3.49) of the stochastic model. The general case is proven in *Step 4*.

With $n = 2$ we have $\boldsymbol{t} = (t_1, t_2)$, $\boldsymbol{\Lambda} = \Lambda_1 \times \Lambda_2$, and $|\boldsymbol{\lambda}| = (|\lambda_1|, |\lambda_2|) = (j_1, j_2)$ for $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$, as well as $a_{\boldsymbol{\lambda}} = a_{\lambda_1} a_{\lambda_2}$ in (3.47). Furthermore, we have $\lambda_m = (j_m, k_m)$ with $j_m \in \mathbb{N}_0$ and $k_m \in \nabla_{j_m}$, where ∇_{j_m} are finite sets with $\#\nabla_{j_m} \asymp 2^{j_m d_m}$, $m = 1, 2$, cf. (T2). Hence, we have to check under which conditions

$$\begin{aligned}
\sum_{\boldsymbol{\lambda} \in \boldsymbol{\Lambda}} |a_{\boldsymbol{\lambda}}|^2\, 2^{2\boldsymbol{t}|\boldsymbol{\lambda}| + 2\ell \||\boldsymbol{\lambda}|\|_\infty} &= \sum_{\lambda_1 \in \Lambda_1} \sum_{\lambda_2 \in \Lambda_2} |a_{\lambda_1}|^2 |a_{\lambda_2}|^2\, 2^{2(t_1|\lambda_1|,\, t_2|\lambda_2|) + 2\ell\|(|\lambda_1|,\, |\lambda_2|)\|_\infty} \\
&= \sum_{j_1 \in \mathbb{N}_0} \sum_{k_1 \in \nabla_{j_1}} \sum_{j_2 \in \mathbb{N}_0} \sum_{k_2 \in \nabla_{j_2}} |a_{j_1,k_1}|^2 |a_{j_2,k_2}|^2\, 2^{2t_1 j_1 + 2t_2 j_2 + 2\ell\|(j_1, j_2)\|_\infty} \\
&< \infty, \quad \text{P-a.s.}
\end{aligned}$$

That is, inserting (3.48) and the abbreviation (3.55) we have to show under which conditions on α_m, β_m, γ_m, $m = 1, 2$, we get

$$\sum_{j_1 \in \mathbb{N}_0} 2^{2t_1 j_1} \sigma_{j_1}^2\, S_{j_1,2} \sum_{j_2 \in \mathbb{N}_0} 2^{2t_2 j_2 + 2\ell\|(j_1, j_2)\|_\infty} \sigma_{j_2}^2\, S_{j_2,2} < \infty, \quad \text{P-a.s.} \tag{3.58}$$

Applying Lemma 3.7 and Lemma 3.8, (3.58) is equivalent to

$$\sum_{j_1 \in \mathbb{N}_0} 2^{2t_1 j_1} \sigma_{j_1}^2\, \#\nabla_{j_1} \rho_{j_1} \sum_{j_2 \in \mathbb{N}_0} 2^{2t_2 j_2 + 2\ell\|(j_1, j_2)\|_\infty} \sigma_{j_2}^2\, \#\nabla_{j_2} \rho_{j_2} < \infty.$$

Using (T2) and (3.49) with $\gamma_1 = \gamma_2 = 0$, as well as (3.50) we continue with the calculation

$$\begin{aligned}
&\sum_{j_1 \in \mathbb{N}_0} 2^{2t_1 j_1} \sigma_{j_1}^2\, \#\nabla_{j_1} \rho_{j_1} \sum_{j_2 \in \mathbb{N}_0} 2^{2t_2 j_2 + 2\ell\|(j_1, j_2)\|_\infty} \sigma_{j_2}^2\, \#\nabla_{j_2} \rho_{j_2} \\
&\quad \asymp \sum_{j_1 \in \mathbb{N}_0} 2^{j_1(2t_1 - d_1(\alpha_1 + \beta_1 - 1))} \sum_{j_2 \in \mathbb{N}_0} 2^{j_2(2t_2 - d_2(\alpha_2 + \beta_2 - 1)) + 2\ell\|(j_1, j_2)\|_\infty} \\
&\quad = \sum_{j_1 \in \mathbb{N}_0} 2^{j_1(2t_1 - d_1(\alpha_1 + \beta_1 - 1))} \\
&\qquad\qquad \left(\sum_{j_2 < j_1} 2^{j_2(2t_2 - d_2(\alpha_2 + \beta_2 - 1)) + 2\ell j_1} + \sum_{j_2 \geq j_1} 2^{j_2(2t_2 - d_2(\alpha_2 + \beta_2 - 1) + 2\ell)} \right) \\
&\quad =: \sum_{j_1 \in \mathbb{N}_0} \mathcal{A}_{j_1} \left(\sum_{j_2 < j_1} \mathcal{B}_{j_1,j_2} + \sum_{j_2 \geq j_1} \mathcal{C}_{j_1,j_2} \right) =: \sum_{j_1 \in \mathbb{N}_0} \mathcal{A}_{j_1} \big(\mathcal{B}_{j_1} + \mathcal{C}_{j_1} \big).
\end{aligned} \tag{3.59}$$

Now, observe that $\sum_{j_2 \geq j_1} \mathcal{C}_{j_1,j_2} < \infty$ if and only if

$$t_2 + \ell < d_2 \left(\frac{\alpha_2 + \beta_2 - 1}{2} \right). \tag{3.60}$$

Applying the geometric series formula we obtain

$$\sum_{j_2 \geq j_1} \mathcal{C}_{j_1,j_2} \asymp 2^{j_1(2t_2 - d_2(\alpha_2 + \beta_2 - 1) + 2\ell)}$$

and hence, we have

$$\sum_{j_1\in\mathbb{N}_0} \mathcal{A}_{j_1}\mathcal{C}_{j_1} \asymp \sum_{j_1\in\mathbb{N}_0} 2^{j_1(2t_1-d_1(\alpha_1+\beta_1-1))}\, 2^{j_1(2t_2-d_2(\alpha_2+\beta_2-1)+2\ell)},$$

which is finite if and only if

$$t_1+t_2+\ell < d_1\left(\frac{\alpha_1+\beta_1-1}{2}\right)+d_2\left(\frac{\alpha_2+\beta_2-1}{2}\right). \tag{3.61}$$

To show $\sum_{j_1\in\mathbb{N}_0}\mathcal{A}_{j_1}\mathcal{B}_{j_1}<\infty$ in (3.59), we again apply the geometric series formula on $\sum_{j_2<j_1}\mathcal{B}_{j_1,j_2}$, and we are left to determine when

$$\sum_{j_1\in\mathbb{N}_0} 2^{j_1(2t_1-d_1(\alpha_1+\beta_1-1)+2\ell)} - 2^{j_1(2t_1-d_1(\alpha_1+\beta_1-1)+2t_2-d_2(\alpha_2+\beta_2-1)+2\ell)} < \infty. \tag{3.62}$$

Summing the difference in formula (3.62) separately, we see that (3.62) is finite if and only if (3.61) and

$$t_1+\ell < d_1\left(\frac{\alpha_1+\beta_1-1}{2}\right) \tag{3.63}$$

hold, or the exponents are both zero or equal. The latter cases yield the condition $t_2 = d_2(\alpha_2+\beta_2-1)/2$ which contradicts (3.60). Observe that (3.60) and (3.63) imply (3.61).

Step 3. In order to generalize *Step 2* to the case $n>2$ and to show (3.51), we iterate the ideas of *Step 2*. In particular, observe that the conditions on α_m, β_m, $m=1,\dots,n$, such that (3.57) holds are determined by splitting the appearing sums iteratively as in (3.59). Proceeding with analogous arguments leads to the conditions (3.51).

Step 4. Now, we show (3.52). Using (T2) and inserting (3.49) and (3.50) into (3.57) yields that we have to determine when

$$\sum_{j_1\in\mathbb{N}_0} j_1^{\gamma_1 d_1}\, 2^{j_1(2t_1-d_1(\alpha_1+\beta_1-1))} \cdots \sum_{j_n\in\mathbb{N}_0} j_n^{\gamma_n d_n}\, 2^{j_n(2t_n-d_n(\alpha_n+\beta_n-1))\,+\,2\ell\|(j_1,\dots j_n)\|_\infty} < \infty.$$

We proceed with the same idea as in *Step 3* by iteratively splitting these sums. Then, using that $\gamma_n d_n < -1$ the sum over j_n, where $j_n = \|(j_1,\dots,j_n)\|_\infty$, converges also if we choose the parameters α_n,β_n such that $t_n+\ell = d_n(\alpha_n+\beta_n-1)/2$ holds. Using that $j_n^{\gamma_n d_n}\le 1$ in all other places it appears, we proceed to the sum where $j_{n-1} = \|(j_1,\dots,j_n)\|_\infty$. Analogously, using $\gamma_{n-1}d_{n-1}<-1$ this sum converges if we have $t_{n-1}+\ell = d_{n-1}(\alpha_{n-1}+\beta_{n-1}-1)/2$. Repeating these arguments yields (3.52).

Finally we prove (3.53) provided that (3.57) is satisfied. Using the Riesz basis property in Theorem 2.35, as well as Lemma 3.7 and Lemma 3.8 together with the independence of the $S_{j_m,2}$, $m=1,\dots,n$, we have

$$\mathrm{E}[\|X\|^2_{H^{\boldsymbol{t},\ell}(\mathcal{O})}] \preceq \sum_{\boldsymbol{j}\in\mathbb{N}_0^n} a_{\boldsymbol{j}}\mathrm{E}\left[\prod_{m=1}^n S_{j_m,2}\right] \preceq \sum_{\boldsymbol{j}\in\mathbb{N}_0^n} a_{\boldsymbol{j}}\prod_{m=1}^n \#\nabla_{j_m}\rho_{j_m} < \infty. \qquad \square$$

Remark 3.32. Observe that, in the case $\boldsymbol{t}=0$, where $H^{\boldsymbol{0},\ell}(\mathcal{O})$ is isomorphic to the standard Sobolev space, for $p=q=2$ the conditions (3.51) and (3.52) of Theorem 3.31 coincide with (3.8) and (3.9) of Theorem 3.10.

Chapter 4

Application to the stochastic Poisson equation

We consider the stochastic Poisson equation, where the right-hand side is a random function which is given by the stochastic model that is analyzed in Section 3.1. In order to obtain approximations to the realizations of the solution, we employ asymptotically optimal adaptive wavelet algorithms as they asymptotically realize the approximation rate of best N-term wavelet approximation. Since the related convergence analysis of these adaptive wavelet algorithms relies on the energy norm, which is equivalent to the norm in H^1, we approximate the realizations of the solutions in H^1. See Section 1.4 for the complete introduction.

In Section 4.1, we analyze best N-term wavelet approximation for the considered class of stochastic Poisson equations under different assumptions on the bounded domain $\mathcal{O} \subset \mathbb{R}^d$, see Theorems 4.1, 4.5, and 4.6. These asymptotic results are matched by numerical experiments in Section 4.2.

The results of this chapter have been partly worked out by the author and collaborators in [24].

4.1 Best N-term wavelet approximation

First, we consider the Poisson equation on bounded Lipschitz domains $\mathcal{O} \subset \mathbb{R}^d$,

$$\begin{aligned} -\Delta U(\omega) &= X(\omega) \quad \text{in } \mathcal{O}, \\ U(\omega) &= 0 \qquad\quad \text{on } \partial\mathcal{O}, \end{aligned} \tag{4.1}$$

with $\omega \in \Omega$ and a random right-hand side $X : \Omega \to L_2(\mathcal{O}) \subset H^{-1}(\mathcal{O})$ that is defined as the class of random functions in Section 3.1: Let

$$\begin{gathered} \alpha, \gamma \in \mathbb{R}, \quad \beta \in [0,1], \\ \rho_j := \min\left\{1,\, C_1 2^{-\beta j d}\right\}, \quad \text{and} \quad \sigma_j^2 := \begin{cases} C_2 j^{\gamma d} 2^{-\alpha j d} & : j > j_0, \\ 1 & : j = 0, \end{cases} \end{gathered} \tag{4.2}$$

where $j_0 \in \mathbb{N}_0$ and $C_1, C_2 > 0$. Furthermore, let $(Z_{j,k}, Y_{j,k})$ for $j \geq j_0$ and $k \in \nabla_j$ be an independent family of random variables on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$, where

$$Z_{j,k} \sim \mathcal{N}(0,1), \quad \text{and} \quad \mathrm{P}(Y_{j,k} = 1) = 1 - \mathrm{P}(Y_{j,k} = 0) = \rho_j. \tag{4.3}$$

Given biorthogonal wavelet bases Ψ, $\widetilde{\Psi}$ which satisfy the underlying assumptions (W1)–(W6) stated in Section 2.3.3, we define the random functions

$$X := \sum_{j=j_0}^{\infty} \sum_{k \in \nabla_j} \sigma_j \, Y_{j,k} \, Z_{j,k} \, \widetilde{\psi}_{j,k}, \tag{4.4}$$

cf. (3.3). Here, X is given as decomposition in the dual wavelet basis $\widetilde{\Psi}$, since this way the approximation to the solution U of (4.1) is a decomposition with respect to Ψ.

In order to analyze best N-term wavelet approximations of the random function

$$U : \Omega \to H^1(\mathcal{O})$$

in $H^1(\mathcal{O})$, we introduce the deterministic error of best N-term wavelet approximation with respect to $H^1(\mathcal{O})$ by

$$e^{\text{det}}_{N,H^1(\mathcal{O})}(u) := \inf \left\{ \|u - \widehat{u}\|_{H^1(\mathcal{O})} \, : \, \widehat{u} \in H^1(\mathcal{O}), \; \eta(\widehat{u}) \leq N \right\},$$

cf. Section 2.3.2. Again,

$$\eta(g) := \# \left\{ \lambda \in \nabla \, : \, g = \sum_{\lambda \in \nabla} c_\lambda \, \psi_\lambda, \; c_\lambda \neq 0 \right\}$$

denotes the number of non-zero wavelet coefficients of g. The quantity $e^{\text{det}}_{N,H^1(\mathcal{O})}(U(\omega))$, where $U(\omega)$, $\omega \in \Omega$, is the exact solution of (4.1), serves as benchmark for the performance of the adaptive algorithms. In the stochastic setting, we investigate the error

$$e^{\text{best}}_{N,H^1(\mathcal{O})}(U) := \inf \left(\mathrm{E} \, [\|U - \widehat{U}\|^2_{H^1(\mathcal{O})}] \right)^{1/2},$$

with the infimum taken over all measurable mappings $\widehat{U} : \Omega \to H^1(\mathcal{O})$ such that

$$\eta(\widehat{X}(\omega)) \leq N \quad \text{P-a.s.}$$

Clearly, we have

$$e^{\text{best}}_{N,H^1(\mathcal{O})}(U) = \left(\mathrm{E} \, [e^{\text{det}}_{N,H^1(\mathcal{O})}(U)^2] \right)^{1/2}.$$

Theorem 4.1. *Suppose that $d \in \{2,3\}$ and that the right-hand side X in (4.1) is of the form (4.4) with $\alpha + \beta > 1 - 4/d$. We set*

$$\varrho := \min \left\{ \frac{1}{2(d-1)}, \; \frac{\alpha + \beta - 1}{6} + \frac{2}{3d} \right\}.$$

Then, for every $\varepsilon > 0$, the error of the best N-term wavelet approximation with respect to $H^1(\mathcal{O})$ satisfies

$$e^{\text{best}}_{N,H^1(\mathcal{O})}(U) \preceq N^{-\varrho+\varepsilon}.$$

Proof. Let $r > 1$. By Remark 2.26 in Section 2.3.3 we have

$$e^{\text{det}}_{N,H^1(\mathcal{O})}(u) \preceq \|u\|_{B^r_\tau(L_\tau(\mathcal{O}))} \, N^{-(r-1)/d}, \tag{4.5}$$

for all $d \geq 1$ in the scale

$$\frac{1}{\tau} = \frac{r-1}{d} + \frac{1}{2}.$$

The next step is to control the norm of a solution u in the Besov space $B^r_\tau(L_\tau(\mathcal{O}))$ in terms of the regularity of the right-hand side x of the Poisson equation. Let

$$-\frac{1}{2} < r^* < \frac{4-d}{2(d-1)},$$

and assume that $x \in H^{r^*}(\mathcal{O})$. We may apply the results from Dahlke et al. [45], Dahlke, DeVore [47] to conclude that $u \in B^{r-\delta}_\tau(L_\tau(\mathcal{O}))$ for sufficiently small $\delta > 0$, where

$$r = \frac{r^*+5}{3} \quad \text{and} \quad \frac{1}{\tau} = \frac{r-\delta-1}{d} + \frac{1}{2}.$$

Moreover,

$$\|u\|_{B^{r-\delta}_\tau(L_\tau(\mathcal{O}))} \preceq \|x\|_{B^{r^*}_2(L_2(\mathcal{O}))}$$

and we can use (4.5), with r replaced by $r-\delta$, to derive

$$e^{\text{det}}_{N,H^1(\mathcal{O})}(u) \preceq \|x\|_{B^{r^*}_2(L_2(\mathcal{O}))} N^{-(r^*+2-3\delta)/(3d)}.$$

If, in addition, $r^*/d < (\alpha+\beta-1)/2$, then

$$e_{N,H^1(\mathcal{O})}(U) \preceq N^{-(r^*+2-3\delta)/(3d)}$$

follows from the regularity result of Theorem 3.10. □

Remark 4.2. In Theorem 4.1, we have concentrated on nonlinear approximations in $H^1(\mathcal{O})$, which is the most important one from the numerical point of view, as we have briefly outlined in the introductory Section 1.4. In the deterministic setting, similar results for approximations in other norms, e.g., in L_2 or even weaker norms, also exist, see, e.g., Dahlke et al. [54] for details.

Remark 4.3. The convergence order of nonadaptive uniform methods does not depend on the Besov regularity of the exact solution but on its Sobolev smoothness, see Remark 2.26. However, on a Lipschitz domain, due to singularities at the boundary, the best one can expect is $U \in H^{3/2}(\mathcal{O})$, even for very smooth right-hand sides, see Grisvard [89], Jerison, Kenig [104]. Therefore, an uniform approximation scheme can only give at best the order $N^{-1/(2d)}$. In our setting, see Theorem 4.1, we have

$$\varrho > \frac{1}{2d},$$

so that for the problem (4.1) optimal adaptive wavelet schemes are always superior when compared with uniform schemes.

Remark 4.4. With increasing values of α and β the smoothness of X increases, see the regularity result of Theorem 3.10. On a general Lipschitz domain, however, this does not necessarily increase the Besov regularity of the corresponding solution. This is reflected by the fact that the upper bound in Theorem 4.1 is at most of order $N^{-1/(2(d-1))}$.

For more specific domains better results are available. For instance, suppose $\mathcal{O}$ is a simply connected polygonal domain in $\mathbb{R}^2$. Then, it is well known that if the right-hand side X in (4.1) is contained in $H^{r-1}(\mathcal{O})$ for some $r \geq 0$, the solution U can be uniquely decomposed into a regular part U_R and a singular part U_S, i.e., $U = U_R + U_S$, where $U_R \in H^{r+1}(\mathcal{O})$ and U_S belongs to a finite-dimensional space that only depends on the shape of the domain. This result has been established by Grisvard [87], see also [88, Chapter 4, 5], or [89, Section 2.7] for details.

Theorem 4.5. *Suppose that $\mathcal{O}$ is a simply connected polygonal domain in $\mathbb{R}^2$ and that the right-hand side X in* (4.1) *is of the form* (4.4) *with $\alpha + \beta > 1/2$. We set*

$$\varrho = \frac{\alpha + \beta}{2}.$$

Then, for every $\varepsilon > 0$, the error of the best N-term wavelet approximation with respect to $H^1(\mathcal{O})$ satisfies

$$e^{\text{best}}_{N,H^1(\mathcal{O})}(U) \preceq N^{-\varrho+\varepsilon}.$$

Proof. We apply the results from Grisvard [87, 88, 89]. Let us denote the segments of $\partial\mathcal{O}$ by $\overline{\Gamma}_1, \ldots, \overline{\Gamma}_M$, $M \in \mathbb{N}$, with open sets Γ_ℓ, $\ell = 1, \ldots, M$, numbered in positive orientation. Furthermore, let Υ_ℓ denote the endpoint of Γ_ℓ and let χ_ℓ denote the measure of the interior angle at Υ_ℓ, $\ell = 1, \ldots, M$. We introduce polar coordinates $(\kappa_\ell, \theta_\ell)$ in the vicinity of each vertex Υ_ℓ, and for $n \in \mathbb{N}$ and $\ell = 1, \ldots, M$ we introduce the functions

$$\mathcal{S}_{\ell,n}(\kappa_\ell, \theta_\ell) := \zeta_\ell(\kappa_\ell)\, \kappa_\ell^{\lambda_{\ell,n}} \sin(n\pi\theta_\ell/\chi_\ell),$$

when $\lambda_{\ell,n} = n\pi/\chi_\ell$ is not an integer, and

$$\mathcal{S}_{\ell,n}(\kappa_\ell, \theta_\ell) := \zeta_\ell(\kappa_\ell)\, \kappa_\ell^{\lambda_{\ell,n}} \left[\log \kappa_\ell \sin(n\pi\theta_\ell/\chi_\ell) + \theta_\ell \cos(n\pi\theta_\ell/\chi_\ell)\right]$$

otherwise. Here ζ_ℓ denotes a suitable C^∞ truncation function.

Consider the solution $u = u_R + u_S$ of the Poisson equation with the right-hand side $x \in H^{r-1}(\mathcal{O})$, and assume that

$$r \notin \{\lambda_{\ell,n} : n \in \mathbb{N},\ \ell = 1, \ldots, M\}.$$

Then one has $u_R \in H^{r+1}(\mathcal{O})$ and $u_S \in \mathcal{S}_{\text{span}}$ for $\mathcal{S}_{\text{span}} := \operatorname{span}\{\mathcal{S}_{\ell,n} \,:\, 0 < \lambda_{n,l} < r\}$. We have to estimate the Besov regularity of both, u_S and u_R, in the scale

$$\frac{1}{\tau} = \frac{s}{2},$$

which is (3.15) with $d = 2$, $p = 2$, and $\nu = 1$, i.e., the regularity in this scale is related with nonlinear wavelet approximation in $H^1(\mathcal{O})$. Classical embeddings of Besov spaces imply that $u_R \in B^s_\tau(L_\tau(\mathcal{O}))$ for every $s < r+1$. Moreover, it has been shown in Dahlke [42] that $\mathcal{S}_{\text{span}} \subset B^s_\tau(L_\tau(\mathcal{O}))$ for every $s > 0$. We conclude that $u_S \in B^s_\tau(L_\tau(\mathcal{O}))$ for every $s < r + 1$.

To estimate u, we argue as follows. Let γ_ℓ be the trace operator with respect to the segment Γ_ℓ, $\ell = 1, \ldots, M$. Grisvard [88, Theorem 5.1.3.5] has shown that the Laplacian Δ maps the direct sum

$$H = \left\{u \in H^{r+1}(\mathcal{O}) \,:\, \gamma_\ell u = 0,\ \ell = 1, \ldots, M\right\} + \mathcal{S}_{\text{span}}$$

onto $H^{r-1}(\mathcal{O})$. Note that $(H, \|\cdot\|_H)$ is a Banach space where

$$\|u\|_H = \|u_R\|_{H^{r+1}(\mathcal{O})} + \sum_{\ell=1}^{M} \sum_{0<\lambda_{\ell,n}<r} |c_{\ell,n}| \quad \text{and} \quad u_S = \sum_{\ell=1}^{M} \sum_{0<\lambda_{\ell,n}<r} c_{\ell,n}\, \mathcal{S}_{\ell,n}.$$

It has been shown in Dahlke et al. [52] that the solution operator Δ^{-1} is continuous as a mapping from $H^{r-1}(\mathcal{O})$ onto H. Therefore

$$\|u\|_{B^s_\tau(L_\tau(\mathcal{O}))} \preceq \|u_R\|_{H^{r+1}(\mathcal{O})} + \sum_{\ell=1}^{M} \sum_{0<\lambda_{\ell,n}<t} |c_{\ell,n}| = \|u\|_H \preceq \|x\|_{H^{r-1}(\mathcal{O})}$$

for every $s < r+1$.

Finally, by Theorem 3.10, $X \in H^{r-1}(\mathcal{O})$ with probability one and $\mathrm{E}[\|X\|^2_{H^{r-1}(\mathcal{O})}]$ is finite if $1/2 < r < \alpha + \beta$. Now the upper bound for $e^{\text{best}}_{N,H^1(\mathcal{O})}(U)$ follows by proceeding as in the proof of Theorem 4.1. □

In case $\mathcal{O}$ is a C^∞-domain, no singularities induced by the shape of the domain can occur. However, similar to Corollary 3.12, it is a remarkable fact that for β close to one an arbitrarily high order of convergence can be realized.

Theorem 4.6. *Suppose that $\mathcal{O}$ is a bounded C^∞-domain in $\mathbb{R}^d$ and that the right-hand side X in* (4.1) *is of the form* (4.4) *with $\alpha/2 + \beta > 1/2$. Moreover, we assume that $\beta < 1$ and we set*

$$\varrho = \frac{1}{1-\beta}\left(\frac{\alpha-1}{2} + \beta\right) + \frac{1}{d}.$$

Then, for every $\varepsilon > 0$, the error of the best N-term wavelet approximation with respect to $H^1(\mathcal{O})$ satisfies

$$e^{\text{best}}_{N,H^1(\mathcal{O})}(U) \preceq N^{-\varrho+\varepsilon}.$$

Proof. An application of Corollary 3.12 with $p = 2$ and $\nu = -d/2 + \delta d$ for a sufficiently small $\delta > 0$, yields $X \in B^s_\tau(L_\tau(\mathcal{O}))$ in the scale

$$\frac{1}{\tau} = \frac{s}{d} + 1 - \delta \tag{4.6}$$

with probability one and $\mathrm{E}[\|X\|^2_{B^s_\tau(L_\tau(\mathcal{O}))}]$ is finite for all

$$-\frac{d}{2} < s < \frac{d}{1-\beta}\left(\frac{\alpha-1}{2} + \beta(1-\delta)\right) = s^*.$$

If $\alpha/2 + \beta > 1/2 + \delta$, we have $s^* > \delta d$ and we can argue as follows. Since the problem is regular, the solution u of the Poisson equation with right-hand side $x \in B^s_\tau(L_\tau(\mathcal{O}))$, $s > \delta > 0$ satisfies $u \in B^{s+2}_\tau(L_\tau(\mathcal{O}))$ with

$$\|u\|_{B^{s+2}_\tau(L_\tau(\mathcal{O}))} \preceq \|x\|_{B^s_\tau(L_\tau(\mathcal{O}))},$$

see Runst, Sickel [143, Chapter 3] or Triebel [157, Theorem 4.3]. By the embedding of Besov spaces given in Theorem 2.6 (iv), we obtain

$$\|u\|_{B^{s+2}_{\tau^*}(L_{\tau^*}(\mathcal{O}))} \preceq \|u\|_{B^{s-\delta+2}_\tau(L_\tau(\mathcal{O}))}$$

for the approximation scale

$$\frac{1}{\tau^*} = \frac{(s+2)-1}{d} + \frac{1}{2}.$$

An application of (4.5) yields

$$e^{\mathrm{det}}_{N,H^1(\mathcal{O})}(u) \preceq \|u\|_{B^{s+2}_{\tau^*}(L_{\tau^*}(\mathcal{O}))} \, N^{-(s+1)/d}.$$

Therefore, we conclude that

$$e^{\mathrm{best}}_{N,H^1(\mathcal{O})}(U) \preceq N^{-(s+1)/d}.$$

Let δ tend to zero to obtain the result as claimed. □

Remark 4.7. For $\beta = 1$ the estimate from Theorem 4.6 is valid for arbitrarily large ϱ, provided that the primal and dual wavelet bases are sufficiently smooth, so that they can characterize the considered Besov spaces.

4.2 Numerical experiments using adaptive wavelet methods

We employ an asymptotically optimal and efficient adaptive wavelet algorithm and present numerical experiments on $[0, 1]$ to complement our analysis of Section 4.1. We focus on the impact of the primary parameters $\alpha \in \mathbb{R}$ and $\beta \in [0, 1)$ of the stochastic model and set $\gamma = 0$. On $\mathcal{O} = [0, 1]$ the considered class of Poisson equations is given by

$$\begin{aligned} -U''(\cdot, \omega) &= X(\cdot, \omega) \qquad \text{in } (0, 1), \\ U(0, \omega) &= U(1, \omega) = 0, \end{aligned} \tag{4.7}$$

where $\omega \in \Omega$ and X as in (4.4). The numerical experiments were performed by using a stable biorthogonal spline wavelet basis as constructed in Primbs [136]. The primal wavelets consist of cardinal splines of order $m = 3$, i.e., they are piecewise quadratic polynomials, and the condition (W4) is satisfied with $\widetilde{m} = 5$. The wavelet basis satisfies (W6) along the nonlinear approximation line with $s_1 = 3$, while $s_1 = 2.5$ along the linear approximation line, see Section 2.3.2. Moreover, $j_0 = 2$ and $\#\nabla_{j_0} = 10$, while $\#\nabla_j = 2^j$ for $j > j_0$. In the stochastic model (4.2) for the right-hand side X we set

$$C_1 = 2^{\beta j_0}, \quad C_2 = 2^{\alpha j_0},$$

which means that sparsity is only induced at levels $j > j_0$ and the coefficients at level j_0 are standard normally distributed. This ensures that we keep the entire polynomial part of X.

In the last several years, much effort has been spent to design adaptive numerical algorithms based on wavelets. For elliptic problems, adaptive wavelet schemes have been derived that are guaranteed to converge with optimal order in the sense that they realize the approximation order of best N-term wavelet approximation Cohen et al. [29, 30], Dahlke et al. [45], Gantumur et al. [81]. Moreover, it has been possible to generalize the algorithms also to the case of wavelet frames Dahlke et al.

[50], Stevenson [150]. In this subsection we apply these algorithms to the numerical treatment of (4.7).

The aim is to investigate if for this model problem adaptive wavelet algorithms are superior when compared with uniform schemes. As we know the order of convergence of linear uniform approximation methods in L_2 is determined by the L_2-Sobolev regularity of the exact solution whereas the order of convergence of adaptive schemes is determined by the Besov smoothness, cf. (4.5) and Section 2.3.2.

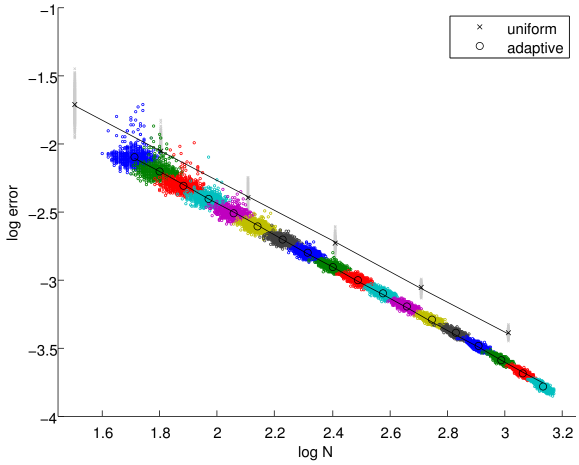
We therefore study an example where the Sobolev smoothness of the right-hand side X stays fixed while the Besov regularity changes. This is achieved by choosing parameter values α and β such that the sum $\alpha+\beta$, which determines the L_2-Sobolev regularity, is kept constant, cf. Section 3.1.4. Then, letting β tend to one increases the Besov smoothness significantly, see Corollary 3.12. Since the problem is completely regular, these interrelations immediately carry over to the exact solution, see Triebel [157, Theorem 4.3].

The numerical experiment is carried out and evaluated as follows. On input $\delta > 0$ the adaptive wavelet scheme computes an N-term wavelet approximation $\widehat{U}(\cdot,\omega)$ to $U(\cdot,\omega)$, whose error with respect to the H^1-norm is at most δ. The number N, $N \in \mathbb{N}$, of terms depends on δ as well as on ω via the right-hand side $X(\cdot,\omega)$, and only a finite number of wavelet coefficients of $X(\cdot,\omega)$ are used to compute $\widehat{U}(\cdot,\omega)$. We determine $U(\cdot,\omega)$ in a master computation with a very high accuracy and then use the norm equivalence (W6) for the space $H^1(\mathcal{O})$. The master computation employs a uniform approximation with truncation level $j_1 = 11$ for the right-hand side. To get a reliable estimate for the average number $\mathrm{E}[\eta(\widehat{U})]$ of non-zero wavelet coefficients of $\widehat{U}$ and for the error $(\mathrm{E}[\|U-\widehat{U}\|^2_{H^1(\mathcal{O})}])^{1/2}$ we use 1000 independent samples of truncated right-hand sides. This procedure is carried out for 18 different values of δ; the results are presented together with a regression line, whose slope yields an estimate for the order of convergence. For the uniform approximation we proceed with only one difference, instead of δ, a fixed truncation level for the approximation of the left-hand side is used, and therefore no estimate is needed for the number of non-zero coefficients. As for the adaptive scheme we use 1000 independent samples for six different truncation levels, $j = 4, \dots, 9$. We add that confidence intervals for the level 0.95 are of length less than three percent of the estimate in all cases.

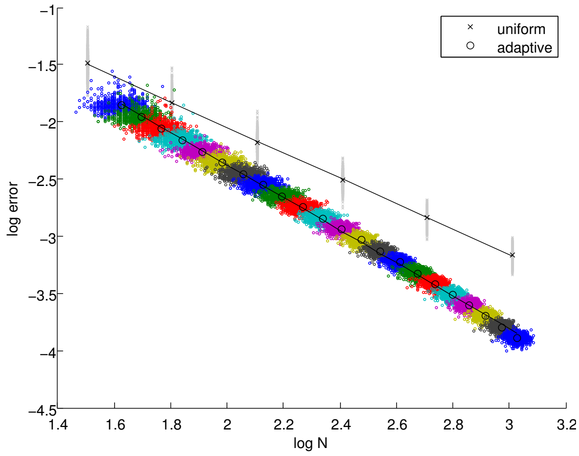
In the first experiment we choose

$$\alpha = 0.9, \quad \beta = 0.2, \tag{4.8}$$

i.e., the right-hand side is contained in $H^s(\mathcal{O})$ only for $s < 0.05$. Consequently, the solution is contained in $H^s(\mathcal{O})$ with $s < 2.05$. An optimal uniform approximation scheme with respect to the H^1-norm yields the approximation order $1.05-\varepsilon$ for every $\varepsilon > 0$. This is observed in Figure 4.1(a), where the empirical order of convergence for the uniform approximation is 1.113. For the relatively small value of $\beta = 0.2$, the Besov smoothness, and therefore the order of best N-term approximation, is not much higher. In fact, by inserting the parameters into Theorem 4.6 with $d = 1$, we get the approximation order $\varrho - \varepsilon$ with $\varrho = 19/16 = 1.1875$. This is also reflected in Figure 4.1(a), where the empirical order of convergence for the adaptive wavelet scheme is 1.164. In both cases the numerical results match very well the asymptotic error analysis, and both methods exhibit almost the same order of convergence. Anyhow, even in this



(a) $\alpha = 0.9$, $\beta = 0.2$



(b) $\alpha = 0.4$, $\beta = 0.7$

Figure 4.1: Error and (expected) number of non-zero coefficients

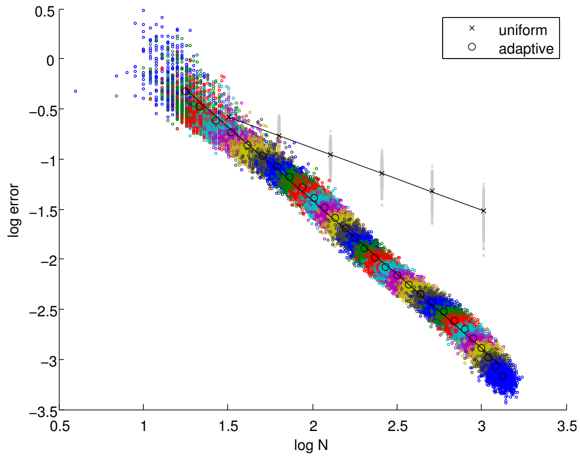
case adaptivity slightly pays off for the same regularity parameter, since the Besov norm is smaller than the Sobolev norm, which yields smaller constants.

The picture changes for higher values of β. As a second test case, we choose

$$\alpha = 0.4, \quad \beta = 0.7. \tag{4.9}$$

Then, the Besov regularity is considerably higher. In fact, from Theorem 4.6 with $d = 1$ we would expect the convergence rate $\varrho - \varepsilon$ with $\varrho = 7/3$, provided that the wavelet basis indeed characterizes the corresponding Besov spaces. It is well known that a tensor product spline wavelet basis of order m in dimension d has this very property for $B^s_\tau(L_\tau)$ with $1/\tau = s - 1/2$ and $s < s_1 = m/d$, see Cohen [28, Theorem 3.7.7]. In our case, $s_1 = 3$, so $\varrho = 2$ is the best we can expect. From Figure 4.1(b), we observe that the empirical order of convergence is slightly lower, namely 1.425. The reason is that the Besov smoothness of the solution is only induced by the right-hand side which, in a Galerkin approach, is expanded in the *dual* wavelet basis. Estimating the Hölder regularity of the dual wavelet basis $\widetilde{\Psi}$, see Villemoes [169], it turns out that this wavelet basis is only contained in $W^s(L_\infty)$ for $s < 0.55$. Therefore, by using classical embeddings of Besov spaces, it is only ensured that this wavelet basis characterizes Besov spaces $B^s_\tau(L_\tau)$, with the same smoothness parameter. Consequently, the solution U is only contained in the spaces $B^s_\tau(L_\tau)$ with $1/\tau = s - 1/2$ and $s < 2.55$ which gives an approximation order $\varrho - \varepsilon$ with $\varrho = 1.55$. This is matched very well in Figure 4.1(b). For uniform approximation the empirical order of convergence is 1.115 and thus does not differ from the result in the first experiment.

Both of the numerical experiments (4.8) and (4.9) already indicate that adaptivity really pays off for the problem (4.7). Nevertheless, we still observe a bottleneck. So far, we only discussed random right-hand sides with realizations being in smoothness spaces with *positive* smoothness parameters. Then, in the univariate case, the solution immediately possesses Sobolev smoothness larger than two, so that uniform schemes already perform quite well. The picture changes for right-hand sides with *negative* smoothness. Indeed, choosing $X \in H^{-1+\varepsilon}(\mathcal{O})$ yields $U \in H^{1+\varepsilon}(\mathcal{O})$ for $\varepsilon > 0$, and the convergence order of uniform schemes in $H^1(\mathcal{O})$ is only ε. Then, by choosing X in the



$\alpha = -0.87$, $\beta = 0.97$

Figure 4.2: Error and (expected) number of non-zero coefficients

Besov spaces as in Theorem 4.6, adaptive schemes still show the same approximation order as before, so that a difference in the performance of uniform and adaptive algorithms is even more noticeable. Since Theorem 3.10 also holds for negative values of α, we can choose

$$\alpha = -48/55, \quad \beta = 107/110,$$

and the corresponding right-hand side X is contained in $H^{-0.45}(\mathcal{O})$. In this case, a uniform scheme has approximation order $0.55 - \varepsilon$, which is reflected in Figure 4.2 an empirical order 0.619. Moreover, by Theorem 4.6, adaptive schemes still obtain approximation order $1.55 - \varepsilon$. Indeed, Figure 4.2 shows almost exactly this order, namely 1.476.

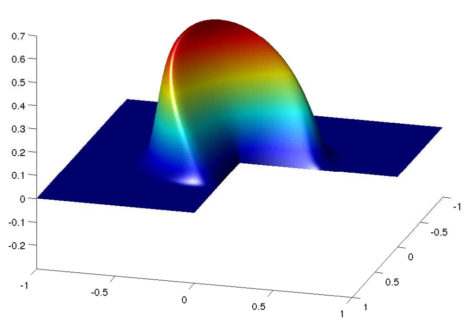
Our sample problem in the two-dimensional case is the Poisson equation (4.1) with zero-Dirichlet boundary conditions on the L-shaped domain

$$\mathcal{O} = (-1, 1)^2 \setminus [0, 1)^2.$$

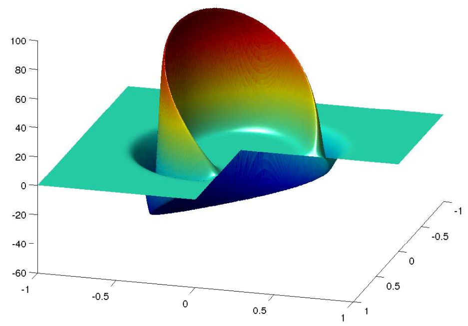
Here we are going to apply domain decomposition methods based on wavelet frames as discussed in Section 2.4.2. We consider the right-hand side $X = x + \widetilde{X}$, where x is a known deterministic function and $\widetilde{X}$ is generated by the stochastic model (4.2), (4.3), and (4.4), but based on frame decompositions. This means that we add a noise term to a deterministic right-hand side. Specifically, we consider perturbed versions of the well-known equation that is defined by the exact solution

$$u(r, \theta) = \zeta(r) r^{2/3} \sin\left(\frac{2}{3}\theta\right),$$

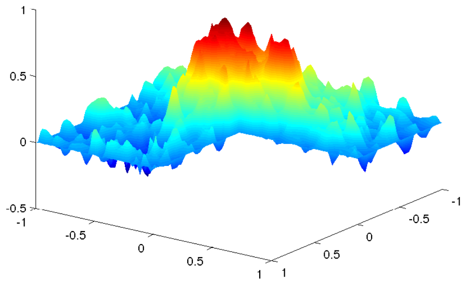
where (r, θ) are polar coordinates with respect to the re-entrant corner, see Figure 4.3 (a,b). Then, u is one of the singularity functions as introduced in the proof of Theorem 4.5. It has a relatively low Sobolev regularity while its Besov smoothness is arbitrary high, see again the proof of Theorem 4.5 for details. For functions of this type we expect that adaptivity pays off. In Figure 4.3 (c,d) we show two solutions to realisations of X for the parameter combination $\alpha = 1$ together with $\beta = 0.1$ and its sparse counterpart $\beta = 0.9$.



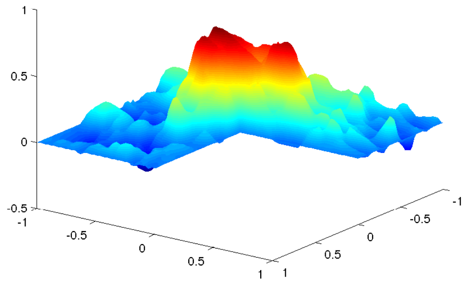
(a) exact solution



(b) exact right-hand side



(c) $\alpha = 1.0$, $\beta = 0.1$



(d) $\alpha = 1.0$, $\beta = 0.9$

Figure 4.3: (a,b) exact equation; (c,d) solutions to perturbed right-hand sides

Chapter 5

On the convergence of the inexact linearly implicit Euler scheme

We investigate the error propagation and analyze the convergence of Rothe's method for evolution equations of the parabolic type with focus on linearly implicit one-step methods. We use uniform discretizations in time and non-uniform (adaptive) discretizations in space. The space discretization methods are assumed to converge up to a given tolerance ε when applied to the resulting elliptic subproblems. Typical examples are adaptive finite element or wavelet methods. We investigate how the tolerances ε in each time step have to be tuned so that the overall scheme converges with the same order as in the case of exact evaluations of the elliptic subproblems. See Section 1.5 for the complete introduction.

In Section 5.1, we take an abstract point of view on Rothe's method and derive sufficient conditions for convergence in the case of inexact operator evaluations, see Theorems 5.21 and 5.26. In Section 5.2, we show that also stochastic evolution equations can be treated if the linearly-implicit Euler scheme is the method of choice, which we apply to a class of semi-linear parabolic SPDEs of the form

$$\mathrm{d}u(t) = Au(t)\mathrm{d}t + f(u(t))\mathrm{d}t + B(u(t))\mathrm{d}W(t), \quad u(0) = u_0, \quad t \in [0, T],$$

see Observation 5.57 and Theorem 5.63. Note that formally both equations are special cases of the general problem

$$\mathrm{d}u(t) = F(t, u(t))\mathrm{d}t + B(u(t))\mathrm{d}W(t), \quad u(0) = u_0, \quad t \in [0, T], \tag{5.1}$$

where $B \equiv 0$ indicates the deterministic case. Finally, in Section 5.3 we substantiate our analysis further and combine the analysis presented in Section 5.1 with complexity estimates for optimal adaptive wavelet solvers in order to obtain complexity estimates for spatially adaptive Rothe methods, see Theorems 5.71, 5.73, 5.78, and 5.79.

The results of this chapter have been partly worked out by the author and collaborators in [22] and [23], see also [21].

5.1 Abstract description of Rothe's method

We state the abstract setting of Rothe's method and derive sufficient conditions for convergence in the case of inexact operator evaluations.

5.1.1 Motivation

We begin with an example that motivates our perspective on the analysis of Rothe's method. To introduce our abstract setting, let us consider the heat equation

$$\left.\begin{aligned} u'(t) &= \Delta u(t) + f(t, u(t)) && \text{on } \mathcal{O},\ t \in (0, T], \\ u(0) &= u_0 && \text{on } \mathcal{O}, \\ u &= 0 && \text{on } \partial\mathcal{O},\ t \in (0, T], \end{aligned}\right\} \tag{5.2}$$

where $\mathcal{O} \subset \mathbb{R}^d$, $d \geq 1$, denotes a bounded Lipschitz domain. We discretize this equation by means of a linearly-implicit Euler scheme with uniform time steps. Let $K \in \mathbb{N}$ be the number of subdivisions of the time interval $[0, T]$, while the step size is denoted by $\tau := T/K$, and the k-th point in time is denoted by $t_k := \tau k$, $k \in \{0, \ldots, K\}$. The linearly-implicit Euler scheme, starting at u_0, is given by

$$\frac{u_{k+1} - u_k}{\tau} = \Delta u_{k+1} + f(t_k, u_k),$$

i.e.,

$$(I - \tau\Delta)u_{k+1} = u_k + \tau f(t_k, u_k), \tag{5.3}$$

for $k = 0, \ldots, K-1$. If we assume that the elliptic problem

$$L_\tau v := (I - \tau\Delta)v = w \quad \text{on } \mathcal{O}, \quad v|_{\partial\mathcal{O}} = 0,$$

can be solved exactly, then one step of the scheme (5.3) can be written as

$$u_{k+1} = L_\tau^{-1} R_{\tau,k}(u_k), \tag{5.4}$$

where

$$R_{\tau,k}(v) := v + \tau f(t_k, v)$$

and L_τ is a boundedly invertible operator between suitable Hilbert spaces. That is, we can look at this equation in a Gel'fand triple setting $(H_0^1(\mathcal{O}),\ L_2(\mathcal{O}),\ H^{-1}(\mathcal{O}))$ with L_τ as an operator from $H_0^1(\mathcal{O})$ to $H^{-1}(\mathcal{O})$. Recall that $H_0^1(\mathcal{O})$ denotes the Sobolev space with Dirichlet boundary conditions, $H^{-1}(\mathcal{O})$ its normed dual, and $L_2(\mathcal{O})$ the Lebesgue space of quadratic integrable functions, cf. Section 2.2.1. We may also consider (5.4) in $L_2(\mathcal{O})$, since $H_0^1(\mathcal{O})$ is embedded in $L_2(\mathcal{O})$ and $L_2(\mathcal{O})$ is embedded in $H^{-1}(\mathcal{O})$, provided that $R_{\tau,k} : L_2(\mathcal{O}) \to L_2(\mathcal{O})$ is well defined.

Having the above simple example in mind, we observe that the fundamental form of (5.4) essentially remains the same even if we introduce more sophisticated discretizations in time, e.g., as outlined below and in Section 5.1.4.

5.1.2 Setting and assumptions

In many applications not only one-stage approximation methods, such as the linearly-implicit Euler scheme, are used, but also more sophisticated S-stage schemes. The reason is, S-stage schemes can lead to higher temporal convergence orders, see Section 5.1.4 for further details. Therefore, in this section we state a scheme with the same form as in (5.4) that provides an abstract interpretation of linearly-implicit S-stage schemes, where $S \in \mathbb{N}$.

As above, we begin with a uniform discretization of the time interval $[0, T]$ with $K \in \mathbb{N}$ subdivisions, step size $\tau := T/K$, and $t_k := k\tau$ for $k \in \{0, \ldots, K\}$. Taking an abstract point of view, we introduce separable real Hilbert spaces $\mathcal{H}$, $\mathcal{G}$, and consider a mapping $u : [0, T] \to \mathcal{H}$. Furthermore, let $L_{\tau,i}$ be a family of, possibly unbounded, linear operators which have bounded inverses

$$L_{\tau,i}^{-1} : \mathcal{G} \to \mathcal{H},$$

and let

$$R_{\tau,k,i} : \underbrace{\mathcal{H} \times \cdots \times \mathcal{H}}_{i} \to \mathcal{G} \tag{5.5}$$

be a family of (nonlinear) evaluation operators for $k \in \{0, \ldots, K-1\}$ and $i = 1, \ldots, S$. As the norm on the Cartesian product in (5.5) we set

$$\big\|(v_1, \ldots, v_i)\big\|_{\mathcal{H}\times\cdots\times\mathcal{H}} := \sum_{l=1}^{i} \|v_l\|_{\mathcal{H}}.$$

Remark 5.1. The mapping $u : [0, T] \to \mathcal{H}$ is understood to be implicitly given as a solution of a (deterministic or stochastic) parabolic partial differential equation of the form (5.1).

Remark 5.2. In most cases the operators $L_{\tau,i}^{-1}$ are not given explicitly and, for this reason, we need an efficient numerical scheme for their evaluation. The situation is completely different with the $R_{\tau,k,i}$, which are usually given explicitly and do not require the solution of operator equations for their evaluation. Concrete examples of these operators are presented and studied in Section 5.1.4 below.

Remark 5.3. In a Gel'fand triple setting (V, U, V^*) typical choices for the spaces $\mathcal{H}$ and $\mathcal{G}$ are $\mathcal{H} = V$, $\mathcal{G} = V^*$ or $\mathcal{H} = \mathcal{G} = U$. However, also a more general setting such as

$$V \subseteq \mathcal{H} \subseteq U \subseteq V^* \subseteq \mathcal{G}$$

is possible. Observe that our motivating example from Section 5.1.1 fits in this setting with $H_0^1(\mathcal{O}) = \mathcal{H} \subseteq L_2(\mathcal{O})$ and $\mathcal{G} = H^{-1}(\mathcal{O})$.

Starting from the given value $u_0 := u(0) \in \mathcal{H}$, we define the *abstract exact S-stage scheme* iteratively by

$$\left.\begin{aligned} u_{k+1} &:= \sum_{i=1}^{S} w_{k,i}, \\ w_{k,i} &:= L_{\tau,i}^{-1} R_{\tau,k,i}(u_k, w_{k,1}, \ldots, w_{k,i-1}), \quad i = 1, \ldots, S, \end{aligned}\right\} \tag{5.6}$$

for $k = 0, \ldots, K-1$. One step of this iteration can be described as an application of the operator

$$\left.\begin{aligned} E_{\tau,k,k+1} : \mathcal{H} &\to \mathcal{H}, \\ v &\mapsto \sum_{i=1}^{S} w_{k,i}(v), \\ w_{k,i}(v) &:= L_{\tau,i}^{-1} R_{\tau,k,i}\big(v, w_{k,1}(v), \ldots, w_{k,i-1}(v)\big), \quad i = 1, \ldots, S. \end{aligned}\right\} \tag{5.7}$$

If we define the family of operators

$$E_{\tau,j,k} := \begin{cases} E_{\tau,k-1,k} \circ \ldots \circ E_{\tau,j,j+1}, & j < k \\ I, & j = k, \end{cases} \tag{5.8}$$

where I denotes the identity mapping, then the output of the exact S-stage scheme (5.6) is given by the sequence

$$u_k = E_{\tau,0,k}(u_0), \quad k = 0, \ldots, K. \tag{5.9}$$

The convergence analysis which we present relies on a crucial technical assumption on the operators defined in (5.8).

Assumption 5.4. For all $0 \le j, k \le K$ the operators

$$E_{\tau,j,k} : \mathcal{H} \to \mathcal{H} \ \text{ are globally Lipschitz continuous}$$

with Lipschitz constants $C^{\mathrm{Lip}}_{\tau,j,k}$.

Remark 5.5. Assumption 5.4 is relatively mild, as it is usually fulfilled in the applications we have in mind. Concrete examples are given at the end of this section, as well as in Section 5.1.4 below.

We call the sequence (5.9) the output of the *exact* S-stage scheme, since the operators involved in the definition of $E_{\tau,0,k}$ are evaluated exactly. In practical applications this is very often not possible; the operators $L^{-1}_{\tau,i}$ and $R_{\tau,k,i}$ can only be evaluated up to a prescribed accuracy. Therefore, as a start, we make the following assumption.

Assumption 5.6. For all τ, $k \in \{0, \ldots, K-1\}$, and for any prescribed tolerance $\varepsilon_k > 0$ and arbitrary $v \in \mathcal{H}$, we have an approximation $\widetilde{E}_{\tau,k,k+1}(v)$ of $E_{\tau,k,k+1}(v)$ at hand, such that

$$\left\| E_{\tau,k,k+1}(v) - \widetilde{E}_{\tau,k,k+1}(v) \right\|_{\mathcal{H}} \le \varepsilon_k$$

with a known upper bound $M_{\tau,k}(\varepsilon_k, v) < \infty$ for the degrees of freedom needed to achieve the prescribed tolerance ε_k.

Remark 5.7. In this abstract setting the term degrees of freedom is a bit vague, since the precise meaning of this term depends on the concrete form of the applied approximation scheme. For instance, in the finite element and the wavelet setting, the degrees of freedom refer to the number of basis functions, which are needed for the approximant to achieve the tolerance, cf. Section 2.3.2.

For simplicity, we make the following assumption.

Assumption 5.8. The initial value is given exactly, i.e.,

$$\widetilde{u}_0 := u(0).$$

Remark 5.9. The case where Assumption 5.8 does not hold, i.e., $\widetilde{u}_0 \neq u(0)$, can be handled in a similar way. However, this only increases *notational* complexity.

Given an approximation scheme satisfying Assumption 5.6 and using Assumption 5.8, the *abstract inexact* variant of (5.6) is defined by

$$\left.\begin{aligned} \widetilde{u}_0 &:= u(0), \\ \widetilde{u}_{k+1} &:= \widetilde{E}_{\tau,k,k+1}(\widetilde{u}_k) \quad \text{for } k = 0, \ldots, K-1. \end{aligned}\right\} \tag{5.10}$$

In Theorem 5.21 below we show how to tune the tolerances $(\varepsilon_k)_{k=0,\dots,K-1}$ in such a way that the abstract inexact scheme (5.10) has the same qualitative properties as the exact scheme (5.6). As in (5.8), we define

$$\widetilde{E}_{\tau,j,k} := \begin{cases} \widetilde{E}_{\tau,k-1,k} \circ \ldots \circ \widetilde{E}_{\tau,j,j+1}, & j < k \\ I, & j = k. \end{cases}$$

Consequently, the output of the inexact S-stage scheme (5.10) is given by

$$\widetilde{u}_k = \widetilde{E}_{\tau,0,k}\big(u(0)\big), \quad k = 0, \dots, K.$$

Now, we are faced with the following problems. In practice, the Lipschitz constants $C^{\mathrm{Lip}}_{\tau,j,k}$ of $E_{\tau,j,k}$, given in Assumption 5.4, might be hard to estimate directly and are only available in very specific situations. As we shall see in Section 5.3, the individual operators $L^{-1}_{\tau,i} R_{\tau,k,i}$, $i = 1, \dots, S$, are much easier to handle. Moreover, a direct approximation scheme for $E_{\tau,k,k+1}$, as required by Assumption 5.6, might also be hard to get directly. Nevertheless, very often, one has convergent numerical schemes for the individual operators $L^{-1}_{\tau,i} R_{\tau,k,i}$. Therefore, with these observations in mind, we are now going to state the corresponding assumptions for these individual operators and specify Assumption 5.4 in the following way.

Assumption 5.10. For $k = 0, \dots, K-1$ and $i = 1, \dots, S$ the operators

$$L^{-1}_{\tau,i} R_{\tau,k,i} : \underbrace{\mathcal{H} \times \cdots \times \mathcal{H}}_{i} \to \mathcal{H}$$

are globally Lipschitz continuous with Lipschitz constants $C^{\mathrm{Lip}}_{\tau,k,(i)}$.

Remark 5.11. Note that, on the one hand, Assumption 5.4 is slightly more general than Assumption 5.10, since it is easy to see that a composition of non-Lipschitz continuous operators can be Lipschitz continuous. On the other hand, Assumption 5.10 implies Assumption 5.4. This is a consequence of the fact, that, if we introduce the constants

$$C'_{\tau,k,(i)} := \prod_{l=i+1}^{S} (1 + C^{\mathrm{Lip}}_{\tau,k,(l)}) \tag{5.11}$$

for $k = 0, \dots, K-1$ and $i = 0, \dots, S$, we can estimate the Lipschitz constants $C^{\mathrm{Lip}}_{\tau,j,k}$ of $E_{\tau,j,k}$ as follows:

$$C^{\mathrm{Lip}}_{\tau,j,k} \le \prod_{r=j}^{k-1} \left(C'_{\tau,r,(0)} - 1\right), \quad 0 \le j \le k \le K. \tag{5.12}$$

This is worked out in detail in the proof of Theorem 5.24 below.

The analogue to Assumption 5.6 is given by the following assumption.

Assumption 5.12. For all τ, $k \in \{0, \dots, K-1\}$, $i \in \{1, \dots, S\}$, there exists a numerical scheme that, for any prescribed tolerance $\varepsilon_{k,i} > 0$ and arbitrary $v_0, \dots, v_{i-1} \in \mathcal{H}$, yields an approximation $[v]_{\varepsilon_{k,i}}$ of

$$v := L^{-1}_{\tau,i} R_{\tau,k,i}(v_0, \dots, v_{i-1}),$$

such that

$$\left\|v - [v]_{\varepsilon_{k,i}}\right\|_{\mathcal{H}} \leq \varepsilon_{k,i}$$

with a known upper bound $M_{\tau,k,i}(\varepsilon_{k,i}, v) < \infty$ for the degrees of freedom needed to achieve the prescribed accuracy $\varepsilon_{k,i}$.

Remark 5.13. We do not specify the numerical scheme $[\,\cdot\,]_\varepsilon$ at this point. It might be based on, e.g., a spectral method, an (adaptive) finite element scheme, or an adaptive wavelet solver. The latter case are discussed in detail in Section 5.3. There, $M_{\tau,k,i}(\varepsilon, v)$ is an upper bound for the number of elements of the spatial wavelet system that is needed to achieve the desired tolerance.

For any numerical scheme satisfying Assumption 5.12, and given tolerances $\varepsilon_{k,i} > 0$, $k = 0, \ldots, K-1$, $i = 1, \ldots, S$, the corresponding *inexact* variant of (5.6) is defined by

$$\left.\begin{aligned} \widetilde{u}_0 &:= u(0), \\ \widetilde{u}_{k+1} &:= \sum_{i=1}^{S} \widetilde{w}_{k,i}, \\ \widetilde{w}_{k,i} &:= \left[L_{\tau,i}^{-1} R_{\tau,k,i}(\widetilde{u}_k, \widetilde{w}_{k,1}, \ldots, \widetilde{w}_{k,i-1})\right]_{\varepsilon_{k,i}}, \quad i = 1, \ldots, S, \end{aligned}\right\} \tag{5.13}$$

for $k = 0, \ldots, K-1$. Note that (5.13) is consistent with (5.10), since it corresponds to the specific choice

$$\left.\begin{aligned} \widetilde{E}_{\tau,k,k+1} : \mathcal{H} &\to \mathcal{H}, \\ v &\mapsto \sum_{i=1}^{S} \widetilde{w}_{k,i}(v), \\ \widetilde{w}_{k,i}(v) &:= \left[L_{\tau,i}^{-1} R_{\tau,k,i}(v, \widetilde{w}_{k,1}(v), \ldots, \widetilde{w}_{k,i-1}(v))\right]_{\varepsilon_{k,i}}, \quad i = 1, \ldots, S. \end{aligned}\right\} \tag{5.14}$$

In Theorem 5.26 below we show how to tune the tolerances in the scheme (5.13) in such a way that the approximation of u in $\mathcal{H}$ has the same qualitative properties as the exact scheme (5.6).

Remark 5.14. For $\widetilde{E}_{\tau,k,k+1}$ as in (5.14) and arbitrary $v \in \mathcal{H}$, the estimate

$$\left\|E_{\tau,k,k+1}(v) - \widetilde{E}_{\tau,k,k+1}(v)\right\|_{\mathcal{H}} \leq \sum_{i=1}^{S} C'_{\tau,k,(i)} \varepsilon_{k,i} \tag{5.15}$$

holds with $C'_{\tau,k,(i)}$ given by (5.11). Thus, for any prescribed tolerance ε_k, if Assumptions 5.10 and 5.12 are fulfilled, we can choose $\varepsilon_{k,i}$, $i = 1, \ldots, S$, in such a way that the error we make by applying $\widetilde{E}_{\tau,k,k+1}$ from (5.14) instead of $E_{\tau,k,k+1}$ is bounded by ε_k, uniformly in $\mathcal{H}$. In this sense Assumption 5.12 implies Assumption 5.6. Detailed arguments for the validity of estimate (5.15) are given in the proof of Theorem 5.24.

5.1.3 Controlling the error of the inexact schemes

We want to use the schemes described in the previous section to compute approximations to a solution $u : [0, T] \to \mathcal{H}$ of a parabolic partial differential equation. The analysis presented in this section is based on the assumption that the exact scheme (5.6) converges to u with a given approximation order δ. We then state conditions how to tune the tolerances in the inexact schemes (5.10) and (5.13), so that they also converge to u and inherit the approximation order δ of the exact scheme, see Theorem 5.21 and Theorem 5.26. We start with a natural assumption.

Assumption 5.15. There exists a unique solution $u : [0,T] \to \mathcal{H}$ to the problem under consideration, i.e., to (5.1).

Remark 5.16. Of course, the type of such solutions depends on the form of the specific parabolic partial differential equation. We avoid, on purpose, a detailed discussion of this aspect in this section and postpone further information to Remark 5.38.

The analysis presented in this section is based on the following central assumption.

Assumption 5.17. The exact scheme (5.6) converges to $u(T)$ with order $\delta > 0$, i.e., for some constant $C_{\text{exact}} > 0$,

$$\left\| u(T) - E_{\tau,0,K}(u(0)) \right\|_{\mathcal{H}} \leq C_{\text{exact}}\, \tau^{\delta},$$

where the constant may depend on f, T, and u_0, but not on τ.

Remark 5.18. Error estimates as in Assumption 5.17 are quite natural and hold very often, see Section 5.1.4 and the references therein, in particular, Lubich, Ostermann [126, Theorem 6.2], which we quote as Theorem 5.37 below.

At first, we give an estimate for the error propagation of the abstract inexact scheme (5.10) measured in the norm of $\mathcal{H}$.

Theorem 5.19. *Suppose that Assumptions 5.4, 5.6, 5.8, and 5.15 hold. Let $(u_k)_{k=0}^K$, $K \in \mathbb{N}$, be the output of the exact scheme* (5.6)*, and let $(\widetilde{u}_k)_{k=0}^K$ be the output of the scheme* (5.10) *with given tolerances ε_k, $k = 0, \ldots, K-1$. Then, for all $0 \leq k \leq K$,*

$$\|u(t_k) - \widetilde{u}_k\|_{\mathcal{H}} \leq \|u(t_k) - u_k\|_{\mathcal{H}} + \sum_{j=0}^{k-1} C^{\text{Lip}}_{\tau,j+1,k}\, \varepsilon_j.$$

Proof. The triangle inequality yields

$$\|u(t_k) - \widetilde{u}_k\|_{\mathcal{H}} \leq \left\| u(t_k) - u_k \right\|_{\mathcal{H}} + \|u_k - \widetilde{u}_k\|_{\mathcal{H}},$$

so it remains to estimate the second term. Using $u_0 = \widetilde{u}_0$ and writing $u_k - \widetilde{u}_k$ as a telescopic sum, we get

$$\begin{aligned} u_k - \widetilde{u}_k &= \big(E_{\tau,0,k}(\widetilde{u}_0) - E_{\tau,1,k}\widetilde{E}_{0,1}(\widetilde{u}_0)\big) \\ &\qquad + \big(E_{\tau,1,k}\widetilde{E}_{0,1}(\widetilde{u}_0) - E_{\tau,2,k}\widetilde{E}_{0,2}(\widetilde{u}_0)\big) \\ &\qquad \cdots \\ &\qquad + \big(E_{\tau,k-1,k}\widetilde{E}_{0,k-1}(\widetilde{u}_0) - \widetilde{E}_{0,k}(\widetilde{u}_0)\big) \\ &= \sum_{j=0}^{k-1} \big(E_{\tau,j,k}\widetilde{E}_{0,j}(u_0) - E_{\tau,j+1,k}\widetilde{E}_{0,j+1}(u_0)\big). \end{aligned}$$

Another application of the triangle inequality yields

$$\|u_k - \widetilde{u}_k\|_{\mathcal{H}} \leq \sum_{j=0}^{k-1} \left\| E_{\tau,j,k}\widetilde{E}_{0,j}(u_0) - E_{\tau,j+1,k}\widetilde{E}_{0,j+1}(u_0) \right\|_{\mathcal{H}}.$$

Due to the Lipschitz continuity of $E_{\tau,j,k}$, cf. Assumption 5.4, each term in the sum can be estimated from above by

$$\begin{aligned}
&\big\|E_{\tau,j,k}\widetilde{E}_{0,j}(u_0) - E_{\tau,j+1,k}\widetilde{E}_{0,j+1}(u_0)\big\|_{\mathcal{H}} \\
&\qquad = \big\|E_{\tau,j+1,k}E_{\tau,j,j+1}\widetilde{E}_{0,j}(u_0) - E_{\tau,j+1,k}\widetilde{E}_{0,j+1}(u_0)\big\|_{\mathcal{H}} \\
&\qquad \le C^{\mathrm{Lip}}_{\tau,j+1,k}\big\|E_{\tau,j,j+1}\widetilde{E}_{0,j}(u_0) - \widetilde{E}_{0,j+1}(u_0)\big\|_{\mathcal{H}}.
\end{aligned} \tag{5.16}$$

With $\widetilde{E}_{0,j}(u_0) = \widetilde{u}_j$ and using Assumption 5.6, we observe

$$\big\|E_{\tau,j,j+1}\widetilde{E}_{0,j}(u_0) - \widetilde{E}_{0,j+1}(u_0)\big\|_{\mathcal{H}} = \big\|E_{\tau,j,j+1}(\widetilde{u}_j) - \widetilde{E}_{j,j+1}(\widetilde{u}_j)\big\|_{\mathcal{H}} \le \varepsilon_j. \qquad \square$$

Remark 5.20. In the description of our abstract setting we have chosen the spaces $\mathcal{H}$ and $\mathcal{G}$ to be the same in all time steps. However, at the expense of a slightly more involved notation, the result of Theorem 5.19 stays true with $\mathcal{H}$ replaced by variable spaces $\mathcal{H}_k$, $k = 0, \dots, K-1$, as long as we can guarantee the Lipschitz continuity of the mappings $E_{\tau,j,k} : \mathcal{H}_j \to \mathcal{H}_k$ with corresponding Lipschitz constants $C^{\mathrm{Lip}}_{\tau,j,k}$, $1 \le j \le k$.

Based on Theorem 5.19 we are now able to state the conditions on the tolerances $(\varepsilon_k)_{k=0,\dots,K-1}$ such that the abstract inexact scheme (5.10) also exhibits the approximation order δ.

Theorem 5.21. *Suppose that Assumptions 5.4, 5.6, 5.8, and 5.15 hold. Let Assumption 5.17 hold for some $\delta > 0$. If we consider the case of inexact operator evaluations as described in* (5.10) *and choose*

$$0 < \varepsilon_k \le \big(C^{\mathrm{Lip}}_{\tau,k+1,K}\big)^{-1}\,\tau^{1+\delta},$$

$k = 0, \dots, K-1$, then we get

$$\big\|u(T) - \widetilde{E}_{\tau,0,K}(u(0))\big\|_{\mathcal{H}} \le (C_{\mathrm{exact}} + T)\,\tau^{\delta}.$$

Proof. Applying Theorem 5.19, Assumption 5.17 and $K = T/\tau$, we obtain

$$\begin{aligned}
\big\|u(t_K) - \widetilde{u}_K\big\|_{\mathcal{H}} &\le \big\|u(t_K) - u_K\big\|_{\mathcal{H}} + \sum_{k=0}^{K-1} C^{\mathrm{Lip}}_{\tau,k+1,K}\,\varepsilon_k \\
&\le C_{\mathrm{exact}}\,\tau^{\delta} + \sum_{k=0}^{K-1} C^{\mathrm{Lip}}_{\tau,k+1,K}\,\big(C^{\mathrm{Lip}}_{\tau,k+1,K}\big)^{-1}\,\tau^{1+\delta} \\
&= C_{\mathrm{exact}}\,\tau^{\delta} + K\tau^{1+\delta} = (C_{\mathrm{exact}} + T)\,\tau^{\delta}. \qquad \square
\end{aligned}$$

One of the final goals of our analysis is to provide upper estimates for the overall complexity of the resulting scheme. As a first step, in this direction, we provide a quite abstract version, which is a direct consequence of Theorem 5.21.

Corollary 5.22. *Suppose that the assumptions of Theorem 5.21 are satisfied. Choose*

$$\varepsilon_k := \big(C^{\mathrm{Lip}}_{\tau,k+1,K}\big)^{-1}\,\tau^{1+\delta},$$

for $k = 0, \dots, K-1$, then the realization of $\widetilde{E}_{\tau,0,K}(u_0)$ requires at most

$$M_{\tau,T}(\delta, (\varepsilon_k)) := \sum_{k=0}^{K-1} M_{\tau,k}(\varepsilon_k, E_{\tau,k,k+1}(\widetilde{u}_k))$$

degrees of freedom.

Remark 5.23. At this point, without specifying an approximation scheme and therefore without a concrete knowledge of $M_{\tau,k}(\varepsilon,\cdot)$, Corollary 5.22 might not look very deep. Nevertheless, it is filled with content in Section 5.3, where we discuss the specific case of adaptive wavelet solvers for which concrete estimates for $M_{\tau,k}(\varepsilon,\cdot)$ are available.

The next step is to play the same game for the inexact scheme (5.13). We start again by controlling the error propagation.

Theorem 5.24. *Suppose that Assumptions 5.8, 5.10, 5.12, and 5.15 hold. Let* $(u_k)_{k=0}^K$, $K\in\mathbb{N}$, *be the output of the exact scheme* (5.6), *and let* $(\widetilde{u}_k)_{k=0}^K$ *be the output of the inexact scheme* (5.13) *with prescribed tolerances* $\varepsilon_{k,i}$, $k=0,\dots,K-1$, $i=1,\dots,S$. *Then, for all* $0\le k\le K$,

$$\big\|u(t_k)-\widetilde{u}_k\big\|_{\mathcal{H}} \le \big\|u(t_k)-u_k\big\|_{\mathcal{H}} + \sum_{j=0}^{k-1}\left(\prod_{l=j+1}^{k-1}(C'_{\tau,l,(0)}-1)\right)\sum_{i=1}^{S} C'_{\tau,j,(i)}\varepsilon_{j,i}.$$

Proof. We just have to repeat the proof of Theorem 5.19 with the special choice (5.14) for $\widetilde{E}_{\tau,k,k+1}$, and to include two modifications. First, instead of the exact Lipschitz constants $C_{\tau,j+1,k}$ in (5.16), we can use their estimates (5.12) announced in Remark 5.11. Second, in the last step of the proof of Theorem 5.19, we may estimate the error we make when using $\widetilde{E}_{\tau,j,j+1}$ instead of $E_{\tau,j,j+1}$ as in (5.15) of Remark 5.14. Thus, to finish the proof we have to show that the estimates (5.12) and (5.15) hold.

We start with (5.12). Note that it is enough to show that

$$C^{\mathrm{Lip}}_{\tau,k,k+1} \le C'_{\tau,k,(0)}-1, \quad 0\le k\le K-1, \tag{5.17}$$

since, obviously,

$$C^{\mathrm{Lip}}_{\tau,j,k} \le \prod_{r=j}^{k-1} C^{\mathrm{Lip}}_{\tau,r,r+1}, \quad 0\le j\le k\le K.$$

Thus, let us prove that (5.17) is true, if Assumption 5.10 holds. To this end, we fix $k\in\{0,\dots,K-1\}$ as well as arbitrary $u,v\in\mathcal{H}$. Using (5.7) and the triangle inequality, we obtain

$$\big\|E_{\tau,k,k+1}(u)-E_{\tau,k,k+1}(v)\big\|_{\mathcal{H}} \le \sum_{i=1}^{S}\|w_{k,i}(u)-w_{k,i}(v)\|_{\mathcal{H}}. \tag{5.18}$$

Applying Assumption 5.10, we get for each $i\in\{1,\dots,S\}$:

$$\big\|w_{k,i}(u)-w_{k,i}(v)\big\|_{\mathcal{H}} \le C^{\mathrm{Lip}}_{\tau,k,(i)}\Big(\|u-v\|_{\mathcal{H}}+\sum_{l=1}^{i-1}\big\|w_{k,l}(u)-w_{k,l}(v)\big\|_{\mathcal{H}}\Big).$$

Hence, for $r=0,\dots,S-1$, we have

$$\begin{aligned}\sum_{i=1}^{r+1}\big\|w_{k,i}(u)-w_{k,i}(v)\big\|_{\mathcal{H}} \le \big(1+C^{\mathrm{Lip}}_{\tau,k,(r+1)}\big)\sum_{i=1}^{r}\big\|w_{k,i}(u)-w_{k,i}(v)\big\|_{\mathcal{H}}\\ + C^{\mathrm{Lip}}_{\tau,k,(r+1)}\|u-v\|_{\mathcal{H}}.\end{aligned} \tag{5.19}$$

By induction, it is easy to show that $e_{r+1} \leq a_r e_r + b_r$ and $e_0 = 0$ imply

$$e_r \leq \sum_{j=1}^{r} b_{j-1} \prod_{l=j+1}^{r} a_{l-1}. \tag{5.20}$$

In our situation, this fact leads to the estimate

$$\sum_{i=1}^{S} \big\| w_{k,i}(u) - w_{k,i}(v) \big\|_{\mathcal{H}} \leq \sum_{i=1}^{S} C^{\mathrm{Lip}}_{\tau,k,(i)} \|u-v\|_{\mathcal{H}} \prod_{l=i+1}^{S} \big(1 + C^{\mathrm{Lip}}_{\tau,k,(l)}\big),$$

since (5.19) holds for $r = 0, \ldots, S-1$. Furthermore, we can use the equality

$$\sum_{i=1}^{S} C^{\mathrm{Lip}}_{\tau,k,(i)} \prod_{l=i+1}^{S} (1 + C^{\mathrm{Lip}}_{\tau,k,(l)}) = \prod_{i=1}^{S} (1 + C^{\mathrm{Lip}}_{\tau,k,(i)}) - 1 = C'_{\tau,k,(0)} - 1$$

to obtain

$$\sum_{i=1}^{S} \big\| w_{k,i}(u) - w_{k,i}(v) \big\|_{\mathcal{H}} \leq \big(C'_{\tau,k,(0)} - 1\big) \|u-v\|_{\mathcal{H}}.$$

Together with (5.18), this proves (5.17).

Now, we show the estimate (5.15). Fix $k \in \{0, \ldots, K-1\}$ and let $\widetilde{E}_{\tau,k,k+1}$ be given by (5.14) with the prescribed tolerances $\varepsilon_{k,i}$, $i = 1, \ldots, S$, from our assertion. Then, for arbitrary $v \in \mathcal{H}$, we have

$$\big\| E_{\tau,k,k+1}(v) - \widetilde{E}_{\tau,k,k+1}(v) \big\|_{\mathcal{H}} \leq \sum_{i=1}^{S} \big\| w_{k,i}(v) - \widetilde{w}_{k,i}(v) \big\|_{\mathcal{H}}. \tag{5.21}$$

Using the triangle inequality and Assumption 5.10 we obtain, for each $i = 1, \ldots, S$,

$$\begin{aligned}
&\big\| w_{k,i}(v) - \widetilde{w}_{k,i}(v) \big\|_{\mathcal{H}} \\
&\qquad = \Big\| L^{-1}_{\tau,i} R_{\tau,k,i}(v, w_{k,1}(v), \ldots, w_{k,i-1}(v)) \\
&\qquad\qquad\qquad - \big[L^{-1}_{\tau,i} R_{\tau,k,i}(v, \widetilde{w}_{k,1}(v), \ldots, \widetilde{w}_{k,i-1}(v)) \big]_{\varepsilon_{k,i}} \Big\|_{\mathcal{H}} \\
&\qquad \leq \Big\| L^{-1}_{\tau,i} R_{\tau,k,i}(v, w_{k,1}(v), \ldots, w_{k,i-1}(v)) - L^{-1}_{\tau,i} R_{\tau,k,i}(v, \widetilde{w}_{k,1}(v), \ldots, \widetilde{w}_{k,i-1}(v)) \Big\|_{\mathcal{H}} \\
&\qquad\quad + \Big\| L^{-1}_{\tau,i} R_{\tau,k,i}(v, \widetilde{w}_{k,1}(v), \ldots, \widetilde{w}_{k,i-1}(v)) \\
&\qquad\qquad\qquad - \big[L^{-1}_{\tau,i} R_{\tau,k,i}(v, \widetilde{w}_{k,1}(v), \ldots, \widetilde{w}_{k,i-1}(v)) \big]_{\varepsilon_{k,i}} \Big\|_{\mathcal{H}} \\
&\qquad \leq C^{\mathrm{Lip}}_{\tau,k,(i)} \sum_{l=1}^{i-1} \big\| w_{k,l}(v) - \widetilde{w}_{k,l}(v) \big\|_{\mathcal{H}} + \varepsilon_{k,i}.
\end{aligned}$$

Thus, for $r = 0, \ldots, S-1$,

$$\sum_{i=1}^{r+1} \big\| w_{k,i}(v) - \widetilde{w}_{k,i}(v) \big\|_{\mathcal{H}} \leq \big(1 + C^{\mathrm{Lip}}_{\tau,k,(r+1)}\big) \sum_{i=1}^{r} \big\| w_{k,i}(v) - \widetilde{w}_{k,i}(v) \big\|_{\mathcal{H}} + \varepsilon_{k,i}.$$

Arguing as above, cf. (5.20), we get

$$\sum_{i=1}^{S} \big\|w_{k,i}(v) - \widetilde{w}_{k,i}(v)\big\|_{\mathcal{H}} \leq \sum_{i=1}^{S} \varepsilon_{k,i} \prod_{l=i+1}^{S} (1 + C^{\mathrm{Lip}}_{\tau,k,(l)}) = \sum_{i=1}^{S} C'_{\tau,k,(i)} \varepsilon_{k,i}.$$

Together with (5.21), this proves (5.15). □

Remark 5.25. By construction, Theorem 5.24 is slightly weaker than Theorem 5.19, but from the practical point of view Theorem 5.24 is more realistic. As already outlined in Section 5.1.2, in many cases, estimates for the Lipschitz constants according to Assumption 5.10 and convergent numerical schemes according to Assumption 5.12 are available.

Based on Theorem 5.24, we are able to state the conditions on the tolerances $\varepsilon_{k,i}$, $k = 0, \dots, K-1$, $i = 1, \dots, S$, such that the scheme (5.13) converges with the desired order. We set

$$C''_{\tau,k} := \prod_{l=k+1}^{K-1} \big(C'_{\tau,l,(0)} - 1\big) \tag{5.22}$$

for $k = 0, \dots, K-1$, where $C'_{\tau,l,(0)}$ is given by (5.11).

Theorem 5.26. *Suppose that Assumptions 5.8, 5.10, 5.12, and 5.15 hold. Let Assumption 5.17 hold for some $\delta > 0$. If we consider the case of inexact operator evaluations as described in* (5.13) *and choose*

$$0 < \varepsilon_{k,i} \leq \frac{1}{S} \Big(C''_{\tau,k} C'_{\tau,k,(i)} \Big)^{-1} \tau^{1+\delta}, \tag{5.23}$$

then we get

$$\big\|u(T) - \widetilde{u}_K\big\|_{\mathcal{H}} \leq (C_{\text{exact}} + T)\, \tau^{\delta}. \tag{5.24}$$

Proof. Applying Theorem 5.24, Assumption 5.17, and choosing $\varepsilon_{k,i}$ as in (5.23), we obtain

$$\begin{aligned} \big\|u(t_K) - \widetilde{u}_K\big\|_{\mathcal{H}} &\leq \big\|u(t_K) - u_K\big\|_{\mathcal{H}} + \sum_{k=0}^{K-1} \sum_{i=1}^{S} C''_{\tau,k} C'_{\tau,k,(i)}\, \varepsilon_{k,i} \\ &= (C_{\text{exact}} + T)\, \tau^{\delta}. \end{aligned}$$

□

Remark 5.27. Let us take a closer look at condition (5.23). The number of factors in $C''_{\tau,k}$ is proportional to $K-k$, so that the tolerances are allowed to grow with k (if all factors in $C''_{\tau,k}$ are greater than or equal to 1). In this case, this means that the stage equations at earlier time steps have to be solved with higher accuracy compared to those towards the end of the iteration. Furthermore, the number of factors in $C'_{\tau,k,(i)}$ is proportional to $S-i$, but independent of k. Consequently, also the early stages have to be solved with higher accuracy compared to the later ones.

Remark 5.28. In Theorem 5.26, (5.23) is a specific choice for the tolerances $\varepsilon_{k,i}$, $k = 0, \dots, K-1$, $i = 1, \dots, S$. Essentially, it is an equilibrium strategy. However, also alternative choices are possible. Indeed, an inspection of the proof shows that any choice of $\varepsilon_{k,i}$ satisfying

$$\sum_{i=1}^{S} C'_{\tau,k,(i)} \varepsilon_{k,i} \leq (C''_{\tau,k})^{-1}\, \tau^{1+\delta}$$

would also be sufficient.

Remark 5.29. In practical applications, it would be natural to use the additional flexibility for the choice of $\varepsilon_{k,i}$ as outlined in Remark 5.28 to minimize the overall number of degrees of freedom of the method, given by

$$M_{\tau,T}(\delta) := M_{\tau,T}(\delta, (\varepsilon_{k,i})_{k,i}) := \sum_{k=0}^{K-1} \sum_{i=1}^{S} M_{\tau,k,i}(\varepsilon_{k,i}, \widehat{w}_{k,i}), \tag{5.25}$$

where for $k = 0, \dots, K-1$, $i = 1, \dots, S$,

$$\widehat{w}_{k,i} := L_{\tau,i}^{-1} R_{\tau,k,i}(\widetilde{u}_k, \widetilde{w}_{k,1}, \dots, \widetilde{w}_{k,i-1}), \tag{5.26}$$

and $M_{\tau,k,i}(\varepsilon_{k,i}, \widehat{w}_{k,i})$ as in Assumption 5.12. We omit the dependency on $(\varepsilon_{k,i})_{k,i}$ whenever the tolerances are clear from the context. This leads to the abstract minimization problem

$$\min_{(\varepsilon_{k,i})_{k,i}} \sum_{k=0}^{K-1} \sum_{i=1}^{S} M_{\tau,k,i}(\varepsilon_{k,i}, \widehat{w}_{k,i}) \quad \text{subject to} \quad \sum_{k=0}^{K-1} \sum_{i=1}^{S} C''_{\tau,k} C'_{\tau,k,(i)}\, \varepsilon_{k,i} \le T\tau^{\delta}.$$

We conclude this section with first applications of Theorem 5.21.

Example 5.30. Let us continue the example from the very beginning of this section and consider Eq. (5.2) in the Gel'fand triple $(H_0^1(\mathcal{O}), L_2(\mathcal{O}), H^{-1}(\mathcal{O}))$. We want to interpret the linearly-implicit Euler scheme as an abstract one-stage method with $\mathcal{H} = \mathcal{G} = L_2(\mathcal{O})$. To this end, let

$$\Delta_{\mathcal{O}}^{D} : D(\Delta_{\mathcal{O}}^{D}) \subseteq L_2(\mathcal{O}) \to L_2(\mathcal{O}),$$

denote the *Dirichlet Laplacian* with domain

$$D(\Delta_{\mathcal{O}}^{D}) := \Big\{ u \in H_0^1(\mathcal{O}) : \Delta u := \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} u \in L_2(\mathcal{O}) \Big\},$$

which is defined as variational operator, see (A.8) in Appendix A.1.3, starting with the symmetric, bounded, and elliptic bilinear form, cf. (A.6),

$$\begin{aligned} a : H_0^1(\mathcal{O}) \times H_0^1(\mathcal{O}) &\to \mathbb{R} \\ (u, v) \mapsto a(u, v) &:= \int_{\mathcal{O}} \langle \nabla u, \nabla v \rangle \, \mathrm{d}x. \end{aligned} \tag{5.27}$$

In this example, we pick a smooth initial value $u_0 \in D(\Delta_{\mathcal{O}}^{D})$, and consider a continuously differentiable function

$$f : [0, T] \times L_2(\mathcal{O}) \to L_2(\mathcal{O}),$$

which is Lipschitz continuous in the second variable, uniformly in $t \in [0, T]$. We denote the Lipschitz constant by $C^{\mathrm{Lip}, f}$. Observe that Assumption 5.15 is satisfied, since $\Delta_{\mathcal{O}}^{D}$ generates a strongly continuous contraction semi-group on $L_2(\mathcal{O})$, cf. Appendix A.1.3, and therefore Eq. (5.2) has a unique classical solution, see, e.g., Pazy [133, Theorems 6.1.5 and 6.1.7]. Thus, there exists a unique continuous function $u : [0, T] \to L_2(\mathcal{O})$, continuously differentiable in $(0, T]$, taking only values in $D(\Delta_{\mathcal{O}}^{D})$, and fulfilling

$$u(0) = u_0, \quad \text{as well as} \quad u'(t) = \Delta_{\mathcal{O}}^{D} u(t) + f(t, u(t)), \quad \text{for } t \in (0, T).$$

In this setting, we can state the exact linearly-implicit Euler scheme (5.3) in the form of an abstract one-stage scheme as follows: With $\mathcal{H} = \mathcal{G} = L_2(\mathcal{O})$ and $\tau = T/K$, we define the operators

$$\begin{aligned} L_{\tau,1}^{-1} : L_2(\mathcal{O}) &\to L_2(\mathcal{O}) \\ v &\mapsto L_{\tau,1}^{-1} v := (I - \tau \Delta_{\mathcal{O}}^{D})^{-1} v, \end{aligned}$$

as well as

$$\begin{aligned} R_{\tau,k,1} : L_2(\mathcal{O}) &\to L_2(\mathcal{O}) \\ v &\mapsto R_{\tau,k,1}(v) := v + \tau f(t_k, v), \end{aligned}$$

for $k = 0, \ldots, K-1$. Then the exact linearly-implicit Euler scheme fits perfectly into the abstract exact scheme (5.6) with $S = 1$.

Under our assumptions on the initial value u_0 and the forcing term f, this scheme converges to the exact solution of Eq. (5.2) with order $\delta = 1$, i.e., there exists a constant $C_{\mathrm{exact}} > 0$, such that

$$\big\| u(T) - u_K \big\|_{L_2(\mathcal{O})} \leq C_{\mathrm{exact}} \tau^1,$$

see for instance Crouzeix, Thomée [39]. Therefore, Assumption 5.17 is satisfied.

Assumption 5.4 can be verified by the following argument: It is well known that for any $\tau > 0$, the operator $L_{\tau,1}^{-1}$ defined above is bounded with norm less than or equal to one, cf. Appendix A.1.3. Because of the Lipschitz continuity of f, for each $k \in \{0, \ldots, K-1\}$, the composition

$$E_{\tau,k,k+1} := L_{\tau,1}^{-1} R_{\tau,k,1} : L_2(\mathcal{O}) \to L_2(\mathcal{O})$$

is Lipschitz continuous with Lipschitz constant

$$C_{\tau,k,k+1}^{\mathrm{Lip}} \leq 1 + \tau C^{\mathrm{Lip},f}.$$

Thus, if we define $E_{\tau,j,k} : L_2(\mathcal{O}) \to L_2(\mathcal{O})$ for $0 \leq j \leq k \leq K$ as in (5.8), these operators are Lipschitz continuous with Lipschitz constants

$$C_{\tau,j,k}^{\mathrm{Lip}} \leq (1 + \tau C^{\mathrm{Lip},f})^{k-j},$$

i.e., Assumption 5.4 is fulfilled. Furthermore, these constants can be estimated uniformly for all j, k and τ, since

$$1 \leq C_{\tau,j,k}^{\mathrm{Lip}} \leq (1 + \tau C^{\mathrm{Lip},f})^{K} \leq \exp(T C^{\mathrm{Lip},f}).$$

Now, let us assume that we have an approximation $\widetilde{E}_{\tau,k,k+1}(v)$, $v \in L_2(\mathcal{O})$, such that Assumption 5.6 is satisfied. Then, we can apply the results from above to choose the tolerances $(\varepsilon_k)_{k=0}^{K-1}$, so that the output $(\widetilde{u}_k)_{k=0}^{K}$ of the inexact scheme (5.10) converges to the exact solution with the same order $\delta = 1$. Indeed, if we choose

$$\varepsilon_k \leq \frac{\tau^2}{\exp(T C^{\mathrm{Lip},f})} \qquad \text{for } k = 0, \ldots, K-1,$$

we can conclude from Theorem 5.21 that the inexact linearly-implicit Euler-scheme (5.10) converges to the exact solution of Eq. (5.2) with order $\delta = 1$, i.e.,

$$\| u(T) - \widetilde{u}_K \|_{L_2(\mathcal{O})} \leq (C_{\mathrm{exact}} + T)\, \tau^1,$$

for all $K \in \mathbb{N}$.

Example 5.31. In the situation from Example 5.30, let us consider a specific form of $f : (0,T] \times L_2(\mathcal{O}) \to L_2(\mathcal{O})$, namely

$$(t,v) \mapsto f(t,v) := \bar{f}(v),$$

where $\bar{f} : \mathbb{R} \to \mathbb{R}$ is continuously differentiable with bounded, strictly negative derivative, i.e., there exists a constant $\bar{B} > 0$, so that

$$-\bar{B} < \frac{\mathrm{d}}{\mathrm{d}x}\bar{f}(x) < 0 \quad \text{for all} \quad x \in \mathbb{R}.$$

Then, for arbitrary $v_1, v_2 \in L_2(\mathcal{O})$ we get for any $k = 0, \ldots, K-1$,

$$\begin{aligned}
\big\| L_{\tau,1}^{-1} R_{\tau,k,1}(v_1) - L_{\tau,1}^{-1} R_{\tau,k,1}(v_2) \big\|_{L_2(\mathcal{O})} \\
&\leq \big\| R_{\tau,k,1}(v_1) - R_{\tau,k,1}(v_2) \big\|_{L_2(\mathcal{O})} \\
&= \big\| v_1 + \tau \bar{f}(v_1) - (v_2 + \tau \bar{f}(v_2)) \big\|_{L_2(\mathcal{O})} \\
&\leq \sup_{x\in\mathbb{R}} \left| 1 + \tau \frac{\mathrm{d}}{\mathrm{d}x} \bar{f}(x) \right| \| v_1 - v_2 \|_{L_2(\mathcal{O})}.
\end{aligned}$$

Thus, if $\tau < 2/\bar{B}$, we have a contraction. For $K \in \mathbb{N}$ big enough, and $\varepsilon_k \leq \tau^2$, $k = 0, \ldots, K-1$, we can argue as in Example 5.30 to show that

$$\big\| u(T) - \widetilde{u}_K \big\|_{L_2(\mathcal{O})} \leq (C_{\mathrm{exact}} + T)\, \tau^1,$$

i.e., the inexact linearly-implicit Euler scheme (5.10) again converges to the exact solution of Eq. (5.2) with order $\delta = 1$, but, in this example, for much larger values of ε_k, thus, with far fewer degrees of freedom.

5.1.4 Applicability of linearly-implicit 1-step S-stage schemes

We substantiate our abstract convergence analysis to the case when Rothe's method is induced by a linearly-implicit S-stage time integrator. We want to compute solutions $u : (0,T] \to V$ to initial value problems of the form

$$u'(t) = F(t, u(t)), \quad u(0) = u_0, \quad t \in [0,T], \tag{5.28}$$

where $F : [0,T] \times V \to V^*$ is a nonlinear right-hand side and $u_0 \in V$ is some initial value. Consequently, we consider the Gel'fand triple setting (V, U, V^*). Typical examples are, for instance, semi-linear equations, i.e.,

$$F(t, u(t)) := Au(t) + f(t, u(t)).$$

In practical applications usually A is a differential operator and f a linear or nonlinear drift term.

Essentially this section consists of two parts. First, we show that linearly-implicit S-stage schemes fit nicely into the abstract setting as outlined in Section 5.1 with $\mathcal{H} = V$ and $\mathcal{G} = V^*$, see Observation 5.33. In the second part, see Observation 5.41, we analyze the case $\mathcal{H} = \mathcal{G} = U$, since error estimates for the discretization in time are often formulated in the norm of U and then a higher order of convergence might be achieved, cf. Theorem 5.37.

In their general form, linearly-implicit S-stage methods are given by

$$u_{k+1} = u_k + \tau \sum_{i=1}^{S} m_i y_{k,i}, \quad k = 0, 1, \ldots, K-1, \tag{5.29}$$

with S linear *stage equations*

$$(I - \tau\gamma_{i,i}J)y_{k,i} = F\Big(t_k + a_i\tau, u_k + \tau \sum_{j=1}^{i-1} a_{i,j} y_{k,j}\Big) + \sum_{j=1}^{i-1} c_{i,j} y_{k,j} + \tau\gamma_i g, \tag{5.30}$$

and

$$a_i := \sum_{j=1}^{i-1} a_{i,j} \sum_{l=1}^{j} \frac{\gamma_{j,l}}{\gamma_{j,j}}, \quad \gamma_i := \sum_{l=1}^{i} \gamma_{i,l},$$

for $i = 1, \ldots, S$. By J and g we denote (approximations of) the partial derivatives $F_u(t_k, u_k)$ and $F_t(t_k, u_k)$, respectively. This particular choice for a_i ensures that J does not enter the right-hand side of (5.30). The parameters $a_{i,j}$, $c_{i,j}$, $\gamma_{i,j}$ and $m_i \neq 0$ have to be suitably chosen according to the desired properties of the scheme.

Remark 5.32. If $J = F_u(t_k, u_k)$ and $g = F_t(t_k, u_k)$ are the exact derivatives, the corresponding scheme is also known as a method of *Rosenbrock* type. However, this specific choice of J and g is not needed to derive a convergent time discretization scheme. In the larger class of *W–methods*, J and g are allowed to be approximations to the exact Jacobians. Often one chooses $g = 0$, which is done at the price of a significantly lower order of convergence and a substantially more complicated stability analysis.

First, we consider the case $\mathcal{H} = V$, $\mathcal{G} = V^*$. The scheme (5.29) immediately fits into the abstract setting of Section 5.1, as long as we interpret the term u_k as the solution to an additional 0th stage equation given by the identity operator I on V. Now, if we define

$$\begin{aligned} L_{\tau,i} : V &\to V^*, \\ v &\mapsto (I - \tau\gamma_{i,i}J)v \end{aligned} \tag{5.31}$$

and use the right-hand side of the stage equations (5.30) to define the operators

$$\begin{aligned} &R_{\tau,k,i} : V \times \cdots \times V \to V^*, \\ &(v_0, \ldots, v_{i-1}) \mapsto \tau m_i \Big(F\big(t_k + a_i\tau, v_0 + \sum_{j=1}^{i-1} \frac{a_{i,j}}{m_j} v_j\big) + \sum_{j=1}^{i-1} \frac{c_{i,j}}{\tau m_j} v_j + \tau\gamma_i g\Big), \end{aligned} \tag{5.32}$$

for $k = 0, \ldots, K-1$ and $i = 1, \ldots, S$, then the scheme (5.29) is related to the abstract Rothe method (5.6) as follows.

Observation 5.33. *For $k = 0, \ldots, K-1$ and $i = 1, \ldots, S$ let $L_{\tau,i}$ and $R_{\tau,k,i}$ be defined by* (5.31) *and* (5.32)*, respectively, and set $L_{\tau,0}^{-1} R_{\tau,k,0} := I_{V \to V}$. Then the linearly-implicit S-stage scheme* (5.29) *is an abstract $(S+1)$-stage scheme in the sense of* (5.6) *with*

$\mathcal{H} = V$, $\mathcal{G} = V^*$. We have

$$\begin{aligned} u_{k+1} &:= \sum_{i=0}^{S} w_{k,i}, \\ w_{k,i} &:= L_{\tau,i}^{-1} R_{\tau,k,i}(u_k, w_{k,1}, \ldots, w_{k,i-1}), \quad i = 0, \ldots, S, \end{aligned}$$

for $k = 0, \ldots, K-1$.

Remark 5.34. Of course, since the operators $R_{\tau,k,i}$ are derived from the right-hand side F, it might happen that they contain, e.g., nontrivial partial differential operators. Nevertheless, even in this case these differential operators are only *applied* to the current iterate and do not require the numerical solution of an operator equation, and that is why the operators $R_{\tau,k,i}$ can still be interpreted as evaluation operators.

Let us now look at an example, where a simple one-stage scheme of the form (5.29) with $\mathcal{H} = V$ and $\mathcal{G} = V^*$ is translated into a scheme with $\mathcal{H} = \mathcal{G} = U$.

Example 5.35. Let $\mathcal{O} \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain. Consider the heat equation (5.2) in the Gel'fand triple $(H_0^1(\mathcal{O}), L_2(\mathcal{O}), H^{-1}(\mathcal{O}))$ with

$$\Delta_{\mathcal{O}}^D u + f(t,u) =: F(t,u)$$

and $F : [0,T] \times H_0^1(\mathcal{O}) \to H^{-1}(\mathcal{O})$, where we assume again that f fulfills the conditions from Example 5.30. The scheme (5.29) with

$$S = 1, \quad \gamma_{1,1} = m_1 = 1, \quad J = \Delta_{\mathcal{O}}^D : H_0^1(\mathcal{O}) \to H^{-1}(\mathcal{O}), \quad \text{and} \quad g = 0$$

leads to

$$u_{k+1} = u_k + \tau\big(I - \tau\Delta_{\mathcal{O}}^D\big)^{-1}\big(\Delta_{\mathcal{O}}^D u_k + f(t_k, u_k)\big), \quad k = 0, \ldots, K-1,$$

which fits perfectly into the setting of Section 5.1. It can be rewritten as a 2-stage scheme of the form (5.6) with $\mathcal{H} = V$ and $\mathcal{G} = V^*$, cf. Observation 5.33. However, since the Dirichlet-Laplacian is not bounded on $L_2(\mathcal{O})$, it can not be understood directly as an S-stage scheme of the form (5.6) with $\mathcal{H} = \mathcal{G} = L_2(\mathcal{O})$, but a short computation shows that it can be rewritten as

$$u_{k+1} = \big(I - \tau\Delta_{\mathcal{O}}^D\big)^{-1}\big(u_k + \tau f(t_k, u_k)\big), \quad k = 0, \ldots, K-1.$$

Thus, if we start with $u_0 \in D(\Delta_{\mathcal{O}}^D)$, and consider the Dirichlet Laplacian as an unbounded operator on $L_2(\mathcal{O})$, this scheme can be interpreted as an abstract one-stage scheme of the form (5.6) with $\mathcal{H} = \mathcal{G} = U$, see Example 5.30. It is worth noting that this result stays true for a wider class of operators A instead of $\Delta_{\mathcal{O}}^D$, see Crouzeix, Thomée [39] for details.

The next step is to discuss the case $\mathcal{H} = \mathcal{G} = U$ in detail, where we restrict the discussion to the case of semi-linear problems (5.28) with a right-hand side of the form

$$F : [0,T] \times V \to V^*, \quad F(t,u) := A(t)u + f(t,u), \tag{5.33}$$

where $A(t)$ is given for all $t \in (0,T)$ in the sense of Appendix A.1.3. Furthermore, we focus on W-methods with the specific choice

$$J(t_k) := A(t_k), \quad g := 0, \tag{5.34}$$

in (5.30). We restrict our analysis to these methods for the following reasons. First, the linearly-implicit Euler scheme, which is the most important example, is a W-method and not a Rosenbrock method. Second, the choice of J in (5.34) avoids the evaluation of the Jacobian in each time step, which might be numerically costly.

In our setting, the overall convergence rate that can be expected is limited by the convergence rate of the exact scheme, cf. Theorem 5.26 and Assumption 5.17. Therefore, to obtain a reasonable result, it is clearly necessary to discuss the approximation properties of the exact S-stage scheme. To the best of our knowledge, the most far reaching results concerning the convergence of S-stage W-methods for evolution problems have been derived by Lubich, Ostermann [126]. For the reader's convenience, we discuss their results as far as it is needed for our purposes. To do so, we need the following definitions and assumptions.

The method (5.29) is called *$A(\theta)$-stable* if the related stability function

$$R(z) := 1 + z\,\mathbf{m}^\top \Big(\mathbf{I} - (c_{i,j})_{i,j=1}^S - z\big(\operatorname{diag}(\gamma_{i,i}) + (a_{i,j})_{i,j=1}^S\big)\Big)^{-1}\mathbf{1},$$

where $\mathbf{1}^\top := (1,\dots,1)^\top$ and $\mathbf{m}^\top := (m_1,\dots,m_S)^\top$, fulfills

$$|R(z)| \le 1 \quad \text{for all } z \in \mathbb{C} \text{ with } |\arg(z)| \ge \pi - \theta.$$

If, additionally, the limit $|R(\infty)| := \lim_{|z|\to\infty}|R(z)| < 1$, then the method is called *strongly $A(\theta)$-stable*.

We say that the scheme (5.29) is of *order* $p \in \mathbb{N}$, if the error of the method, when applied to ordinary differential equations defined on open subsets of $\mathbb{R}^d$ with sufficiently smooth right-hand sides, satisfies

$$\|u(t_k) - u_k\|_{\mathbb{R}^d} \le C_{\mathrm{ord}}\,\tau^p,$$

uniformly on bounded time intervals.

Assumption 5.36. Let $C_{\mathrm{offset}} \ge 0$ and denote $\widehat{J}(t) := A(t) + C_{\mathrm{offset}} I$.

(i) For both instances $G(t) := F_u(t,u(t))$ and $G(t) := \widehat{J}(t)$ it holds that $G(t) : V \to V^*$, $t \in [0,T]$, is a uniformly bounded family of linear operators in $\mathcal{L}(V, V^*)$. Each $G(t)$ is boundedly invertible and the family $G(t)^{-1}$, $t \in [0,T]$, is uniformly bounded in $\mathcal{L}(V^*, V)$.

(ii) There exist constants $\phi < \pi/2$, $C_i^{\mathrm{sect}} > 0$, $i = 1,2$ such that for all $t \in [0,T]$ and $z \in \mathbb{C}$ with $|\arg(z)| \le \pi - \phi$ the operators $zI - F_u(t,u(t))$ and $zI - \widehat{J}(t)$ are invertible, and their resolvents are bounded on V, i.e.,

$$\big\|(zI - F_u(t,u(t)))^{-1}\big\|_{\mathcal{L}(V,V)} \le \frac{C_1^{\mathrm{sect}}}{|z|}, \quad \big\|(zI - \widehat{J}(t))^{-1}\big\|_{\mathcal{L}(V,V)} \le \frac{C_2^{\mathrm{sect}}}{|z|}.$$

(iii) The mapping $t \mapsto F_u(t,u(t)) \in \mathcal{L}(V, V^*)$ is sufficiently often differentiable on $[0,T]$ and fulfills the Lipschitz condition

$$\big\|F_u(t,u(t)) - F_u(t',u(t'))\big\|_{\mathcal{L}(V,V^*)} \le C_u^F |t - t'| \quad \text{for } 0 \le t \le t' \le T.$$

(iv) The following bounds hold uniformly for v varying in bounded subsets of V and $0 \le t \le T$:

$$\big\|F_{tu}(t,v)w\big\|_{V^*} \le C_{tu}^F \|w\|_V, \quad \big\|F_{uu}(t,v)[w_1,w_2]\big\|_{V^*} \le C_{uu}^F \|w_1\|_V \|w_2\|_V.$$

(v) There exists a splitting

$$f_u(t, u(t)) =: S_k^{(l)} + S_k^{(r)} \tag{5.35}$$

and constants $\mu < 1$, $\beta \geq \mu$ (positive), $C_k^{(l)}$ (sufficiently small) as well as $C_{k,\mu}^{(r)}$, $C_{k,\beta}$, $C_k^{(l)}$, and $C_{k,\beta}^{(r)}$, such that

$$\begin{aligned} \|S_k^{(l)}\|_{\mathcal{L}(V,V^*)} &\leq C_k^{(l)}, \\ \big\|S_k^{(r)} \widehat{J}^{-\mu}(t_k)\big\|_{\mathcal{L}(V^*,V^*)} &\leq C_{k,\mu}^{(r)}, \\ \big\|\widehat{J}^{\beta}(t_k)(F_u(t_k, u(t_k)))^{-\beta}\big\|_{\mathcal{L}(V,V)} &\leq C_{k,\beta}, \\ \big\|\widehat{J}^{\beta}(t_k) S_k^{(l)} \widehat{J}^{-\beta}(t_k)\big\|_{\mathcal{L}(V,V^*)} &\leq C_k^{(l)}, \\ \big\|S_k^{(r)} \widehat{J}^{-\beta}(t_k)\big\|_{\mathcal{L}(V^*,V^*)} &\leq C_{k,\beta}^{(r)}. \end{aligned}$$

With above definitions and assumptions at hand, we quote LUBICH, OSTERMANN [126] concerning the convergence of exactly evaluated S-stage W-methods.

Theorem 5.37 [126, Theorem 6.2]**.** *Suppose that the solution u of Eq.* (5.28)*, together with* (5.33)*, is unique and has sufficiently regular temporal derivatives. Let Assumption 5.36 hold. Suppose that the scheme* (5.29) *is a W-method of order $p \geq 2$ that is strongly $A(\theta)$-stable with $\theta > \phi$ and $\phi < \pi/2$, cf. 5.36(ii). Let $\beta \in [0,1]$ be as in 5.36(v) such that $D(A(t)^\beta)$ is independent of t (with uniformly equivalent norms), $A^\beta u' \in L_2(0,T;V)$. Then the error provided by the numerical solution u_k, $k = 0, \ldots, K$ is bounded in $\tau \leq \tau_0$ by*

$$\begin{aligned} &\left(\tau \sum_{k=0}^{K} \|u_k - u(t_k)\|_V^2\right)^{1/2} + \max_{0\leq k\leq K} \|u_k - u(t_k)\|_U \\ &\quad \leq C_1^{\text{conv}} \tau^{1+\beta} \left(C_2^{\text{conv}} + C_1^{\text{conv}} C_k^{(l)}\right) C_k^{(l)} \left(\int_0^T \big\|A^\beta u'(t)\big\|_V^2 \,\mathrm{d}t\right)^{1/2} \\ &\quad + C_1^{\text{conv}} \tau^2 \left(\int_0^T \big\|A^\beta u'(t)\big\|_V^2 \,\mathrm{d}t + \int_0^T \|u''(t)\|_V^2 \,\mathrm{d}t + \int_0^T \|u'''(t)\|_{V^*}^2 \,\mathrm{d}t\right)^{1/2}. \end{aligned} \tag{5.36}$$

The constants C_1^{conv}, C_2^{conv}, and τ_0 depend on the concrete choice of the W-method, the constants in the assumptions, and on T. The maximal time step size τ_0 depends in addition on the size of the integral terms in (5.36)*.*

Remark 5.38. As in Theorem 5.37, and throughout this section, we assume that a unique solution exists, i.e., Assumption 5.15 holds. This is the starting point for our convergence analysis of inexact S-stage schemes. Thus, we do not discuss the solvability and uniqueness theory for PDEs in detail. However, since in the forthcoming examples we use the results from LUBICH, OSTERMANN [126], let us briefly recall which solution concept is used in the following standard situation: Consider a linear operator $A : V \to V^*$ fulfilling the conditions from Appendix A.1.3, and assume that F in (5.28) has the form $F(t,u) := Au + f(t)$. Then, a *weak formulation* of Eq. (5.28) is: find

$$u \in C([0,T];U) \cap L_2(0,T;V),$$

such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u(t), v\rangle_U = \langle Au(t), v\rangle_{V^*\times V} + \langle f(t), v\rangle_U \quad \text{for all} \quad v \in V, t \in (0, T].$$

Before we continue our analysis, let us present a well-known W-method which fulfills the assumptions of Theorem 5.37.

Example 5.39. For $S = 2$, we present the following scheme taken from VERWER ET AL. [168], which is a strongly $A(\theta)$-stable ($\theta = \pi/2$) W-method of order $p = 2$. It is sometimes called *ROS2* in the literature and is given by

$$u_{k+1} = u_k + \frac{3}{2}\tau y_{k,1} + \frac{1}{2}\tau y_{k,2},$$

where

$$\begin{aligned} y_{k,1} &= \left(I - \tau\frac{1}{2+\sqrt{2}}A(t_k)\right)^{-1}\Big(A(t_k)u_k + f(t_k, u_k)\Big), \\ y_{k,2} &= \left(I - \tau\frac{1}{2+\sqrt{2}}A(t_k)\right)^{-1}\Big(A(t_k+\tau)(u_k + \tau y_{k,1}) + f(t_k+\tau, u_k + \tau y_{k,1}) - 2y_{k,1}\Big). \end{aligned}$$

It fits into the setting of (5.29) with

$$m_1 = 3/2, \quad m_2 = 1/2, \quad \gamma_{1,1} = \gamma_{2,2} = (2+\sqrt{2})^{-1}, \quad a_{2,1} = 1, \quad \text{and} \quad c_{2,1} = -2.$$

For the remainder of this section, we restrict the setting of (5.33) to the special case

$$F : [0, T] \times V \to V^*, \quad F(t, u) := Au + f(t), \tag{5.37}$$

where $A : V \to V^*$ is given in the sense of Appendix A.1.3, and $f : [0, T] \to U$ is a continuously differentiable function. In this case, as already mentioned in Example 5.30, Eq. (5.1.4) has a unique classical solution, provided $u_0 \in D(A; U)$, see e.g., PAZY [133, Corollary 2.5]. It is worth noting that this unique solution is a also a weak solution in the sense of LUBICH, OSTERMANN [126], as addressed in Remark 5.38.

Using the abstract results from Section 5.1, we analyze the inexact S-stage method corresponding to the W-method with

$$J := A \quad \text{and} \quad g := 0. \tag{5.38}$$

Furthermore, we restrict the discussion to the case $S = 2$. This is not a major restriction for the following reason. According to Theorem 5.37, the maximal convergence order of W-methods is bounded by $\delta = 1 + \beta$, where $\beta \in [0, 1]$. In Example 5.42 below, we show that an F of the form (5.37) fulfills Assumption 5.36(v) with $\beta = 1$. If we additionally impose the asserted regularity assumptions with $\beta = 1$, cf. (5.44) in Example 5.42, then we can apply Theorem 5.37 with $\beta = 1$ to the *ROS*2-method given in Example 5.39 (which is a 2-stage method), and get the optimal order in this context.

The structure (5.37) of the right-hand side F in Eq. (5.1.4), allows the following reformulation of the W-method with (J, g) as in (5.38).

Lemma 5.40. *Consider the S-stage W-method given by* (5.29) *with $S = 2$ and F and (J, g) as in* (5.37) *and* (5.38)*, respectively. Then, if $\gamma_{i,i} \neq 0$, for $i = 1, 2$, we have*

$$u_{k+1} = \left(1 - \frac{m_1}{\gamma_{1,1}} - \frac{m_2}{\gamma_{2,2}}\left(1 - \frac{a_{2,1}}{\gamma_{1,1}}\right)\right) u_k + \left(\tau m_1 - \tau m_2 \frac{a_{2,1}}{\gamma_{2,2}}\right) v_{k,1} + \tau m_2 v_{k,2},$$

where

$$v_{k,1} = L_{\tau,1}^{-1}\left(\frac{1}{\tau\gamma_{1,1}} u_k + f(t_k)\right),$$
$$v_{k,2} = L_{\tau,2}^{-1}\left(\left(\frac{1}{\tau\gamma_{2,2}}\left(1 - \frac{a_{2,1}}{\gamma_{1,1}}\right) - \frac{c_{2,1}}{\tau\gamma_{1,1}}\right) u_k + \left(\frac{a_{2,1}}{\gamma_{2,2}} + c_{2,1}\right) v_{k,1} + f(t_k + a_2\tau)\right).$$

Proof. See Appendix B.5. □

As an immediate consequence of Lemma 5.40, we obtain the following observation.

Observation 5.41. *If $\gamma_{i,i} \neq 0$, for $i = 1, 2$, and $m_1\gamma_{2,2} \neq m_2 a_{2,1}$, then the scheme under consideration perfectly fits into the setting of Section 5.1 with $\mathcal{H} = \mathcal{G} = U$. It can be written in the form of the abstract Rothe method* (5.6)*. More precisely, we have*

$$\left.\begin{aligned} u_{k+1} &= \sum_{i=0}^{2} w_{k,i}, \\ w_{k,i} &:= L_{\tau,i}^{-1} R_{\tau,k,i}(u_k, w_{k,1}, \ldots, w_{k,i-1}), \quad i = 0, 1, 2, \end{aligned}\right\} \tag{5.39}$$

with

$$\begin{aligned} L_{\tau,i}^{-1} &: U \to U, \\ & v \mapsto \big(I - \tau\gamma_{i,i} A\big)^{-1} v, \quad \text{for } i = 1, 2, \end{aligned} \tag{5.40}$$

as well as the evaluation operators

$$\begin{aligned} R_{\tau,k,1} &: U \to U, \\ & v \mapsto \left(\frac{m_1}{\gamma_{1,1}} - \frac{m_2 a_{2,1}}{\gamma_{2,2}\gamma_{1,1}}\right) v + \tau\left(m_1 - m_2\frac{a_{2,1}}{\gamma_{2,2}}\right) f(t_k), \end{aligned} \tag{5.41}$$

and

$$\begin{aligned} R_{\tau,k,2} &: U \times U \to U, \\ & (v_0, v_1) \mapsto \left(\frac{m_2}{\gamma_{2,2}}\left(1 - \frac{a_{2,1}}{\gamma_{1,1}}\right) - \frac{c_{2,1} m_2}{\gamma_{1,1}}\right) v_0 \\ & \qquad + \frac{m_2 a_{2,1} + m_2\gamma_{2,2} c_{2,1}}{m_1\gamma_{2,2} - m_2 a_{2,1}} v_1 + \tau m_2 f(t_k + a_2\tau). \end{aligned} \tag{5.42}$$

Furthermore, a 0*th step given by*

$$\begin{aligned} L_{\tau,0}^{-1} R_{\tau,k,0} &: U \to U, \\ & v \mapsto \left(1 - \frac{m_1}{\gamma_{1,1}} - \frac{m_2}{\gamma_{2,2}}\left(1 - \frac{a_{2,1}}{\gamma_{1,1}}\right)\right) v. \end{aligned} \tag{5.43}$$

An easy computation, together with the fact that $L_{\tau,1}^{-1}$ and $L_{\tau,2}^{-1}$ are contractions on U, cf. Appendix A.1.3, yield the Lipschitz constant

$$C_{\tau,k,(1)}^{\mathrm{Lip}} = \left|\frac{m_1}{\gamma_{1,1}} - \frac{m_2 a_{2,1}}{\gamma_{2,2}\gamma_{1,1}}\right|$$

of $L_{\tau,1}^{-1}R_{\tau,k,1}$*. Simultaneously, the Lipschitz constant of* $L_{\tau,2}^{-1}R_{\tau,k,2}$ *can be estimated as follows:*

$$C_{\tau,k,(2)}^{\mathrm{Lip}} \leq \max\left\{\left|\frac{m_2}{\gamma_{2,2}}\left(1-\frac{a_{2,1}}{\gamma_{1,1}}\right)-\frac{m_2c_{2,1}}{\gamma_{1,1}}\right|,\ \left|\frac{m_2a_{2,1}+m_2\gamma_{2,2}c_{2,1}}{m_1\gamma_{2,2}-m_2a_{2,1}}\right|\right\}.$$

Note that both constants are independent of k *and* τ*.*

Example 5.42. As a first step towards the case of inexact operator evaluations we need to check Assumption 5.36 for the applicability of Theorem 5.37 in the current setting (5.37), (5.38). We begin by choosing $C_{\text{offset}} = 0$. As a consequence it holds that $\widehat{J} = F_u(t, u(t)) = A$, independently of t. Assumption 5.36(i) holds by the assumptions on A, see Appendix A.1.3. This, together with the ellipticity assumption, given in (A.6), already implies Assumption 5.36(ii), see KATO [108]. Further, $A = F_u(t, v)$ is independent of (t, v), and as a consequence Assumptions 5.36(iii) and (iv) hold with $C_u^F = C_{tu}^F = C_{uu}^F = 0$. Finally, since J is the exact Jacobian, it is possible to choose $S_k^{(l)} = S_k^{(r)} = 0$ in (5.35), such that Assumption 5.36(v) holds with $C_k^{(l)} = C_{k,\mu}^{(r)} = C_{k,\beta}^{(r)} = 0$, $C_{k,\beta} = 1$ and $\beta = 1$. Concerning the W-method (5.29) we assume it to be of order $p \geq 2$ and strongly $A(\theta)$-stable with $\theta > \phi$, where ϕ is as in Assumption 5.36(ii). Therefore, e.g., the *ROS2* method from Example 5.39 could be employed. If for the solution of Eq. (5.1.4) with F as in (5.37) the regularity assumptions

$$Au', u'' \in L_2(0,T;V),\ u''' \in L_2(0,T;V^*) \tag{5.44}$$

hold, then we can apply Theorem 5.37. Using $C_k^{(l)} = 0$ and $\beta = 1$, the convergence result (5.36) reads as

$$\begin{aligned}&\left(\tau\sum_{k=0}^{K}\|u_k-u(t_k)\|_V^2\right)^{1/2}+\max_{0\leq k\leq K}\|u_k-u(t_k)\|_U\\&\quad\leq C_1^{\mathrm{conv}}\tau^2\left(\int_0^T\|Au'(t)\|_V^2\,\mathrm{d}t+\int_0^T\|u''(t)\|_V^2\,\mathrm{d}t+\int_0^T\|u'''(t)\|_{V^*}^2\,\mathrm{d}t\right)^{1/2}.\end{aligned}$$

That means, the error measured in the norm $\|\cdot\|_U$ is of order $\delta = 2$.

Example 5.43. We employ the method *ROS2* from Example 5.39 to our general convergence results of Theorem 5.26 for the case of inexact solution of the stage equations. First, we present the method in its reformulation on $\mathcal{H} = \mathcal{G} = U$, as given in Observation 5.41. Inserting the coefficients

$$m_1 = \frac{3}{2},\quad m_2 = \frac{1}{2},\quad \gamma_{1,1} = \gamma_{2,2} = (2+\sqrt{2})^{-1},\quad a_{2,1} = 1,\quad \text{and}\quad c_{2,1} = -2$$

into (5.39), (5.40), (5.41), (5.42), and (5.43) yields

$$\begin{aligned}u_{k+1} &= \sum_{i=0}^{2} w_{k,i},\\ w_{k,i} &:= L_{\tau,i}^{-1}R_{\tau,k,i}(u_k, w_{k,1},\ldots,w_{k,i-1}),\quad i = 0,1,2,\end{aligned}$$

where the 0th stage vanishes, i.e., $L_{\tau,0}^{-1}R_{\tau,k,0} \equiv 0$,

$$\begin{aligned}L_{\tau,1}^{-1} = L_{\tau,2}^{-1} &: U \to U,\\ &v \mapsto \left(I - \tau\frac{1}{2+\sqrt{2}}A\right)^{-1} v,\end{aligned}$$

and the evaluation operators are given by

$$\begin{aligned} R_{\tau,k,1} &: U \to U, \\ &v \mapsto -\frac{\sqrt{2}}{2} v + \tau \frac{1-\sqrt{2}}{2} f(t_k), \end{aligned}$$

and

$$\begin{aligned} R_{\tau,k,2} &: U \times U \to U, \\ &(v_0, v_1) \mapsto -\frac{\sqrt{2}}{2} v_0 + \frac{\sqrt{2}}{1-\sqrt{2}} v_1 + \tau \frac{1}{2} f(t_k + \tau). \end{aligned}$$

This scheme fits perfectly into the abstract Rothe method (5.6) with $S = 2$. By Observation 5.41, we get the following estimates of the Lipschitz constants of $L_{\tau,i}^{-1} R_{\tau,k,i}$, $i = 1, 2$:

$$C_{\tau,k,(1)}^{\mathrm{Lip}} = \frac{\sqrt{2}}{2}, \quad \text{and} \quad C_{\tau,k,(2)}^{\mathrm{Lip}} \leq \max\left\{ \frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{1-\sqrt{2}} \right\} \leq \frac{\sqrt{2}}{2}.$$

As in Example 5.42, we assume that the exact solution u satisfies (5.44). Furthermore, we assume we have a numerical scheme at hand, such that Assumption 5.12 is satisfied. Then, by Theorem 5.37 and Theorem 5.26, if we choose the tolerances $\varepsilon_{k,i}$, for $k = 0, \ldots, K-1$ and $i = 1, 2$, so that they satisfy

$$0 < \varepsilon_{k,i} \leq \frac{1}{2} \tau^3 \left(\frac{1}{2} + \sqrt{2} \right)^{K-k-1} \prod_{l=i+1}^{2} \left(1 + \frac{\sqrt{2}}{2} \right),$$

the corresponding inexact 2-stage scheme (5.13) converges with order $\delta = 2$. The computational cost can be estimated by

$$\sum_{k=0}^{K-1} \big(M_{\tau,k,1}(\varepsilon_{k,1}, \widehat{w}_{k,1}) + M_{\tau,k,2}(\varepsilon_{k,2}, \widehat{w}_{k,2}) \big)$$

with $M_{\tau,k,i}(\cdot, \cdot)$ as in Assumption 5.12 and $\widehat{w}_{k,i}$ as in Remark 5.29.

Remark 5.44. For methods of Rosenbrock type, i.e., under the assumption that we use exact Jacobians J and g, a result similar to Theorem 5.37 holds. In Lubich, Ostermann [126, Theorem 5.2] it is shown that for methods of order $p \geq 3$ and under certain additional regularity assumptions on the exact solution u of Eq. (5.1.4) the error can be bounded similar to (5.36) with rate $\tau^{2+\beta}$, $\beta \in [0, 1]$.

5.2 Application to stochastic evolution equations

We apply Rothe's method to a class of semi-linear parabolic SPDEs and derive sufficient conditions for convergence in the case of inexact operator evaluations. We use the stochastic analogue of the linearly-implicit Euler scheme (5.3) in time, and the spatial discretization in every time step is a (possibly nonlinear) black box-solver $[\,\cdot\,]_\varepsilon$, e.g., an adaptive wavelet solver as described in Section 5.3.3. As before, we interpret parabolic SPDEs as ordinary SDEs in a suitable function space U. We consider a separable real Hilbert space U and the U-valued SDE

$$\mathrm{d}u(t) = Au(t)\mathrm{d}t + F(u(t))\mathrm{d}t + B(u(t))\mathrm{d}W(t), \quad u(0) = u_0, \quad t \in [0, T], \tag{5.45}$$

driven by a cylindrical Wiener process $W = (W(t))_{t\in[0,T]}$ on the sequence space ℓ_2. Here, $u = (u(t))_{t\in[0,T]}$ is a U-valued stochastic process,

$$A : D(A) \subset U \to U$$

is a densely defined, strictly negative definite, self-adjoint, linear operator such that zero belongs to the resolvent set, and A^{-1} is compact on U. The drift term

$$F : D((-A)^{\varrho}) \to D((-A)^{\varrho-\varrho_F})$$

and the diffusion term

$$B : D((-A)^{\varrho}) \to \mathcal{L}(\ell_2,\, D((-A)^{\varrho-\varrho_B}))$$

are Lipschitz continuous maps for suitable constants ϱ, ϱ_F, and ϱ_B. Details are given in Section 5.2.1. Our setting is based on the one considered in Printems [137] where the convergence of semi-discretizations in time is investigated. This is why, in contrast to the previous sections, the forcing term F may not depend on the time variable t. Compared with Printems [137] we allow the spatial regularity of the whole setting to be 'shifted' in terms of the additional parameter ϱ. In concrete applications to parabolic SPDEs, this leads to estimates of the discretization error in terms of the numerically important energy norm, cf. Example 5.55, provided that the initial condition u_0 and the forcing terms F and B are sufficiently regular.

5.2.1 Setting and assumptions

Let us describe the setting for Eq. (5.45) systematically and in detail.

Assumption 5.45. The operator $A : D(A) \subset U \to U$ is linear, densely defined, strictly negative definite, and self-adjoint. Zero belongs to the resolvent set of A and the inverse $A^{-1} : U \to U$ is compact. There exists an $\varrho_A > 0$ such that $(-A)^{-\varrho_A}$ is a trace class operator on U.

To simplify notation, the separable real Hilbert space U is always assumed to be infinite-dimensional. It follows that A enjoys a spectral decomposition of the form

$$Av = \sum_{j\in\mathbb{N}} \lambda_j \langle v, e_j\rangle_U e_j, \quad v \in D(A),$$

where $(e_j)_{j\in\mathbb{N}}$ is an orthonormal basis of U consisting of eigenvectors of A with strictly negative eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ such that

$$0 > \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_j \to -\infty, \quad j \to \infty.$$

For $s \geq 0$ we set

$$D((-A)^s) := \left\{ v \in U \,:\, \sum_{j\in\mathbb{N}} \left|(-\lambda_j)^s \langle v, e_j\rangle_U\right|^2 < \infty \right\}, \tag{5.46}$$

$$(-A)^s v := \sum_{j\in\mathbb{N}} (-\lambda_j)^s \langle v, e_j\rangle_U e_j, \quad v \in D((-A)^s), \tag{5.47}$$

so that $D((-A)^s)$, endowed with the norm $\|\cdot\|_{D((-A)^s)} := \|(-A)^s \cdot\|_U$, is a Hilbert space. For $s < 0$ we define $D((-A)^s)$ as the completion of U with respect to the norm $\|\cdot\|_{D((-A)^s)}$, defined on U by

$$\|v\|^2_{D((-A)^s)} := \sum_{j\in\mathbb{N}} \big|(-\lambda_j)^s \langle v, e_j\rangle_U\big|^2.$$

Thus, $D((-A)^s)$ can be considered as a space of formal sums

$$v = \sum_{j\in\mathbb{N}} v^{(j)} e_j \quad \text{such that} \quad \sum_{j\in\mathbb{N}} \big|(-\lambda_j)^s v^{(j)}\big|^2 < \infty$$

with coefficients $v^{(j)} \in \mathbb{R}$. Generalizing (5.47), we obtain operators $(-A)^s$, $s \in \mathbb{R}$, which map $D((-A)^r)$ isometrically onto $D((-A)^{r-s})$ for all $r \in \mathbb{R}$. Now, the trace class condition in Assumption 5.45 can be reformulated as the requirement that there exists an $\varrho_A > 0$ such that

$$\operatorname{Tr}(-A)^{-\varrho_A} = \sum_{j\in\mathbb{N}} (-\lambda_j)^{-\varrho_A} < \infty. \tag{5.48}$$

The Dirichlet-Laplacian from Example 5.30 fulfills Assumption 5.45.

Example 5.46. Let $\mathcal{O} \subset \mathbb{R}^d$ be bounded and open, $U := L_2(\mathcal{O})$, and let $A := \Delta^D_{\mathcal{O}}$ be the Dirichlet-Laplacian on $\mathcal{O}$ from Example 5.30, i.e.,

$$\Delta^D_{\mathcal{O}} : D(\Delta^D_{\mathcal{O}}) \subseteq L_2(\mathcal{O}) \to L_2(\mathcal{O})$$

with domain

$$D(\Delta^D_{\mathcal{O}}) := \left\{ u \in H^1_0(\mathcal{O}) \,:\, \Delta u := \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} u \in L_2(\mathcal{O}) \right\}.$$

Note that this definition of the domain of the Dirichlet-Laplacian is consistent with the definition of $D((-\Delta^D_{\mathcal{O}})^s)$ for $s = 1$ in (5.46), see e.g., Lindner [124, Remark 1.13] for details. This linear operator fulfills Assumption 5.45 for all $\varrho_A > d/2$: It is densely defined, self-adjoint, and strictly negative definite, since it has been introduced in complete analogy to the variational operator $\widetilde{A}$ from Appendix A.1.3, starting with the symmetric, bounded, and elliptic bilinear form (5.27). Furthermore, due to the Rellich-Kondrachov theorem (see, e.g., Adams, Fournier [2, Chapter VI]), it possesses a compact inverse $(\Delta^D_{\mathcal{O}})^{-1} : L_2(\mathcal{O}) \to L_2(\mathcal{O})$. Moreover, Weyl's law states that

$$-\lambda_j \asymp j^{2/d}, \qquad j \in \mathbb{N},$$

see Birman, Solomyak [11], implying that (5.48) holds for all $\varrho_A > d/2$.

Concerning the forcing terms F and B of Eq. (5.45) we assume the following.

Assumption 5.47. For certain smoothness parameters

$$\varrho \geq 0, \quad \varrho_F < 1, \quad \text{and} \quad \varrho_B < \frac{1-\varrho_A}{2} \tag{5.49}$$

(ϱ_A as in Assumption 5.45) F and B map $D((-A)^\varrho)$ to $D((-A)^{\varrho-\varrho_F})$ and $D((-A)^\varrho)$ to $\mathcal{L}(\ell_2, D((-A)^{\varrho-\varrho_B}))$, respectively. Furthermore, they are globally Lipschitz continuous, that is, there exist positive constants C^{Lip}_F and C^{Lip}_B such that

$$\|F(v) - F(w)\|_{D((-A)^{\varrho-\varrho_F})} \leq C^{\text{Lip}}_F \|v - w\|_{D((-A)^\varrho)}$$

and

$$\|B(v) - B(w)\|_{\mathcal{L}(\ell_2,\, D((-A)^{\varrho-\varrho_B}))} \leq C_B^{\mathrm{Lip}} \|v - w\|_{D((-A)^{\varrho})}$$

for all $v, w \in D((-A)^{\varrho})$.

Remark 5.48. The parameters ϱ_F and ϱ_B in Assumption 5.47 are allowed to be negative.

Remark 5.49. Assumption 5.47 goes along the lines of PRINTEMS [137] ('shifted' by $\varrho \geq 0$). The linear growth conditions [137, (3.5), (3.7)] follow from the (global) Lipschitz continuity of the mappings F and B.

Finally, we describe the noise and the initial condition in Eq. (5.45). For the notion of a normal filtration, see Appendix A.4.

Assumption 5.50. The noise $W = (W(t))_{t\in[0,T]}$ is a cylindrical Wiener process on ℓ_2 with respect to a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$. The underlying probability space $(\Omega, \mathcal{F}, \mathrm{P})$ is complete. For ϱ as in Assumption 5.47, the initial condition u_0 in Eq. (5.45) belongs to the space $L_2(\Omega, \mathcal{F}_0, \mathrm{P}; D((-A)^{\varrho}))$.

Let $(e^{At})_{t\geq 0}$ be the strongly continuous semi-group of contractions on U which is generated by A. We call a *mild solution* to Eq. (5.45) a predictable process

$$u : \Omega \times [0, T] \to D((-A)^{\varrho})$$

with

$$\sup_{t\in[0,T]} \mathrm{E}\left[\|u(t)\|^2_{D((-A)^{\varrho})}\right] < \infty \tag{5.50}$$

such that for every $t \in [0, T]$ the equality

$$u(t) = e^{At}u_0 + \int_0^t e^{A(t-s)} F(u(s))\mathrm{d}s + \int_0^t e^{A(t-s)} B(u(s))\mathrm{d}W(s) \tag{5.51}$$

holds P-almost surely in $D((-A)^{\varrho})$. The first integral in (5.51) is a $D((-A)^{\varrho})$-valued Bochner integral for P-almost every $\omega \in \Omega$, cf. Appendix A.2, while the second integral is a $D((-A)^{\varrho})$-valued stochastic integral as defined in Appendix A.5.

Remark 5.51. Both integrals in (5.51) exist due to (5.50) and Assumptions 5.45, 5.47. For example, considering the stochastic integral in (5.51), we know that it exists as an element of $L_2(\Omega, \mathcal{F}_t, \mathrm{P}; D((-A)^{\varrho}))$ if the integral

$$\int_0^t \mathrm{E}\left[\left\|e^{A(t-s)} B(u(s))\right\|^2_{\mathcal{L}_{\mathrm{HS}}(\ell_2,\, D((-A)^{\varrho}))}\right] \mathrm{d}s$$

is finite, where $\mathcal{L}_{\mathrm{HS}}(\ell_2,\ D((-A)^{\varrho}))$ denotes the space of Hilbert-Schmidt operators from ℓ_2 to $D((-A)^{\varrho})$. The integrand of the last integral can be estimated from above by

$$\mathrm{Tr}(-A)^{-\varrho_A} \left\|(-A)^{\varrho_B+\varrho_A/2} e^{A(t-s)}\right\|^2_{\mathcal{L}(D((-A)^{\varrho}))} \mathrm{E}\left[\left\|(-A)^{-\varrho_B} B(u(s))\right\|^2_{\mathcal{L}(\ell_2,\, D((-A)^{\varrho}))}\right],$$

and we have

$$\left\|(-A)^{\varrho_B+\varrho_A/2} e^{A(t-s)}\right\|^2_{\mathcal{L}(D((-A)^{\varrho}))} \leq C(t-s)^{-(2\varrho_B+\varrho_A)}$$

with $2\varrho_B + \varrho_A < 1$. Moreover,

$$\mathrm{E}\left[\left\|(-A)^{-\varrho_B} B(u(s))\right\|^2_{\mathcal{L}(\ell_2,\, D((-A)^{\varrho}))}\right] \leq C\left(1 + \sup_{r\in[0,T]} \mathrm{E}\left[\|u(r)\|^2_{D((-A)^{\varrho})}\right]\right).$$

The last estimate follows from the global Lipschitz property of the mapping

$$B : D((-A)^{\varrho}) \to \mathcal{L}(\ell_2,\, D((-A)^{\varrho-\varrho_B})).$$

Proposition 5.52. *Let Assumptions 5.45, 5.47, and 5.50 be satisfied. Then, Eq.* (5.45) *has a unique mild solution, i.e., there exists a unique (up to modifications) predictable stochastic process* $u : \Omega \times [0,T] \to D((-A)^{\varrho})$ *with* $\sup_{t\in[0,T]} \mathrm{E}\left[\|u(t)\|^2_{D((-A)^{\varrho})}\right] < \infty$ *such that, for every* $t \in [0,T]$*, Eq.* (5.51) *holds* P*-almost surely.*

Proof. For the case $\varrho = 0$ existence and uniqueness of a mild solution to Eq. (5.45) has been given in Printems [137, Proposition 3.1]. The proof is a modification of the proof of Da Prato, Zabczyk [40, Theorem 7.4] by using a contraction argument in $L_\infty([0,T]; L_2(\Omega; U))$. For the general case $\varrho \geq 0$ the existence and uniqueness can been proven analogously, see Jentzen, Kloeden [102, Theorem 5.1]. Alternatively, the case $\varrho > 0$ can be traced back to the case $\varrho = 0$. Suppose that Assumptions 5.45, 5.47, and 5.50 hold for some $\varrho > 0$. We set

$$\widehat{U} := D((-A)^{\varrho}), \quad D(\widehat{A}) := D((-A)^{\varrho+1})$$

and consider the unbounded operator $\widehat{A}$ on $\widehat{U}$ given by

$$\widehat{A} : D(\widehat{A}) \subset \widehat{U} \to \widehat{U}, \quad v \mapsto \widehat{A}v := Av.$$

Note that $\widehat{A}$ fulfills Assumption 5.45 with A, $D(A)$, and U replaced by $\widehat{A}$, $D(\widehat{A})$ and $\widehat{U}$, respectively. Defining the spaces $D((-\widehat{A})^s)$ analogously to the spaces $D((-A)^s)$, we have $D((-A)^{\varrho+s}) = D((-\widehat{A})^s)$, $s \in \mathbb{R}$, so that Assumptions 5.47 and 5.50 can be reformulated with ϱ, $D((-A)^{\varrho})$, $D((-A)^{\varrho-\varrho_F})$ and $D((-A)^{\varrho-\varrho_B})$ replaced by $\widehat{\varrho} := 0$, $D((-\widehat{A})^{\widehat{\varrho}})$, $D((-\widehat{A})^{\widehat{\varrho}-\varrho_F})$ and $D((-\widehat{A})^{\widehat{\varrho}-\varrho_B})$, respectively. Thus, the equation

$$\mathrm{d}u(t) = \big(\widehat{A}u(t) + F(u(t))\big)\mathrm{d}t + B(u(t))\mathrm{d}W(t), \quad u(0) = u_0, \tag{5.52}$$

fits into the setting of [137], so that, by [137, Proposition 3.1], there exists a unique mild solution u to Eq. (5.52). Since the operators $e^{At} \in \mathcal{L}(U)$ and $e^{\widehat{A}t} \in \mathcal{L}(\widehat{U})$ coincide on $\widehat{U} \subset U$, it is clear that any mild solution to Eq. (5.52) is a mild solution to Eq. (5.45) and vice versa. □

Remark 5.53. If the initial condition u_0 belongs to

$$L_p(\Omega, \mathcal{F}_0, \mathrm{P}; D((-A)^{\varrho})) \subset L_2(\Omega, \mathcal{F}_0, \mathrm{P}; D((-A)^{\varrho}))$$

for some $p > 2$, then the solution u even satisfies $\sup_{t\in[0,T]} \mathrm{E}\left[\|u(t)\|^p_{D((-A)^{\varrho})}\right] < \infty$. This is a consequence of the Burkholder-Davis-Gundy inequalities, cf. Da Prato, Zabczyk [40, Theorem 7.4] or Printems [137, Proposition 3.1]. Analogous improvements are valid for the estimates in Propositions 5.60 and 5.65 below.

In the following example we state concrete examples for stochastic PDEs that fit into our setting.

Example 5.54. Let $\mathcal{O} \subset \mathbb{R}^d$ be bounded and open, $U := L_2(\mathcal{O})$, and let $A := \Delta_{\mathcal{O}}^D$ be the Dirichlet-Laplacian on $\mathcal{O}$ as described in Example 5.46. We consider examples for stochastic PDEs in dimension $d = 1$ and $d \geq 2$.

First, let $\mathcal{O} \subset \mathbb{R}^1$ be one-dimensional and consider the problem

$$\left.\begin{aligned} \mathrm{d}u(t,x) &= \Delta_x u(t,x)\mathrm{d}t + g(u(t,x))\mathrm{d}t + h(u(t,x))\,\mathrm{d}W_1(t,x), \\ &\qquad\qquad (t,x) \in [0,T] \times \mathcal{O}, \\ u(t,x) &= 0, \qquad\quad (t,x) \in [0,T] \times \partial\mathcal{O}, \\ u(0,x) &= u_0(x), \quad\ x \in \mathcal{O}. \end{aligned}\right\} \tag{5.53}$$

where $u_0 \in L_2(\mathcal{O})$, $g : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous, $h : \mathbb{R} \to \mathbb{R}$ is bounded and globally Lipschitz continuous, and $W_1 = (W_1(t))_{t\in[0,T]}$ is a Wiener process (with respect to a normal filtration on a complete probability space) whose Cameron-Martin space is some space of functions on $\mathcal{O}$ that is continuously embedded in $L_2(\mathcal{O})$, e.g., W_1 is a cylindrical Wiener process on $L_2(\mathcal{O})$. Let $(\psi_k)_{k\in\mathbb{N}}$ be an arbitrary orthonormal basis of the Cameron-Martin space of W_1 and set

$$\begin{aligned} F(v)(x) &:= g(v(x)), && v \in L_2(\mathcal{O}),\ x \in \mathcal{O}, \\ \big(B(v)\mathbf{a}\big)(x) &:= h(v(x)) \sum_{k\in\mathbb{N}} a_k \psi_k(x), && v \in L_2(\mathcal{O}),\ \mathbf{a} = (a_k)_{k\in\mathbb{N}} \in \ell_2,\ x \in \mathcal{O}. \end{aligned} \tag{5.54}$$

Then, Eq. (5.45) is an abstract version of problem (5.53), and the mappings F and B are globally Lipschitz continuous (and thus linearly growing) from $D((-A)^0) = L_2(\mathcal{O})$ to $L_2(\mathcal{O})$ and from $D((-A)^0)$ to $\mathcal{L}(\ell_2, L_2(\mathcal{O}))$, respectively. It follows that Assumptions 5.45, 5.47, and 5.50 are satisfied for $1/2 < \varrho_A < 1$ (compare Example 5.46) and $\varrho = \varrho_F = \varrho_B = 0$.

Now, let $\mathcal{O} \subset \mathbb{R}^d$ be d-dimensional, $d \geq 2$, and consider the problem (5.53) where $u_0 \in L_2(\mathcal{O})$, $g : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous, $h : \mathbb{R} \to \mathbb{R}$ is constant (additive noise), and $W_1 = (W_1(t))_{t\in[0,T]}$ is a Wiener process whose Cameron-Martin space is some space of functions on $\mathcal{O}$ that is continuously embedded in $D((-A)^{-\varrho_B})$ for some $\varrho_B < 1/2 - d/4$. One easily sees that the mappings F and B, defined as in (5.54), are globally Lipschitz continuous (and thus linearly growing) from $D((-A)^0) = L_2(\mathcal{O})$ to $L_2(\mathcal{O})$ and from $D((-A)^0)$ to $\mathcal{L}(\ell_2, D((-A)^{-\varrho_B}))$, respectively. It follows that Assumptions 5.45, 5.47, and 5.50 are satisfied for $\varrho_B < 1/2 - d/4$, $d/2 < \varrho_A < 1 - 2\varrho_B$, and $\varrho = \varrho_F = 0$. Alternatively, we could assume h to be sufficiently smooth and replace $h(u(t,x))$ in problem (5.53) by, e.g., $h\big(\int_{\mathcal{O}} k(x,y)u(y)\mathrm{d}y\big)$ with a sufficiently smooth kernel $k : \mathcal{O} \times \mathcal{O} \to \mathbb{R}$.

Example 5.55. As in Examples 5.46 and 5.54, let $A := \Delta_{\mathcal{O}}^D$ be the Dirichlet-Laplacian on a bounded and open domain $\mathcal{O} \subset \mathbb{R}^d$. From the numerical point of view, we are especially interested in stochastic PDEs of type (5.45) with $\varrho = 1/2$. In this case the solution process takes values in the space $D((-A)^{1/2}) = H_0^1(\mathcal{O})$, and, in Proposition 5.60 and Theorem 5.63 below, we obtain estimates for the approximation error in terms of the energy norm

$$\|v\|_{D((-\Delta_{\mathcal{O}}^D)^{1/2})} = \langle \nabla v, \nabla v \rangle_{L_2(\mathcal{O})}^{1/2}, \quad v \in H_0^1(\mathcal{O}).$$

The energy norm is crucial because error estimates for numerical solvers of elliptic problems (which we want to apply in each time step) are usually expressed in terms of this norm, cf. Section 5.3, where we consider adaptive wavelet solvers with optimal convergence rates.

First, let $\mathcal{O} \subset \mathbb{R}^1$ be one-dimensional, and consider the problem (5.53) where $u_0 \in H^1_0(\mathcal{O})$, $g : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous, $h : \mathbb{R} \to \mathbb{R}$ is linear or constant, and $W_1 = (W_1(t))_{t\in[0,T]}$ is a Wiener process whose Cameron-Martin space is some space of functions on $\mathcal{O}$ that is continuously embedded in $D((-A)^{1/2-\varrho_B})$ for some nonnegative $\varrho_B < 1/4$, so that W_1 takes values in a bigger Hilbert space, say, in $D((-A)^{-1/4})$. (The embedding $D((-A)^{1/2-\varrho_B}) \hookrightarrow D((-A)^{-1/4})$ is Hilbert-Schmidt since (5.48) is fulfilled for $\varrho_A > 1/2$, compare Example 5.46.) Take an arbitrary orthonormal basis $\{\psi_k\}_{k\in\mathbb{N}}$ of the Cameron-Martin space of W_1, and define $F(v)$ and $B(v)$ for $v \in H^1_0(\mathcal{O})$ analogously to (5.54), i.e.,

$$\begin{aligned} F(v)(x) &:= g(v(x)), && v \in H^1_0(\mathcal{O}),\ x \in \mathcal{O}, \\ \big(B(v)\mathbf{a}\big)(x) &:= h(v(x)) \sum_{k\in\mathbb{N}} a_k \psi_k(x), && v \in H^1_0(\mathcal{O}),\ \mathbf{a} = (a_k) \in \ell_2,\ x \in \mathcal{O}. \end{aligned} \tag{5.55}$$

Then, Eq. (5.45) is an abstract version of problem (5.53), and the mappings

$$F : D((-A)^{1/2}) = H^1_0(\mathcal{O}) \to D((-A)^0) = L_2(\mathcal{O})$$

and

$$B : D((-A)^{1/2}) \to \mathcal{L}(\ell_2, D((-A)^{1/2-\varrho_B}))$$

are globally Lipschitz continuous (and thus linearly growing). The mapping properties of B follow from the inequalities

$$\|vw\|_{L_2(\mathcal{O})} \leq \|v\|_{H^1_0(\mathcal{O})}\|w\|_{L_2(\mathcal{O})} \quad \text{and} \quad \|vw\|_{H^1_0(\mathcal{O})} \leq C\|v\|_{H^1_0(\mathcal{O})}\|w\|_{H^1_0(\mathcal{O})},$$

which are a consequence of the Sobolev embedding $H^1(\mathcal{O}) \hookrightarrow L_\infty(\mathcal{O})$ in dimension 1 and interpolation since $D((-A)^{1/2-\varrho_B}) = [L_2(\mathcal{O}), D((-A)^{1/2})]_{1-2\varrho_B}$. Thus, Assumptions 5.45, 5.47, and 5.50 are fulfilled for $\varrho = \varrho_F = 1/2$, $0 \leq \varrho_B < 1/4$ and $1/2 < \varrho_A < 1-2\varrho_B$.

Now, let $\mathcal{O} \subset \mathbb{R}^d$ be d-dimensional, $d \geq 2$, and consider problem (5.53) where $u_0 \in H^1_0(\mathcal{O})$, $g : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous, $h : \mathbb{R} \to \mathbb{R}$ is constant, and $W_1 = (W_1(t))_{t\in[0,T]}$ is a Wiener process whose Cameron-Martin space is continuously embedded in $D((-A)^{1/2-\varrho_B})$ for some $\varrho_B < 1/2 - d/4$. Then, the mappings F and B, defined analogously to the one dimensional case, are globally Lipschitz continuous (and thus linearly growing) from $D((-A)^{1/2}) = H^1_0(\mathcal{O})$ to $D((-A)^0) = L_2(\mathcal{O})$ and from $D((-A)^{1/2})$ to $\mathcal{L}(\ell_2, D((-A)^{1/2-\varrho_B}))$ respectively. It follows that Assumptions 5.45, 5.47, and 5.50 are fulfilled for $\varrho = \varrho_F = 1/2$, $\varrho_B < 1/2 - d/4$ and $1 < \varrho_A < 1 - 2\varrho_B$.

5.2.2 Semi-discretization in time

From now on, let Assumptions 5.45, 5.47, and 5.50 be satisfied.

For the time discretization of the (mild) solution $u = (u(t))_{t\in[0,T]}$ to Eq. (5.45) we use the stochastic analogue of the linearly-implicit Euler scheme (5.3), i.e., for $K \in \mathbb{N}$ and

$$\tau := \frac{T}{K}$$

we consider discretizations $(u_k)_{k=0}^K$ given by the initial condition u_0 in Eq. (5.45) and

$$\left.\begin{aligned} u_{k+1} = (I-\tau A)^{-1}\big(u_k + \tau F(u_k) + \sqrt{\tau} B(u_k)\chi_k\big)\,, \\ k=0,\ldots,K-1, \end{aligned}\right\} \tag{5.56}$$

where

$$\chi_k := \chi_k^K := \frac{1}{\sqrt{\tau}}\left(W\big(t_{k+1}^K\big) - W\big(t_k^K\big)\right).$$

Note that each χ_k, $k=0,\ldots,K-1$, is an $\mathcal{F}_{t_{k+1}}$-measurable Gaussian white noise on ℓ_2, i.e., a linear isometry from ℓ_2 to $L_2(\Omega,\mathcal{F}_{t_{k+1}},\mathrm{P})$ such that for each $\mathbf{a}\in\ell_2$ the real valued random variable $\chi_k(\mathbf{a})$ is centered Gaussian with variance $\|\mathbf{a}\|_{\ell_2}^2$. We write $\chi_k(\mathbf{a}) \sim \mathcal{N}(0,\|\mathbf{a}\|_{\ell_2})$ for short. Moreover, for each $k=0,\ldots,K-1$, the sub-σ-algebra of $\mathcal{F}$ generated by $\{\chi_k(\mathbf{a}) : \mathbf{a}\in\ell_2\}$ is independent of $\mathcal{F}_{t_k}$.

We explain in which way the scheme (5.56) has to be understood. Let G be a separable real Hilbert space such that $D((-A)^{\varrho-\varrho_B})$ is embedded into G via a Hilbert-Schmidt embedding. Then, for all $k=0,\ldots,K-1$ and for all $\mathcal{F}_{t_k}$-measurable, $D((-A)^{\varrho})$-valued, square integrable random variables $v\in L_2(\Omega,\mathcal{F}_{t_k},\mathrm{P};D((-A)^{\varrho}))$, the term $B(v)\chi_k$ can be interpreted as an $\mathcal{F}_{t_{k+1}}$-measurable, square integrable, G-valued random variable in the sense

$$B(v)\chi_k := L_2(\Omega,\mathcal{F}_{t_{k+1}},\mathrm{P};G)\text{-}\lim_{J\to\infty}\sum_{j=1}^J \chi_k(\mathbf{b}_j)B(v)\mathbf{b}_j \tag{5.57}$$

where $\{\mathbf{b}_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of ℓ_2. This definition is independent of the specific choice of the orthonormal basis $\{\mathbf{b}_j\}_{j\in\mathbb{N}}$. Note that the stochastic independence of $\{\chi_k(\mathbf{a}) : \mathbf{a}\in\ell_2\}$ and $\mathcal{F}_{t_k}$ is important at this point. We have

$$\mathrm{E}\left[\|B(v)\chi_k\|_G^2\right] = \mathrm{E}\left[\|B(v)\|_{\mathcal{L}_{\mathrm{HS}}(\ell_2,G)}^2\right], \tag{5.58}$$

the last term being finite due to the Lipschitz continuity of B by Assumption 5.47 (see also Remark 5.49) and the fact that the embedding $D((-A)^{\varrho-\varrho_B}) \hookrightarrow G$ is Hilbert-Schmidt. Let us explicitly set

$$G := D((-A)^{\varrho-\max\{\varrho_F,\,\varrho_B+\varrho_A/2\}}).$$

The condition $\mathrm{Tr}(-A)^{-\varrho_A}<\infty$ in Assumption 5.45 yields that the embedding

$$D((-A)^{\varrho-\varrho_B}) \hookrightarrow D((-A)^{\varrho-\varrho_B-\varrho_A/2})$$

is Hilbert-Schmidt, and the embedding

$$D((-A)^{\varrho-\varrho_B-\varrho_A/2}) \hookrightarrow D((-A)^{\varrho-\max\{\varrho_F,\,\varrho_B+\varrho_A/2\}})$$

is clearly continuous. Thus, we have indeed a Hilbert-Schmidt embedding

$$D((-A)^{\varrho-\varrho_B}) \hookrightarrow G.$$

For all $k=0,\ldots,K-1$ and $v\in L_2(\Omega,\mathcal{F}_{t_k},\mathrm{P};D((-A)^{\varrho}))$ we consider the term $B(v)\chi_k$ as an element in the space

$$L_2(\Omega,\mathcal{F}_{t_{k+1}},\mathrm{P};G) = L_2\big(\Omega,\mathcal{F}_{t_{k+1}},\mathrm{P};D((-A)^{\varrho-\max\{\varrho_F,\,\varrho_B+\varrho_A/2\}})\big).$$

Next, due to the Lipschitz continuity of F by Assumption 5.47 (see also Remark 5.49), we also know that for all $v \in L_2(\Omega, \mathcal{F}_{t_k}, \mathrm{P}; D((-A)^{\varrho}))$ the term $F(v)$ is an element in $L_2(\Omega, \mathcal{F}_{t_k}, \mathrm{P}; G)$. Finally, as a consequence of Lemma 5.58 below and the fact that $\max\{\varrho_F,\, \varrho_B + \varrho_A/2\} < \max\{\varrho_F,\, 1/2\} < 1$ due to (5.49), the operator $(I - \tau A)^{-1}$ is continuous from G to $D((-A)^{\varrho})$. It follows that the discretizations $(u_k)_{k=0}^{K}$ are uniquely determined by (5.56) and that every u_k belongs to the space $L_2(\Omega, \mathcal{F}_{t_k}, \mathrm{P}; D((-A)^{\varrho}))$.

Remark 5.56. In practice, one has to truncate the noise decomposition (5.57) and one may approximate $B(v)\chi_k \in L_2(\Omega, \mathcal{F}_{t_{k+1}}, \mathrm{P}; G)$ by a finite sum

$$\sum_{j=1}^{J} \chi_k(\mathbf{b}_j) B(v)\mathbf{b}_j \in L_2(\Omega, \mathcal{F}_{t_{k+1}}, \mathrm{P}; D((-A)^{\varrho-\varrho_B})), \quad J \in \mathbb{N}.$$

However, in Section 5.3 we are going to assume that the right hand sides of the elliptic equations in each time step are given exactly, cf. Assumption 5.72(ii).

Now, we can embed the scheme (5.56) into the abstract setting of Section 5.1. For measurability reasons the spaces $\mathcal{H}$ and $\mathcal{G}$ have to depend on the time step k, i.e., we consider spaces $\mathcal{H} = \mathcal{H}_k$ and $\mathcal{G} = \mathcal{G}_k$.

Observation 5.57. *Let*

$$\left.\begin{aligned}
\mathcal{H}_k &:= L_2(\Omega, \mathcal{F}_{t_k}, \mathrm{P}; D((-A)^{\varrho})), && k = 0, \ldots, K,\\
\mathcal{G}_k &:= L_2(\Omega, \mathcal{F}_{t_k}, \mathrm{P}; G), && k = 1, \ldots, K,\\
R_{\tau,k} &: \mathcal{H}_k \to \mathcal{G}_{k+1} && \\
&\quad v \mapsto R_{\tau,k}(v) := v + \tau F(v) + \sqrt{\tau} B(v)\chi_k, && k = 0, \ldots, K-1,\\
L_\tau^{-1} &: \mathcal{G}_k \to \mathcal{H}_k && \\
&\quad v \mapsto L_\tau^{-1} v := (I - \tau A)^{-1} v, && k = 1, \ldots, K.
\end{aligned}\right\} \tag{5.59}$$

With these definitions at hand, the linearly-implicit Euler scheme (5.56) *can be written in the form of the abstract S-stage scheme* (5.6) *with $S = 1$, $L_{\tau,1}^{-1} := L_\tau^{-1}$, and $R_{\tau,k,1} := R_{\tau,k}$, for $k = 0, \ldots, K-1$. We have*

$$u_{k+1} = L_\tau^{-1} R_{\tau,k}(u_k), \quad k = 0, \ldots, K-1. \tag{5.60}$$

The fact that the spaces $\mathcal{H} = \mathcal{H}_k$ and $\mathcal{G} = \mathcal{G}_k$ depend on the time step k does not cause any problems when using results from Section 5.1, as far as the corresponding assumptions are fulfilled, see also Remark 5.20.

The following Lemma is helpful not only for the preceding argument but also for estimates further down.

Lemma 5.58. *Let $\tau > 0$ and $r \in \mathbb{R}$. The operator $I - \tau A$ is a homeomorphism from $D((-A)^r)$ to $D((-A)^{r-1})$. For $n \in \mathbb{N}$ we have the following operator norm estimates for $(I - \tau A)^{-n}$, considered as an operator from $D((-A)^{r-s})$ to $D((-A)^r)$, $s \leq 1$:*

$$\|(I - \tau A)^{-n}\|_{\mathcal{L}(D((-A)^{r-s}), D((-A)^r))} \leq \begin{cases} s^s \left(1 - \frac{s}{n}\right)^{(n-s)} (n\tau)^{-s} & : \ 0 < s \leq 1,\\ (-\lambda_1)^s (1 - \tau\lambda_1)^{-n} & : \ s \leq 0. \end{cases}$$

Proof. See [21, Lemma 4.13]. □

Remark 5.59. Without additional assumptions on B or a truncation of the noise decomposition (5.57), the operator $R_{\tau,k}$ cannot easily be traced back to a family of operators

$$R_{\tau,k,\omega} : D((-A)^{\varrho}) \to G, \quad \omega \in \Omega,$$

in the sense that for $v \in \mathcal{H}_k = L_2(\Omega, \mathcal{F}_{t_k}, \mathrm{P}; D((-A)^{\varrho}))$ the image $R_{\tau,k}(v)$ is determined by

$$(R_{\tau,k}(v))(\omega) = R_{\tau,k,\omega}(v(\omega)) \quad \text{for P-almost all } \omega \in \Omega. \tag{5.61}$$

However, this is possible, for instance, if the operator $B(v) : \ell_2 \to D((-A)^{\varrho-\varrho_B})$ for all $v \in D((-A)^{\varrho})$ has a continuous extension $B(v) : U_0 \to D((-A)^{\varrho-\varrho_B})$ to a bigger Hilbert space U_0 such that ℓ_2 is embedded into U_0 via a Hilbert-Schmidt embedding. Another instance where a representation of the form (5.61) is possible is the case where the mapping $B : D((-A)^{\varrho}) \to \mathcal{L}(\ell_2, D((-A)^{\varrho-\varrho_B}))$ is constant, i.e., the case of additive noise. We take a closer look at the latter case, writing $B \in \mathcal{L}(\ell_2, D((-A)^{\varrho-\varrho_B}))$ for short. We fix a version of each of the P-almost surely determined, G-valued random variables $B\chi_k = B\chi_k^K$, $k = 0, \ldots, K-1$, $K \in \mathbb{N}$, and set

$$R_{\tau,k,\omega}(v) := v + F(v) + (B\chi_k)(\omega), \quad \omega \in \Omega, \; v \in D((-A)^{\varrho}).$$

It is clear that (5.61) holds for all $v \in L_2(\Omega, \mathcal{F}_{t_k}, \mathrm{P}; D((-A)^{\varrho}))$, and we have the following alternative interpretation of the scheme (5.56) in the case of additive noise within the abstract setting of Section 5.1.

$$\left.\begin{aligned} \mathcal{H} &:= D((-A)^{\varrho})), \\ \mathcal{G} &:= G = D((-A)^{\varrho-\max\{\varrho_F,\, \varrho_B+\varrho_A/2\}}), \\ R_{\tau,k,\omega} &: \mathcal{H} \to \mathcal{G} \\ &\quad v \mapsto R_{\tau,k,\omega}(v) := v + \tau F(v) + \sqrt{\tau}(B\chi_k)(\omega), \\ L_\tau^{-1} &: \mathcal{G} \to \mathcal{H} \\ &\quad v \mapsto L_\tau^{-1} v := (I - \tau A)^{-1} v, \end{aligned}\right\} \tag{5.59$_\omega$}$$

$k = 0, \ldots, K-1$. With these definitions, the abstract scheme (5.6) in Section 5.1 with $S = 1$ describes the stochastic scheme (5.56) in an ω-wise sense, $\omega \in \Omega$.

Now, we verify Assumption 5.17 for the scheme (5.56) in its abstract form (5.60), see Remark 5.61 below. Therefore, we state an extension of the error estimate for the linearly-implicit Euler scheme given in Printems [137].

Proposition 5.60. *Let Assumptions 5.45, 5.47, and 5.50 be satisfied. Let* $(u_k)_{k=0}^K$ *be the time discretization of the mild solution* $(u(t))_{t\in[0,T]}$ *to Eq.* (5.45), *given by the linearly-implicit Euler scheme* (5.56). *Then, for every*

$$\delta < \min\left\{1 - \varrho_F, \frac{1-\varrho_A}{2} - \varrho_B\right\},$$

we have for all $1 \leq k \leq K$

$$\Big(\mathrm{E}\left[\|u(t_k) - u_k\|^2_{D((-A)^{\varrho})}\right]\Big)^{1/2} \leq C\left(\tau^{\delta} + \frac{1}{k}\Big(\mathrm{E}\left[\|u_0\|^2_{D((-A)^{\varrho})}\right]\Big)^{1/2}\right),$$

where the constant $C > 0$ *depends only on* δ, A, B, F, ϱ_A, ϱ_B, ϱ_F, *and* T.

Proof. See [21, Proposition 4.15]. □

Remark 5.61. If $k \geq K^\delta$, $\delta > 0$, then $1/k \leq T^{-\delta}\tau^\delta$, and we obtain

$$\left(\mathrm{E}\left[\|u(t_k) - u_k\|^p_{D((-A)^\varrho)}\right]\right)^{1/p} \leq C_{\mathrm{exact}}\, \tau^\delta \tag{5.62}$$

with a constant $C_{\mathrm{exact}} > 0$ that depends only on δ, u_0, A, B, F, ϱ_A, ϱ_B, ϱ_F, and T. Since δ is always smaller than 1, it follows in particular that (5.62) holds for $k = K$. Using the definitions in (5.59) and the notation introduced in Section 5.1 this means that Assumption 5.17 is satisfied, i.e., we have

$$\|u(T) - E_{\tau,0,K}(u_0)\|_{\mathcal{H}_K} \leq C_{\mathrm{exact}}\, \tau^\delta,$$

where the Euler scheme operator $E_{\tau,0,K} : \mathcal{H}_0 \to \mathcal{H}_K$ is given by the composition

$$(L_\tau^{-1} R_{\tau,K-1}) \circ (L_\tau^{-1} R_{\tau,K-2}) \circ \cdots \circ (L_\tau^{-1} R_{\tau,0}).$$

5.2.3 Discretization in time and space

So far we have verified the existence and uniqueness of a mild solution to Eq. (5.45) as well as the convergence of the exactly evaluated Euler scheme (5.56) with rate $\delta < \min\{1 - \varrho_F,\, (1 - \varrho_A)/2 - \varrho_B\}$. We now turn to the corresponding inexact scheme where the iterated stationary problems in (5.56), respectively (5.60), are solved only approximately. As above, we are interested in how the tolerances for the spatial approximation errors in each time step have to be chosen to achieve the same order of convergence as for the exact scheme. Throughout this section we use the definitions given in Observation 5.57 to interpret the setting described in the previous Sections 5.2.1 and 5.2.2 in terms of the abstract framework of Section 5.1.

As in Section 5.1, Assumption 5.12, we assume that we have a numerical scheme $[\,\cdot\,]_\varepsilon$ at hand which, for all $w \in \mathcal{H}_k$, $k = 0, \ldots, K-1$, and for every prescribed tolerance $\varepsilon > 0$, provides us with an approximation $[v]_\varepsilon$ of

$$v = L_\tau^{-1} R_{\tau,k}(w)$$

such that

$$\|v - [v]_\varepsilon\|_{\mathcal{H}_{k+1}} = \left(\mathrm{E}[\|v - [v]_\varepsilon\|^2_{D((-A)^\varrho)}]\right)^{1/2} \leq \varepsilon.$$

We think of $[v]_\varepsilon$ as the result of an ω-wise application of some deterministic solver for elliptic equations with error at most ε in $D((-A)^\varrho)$. For instance, an adaptive wavelet solver as described in Section 5.3.3 with a proper evaluation of the nonlinearities F and B, see, e.g., Cohen et al. [31], Dahmen et al. [61], Kappei [107], and an adequate truncation of the noise.

Remark 5.62. Here, we concentrate on the convergence analysis and do not discuss the number of degrees of freedom involved. However, all results from Section 5.1 that do not involve assertions concerning the number of degrees of freedom remain valid in the present setting with obvious modifications.

Given prescribed tolerances ε_k, $k = 0, \dots, K-1$, for the spatial approximation errors in each time step, we consider the inexact version of the linearly-implicit Euler scheme (5.60)

$$\left.\begin{aligned} \widetilde{u}_0 &= u_0, \\ \widetilde{u}_{k+1} &= [L_\tau^{-1} R_{\tau,k}(\widetilde{u}_k)]_{\varepsilon_k}, \quad k = 0, \dots, K-1, \end{aligned}\right\} \tag{5.63}$$

which is the analogue to scheme (5.13) from Section 5.1 with $S = 1$ and $\varepsilon_{k,1} := \varepsilon_k$. We already know from the considerations in Section 5.1 that sufficient conditions how to tune the tolerances ε_k in the inexact scheme (5.63) to obtain the same order of convergence as for the exact scheme (5.60) can be described in terms of the Lipschitz constants $C^{\mathrm{Lip}}_{\tau,j,k}$ of the operators

$$E_{\tau,j,k} = (L_\tau^{-1} R_{\tau,k-1}) \circ (L_\tau^{-1} R_{\tau,k-2}) \circ \cdots \circ (L_\tau^{-1} R_{\tau,j}) : \mathcal{H}_j \to \mathcal{H}_k,$$

$1 \le j \le k \le K$, $K \in \mathbb{N}$. In the present setting we are able to show that the constants $C^{\mathrm{Lip}}_{\tau,j,k}$ are bounded uniformly in j, k, and τ, see Lemma 5.64 below. Together with the arguments in Section 5.1 and Proposition 5.60 this leads to the main result of this section.

Theorem 5.63. *Let Assumptions 5.45, 5.47, and 5.50 be satisfied. Let $(u(t))_{t\in[0,T]}$ be the unique mild solution to Eq.* (5.45) *and let*

$$\delta < \min\left\{1 - \varrho_F,\, \frac{1-\varrho_A}{2} - \varrho_B\right\}.$$

If one chooses

$$\varepsilon_k \le \tau^{1+\delta}$$

for all $k = 0, \dots, K-1$, $K \in \mathbb{N}$, then the output $\widetilde{u}_K$ of the inexact linearly-implicit Euler scheme (5.63) *converges to $u(T)$ with rate δ, i.e., we have*

$$\left(\mathrm{E}[\|u(T) - \widetilde{u}_K\|^2_{D((-A)^\varrho)}]\right)^{1/2} \le C\tau^\delta$$

with a constant C depending only on u_0, δ, A, B, F, ϱ_A, ϱ_B, ϱ_F, and T.

The verification of Theorem 5.63 is based on the estimate of the Lipschitz constants $C^{\mathrm{Lip}}_{\tau,j,k}$ given in the following lemma.

Lemma 5.64. *Let Assumptions 5.45, 5.47, and 5.50 be fulfilled. There exists a finite constant $C > 0$, depending only on A, B, F, ϱ_A, ϱ_B, ϱ_F, and T, such that*

$$C^{\mathrm{Lip}}_{\tau,j,k} \le C \quad \text{for all } 1 \le j \le k \le K,\ K \in \mathbb{N}.$$

Proof. See Appendix B.6. □

With this result at hand we obtain, as a next step towards the verification of Theorem 5.63, the following error estimate for the inexact scheme.

Proposition 5.65. *Let Assumptions 5.45, 5.47, and 5.50 be satisfied. Let $(u(t))_{t\in[0,T]}$ be the unique mild solution to Eq.* (5.45). *Let $(\widetilde{u}_k)_{k=0}^K$ be the discretization of $(u(t))_{t\in[0,T]}$ in time and space given by the inexact linearly-implicit Euler scheme*

(5.63), *where* ε_k, $k = 0, \dots, K-1$, *are prescribed tolerances for the spatial approximation errors in each time step. Then, for every* $1 \le k \le K$, $K \in \mathbb{N}$, *and for every*

$$\delta < \min\left\{1 - \varrho_F,\ \frac{1-\varrho_A}{2} - \varrho_B\right\}.$$

we have

$$\left(\mathrm{E}\left[\|u(t_k) - \widetilde{u}_k\|^2_{D((-A)^\varrho)}\right]\right)^{1/2} \le C\left(\tau^\delta + \frac{1}{k}\left(\mathrm{E}\left[\|u_0\|^2_{D((-A)^\varrho)}\right]\right)^{1/2} + \sum_{j=0}^{k-1}\varepsilon_j\right)$$

with a constant C *that depends only on* δ, A, B, F, ϱ_A, ϱ_B, ϱ_F, *and* T.

Proof. Arguing as in the proof of Theorem 5.19, cf. Remark 5.20, we obtain the general error estimate

$$\begin{aligned}\left(\mathrm{E}\left[\|u(t_k) - \widetilde{u}_k\|^2_{D((-A)^\varrho)}\right]\right)^{1/2} &= \|u(t_k) - \widetilde{u}_k\|_{\mathcal{H}_k} \\ &\le \|u(t_k) - u_k\|_{\mathcal{H}_k} + \sum_{j=0}^{k-1} C^{\mathrm{Lip}}_{\tau,j+1,k}\varepsilon_j,\end{aligned} \tag{5.64}$$

where $(u_k)_{k=0}^K$ is the discretization of $(u(t))_{t\in[0,T]}$ in time given by the exact linearly-implicit Euler scheme (5.56), respectively (5.60). Note that the analogues to Assumptions 5.6 and 5.8 in Theorem 5.19 are fulfilled due to the construction of the inexact scheme (5.63); the analogues to Assumptions 5.4 and 5.15 follow from Assumptions 5.45, 5.47, and 5.50 via Proposition 5.52 and Lemma 5.64. Alternatively, we could have used a modified version of Theorem 5.24 with $S = 1$ and $\varepsilon_{j,1} := \varepsilon_j$. The assertion of the proposition follows directly from (5.64), Proposition 5.60, and Lemma 5.64. □

The proof of Theorem 5.63 is now straightforward, similar to the argumentation in the proofs of Theorems 5.21 and 5.26.

Proof of Theorem 5.63. The assertion follows from Proposition 5.65 and the elementary estimates

$$\frac{1}{K} \le \frac{1}{K^\delta} = T^{-\delta}\tau^\delta \quad \text{and} \quad \sum_{j=0}^{K-1}\varepsilon_k \le T\tau^\delta.$$ □

5.3 Spatial approximation by wavelet methods

We combine the analysis presented in Section 5.1 with complexity estimates for optimal adaptive wavelet solvers in order to obtain complexity results for the inexact Rothe method. In Section 5.1, we assumed, cf. Assumption 5.12, that we have a numerical solver at hand which enables us to compute the solution of the subproblem arising at the k-th time step and i-th stage up to a prescribed tolerance $\varepsilon_{k,i}$. In practice, this goal can be achieved by employing *adaptive* discretization strategies with a posteriori error control and guaranteed convergence properties. We conclude in Section 5.3.3 by discussing adaptive strategies based on wavelets that are guaranteed to converge for a large range of problems and are asymptotically optimal, i.e., cf. Section 2.3.2, they asymptotically realize the same convergence order as best N-term wavelet approximation and the computational cost is proportional to the number of degrees of freedom N.

Again, we assume that the underlying domain $\mathcal{O} \subseteq \mathbb{R}^d$ enables us to construct biorthogonal wavelet bases $\Psi = \{\psi_\mu : \mu \in \nabla\}$ and $\widetilde{\Psi} = \{\widetilde{\psi}_\mu : \mu \in \nabla\}$, which satisfy the properties (W1)–(W6) stated in Section 2.3.3.

5.3.1 Complexity estimates using adaptive wavelet solvers

We derive estimates on the number of degrees of freedom within the wavelet setting, which are needed to guarantee that the inexact scheme (5.13) converges with the same order as the exact scheme (5.6). As it turns out, among other things, regularity estimates for the exact solution in specific scales of Besov spaces are essential.

To keep the technicalities at a reasonable level, we focus our analysis on parabolic evolution equations of the form

$$u'(t) = A(t)u(t) + f(t, u(t)), \quad u(0) = u_0, \quad t \in (0, T], \tag{5.65}$$

in a Gel'fand triple setting (V, U, V^*) with $V = H_0^{\widehat{s}}(\mathcal{O})$, $U = L_2(\mathcal{O})$, and $V^* = H^{-\widehat{s}}(\mathcal{O})$ for some $\widehat{s} > 0$, i.e., $A : (0, T] \times V \to V^*$ and $f : (0, T] \times U \to U$. This way, we are in the setting of Section 5.1 with $\mathcal{H} = H^\nu(\mathcal{O})$ for some smoothness parameter $0 \leq \nu \leq \widehat{s}$ and $\mathcal{G} \supseteq H^{-\widehat{s}}(\mathcal{O})$. Recall that we assume 5.8 and 5.15, i.e., the initial value is given exactly and (5.65) has a unique solution. Furthermore, we assume that an exact scheme (5.6) is given which satisfies Assumption 5.10 on the global Lipschitz continuity of its operators, as well as Assumption 5.17, i.e., it exhibits convergence order δ.

We split our analysis into two parts. In the first part, we concentrate on the (rather theoretical) case, where the solutions of the stage equations are approximated by using best N-term wavelet approximation; and the complexity estimate is given in Theorem 5.71. Unfortunately, best N-term wavelet approximation is not implementable in our case, since the solutions to the subproblems are not known explicitly, so the N largest wavelet coefficients cannot be extracted directly. Therefore, in the second part, we turn our attention to the case where the stage equations are solved numerically by using an implementable wavelet solver which is asymptotically optimal. In Theorem 5.73 we show that the complexity estimate, derived in Theorem 5.71, immediately extends to this case.

Now to the first part. We consider the inexact scheme (5.13) and apply best N-term wavelet approximation in each stage as an approximation scheme in place of Assumption 5.12.

Remark 5.66. In the case that Ψ is an orthonormal wavelet basis, a best N-term wavelet approximation to a function v can be derived by thresholding, i.e., selecting N wavelet coefficients that are largest in absolute value in the wavelet decomposition of v. In the biorthogonal case, thresholding yields a best N-term wavelet approximation up to a constant, cf. Section 2.3.2. Therefore, in this sense best N-term wavelet approximation is an approximation scheme that fulfills Assumption 5.12.

The error of best N-term wavelet approximation in $H^\nu(\mathcal{O})$ is defined as

$$e_{N,\nu}^{\det}(v) := \inf \left\{ \Big\| v - \sum_{\mu \in \Lambda} c_\mu \psi_\mu \Big\|_{H^\nu(\mathcal{O})} : c_\mu \in \mathbb{R},\, \Lambda \subset \nabla,\, \#\Lambda = N \right\},$$

cf. Section 2.3.2. Furthermore, recall that for $v \in B^s_q(L_q(\mathcal{O}))$, where

$$\frac{1}{q} = \frac{s-\nu}{d} + \frac{1}{2}, \quad 0 \le \nu < s < s_1, \tag{5.66}$$

and under the assumptions (W1)–(W6), we have the estimate

$$e^{\det}_{N,H^\nu(\mathcal{O})}(v) \le C_{\mathrm{nlin}} \, \|v\|_{B^s_q(L_q(\mathcal{O}))} \, N^{-\frac{s-\nu}{d}}, \tag{5.67}$$

with a constant $C_{\mathrm{nlin}} > 0$, which does not depend on v or N, see Remark 2.26.

We apply Theorem 5.26 and derive an estimate for the number of degrees of freedom needed to compute a solution up to a tolerance $(C_{\mathrm{exact}} + T)\,\tau^\delta$.

Lemma 5.67. *Suppose that (W1)–(W6) and Assumptions 5.8, 5.10, and 5.15 hold. Let Assumption 5.17 hold for some $\delta > 0$ and let the inexact scheme* (5.13) *be based on best N-term wavelet approximation with the tolerances given by*

$$\varepsilon_{k,i} := \big(S\, C''_{\tau,k} C'_{\tau,k,(i)}\big)^{-1} \tau^{1+\delta} \tag{5.68}$$

with $C'_{\tau,k,(i)}$ as in (5.11) *and $C''_{\tau,k}$ as in* (5.22)*. Let the exact solutions $\widehat{w}_{k,i}$ of the stage equations in* (5.13)*, be given by* (5.26)*, and assume that all $\widehat{w}_{k,i}$ are contained in the same Besov space $B^s_q(L_q(\mathcal{O}))$ with* (5.66)*. Then we have* (5.24)*, i.e.,*

$$\big\|u(T) - \widetilde{u}_K\big\|_{H^\nu(\mathcal{O})} \le (C_{\mathrm{exact}} + T)\,\tau^\delta,$$

and the number of the degrees of freedom $M_{\tau,T}(\delta)$, given by (5.25)*, that are needed for the computation of $\{\widetilde{u}_k\}_{k=0}^K$ is bounded from above by*

$$M_{\tau,T}(\delta) \le \sum_{k=0}^{K-1} \sum_{i=1}^{S} \left\lceil C_{\mathrm{nlin}}^{\frac{d}{s-\nu}} \, \|\widehat{w}_{k,i}\|^{\frac{d}{s-\nu}}_{B^s_q(L_q(\mathcal{O}))} \Big(\big(S\, C''_{\tau,k} C'_{\tau,k,(i)}\big)^{-1} \tau^{1+\delta} \Big)^{-\frac{d}{s-\nu}} \right\rceil,$$

with C_{nlin} as in (5.67)*, and where $\lceil\cdot\rceil$ denotes the upper Gauss-bracket.*

Proof. We are in the setting of Theorem 5.26. By (5.67) we may, for each stage equation, choose $N \in \mathbb{N}_0$ as the smallest possible integer, such that

$$e^{\det}_{N,H^\nu(\mathcal{O})}(\widehat{w}_{k,i}) \le C_{\mathrm{nlin}} \|\widehat{w}_{k,i}\|_{B^s_q(L_q(\mathcal{O}))} \, N^{-\frac{s-\nu}{d}} \le \varepsilon_{k,i},$$

holds, that is

$$N = \left\lceil \Big(C_{\mathrm{nlin}} \|\widehat{w}_{k,i}\|_{B^s_q(L_q(\mathcal{O}))} \Big)^{\frac{d}{s-\nu}} \varepsilon_{k,i}^{-\frac{d}{s-\nu}} \right\rceil.$$

Using (5.68) and summing over k and i completes the proof. □

Lemma 5.67 shows that we need estimates for the Besov norms of the exact solutions $\widehat{w}_{k,i}$ of the stage equations in (5.13). We can provide an estimate in the following setting.

Lemma 5.68. *Suppose $L^{-1}_{\tau,i} \in \mathcal{L}(L_2(\mathcal{O}), B^s_q(L_q(\mathcal{O})))$ with* (5.66)*, $i = 1, \ldots, S$, and assume that the operators $R_{\tau,k,i} : L_2(\mathcal{O}) \times \cdots \times L_2(\mathcal{O}) \to L_2(\mathcal{O})$ are Lipschitz continuous with Lipschitz constants $C^{\mathrm{Lip},R}_{\tau,k,(i)}$ for all $k = 0, \ldots, K-1$, $i = 1, \ldots S$. With $C'_{\tau,j,(i)}$ as in* (5.11)*, we define*

$$C^{\mathrm{Bes}}_{k,i} := \Bigg(\prod_{l=1}^{i-1} \big(1 + \max\{C^{\mathrm{Lip}}_{\tau,k,(l)}, \|L^{-1}_{\tau,l} R_{\tau,k,l}(0,\ldots,0)\|_{L_2(\mathcal{O})}\}\big) \big(1 + \|u_k\|_{L_2(\mathcal{O})}\big)$$

$$
\begin{aligned}
&+\prod_{l=1}^{i-1}\big(1+C^{\mathrm{Lip}}_{\tau,k,(l)}\big)\sum_{j=0}^{k-1}\bigg(\prod_{n=j+1}^{k-1}\big(C'_{\tau,n,(0)}-1\big)\bigg)\sum_{r=1}^{S}C'_{\tau,j,(r)}\varepsilon_{j,r}\\
&+\sum_{j=1}^{i-1}\varepsilon_{k,j}\prod_{l=j+1}^{i-1}\big(1+C^{\mathrm{Lip}}_{\tau,k,(l)}\big)\bigg).
\end{aligned}
\tag{5.69}
$$

Then all $\widehat{w}_{k,i}$, as defined in (5.26)*, are contained in the same Besov space $B^s_q(L_q(\mathcal{O}))$ with* (5.66)*, and their norms can be estimated by*

$$
\begin{aligned}
\|\widehat{w}_{k,i}\|_{B^s_q(L_q(\mathcal{O}))} \le{}& \|L^{-1}_{\tau,i}\|_{\mathcal{L}(L_2(\mathcal{O}),\,B^s_q(L_q(\mathcal{O})))}\\
&\times \max\Big\{C^{\mathrm{Lip},R}_{\tau,k,(i)},\|R_{\tau,k,i}(0,\ldots,0)\|_{L_2(\mathcal{O})}\Big\}\,C^{\mathrm{Bes}}_{k,i}.
\end{aligned}
\tag{5.70}
$$

Proof. The proof is similar to the one of Theorem 5.24. It is given in Appendix B.7. □

Remark 5.69. In Lemma 5.68, the assumption $L^{-1}_{\tau,i}\in\mathcal{L}(L_2(\mathcal{O}),\,B^s_q(L_q(\mathcal{O})))$ with (5.66), and the Lipschitz continuity of $R_{\tau,k,i}$ imply Assumption 5.10 with $\mathcal{H}=H^\nu(\mathcal{O})$. However, this Lipschitz constant may not be optimal.

Remark 5.70. If $R_{\tau,k,i}$ is bounded, then we can prove a similar result as in Lemma 5.68.

The combination of Lemma 5.67 and 5.68 yields the main result of the first part, i.e., the complexity estimate for the case that best N-term wavelet approximations are used for the solution of the stage equations.

Theorem 5.71. *Let the assumptions of the Lemmas 5.67 and 5.68 be satisfied. With $C'_{\tau,k,(i)}$ as in* (5.11) *and $C''_{\tau,k}$ as in* (5.22)*, we have*

$$
\begin{aligned}
&M_{\tau,T}(\delta)\\
&\le\sum_{k=0}^{K-1}\sum_{i=1}^{S}\bigg\lceil C_{\mathrm{nlin}}^{\frac{d}{s-\nu}}\Big(\max\big\{C^{\mathrm{Lip},R}_{\tau,k,(i)},\|R_{\tau,k,i}(0,\ldots,0)\|_{L_2(\mathcal{O})}\big\}\,C^{\mathrm{Bes}}_{k,i}\Big)^{\frac{d}{s-\nu}}\\
&\quad\times\big(\|L^{-1}_{\tau,i}\|_{\mathcal{L}(L_2(\mathcal{O}),\,B^s_q(L_q(\mathcal{O})))}\big)^{\frac{d}{s-\nu}}\Big(\big(S\,C''_{\tau,k}C'_{\tau,k,(i)}\big)^{-1}\tau^{1+\delta}\Big)^{-\frac{d}{s-\nu}}\bigg\rceil.
\end{aligned}
\tag{5.71}
$$

As outlined above, the next step is to discuss the complexity of Rothe's method in the case that *implementable* numerical wavelet schemes instead of the best N-term wavelet approximation are employed for the stage equations. We make the following assumptions, cf. Assumption 5.12.

Assumption 5.72. (i) There exists an implementable asymptotically optimal numerical wavelet scheme for the stage equations arising in (5.13). That is, if the best N-term wavelet approximation in $H^\nu(\mathcal{O})$ converges with rate

$$
N^{-\frac{s-\nu}{d}},\quad\text{for some}\quad s>\nu>0,
$$

then the scheme computes finite index sets $\Lambda_l\subset\nabla$ and coefficients $(c_\mu)_{\mu\in\Lambda_l}$ with

$$
\Big\|L^{-1}_{\tau,i}v-\sum_{\mu\in\Lambda_l}c_\mu\psi_\mu\Big\|_{H^\nu(\mathcal{O})}\le C^{\mathrm{asym}}_{\tau,i,s,\nu}(L^{-1}_{\tau,i}v)\,(\#\Lambda_l)^{-\frac{s-\nu}{d}}
\tag{5.72}
$$

for some constant $C^{\mathrm{asym}}_{\tau,i,s,\nu}(L^{-1}_{\tau,i}v)$. Further, for all $\varepsilon > 0$ there exists an $l(\varepsilon)$ such that

$$\Big\| L^{-1}_{\tau,i}v - \sum_{\mu\in\Lambda_l} c_\mu \psi_\mu \Big\|_{H^\nu(\mathcal{O})} \leq \varepsilon, \quad l \geq l(\varepsilon),$$

and such that

$$\#\Lambda_{l(\varepsilon)} \leq C^{\mathrm{asym}}_{\tau,i,s,\nu}(L^{-1}_{\tau,i}v)\, \varepsilon^{-\frac{d}{s-\nu}}.$$

(ii) The operators $R_{\tau,k,i}$ can be evaluated exactly.

In Section 5.3.3 below, we discuss a prototype of an adaptive wavelet method, fulfilling Assumption 5.72(i), which has been derived in Cohen et al. [29]. It satisfies an optimality estimate of the form (5.72) for the energy norm (2.31), Section 2.4.1. However, since the energy norm is equivalent to some Sobolev norm $\|\cdot\|_{H^\nu(\mathcal{O})}$, cf. (2.32), the estimate (5.72) also holds for this case. Moreover, it has been shown in Cohen et al. [29] that the constant is of a specific form, which is similar to (5.67). Therefore, we specify Assumption 5.72(i) in the following way.

Assumption 5.72. **(iii)** The constant $C^{\mathrm{asym}}_{\tau,i,s,\nu}(L^{-1}_{\tau,i}v)$ in (5.72) is of the form

$$C^{\mathrm{asym}}_{\tau,i,s,\nu}(L^{-1}_{\tau,i}v) = \widehat{C}^{\mathrm{asym}}_{\tau,i} \|L^{-1}_{\tau,i}v\|_{B^s_q(L_q(\mathcal{O}))}, \quad \frac{1}{q} = \frac{s-\nu}{d} + \frac{1}{2},$$

with a constant $\widehat{C}^{\mathrm{asym}}_{\tau,i}$ independent of $L^{-1}_{\tau,i}v$.

In this setting we are immediately able to state our main result.

Theorem 5.73. *Let the assumptions of the Lemmas 5.67 and 5.68 be satisfied. If an optimal numerical wavelet scheme, that satisfies Assumption 5.72, is used for the numerical solution of the stage equations, then the necessary number of degrees of freedom can be estimated as in Theorem 5.71 with* $\widehat{C}^{\mathrm{asym}}_{\tau,i}$ *instead of* C_{nlin}*, i.e.,*

$$\begin{aligned} &M_{\tau,T}(\delta) \\ &\leq \sum_{k=0}^{K-1}\sum_{i=1}^{S} \Big[\big(\widehat{C}^{\mathrm{asym}}_{\tau,i}\big)^{\frac{d}{s-\nu}} \Big(\max\big\{C^{\mathrm{Lip},R}_{\tau,k,(i)}, \|R_{\tau,k,i}(0,\ldots,0)\|_{L_2(\mathcal{O})}\big\}\, C^{\mathrm{Bes}}_{k,i}\Big)^{\frac{d}{s-\nu}} \\ &\quad \times \big(\|L^{-1}_{\tau,i}\|_{\mathcal{L}(L_2(\mathcal{O}),\, B^s_q(L_q(\mathcal{O})))}\big)^{\frac{d}{s-\nu}} \Big(\big(S\, C''_{\tau,k} C'_{\tau,k,(i)}\big)^{-1} \tau^{1+\delta}\Big)^{-\frac{d}{s-\nu}} \Big]. \end{aligned} \tag{5.73}$$

Remark 5.74. The constant $\widehat{C}^{\mathrm{asym}}_{\tau,i}$ depends on the concrete design of the adaptive method at hand. As an example this constant may depend on the design of the routines **APPLY**, **RHS**, and **COARSE**, which are further discussed in Section 5.3.3 below. Moreover, the value of $\widehat{C}^{\mathrm{asym}}_{\tau,i}$ depends on the equivalence constants of the energy norm and the Sobolev norm in (2.32). Therefore this constant may grow as τ gets small. However, this is an intrinsic problem and not caused by our approach.

At this point the question remains if and how the Besov norms of the exact solutions of the stage equations $\widehat{w}_{k,i}$, cf. (5.70) can be specified, and moreover how all the constants involved in (5.71) and (5.73) can be estimated. Therefore, in the next section we present a detailed study for the most important model problem, that is the linearly-implicit Euler scheme applied to the heat equation.

5.3.2 Complexity estimates for the heat equation

We conclude the discussion of Example 5.30, i.e., we substantiate the analysis further and discuss regularity estimates for the heat equation. It turns out, that in this case concrete Besov regularity estimates and an explicit estimate of the overall complexity can be derived. Recall the heat equation

$$\begin{aligned} u'(t) &= \Delta u(t) + f(t,u(t)) && \text{on } \mathcal{O},\ t \in (0,T], \\ u(0) &= u_0 && \text{on } \mathcal{O}, \\ u &= 0 && \text{on } \partial\mathcal{O},\ t \in (0,T], \end{aligned}$$

on a bounded Lipschitz domain $\mathcal{O} \subset \mathbb{R}^d$, and consider the case $\mathcal{H} = \mathcal{G} = U = L_2(\mathcal{O})$. The operators $L_{\tau,1}^{-1}$ and $R_{\tau,k,1}$ are given by

$$L_{\tau,1}^{-1} = (I - \tau\Delta_{\mathcal{O}}^D)^{-1} \quad \text{and} \quad R_{\tau,k,1} = I + \tau f(t_k, \cdot). \tag{5.74}$$

The first step is to estimate the Besov regularity of the solutions to the stage equations. To this end, the mapping properties of $L_{\tau,1}^{-1}$ with respect to the adaptivity scale of Besov spaces (5.66) have to be analyzed. Recall that for special cases bounds for the Lipschitz constant of $R_{\tau,k,1} : L_2(\mathcal{O}) \to L_2(\mathcal{O})$ have already been proven in Section 5.1, i.e.,

$$C_{\tau,k,(1)}^{\mathrm{Lip},R} \le 1 + \tau C^{\mathrm{Lip},f} \quad \text{and} \quad C_{\tau,k,(1)}^{\mathrm{Lip},R} \le \sup_{x\in\mathbb{R}} \left| 1 + \tau \frac{\mathrm{d}}{\mathrm{d}x} \bar{f}(x) \right|$$

are shown in Example 5.30 and Example 5.31, respectively. We set

$$C_{\mathrm{Bes},\varepsilon}^{\mathrm{Lap}} := \left\| (\Delta_{\mathcal{O}}^D)^{-1} \right\|_{\mathcal{L}(L_2(\mathcal{O}),\, B_1^{2-\varepsilon}(L_1(\mathcal{O})))} \tag{5.75}$$

and

$$C_{\mathrm{Sob}}^{\mathrm{Lap}} := \left\| (\Delta_{\mathcal{O}}^D)^{-1} \right\|_{\mathcal{L}(L_2(\mathcal{O}),\, H^{3/2}(\mathcal{O}))},$$

where $(\Delta_{\mathcal{O}}^D)^{-1} \in \mathcal{L}(L_2(\mathcal{O}),\, B_1^{2-\varepsilon}(L_1(\mathcal{O})))$ has been shown in Dahlke, DeVore [47], see also Dahlke, Sickel [56, Corollary 1] for details. The fundamental result

$$(\Delta_{\mathcal{O}}^D)^{-1} \in \mathcal{L}(L_2(\mathcal{O}),\, H^{3/2}(\mathcal{O}))$$

has been shown in Jerison, Kenig [104, Theorem B].

Lemma 5.75. *Let $\varepsilon > 0$. Then the operator $(I - \tau\Delta_{\mathcal{O}}^D)^{-1}$ is contained in $\mathcal{L}(L_2(\mathcal{O}),\, B_1^{2-\varepsilon}(L_1(\mathcal{O})))$ and in $\mathcal{L}(L_2(\mathcal{O}),\, H^{3/2}(\mathcal{O}))$. The respective operator norms can be estimated by*

$$\left\| (I - \tau\Delta_{\mathcal{O}}^D)^{-1} \right\|_{\mathcal{L}(L_2(\mathcal{O}),\, B_1^{2-\varepsilon}(L_1(\mathcal{O})))} \le \frac{1}{\tau} C_{\mathrm{Bes},\varepsilon}^{\mathrm{Lap}} \tag{5.76}$$

and

$$\left\| (I - \tau\Delta_{\mathcal{O}}^D)^{-1} \right\|_{\mathcal{L}(L_2(\mathcal{O}),\, H^{3/2}(\mathcal{O}))} \le \frac{1}{\tau} C_{\mathrm{Sob}}^{\mathrm{Lap}}, \tag{5.77}$$

respectively.

Proof. We start by proving (5.76). The observation

$$(I-\tau\Delta_{\mathcal{O}}^{D})^{-1}=(-\tau\Delta_{\mathcal{O}}^{D})^{-1}\big(I-(I-\tau\Delta_{\mathcal{O}}^{D})^{-1}\big)$$

leads to

$$\big\|(I-\tau\Delta_{\mathcal{O}}^{D})^{-1}\big\|_{\mathcal{L}(L_2(\mathcal{O}),\,B_1^{2-\varepsilon}(L_1(\mathcal{O})))}\le\tau^{-1}\,C_{\mathrm{Bes},\varepsilon}^{\mathrm{Lap}}\,\big\|I-(I-\tau\Delta_{\mathcal{O}}^{D})^{-1}\big\|_{\mathcal{L}(L_2(\mathcal{O}))};$$

and the last term can be bounded from above by

$$\begin{aligned}\big\|I-(I-\tau\Delta_{\mathcal{O}}^{D})^{-1}\big\|_{\mathcal{L}(L_2(\mathcal{O}))}^2&=\sup_{\|v\|_{L_2(\mathcal{O})}=1}\sum_{k\in\mathbb{N}}\big|(1-(1-\tau\lambda_k)^{-1})\langle v,e_k\rangle_{L_2(\mathcal{O})}\big|^2\\&=\sup_{\|v\|_{L_2(\mathcal{O})}=1}\sum_{k\in\mathbb{N}}\left|\frac{-\tau\lambda_k}{1-\tau\lambda_k}\langle v,e_k\rangle_{L_2(\mathcal{O})}\right|^2\\&\le\sup_{\|v\|_{L_2(\mathcal{O})}=1}\sum_{k\in\mathbb{N}}\left|\langle v,e_k\rangle_{L_2(\mathcal{O})}\right|^2\\&=1.\end{aligned}$$

The estimate (5.77) follows in a similar fashion. □

With Lemma 5.75 at hand, we are now ready to prove the desired mapping properties for $L_{\tau,1}^{-1}:L_2(\mathcal{O})\to B_q^s(L_q(\mathcal{O}))$, where the scale (5.66) holds. We set

$$C_{\mathrm{inter}}^{\mathrm{Lap}}(\theta):=(C_{\mathrm{Sob}}^{\mathrm{Lap}})^{1-\theta}\,(C_{\mathrm{Bes},\varepsilon}^{\mathrm{Lap}})^{\theta},\quad\theta\in(0,1).\tag{5.78}$$

Lemma 5.76. *Let $\varepsilon>0$, $d\ge2$, $\nu\ge0$.* ***(i)*** *For $(2-\varepsilon)-\frac{d}{2}<\nu<\frac{3}{2}$, that is*

$$\theta:=\frac{3-2\nu}{d-1+2\varepsilon}\in(0,1),\tag{5.79}$$

we have

$$(I-\tau\Delta_{\mathcal{O}}^{D})^{-1}\in\mathcal{L}(L_2(\mathcal{O}),\,B_q^s(L_q(\mathcal{O})))\quad\textit{with}\quad s=\frac{3d-2\nu+4\varepsilon\nu}{2d-2+4\varepsilon}$$

and $1/q=(s-\nu)/d+1/2$. Its norm can be bounded in the following way

$$\big\|(I-\tau\Delta_{\mathcal{O}}^{D})^{-1}\big\|_{\mathcal{L}(L_2(\mathcal{O}),\,B_q^s(L_q(\mathcal{O})))}\le\frac{1}{\tau}C_{\mathrm{inter}}^{\mathrm{Lap}}(\theta).\tag{5.80}$$

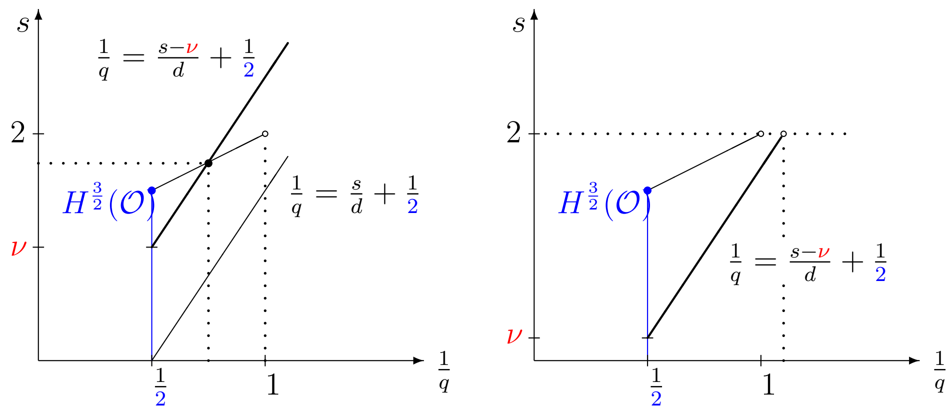
(ii) *For $0\le\nu\le(2-\varepsilon)-\frac{d}{2}$, we have*

$$(I-\tau\Delta_{\mathcal{O}}^{D})^{-1}\in\mathcal{L}(L_2(\mathcal{O}),\,B_q^{2-\varepsilon}(L_q(\mathcal{O})))\quad\textit{with}\quad q=\frac{2d}{4-2\varepsilon-2\nu+d}.$$

and its norm can be bounded by $\tau^{-1}\,C_{\mathrm{Bes},\varepsilon}^{\mathrm{Lap}}$ as in (5.76).

Proof. **(i)** The proof is based on interpolation properties of Besov spaces. See Appendix A.1.4 for the definition and Bergh, Löfström [9] for details on interpolation properties of Besov spaces. For real interpolation it holds that

$$\big(B_{p_0}^{s_0}(L_{p_0}(\mathcal{O})),B_{p_1}^{s_1}(L_{p_1}(\mathcal{O}))\big)_{\bar\theta,p}=B_p^s(L_p(\mathcal{O}))$$



(a) $\nu \geq 0$, $(2-\varepsilon) - \frac{d}{2} < \nu < \frac{3}{2}$ (b) $0 \leq \nu \leq (2-\varepsilon) - \frac{d}{2}$

Figure 5.1: DeVore-Triebel diagrams: Illustrating Lemma 5.76, $d = 3$

in the sense of equivalent (quasi-)norms, provided that the parameters satisfy

$$0 < \bar{\theta} < 1, \quad s = (1-\bar{\theta})s_0 + \bar{\theta}s_1, \quad \frac{1}{p} = \frac{1-\bar{\theta}}{p_0} + \frac{\bar{\theta}}{p_1} \tag{5.81}$$

and $s_0, s_1 \in \mathbb{R}$, $0 < p_0, p_1 < \infty$. Furthermore, if (5.81) holds, a linear operator $\mathcal{T}$ that is contained in $\mathcal{L}(L_2(\mathcal{O}), B^{s_0}_{p_0}(L_{p_0}(\mathcal{O})))$ and $\mathcal{L}(L_2(\mathcal{O}), B^{s_1}_{p_1}(L_{p_1}(\mathcal{O})))$ is also an element of $\mathcal{L}(L_2(\mathcal{O}), B^s_p(L_p(\mathcal{O})))$. Its norm can be estimated by

$$\|\mathcal{T}\|_{\mathcal{L}(L_2(\mathcal{O}),\, B^s_p(L_p(\mathcal{O})))} \leq \|\mathcal{T}\|^{1-\bar{\theta}}_{\mathcal{L}(L_2(\mathcal{O}),\, B^{s_0}_{p_0}(L_{p_0}(\mathcal{O})))} \|\mathcal{T}\|^{\bar{\theta}}_{\mathcal{L}(L_2(\mathcal{O}),\, B^{s_1}_{p_1}(L_{p_1}(\mathcal{O})))}.$$

Observe that $H^{3/2}(\mathcal{O}) = B^{3/2}_2(L_2(\mathcal{O}))$ and that we can apply Lemma 5.75. We need to determine the value for $\bar{\theta}$, such that the resulting interpolation space lies on the nonlinear approximation line $1/p = (s-\nu)/d + 1/2$. This is the case for

$$\bar{\theta} = \frac{3-2\nu}{d-1+2\varepsilon},$$

cf. Figure 5.1(a).

(ii) See the proof of (5.76) in Lemma 5.75 together with the continuous embedding of $B^{2-\varepsilon}_1(L_1(\mathcal{O})) \hookrightarrow B^{2-\varepsilon}_q(L_q(\mathcal{O}))$. In Figure 5.1(b) the upper bound is given by the intersection of the lines $s = (2-\varepsilon)$ and $1/p = (s-\nu)/d + 1/2$. □

Remark 5.77. Our findings for the discretization of the heat equation by means of the linearly-implicit Euler scheme carry over to discretizations with $S > 1$ stages. For the case $S = 2$ the operators $L^{-1}_{\tau,i}$, $R_{\tau,k,i}$, $i = 1, 2$, are provided by Observation 5.41 and are similar to (5.74), e.g.,

$$L^{-1}_{\tau,i} = (I - \tau\gamma_{i,i}\Delta^D_{\mathcal{O}})^{-1}, \quad i = 1, 2.$$

Lemma 5.76 can be reformulated with $\tau\gamma_{i,i}$ replacing τ, and the Lipschitz continuity of $R_{\tau,k,i}$ can be established directly as before.

We are now able to give specific bounds for the number of degrees of freedom needed to compute the solution of the heat equation by means of the linearly-implicit Euler scheme. Again, we split our analysis into two parts. First, we apply Theorem 5.26 to the case when best N-term wavelet approximation (with respect to the $H^\nu(\mathcal{O})$ norm, $\nu \geq 0$) is used in each step of the inexact scheme (5.13).

Theorem 5.78. *Let the assumptions of the Lemmas 5.67, 5.68 and 5.76 hold. Let τ be small enough such that*

$$(1+\tau C^{\mathrm{Lip},f})^{-1}\tau\big\|f(0)\big\|_{L_2(\mathcal{O})}\le 1.$$

We set $C_{\mathrm{sup,u}} := \sup_{t\in[0,T]}\|u(t)\|_{L_2(\mathcal{O})}$ and

$$C_{\mathrm{short}}(\tau):=\begin{cases} C_{\mathrm{nlin}}\,C^{\mathrm{Lap}}_{\mathrm{Bes},\varepsilon}\,(1+\tau C^{\mathrm{Lip},f}) & :\ 0\le\nu\le(2-\varepsilon)-\frac{d}{2},\\ C_{\mathrm{nlin}}\,C^{\mathrm{Lap}}_{\mathrm{inter}}(\theta)\,(1+\tau C^{\mathrm{Lip},f}) & :\ (2-\varepsilon)-\frac{d}{2}<\nu<\frac{3}{2},\ \nu>0,\end{cases}$$

where C_{nlin}, $C^{\mathrm{Lap}}_{\mathrm{Bes},\varepsilon}$, $C^{\mathrm{Lap}}_{\mathrm{inter}}$, and θ are given by (5.67), (5.75), (5.78), *and* (5.79), *respectively. Let C_{exact} be given as in Assumption 5.17. In the setting of Example 5.30, if best N-term wavelet approximation for the spatial approximation of the stage equations is applied, then the number of degrees of freedom $M_{\tau,T}$ needed to compute a solution up to a tolerance $(C_{\mathrm{exact}}+T)\,\tau$ can be estimated by*

$$M_{\tau,T}\le T\tau^{-1}+\frac{1}{2}\Big(2C_{\mathrm{short}}(\tau)\Big)^{\frac{2}{\theta}}\Big(T^{\frac{2}{\theta}+1}\tau^{-(\frac{2}{\theta}+1)}+C_{\mathrm{lim}}(\tau)\tau^{-(\frac{6}{\theta}+1)}\Big),$$

with

$$C_{\mathrm{lim}}(\tau):=\big(1+C_{\mathrm{sup,u}}+C_{\mathrm{exact}}\tau\big)^{\frac{2}{\theta}}\tau\frac{(1+\tau C^{\mathrm{Lip},f})^{\frac{2}{\theta}T\tau^{-1}}-1}{1-(1+\tau C^{\mathrm{Lip},f})^{-\frac{2}{\theta}}}.$$

Furthermore,

$$\lim_{\tau\to 0}C_{\mathrm{lim}}(\tau)=\frac{\theta}{2}\big(1+C_{\mathrm{sup,u}}\big)^{\frac{2}{\theta}}(C^{\mathrm{Lip},f})^{-1}\big(\exp\big(C^{\mathrm{Lip},f}\frac{2}{\theta}T\big)-1\big).$$

Proof. We apply Theorem 5.71 with $S=1$ and $\delta=1$. In the setting of Example 5.30 it holds that

$$C^{\mathrm{Lip},R}_{\tau,k,(1)}=1+\tau C^{\mathrm{Lip},f},\quad C'_{\tau,k,(1)}=1,\quad C'_{\tau,k,(0)}=2+\tau C^{\mathrm{Lip},f},$$

independently of k. Thus (5.22) reads as $C''_{\tau,k}=(1+\tau C^{\mathrm{Lip},f})^{K-k-1}$ and (5.69) can be simplified to

$$C^{\mathrm{Bes}}_{k,1}=1+\|u_k\|_{L_2(\mathcal{O})}+k(C^{\mathrm{Lip},R}_{\tau,k,(1)})^{k-K}\tau^2.$$

The norm of $\|u_k\|_{L_2(\mathcal{O})}$ can be bounded as follows. By Assumption 5.17 we have

$$\|u(t_k)-u_k\|_{L_2(\mathcal{O})}\le C_{\mathrm{exact}}\tau$$

and as a consequence

$$\|u_k\|_{L_2(\mathcal{O})}\le\big\|u(t_k)-u_k\big\|_{L_2(\mathcal{O})}+\|u(t_k)\|_{L_2(\mathcal{O})}\le C_{\mathrm{exact}}\tau+C_{\mathrm{sup,u}},$$

where $C_{\mathrm{sup,u}}$ is finite since $[0,T]$ is compact and u is continuous. Using the bound (5.80) of Lemma 5.76(i) in the estimate (5.71) we obtain

$$\begin{aligned} M_{\tau,T}&\le\sum_{k=0}^{K-1}\left\lceil\Big(C_{\mathrm{short}}\big((C^{\mathrm{Lip},R}_{\tau,k,(1)})^{K-k}\tau^{-3}(1+C_{\mathrm{sup,u}}+C_{\mathrm{exact}}\tau)+k\big)\Big)^{\frac{2}{\theta}}\right\rceil\\ &\le K+\sum_{k=0}^{K-1}\Big(C_{\mathrm{short}}\big((C^{\mathrm{Lip},R}_{\tau,k,(1)})^{K-k}\tau^{-3}(1+C_{\mathrm{sup,u}}+C_{\mathrm{exact}}\tau)+k\big)\Big)^{\frac{2}{\theta}}.\end{aligned}$$

An application of Jensen's inequality and the geometric series formula yield

$$
\begin{aligned}
M_{\tau,T} &\leq K + C_{\text{short}}^{\frac{2}{\theta}} 2^{\frac{2}{\theta}-1} \\
&\quad \times \sum_{k=0}^{K-1} \left(\left((C_{\tau,k,(1)}^{\text{Lip},R})^{K-k} \tau^{-3} (1 + C_{\text{sup,u}} + C_{\text{exact}}\tau) \right)^{\frac{2}{\theta}} + k^{\frac{2}{\theta}} \right) \\
&\leq K \left(1 + \frac{1}{2} (2 C_{\text{short}} K)^{\frac{2}{\theta}} \right) \\
&\quad + \tau^{-\frac{6}{\theta}} \frac{1}{2} \Big(2 C_{\text{short}} (1 + C_{\text{sup,u}} + C_{\text{exact}}\tau) \Big)^{\frac{2}{\theta}} \frac{(1 + \tau C^{\text{Lip},f})^{\frac{2}{\theta}K} - 1}{1 - (1 + \tau C^{\text{Lip},f})^{-\frac{2}{\theta}}}.
\end{aligned}
$$

The proof is finalized by the insertion of $K = T\tau^{-1}$ and the observations

$$
\begin{aligned}
&\lim_{\tau \to 0} (1 + \tau C^{\text{Lip},f})^{\frac{2}{\theta} T \tau^{-1}} - 1 = \exp\big(C^{\text{Lip},f} \frac{2}{\theta} T \big) - 1, \\
&\lim_{\tau \to 0} \frac{\tau}{1 - (1 + \tau C^{\text{Lip},f})^{-\frac{2}{\theta}}} = \frac{1}{\frac{2}{\theta} C^{\text{Lip},f}}.
\end{aligned}
$$

The case, where Lemma 5.76(ii) is applied to (5.71), is analogous. □

Now, we turn to the case when an optimal numerical wavelet scheme is used for the numerical solution of the stage equations in (5.13). The wavelet schemes we have in mind, cf. Section 5.3.3, are optimal with respect to the energy norm (2.31). In our setting it is induced by L_τ and equivalent to the Sobolev norm $H^1(\mathcal{O})$. For this reason, we now state the estimate for the number of degrees of freedom in the case of the Sobolev norm $H^1(\mathcal{O})$, i.e., $\nu = 1$.

Theorem 5.79. *Let the assumptions of Theorem 5.78 hold, whereas we now employ an implementable asymptotically optimal numerical scheme, such that Assumption 5.72 holds for $\nu = 1$. Using $\widehat{C}_{\text{short}}(\tau) := \widehat{C}_{\tau,1}^{\text{asym}} C_{\text{inter}}^{\text{Lap}}(\theta)\,(1 + \tau C^{\text{Lip},f})$, the number of degrees of freedom needed to compute a solution up to a tolerance $(C_{\text{exact}} + T)\,\tau$ can be estimated by*

$$
M_{\tau,T} \leq T\tau^{-1} + \frac{1}{2} \left(2 \widehat{C}_{\text{short}}(\tau) \right)^{\frac{2}{\theta}} \left(T^{\frac{2}{\theta}+1} \tau^{-(\frac{2}{\theta}+1)} + \widehat{C}_{\text{lim}}(\tau) \tau^{-(\frac{6}{\theta}+1)} \right), \tag{5.82}
$$

with

$$
\widehat{C}_{\text{lim}}(\tau) := \big(1 + C_{\text{sup,u}} + C_{\text{exact}}\tau \big)^{\frac{2}{\theta}} \tau \frac{(1 + \tau C^{\text{Lip},f})^{\frac{2}{\theta} T \tau^{-1}} - 1}{1 - (1 + \tau C^{\text{Lip},f})^{-\frac{2}{\theta}}}
$$

and

$$
\theta := \frac{1}{d - 1 + 2\varepsilon}. \tag{5.83}
$$

Furthermore,

$$
\lim_{\tau \to 0} \widehat{C}_{\text{lim}}(\tau) = \frac{\theta}{2} \big(1 + C_{\text{sup,u}} \big)^{\frac{2}{\theta}} (C^{\text{Lip},f})^{-1} \left(\exp\left(C^{\text{Lip},f} \frac{2}{\theta} T \right) - 1 \right).
$$

Remark 5.80. The calculations above shows that, among other things, the overall complexity of the resulting scheme heavily depends on the Besov smoothness of the exact solutions to the stage equations. Due to the Lipschitz character of the domain

$\mathcal{O}$, and since we are working in the L_2-setting, this Besov regularity is limited by $s = 2$. However, for more specific domains, e.g., polygonal domains in $\mathbb{R}^2$ and smoother right-hand sides, much higher Besov smoothness can be achieved, see, e.g., Dahlke [42], Dahlke et al. [52] for details.

Remark 5.81. Let us further discuss the asymptotic behavior of $M_{\tau,T}$ as τ tends to zero. For simplicity, let us consider the case $d = 2$, then we can choose θ arbitrary close to 1. Asymptotically optimal schemes are usually described in the energy norm induced by the operator $L_{\tau,1}$, with a constant analogous to (5.72) that is independent of $L_{\tau,1}$, see, e.g., Cohen et al. [29]. With the notation as (2.32), the following consideration for the energy norm induced by $L_{\tau,1}$

$$\big\langle (I + \tau\Delta_{\mathcal{O}}^{D})u, u\big\rangle_{L_2(\mathcal{O})} \geq \langle u, u\rangle_{L_2(\mathcal{O})} + \tau\, c_{\text{energy}}^2(\Delta_{\mathcal{O}}^{D})\, \|u\|_{H^1(\mathcal{O})}^2,$$

implies $c_{\text{energy}}(I + \tau\Delta_{\mathcal{O}}^{D}) \geq \tau^{\frac{1}{2}} c_{\text{energy}}(\Delta_{\mathcal{O}}^{D})$, so that we can conclude

$$\widehat{C}_{\tau,1}^{\,\text{asym}} = \widehat{C}_1\, \tau^{-\frac{1}{2}}$$

with some constant $\widehat{C}_1$ independent of τ. In this case (5.82) reads as

$$M_{\tau,T} \leq T\tau^{-1} + \frac{1}{2}\left(2\widehat{C}_1 C_{\text{inter}}^{\text{Lap}}(1 + \tau C^{\text{Lip},f})\right)^2 \left(T^3\tau^{-4} + \widehat{C}_{\text{lim}}(\tau)\tau^{-8+\varepsilon'}\right),$$

i.e., for small τ the last term is dominating and therefore the number of degrees of freedom behaves as $\tau^{-8+\varepsilon'}$.

5.3.3 Adaptive wavelet methods for elliptic problems

To complement our analysis, we summarize some basic ideas on how wavelets can be used for the adaptive numerical treatment of elliptic operator equations. We consider equations of the form

$$Au = f, \tag{5.84}$$

where we assume A to be a boundedly invertible operator from some Hilbert space V into its normed dual V^* in a Gel'fand triple setting (V, U, V^*), cf. Section 2.4.1. In our approach of the Rothe method, the operator A is one of the operators $L_{\tau,i}$ that arise in the treatment of the elliptic stage equations. Therefore, in applications V is usually one of the Sobolev spaces $H^\nu(\mathcal{O})$ or $H_0^\nu(\mathcal{O})$.

Operator equations of the form (5.84) can be solved by a *Galerkin scheme*. One defines an increasing sequence of finite dimensional approximation spaces

$$S_{\Lambda_l} := \operatorname{span}\{\eta_\mu \,:\, \mu \in \Lambda_l\},$$

where $S_{\Lambda_l} \subset S_{\Lambda_{l+1}}$, and projects the problem onto these spaces, i.e.,

$$\langle Au_{\Lambda_l}, v\rangle_{V^*\times V} = \langle f, v\rangle_{V^*\times V} \quad \text{for all} \quad v \in S_{\Lambda_l}.$$

To compute the current Galerkin approximation, one has to solve a linear system

$$\mathbf{G}_{\Lambda_l}\mathbf{c}_{\Lambda_l} = \mathbf{f}_{\Lambda_l},$$

with $\mathbf{G}_{\Lambda_l} := (\langle A\eta_{\mu'}, \eta_\mu\rangle_{V^*\times V})_{\mu,\mu'\in\Lambda_l}$ and $(\mathbf{f}_\Lambda)_\mu := \langle f, \eta_\mu\rangle_{V^*\times V}, \mu \in \Lambda_l$.

Choosing the approximation spaces in an arbitrary way might result in a very inefficient scheme. A natural idea is to use an *adaptive* scheme, i.e., an updating strategy which essentially consists of the steps

solve	–	estimate	–	refine
$\mathbf{G}_{\Lambda_l}\mathbf{c}_{\Lambda_l} = \mathbf{f}_{\Lambda_l}$		$\|u - u_{\Lambda_l}\| = ?$ a posteriori error estimator		add functions if necessary.

The second step is highly nontrivial since the exact solution u is unknown, so that clever *a posteriori* error estimators are needed. An equally challenging task is to show that the refinement strategy leads to a convergent scheme and to estimate its order of convergence, if possible. It has been shown, see, e.g., Cohen et al. [29, 30], Dahlke et al. [46], that both tasks can be solved if wavelets are used as basis functions for the Galerkin scheme:

First, (5.84) is transformed into a discrete problem, cf. Section 2.4.1. From the norm equivalences (W6) it is easy to see that (5.84) is equivalent to

$$\mathbf{A}\mathbf{u} = \mathbf{f},$$

where $\mathbf{A} := \mathbf{D}^{-1}\langle A\Psi, \Psi\rangle^\top_{V^*\times V}\mathbf{D}^{-1}$, $\mathbf{u} := \mathbf{D}\mathbf{c}$, $\mathbf{f} := \mathbf{D}^{-1}\langle f, \Psi\rangle^\top_{V^*\times V}$, and $\mathbf{D} := (2^{-s|\mu|}\,\delta_{\mu,\mu'})_{\mu,\mu'\in\nabla}$. Computing a Galerkin approximation amounts to solving the system

$$\mathbf{A}_\Lambda\mathbf{u}_\Lambda = \mathbf{f}_\Lambda := \mathbf{f}|_\Lambda, \qquad \mathbf{A}_\Lambda := (2^{-s(|\mu|+|\nu|)}\langle\psi_\mu, A\psi_\nu\rangle_{V^*\times V})_{\mu,\nu\in\Lambda}.$$

Now, ellipticity and the norm equivalences (W6) yield

$$\begin{aligned}\|\mathbf{u} - \mathbf{u}_\Lambda\|_{\ell_2(\nabla)} &\preceq \|\mathbf{A}(\mathbf{u} - \mathbf{u}_\Lambda)\|_{\ell_2(\nabla)}\\ &\preceq \|\mathbf{f} - \mathbf{A}(\mathbf{u}_\Lambda)\|_{\ell_2(\nabla)}\\ &= \|\mathbf{r}_\Lambda\|_{\ell_2(\nabla)},\end{aligned}$$

so that the $\ell_2(\nabla)$-norm of the *residual* $\mathbf{r}_\Lambda$ serves as an *a posteriori* error estimator. Each individual coefficient $(\mathbf{r}_\Lambda)_\mu$ can be viewed as a local error indicator. Therefore, a natural adaptive strategy would consist in catching the bulk of the residual, i.e., to choose the new index set $\widehat{\Lambda}$ such that

$$\|\mathbf{r}_\Lambda|_{\widehat{\Lambda}}\|_{\ell_2(\nabla)} \geq \zeta\|\mathbf{r}_\Lambda\|_{\ell_2(\nabla)}, \quad \text{for some} \quad \zeta \in (0,1).$$

However, such a scheme cannot be implemented since the residual involves infinitely many coefficients. To transform this idea into an implementable scheme, the following three subroutines can be utilized:

(S1) $\mathbf{RHS}[\varepsilon, \mathbf{g}] \to \mathbf{g}_\varepsilon$ determines for $\mathbf{g} \in \ell_2(\nabla)$ a finitely supported $\mathbf{g}_\varepsilon \in \ell_2(\nabla)$ such that

$$\|\mathbf{g} - \mathbf{g}_\varepsilon\|_{\ell_2(\nabla)} \leq \varepsilon.$$

(S2) $\mathbf{APPLY}[\varepsilon, \mathbf{G}, \mathbf{v}] \to \mathbf{w}_\varepsilon$ determines for $\mathbf{G} \in \mathcal{L}(\ell_2(\nabla))$ and for a finitely supported $\mathbf{v} \in \ell_2(\nabla)$ a finitely supported $\mathbf{w}_\varepsilon \in \ell_2(\nabla)$ such that

$$\|\mathbf{G}\mathbf{v} - \mathbf{w}_\varepsilon\|_{\ell_2(\nabla)} \leq \varepsilon.$$

(S3) **COARSE**$[\varepsilon, \mathbf{v}] \to \mathbf{v}_\varepsilon$ determines for a finitely supported $\mathbf{v} \in \ell_2(\nabla)$ a finitely supported $\mathbf{v}_\varepsilon \in \ell_2(\nabla)$ with at most m significant coefficients, such that

$$\|\mathbf{v} - \mathbf{v}_\varepsilon\|_{\ell_2(\nabla)} \leq \varepsilon. \tag{5.85}$$

Moreover, $m \preceq m_{\min}$ holds, $m_{\min}$ being the minimal number of entries for which (5.85) is valid.

Employing the key idea outlined above leads to the following adaptive algorithm.

Algorithm 5.82. SOLVE$[\varepsilon, \mathbf{A}, \mathbf{f}] \to \mathbf{u}_\varepsilon$

$\Lambda_0 := \emptyset$; $\mathbf{r}_{\Lambda_0} := \mathbf{f}$; $\varepsilon_0 := \|\mathbf{f}\|_{\ell_2(\nabla)}$; $j := 0$; $u_0 := 0$;
while $\varepsilon_j > \varepsilon$ **do**
  $\varepsilon_{j+1} := 2^{-(j+1)}\|\mathbf{f}\|_{\ell_2(\nabla)}$; $\Lambda_{j,0} := \Lambda_j$; $\mathbf{u}_{j,0} := \mathbf{u}_j$;
  for $l = 1, \ldots, L$ **do**
    Compute Galerkin approximation $\mathbf{u}_{\Lambda_{j,l-1}}$ for $\Lambda_{j,l-1}$;
    Compute
      $\widetilde{\mathbf{r}}_{\Lambda_{j,l-1}} := \mathbf{RHS}[C_1^{\mathrm{tol}}\varepsilon_{j+1}, \mathbf{f}] - \mathbf{APPLY}[C_1^{\mathrm{tol}}\varepsilon_{j+1}, \mathbf{A}, \mathbf{u}_{\Lambda_{j,l-1}}]$;
    Compute smallest set $\Lambda_{j,l}$,
      such that, $\|\widetilde{\mathbf{r}}_{\Lambda_{j,l-1}}|_{\Lambda_{j,l}}\|_{\ell_2(\nabla)} \geq \frac{1}{2}\|\widetilde{\mathbf{r}}_{\Lambda_{j,l-1}}\|_{\ell_2(\nabla)}$;
  end for
  COARSE$[C_2^{\mathrm{tol}}\varepsilon_{j+1}, \mathbf{u}_{\Lambda_{j,L}}] \to (\Lambda_{j+1}, \mathbf{u}_{j+1})$;
  $j := j + 1$;
end while

In Cohen et al. [29], it has been shown that Algorithm 5.82 exactly fits into the setting of Assumption 5.72(i). Let us denote by $\Lambda_\varepsilon \subset \nabla$ the final index set when Algorithm 5.82 terminates (the method of updating ε_j ensures termination). Then Algorithm 5.82 has the properties:

(P1) Algorithm 5.82 is guaranteed to converge for a huge class of problems, in particular for the differential operators $L_{\tau,i}$ that we have in mind. Denoting with $H^\nu(\mathcal{O})$ the Sobolev space according to (2.32), we have

$$\Big\| u - \sum_{\mu \in \Lambda_\varepsilon} c_\mu \psi_\mu \Big\|_{H^\nu(\mathcal{O})} \leq C(u)\varepsilon.$$

(P2) Algorithm 5.82 is asymptotically optimal in the sense of Assumption 5.72, i.e., with $1/q = (s-\nu)/d + 1/2$, we have

$$\Big\| u - \sum_{\mu \in \Lambda_\varepsilon} c_\mu \psi_\mu \Big\|_{H^\nu(\mathcal{O})} \leq \widehat{C}^{\mathrm{asym}} \, \|u\|_{B^s_q(L_q(\mathcal{O}))} (\#\Lambda_\varepsilon)^{-(s-\nu)/d}.$$

(P3) Additionally, the number of arithmetic operations in Algorithm 5.82 stays proportional to the number of unknowns, i.e., the number of floating point operations needed to compute $\mathbf{u}_\varepsilon$ is bounded by a constant times $\#\mathrm{supp}\, \mathbf{u}_\varepsilon$.

Remark 5.83. In Algorithm 5.82, C_1^{tol} and C_2^{tol} denote some suitably chosen constants whose concrete values depend on the problem under consideration. Also, the parameter L has to be chosen in a suitable way. We refer again to Cohen et al. [29] for details.

Remark 5.84. In the numerical realization of the three fundamental subroutines, the routine **COARSE** consists of a thresholding step, whereas **RHS** essentially requires the computation of a best N-term approximation. The most complicated building block is **APPLY**. Let us just mention that its existence can be established for elliptic operators with Schwartz kernels by using the cancellation property of wavelets. For isotropic wavelets, we refer to Cohen et al. [29, 30], Stevenson [150], and to Dijkema et al. [69] for the anisotropic case using L_2-orthogonal wavelets.

Remark 5.85. In Gantumur et al. [81] it has been shown that a coarsing routine is not necessary to proof optimality. However, since the implementation of a **COARSE** routine is usually simple, it is often beneficial for the performance of numerical experiments to remove small coefficients.

Appendix

A Fundamentals

This appendix outlines the fundamental structures useful to support the understanding of the subject of this thesis. In order to provide a concise overview of the considered setting we omit many details and instead refer to the relevant literature on the respective topics.

Appendix A.1 states the specific functional analytical setting considered in this thesis. In Appendix A.2 we give the definition of the employed spaces of integrable mappings. An introduction to the concept of distributions generalizing the notion of functions is given in Appendix A.3. Appendix A.4 states the fundamental probabilistic setting and terms which are used, while Appendix A.5 gives an overview of the construction of Hilbert space-valued stochastic integrals with respect to a fixed cylindrical Wiener process.

A.1 Fundamental spaces

We state the specific functional analytical setting considered in this thesis.

Let $(G, \|\cdot\|_G)$ be a normed vector space over the field $\mathbb{R}$. The *algebraic dual space*, i.e., the set of all *linear functionals* $x' : G \to \mathbb{R}$ is denoted by G'. It itself is a vector space over $\mathbb{R}$ when equipped with the arithmetic operations $(x' + y')(x) := x'(x) + y'(x)$ and $(cx')(x) := x'(cx)$, where $x', y' \in G'$, $x \in G$, $c \in \mathbb{R}$. The mapping

$$\begin{aligned} \langle \cdot, \cdot \rangle_{G' \times G} : G' \times G &\to \mathbb{R}, \\ x' \times x &\mapsto x'(x), \end{aligned}$$

is called *dual pairing*. Let $(G_1, \|\cdot\|_{G_1})$ and $(G_2, \|\cdot\|_{G_2})$ be two normed vector spaces with the underlying metric being induced by the norm. The *adjoint operator* $T' : G_2' \to G_1'$ of a linear operator $T : G_1 \to G_2$ is uniquely given by

$$\langle T'y', x \rangle_{G_1' \times G_1} = \langle y', Tx \rangle_{G_2' \times G_2}, \quad x \in G_1,\, y' \in G_2',$$

see, e.g., Rudin [142, Theorem 4.10]. The space $\mathcal{L}(G_1, G_2)$ of all linear *and continuous* operators $T : G_1 \to G_2$ is itself a normed vector space over $\mathbb{R}$ together with

$$\|T\|_{\mathcal{L}(G_1,G_2)} := \sup_{x \in G_1,\, \|x\|_{G_1} \leq 1} \|Tx\|_{G_2}, \quad T \in \mathcal{L}(G_1, G_2). \tag{A.1}$$

The space $\mathcal{L}(G_1, \mathbb{R}) := G_1^*$ is called *topological dual space* of G_1, i.e., it consists of all linear and continuous functionals on G_1. In this case the adjoint of a $T \in \mathcal{L}(G_1, G_2)$ is also denoted by T^* in place of T'. For $G_1 = G_2 = G$, we write $\mathcal{L}(G) := \mathcal{L}(G, G)$.

A real *Banach space* $(G, \|\cdot\|_G)$ is a normed vector space over $\mathbb{R}$, which is complete with respect to the metric induced by the norm – usually the canonical metric $\operatorname{dist}(x, y) = \|x - y\|_G$, $x, y \in G$. It is called *separabel* if it contains a countable dense subset. A real *Hilbert space* is an inner product vector space $(U, \langle\cdot,\cdot\rangle_U)$ over $\mathbb{R}$ which is complete with respect to the metric induced by the inner product $\langle\cdot,\cdot\rangle_U : U \times U \to \mathbb{R}$. It is a Banach space with respect to the norm $\|\cdot\|_U := \sqrt{\langle\cdot,\cdot\rangle_U}$, which resembles in many ways the Euclidean norm in $\mathbb{R}^d$, e.g., on can define *orthogonality* of two functions

$$f \perp g \quad :\Longleftrightarrow \quad \langle f, g\rangle_U = 0, \quad f, g \in U,$$

see, e.g., Schilling [145] for more details on inner product and Hilbert spaces.

Remark. The space $(\mathcal{L}(G_1, G_2), \|\cdot\|_{\mathcal{L}(G_1,G_2)})$ is a Banach space if $(G_2, \|\cdot\|_{G_2})$ is a Banach space, see, e.g., Rudin [142, Theorem 4.1]. Therefore, a topological dual space is always a Banach space.

Remark. In the case that only the *generalized triangle inequality* holds, i.e.,

$$\|x_1 + x_2\|_G \le C\left(\|x_1\|_G + \|x_2\|_G\right), \quad x_1, x_2 \in G, \text{ for some } C \ge 1,$$

we call $\|\cdot\|_G$ a *quasi-norm* and speak of *quasi-Banach spaces*. In contrast to Banach spaces, quasi-Banach spaces in general are not locally convex, the quasi-norm may not be continuous, and the topological dual space may be empty. Nevertheless, certain quasi-Banach spaces are essential in the study of nonlinear approximation methods. Therefore, the quasi-Banach spaces considered in this thesis satisfy

$$\|x_1 + x_2\|_G^\mu \le \|x_1\|_G^\mu + \|x_2\|_G^\mu, \quad x_1, x_2 \in G, \text{ for some } \mu > 0,$$

which implies the generalized triangle inequality, and are always embedded in some Banach space in such a way that their dual spaces are rich enough.

Especially comprehensive is the study of the topological dual spaces of Hilbert spaces $(U, \langle\cdot,\cdot\rangle_U)$, since U^* can be identified with U by the *Riesz isometric isomorphism*

$$\begin{aligned} \Phi_{\text{iso}} : U &\to U^*, \\ u &\mapsto \langle u, \cdot\rangle_U, \end{aligned} \tag{A.2}$$

i.e., in particular the dual pairing coincides with the inner product. We refer to, e.g., Yosida [177, Section III.6] for details.

A.1.1 Frames and Riesz bases of separable Hilbert spaces

A countable set $\{e_k\}_{k\in\mathcal{I}} \subset U$ is by definition a *frame* of a separable Hilbert space $(U, \langle\cdot,\cdot\rangle_U)$, if and only if there exist positive constants c_s, C_s such that

$$c_s\|u\|_U^2 \le \sum_{k\in\mathcal{I}} |\langle u, e_k\rangle_U|^2 \le C_s\|u\|_U^2, \quad u \in U.$$

A frame is called *Riesz basis* if its vectors are linearly independent. In this case

$$c_s \sum_{k\in\mathcal{I}} |a_k|^2 \le \Big\| \sum_{k\in\mathcal{I}} a_k e_k \Big\|_U^2 \le C_s \sum_{k\in\mathcal{I}} |a_k|^2 \tag{A.3}$$

holds for all sequences $(a_k)_{k\in\mathcal{I}} \in \ell_2(\mathcal{I})$ and $\operatorname{clos}(\operatorname{span} e_k) = U$. Frames allow for *stable* analysis and synthesis representations of vectors. An analysis representation of an element $u \in U$ is given by the *frame analysis operator*

$$\begin{aligned}\Phi_{\text{frame}} : U &\to \ell_2(\mathcal{I}),\\ u &\mapsto (\langle u, e_k\rangle_U)_{k\in\mathcal{I}},\end{aligned}$$

resulting in a sequence of *frame coefficients* of u. The adjoint $\Phi^*_{\text{frame}} : \ell_2(\mathcal{I}) \to U$ of Φ_{frame}, note (A.2), is the *frame synthesis operator* since

$$\Phi^*_{\text{frame}}(a_k)_{k\in\mathcal{I}} = \sum_{k\in\mathcal{I}} a_k e_k, \quad (a_k)_{k\in\mathcal{I}} \in \ell_2(\mathcal{I}).$$

The reconstruction of an element $u \in U$ from its frame coefficients is computed with the help of a *dual frame* $\{\widetilde{e}_k\}_{k\in\mathcal{I}}$. Such a dual frame can be defined by

$$\widetilde{e}_k := (\Phi^*_{\text{frame}}\Phi_{\text{frame}})^{-1} e_k, \quad k \in \mathcal{I}, \tag{A.4}$$

so that we obtain the stable decompositions

$$u = \sum_{k\in\mathcal{I}} \langle u, e_k\rangle \widetilde{e}_k = \sum_{k\in\mathcal{I}} \langle u, \widetilde{e}_k\rangle e_k$$

of an $u \in U$, see Mallat [128, Theorem 5.5]. If $\{e_k\}_{k\in\mathcal{I}}$ is a Riesz basis, then its dual frame $\{\widetilde{e}_k\}_{k\in\mathcal{I}}$, defined by (A.4), is also linearly independent and we have

$$\langle e_k, \widetilde{e}_{k'}\rangle_U = \delta_{k,k'},$$

where $\delta_{k,k'}$ is the Kronecker delta, i.e., these dual Riesz bases are biorthogonal families of vectors. For a detailed discussion of frames, we refer to Mallat [128, Chapter 5].

A.1.2 Trace-class and Hilbert-Schmidt operators

Let $(U_1, \langle\cdot,\cdot\rangle_{U_1})$ and $(U_2, \langle\cdot,\cdot\rangle_{U_2})$ be two separable Hilbert spaces. We call a linear and continuous operator $T \in \mathcal{L}(U_1, U_2)$ *nuclear* or of *trace-classtrace-class operator* if there exist three sequences: $(f_n)_{n\in\mathbb{N}} \subset U_1$ which is orthonormal with respect to $\langle\cdot,\cdot\rangle_{U_1}$, i.e., for all $n, m \in \mathbb{N}$ we have $\langle f_n, f_m\rangle_{U_1} = \delta_{n,m}$, and $(g_n)_{n\in\mathbb{N}} \subset U_2$ which is orthonormal with respect to $\langle\cdot,\cdot\rangle_{U_2}$, as well as $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ with $\sum_{n=1}^\infty |a_n| < \infty$ such that

$$T = \sum_{n=1}^\infty a_n \langle f_n, \cdot\rangle_{U_1} g_n.$$

The *trace* of T is well-defined as $\operatorname{Tr}(T) := \sum_{n=1}^\infty |a_n|$. A trace-class operator is of *finite rank* if there exists a finite number $N \in \mathbb{N}$ such that for all $n \geq N$ we have $a_n = 0$. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of U_1. An operator $T \in \mathcal{L}(U_1, U_2)$ is called *Hilbert-Schmidt* if $\sum_{n=1}^\infty \|Te_n\|^2_{U_2} < \infty$. The space of all Hilbert-Schmidt operators is denoted by $\mathcal{L}_{\text{HS}}(U_1, U_2)$ and

$$\|T\|_{HS} := \left(\sum_{n=1}^\infty \|Te_n\|_{U_2}\right)^{1/2} \tag{A.5}$$

defines a norm on $\mathcal{L}_{\mathrm{HS}}(U_1, U_2)$, in particular since (A.5) is independent of the choice of $\{e_n\}_{n\in\mathbb{N}}$. The space $\mathcal{L}_{\mathrm{HS}}(U_1, U_2)$ contains all trace-class operators and the space of all finite rank operators is dense in $\mathcal{L}_{\mathrm{HS}}(U_1, U_2)$. An inner product on $\mathcal{L}_{\mathrm{HS}}(U_1, U_2)$ can be defined by

$$\langle T_1, T_2\rangle_{HS} := \sum_{n=1}^{\infty} \langle T_1 e_n, T_2 e_n\rangle_{U_2}, \quad T_1, T_2 \in \mathcal{L}_{\mathrm{HS}}(U_1, U_2),$$

in particular, $(\mathcal{L}_{\mathrm{HS}}(U_1, U_2), \langle\cdot,\cdot\rangle_{HS})$ is a separable Hilbert space. For details, see, e.g., Prévôt, Röckner [135, Appendix B].

A.1.3 Gel'fand triple and variational operators

Let $(U, \langle\cdot,\cdot\rangle_U)$ be a separable real Hilbert space and let V be a topological vector space densely embedded in U via a continuous inclusion map $j : V \hookrightarrow U$. We also write

$$V \overset{j}{\hookrightarrow} U.$$

Then, the adjoint operator $j^* : U^* \hookrightarrow V^*$ of j embeds U^* densely into the topological dual V^*. By the Riesz isomorphism $\Phi_{\mathrm{iso}} : U \to U^*$, see (A.2), we can identify U with its topological dual U^*. Thus, we obtain

$$V \overset{j}{\hookrightarrow} U \overset{\Phi_{\mathrm{iso}}}{\equiv} U^* \overset{j^*}{\hookrightarrow} V^*$$

and

$$\langle j^*\Phi_{\mathrm{iso}} j(v_1), v_2\rangle_{V^*\times V} = \langle j(v_1), j(v_2)\rangle_U, \quad \text{for all } v_1, v_2 \in V.$$

The triple (V, U, V^*) is called *Gel'fand triple* or *rigged Hilbert space*.

Suppose $(V, \langle\cdot,\cdot\rangle_V)$ is itself a real and separable Hilbert space. Furthermore, let

$$a : V \times V \to \mathbb{R}$$

be a *symmetric*, *continuous* (or *bounded*), and *coercive* (or *elliptic*) bilinear form on V, i.e., there exist positive constants C_{bound} and C_{ell} such that for all $u, v \in V$ we have

$$a(u,v) = a(v,u), \quad |a(u,v)| \le C_{\mathrm{bound}}\|u\|_V\|v\|_V, \quad \text{and} \quad a(v,v) \ge C_{\mathrm{ell}}\|v\|_V^2. \tag{A.6}$$

Then, by the Lax-Milgram theorem, see, e.g., Yosida [177], the operator

$$\begin{aligned} A : V &\to V^* \\ v &\mapsto Av := -a(v,\cdot) \end{aligned} \tag{A.7}$$

is boundedly invertible. In the Gel'fand triple setting, we can consider $A : V \to V^*$ as an unbounded operator on the intermediate Hilbert space U. We set

$$D(A,U) := \{v \in V \;:\; Av \in j^*\Phi_{\mathrm{iso}}(U)\},$$

and define the unbounded *variational operator* to A by

$$\begin{aligned} \bar{A} : j(D(A,U)) \subseteq U &\to U \\ u &\mapsto \bar{A}u := \Phi_{\mathrm{iso}}^{-1} j^{*-1} A j^{-1} u. \end{aligned} \tag{A.8}$$

The operator $\bar{A}$ is densely defined, since U^* is densely embedded in V^*, the symmetry of the bilinear form $a(\cdot,\cdot)$ implies that $\bar{A}$ is self-adjoint, and $\bar{A}$ is strictly negative definite, because $a(\cdot,\cdot)$ is assumed to be coercive. Furthermore, since $A : V \to V^*$ is boundedly invertible, the operator $\bar{A}^{-1} : U \to U$, defined by $\bar{A}^{-1} := jA^{-1}j^*\Phi_{\text{iso}}$ is the bounded inverse of $\bar{A}$, and $\bar{A}^{-1}$ is compact if the embedding j is compact.

Let $\tau > 0$, $a(\cdot,\cdot)$, and $A : V \to V^*$ be given as above. We consider the bilinear form

$$\begin{aligned} a_\tau : V \times V &\to \mathbb{R} \\ (u,v) &\mapsto a_\tau(u,v) := \tau\langle j(u), j(v)\rangle_U + a(u,v), \end{aligned}$$

which is also symmetric, continuous, and coercive in the sense of (A.6). For $u, v \in V$, the equalities

$$\begin{aligned} a_\tau(u,v) &= \tau\langle j^*\Phi_{\text{iso}}j(u), v\rangle_{V^*\times V} - \langle Au, v\rangle_{V^*\times V} \\ &= \langle(\tau j^*\Phi_{\text{iso}}j - A)u, v\rangle_{V^*\times V} \end{aligned}$$

hold and, by application of the Lax-Milgram theorem, we conclude that the operator

$$(\tau j^*\Phi_{\text{iso}}j - A) : V \to V^*$$

is boundedly invertible. Therefore, the operator

$$\begin{aligned} (\tau I - \bar{A}) : j(D(A,U)) \subseteq U &\to U \\ u &\mapsto (\tau I - \bar{A})u := \tau u - \bar{A}u, \end{aligned}$$

which coincides with $\Phi_{\text{iso}}^{-1}j^{*-1}(\tau j^*\Phi_{\text{iso}}j - A)j^{-1}$ on $j(D(A,U))$, possesses a bounded inverse

$$(\tau I - \bar{A})^{-1} = j(\tau j^*\Phi_{\text{iso}}j - A)^{-1}j^*\Phi_{\text{iso}} : U \to U.$$

Thus, the resolvent set $\rho(\bar{A})$ of $\bar{A}$ contains all $\tau \geq 0$. In particular, for any $\tau > 0$, the range of the operator $(\tau I - \bar{A})$ is the whole space U. Since, furthermore, $\bar{A}$ is dissipative, the Lumer-Phillips theorem implies that $\bar{A}$ generates a strongly continuous semi-group $\{e^{\bar{A}t}\}_{t\geq 0}$ of contractions on U, see, e.g., Pazy [133, Theorem 1.4.3]. Thus, an application of the Hille-Yosida theorem (see, e.g., [133, Theorem 1.3.1]) shows that the operator $L_\tau^{-1} := (I - \tau\bar{A})^{-1} = \tau(\tau I - \bar{A})^{-1} : U \to U$ is a contraction for each $\tau > 0$. Note that with a slight abuse of notation, we sometimes write A instead of $\bar{A}$.

A.1.4 The considered θ, q-interpolation spaces

Roughly speaking, interpolation determines intermediate spaces Z of two spaces X and Y, for which all linear operators that map X and Y continuously into themselves also map Z continuously into itself. Here, we consider the scale of real-valued θ, q-interpolation spaces $Z = (X,Y)_{\theta,q}$ based on the real method of Lions and Peetre, by using Peetres K-functional. We assume that $(X, \|\cdot\|_X)$ is a (quasi-)normed vector space and $(Y, |\cdot|_Y)$ a (quasi-)semi-normed vector space which is continuously embedded in X, that is $Y \hookrightarrow X$ and $\|\cdot\|_X \preceq |\cdot|_Y$.

The *K-functional* $K(f,t)$ is defined by

$$K(f,t) := K(f,t,X,Y) := \inf_{g\in Y} \|f-g\|_X + t|g|_Y, \quad f \in X,\ t > 0.$$

The term $t|g|_Y$ can be understood as a penalty term to the approximation of f by the function g from Y. In this setting, the θ, q-*interpolation space* $(X,Y)_{\theta,q}$, $\theta \in (0,1)$, $q \in (0,\infty)$, is defined as the set of all functions $f \in X$ such that the term

$$|f|_{(X,Y)_{\theta,q}} := \left(\int_0^\infty \left(t^{-\theta} K(f,t)\right)^q \frac{\mathrm{d}t}{t} \right)^{1/q}$$

is finite. We refer to, e.g., Bergh, Löfström [9, Chapter 3] and DeVore, Lorentz [67, Chapter 6] for a detailed discussion on the K-functional and interpolation spaces.

A.2 Spaces of integrable mappings

We consider integrability of measurable mappings defined on a σ-finite measure space $(X, \mathcal{A}, \mu)$ with values in a separable Banach space $(G, \|\cdot\|_G)$ over $\mathbb{R}$. A mapping f is called *measurable* if the preimage of every measurable set in G under f is an element of $\mathcal{A}$, where in this setting usually the strong Borel σ-algebra $\mathcal{B}(G)$ is considered, i.e., the smallest σ-algebra which contains all open subsets of G.

A mapping $s : X \to G$ is called *simple* or *elementary* if there exists a finite number $N \in \mathbb{N}$, as well as for $n \in \{1, \dots, N\}$, there exists $b_n \in G$ and mutually disjoint events $X_n \in \mathcal{A}$ such that

$$s(x) = \sum_{n=1}^N b_n \mathbb{1}_{X_n}(x), \quad x \in X.$$

If $\mu(X_n)$ is finite whenever $b_n \neq 0$, then the simple mapping s is *integrable*, and its *integral* is well-defined by

$$\int_X s(x)\, \mu(\mathrm{d}x) := \sum_{n=1}^N \mu(X_n) b_n.$$

Since we assume G to be separable, we have that for every measurable mapping $f : X \to G$ there exists a sequence $(s_i)_{i\in\mathbb{N}}$ of simple mappings with $\lim_{i\to\infty} s_i(x) = f(x)$ for μ-almost all $x \in X$. If additionally

$$\lim_{i\to\infty} \int_X \|s_i(x) - f(x)\|_G\, \mu(\mathrm{d}x) = 0 \tag{A.9}$$

for μ-almost all $x \in X$, then f is called *Bochner integrable* and its integral is well-defined by

$$\int_A f(x)\, \mu(\mathrm{d}x) := \lim_{i\to\infty} \int_X s_i(x)\, \mathbb{1}_A(x)\, \mu(\mathrm{d}x), \quad A \in \mathcal{A}.$$

Note, a measurable mapping $f : X \to G$ is Bochner integrable if and only if

$$\int_X \|f(x)\|_G\, \mu(\mathrm{d}x) < \infty.$$

The proofs can be found in Yosida [177, Section V.5] and we refer to Schilling [145, Chapter 10] for details on the abstract Lebesgue integral used in, e.g., (A.9).

Within the *Lebesgue-Bochner spaces* $L_p(X, \mathcal{A}, \mu; G)$, $1 \leq p < \infty$, we collect all μ-equivalence classes $[f]$ which contain a measurable mapping $f : X \to G$ such that

$$|f|_{L_p(X,\mathcal{A},\mu;G)} := \left(\int_X \|f(x)\|_G^p \, \mu(\mathrm{d}x) \right)^{1/p} < \infty. \tag{A.10}$$

Two mappings $f_1, f_2 : X \to G$ are called *equivalent* or $f_1 \sim f_2$ if they only differ on a μ-null set, i.e., $\mu(\{x \in X \,:\, f_1(x) \neq f_2(x)\}) = 0$.

The *Lebesgue-Bochner space* $L_\infty(X, \mathcal{A}, \mu; G)$ denotes the space of all μ-equivalence classes $[f]$ containing a measurable mapping $f : X \to G$ for which there exists a $r \geq 0$ such that $\mu(\{x \in X \,:\, \|f(x)\|_G > r\}) = 0$. For this space we define

$$|f|_{L_\infty(X,\mathcal{A},\mu;G)} := \inf \{ r \geq 0 \,:\, \mu(\{x \in X \,:\, \|f(x)\|_G > r\}) = 0 \}. \tag{A.11}$$

Note that, with respect to

$$\|[f]\|_{L_p(X,\mathcal{A},\mu;G)} := \inf \left\{ |g|_{L_p(X,\mathcal{A},\mu;G)} \,:\, g \sim f \right\}, \quad 1 \leq p \leq \infty, \tag{A.12}$$

the spaces $(L_p(X, \mathcal{A}, \mu; G), \|\cdot\|_{L_p(X,\mathcal{A},\mu;G)})$ are Banach spaces if $(G, \|\cdot\|_G)$ is a Banach space, while for $p = 2$ and $(G, \langle \cdot, \cdot \rangle_G)$ a Hilbert space, $L_2(X, \mathcal{A}, \mu; G)$ are also Hilbert spaces with respect to the inner product

$$\langle f, g \rangle_{L_2(X,\mathcal{A},\mu;G)} := \int_X \langle f(x), g(x) \rangle_G \, \mu(\mathrm{d}x).$$

In this context of integrability, instead of the equivalence class $[f]$ it is common to speak of *the mapping* f and write $f \in L_p(X, \mathcal{A}, \mu; G)$, as well as $\|f\|_{L_p(X,\mathcal{A},\mu;G)}$, in particular since $\|[f]\|_{L_p(X,\mathcal{A},\mu;G)} = |f|_{L_p(X,\mathcal{A},\mu;G)}$. Furthermore, two elements $f, g \in L_p(X, \mathcal{A}, \mu; G)$ can only be compared up to sets of measure zero, e.g., $f \leq g$ means $f(x) \leq g(x)$ for all x outside a μ-null set. If (A.10) holds, f is called *p-integrable* and if (A.11) is finite, f is called *bounded*.

Remark. In the case of $G = \mathbb{R}$ the above integral construction and (A.10) can readily be extended to $0 < p < 1$ turning (A.12) into a quasi-norm and $L_p(X, \mathcal{A}, \mu; \mathbb{R})$ into a quasi-Banach space. However, the situation is much more involved if G is a general Banach space or quasi-Banach space, see Albiac, Ansorena [3] for details.

Example. If $\mu = \lambda^d|_{\mathcal{O}}$ is the Lebesgue measure on a Lebesgue-measurable set $\mathcal{O} \subseteq \mathbb{R}^d$ and $G = \mathbb{R}$, then

$$L_p(\mathcal{O}) := L_p(\mathcal{O}, \mathcal{B}(\mathcal{O}), \lambda^d|_{\mathcal{O}}; \mathbb{R}), \quad 0 < p \leq \infty,$$

are known as *Lebesgue spaces*. We refer to Schilling [145] for details on the Lebesgue measure λ^d and the case $1 \leq p \leq \infty$.

Example. Let $\mathcal{I}$ be a countable set and $\sum_{i \in \mathcal{I}} \delta_i$ be the counting measure, i.e., the sum of the Dirac measures δ_i at the points $i \in \mathcal{I}$. The spaces

$$\ell_p(\mathcal{I}) := L_p(\mathcal{I}, 2^{\mathcal{I}}, \sum_{i \in \mathcal{I}} \delta_i; \, \mathbb{R}), \quad 1 \leq p \leq \infty,$$

are called *p-summable sequence spaces* over $\mathcal{I}$ for $p < \infty$, while $\ell_\infty(\mathcal{I})$ is the space of *bounded* sequences, cf. Schilling [145, Example 12.12]. We set $\ell_p := \ell_p(\mathbb{N})$.

A.3 Distributions, generalized derivatives, and the Fourier transform

We give an introduction to distributions which generalize the notion of functions and the Fourier transform. In particular, the concept of generalized derivatives for locally integrable functions is based on the integration by parts formula. For the proofs of the following statements, we refer to Adams, Fournier [2] and in particular to Rudin [142, Chapters 6, 7].

Let $\mathcal{O} \subseteq \mathbb{R}^d$ be a domain and let $\mathcal{D}(\mathcal{O})$ be the set of all *test functions* on $\mathcal{O}$, that is the set of compactly supported infinitely often differentiable functions $\mathcal{C}_0^\infty(\mathcal{O})$ equipped with a final locally convex topology. A linear functional on $\mathcal{D}(\mathcal{O})$ which is continuous with respect to this topology is called *(Schwartz) distribution* and $\mathcal{D}'(\mathcal{O})$ denotes the space of all of these distributions.

Here, two properties of distributions are of special interest, namely differentiation and the Fourier transform. The derivative $D^\alpha T$, $\alpha \in \mathbb{N}_0^d$, $|\alpha| := \alpha_1 + \cdots + \alpha_d$, of a distribution $T \in \mathcal{D}'(\mathcal{O})$ is defined by

$$D^\alpha T(\varphi) := (-1)^{|\alpha|} T(D^\alpha \varphi) \quad \text{for all } \varphi \in \mathcal{D}(\mathcal{O}), \tag{A.13}$$

where $D^\alpha \varphi$ is the classical derivative of φ. Of course, D^0 is the identity mapping. It turns out that $D^\alpha T$ is a continuous linear functional on $\mathcal{D}(\mathcal{O})$ and $D^\alpha T \in \mathcal{D}'(\mathcal{O})$ for any $\alpha \in \mathbb{N}_0^d$, $T \in \mathcal{D}'(\mathcal{O})$. Furthermore, $D^\alpha : \mathcal{D}'(\mathcal{O}) \to \mathcal{D}'(\mathcal{O})$ is a continuous operator.

The space of distributions contains and generalizes the space of all locally integrable functions $L_{1,\mathrm{loc}}(\mathcal{O})$, where a Lebesgue measurable function $f : \mathcal{O} \to \mathbb{R}$ is, by definition, *locally integrable* if $\int_K |f| \,\mathrm{d}x < \infty$ for all K compactly embedded in $\mathcal{O}$. The topology on $\mathcal{D}(\mathcal{O})$ is chosen in such a way that any $f \in L_{1,\mathrm{loc}}(\mathcal{O})$ yields a *regular* distribution $T_f \in \mathcal{D}'(\mathcal{O})$ whose value on the test functions is given by the Lebesgue integral, i.e.,

$$T_f(\varphi) = \int_{\mathcal{O}} f\varphi \,\mathrm{d}x, \quad \varphi \in \mathcal{D}(\mathcal{O}). \tag{A.14}$$

Since two locally integrable functions yield the same element in $\mathcal{D}'(\mathcal{O})$ if and only if they are equal almost everywhere, one can identify T_f with f. Note that not every distribution is of the form (A.14), e.g., the Dirac distribution defined by $\delta(\varphi) := \varphi(0)$, $0 \in \mathcal{O}$, is not a regular distribution.

Above definitions (A.13) and (A.14) allow us to define generalized derivatives of locally integrable functions. A locally integrable function $f_\alpha \in L_{1,\mathrm{loc}}(\mathcal{O})$, $\alpha \in \mathbb{N}_0^d$, is called *generalized derivative* of an $f \in L_{1,\mathrm{loc}}(\mathcal{O})$ if

$$T_{f_\alpha} = D^\alpha T_f \quad \text{in } \mathcal{D}'(\mathcal{O}).$$

In this case f_α is unique up to sets of measure zero. In particular, in the case the derivative $D^\alpha f$ also exists in the classical sense and is locally integrable, it coincides with the generalized derivative. This allows us to also use the notation $D^\alpha f$ to denote the generalized derivative and $\partial^\alpha f$ for the generalized partial derivative.

Remark. It has been shown that $L_p(\mathcal{O}) \subset L_{1,\mathrm{loc}}(\mathcal{O})$ for $1 \le p \le \infty$ on any domain, see, e.g., Adams, Fournier [2, Corollary 2.15].

Now, we turn to the Fourier transform and its generalization to distributions. The space for the Fourier transform $\mathfrak{F}$ of complex-valued functions f, which is given by

$$\mathfrak{F}f(\xi) := \int_{\mathbb{R}^d} \exp(-i\langle x, \xi\rangle)\, f(x)\, \mathrm{d}x, \quad \xi \in \mathbb{R}^d,$$

and its inverse

$$\mathfrak{F}^{-1}f(x) := (2\pi)^{-d} \int_{\mathbb{R}^d} \exp(i\langle x, \xi\rangle)\, f(\xi)\, \mathrm{d}\xi, \quad x \in \mathbb{R}^d,$$

is the *Schwartz space of rapid decrease* $\mathcal{S}(\mathbb{R}^d)$. It consists of all functions $\varphi : \mathbb{R}^d \to \mathbb{C}$ such that

$$\sup_{x\in\mathbb{R}^d} |x^\beta D^\alpha \varphi(x)| < \infty \quad \text{for all } \alpha, \beta \in \mathbb{N}_0^d,$$

and it is the right space for $\mathfrak{F}$, since it has been shown that $\mathfrak{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is an isomorphism with inverse $\mathfrak{F}^{-1}$. The space $\mathcal{S}(\mathbb{R}^d)$ is also equipped with a final locally convex topology and $\mathcal{D}(\mathbb{R}^d)$ is densely and continuously embedded in $\mathcal{S}(\mathbb{R}^d)$, see, e.g., Rudin [142, Theorem 7.10]. Therefore, with $\iota : \mathcal{D}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ the identity mapping and $L \in \mathcal{S}'(\mathbb{R}^d)$, one obtains a so-called *tempered distribution* $u_L = L \circ \iota \in \mathcal{D}'(\mathbb{R}^d)$, which, by identifying u_L and L, implies that the space $\mathcal{S}'(\mathbb{R}^d)$ coincides with the set of all tempered distributions, cf. Rudin [142, Definition 7.11].

It is possible to extend $\mathfrak{F}$ to a map on $\mathcal{S}'(\mathbb{R}^d)$. Since $(2\pi)^d\mathfrak{F}^{-1}$ is the adjoint of $\mathfrak{F}$ with respect to the duality pairing, the Fourier transform $\mathfrak{F}$ on $\mathcal{S}'(\mathbb{R}^d)$ can be defined by

$$\mathfrak{F}f(\overline{\varphi}) := f\left((2\pi)^d\, \overline{\mathfrak{F}^{-1}\varphi}\right) \quad \text{for all } f \in \mathcal{S}'(\mathbb{R}^d),\ \varphi \in \mathcal{S}(\mathbb{R}^d).$$

Therefore, by continuity $\mathfrak{F}$ extends to an isomorphism $\mathfrak{F} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ with inverse $\mathfrak{F}^{-1}$.

Remark. Given the above setting for the Fourier transform, we restrict our analysis to quasi-normed spaces of distributions which are continuously embedded in $\mathcal{S}'(\mathbb{R}^d)$, e.g., the spaces $L_p(\mathbb{R}^d)$ with $1 \le p \le \infty$. Note that the L_p-spaces with $0 < p < 1$ can not be understood as subspaces of $\mathcal{S}'(\mathbb{R}^d)$, cf. Triebel [158, §2.2.3].

A.4 Probabilistic setting

We state the fundamental probabilistic setting and terms which are employed. For details we refer to Hairer [95] and Prévôt, Röckner [135].

A.4.1 Probability space, random variable, and stochastic process

A *probability space* is a measure space $(\Omega, \mathcal{F}, \mathrm{P})$, where Ω is called *sample space*, the σ-algebra $\mathcal{F}$ is the *set of events*, and $\mathrm{P} : \mathcal{F} \to [0, 1]$ is a *probability measure*, i.e., P is countably additive on pairwise disjoint events and $\mathrm{P}(\Omega) = 1$. Note that, we assume $(\Omega, \mathcal{F}, \mathrm{P})$ to be *complete*, i.e., all subsets of events with measure zero are also events.

Let (S, Σ) be a measurable space. An S-valued *random variable* is a mapping $Y : \Omega \to S$ which is $(\mathcal{F}, \Sigma)$-measurable, i.e., the preimage of every set of Σ under Y is an element of $\mathcal{F}$. The *law* or *distribution* of an S-valued random variable Y is the push-forward measure $Y_*\mathrm{P} := \mathrm{P} \circ Y^{-1}$ of Y on (S, Σ).

Two random variables $Y : \Omega \to S$, $Z : \Omega \to S$ are *independent* if they generate independent σ-algebras, i.e., any event, e.g., $\{\omega \in \Omega \,:\, Y(\omega) = \cdot\} \in \mathcal{F}$, in terms of Y is independent of any event defined in terms of Z.

An S-valued *stochastic process* $X : \Omega \times \mathcal{T} \to S$ is a collection

$$X = \{X_t\}_{t\in\mathcal{T}} = \{X(t)\}_{t\in\mathcal{T}},$$

indexed by a measurable space $(\mathcal{T}, \Upsilon)$, of S-valued random variables $X(t) : \Omega \to S$. A map $X(\omega, \cdot) : \mathcal{T} \to S$, $\omega \in \Omega$, is called a *path* or *realization* of X. A stochastic process $X_{\mathrm{mod}} : \Omega \times \mathcal{T} \to S$ is called a *modification of* X if

$$\mathrm{P}\left(\{\omega \in \Omega \,:\, X_{\mathrm{mod}}(\omega, t) = X(\omega, t)\}\right) = 1 \quad \text{for each } t \in \mathcal{T}.$$

In case $X : \Omega \times \mathcal{T} \to S$ is also $(\mathcal{F} \otimes \Upsilon, \Sigma)$-measurable it is called *product measurable*. A stochastic process X induces a $(S^{\mathcal{T}}, \Upsilon \otimes \Sigma)$-valued random variable

$$\Phi_X : \Omega \to \{f : \mathcal{T} \to S\} \quad \text{via} \quad (\Phi_X(\omega))(t) := X_t(\omega), \quad \omega \in \Omega,\ t \in \mathcal{T},$$

so that the *law* or *distribution* of the stochastic process X can be defined as the law of Φ_X, i.e., the push-forward measure $(\Phi_X)_*\mathrm{P}$ of P along Φ_X on $(S^{\mathcal{T}}, \Upsilon \otimes \Sigma)$.

A stochastic process is also called *random function* or *random mapping* depending on S, in particular if the index set $\mathcal{T}$ is one-dimensional, e.g., $\mathcal{T}$ is a time interval $[0, T] \subset \mathbb{R}$ and Υ is the Borel σ-algebra $\mathcal{B}([0, T])$. For multivariate $\mathcal{T}$ with adequate Borel measurability, also the terms *random field* or *stochastic field* are used, e.g., $\mathcal{T}$ is a (spatial) domain $\mathcal{O} \subseteq \mathbb{R}^d$, $d > 1$, or $\mathcal{T} = [0, T] \times \mathcal{O}$. However, for the most part we simply stick to the term random function.

Given an ordered index set, e.g., $([0, T], \leq)$, one can introduce a *filtration* on $(\Omega, \mathcal{F}, \mathrm{P})$, which is an increasing family $(\mathcal{F}_t)_{t\in[0,T]}$ of sub-σ-algebras of $\mathcal{F}$. Note, we assume the filtration to be *normal*, i.e., $\mathcal{F}_0$ contains all events with measure zero and $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ for all $t \in [0, T]$. A stochastic process $X : \Omega \times [0, T] \to S$ is called *non-anticipating* or *adapted* to a filtration $(\mathcal{F}_t)_{t\in[0,T]}$ if for all $t \in [0, T]$ the random variable $X(t) : \Omega \to S$ is also $(\mathcal{F}_t, \Sigma)$-measurable. Given the σ-algebra

$$\mathcal{P}_{[0,T]} := \sigma\Big(\big\{(s, t] \times F_s \,:\, 0 \leq s < t \leq T,\ F_s \in \mathcal{F}_s\big\} \cup \big\{\{0\} \times F_0 \,:\, F_0 \in \mathcal{F}_0\big\}\Big),$$

a stochastic process $X : \Omega \times [0, T] \to S$ is *predictable* if it is $(\mathcal{P}_{[0,T]}, \Sigma)$-measurable.

If the space S is, e.g., a separable Banach space $(G, \|\cdot\|_G)$ with $\Sigma = \mathcal{B}(G)$, we can consider integrability of random variables, i.e., $Y \in L_1(\Omega, \mathcal{F}, \mathrm{P}; G)$, cf. Appendix A.2. Here, we call the integral

$$\mathrm{E}[Y] := \int_\Omega Y(\omega)\mathrm{P}(\mathrm{d}\omega)$$

the *expectation* or *expected value* of Y. Given a sub-σ-algebra $\mathcal{F}_{\mathrm{sub}}$ of $\mathcal{F}$, then there exists a unique (up to P-null sets) integrable random variable $Z : \Omega \to G$ such that

$$\int_A Y(\omega)\mathrm{P}(\mathrm{d}\omega) = \int_A Z(\omega)\mathrm{P}(\mathrm{d}\omega), \quad \text{for all } A \in \mathcal{F}_{\mathrm{sub}},$$

called *conditional expectation* of Y with respect to $\mathcal{F}_{\mathrm{sub}}$, see, e.g., Prévôt, Röckner [135, Proposition 2.2.1] for the proof. It is denoted by $Z = \mathrm{E}[Y|\mathcal{F}_{\mathrm{sub}}]$. In this setting, a stochastic process $X : \Omega \times [0, T] \to G$ is called L_p*-martingale*, $p \in [1, \infty)$, with respect to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ if it is adapted to it and we have $X(t) \in L_p(\Omega, \mathcal{F}, \mathrm{P}; G)$ for all $t \in [0, T]$, as well as

$$\mathrm{E}\left[X(t) \,|\, \mathcal{F}_s\right] = X(s) \quad \text{P-a.s.}, \quad \text{for all } s, t \in [0, T] \text{ with } s \leq t.$$

A.4.2 Gaussian measure

Let $(G, \|\cdot\|_G)$ be a separable (reflexive) Banach space equipped with the strong Borel σ-algebra $\mathcal{B}(G)$. A *Gaussian measure* μ on G is a Borel measure such that for every linear functional $b : G \to \mathbb{R}$ the push-forward measure $b_*\mu = \mu \circ b^{-1}$ is a Gaussian probability measure $\mathcal{N}(m, \sigma^2)$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathrm{d}x)$, i.e.,

$$\forall b \in G^* \;\; \exists m_b \in \mathbb{R} \;\; \exists \sigma_b \geq 0 \;\; \forall A \in \mathcal{B}(\mathbb{R}) \; : \; (\mu \circ b^{-1})(A) = \frac{1}{\sqrt{2\pi\sigma_b^2}} \int_A e^{-\frac{(x-m_b)^2}{2\sigma_b}} \mathrm{d}x,$$

if $\sigma_b > 0$ and, if $\sigma_b = 0$, then $\mu \circ b^{-1} = \delta_{m_b}$. Here, $\mathrm{d}x$ denotes the Lebesgue measure and $\mathcal{N}(m, \sigma^2)$ is also called *normal distribution*. A Gaussian measure is *centered* if $m_b = 0$ for every $b \in G^*$. It is common to consider the centered case and to use simple translations to obtain the general case. With G being separable, μ is well-defined since the one-dimensional projections $b_*\mu$ carry sufficient information to characterize it, see, e.g., Hairer [95, Proposition 3.6]. Furthermore, since the *mean* $m : G^* \to \mathbb{R}$ of μ, defined by

$$m(b) := \int_G b(x)\, \mu(\mathrm{d}x), \quad b \in G^*,$$

is an element of G^{**} it is common to require G to be reflexive, i.e., $G = G^{**}$. The mapping $\mathrm{Cov}_\mu : G^* \times G^* \to \mathbb{R}$ defined by

$$\mathrm{Cov}_\mu(b_1, b_2) := \int_G b_1(x) b_2(x) \; \mu(\mathrm{d}x), \quad b_1, b_2 \in G^*,$$

of a centered Gaussian measure μ is called *covariance operator* of μ. Note, by definition Cov_μ is bilinear and non-negative definite. The name is due to the fact that Cov_μ can be understood as an operator $\bar{\mathrm{C}}\mathrm{ov}_\mu : G^* \to G^{**}$ by $\bar{\mathrm{C}}\mathrm{ov}_\mu(b_1)(b_2) = \mathrm{Cov}_\mu(b_1, b_2)$.

In the Hilbert space case, above considerations can be summarized as follows. Let $(H, \langle \cdot, \cdot \rangle_H)$ be a separable Hilbert space. A finite measure μ on $(H, \mathcal{B}(H))$ is Gaussian if and only if

$$\int_H e^{i\langle u,v\rangle_H} \mu(\mathrm{d}v) = e^{i\langle m,u\rangle_H - \frac{1}{2}\langle Qu,u\rangle_H}, \quad u \in H,$$

where $m \in H$ is its mean and $Q \in \mathcal{L}(H)$ is its symmetric, non-negative covariance operator of finite trace. Moreover, the measure μ is uniquely determined by m and Q. It is denoted by $\mu = \mathcal{N}(m, Q)$. We refer to Prévôt, Röckner [135, Theorem 2.1.2] or Da Prato, Zabczyk [40] for details and the proof.

Suppose Ω is some space of G-valued continuous functions, where $(G, \|\cdot\|_G)$ is a separable (reflexive) Banach space, e.g., $\Omega = \mathcal{C}([0,T], G)$, which permits a Gaussian measure μ. Then the G-valued canonical stochastic process $X = \{X(t)\}_t$, i.e.,

$$X(t)(\omega) := \omega(t), \quad \omega \in \Omega,$$

is called *Gaussian stochastic process.*

Example. The canonical stochastic process B for the Gaussian measure on $\mathcal{C}([0,T], \mathbb{R})$ with $B(0) = 0$ and $\mathrm{E}[|B(t) - B(s)|^2] = |t - s|$, $t, s \in [0,T]$, is called *Brownian motion* or standard one-dimensional Wiener process, cf. Hairer [95, Section 3.4]. For a construction of such a process, we refer to Schilling [145, §24.29].

Example (Wiener measure). Let

$$\mathcal{C}_{\mathrm{W}}(\mathbb{R}_+,\mathbb{R}) := \left\{ f \in \mathcal{C}(\mathbb{R}_+,\mathbb{R}) \,:\, \lim_{t\to\infty} f(t)/(1+t^2) < \infty \right\}$$

with $\|f\|_{\mathcal{C}_{\mathrm{W}}} := \sup_{t\in\mathbb{R}} |f(t)|/(1+t^2)$. Then there exists a Gaussian measure, called *Wiener measure*, on $\mathcal{C}_{\mathrm{W}}$ with covariance function $\mathrm{Cov}(s,t) = \min\{s,t\}$, $s,t \in \mathbb{R}_+$, see Hairer [95, Proposition 3.53].

Example (Cylindrical Wiener process). Let $(\bar{H}, \langle\cdot,\cdot\rangle_{\bar{H}})$ be a Hilbert space containing a separable Hilbert space $(H, \langle\cdot,\cdot\rangle_H)$ as a dense subset and such that the inclusion map $\iota : H \to \bar{H}$ is Hilbert-Schmidt. A Gaussian stochastic process $W : \Omega \times [0,T] \to \bar{H}$, such that

$$\mathrm{E}\left[\langle u, W(s)\rangle_{\bar{H}} \langle W(t), v\rangle_{\bar{H}}\right] = \min\{s,t\}\langle \iota\iota^* u, v\rangle_{\bar{H}}$$

for any two times $s,t \in [0,T]$ and any two elements $u,v \in \bar{H}$, is called *cylindrical Wiener process* over H. Since ι is Hilbert-Schmidt the law of W does not depend on $\bar{H}$, which justifies the denotation of W being a cylindrical Wiener process on H. Such a process can be realized as the canonical stochastic process for some Gaussian measure on $\mathcal{C}_{\mathrm{W}}(\mathbb{R}_+, \bar{H})$. Again, we refer to Hairer [95, Section 3.4] for details.

A.5 Cylindrical Wiener process and stochastic integration

We give an overview of the construction of Hilbert space-valued stochastic integrals with respect to a fixed cylindrical Wiener process. Therefore, let $[0,T]$, $T > 0$, be understood as the time horizon and let $(\Omega, \mathcal{F}, \mathrm{P})$ be a complete probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$. We start by summarizing the construction of a cylindrical Wiener process and then proceed based on the Itō calculus for stochastic integration. We refer to Hairer [95] and Prévôt, Röckner [135] for details.

A.5.1 Cylindrical Wiener process

Wiener processes have applications throughout the mathematical fields, as they represent the integrals of Gaussian noise. It is a common and accepted driving process in numerical modeling of stochastic equations.

Let $(H, \langle\cdot,\cdot\rangle_H)$ be a separable real Hilbert space, and let $Q \in \mathcal{L}(H)$ be non-negative definite, symmetric, and of finite trace. A stochastic process $W : \Omega \times [0,T] \to H$ is called a *Q-Wiener process* if

(Q1) $W(0) = 0$ and W has P-a.s. continuous paths $t \mapsto W(t)$, $t \in [0,T]$,

(Q2) W has independent increments, i.e., $\{W(t_1) - E(t_2), \dots, W(t_n) - W(t_{n-1})\}$ for $0 \le t_1 < t_2 < \cdots < t_{n-1} < t_n \le T$ is an independent family of random variables,

(Q3) the increments are Gaussian with $\mathrm{P} \circ (W(t) - W(s))^{-1} = \mathcal{N}(0, (t-s)Q)$ for all $0 \le s \le t \le T$.

Note, a real-valued Wiener process is a Brownian motion $\{\beta(t)\}_{t\in[0,T]}$ if, in particular, $\mathrm{P} \circ (\beta(t) - \beta(s))^{-1} = \mathcal{N}(0, t-s)$, $0 \le s \le t \le T$, cf. Appendix A.4.2.

Let $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of H consisting of eigenvectors of Q and corresponding non-negative eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$ numbered in decreasing order,

$$Qe_k = \lambda_k e_k, \quad \lambda_k \geq 0, \quad k \in \mathbb{N}.$$

In this setting, we have the following representation. A stochastic process W is a Q-Wiener process, if and only if

$$W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\beta_k(t)e_k, \qquad t \in [0,T], \tag{A.15}$$

where $\beta_k : \Omega \times [0,T] \to \mathbb{R}$, $k \in \{n \in \mathbb{N} : \lambda_n > 0\}$ are independent Brownian motions on $(\Omega, \mathcal{F}, \mathrm{P})$. Moreover, the series converges in $L_2(\Omega, \mathcal{F}, \mathrm{P}; \mathcal{C}([0,T], H))$, and in particular, for any Q as above there exists a Q-Wiener process, see Prévôt, Röckner [135, Proposition 2.1.10]. Note that the convergence of (A.15) in $L_2(\Omega; H)$ depends on $\mathrm{Tr}(Q)$ being finite, i.e.,

$$\mathrm{E}\left[\Big\|\sum_{k=0}^{\infty} \sqrt{\lambda_k}\beta_k(t)e_k\Big\|_H^2\right] = \sum_{k=0}^{\infty} \lambda_k \mathrm{E}[\beta_k(t)^2] = t\sum_{k=0}^{\infty}\lambda_k = t\,\mathrm{Tr}(Q) < \infty.$$

If Q is not of finite trace, e.g., the case where $Q = \mathrm{Id}$ is the identity operator, it is also possible to construct a Q-Wiener process. To this end, let $Q^{1/2}(H)$ together with the inner product

$$\langle u, v\rangle_{Q^{1/2}(H)} := \langle Q^{-1/2}u, Q^{-1/2}v\rangle_H, \quad u, v \in Q^{1/2}(H),$$

be a subspace of H — called *Cameron-Martin space* of H. Let $(\bar{H}, \langle\cdot,\cdot\rangle_{\bar{H}})$ be a larger Hilbert space containing H as dense subset, and let the inclusion map

$$J : Q^{1/2}(H) \to \bar{H}$$

be Hilbert-Schmidt. Note, $Q^{-1/2}$ is the pseudo-inverse of $Q^{1/2}$ if Q is not one-to-one. For the definition and details on pseudo-inverse we refer to Prévôt, Röckner [135, Appendix C].

Example. Let $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of $Q^{1/2}(H)$ and let $(a_k)_{k\in\mathbb{N}}$ be a square-summable sequence of positive real numbers. Then

$$\begin{aligned} J_{(a_k)} : Q^{1/2}(H) &\to \bar{H} := H \\ u &\mapsto \sum_{k=1}^{\infty} a_k \langle u, e_k\rangle_{Q^{1/2}(H)} e_k \end{aligned}$$

is one-to-one and Hilbert-Schmidt: Let $u, v \in Q^{1/2}(H)$ such that $J_{(a_k)}(u) = J_{(a_k)}(v)$. Then

$$J_{(a_k)}(u) - J_{(a_k)}(v) = \sum_{k=1}^{\infty} a_k \langle u - v, e_k\rangle_{Q^{1/2}(H)} e_k = 0$$

implies $u = v$ since $a_k > 0$, $k \in \mathbb{N}$, and $\{e_k\}_{k\in\mathbb{N}}$ is a basis, and so $J_{(a_k)}$ is one-to-one. Furthermore,

$$\sum_{i=1}^{\infty} \|J_{(a_k)}(e_i)\|_H^2 = \sum_{i=1}^{\infty} \Big\|\sum_{k=1}^{\infty} a_k\langle e_i, e_k\rangle_{Q^{1/2}(H)} e_k\Big\|_H^2 = \sum_{k=1}^{\infty} \|a_k e_k\|_H^2 = \sum_{k=1}^{\infty} a_k^2 < \infty$$

since $(a_k)_{k\in\mathbb{N}}$ is square-summable, and so $J_{(a_k)}$ is Hilbert-Schmidt.

Now, $\bar{Q} := JJ^* \in \mathcal{L}(\bar{H}, \bar{H})$ is non-negative, symmetric, and has finite trace. The series

$$W(t) = \sum_{k=1}^{\infty} \beta_k(t) J e_k, \quad t \in [0,T],$$

defines a $\bar{Q}$-Wiener process $W : \Omega \times [0,T] \to \bar{H}$ which is a cylindrical Wiener process on H, cf. Appendix A.4.2, since

$$J : Q^{1/2}(H) \to \bar{Q}^{1/2}(\bar{H})$$

is an isometry, see Prévôt, Röckner [135, Proposition 2.5.2].

A.5.2 Stochastic integration

Let $(H, \langle\cdot,\cdot\rangle_H)$ and $(U, \langle\cdot,\cdot\rangle_U)$ be two separable real Hilbert spaces, and let

$$W : \Omega \times [0,T] \to \bar{H}$$

be a cylindrical Wiener process on $H \subset \bar{H}$ with respect to the normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$. We turn to the construction of the U-valued *stochastic Itō integral*

$$\int_0^t \Phi(s) \mathrm{d}W(s), \quad t \in [0,T],$$

over certain $\mathcal{L}_{\mathrm{HS}}(H,U)$-valued stochastic processes Φ.

Analogously to the construction of the integral in Appendix A.2, one first considers a class $\mathcal{E}$ of elementary stochastic processes, where $\Phi : \Omega \times [0,T] \to \mathcal{L}_{\mathrm{HS}}(H,U)$ is *elementary* if there exists a finite number $n \in \mathbb{N}$ and $0 = t_0 < \cdots < t_n = T$ such that

$$\Phi(\omega, t) = \sum_{k=0}^{n-1} \Phi_k(\omega)\, \mathbb{1}_{(t_k, t_{k-1}]}(t), \quad \omega \in \Omega,\ t \in [0,T],$$

where $\Phi_k : \Omega \to \mathcal{L}_{\mathrm{HS}}(H,U)$, $0 \le k \le n-1$, are $\mathcal{F}_{t_k}$-measurable with respect to the strong Borel σ-algebra on the space of Hilbert-Schmidt operators $\mathcal{L}_{\mathrm{HS}}(H,U)$. Thus, Φ is an $L_2(\Omega \times [0,T], \mathcal{F} \otimes \mathcal{B}([0,T]), \mathrm{P} \otimes \mathrm{d}t;\ \mathcal{L}_{\mathrm{HS}}(H,U))$-valued stochastic process.

For elements $\Phi \in \mathcal{E}$ one defines the stochastic integral Int with respect to W by

$$\mathrm{Int}(\Phi)(t) := \int_0^t \Phi(s) \mathrm{d}W(s) := \sum_{k=0}^{n-1} \Phi_k(W) \left(W(t_{k+1} \wedge t) - W(t_k \wedge t) \right), \quad t \in [0,T],$$

which is independent of the representation of Φ: Since Φ_k is $\mathcal{F}_{t_k}$-measurable we have that $\Phi_k(W)$ is independent of $W(t_{k+1}) - W(t_k)$ and the right-hand side in the definition above makes sense. Furthermore, the value of Int is independent of the choice of $\bar{H}$ on which W can be realized, cf. Hairer [95, Chapter 3.4].

Since the following extension

$$\mathrm{E}\big[\|\mathrm{Int}(\Phi)(T)\|_U^2\big] = \sum_{k=0}^{n-1} \mathrm{E}\big[\,\mathrm{Tr}(\Phi_k(W) - \Phi_k^*(W))(t_{k+1} - t_k)\big] = \mathrm{E}\bigg[\int_0^T \mathrm{Tr}\ \Phi(t)\Phi^*(t)\mathrm{d}t\bigg]$$

of the *Itō isometry* to the Hilbert space setting can be shown, it turns out that

$$\mathrm{Int} : \mathcal{E} \to L_2(\Omega, \mathcal{F}, \mathrm{P}; U)$$

is an isometry. Using this isometry, the completeness of $L_2(\Omega, \mathcal{F}, \mathrm{P}; U)$, and the observation that $\mathcal{E}$ is dense in the space $L_2^{\mathrm{pr}}(\Omega \times [0,T], \mathcal{P}_{[0,T]}, \mathrm{P} \otimes \mathrm{d}t;\, \mathcal{L}_{\mathrm{HS}}(H,U))$ of all predictable $\mathcal{L}_{\mathrm{HS}}(H,U)$-valued processes, see Hairer [95, Proposition 3.59], it can be concluded that the stochastic integral Int can be uniquely defined for every process $\Phi \in L_2^{\mathrm{pr}}(\Omega \times [0,T], \mathcal{P}_{[0,T]}, \mathrm{P} \otimes \mathrm{d}t;\, \mathcal{L}_{\mathrm{HS}}(H,U))$, see Hairer [95, Corollary 3.60].

B Proofs

B.1 Proof of Lemma 3.5

On the one hand, let $\sum_{i=1}^\infty X_i < \infty$, P-a.s., with $\mathbb{1}_A$ being the indicator function of A, we have

$$\sum_{i=1}^\infty \mathrm{E}\left[\frac{X_i}{1+X_i}\right] \le \sum_{i=1}^\infty \mathrm{E}[X_i\, \mathbb{1}_{\{X_i \le 1\}} + \mathbb{1}_{\{X_i > 1\}}] = \sum_{i=1}^\infty \mathrm{E}[X_i\, \mathbb{1}_{\{X_i \le 1\}}] + \sum_{i=1}^\infty P(X_i > 1).$$

Both sums on the right-hand side are finite, due to Kolmogorov's three-series theorem, see, e.g., Shiryayev [148]. On the other hand, let $\sum_{i=1}^\infty \mathrm{E}\left[\frac{X_i}{1+X_i}\right] < \infty$, then we get

$$\begin{aligned}\sum_{i=1}^\infty P(X_i > 1) &= 2\mathrm{E}\left[\sum_{i=1}^\infty \left(\frac{1}{2}\, \mathbb{1}_{\{X_i > 1\}}\right)\right] \le 2\sum_{i=1}^\infty \mathrm{E}\left[\frac{X_i}{1+X_i}\, \mathbb{1}_{\{X_i > 1\}}\right] \\ &\le 2\sum_{i=1}^\infty \mathrm{E}\left[\frac{X_i}{1+X_i}\right] < \infty\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^\infty \mathrm{E}\left[X_i\, \mathbb{1}_{\{X_i \le 1\}}\right] &= 2\mathrm{E}\left[\sum_{i=1}^\infty \left(\frac{1}{2}\, X_i\, \mathbb{1}_{\{X_i \le 1\}}\right)\right] \le 2\sum_{i=1}^\infty \mathrm{E}\left[\frac{X_i}{1+X_i}\, \mathbb{1}_{\{X_i \le 1\}}\right] \\ &\le 2\sum_{i=1}^\infty \mathrm{E}\left[\frac{X_i}{1+X_i}\right] < \infty.\end{aligned}$$

This yields

$$\begin{aligned}\sum_{i=1}^\infty \mathrm{var}(X_i\, \mathbb{1}_{\{X_i \le 1\}}) &= \sum_{i=1}^\infty \mathrm{E}[X_i^2\, \mathbb{1}_{\{X_i \le 1\}}] - \sum_{i=1}^\infty \mathrm{E}[X_i\, \mathbb{1}_{\{X_i \le 1\}}]^2 \\ &\le \underbrace{2\sum_{i=1}^\infty \mathrm{E}\left[\frac{X_i}{1+X_i}\right]}_{<\infty} - \underbrace{\sum_{i=1}^\infty \mathrm{E}[X_i\, \mathbb{1}_{\{X_i \le 1\}}]^2}_{<\infty} < \infty\end{aligned}$$

which is equivalent to $\sum_{i=1}^\infty X_i < \infty$, P-a.s., again due to Kolmogorov's three-series theorem. $\square$

B.2 Proof of Lemma 3.6

For $r = \frac{s}{t}$ with $s, t \in \mathbb{N}$ we have

$$\mathrm{E}[X_{n,p}^r] = \mathrm{E}[X_{n,p}^{s/t}] \leq (\mathrm{E}[X_{n,p}^s])^{1/t}$$

using Jensen's inequality, see, e.g., SCHILLING [145, Theorem 12.14]. Furthermore, we have $\mathrm{E}[X_{1,p}^s] = p \leq 1$, and for $n \geq 2$

$$\begin{aligned}\mathrm{E}[X_{n,p}^s] &= \sum_{k=1}^{n} k^s \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=0}^{n-1} (k+1)^{s-1} n \binom{n-1}{k} p^{k+1} (1-p)^{n-1-k} \\ &= np\, \mathrm{E}[1 + X_{n-1,s}]^{s-1}.\end{aligned}$$

Inductively over s, we obtain

$$\mathrm{E}[X_{n,p}^s] \leq np\,(\mathrm{E}[1] + \mathrm{E}[X_{n-1,s}]^{s-1}) \leq \widetilde{c}(np + (np)^s) \leq c(1 + (np)^s),$$

for positive constants $\widetilde{c}$ and $c = c(n)$ independent of p and s. For $r \in \mathbb{Q}$ this leads to

$$\mathrm{E}[X_{n,p}^r] \leq np\,(\mathrm{E}[1] + \mathrm{E}[X_{n-1,s}]^{s-1})^{1/t} \leq \widetilde{c}(np + (np)^s)^{1/t} \leq c(1 + (np)^r)$$

and using the density of $\mathbb{Q}$ in $\mathbb{R}$ we get the result for all $r \in \mathbb{R}$. □

B.3 Proof of Lemma 3.7

It is $\mathrm{E}[S_{j,p}] = \#\nabla_j \rho_j \nu_p$ and $\mathrm{var}(S_{j,p}) = \#\nabla_j \rho_j (\nu_{2p} - \rho_j \nu_p^2)$. Using Chebyshev's inequality, see, e.g., SCHILLING [145], we get

$$P(|S_{j,p}/(\#\nabla_j \rho_j) - \nu_p| \geq \varepsilon) \leq \varepsilon^{-2} (\#\nabla_j \rho_j)^{-1} (\nu_{2p} - \rho_j \nu_p^2) \leq c(\varepsilon^{-2} 2^{-(1-\beta)jd}).$$

Since $\beta < 1$, applying the Borel-Cantelli Lemma, see, e.g., SCHILLING [145], yields almost sure convergence. Let $r > 0$ and $y_{j,k} \in \{0, 1\}$. By the equivalence of moments of Gaussian measures there exists a constant $c_1 > 0$ such that

$$\mathrm{E}\left[\left(\sum_{k\in\nabla_j} |y_{j,k} Z_{j,k}|^p\right)^r\right] \leq c_1 \left(\mathrm{E}\left[\sum_{k\in\nabla_j} |y_{j,k} Z_{j,k}|^p\right]\right)^r = c_1\, \nu_p^r \left(\sum_{k\in\nabla_j} y_{j,k}\right)^r.$$

Since $(Y_{j,k})_{k\in\nabla_j}$ and $(Z_{j,k})_{k\in\nabla_j}$ are independent this yields $\mathrm{E}[S_{j,p}^r] \leq c_1 (\nu_p)^r\, \mathrm{E}[S_{j,0}^r]$. Using Lemma 3.6 there exists a constant $c_2 > 0$ such that $\mathrm{E}[S_{j,0}^r] \leq c_2\, (\#\nabla_j \rho_j)^r$. □

B.4 Proof of Lemma 3.8

Let Z be $\mathcal{N}(0,1)$-distributed. The characteristic function φ_{S_p} of S_p is given by

$$\varphi_{S_p}(t) = \mathrm{E}[\exp(itS_p)] = \exp(\varphi_{|Z|^p}(t) - 1).$$

Furthermore, for the characteristic function $\varphi_{S_{j,p}}$ of $S_{j,p}$,

$$\begin{aligned}\varphi_{S_{j,p}}(t) &= \left(\rho_j \varphi_{|Z|^p}(t) + 1 - \rho_j\right)^{\#\nabla_j} \\ &= \left(1 + \frac{1}{\#\nabla_j}\,\rho_j \#\nabla_j\left(\varphi_{|Z|^p}(t) - 1\right)\right)^{\#\nabla_j}.\end{aligned}$$

We use (3.6) to conclude that

$$\lim_{j\to\infty} \varphi_{S_{j,p}}(t) = \varphi_{S_p}(t),$$

which yields the convergence in distribution as claimed. Suppose that $p \geq 1$ and $r > 0$. Then we take $c_1 > 0$ such that $z^{rp} \leq c_1 \exp(z)$ for every $z \geq 0$ and we put $c_2 = \mathrm{E}[\exp(|Z|)]$ to obtain

$$\mathrm{E}[S_{j,p}^r] \leq \mathrm{E}[S_{j,1}^{rp}] \leq c_1\,\mathrm{E}[\exp(S_{j,1})] = c_1(1 + \rho_j(c_2 - 1))^{\#\nabla_j}.$$

Note that the upper bound converges to $c_1 \exp(c_2 - 1)$. In the case $0 < p < 1$ we have $S_{j,p} \leq S_{j,0} + S_{j,1}$. Hence it remains to observe that $\sup_{j\geq j_0} \mathrm{E}[S_{j,0}^r] < \infty$, which follows from Lemma 3.6. □

B.5 Proof of Lemma 5.40

By (5.37) and (5.38) the stage equations (5.30) read as

$$\begin{aligned}(I - \tau\gamma_{1,1}A)w_{k,1} &= Au_k + f(t_k), \\ (I - \tau\gamma_{2,2}A)w_{k,2} &= A(u_k + \tau a_{2,1}w_{k,1}) + f(t_k + a_2\tau) + c_{2,1}w_{k,1}.\end{aligned}$$

We begin with an application of the following basic observation, that

$$I = (I - CA)^{-1}(I - CA)$$

implies

$$(I - CA)^{-1}A = -\frac{1}{C}I + \frac{1}{C}(I - CA)^{-1}.$$

It follows that

$$\begin{aligned}w_{k,1} &= \left(-\frac{1}{\tau\gamma_{1,1}}I + \frac{1}{\tau\gamma_{1,1}}(I - \tau\gamma_{1,1}A)^{-1}\right)u_k + (I - \tau\gamma_{1,1}A)^{-1}f(t_k) \\ &= -\frac{1}{\tau\gamma_{1,1}}u_k + L_{\tau,1}^{-1}\left(\frac{1}{\tau\gamma_{1,1}}u_k + f(t_k)\right).\end{aligned}$$

We denote

$$v_{k,1} = L_{\tau,1}^{-1}\left(\frac{1}{\tau\gamma_{1,1}}u_k + f(t_k)\right).$$

A similar computation for the second stage equation yields

$$\begin{aligned}
w_{k,2} &= \left(-\frac{1}{\tau\gamma_{2,2}}I + \frac{1}{\tau\gamma_{2,2}}(I-\tau\gamma_{2,2}A)^{-1}\right)(u_k + \tau a_{2,1}w_{k,1}) \\
&\qquad + (I-\tau\gamma_{2,2}A)^{-1}\left(f(t_k + a_2\tau) + c_{2,1}w_{k,1}\right) \\
&= -\frac{1}{\tau\gamma_{2,2}}\left(\left(1-\frac{a_{2,1}}{\gamma_{1,1}}\right)u_k + \tau a_{2,1}v_{k,1}\right) \\
&\qquad + L_{\tau,2}^{-1}\biggl(\frac{1}{\tau\gamma_{2,2}}\left(\left(1-\frac{a_{2,1}}{\gamma_{1,1}}\right)u_k + \tau a_{2,1}v_{k,1}\right) \\
&\qquad\qquad + f(t_k + a_2\tau) + c_{2,1}\left(-\frac{1}{\tau\gamma_{1,1}}u_k + v_{k,1}\right)\biggr) \\
&= -\frac{1}{\tau\gamma_{2,2}}\left(1-\frac{a_{2,1}}{\gamma_{1,1}}\right)u_k - \frac{a_{2,1}}{\gamma_{2,2}}v_{k,1} \\
&\qquad + L_{\tau,2}^{-1}\biggl(\left(\frac{1}{\tau\gamma_{2,2}}\left(1-\frac{a_{2,1}}{\gamma_{1,1}}\right) - \frac{c_{2,1}}{\tau\gamma_{1,1}}\right)u_k \\
&\qquad\qquad + \left(\frac{a_{2,1}}{\gamma_{2,2}} + c_{2,1}\right)v_{k,1} + f(t_k + a_2\tau)\biggr).
\end{aligned}$$

We denote

$$v_{k,2} = L_{\tau,2}^{-1}\left(\left(\frac{1}{\tau\gamma_{2,2}}\left(1-\frac{a_{2,1}}{\gamma_{1,1}}\right) - \frac{c_{2,1}}{\tau\gamma_{1,1}}\right)u_k + \left(\frac{a_{2,1}}{\gamma_{2,2}} + c_{2,1}\right)v_{k,1} + f(t_k + a_2\tau)\right)$$

and arrive at

$$\begin{aligned}
u_{k+1} &= u_k + \tau m_1\left(-\frac{1}{\tau\gamma_{1,1}}u_k + v_{k,1}\right) \\
&\qquad + \tau m_2\left(-\frac{1}{\tau\gamma_{2,2}}\left(1-\frac{a_{2,1}}{\gamma_{1,1}}\right)u_k - \frac{a_{2,1}}{\gamma_{2,2}}v_{k,1} + v_{k,2}\right) \\
&= \left(1 - \frac{m_1}{\gamma_{1,1}} - \frac{m_2}{\gamma_{2,2}}\left(1-\frac{a_{2,1}}{\gamma_{1,1}}\right)\right)u_k + \left(\tau m_1 - \tau m_2\frac{a_{2,1}}{\gamma_{2,2}}\right)v_{k,1} + \tau m_2 v_{k,2},
\end{aligned}$$

which is the claim. □

B.6 Proof of Lemma 5.64

The proof is based on a Gronwall argument. Fix $1 \le j \le k \le K$ and observe that, by induction over k,

$$\begin{aligned}
E_{\tau,j,k}(v) &= L_\tau^{-(k-j)}v \\
&\qquad + \sum_{i=0}^{k-j-1} L_\tau^{-(k-j)+i}\Bigl(\tau F\bigl(E_{\tau,j,j+i}(v)\bigr) + \sqrt{\tau}B\bigl(E_{\tau,j,j+i}(v)\bigr)\chi_{j+i}\Bigr)
\end{aligned}$$

for all $v \in \mathcal{H}_j$. Therefore, for all $v, w \in \mathcal{H}_j$, we have

$$\begin{aligned}
&\|E_{\tau,j,k}(v) - E_{\tau,j,k}(w)\|_{\mathcal{H}_k} \\
&\qquad \le \|L_\tau^{-(k-j)}v - L_\tau^{-(k-j)}w\|_{\mathcal{H}_k}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=0}^{k-j-1}\tau\left\|L_\tau^{-(k-j)+i}\Big(F\big(E_{\tau,j,j+i}(v)\big)-F\big(E_{\tau,j,j+i}(w)\big)\Big)\right\|_{\mathcal{H}_k}\\
&+\left\|\sum_{i=0}^{k-j-1}\sqrt{\tau}L_\tau^{-(k-j)+i}\Big(B\big(E_{\tau,j,j+i}(v)\big)-B\big(E_{\tau,j,j+i}(w)\big)\Big)\chi_{j+i}\right\|_{\mathcal{H}_k}\\
&=:(I)+(II)+(III).
\end{aligned}
\tag{B.1}
$$

We estimate each of the terms (I), (II), and (III) separately.

By Lemma 5.58 and the trivial fact that $\|v-w\|_{\mathcal{H}_k}=\|v-w\|_{\mathcal{H}_j}$ for all $v,w\in\mathcal{H}_j$, we have

$$
\begin{aligned}
(I)&\leq\big\|L_\tau^{-1}\big\|^{k-j}_{\mathcal{L}(D((-A)^\varrho))}\|v-w\|_{\mathcal{H}_k}\\
&\leq(1-\tau\lambda_1)^{-(k-j)}\|v-w\|_{\mathcal{H}_k}\\
&\leq\|v-w\|_{\mathcal{H}_j}.
\end{aligned}
\tag{B.2}
$$

Concerning the term (II) in (B.1), let us first concentrate on the case $\varrho_F\in(0,1)$. We use the Lipschitz condition on F in Assumption 5.47 and Lemma 5.58 to obtain

$$
\begin{aligned}
(II)&\leq\sum_{i=0}^{k-j-1}\tau\big\|L_\tau^{-(k-j)+i}(-A)^{\varrho_F}\big\|_{\mathcal{L}(D((-A)^\varrho))}\\
&\qquad\qquad\times C_F^{\mathrm{Lip}}\big\|E_{\tau,j,j+i}(v)-E_{\tau,j,j+i}(w)\big\|_{\mathcal{H}_{j+i}}\\
&\leq\sum_{i=0}^{k-j-1}\tau\frac{{\varrho_F}^{\varrho_F}}{(\tau(k-j-i))^{\varrho_F}}C_F^{\mathrm{Lip}}C^{\mathrm{Lip}}_{\tau,j,j+i}\,\|v-w\|_{\mathcal{H}_j}\\
&\leq C_F^{\mathrm{Lip}}\sum_{i=0}^{k-j-1}\frac{\tau}{(\tau(k-j-i))^{\varrho_F}}C^{\mathrm{Lip}}_{\tau,j,j+i}\,\|v-w\|_{\mathcal{H}_j}\,.
\end{aligned}
\tag{B.3}
$$

For the case that $\varrho_F\leq 0$ we get with similar arguments

$$
\begin{aligned}
(II)&\leq C_F^{\mathrm{Lip}}\sum_{i=0}^{k-j-1}\frac{\tau(-\lambda_1)^{\varrho_F}}{(1-\tau\lambda_1)^n}C^{\mathrm{Lip}}_{\tau,j,j+i}\,\|v-w\|_{\mathcal{H}_j}\\
&\leq C_F^{\mathrm{Lip}}(-\lambda_1)^{\varrho_F}\sum_{i=0}^{k-j-1}\tau C^{\mathrm{Lip}}_{\tau,j,j+i}\,\|v-w\|_{\mathcal{H}_j}\,.
\end{aligned}
\tag{B.4}
$$

Let us now look at the term (III) in (B.1). Using the independence of the stochastic increments χ_{j+i} and Eq. (5.58), we get

$$
\begin{aligned}
(\mathit{III})^2 &= \sum_{i=0}^{k-j-1} \tau \, \mathrm{E}\left[\left\| L_\tau^{-(k-j)+i}\Big(B\big(E_{\tau,j,j+i}(v)\big) - B\big(E_{\tau,j,j+i}(w)\big)\Big)\chi_{j+i}\right\|^2_{D((-A)^\varrho)}\right] \\
&\le \sum_{i=0}^{k-j-1} \tau \left\| L_\tau^{-(k-j)+i}\right\|^2_{\mathcal{L}(D((-A)^{\varrho-\varrho_B-\varrho_A/2}),\, D((-A)^\varrho))} \\
&\qquad\qquad \times \mathrm{E}\left[\left\|\Big(B\big(E_{\tau,j,j+i}(v)\big) - B\big(E_{\tau,j,j+i}(w)\big)\Big)\chi_{j+i}\right\|^2_{D((-A)^{\varrho-\varrho_B-\varrho_A/2})}\right] \\
&\le \sum_{i=0}^{k-j-1} \tau \left\| L_\tau^{-(k-j)+i}\right\|^2_{\mathcal{L}(D((-A)^{\varrho-\varrho_B-\varrho_A/2}),\, D((-A)^\varrho))} \\
&\qquad\qquad \times \mathrm{E}\left[\left\| B\big(E_{\tau,j,j+i}(v)\big) - B\big(E_{\tau,j,j+i}(w)\big)\right\|^2_{\mathcal{L}_{\mathrm{HS}}(\ell_2,\, D((-A)^{\varrho-\varrho_B-\varrho_A/2}))}\right].
\end{aligned}
$$

Concentrating first on the case $\varrho_B + \varrho_A/2 > 0$, we continue by using the Lipschitz condition on B in Assumption 5.47 and Lemma 5.58 to obtain

$$
\begin{aligned}
(\mathit{III})^2 \le& \sum_{i=0}^{k-j-1} \tau \frac{(\varrho_B + \varrho_A/2)^{2\varrho_B+\varrho_A}}{(\tau(k-j-i))^{2\varrho_B+\varrho_A}} \mathrm{Tr}(-A)^{-\varrho_A} \\
&\qquad \times (C_B^{\mathrm{Lip}})^2 \, \mathrm{E}\left[\| E_{\tau,j,j+i}(v) - E_{\tau,j,j+i}(w)\|^2_{D((-A)^\varrho)}\right] \\
\le& (C_B^{\mathrm{Lip}})^2 \, \mathrm{Tr}(-A)^{-\varrho_A} \\
&\qquad \times \sum_{i=0}^{k-j-1} \frac{\tau}{(\tau(k-j-i))^{2\varrho_B+\varrho_A}} (C^{\mathrm{Lip}}_{\tau,j,j+i})^2 \|v-w\|^2_{\mathcal{H}_j}.
\end{aligned}
\tag{B.5}
$$

In the case $\varrho_B + \varrho_A/2 \le 0$ the same arguments lead to

$$
\begin{aligned}
(\mathit{III})^2 \le& \sum_{i=0}^{k-j-1} \tau \frac{(-\lambda_1)^{2\varrho_B+\varrho_A}}{(1-\tau\lambda_1)^{2n}} \mathrm{Tr}(-A)^{-\varrho_A} \\
&\qquad \times (C_B^{\mathrm{Lip}})^2 \, \mathrm{E}\left[\| E_{\tau,j,j+i}(v) - E_{\tau,j,j+i}(w)\|^2_{D((-A)^\varrho)}\right] \\
\le& (C_B^{\mathrm{Lip}})^2 \, \mathrm{Tr}(-A)^{-\varrho_A} (-\lambda_1)^{2\varrho_B+\varrho_A} \sum_{i=0}^{k-j-1} \tau (C^{\mathrm{Lip}}_{\tau,j,j+i})^2 \|v-w\|^2_{\mathcal{H}_j}.
\end{aligned}
\tag{B.6}
$$

Now we have to consider four different cases.

Case 1. $\varrho_F \in (0,1)$ and $\varrho_B + \varrho_A/2 \in (0, 1/2)$. The combination of (B.1), (B.2), (B.3), and (B.5) yields

$$
\begin{aligned}
C^{\mathrm{Lip}}_{\tau,j,k} \le& 1 + C_F^{\mathrm{Lip}} \sum_{i=0}^{k-j-1} \frac{\tau}{(\tau(k-j-i))^{\varrho_F}} C^{\mathrm{Lip}}_{\tau,j,j+i} \\
&+ C_B^{\mathrm{Lip}} (\mathrm{Tr}(-A)^{-\varrho_A})^{1/2} \left(\sum_{i=0}^{k-j-1} \frac{\tau}{(\tau(k-j-i))^{2\varrho_B+\varrho_A}} (C^{\mathrm{Lip}}_{\tau,j,j+i})^2 \right)^{1/2}.
\end{aligned}
\tag{B.7}
$$

Note that inductively we obtain in particular the finiteness of all $C^{\text{Lip}}_{\tau,j,j+i}$. Next, we estimate the two sums over i on the right hand side of (B.7) via Hölder's inequality. Set

$$q := \frac{1}{\min\{1-\varrho_F,\,(1-\varrho_A)/2-\varrho_B\}} + 2 > 2.$$

Hölder's inequality with exponents $q/(q-1)$ and q yields

$$\begin{aligned}
&\sum_{i=0}^{k-j-1} \frac{\tau}{(\tau(k-j-i))^{\varrho_F}}\, C^{\text{Lip}}_{\tau,j,j+i} \\
&\quad\le \left(\sum_{i=0}^{k-j-1} \frac{\tau}{(\tau(k-j-i))^{\frac{\varrho_F q}{q-1}}}\right)^{\frac{q-1}{q}} \left(\sum_{i=0}^{k-j-1} \tau (C^{\text{Lip}}_{\tau,j,j+i})^q\right)^{\frac{1}{q}} \\
&\quad\le \left(\sum_{i=1}^{K} \frac{\tau}{(\tau i)^{\frac{\varrho_F q}{q-1}}}\right)^{\frac{q-1}{q}} \left(\sum_{i=0}^{k-j-1} \tau (C^{\text{Lip}}_{\tau,j,j+i})^q\right)^{\frac{1}{q}} \\
&\quad\le \left(\int_0^T t^{-\frac{\varrho_F q}{q-1}} \mathrm{d}t\right)^{\frac{q-1}{q}} \left(\sum_{i=0}^{k-j-1} \tau (C^{\text{Lip}}_{\tau,j,j+i})^q\right)^{\frac{1}{q}},
\end{aligned} \tag{B.8}$$

where the integral in the last line is finite since $\frac{\varrho_F q}{q-1} = \frac{\varrho_F}{1-1/q} < \frac{\varrho_F}{1-(1-\varrho_F)} = 1$. Similarly, applying Hölder's inequality with exponents $q/(q-2)$ and $q/2$,

$$\begin{aligned}
&\sum_{i=0}^{k-j-1} \frac{\tau}{(\tau(k-j-i))^{2\varrho_B+\varrho_A}} (C^{\text{Lip}}_{\tau,j,j+i})^2 \\
&\quad\le \left(\sum_{i=0}^{k-j-1} \frac{\tau}{(\tau(k-j-i))^{\frac{(2\varrho_B+\varrho_A)q}{q-2}}}\right)^{\frac{q-2}{q}} \left(\sum_{i=0}^{k-j-1} \tau (C^{\text{Lip}}_{\tau,j,j+i})^q\right)^{\frac{2}{q}} \\
&\quad\le \left(\int_0^T t^{-\frac{(2\varrho_B+\varrho_A)q}{q-2}} \mathrm{d}t\right)^{\frac{q-2}{q}} \left(\sum_{i=0}^{k-j-1} \tau (C^{\text{Lip}}_{\tau,j,j+i})^q\right)^{\frac{2}{q}}.
\end{aligned} \tag{B.9}$$

The integral in the last line is finite since

$$\frac{(2\varrho_B+\varrho_A)q}{q-2} = \frac{(2\varrho_B+\varrho_A)}{1-2/q} < \frac{(2\varrho_B+\varrho_A)}{1-(1-\varrho_A-2\varrho_B)} = 1.$$

Combining (B.7), (B.8), (B.9) and using the equivalence of norms in $\mathbb{R}^3$, we obtain

$$(C^{\text{Lip}}_{\tau,j,k})^q \le C_0 \left(1 + \sum_{i=0}^{k-j-1} \tau (C^{\text{Lip}}_{\tau,j,j+i})^q\right), \tag{B.10}$$

with a constant C_0 that depends only on A, F, B, ϱ_A, ϱ_B, ϱ_F and T. Since (B.10) holds for arbitrary $K \in \mathbb{N}$ and $1 \le j \le k \le K$, we can apply a discrete version of Gronwall's lemma and obtain

$$(C^{\text{Lip}}_{\tau,j,k})^q \le e^{(k-j)\tau C_0} C_0 \le e^{T C_0} C_0.$$

for all $1 \le j \le k \le K$, $K \in \mathbb{N}$ and $\tau = T/K$. It follows that the assertion of the proposition holds in this first case with

$$C := (e^{TC_0} C_0)^{1/q}.$$

Case 2. $\varrho_F \le 0$ and $\varrho_B + \varrho_A/2 \le 0$. A combination of (B.1) with (B.2), (B.4), and (B.6) leads to

$$\begin{aligned} C^{\mathrm{Lip}}_{\tau,j,k} \le 1 + C^{\mathrm{Lip}}_F (-\lambda_1)^{\varrho_F} \sum_{i=0}^{k-j-1} \tau C^{\mathrm{Lip}}_{\tau,j,j+i} \\ + C^{\mathrm{Lip}}_B (\mathrm{Tr}(-A)^{-\varrho_A})^{1/2} (-\lambda_1)^{\varrho_B+\varrho_A/2} \Big(\sum_{i=0}^{k-j-1} \tau (C^{\mathrm{Lip}}_{\tau,j,j+i})^2 \Big)^{1/2}. \end{aligned}$$

Applying Hölder's inequality with exponent $q_2 := 2$ to estimate the first sum over i on the right hand side, we get

$$\begin{aligned} C^{\mathrm{Lip}}_{\tau,j,k} \le 1 + C^{\mathrm{Lip}}_F (-\lambda_1)^{\varrho_F} T^{1/2} \Big(\sum_{i=0}^{k-j-1} \tau (C^{\mathrm{Lip}}_{\tau,j,j+i})^2 \Big)^{1/2} \\ + C^{\mathrm{Lip}}_B (\mathrm{Tr}(-A)^{-\varrho_A})^{1/2} (-\lambda_1)^{\varrho_B+\varrho_A/2} \Big(\sum_{i=0}^{k-j-1} \tau (C^{\mathrm{Lip}}_{\tau,j,j+i})^2 \Big)^{1/2}, \end{aligned}$$

which leads to

$$(C^{\mathrm{Lip}}_{\tau,j,k})^2 \le C \Big(1 + \sum_{i=0}^{k-j-1} \tau (C^{\mathrm{Lip}}_{\tau,j,j+i})^2 \Big),$$

where the constant $C \in (0,\infty)$ depends only on A, F, B, ϱ_A, ϱ_B, ϱ_F and T. As in Case 1, an application of Gronwall's lemma proves the assertion in this second case.

Case 3. $\varrho_F \in (0,1)$ and $\varrho_B + \varrho_A/2 \le 0$. In this situation, we combine (B.1) with (B.2), (B.3) and (B.6) to get

$$\begin{aligned} C^{\mathrm{Lip}}_{\tau,j,k} \le 1 + C^{\mathrm{Lip}}_F \sum_{i=0}^{k-j-1} \frac{\tau}{(\tau(k-j-i))^{\varrho_F}} C^{\mathrm{Lip}}_{\tau,j,j+i} \\ + C^{\mathrm{Lip}}_B (\mathrm{Tr}(-A)^{-\varrho_A})^{1/2} (-\lambda_1)^{\varrho_B+\varrho_A/2} \Big(\sum_{i=0}^{k-j-1} \tau (C^{\mathrm{Lip}}_{\tau,j,j+i})^2 \Big)^{1/2}. \end{aligned}$$

Setting

$$q_3 := \frac{1}{1-\varrho_F} + 2$$

and following the line of argumentation from the first case with q_3 instead of q we reach our goal also in this situation.

Case 4. $\varrho_F \le 0$ and $\varrho_B + \varrho_A/2 \in (0,1/2)$. Combine (B.1), (B.2), (B.4) and (B.5) to get

$$\begin{aligned} C^{\mathrm{Lip}}_{\tau,j,k} \le 1 + C^{\mathrm{Lip}}_F (-\lambda_1)^{\varrho_F} \sum_{i=0}^{k-j-1} \tau C^{\mathrm{Lip}}_{\tau,j,j+i} \\ + C^{\mathrm{Lip}}_B (\mathrm{Tr}(-A)^{-\varrho_A})^{1/2} \left(\sum_{i=0}^{k-j-1} \frac{\tau}{(\tau(k-j-i))^{2\varrho_B+\varrho_A}} (C^{\mathrm{Lip}}_{\tau,j,j+i})^2 \right)^{1/2}. \end{aligned}$$

Arguing as in the third case with

$$q_4 := \frac{1}{1/2 - (\varrho_B + \varrho_A/2)} + 2$$

instead of q_3, we get the estimate we need to finish the proof. $\square$

B.7 Proof of Lemma 5.68

We start with the estimate

$$\begin{aligned}\|\widehat{w}_{k,i}\|_{B^s_q(L_q(\mathcal{O}))} &= \left\|L^{-1}_{\tau,i} R_{\tau,k,i}(\widetilde{u}_k, \widetilde{w}_{k,1}, ..., \widetilde{w}_{k,i-1})\right\|_{B^s_q(L_q(\mathcal{O}))} \\ &\le \|L^{-1}_{\tau,i}\|_{\mathcal{L}(L_2(\mathcal{O}), B^s_q(L_q(\mathcal{O})))} \left\|R_{\tau,k,i}(\widetilde{u}_k, \widetilde{w}_{k,1}, ..., \widetilde{w}_{k,i-1})\right\|_{L_2(\mathcal{O})}.\end{aligned}$$

The Lipschitz continuity of $R_{\tau,k,i}$ implies the linear growth property

$$\begin{aligned}&\left\|R_{\tau,k,i}(\widetilde{u}_k, \widetilde{w}_{k,1}, ..., \widetilde{w}_{k,i-1})\right\|_{L_2(\mathcal{O})} \\ &\le C^{\mathrm{Lip,R}}_{\tau,k,(i)} \left(\|\widetilde{u}_k\|_{L_2(\mathcal{O})} + \sum_{j=1}^{i-1} \|\widetilde{w}_{k,j}\|_{L_2(\mathcal{O})} \right) + \left\|R_{\tau,k,i}(0, ..., 0)\right\|_{L_2(\mathcal{O})} \\ &\le \max\left\{C^{\mathrm{Lip,R}}_{\tau,k,(i)}, \left\|R_{\tau,k,i}(0, ..., 0)\right\|_{L_2(\mathcal{O})}\right\} \left(1 + \|\widetilde{u}_k\|_{L_2(\mathcal{O})} + \sum_{j=1}^{i-1} \|\widetilde{w}_{k,j}\|_{L_2(\mathcal{O})}\right) \\ &\le \max\left\{C^{\mathrm{Lip,R}}_{\tau,k,(i)}, \left\|R_{\tau,k,i}(0, ..., 0)\right\|_{L_2(\mathcal{O})}\right\} \left(1 + \|u_k\|_{L_2(\mathcal{O})} + \sum_{j=1}^{i-1} \|w_{k,j}\|_{L_2(\mathcal{O})}\right. \\ &\qquad\qquad \left. + \|u_k - \widetilde{u}_k\|_{L_2(\mathcal{O})} + \sum_{j=1}^{i-1} \|w_{k,j} - \widetilde{w}_{k,j}\|_{L_2(\mathcal{O})}\right).\end{aligned}$$

As before, the Lipschitz continuity of $L^{-1}_{\tau,i} R_{\tau,k,i}$ implies

$$\begin{aligned}&\|w_{k,i}\|_{L_2(\mathcal{O})} = \left\|L^{-1}_{\tau,i} R_{\tau,k,i}(u_k, w_{k,1}, \dots, w_{k,i-1})\right\|_{L_2(\mathcal{O})} \\ &\le \max\left\{C^{\mathrm{Lip}}_{\tau,k,(i)}, \left\|L^{-1}_{\tau,i} R_{\tau,k,i}(0, ..., 0)\right\|_{L_2(\mathcal{O})}\right\} \left(1 + \|u_k\|_{L_2(\mathcal{O})} + \sum_{j=1}^{i-1} \|w_{k,j}\|_{L_2(\mathcal{O})}\right).\end{aligned}$$

By induction, we estimate

$$\begin{aligned}&1 + \|u_k\|_{L_2(\mathcal{O})} + \sum_{j=1}^{i-1} \|w_{k,j}\|_{L_2(\mathcal{O})} \\ &\le \prod_{l=1}^{i-1} \left(1 + \max\left\{C^{\mathrm{Lip}}_{\tau,k,(l)}, \left\|L^{-1}_{\tau,l} R_{\tau,k,l}(0, \dots, 0)\right\|_{L_2(\mathcal{O})}\right\}\right) \left(1 + \|u_k\|_{L_2(\mathcal{O})}\right).\end{aligned}$$

Note that

$$\|\widetilde{w}_{k,i} - \widehat{w}_{k,i}\|_{L_2(\mathcal{O})} \le \|\widetilde{w}_{k,i} - \widehat{w}_{k,i}\|_{H^\nu(\mathcal{O})} \le \varepsilon_{k,i}.$$

This enables us to follow similar lines as in the proof of Theorem 5.24. We estimate

$$\begin{aligned}
&\|u_k - \widetilde{u}_k\|_{L_2(\mathcal{O})} + \sum_{j=1}^{i-1} \|w_{k,j} - \widetilde{w}_{k,j}\|_{L_2(\mathcal{O})} \\
&\leq (1 + C^{\mathrm{Lip}}_{\tau,k,(i-1)}) \left(\|u_k - \widetilde{u}_k\|_{L_2(\mathcal{O})} + \sum_{j=1}^{i-2} \|w_{k,j} - \widetilde{w}_{k,j}\|_{L_2(\mathcal{O})} \right) \\
&\qquad + \Big\| L^{-1}_{\tau,i-1} R_{\tau,k,i-1}(\widetilde{u}_k, \widetilde{w}_{k,1}, \ldots, \widetilde{w}_{k,i-2}) \\
&\qquad\qquad - \big[L^{-1}_{\tau,i-1} R_{\tau,k,i-1}(\widetilde{u}_k, \widetilde{w}_{k,1}, \ldots, \widetilde{w}_{k,i-2}) \big]_{\varepsilon_{k,i-1}} \Big\|_{L_2(\mathcal{O})} \\
&\leq (1 + C^{\mathrm{Lip}}_{\tau,k,(i-1)}) \left(\|u_k - \widetilde{u}_k\|_{L_2(\mathcal{O})} + \sum_{j=1}^{i-2} \|w_{k,j} - \widetilde{w}_{k,j}\|_{L_2(\mathcal{O})} \right) + \varepsilon_{k,i-1}
\end{aligned}$$

and conclude by induction

$$\begin{aligned}
&\|u_k - \widetilde{u}_k\|_{L_2(\mathcal{O})} + \sum_{j=1}^{i-1} \|w_{k,j} - \widetilde{w}_{k,j}\|_{L_2(\mathcal{O})} \\
&\qquad \leq \left(\prod_{l=1}^{i-1} (1 + C^{\mathrm{Lip}}_{\tau,k,(l)}) \right) \|u_k - \widetilde{u}_k\|_{L_2(\mathcal{O})} + \sum_{j=1}^{i-1} \varepsilon_{k,j} \prod_{l=j+1}^{i-1} (1 + C^{\mathrm{Lip}}_{\tau,k,(l)}).
\end{aligned}$$

The proof is finished by

$$\|u_k - \widetilde{u}_k\|_{L_2(\mathcal{O})} \leq \sum_{j=0}^{k-1} \left(\prod_{l=j+1}^{k-1} (C'_{\tau,l,(0)} - 1) \right) \sum_{i=1}^{S} C'_{\tau,j,(i)} \varepsilon_{j,i},$$

which is shown as in Theorem 5.24. $\square$

Figures

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Nomenclature

$\mathbb{Z}$	all integer numbers	$\mathbb{R}$	all real numbers
$\mathbb{N}$	all positive integer numbers	$\mathbb{R}_+$	all positive real numbers
$\mathbb{N}_0$	all nonnegative integer numbers	$\mathbb{C}$	all complex numbers
$\mathbb{Q}$	all rational numbers		

d	dimension: $d \in \mathbb{N}$
$\mathbb{R}^d$	Euclidean d-plane
$\mathcal{O}$	domain: open and connected subset of $\mathbb{R}^d$
$\mathbb{1}_M$	indicator function: $\mathbb{1}_M(x) = 1$ if $x \in M$ and $\mathbb{1}_M(x) = 0$ otherwise
$A \preceq B$	$\forall A, B : M \to [0,\infty] \quad \exists c = c(A,B) > 0 \quad \forall m \in M \,:\, A(m) \leq cB(m)$
$A \succeq B$	$B \preceq A$
$A \asymp B$	$B \preceq A$ and $A \preceq B$
$\mathrm{Bin}(n,p)$	Binomial distribution with n trials and success probability p
$\Gamma(t)$	Euler's Gamma function: $\Gamma(t) := \int_{\mathbb{R}_+} x^{t-1}\, e^{-x}\, \lambda(\mathrm{d}x)$, $t > 0$
$\delta_{i,j}$	Kronecker delta symbol: $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise
λ^d, $\mathrm{d}x$	Lebesgue measure
$\mathcal{N}(m,\sigma^2)$	normal distribution with mean m and variance σ^2
Φ_{cdf}	$\Phi_{\mathrm{cdf}}(t) = (2\pi)^{-1/2} \int_{-\infty}^{t} e^{-x^2/2}\, \mathrm{d}x$
$\mathrm{supp}(f)$	support: the complement of the largest open set on which f vanishes
$\mathrm{Tr}(T)$	trace of T
(V, U, V^*)	Gel'fand triple, rigged Hilbert space
$(X, \mathcal{A}, \mu)$	σ-finite measure space
$(\Omega, \mathcal{F}, \mathrm{P})$	complete probability space

$B^s_q(L_p(\mathcal{O}))$	2.2.1	$L_p(\mathcal{O})$	A.2
$B^{s,\mathbf{a}}_q(L_p(\mathcal{O}))$	2.2.2	$L_p(X,\mathcal{A},\mu;G)$	A.2
$\mathcal{C}^\infty_0(\mathcal{O})$	2.2	$\mathcal{L}(G_1,G_2)$	A.1
$\mathcal{C}_{\mathrm{W}}(\mathbb{R}_+,\mathbb{R})$	A.4.2	$\mathcal{L}_{\mathrm{HS}}(U_1,U_2)$	A.1.2
$\mathcal{D}(\mathcal{O})$, $\mathcal{D}'(\mathcal{O})$	A.3	ℓ_p	A.2
$H^s(\mathcal{O})$	2.2.1	$\mathcal{S}(\mathbb{R}^d)$, $\mathcal{S}'(\mathbb{R}^d)$	A.3
$H^{s,\mathbf{a}}(\mathbb{R}^d)$	2.2.2	$W^m_p(\mathcal{O})$	2.2
$H^{\mathbf{t},\ell}(\mathcal{O})$	2.2.3	$W^s_p(\mathcal{O})$	2.2.1
$L_{1,\mathrm{loc}}(\mathcal{O})$	A.3	$\mathring{W}^s_p(\mathcal{O})$	2.2.1

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Erklärung

Ich versichere, dass die vorliegende Arbeit „Adaptive Wavelet Methoden für eine Klasse von stochastischen partiellen Differentialgleichungen“ von mir eigenständig und nur unter Verwendung der angegebenen Quellen und Hilfsmittel angefertigt wurde.

Die vorliegende Arbeit wurde so oder in einer ähnlichen Form bei keiner weiteren Hochschule als Dissertation eingereicht und sie diente keinen sonstigen Prüfungszwecken.

Stefan Kinzel