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Semiclassical Analysis of Schrödinger Operators on Closed Manifolds and Symmetry Reduction

Dissertation

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Zusammenfassung

Sei M eine geschlossene zusammenhängende Riemann'sche Mannigfaltigkeit. Im ersten Teil der vorliegenden Arbeit entwickeln wir einen Funktionalkalkül für Operatoren der Form $f_h(P(h))$ im Rahmen der semiklassischen Pseudodifferentialoperatoren, wobei $\{f_h\}_{h\in(0,1]} \subset \mathrm{C}^\infty_\mathrm{c}(\mathbb{R})$ eine Familie von h-abhängigen Funktionen ist, die gewissen Regularitätsbedingungen genügt, und $P(h)$ entweder einen geeigneten selbstadjungierten semiklassischen Pseudodifferentialoperator in $\mathrm{L}^2(\mathbb{R}^n)$ oder einen Schrödinger-Operator in $\mathrm{L}^2(M)$ bezeichnet. Mit Hilfe unserer Ergebnisse lassen sich semiklassische Spurformeln mit Restgliedabschätzungen beweisen, die gut dafür geeignet sind, Bereiche im Spektrum mit einer Breite der Ordnung h^δ zu untersuchen, wobei $0 \leq \delta < \frac{1}{2}$. Der zweite Teil der Arbeit behandelt die Spektraltheorie und Quantenergodizität von Schrödinger-Operatoren auf M unter der Voraussetzung, dass das zugrunde liegende Hamilton'sche System gewisse Symmetrien besitzt. Genauer gesagt beweisen wir eine verallgemeinerte äquivariante Version des semiklassischen Weyl-Gesetzes mit Restgliedabschätzung unter der Annahme, dass auf M eine kompakte, zusammenhängende Lie-Gruppe G isometrisch und effektiv wirkt. Wir verwenden dazu einen Satz aus dem ersten Teil dieser Arbeit sowie kürzlich erzielte Ergebnisse zu singulären äquivarianten Asymptotiken, und leiten daraus ein äquivariantes Quantenergodizitätstheorem ab, sofern der Symmetrie-reduzierte Hamilton'sche Fluss auf dem Hauptstratum der singulären symplektischen Reduktion von M ergodisch ist. Unter anderem erhalten wir eine äquivariante Version des Shnirelman-Zelditch-Colin-de-Verdière-Theorems, sowie einen darstellungstheoretischen Gleichverteilungssatz. In dem Fall, dass M/G eine Orbifaltigkeit ist, erzielte Kordyukov vor Kurzem ähnliche Ergebnisse. Ist G die triviale Gruppe, so erhalten wir die bekannten klassischen Resultate.

Summary

Let M be a closed connected Riemannian manifold. In the first part of this thesis, we develop a functional calculus for operators of the form $f_h(P(h))$ within the theory of semiclassical pseudodifferential operators, where $\{f_h\}_{h\in(0,1]} \subset \mathrm{C}^\infty_\mathrm{c}(\mathbb{R})$ denotes a family of h-dependent functions satisfying some regularity conditions, and $P(h)$ is either an appropriate self-adjoint semiclassical pseudodifferential operator in $\mathrm{L}^2(\mathbb{R}^n)$ or a Schrödinger operator in $\mathrm{L}^2(M)$. Our results lead to semiclassical trace formulas with remainder estimates that are well-suited for studying spectral windows of width of order h^δ, where $0 \leq \delta < \frac{1}{2}$. In the second part of the thesis, we study the spectral and quantum ergodic properties of Schrödinger operators on M in case that the underlying Hamiltonian system possesses certain symmetries. More precisely, if M carries an isometric and effective action of a compact connected Lie group G, we prove a generalized equivariant version of the semiclassical Weyl law with an estimate for the remainder, using a theorem from the first part of this thesis and relying on recent results on singular equivariant asymptotics. We then deduce an equivariant quantum ergodicity theorem under the assumption that the symmetry-reduced Hamiltonian flow on the principal stratum of the singular symplectic reduction of M is ergodic. In particular, we obtain an equivariant version of the Shnirelman-Zelditch-Colin-de-Verdière theorem, as well as a representation theoretic equidistribution theorem. If M/G is an orbifold, similar results were recently obtained by Kordyukov. When G is trivial, one recovers the classical results.

Semiclassical Analysis of Schrödinger Operators on Closed Manifolds and Symmetry Reduction

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Contents

Introduction

This thesis consists of two parts, *Semiclassical functional calculus for h-dependent functions* and *Semiclassical analysis and symmetry reduction*, each of which has its own introductory chapter named *Overview*. The heart of the thesis is the second part. Part I represents the outcome of a program whose initial goal was solely to provide a detailed explanation and foundation of the methods of proof used in Part II, and which then developed a life of its own. The tools from the first part are not restricted in their applications to those in the second part, which is why Part I is not just a chapter.

The reader who is interested in the spectral theory of invariant Schrödinger operators, group actions and representations, (singular) symplectic reduction, oscillatory integrals, and the combination of these topics in relation with quantum ergodicity is encouraged to begin reading this thesis with the second part. On the other hand, if one prefers to dwell on semiclassical spectral functional calculus and trace formulas, or insists on a logically correct order of introducing the necessary tools and background material, Part I is the natural choice to start with.

The unifying concept that lies behind both Parts I and II is *spectral asymptotics*. This means that all main theorems proved in this thesis are of the form

$$\text{interesting quantity} \;=\; \text{leading term} \;+\; \text{remainder term},$$

where the remainder term becomes negligible when passing to a certain limit related to the spectrum of an operator or a family of operators, and the rate of the convergence is also of interest. Semiclassical analysis was invented to deal precisely with such problems, and therefore we shall use it as our technical framework. As indicated already, the main object of study in both parts is a semiclassical Schrödinger operator on a closed connected Riemannian manifold, and the main difference between the two parts is the fact that we assume only in Part II that the underlying manifold carries a certain Lie group action with respect to which the considered Schrödinger operator is invariant. We point out that the *Overview* chapters in Parts I and II are not just introductory texts, but essential content in the sense that they introduce the general setup and notation for the whole part, respectively. The detailed explanations of the required concepts are then given in the *Background* chapters. The *Background* chapters from both Part I and Part II are relevant to Part II, which combines methods from several different areas of mathematics such as spectral analysis, Lie group theory, ergodic theory, symplectic geometry, and mathematical physics.

The material presented in this thesis is an amalgamation of the contents of the arXiv Preprint 1507.06214 by the author and the arXiv Preprints 1508.03540, 1508.07381 by Pablo Ramacher and the author.

Conventions and notation

A selection of symbols that appear frequently in this thesis can be found in the Glossary on page 135. Not all symbols in the Glossary are exclusively used for the purpose explained there; instead, the Glossary entries hold *unless stated otherwise*. For example, in large parts of this thesis, M denotes a compact connected Riemannian manifold without boundary, but in some paragraphs the same symbol denotes just any smooth manifold, and this is stated explicitly in those paragraphs.

Additional global conventions and notations

Let us collect a few basic notations and conventions that fit neither in the Glossary nor in the Chapters 2 and 6. In this thesis, in agreement with common literature, a *smooth manifold* is a paracompact Hausdorff space locally homeomorphic to real euclidean space by charts whose transition maps are smooth, together with a collection of equivalent atlases. The word *smooth* and the symbol C^∞ mean "infinitely many times (partially) differentiable". A *closed* smooth manifold is a compact smooth manifold without boundary. A Riemannian manifold is a smooth manifold X whose tangent bundle TX is equipped with a Riemannian metric, and the metric is used to identify TX with the co-tangent bundle T^*X. More precisely, if $g(x) : T_xX \times T_xX \to \mathbb{R}$ denotes the symmetric non-degenerate bilinear form defined on T_xX for each $x \in X$ by the Riemannian metric, the identification is given by the isomorphism

$$\begin{aligned} T_xX &\xrightarrow{\cong} T_x^*X, \\ v &\longmapsto \big(v' \mapsto g(x)(v, v')\big). \end{aligned}$$

Under this identification, T^*X carries a Riemannian metric, too, by setting for $\xi, \xi' \in T_x^*X$

$$g^*(x)(\xi, \xi') := g(v, v'), \qquad \text{where } g(x)(v, w) = \xi(w),\ g(x)(v', w) = \xi'(w)\ \forall\ w \in T_xX.$$

Since Riemannian metrics are usually denoted by g or h, but g denotes an element of a Lie group in large parts of this thesis and h is the semiclassical parameter, we shall avoid writing down explicit names for occurring Riemannian metrics. In most cases, the notation

$$\|\xi\|_x^2 := g^*(x)(\xi, \xi), \qquad \xi \in T_x^*X \tag{0.0.1}$$

suffices for our purposes. In case that X is a topological space, a smooth manifold, or a Riemannian manifold, function vector spaces like the continuous functions $\mathrm{C}(X)$, the smooth functions $\mathrm{C}^\infty(X)$, and the equivalence classes of square-integrable functions $\mathrm{L}^2(X)$ with respect to the Riemannian volume density, are supposed to be defined over the field of complex numbers, that is the functions take values in $\mathbb{C}$. To consider a different co-domain E, we write $\mathrm{C}(X, E)$ and similarly $\mathrm{C}^\infty(X, E)$ and so on. Spaces of compactly supported functions are decorated with the subscript $_c$, for example $\mathrm{C}_c(X)$, $\mathrm{C}^\infty_\mathrm{c}(X)$. When it does not cause problems, we sometimes confuse the equivalence classes in L^2-spaces with functions representing them, and similarly we omit the related words *essential* and *almost all*. For example, we consider pointwise multiplication operators in L^2-spaces which should actually be called essential multiplication operators. In general, we make no difference in our notation between a function and the pointwise multiplication operator defined by that function in some operator space. For two vector spaces V, W, we write $\mathcal{L}(V, W)$ for the linear maps $V \to W$. If V, W are normed vector spaces, we write $\mathcal{B}(V, W)$ for the bounded linear operators $V \to W$ and we set

$$\mathcal{B}(V) := \mathcal{B}(V, V).$$

If $\varphi \in \mathrm{C}^\infty_\mathrm{c}(U)$, where U is an open subset of a smooth manifold X, we consider φ as a smooth function on X without mentioning the extension by zero explicitly. Similarly, we consider a function in $\mathrm{C}^\infty(X)$ as an element of $\mathrm{C}^\infty(TX)$ or $\mathrm{C}^\infty(T^*X)$ without explicitly mentioning the composition with the (co-)tangent bundle projection. For a chart $\gamma : U \to V$, $V \subset \mathbb{R}^n$, and a function $f \in \mathrm{C}^\infty(T^*U)$, we write $f \circ (\gamma^{-1}, (\partial\gamma^{-1})^T)$ for the composition of f with γ^{-1} in the manifold variable and the adjoint of its derivative in the co-tangent space variable. In general, 1_S denotes the constant function with value 1 on a set S, and $\mathbf{1}_S$ denotes the identity map $S \to S$. For a subset $T \subset S$, we write $S - T$ for the set of elements contained in S but not in T. If S is a finite set, the number of its elements is denoted by $\#S$. The dot $\cdot$ can denote either scalar multiplication or the standard inner product in $\mathbb{R}^n$. The latter will sometimes be alternatively written as $\langle \cdot, \cdot \rangle$, not to be confused with the notation $\langle \cdot \rangle := \sqrt{1 + \| \cdot \|^2}$ where the argument is only one vector. In remainder estimates, we occasionally write $\mathrm{O}_\bullet(\cdots)$ to emphasize that the implicit constants in the estimate depend on $\bullet$. All partitions of unity occurring in this thesis are supposed to be smooth. Sometimes, redundancies will occur in the text as we repeat attributes that are in fact implicit in our global notation.

Part I

Semiclassical functional calculus for h-dependent functions

Chapter 1

Overview

1.1 Motivation and setup

The functional calculus for unbounded self-adjoint operators combines well with the symbolic calculus for semiclassical pseudodifferential operators, leading to trace formulas that can be used to study the spectrum of such operators within a spectral window of fixed h-independent width. Here, $h \in (0, 1]$ is the *semiclassical parameter*, the ubiquitous global variable in the world of semiclassical analysis, a brief introduction to which is given in Section 2.1. To explain things more precisely, let X be either the euclidean space $\mathbb{R}^n$ or a closed connected Riemannian manifold of dimension n, endowed with the Riemannian measure dX, and $P(h)$ be the self-adjoint extension of an essentially self-adjoint semiclassical pseudodifferential operator in $\mathrm{L}^2(X)$ with real-valued semiclassical principal symbol p. Given a function $f \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R})$, and assuming that p satisfies reasonable technical ellipticity conditions, $f(P(h))$ extends to a bounded semiclassical pseudodifferential operator on $\mathrm{L}^2(X)$ with semiclassical principal symbol $f \circ p$, see [17, 63]. Provided that one has chosen f and p appropriately such that $f(P(h))$ is of trace class, one obtains the asymptotic trace formula

$$(2\pi h)^n \operatorname{tr}_{\mathrm{L}^2(X)} f(P(h)) = \int\limits_{T^*X} f \circ p \; d(T^*X) + \mathrm{O}(h) \qquad \text{as } h \to 0, \tag{1.1.1}$$

where $d(T^*X)$ is the volume form defined by the canonical symplectic form on the co-tangent bundle T^*X. Now, suppose we are in the situation that there are numbers $E \in \mathbb{R}$ and $c, \varepsilon > 0$ such that for each $h \in (0, 1]$ the following holds: $p^{-1}([E-\varepsilon, E+c+\varepsilon])$ is compact, E and $E+c$ are regular values of p, and $P(h)$ has discrete spectrum in $[E-\varepsilon, E+c+\varepsilon]$ consisting of only finitely many eigenvalues $\{E_j(h)\}_{j \in J(h) \subset \mathbb{N}}$. Then, (1.1.1) can be applied for each $f \in \mathrm{C}^\infty_{\mathrm{c}}([E-\varepsilon/2, E+c+\varepsilon/2])$, and by approximating the characteristic function of $[E, E+c]$ with such functions, one gets the semiclassical Weyl law

$$(2\pi h)^n \#\big\{j \in J(h) : E_j(h) \in [E, E+c]\big\} = \operatorname{vol}_{T^*X}\big(p^{-1}([E, E+c])\big) + \mathrm{o}(1) \tag{1.1.2}$$

as $h \to 0$, see [17, Cor. 9.7] and [63, Thm. 14.11]. In general, it is more desirable to study spectral windows of shrinking width of the form

$$[E, E+c(h)], \qquad \lim_{h \to 0} c(h) = 0,$$

since in this case the leading term in Weyl-type formulas as above turns into an integral over the compact hypersurface $p^{-1}(\{E\}) \subset T^*X$ with respect to the induced Liouville measure, and one may drop some technical hypotheses which would be necessary without the localization to such a hypersurface. However, this is not possible in the functional calculus approach sketched above, where the function f is fixed and independent of h.

In this first part of the thesis, we shall develop a semiclassical functional calculus for operators of the form $f_h(P(h))$ within the theory of semiclassical pseudodifferential operators, where $f_h \in \mathrm{C}^\infty_\mathrm{c}(\mathbb{R})$ is explicitly allowed to depend on $h \in (0,1]$.

The largest class of h-dependent functions that we will consider is given by $\bigcup_{\delta\in[0,\frac{1}{2})} \mathcal{S}^{\mathrm{comp}}_\delta$, where for each $\delta \in [0,\frac{1}{2})$ the symbol $\mathcal{S}^{\mathrm{comp}}_\delta$ denotes the set of all families $\{f_h\}_{h\in(0,1]} \subset \mathrm{C}^\infty_\mathrm{c}(\mathbb{R})$ such that

(1) $\{f_h\}_{h\in(0,1]}$ defines an element of the semiclassical symbol class $S_{h;\delta}(1_\mathbb{R})$, meaning that

$$\left\| f_h^{(j)} \right\|_\infty = \mathrm{O}(h^{-\delta j}) \qquad \text{as } h \to 0, \quad j = 0,1,2,\dots,$$

where $f_h^{(j)}$ denotes the j-th derivative of f_h;

(2) the diameter of the support of f_h does not grow faster than polynomially in h^{-1} as $h \to 0$.

The second property means that there is some $N \geq 0$ such that $\mathrm{diam}(\mathrm{supp}\, f_h) = \mathrm{O}(h^{-N})$ as $h \to 0$. This is a very mild technical condition; in usual applications the diameter of the support of f_h will be bounded or even tend to zero as $h \to 0$. It is therefore convenient to introduce also the subset of $\mathcal{S}^{\mathrm{comp}}_\delta$ given by

$$\mathcal{S}^{\mathrm{bcomp}}_\delta := \big\{ \{f_h\}_{h\in(0,1]} \in \mathcal{S}^{\mathrm{comp}}_\delta : \exists \text{ compact interval } I \subset \mathbb{R} \text{ with } \mathrm{supp}\, f_h \subset I \ \forall\, h \in (0,1] \big\}.$$

The class $\mathcal{S}^{\mathrm{bcomp}}_\delta$ is technically easier to handle and we stress again that the loss of generality from $\mathcal{S}^{\mathrm{comp}}_\delta$ to $\mathcal{S}^{\mathrm{bcomp}}_\delta$ seems to be irrelevant to most applications. In the following, we will use a shorter notation and just write $f_h \in \mathcal{S}^{\mathrm{bcomp}}_\delta$ or $f_h \in \mathcal{S}^{\mathrm{comp}}_\delta$.

1.1.1 Goals of Part I

We pursue two main goals. The first is to prove that $f_h(P(h))$ is a semiclassical pseudodifferential operator, provided that $f_h \in \mathcal{S}^{\mathrm{comp}}_\delta$ and $P(h)$ is the self-adjoint extension of an appropriate essentially self-adjoint semiclassical pseudodifferential operator in $\mathbb{R}^n$. This involves relating the abstract functional calculus to the semiclassical symbolic calculus with suitable estimates. Our second goal is to provide a detailed treatise of the semiclassical functional calculus for Schrödinger operators on a closed connected n-dimensional Riemannian manifold M of the form

$$P(h) = -h^2\Delta + V, \qquad V \in \mathrm{C}^\infty(M,\mathbb{R}), \qquad P(h) : \mathrm{H}^2(M) \to \mathrm{L}^2(M), \tag{1.1.3}$$

where V is a real-valued potential, Δ is the unique self-adjoint extension of the Laplace-Beltrami operator $\breve{\Delta} : \mathrm{C}^\infty(M) \to \mathrm{C}^\infty(M) \subset \mathrm{L}^2(M)$, and $\mathrm{H}^2(M)$ denotes the second Sobolev space. Thus, $P(h)$ is the unique self-adjoint extension of the essentially self-adjoint operator

$$\breve{P}(h) := -h^2\breve{\Delta} + V, \qquad \breve{P}(h) : \mathrm{C}^\infty(M) \to \mathrm{C}^\infty(M) \subset \mathrm{L}^2(M).$$

As is well known, the spectrum of $P(h)$ is discrete for each $h \in (0,1]$ and accumulates only at $+\infty$, see [63, Chapter 14]. We write

$$p(x,\xi) := \|\xi\|_x^2 + V(x), \qquad p : T^*M \to \mathbb{R},$$

for the Hamiltonian function associated to $P(h)$, which represents its semiclassical principal symbol. The notations above will be used in all following chapters. Apart from establishing that $f_h(P(h))$ is a semiclassical pseudodifferential operator when $f_h \in \mathcal{S}_\delta^{\mathrm{comp}}$, we are interested in an explicit expression for $f_h(P(h))$ in terms of local quantizations of symbol functions in $\mathbb{R}^n$, which can be used flexibly to prove new semiclassical trace formulas that are well-suited for studying spectral windows of width of order h^δ, where $0 \leq \delta < \frac{1}{2}$. This involves relating the abstract functional calculus to the local semiclassical symbolic calculus with trace norm remainder estimates, which we do under the assumption that $f_h \in \mathcal{S}_\delta^{\mathrm{bcomp}}$. While in $\mathbb{R}^n$ trace norm estimates have been carried out very precisely for appropriate classes of operators in [27] and [17], there seems to be no reference where for a Schrödinger operator $P(h)$ as above the transition from the global operator $f_h(P(h))$ to the locally defined quantizations, obtained by introducing an atlas and a partition of unity, is made in a way such that the trace norm of the remainder operators is precisely controlled, not even if f_h is actually independent of h.

1.1.2 Methods

To achieve our first goal, we generalize existing theorems for a fixed h-independent function $f \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R})$ to the classes $\mathcal{S}_\delta^{\mathrm{comp}}$. The main technical difficulties arise from estimates that are uniform in h for fixed f, but which depend on f and hence are no longer uniform in h when $f = f_h$. To achieve our second goal, we apply well-known methods, but in a more detailed fashion than usually found in literature. Here, a technical difficulty lies in the fact that semiclassical pseudodifferential operators on manifolds are locally not exactly defined by the quantization of a symbol function, but only up to a remainder with operator norm of order h^∞, while we are interested in trace norm estimates.

1.2 Summary of main results

In what follows, the results of this first part of the thesis are presented in a slightly condensed form. First, we consider semiclassical pseudodifferential operators in $\mathrm{L}^2(\mathbb{R}^n)$, with the notation $\overline{A}$ for the self-adjoint extension of an essentially self-adjoint operator A, and Op_h for the Weyl quantization. For the other required definitions, in particular those of *order functions* and the associated notion of ellipticity, the symbol classes $S^k_{h;\delta}(\mathfrak{m})$ and $S^m_{h;\delta}(M)$, and the operator classes $\Psi^m_{h;\delta}(M)$, we refer the reader to Section 2.1. To state our results, let $\delta \in [0, \frac{1}{2})$. The main result of Chapter 3 is

Result 1 (Theorem 3.1.1)**.** *Let $\mathfrak{m} : \mathbb{R}^{2n} \to (0,\infty)$ be an order function with $\mathfrak{m} \geq 1$, and let $s \in S_h(\mathfrak{m})$ be a real-valued symbol function such that $s + i$ is $\mathfrak{m}$-elliptic. Choose $f_h \in \mathcal{S}_\delta^{\mathrm{comp}}$. Then, for small h the operator $f_h\big(\overline{\mathrm{Op}_h(s)}\big) : \mathrm{L}^2(\mathbb{R}^n) \to \mathrm{L}^2(\mathbb{R}^n)$ is a semiclassical pseudodifferential operator. More precisely, there is a symbol function $a \in \bigcap_{k\in\mathbb{N}} S_{h;\delta}(\mathfrak{m}^{-k})$ and a number $h_0 \in (0,1]$ such that for $h \in (0,h_0]$*

$$f_h\Big(\overline{\mathrm{Op}_h(s)}\Big) = \mathrm{Op}_h(a).$$

Moreover, if s has an asymptotic expansion in $S_h(\mathfrak{m})$ of the form $s \sim \sum_{j=0}^{\infty} h^j s_j$, then a has an expansion in $S_{h;\delta}(1/\mathfrak{m})$, with explicitly known coefficients, of the form

$$a \sim \sum_{j=0}^{\infty} a_j, \qquad a_j \in S_{h;\delta}^{j(2\delta-1)}(1/\mathfrak{m}), \qquad a_0(y,\eta,h) = f_h(s_0(y,\eta,h)). \tag{1.2.1}$$

The second result concerns the Schrödinger operator (1.1.3) on a closed connected Riemannian manifold M of dimension n. We obtain in Chapter 4:

Result 2 (Theorem 4.2.1)**.** *Choose a function $\varrho_h \in \mathcal{S}_\delta^{\mathrm{comp}}$. Then, for small h the operator $\varrho_h(P(h))$ is a semiclassical pseudodifferential operator on M of order $(-\infty, \delta)$, and its principal symbol is represented by the function $\varrho_h \circ p$.*

In order to prove trace formulas for a semiclassical pseudodifferential operator on a manifold, it is useful to approximate the operator by pullbacks of semiclassical pseudodifferential operators in $\mathbb{R}^n$ by introducing an atlas and a partition of unity, up to a trace class remainder operator with small trace norm. In addition, one would like to localize the leading term in the obtained trace formulas using an operator $B \in \Psi_{h;\delta}^0(M)$ with principal symbol represented by a symbol function $b \in S_{h;\delta}^0(M)$. Thus, let us introduce a finite atlas

$$\{U_\alpha, \gamma_\alpha\}_{\alpha \in \mathcal{A}}, \qquad \gamma_\alpha : U_\alpha \xrightarrow{\simeq} \mathbb{R}^n, \qquad U_\alpha \subset M \text{ open}.$$

We choose the whole euclidean space $\mathbb{R}^n$ as the image of our charts in order to avoid problems related to the fact that pseudodifferential operators are non-local. Furthermore, in order to state our next result we require a smooth partition of unity $\{\varphi_\alpha\}_{\alpha \in \mathcal{A}}$ on M subordinate to $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ and for each $\alpha \in \mathcal{A}$ an associated triple of cutoff functions $\overline{\varphi}_\alpha, \overline{\overline{\varphi}}_\alpha, \overline{\overline{\overline{\varphi}}}_\alpha \in \mathrm{C_c^\infty}(U_\alpha)$ with

$$\overline{\varphi}_\alpha \equiv 1 \text{ on } \operatorname{supp} \varphi_\alpha, \qquad \overline{\overline{\varphi}}_\alpha \equiv 1 \text{ on } \operatorname{supp} \overline{\varphi}_\alpha, \qquad \overline{\overline{\overline{\varphi}}}_\alpha \equiv 1 \text{ on } \operatorname{supp} \overline{\overline{\varphi}}_\alpha.$$

For each chart, define a local symbol function

$$u_{\alpha,0}(y,\eta,h) := \big((\varrho_h \circ p) \cdot b\big)\big(\gamma_\alpha^{-1}(y), (\partial\gamma_\alpha^{-1})^T\eta, h\big) \cdot \varphi_\alpha\big(\gamma_\alpha^{-1}(y)\big)$$

where $(y,\eta) \in \mathbb{R}^{2n}$, $\alpha \in \mathcal{A}$, $h \in (0,1]$. Then, one has the following

Result 3 (Theorem 4.3.1)**.** *Suppose that $\varrho_h \in \mathcal{S}_\delta^{\mathrm{bcomp}}$. Then, for each $N \in \mathbb{N}$, there is a number $h_0 \in (0,1]$, a collection of symbol functions $\{r_{\alpha,\beta,N}\}_{\alpha,\beta \in \mathcal{A}} \subset S_{h;\delta}^{2\delta-1}(1_{\mathbb{R}^{2n}})$ and an operator $\mathfrak{R}_N(h) \in \mathcal{B}(\mathrm{L}^2(M))$ such that*

- *one has for all $f \in \mathrm{L}^2(M)$, $h \in (0,h_0]$ the relation*

$$\begin{aligned} B \circ \varrho_h(P(h))(f) = &\sum_{\alpha \in \mathcal{A}} \overline{\varphi}_\alpha \cdot \mathrm{Op}_h(u_{\alpha,0})\big((f \cdot \overline{\overline{\overline{\varphi}}}_\alpha) \circ \gamma_\alpha^{-1}\big) \circ \gamma_\alpha \\ &+ \sum_{\alpha,\beta \in \mathcal{A}} \overline{\varphi}_\beta \cdot \mathrm{Op}_h(r_{\alpha,\beta,N})\big((f \cdot \overline{\overline{\varphi}}_\alpha \cdot \overline{\overline{\overline{\varphi}}}_\beta) \circ \gamma_\beta^{-1}\big) \circ \gamma_\beta \;+\; \mathfrak{R}_N(h)(f); \end{aligned} \tag{1.2.2}$$

- *the operator $\mathfrak{R}_N(h) \in \mathcal{B}(\mathrm{L}^2(M))$ is of trace class and its trace norm fulfills*

$$\|\mathfrak{R}_N(h)\|_{\mathrm{tr},\mathrm{L}^2(M)} = \mathrm{O}\big(h^N\big) \quad \text{as } h \to 0;$$

- *for fixed $h \in (0,h_0]$, each symbol function $r_{\alpha,\beta,N}$ is an element of $\mathrm{C_c^\infty}(\mathbb{R}^{2n})$ that fulfills*

$$\operatorname{supp} r_{\alpha,\beta,N} \subset \operatorname{supp} \big((\varrho_h \circ p) \cdot b \cdot \varphi_\alpha\big) \circ (\gamma_\alpha^{-1}, (\partial\gamma_\alpha^{-1})^T).$$

1.3 Discussion

1.3.1 Applications

In general, Result 3 can be used to prove an asymptotic semiclassical trace formula with non-trivial remainder estimates for an operator of the form

$$T \circ B \circ \varrho_h(P(h))$$

where $T : \mathrm{L}^2(M) \to \mathrm{L}^2(M)$ is some bounded, explicitly known operator. For example, T can be defined using an additional structure on the manifold M. Then, by Result 3, one has for each $N \in \mathbb{N}$

$$\mathrm{tr}_{\mathrm{L}^2(M)}\big[T \circ B \circ \varrho_h(P(h))\big] = \mathrm{tr}_{\mathrm{L}^2(M)}\big(T \circ L_N\big) + \mathrm{O}\big(h^N\big) \qquad \text{as } h \to 0,$$

where L_N is the operator defined by the right hand side of (1.2.2) without $\mathfrak{R}_N(h)(f)$. The significance of Result 3 is that proving a trace formula for $T \circ B \circ \varrho_h(P(h))$ with remainder of order h^N immediately reduces to calculating the leading term $\mathrm{tr}_{\mathrm{L}^2(M)}(T \circ L_N)$, and this term involves only pullbacks of semiclassical pseudodifferential operators in $\mathbb{R}^n$, so that in the calculations one can rely on the precise symbolic calculus on $\mathbb{R}^n$ and needs to deal only with compactly supported symbol functions.

In the simplest case where $T = B = \mathbf{1}_{\mathrm{L}^2(M)}$, Corollary 4.3.2 yields the h-dependent analogue of (1.1.1) given by

$$(2\pi h)^n \,\mathrm{tr}_{\mathrm{L}^2(M)}\varrho_h\,(P(h)) = \int\limits_{T^*M} \varrho_h \circ p \; d(T^*M) + \mathrm{O}\Big(h^{1-2\delta}\mathrm{vol}_{\,T^*M}\big(\mathrm{supp}\;\varrho_h \circ p\big)\Big) \quad \text{as } h \to 0.$$

Provided that $\delta < \frac{1}{3}$, this leads directly to an improved version of (1.1.2) given by

$$(2\pi)^n h^{n-\delta} \#\big\{j \in J(h) : E_j(h) \in [E, E + h^\delta]\big\} = \mathrm{vol}\; p^{-1}(\{E\}) + \mathrm{O}\Big(h^\delta + h^{\frac{1}{3}-\delta}\Big), \tag{1.3.1}$$

where the volume is now measured using the induced Liouville measure on $p^{-1}(\{E\})$, compare Lemma 6.3.8 and the proof of Theorem 7.2.1 in Part II. Of course, formula (1.3.1) is far from optimal in terms of its quantitative statement (see Subsection 1.3.2 below), yet it serves as a simple example of the qualitative fact that due to the localization onto the hypersurface $p^{-1}(\{E\})$ it is now enough to assume that $p^{-1}([E - \varepsilon, E + \varepsilon])$ is compact for some small $\varepsilon > 0$ and that E is a regular value of p, i.e. the $c = 0$ version of the assumptions required for (1.1.2).

When choosing $\delta > 0$, one can use the operator B to perform a localization to small subsets (for example, single points or geodesics) of T^*M in the semiclassical limit. This is the small-scale approach, see [25]. By choosing T to be the projection onto a linear subspace V of $\mathrm{L}^2(M)$ and then taking into account the equality

$$\mathrm{tr}\big[T \circ B \circ \varrho_h(P(h))\big] = \mathrm{tr}\big[B \circ \varrho_h(P(h)) \circ T\big] = \mathrm{tr}\big[T \circ B \circ \varrho_h(P(h)) \circ T\big],$$

one can use Result 3 to study the spectral properties of the bi-restriction of $P(h)$ to V, still possibly localizing the problem using B. As a major application, we use Result 3 in Section 7.1 in the second part of this thesis to prove a singular equivariant semiclassical trace formula for Schrödinger operators in case that M carries an isometric effective action of a compact connected Lie group G, see Theorem 7.1.1. There, one has $T = T_\chi$, where

$T_\chi : \mathrm{L}^2(M) \to \mathrm{L}^2(M)$ is the projection onto an isotypic component of the left-regular G-representation in $\mathrm{L}^2(M)$ associated to a character $\chi \in \widehat{G}$. The calculation of $\mathrm{tr}_{\mathrm{L}^2(M)}(T_\chi \circ L_N)$ reduces to the evaluation of certain oscillatory integrals which can be carried out using a formula from [46] whose remainder term is of lower order than that in Result 3. Here, knowing a better remainder estimate in Result 3 would not improve the results, so in this case Result 3 is fully sufficient both qualitatively and quantitatively. The trace formula stated in Theorem 7.1.1 could not be established without a functional calculus for h-dependent functions. It implies a generalized equivariant semiclassical Weyl law with remainder estimate, as well as a symmetry-reduced quantum ergodicity theorem, see Chapters 7 and 8.

1.3.2 Previously known results

An h-dependent functional calculus of the form $f_h(P(h))$ has been used in the literature before, but only in special cases, not systematically, and sometimes only implicitly. For example, consider a Schwartz function $\chi : \mathbb{R} \to \mathbb{R}$ whose Fourier transform has compact support. Then a common approach in the literature is to study the operator $\chi\big(\frac{P(h)-E}{h}\big)$ in the context of semiclassical Fourier integral operators, $E \in \mathbb{R}$ being a fixed regular value of p. Writing $f_h^E(x) := \chi\big(\frac{x-E}{h}\big)$, this amounts to studying $f_h^E(P(h))$, as in the semiclassical Gutzwiller trace formula [16, 24] and in the proof of the semiclassical quantum ergodicity theorem in [20]. Using the same techniques, one can also prove a semiclassical Weyl law for the smallest possible spectral window $[E, E+h]$ with the best possible $\mathrm{O}(h)$-remainder, see [17, 20, 32]. However, these techniques are considerably more involved than the simple semiclassical pseudodifferential operator calculus. The only other way known to the author in which the abstract functional calculus has been used for h-dependent functions in the literature is of a very basic form. Namely, given any family $\{f_h\}_{h\in(0,1]}$ of bounded Borel functions on $\mathbb{R}$ with uniformly bounded supremum norms for $h \in (0,1]$, one can use the fact that $f_h(P(h))$ has uniformly bounded operator norm as $h \to 0$, an estimate which follows directly from the spectral theorem. In particular, one considers $f_h(P(h))$ only abstractly as an h-dependent bounded operator, and not concretely as a semiclassical pseudodifferential operator or Fourier integral operator. See e.g. [20, Proof of Lemma 3.11].

1.3.3 Strengths and weaknesses of methods and outlook

Developing the functional calculus for h-dependent functions within the theory of semiclassical pseudodifferential operators restricts the applications in spectral analysis to spectral windows of width of order h^δ with $\delta < \frac{1}{2}$. In particular, the best possible case $\delta = 1$ cannot be studied. However, qualitatively there is no significant difference between the cases $\delta = 1$ and $\delta > 0$, since for any $\delta > 0$ the spectral window $[E, E+h^\delta]$ shrinks to a point polynomially fast in the semiclassical limit, leading to a localization on an energy hypersurface in Weyl-type formulas, and if the manifold dimension is greater than 1, then by Weyl's law the number of eigenvalues in $[E, E+h^\delta]$ grows as $h \to 0$, regardless whether $\delta = 1$ or just $\delta > 0$. Thus, going beyond the theory of semiclassical pseudodifferential operators could only lead to quantitative improvements, at the expense of losing the simplicity of the symbolic calculus. Although non-optimal, the quantitative results presented here are sufficient for many applications, as outlined above. An obvious possible future line of research consists in developing a functional calculus for h-dependent functions within more general semiclassical frameworks of operators, as the theory of semiclassical Fourier integral operators. This will probably yield improved quantitative results.

Chapter 2

Background I

2.1 Semiclassical analysis

In what follows, we shall briefly recall the theory of semiclassical symbol classes and pseudodifferential operators on $\mathbb{R}^n$ and on general smooth manifolds. For a detailed introduction, we refer the reader to [63, Chapters 9 and 14] and [17, Chapters 7 and 8]. Semiclassical analysis developed out of the theory of pseudodifferential operators, a thorough exposition of which can be found in [53]. An important feature that distinguishes semiclassical analysis from usual pseudodifferential operator theory is that instead of the usual symbol functions and corresponding operators, one considers families of symbol functions and pseudodifferential operators indexed by a global parameter

$$h \in (0, 1].$$

Essentially, the definitions of those families are obtained from the usual definitions by substituting in the symbol functions the co-tangent space variable ξ by $h\xi$. To begin, recall that a Lebesgue-measurable function $\mathfrak{m} : \mathbb{R}^n \to (0, \infty)$ is called *order function* if there are constants $C, N > 0$ such that

$$\mathfrak{m}(v_1) \leq C \langle v_1 - v_2 \rangle^N \mathfrak{m}(v_2) \quad \forall\, v_1, v_2 \in \mathbb{R}^n.$$

Here we used the notation $\langle v \rangle := \sqrt{1 + \|v\|^2}$. If $\mathfrak{m}_1, \mathfrak{m}_2$ are order functions, then $\mathfrak{m}_1\mathfrak{m}_2$ is also an order function. For example, $\mathfrak{m} = 1_{\mathbb{R}^n}$ and $\mathfrak{m}(v) = \langle v \rangle^k$, $k > 0$, are order functions. Let $\mathfrak{m} : \mathbb{R}^n \to (0, \infty)$ be an order function. For $\delta \in [0, \frac{1}{2})$ and $k \in \mathbb{R}$, we define the semiclassical symbol class $S^k_{h;\delta}(\mathfrak{m})$ as the set of all functions $s : \mathbb{R}^n \times (0, 1] \to \mathbb{C}$ such that $s(\cdot, h) \in \mathrm{C}^\infty(\mathbb{R}^n)$ for each $h \in (0, 1]$ and for each non-negative n-dimensional multiindex α, there is a constant $C_{\alpha,\delta,k} > 0$ with

$$|\partial^\alpha_v s(v, h)| \leq C_{\alpha,\delta,k}\, \mathfrak{m}(v)\, h^{-\delta|\alpha|-k} \quad \forall\, (v, h) \in \mathbb{R}^n \times (0, 1]. \tag{2.1.1}$$

We write

$$S^k_h(\mathfrak{m}) := S^k_{h;0}(\mathfrak{m}), \qquad S_{h;\delta}(\mathfrak{m}) := S^0_{h;\delta}(\mathfrak{m}), \qquad S_h(\mathfrak{m}) := S^0_{h;0}(\mathfrak{m}).$$

We call an element of a semiclassical symbol class a *symbol function*. Furthermore, let us define

$$S^{-\infty}_h(\mathfrak{m}) := \bigcap_{k\in\mathbb{R}} S^k_{h;\delta}(\mathfrak{m}), \quad \delta \in [0, 1/2) \text{ arbitrary}.$$

This set is in fact well-defined (the intersection on the right hand side is independent of δ). $S_h^{-\infty}(\mathfrak{m})$ is the set of functions $s : \mathbb{R}^n \times (0,1] \to \mathbb{C}$ such that $s(\cdot,h) \in \mathrm{C}^\infty(\mathbb{R}^n)$ for each $h \in (0,1]$ and for each non-negative n-dimensional multiindex α and each $N \in \mathbb{N}$, there is a constant $C_{\alpha,N} > 0$ with

$$|\partial_v^\alpha s(v,h)| \leq C_{\alpha,N}\, \mathfrak{m}(v)\, h^N \quad \forall\, (v,h) \in \mathbb{R}^n \times (0,1].$$

In order to recall the definition of semiclassical asymptotic series, let $\mathfrak{m} : \mathbb{R}^n \to (0,\infty)$ be an order function. Given $\delta \in [0, \frac{1}{2})$, a sequence $\{k_j\}_{j\in\mathbb{N}} \subset \mathbb{R}$ with $k_j \to -\infty$ as $j \to \infty$, a sequence $\{s_j\}_{j\in\mathbb{N}}$ with $s_j \in S^{k_j}_{h;\delta}(\mathfrak{m})$, and a symbol function $s \in S_{h;\delta}(\mathfrak{m})$, we say that $\{s_j\}_{j\in\mathbb{N}}$ is *asymptotic to* s in $S_{h;\delta}(\mathfrak{m})$, in short

$$s \sim \sum_{j=0}^{\infty} s_j \quad \text{in } S_{h;\delta}(\mathfrak{m}),$$

provided that for each $N \in \mathbb{N}$ one has $s - \sum_{j=0}^N s_j \in S^{k_{N+1}}_{h;\delta}(\mathfrak{m})$. We denote by $\mathcal{S}(\mathbb{R}^n)$ the vector space of Schwartz functions on $\mathbb{R}^n$, equipped with the semi-norms

$$|f|_{\alpha,\beta} := \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta f(x)|$$

and denote by $\mathcal{S}'(\mathbb{R}^n)$ the topological dual space of $\mathcal{S}(\mathbb{R}^n)$, i.e. the space of continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$, equipped with the weak-$*$ topology. Let $\mathfrak{m} : \mathbb{R}^{2n} \to (0,\infty)$ be an order function. For $s \in S^k_{h;\delta}(\mathfrak{m})$, $f \in \mathcal{S}(\mathbb{R}^n)$, and $x \in \mathbb{R}^n$, define

$$\mathrm{Op}_h(s)(f)(x) := \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}(x-y)\cdot\eta} s\Big(\frac{x+y}{2}, \eta, h\Big) f(y)\, dy\, d\eta. \tag{2.1.2}$$

Then, by [63, Theorem 4.16] and [17, Theorem 7.8], the function

$$\mathrm{Op}_h(s)(f) : x \mapsto \mathrm{Op}_h(s)(f)(x)$$

is an element of $\mathcal{S}(\mathbb{R}^n)$ and the map

$$\mathrm{Op}_h(s) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n), \qquad f \mapsto \mathrm{Op}_h(s)(f),$$

is a continuous linear operator. Moreover, by duality $\mathrm{Op}_h(s)$ extends to a continuous linear operator

$$\mathrm{Op}_h(s) : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n).$$

This so-called *Weyl-quantization* is motivated by the fact that the classical Hamiltonian $H(x,\xi) = \xi^2$ should correspond to the quantum Laplacian $-h^2\Delta$, and that real-valued symbol functions should correspond to symmetric or, more desirably, essentially self-adjoint operators. An operator $\mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ of the form (2.1.2) is called a *semiclassical pseudodifferential operator* on $\mathbb{R}^n$. We denote by $\mathrm{Op}_h(S^k_{h;\delta}(\mathfrak{m}))$ the set of semiclassical pseudodifferential operators that are quantizations of symbol functions in $S^k_{h;\delta}(\mathfrak{m})$. For the following important formula, we introduce the standard symplectic form $\sigma : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ given by $\sigma(x,\xi;y,\eta) := \xi\cdot y - x\cdot\eta$.

Theorem 2.1.1 (Composition formula, [17, Theorem 7.9])**.** *Let $\mathfrak{m}_1, \mathfrak{m}_2 : \mathbb{R}^{2n} \to (0,\infty)$ be order functions and $s_j \in S_{h;\delta}(\mathfrak{m}_j)$. Then, there is a symbol function $s \in S_{h;\delta}(\mathfrak{m}_1\mathfrak{m}_2)$ such that $\mathrm{Op}_h(s_1) \circ \mathrm{Op}_h(s_2) = \mathrm{Op}_h(s)$, and*

$$s \sim \sum_{k=0}^{\infty} \frac{1}{k!}\Big(\frac{ih}{2}\sigma(D_x, D_\xi; D_y, D_\eta)\Big)^k s_1(x,\xi,h) s_2(y,\eta,h)|_{y=x,\eta=\xi} \quad \text{in } S_{h;\delta}(\mathfrak{m}_1\mathfrak{m}_2). \tag{2.1.3}$$

Let $s \in S^k_{h;\delta}(1_{\mathbb{R}^{2n}})$. Then, by [17, Theorem 7.11], the operator $\mathrm{Op}_h(s) : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ bi-restricts[1] to a bounded linear operator $\mathrm{Op}_h(s) \in \mathcal{B}(\mathrm{L}^2(\mathbb{R}^n))$ which is essentially given by (2.1.2), and there is a constant $C > 0$ which is independent of h, such that

$$\|\mathrm{Op}_h(s)\|_{\mathcal{B}(\mathrm{L}^2(\mathbb{R}^n))} \leq Ch^{-k} \quad \forall\, h \in (0, 1]. \tag{2.1.4}$$

It will be important for us to know when a semiclassical pseudodifferential operator is of trace class, and to estimate its trace norm. For these tasks, the following results from [17, p. 113, Lemma 9.3, Theorem 9.4] are very useful. Let $s \in S^k_{h;\delta}(1_{\mathbb{R}^{2n}})$ for some $\delta \in [0, \frac{1}{2})$, $k \in \mathbb{R}$, and suppose that the function $s(\cdot, h) : \mathbb{R}^{2n} \to \mathbb{C}$, $(y, \eta) \mapsto s(y, \eta, h)$ fulfills

$$\sum_{|\alpha| \leq 2n+1} \|\partial^\alpha s(\cdot, h)\|_{\mathrm{L}^1(\mathbb{R}^{2n})} < \infty.$$

Then, the operator $\mathrm{Op}_h(s) : \mathrm{L}^2(\mathbb{R}^n) \to \mathrm{L}^2(\mathbb{R}^n)$ is of trace class with trace norm

$$\|\mathrm{Op}_h(s)\|_{\mathrm{tr},\mathrm{L}^2(\mathbb{R}^n)} \leq Ch^{-n} \sum_{|\alpha| \leq 2n+1} \|\partial^\alpha s(\cdot, h)\|_{\mathrm{L}^1(\mathbb{R}^{2n})},$$

where $C > 0$ is independent of h, and its trace is given by

$$\mathrm{tr}_{\,\mathrm{L}^2(\mathbb{R}^n)} \mathrm{Op}_h(s) = \frac{1}{(2\pi h)^n} \int\limits_{\mathbb{R}^{2n}} s(y, \eta, h)\, dy\, d\eta. \tag{2.1.5}$$

Moreover, the integral kernel estimates on [17, p. 113] and the estimate proved there for the relation between standard quantization and Weyl quantization imply the following results. Choose $\phi \in \mathrm{C}^\infty_\mathrm{c}(\mathbb{R}^n)$ with support inside some compact set $K \subset \mathbb{R}^n$, and denote the operator $\mathrm{L}^2(\mathbb{R}^n) \to \mathrm{L}^2(\mathbb{R}^n)$ given by pointwise multiplication with ϕ by Φ.

- Suppose that the function $s(y, \cdot, h) : \eta \mapsto s(y, \eta, h)$ is a Schwartz function for each $y \in \mathbb{R}^n$ and each $h \in (0, 1]$. Then, the operator $\Phi \circ \mathrm{Op}_h(s)$ is of trace class and its trace norm fulfills uniformly for $h \in (0, 1]$ the estimate

$$\|\Phi \circ \mathrm{Op}_h(s)\|_{\mathrm{tr},\mathrm{L}^2(\mathbb{R}^n)} \leq Ch^{-n} \sum_{|\alpha| \leq 2n+1} \|\partial^\alpha (\phi s)(\cdot, h)\|_{\mathrm{L}^1(\mathbb{R}^{2n})}, \tag{2.1.6}$$

  with a constant $C > 0$ that is independent of h.

- As a special case of the previous one, we have in particular: Suppose that the function $s(y, \cdot, h) : \eta \mapsto s(y, \eta, h)$ is compactly supported in $\mathbb{R}^n$ for each $y \in \mathbb{R}^n$ and each $h \in (0, 1]$, and the volume of the support of the function $s(y, \cdot, h)$ is bounded uniformly in $y \in \mathbb{R}^n$ by some h-dependent constant $C_h > 0$. Then,

$$\|\Phi \circ \mathrm{Op}_h(s)\|_{\mathrm{tr},\mathrm{L}^2(\mathbb{R}^n)} \leq C_\phi C_h h^{-n} \sum_{|\alpha| \leq 2n+1} \max_{(y,\eta) \in K \times \mathbb{R}^n} |\partial^\alpha s(y, \eta, h)|, \tag{2.1.7}$$

  with a constant $C_\phi > 0$ that depends on ϕ but not on h.

Very useful in combination with the previous lines is also the following observation, which follows from the statements above and the composition formula (2.1.3), compare [17, Proposition 9.5]. For $i \in \{1, 2\}$, let $s_i \in S^{k_i}_{h;\delta}(1_{\mathbb{R}^{2n}})$ for some $\delta \in [0, \frac{1}{2})$, $k_i \in \mathbb{R}$, and suppose that

[1] Here, we are regarding $\mathrm{L}^2(\mathbb{R}^n)$ as a subset of $\mathcal{S}'(\mathbb{R}^n)$.

for each $h \in (0,1]$ the function $s_1(\cdot,h) : \mathbb{R}^{2n} \to \mathbb{C}$ is compactly supported inside the interior of some h-independent compactum $K \subset \mathbb{R}^{2n}$. Let $s_1 \sharp s_2 \in S^{k_1+k_2}_{h;\delta}$ be the symbol obtained from s_1 and s_2 by the composition formula (2.1.3). Then for each $N \in \mathbb{N}$, $R > 0$, and each non-negative $2n$-dimensional multiindex α, there is a constant $C_{\alpha,N} > 0$ such that for all $(y,\eta) \in \mathbb{R}^{2n}$ with $\operatorname{dist}((y,\eta),K) \geq R$, one has

$$|\partial^\alpha (s_1 \sharp s_2)(y,\eta,h)| \leq C_{\alpha,N} h^{N(1-\delta)-k_1-k_2-\delta|\alpha|} \operatorname{dist}((y,\eta),K)^{-N} \qquad \forall\, h \in (0,1].$$

As the function $(y,\eta) \mapsto \langle \operatorname{dist}((y,\eta),K)\rangle^{-N}$ is in $\mathrm{L}^1(\mathbb{R}^{2n})$ if $N > 2n$, we can combine the preceding results to get

Corollary 2.1.2. *For $i \in \{1,2\}$, let $s_i \in S^{k_i}_{h;\delta}(1_{\mathbb{R}^{2n}})$ for some $\delta \in [0,\frac{1}{2})$, $k_i \in \mathbb{R}$, and suppose that for each $h \in (0,1]$ the function $s_1(\cdot,h) : \mathbb{R}^{2n} \to \mathbb{C}$ is compactly supported inside some h-independent compactum $K \subset \mathbb{R}^{2n}$. Then the operator $\mathrm{Op}_h(s_1 \sharp s_2)$ is of trace class and*

$$\|\mathrm{Op}_h(s_1 \sharp s_2)\|_{\mathrm{tr},\mathrm{L}^2(\mathbb{R}^n)} = \mathrm{O}\big(h^{-n-k_1-k_2-(2n+1)\delta}\big) \qquad \textit{as } h \to 0. \tag{2.1.8}$$

This corollary is important as it tells us that the trace norm of the composition of two semiclassical pseudodifferential operators, one of which has a symbol supported inside a fixed compactum, essentially depends only on the norm of the derivatives of the original two symbols near the compactum. Surely, Corollary 2.1.2 could be generalized to h-dependent compactums $K(h)$, but as we are mainly interested in a functional calculus for h-dependent functions whose support shrinks as $h \to 0$, it is no big loss of generality to assume that the shrinking happens inside a fixed h-independent compactum.

Let $\mathfrak{m} : \mathbb{R}^{2n} \to (0,\infty)$ be an order function and let $s \in S_h(\mathfrak{m})$. We call the symbol function s $\mathfrak{m}$-*elliptic* if there is a constant $\varepsilon > 0$ such that $|s| \geq \varepsilon\, \mathfrak{m}$. Crucial for all what follows is the following result:

Theorem 2.1.3 (Essential self-adjointness [17, Prop. 8.5])**.** *Let $\mathfrak{m} : \mathbb{R}^{2n} \to (0,\infty)$ be an order function with $\mathfrak{m} \geq 1$, and let $s \in S_h(\mathfrak{m})$ be a real-valued symbol function such that $s + i$ is $\mathfrak{m}$-elliptic, where i denotes the imaginary unit $\sqrt{-1}$. Then, there is a number $h_0 \in (0,1]$ such that the operator $(\mathrm{Op}_h(s)+i)^{-1} \in \mathcal{B}(\mathrm{L}^2(\mathbb{R}^n))$ exists for each $h \in (0,h_0]$. Furthermore, the operator $\mathrm{Op}_h(s) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \subset \mathrm{L}^2(\mathbb{R}^n)$ is essentially self-adjoint in $\mathrm{L}^2(\mathbb{R}^n)$ for each $h \in (0,h_0]$, and one obtains the unique self-adjoint extension $\overline{\mathrm{Op}_h(s)}$ by equipping $\mathrm{Op}_h(s)$ with the domain*

$$(\mathrm{Op}_h(s)+i)^{-1}\mathrm{L}^2(\mathbb{R}^n) \subset \mathrm{L}^2(\mathbb{R}^n).$$

For example, if $\mathfrak{m}(y,\eta) = \langle \eta \rangle^2$ and $s(y,\eta,h) = \|\eta\|^2$, then

$$(\mathrm{Op}_h(s)+i)^{-1}\mathrm{L}^2(\mathbb{R}^n) = \mathrm{H}^2_h(\mathbb{R}^n), \tag{2.1.9}$$

where $\mathrm{H}^2_h(\mathbb{R}^n)$ is the semiclassical equivalent of the Sobolev space $\mathrm{H}^2(\mathbb{R}^n)$, see [63, Thm. 8.10]. Now, the known functional calculus in $\mathbb{R}^n$ for fixed h-independent functions is summarized in

Theorem 2.1.4 ([17, Theorem 8.7 and p. 103])**.** *Let s be a symbol function as in Theorem 2.1.3 and let $f \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R})$. Consider the operator $f\big(\overline{\mathrm{Op}_h(s)}\big) : \mathrm{L}^2(\mathbb{R}^n) \to \mathrm{L}^2(\mathbb{R}^n)$ defined by the spectral calculus for unbounded self-adjoint operators. Then, there is a symbol function $a_{s,f} \in \bigcap_{k\in\mathbb{N}} S_h(\mathfrak{m}^{-k})$ and a number $h_0 \in (0,1]$ such that for $h \in (0,h_0]$*

$$f\Big(\overline{\mathrm{Op}_h(s)}\Big) = \mathrm{Op}_h(a_{s,f}).$$

Moreover, if s fulfills $s \sim \sum_{j=0}^{\infty} h^j s_j$ in $S_h(\mathfrak{m})$ for some sequence $\{s_j\}_{j=0,1,2,\ldots} \subset S_h(\mathfrak{m})$, then there is a sequence of polynomials $\{q_{s,j}(y,\eta,t,h)\}_{j=0,1,2,\ldots}$ in one variable $t \in \mathbb{R}$ with coefficients h-dependent functions in $\mathrm{C}^\infty(\mathbb{R}^{2n})$ and with $q_{s,0} \equiv 1$, such that

$$a_{s,f} \sim \sum_{j=0}^{\infty} h^j a_{s,f,j} \text{ in } S_h(1/\mathfrak{m}), \quad a_{s,f,j}(y,\eta,h) = \frac{1}{(2j)!}\Big(\frac{\partial}{\partial t}\Big)^{2j} (q_{s,j}(y,\eta,t,h) f(t))_{t=s_0(y,\eta,h)}.$$

In particular, $a_{s,f,0} = f \circ s_0$.

A corollary is the following trace formula for semiclassical pseudodifferential operators in $\mathbb{R}^n$:

Theorem 2.1.5 ([17, Theorem 9.6])**.** *Let $\mathfrak{m} : \mathbb{R}^{2n} \to (0,\infty)$ be an order function with $\mathfrak{m} \geq 1$, and let $s \in S_h(\mathfrak{m})$ be a real-valued symbol function such that $s + i$ is $\mathfrak{m}$-elliptic, with an asymptotic expansion $s \sim \sum_{j=0}^{\infty} h^j s_j$ in $S_h(\mathfrak{m})$, where $\{s_j\}_{j=0,1,2,\ldots} \subset S_h(\mathfrak{m})$. Let $I \subset \mathbb{R}$ be a bounded open interval with*

$$\liminf_{\|v\| \to +\infty} \operatorname{dist}(s(v,h), I) \geq C \qquad \forall\, h \in (0,1]$$

for a constant $C > 0$ which is independent of h, and let $f \in \mathrm{C}_\mathrm{c}^\infty(I) \subset \mathrm{C}_\mathrm{c}^\infty(\mathbb{R})$ be given. Then, the operator $f\big(\overline{\mathrm{Op}_h(s)}\big) : \mathrm{L}^2(\mathbb{R}^n) \to \mathrm{L}^2(\mathbb{R}^n)$ is of trace class for small h, and as $h \to 0$, its trace is asymptotically given by

$$\operatorname{tr}_{\mathrm{L}^2(\mathbb{R}^n)} f\Big(\overline{\mathrm{Op}_h(s)}\Big) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} f\big(s_0(y,\eta,h)\big)\, dy\, d\eta + \mathrm{O}\big(h^{-n+1}\big).$$

In order to introduce semiclassical pseudodifferential operators on general smooth manifolds, we need the following special type of symbol classes which is invariant under pullbacks along diffeomorphisms. For $m \in \mathbb{R}$ and $\delta \in [0, \frac{1}{2})$, one sets

$$\begin{aligned} S^m_{h;\delta}(\mathbb{R}^n) := \Big\{ & a : \mathbb{R}^{2n} \times (0,1] \to \mathbb{C} :\ a(\cdot,h) \in \mathrm{C}^\infty(\mathbb{R}^{2n})\ \forall\, h \in (0,1], \text{ and } \forall \text{ multiindices } s,t \\ & \exists\, C_{s,t} > 0 : |\partial_x^s \partial_\xi^t a(x,\xi,h)| \leq C_{s,t} \langle \xi \rangle^{m-|t|} h^{-\delta(|s|+|t|)}\ \forall\, x \in \mathbb{R}^n,\ h \in (0,1] \Big\}. \end{aligned} \tag{2.1.10}$$

Note that $S^m_{h;\delta}(\mathbb{R}^n) \subset S_{h;\delta}(\mathfrak{m}_m)$, where $\mathfrak{m}_m : \mathbb{R}^{2n} \to (0,\infty)$ is given by $\mathfrak{m}_m(x,\xi) := \langle \xi \rangle^m$, but the reverse inclusion is not true. The symbol classes (2.1.10) generalize the classical Kohn-Nirenberg classes. In the literature one usually encounters only the case $\delta = 0$. In our context it is natural to allow $\delta > 0$, since the h-dependent functional calculus is primarily useful for functions whose derivatives have growing supremum norms as $h \to 0$. See [20] for more applications of the symbol class (2.1.10). Let now M be a smooth manifold of dimension n, and let $\{(U_\alpha, \gamma_\alpha)\}_{\alpha \in \mathcal{A}}$, $\gamma_\alpha : M \supset U_\alpha \to V_\alpha \subset \mathbb{R}^n$, be an atlas for M. Then one defines

$$\begin{aligned} S^m_{h;\delta}(M) := \Big\{ & a : T^*M \times (0,1] \to \mathbb{C},\ a(\cdot,h) \in \mathrm{C}^\infty(T^*M)\ \forall\, h \in (0,1], \\ & (\gamma_\alpha^{-1})^*(\varphi_\alpha a) \in S^m_{h;\delta}(\mathbb{R}^n)\ \forall\, \alpha \in \mathcal{A},\ \forall\, \varphi_\alpha \in \mathrm{C}_\mathrm{c}^\infty(U_\alpha) \Big\}, \end{aligned} \tag{2.1.11}$$

where $(\gamma_\alpha^{-1})^*$ denotes the pullback[2] along γ_α^{-1}. The definition is independent of the choice of atlas, and we call an element of $S^m_{h;\delta}(M)$ a *symbol function*, similarly to the notion of symbol

[2] The pullback is defined as follows: First, one identifies T^*V_α with $V_\alpha \times \mathbb{R}^n$. Then, given $a : T^*M \times (0,1] \to \mathbb{C}$, the function $(\varphi_\alpha a) \circ \big(\gamma_\alpha^{-1} \times (\partial \gamma_\alpha^{-1})^T \times \mathbf{1}_{(0,1]}\big) : V_\alpha \times \mathbb{R}^n \times (0,1] \to \mathbb{C}$ has compact support inside V_α in the first variable, and hence extends by zero to a function $\mathbb{R}^{2n} \times (0,1] \to \mathbb{C}$ which is smooth for each fixed h. This function is defined to be $(\gamma_\alpha^{-1})^*(\varphi_\alpha a)$.

functions on $\mathbb{R}^n$ defined above. We use the short hand notations

$$S^{-\infty}_{h;\delta}(M) := \bigcap_{m\in\mathbb{R}} S^m_{h;\delta}(M), \qquad S^m_h(M) := S^m_{h;0}(M), \qquad m \in \mathbb{R}\cup\{-\infty\}.$$

For $m \in \mathbb{R}\cup\{-\infty\}$ and $\delta \in [0, \frac{1}{2})$, we call a $\mathbb{C}$-linear map $P : \mathrm{C}^\infty_\mathrm{c}(M) \to \mathrm{C}^\infty(M)$ *semiclassical pseudodifferential operator on M of order (m, δ)* if the following holds:

1. For some (and hence any) atlas $\{(U_\alpha, \gamma_\alpha)\}_{\alpha\in\mathcal{A}}$, $\gamma_\alpha : M \supset U_\alpha \to V_\alpha \subset \mathbb{R}^n$ of M there exists a collection of symbol functions $\{s_\alpha\}_{\alpha\in\mathcal{A}} \subset S^m_{h;\delta}(\mathbb{R}^n)$ such that for any two functions $\varphi_{\alpha,1}, \varphi_{\alpha,2} \in \mathrm{C}^\infty_\mathrm{c}(U_\alpha)$, it holds

$$\varphi_{\alpha,1} P(\varphi_{\alpha,2} f) = \varphi_{\alpha,1}\mathrm{Op}_h(s_\alpha)((\varphi_{\alpha,2} f)\circ\gamma_\alpha^{-1})\circ\gamma_\alpha.$$

2. For all $\varphi_1, \varphi_2 \in \mathrm{C}^\infty_\mathrm{c}(M)$ with $\mathrm{supp}\,\varphi_1 \cap \mathrm{supp}\,\varphi_2 = \emptyset$, one has

$$\|\Phi_1 \circ P \circ \Phi_2\|_{\mathrm{H}^{-N}(M)\to\mathrm{H}^N(M)} = \mathrm{O}(h^\infty) \quad \forall\, N = 0, 1, 2, \ldots,$$

where Φ_j is given by pointwise multiplication with φ_j, and $\mathrm{H}^N(M)$ is the N-th Sobolev space.

When $\delta = 0$, we just say *order m* instead of *order* $(m, 0)$. We denote by $\Psi^m_{h;\delta}(M)$ the $\mathbb{C}$-linear space of all semiclassical pseudodifferential operators on M of order (m, δ), and we write

$$\Psi^m_h(M) := \Psi^m_{h;0}(M), \qquad \Psi^{-\infty}_h(M) = \bigcap_{m\in\mathbb{Z}} \Psi^m_h(M).$$

From the classical theorems about pseudodifferential operators one infers in particular the following relation between symbol functions and semiclassical pseudodifferential operators, see [34, page 86], [63, Theorem 14.1], [20, page 383]. There is a $\mathbb{C}$-linear map

$$\Psi^m_{h;\delta}(M) \to S^m_{h;\delta}(M)/\big(h^{1-2\delta} S^{m-1}_{h;\delta}(M)\big), \quad P \mapsto \sigma(P) \tag{2.1.12}$$

which assigns to a semiclassical pseudodifferential operator its *principal symbol*. Moreover, for each choice of atlas $\{(U_\alpha, \gamma_\alpha)\}_{\alpha\in\mathcal{A}}$ of M and a partition of unity $\{\varphi_\alpha\}_{\alpha\in\mathcal{A}}$ subordinate to $\{U_\alpha\}_{\alpha\in\mathcal{A}}$, there is a $\mathbb{C}$-linear map called *quantization*, written

$$S^m_{h;\delta}(M) \to \Psi^m_{h;\delta}(M), \quad s \mapsto \mathrm{Op}_{h,\{U_\alpha,\varphi_\alpha\}_{\alpha\in\mathcal{A}}}(s). \tag{2.1.13}$$

Any choice of such a map induces the same $\mathbb{C}$-linear bijection

$$\Psi^m_{h;\delta}(M)/\big(h^{1-2\delta}\Psi^{m-1}_{h;\delta}(M)\big) \underset{\mathrm{Op}_h}{\overset{\sigma}{\rightleftarrows}} S^m_{h;\delta}(M)/\big(h^{1-2\delta} S^{m-1}_{h;\delta}(M)\big), \tag{2.1.14}$$

which means in particular that the bijection exists and is independent from the choice of atlas and partition of unity. We will call an element in the quotient set

$$S^m_{h;\delta}(M)/\big(h^{1-2\delta} S^{m-1}_{h;\delta}(M)\big)$$

a *principal symbol*, whereas we call the elements of $S^m_{h;\delta}(M)$ *symbol functions*, as introduced above. Operations on principal symbols such as pointwise multiplication with other principal symbols or smooth functions and composition with smooth functions are defined by performing the corresponding operations on the level of symbol functions. For a semiclassical pseudodifferential operator A on M, we will use the notation

$$\sigma(A) = [a]$$

to express that the principal symbol $\sigma(A)$ is the equivalence class in the quotient set

$$S^m_{h;\delta}(M)/\big(h^{1-2\delta}S^{m-1}_{h;\delta}(M)\big)$$

defined by the symbol function $a \in S^m_{h;\delta}(M)$. Finally, returning to the setup introduced at the beginning of this first part of the thesis, the known functional calculus for our Schrödinger operator $P(h)$ on the closed connected Riemannian manifold M for a fixed h-independent function is summarized in the following

Theorem 2.1.6 ([63, Theorems 14.9 and 14.10])**.** *Let $f \in \mathcal{S}(\mathbb{R})$. Then, the operator $f(P(h))$, defined by the spectral theorem for unbounded self-adjoint operators, is an element of $\Psi_h^{-\infty}(M)$. Furthermore, $f(P(h))$ extends to a bounded operator $f(P(h)) : \mathrm{L}^2(M) \to \mathrm{L}^2(M)$ of trace class, and one has*

$$\sigma\left(f(P(h))\right) = [f \circ p]. \tag{2.1.15}$$

As $h \to 0$, the trace of $f(P(h))$ is asymptotically given by

$$\mathrm{tr}_{\mathrm{L}^2(M)} f(P(h)) = \frac{1}{(2\pi h)^n} \int\limits_{T^*M} f \circ p \; d(T^*M) + \mathrm{O}(h^{-n+1}).$$

□

Remark 2.1.7. Clearly, the habit to decorate the names of the various sets of symbol classes and semiclassical pseudodifferential operators with the lower index h is an abuse of notation because the sets themselves do not directly depend on the semiclassical parameter. For example, it would make no sense to ask the question what happens to the whole sets in the limit $h \to 0$. The h in the names is there to emphasize that the *elements* of the sets depend (pointwise) on h. In particular, if one set h globally to 1, one would recover the usual sets of pseudodifferential operators and symbol functions, compare Chapter 6.

Chapter 3

Results for $\mathbb{R}^n$

In this chapter, we extend Theorems 2.1.4 and 2.1.5 to functions which depend on the semiclassical parameter h.

3.1 Relating the functional and symbolic calculi

We begin with the following

Theorem 3.1.1. *Let $\mathfrak{m} : \mathbb{R}^{2n} \to (0, \infty)$ be an order function with $\mathfrak{m} \geq 1$, and let $s \in S_h(\mathfrak{m})$ be a real-valued symbol function such that $s + i$ is $\mathfrak{m}$-elliptic, where i denotes the imaginary unit $\sqrt{-1}$. Choose $f_h \in \mathcal{S}_\delta^{\mathrm{comp}}$. Then, the operator $f_h\big(\overline{\mathrm{Op}_h(s)}\big) : \mathrm{L}^2(\mathbb{R}^n) \to \mathrm{L}^2(\mathbb{R}^n)$, defined by the spectral calculus for unbounded self-adjoint operators, is a semiclassical pseudodifferential operator for small h. More precisely, there is a symbol function $a \in \bigcap_{k\in\mathbb{N}} S_{h;\delta}(\mathfrak{m}^{-k})$ and a number $h_0 \in (0, 1]$ such that for $h \in (0, h_0]$*

$$f_h\Big(\overline{\mathrm{Op}_h(s)}\Big) = \mathrm{Op}_h(a).$$

Moreover, if s fulfills $s \sim \sum_{j=0}^{\infty} h^j s_j$ in $S_h(\mathfrak{m})$ for some sequence $\{s_j\}_{j=0,1,2,\ldots} \subset S_h(\mathfrak{m})$, then there is an asymptotic expansion in $S_{h;\delta}(1/\mathfrak{m})$

$$a \sim \sum_{j=0}^{\infty} a_j, \qquad a_j \in S_{h;\delta}^{j(2\delta-1)}(1/\mathfrak{m}), \tag{3.1.1}$$

where

$$a_j(y, \eta, h) = \frac{1}{(2j)!}\Big(\frac{\partial}{\partial t}\Big)^{2j}\big(q_j(y, \eta, t, h) f_h(t)\big)_{t=s_0(y,\eta,h)} \tag{3.1.2}$$

for a sequence of polynomials $\{q_j(y, \eta, t, h)\}_{j=0,1,2,\ldots}$ in one variable $t \in \mathbb{R}$ with coefficients being h-dependent functions in $\mathrm{C}^\infty(\mathbb{R}^{2n})$ and satisfying $q_0 \equiv 1$. In particular,

$$a_0(y, \eta, h) = f_h(s_0(y, \eta, h)).$$

Proof. We will adapt the proof of Dimassi and Sjöstrand of Theorem 2.1.4, extending it to h-dependent functions $f_h \in \mathcal{S}_\delta^{\mathrm{comp}}$. Let us briefly recall the main steps in the proof of Theorem 2.1.4. First, one uses the *Helffer-Sjöstrand formula* to express $f\big(\overline{\mathrm{Op}_h(s)}\big)$ as a complex integral which involves the resolvent of the operator $\overline{\mathrm{Op}_h(s)}$ and an almost analytic extension of f. Then one proves that the resolvent is a semiclassical pseudodifferential operator, and that its symbol has a certain asymptotic expansion whose terms are then plugged into the Helffer-Sjöstrand formula. That the resulting term-wise integrals exist and that they define elements of appropriate symbol classes is proved using the properties of the almost analytic extension of f. Finally, the precise algebraic form of the resulting symbol expansion for $f\big(\overline{\mathrm{Op}_h(s)}\big)$ is obtained by replacing the complex integrals in the expansion up to a negligible remainder by integrals that one can evaluate using the Cauchy integral formula. We will now precisely study the relevant steps in the proof of Theorem 2.1.4 and check how they need to be generalized or modified to work also for h-dependent functions f_h satisfying our regularity conditions. As mentioned before, the first step in the proof of Theorem 2.1.4 is the *Helffer-Sjöstrand formula*, see [17, Theorem 8.1]. Thus, let P be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Let $f \in \mathrm{C}^2_c(\mathbb{R})$ and let $\tilde{f} \in \mathrm{C}^1_c(\mathbb{C})$ be an extension of f with $\bar{\partial}_z \tilde{f}(z) = \mathrm{O}(|\mathrm{Im}\ z|)$, where $\bar{\partial}_z = \frac{1}{2}(\partial_x + i\partial_y)$ with the notation $z = x + iy$. Then

$$f(P) = \frac{-1}{\pi} \int\limits_{\mathbb{C}} \bar{\partial}_z \tilde{f}(z)(z-P)^{-1}\, dz, \tag{3.1.3}$$

where dz denotes the Lebesgue measure on $\mathbb{C}$, and the integral is a Riemann integral for functions with values in $\mathcal{B}(\mathcal{H})$. The statement (3.1.3) is of course applicable to $f = f_h \in \mathrm{C}^2_c(\mathbb{R})$ for each $h \in (0,1]$ separately. No generalization is needed here. The second key step in the proof of Theorem 2.1.4 is the following Lemma, see [17, Prop. 8.6]. Let s be a symbol function as in Theorem 2.1.3. Then for each $z \in \mathbb{C}$, $\mathrm{Im}\ z \neq 0$, there is a symbol function $r_z \in S_h(1_{\mathbb{R}^{2n}})$ such that

$$\Big(z - \overline{\mathrm{Op}_h(s)}\Big)^{-1} = \mathrm{Op}_h(r_z). \tag{3.1.4}$$

The family $\{r_z\}$, indexed by z, has the property that for each $2n$-dimensional non-negative multiindex α, there is a constant $C_\alpha > 0$ such that for $|z| \leq$ const., one has

$$|\partial_v^\alpha r_z(v,h)| \leq C_\alpha \max\Big(1, \frac{h^{1/2}}{|\mathrm{Im}\ z|}\Big)^{2n+1} |\mathrm{Im}\ z|^{-|\alpha|-1} \quad \forall\, v \in \mathbb{R}^{2n},\ h \in (0,h_0], \tag{3.1.5}$$

where the number $h_0 \in (0,1]$ is the same as in Theorem 2.1.3. The lemma does not involve the function f at all, so obviously it does not need to be modified when $f = f_h$. The third step, which in combination with the Helffer-Sjöstrand formula and (3.1.4), (3.1.5) yields that $f\big(\overline{\mathrm{Op}_h(s)}\big)$ is a semiclassical pseudodifferential operator for small h, is given by the assertion that

$$\int\limits_{|\mathrm{Im}\ z| \leq h^\gamma} \bar{\partial}_z \tilde{f}(z)\, r_z\, dz \in S_h^{-\infty}(1_{\mathbb{R}^{2n}}) \quad \forall\, \gamma > 0,\ \forall\, f \in \mathrm{C}^\infty_\mathrm{c}(\mathbb{R}). \tag{3.1.6}$$

It is proved using (3.1.5) and a particular choice for the extension map $f \mapsto \tilde{f}$ in the Helffer-Sjöstrand formula. Specifically, one considers the extension[1]

$$\tilde{f}(x+iy) = \frac{\psi_f(x)\chi(y)}{2\pi} \int\limits_{\mathbb{R}} e^{i(x+iy)\xi} \chi(y\xi) \mathcal{F}(f)(\xi)\, d\xi, \tag{3.1.7}$$

where $\mathcal{F}(f)$ is the Fourier transform of f, $\chi \in \mathrm{C}^\infty_\mathrm{c}(\mathbb{R},[0,1])$ is equal to 1 on $[-1,1]$, and $\psi_f \in \mathrm{C}^\infty_\mathrm{c}(\mathbb{R})$ is equal to 1 in a neighborhood of supp f. The main feature of the extension

[1]The multiplication with a cutoff function $\chi(y)$ in front of the integral is only implicitly mentioned at [17, p. 93/94].

map $\mathrm{C}_{\mathrm{c}}^{\infty}(\mathbb{R}) \to \mathrm{C}_{\mathrm{c}}^{\infty}(\mathbb{C})$ defined by (3.1.7) is that the functions $\tilde{f}$ in its image are *almost analytic*, meaning that for each $N \in \mathbb{N}$ there is a constant $C_N > 0$ such that

$$|\bar{\partial}_z \tilde{f}(z)| \leq C_N |\mathrm{Im}\ z|^N \qquad \forall\, z \in \mathbb{C}. \tag{3.1.8}$$

Now suppose that $f = f_h$. This time we cannot just apply the existing results separately to f_h for each h, because (3.1.6) is a statement about uniform estimates in $h \in (0,1]$. Of course, for each individual h, we obtain an almost analytic extension $\tilde{f}_h$ for which (3.1.8) holds. However, the constant C_N appearing in the inequality for some $\tilde{f}_h$ depends on the function f_h, and so in particular $C_N = C_N(h)$ depends on h. Since we need estimates that are uniform for $h \in (0,1]$, we cannot directly use (3.1.8) when $f = f_h$. Instead, we study the proof of (3.1.8) to deduce a more precise estimate with constants that are independent of the function f. In order to do this, let us make the additional assumptions about the function ψ_f that we have $|\psi_f| \leq 1, |\psi_f'| \leq 1$, and that $\psi_f \equiv 1$ in a closed interval $I_f = [m_f, M_f] \subset \mathbb{R}$ whose endpoints have distance 1 to the support of f, and that $\psi_f = 0$ outside $[m_f - 2, M_f + 2]$. As in [17, p. 94], one calculates for each $N \in \mathbb{N}$ and $f \in \mathrm{C}_{\mathrm{c}}^{\infty}(\mathbb{R})$

$$\begin{aligned}\bar{\partial}_z \tilde{f}(x+iy) = y^N \frac{i}{4\pi}\Big(\psi_f(x) \int_{\mathbb{R}} e^{i(x+iy)\xi} \chi_N(y\xi) \xi^{N+1} \mathcal{F}(f)(\xi)\, d\xi \\ + \psi_f'(x) \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i(x-\widetilde{x}+iy)\xi} \frac{\chi_N(y\xi)}{(\xi+i)^2} (i + D_{\widetilde{x}})^2 D_{\widetilde{x}}^N \Big(\frac{f(\widetilde{x})}{x - \widetilde{x} + iy}\Big)\, d\widetilde{x}\, d\xi\Big) \quad \forall\, y \in [-1,1],\end{aligned} \tag{3.1.9}$$

where we used the notation $\chi_N(t) := t^{-N}\chi'(t)$.[2] Due to our assumptions on ψ_f, $|x - \widetilde{x}|$ is bounded from below by 1 on the support of $\psi_f'(x) f(\widetilde{x})$, and we obtain from (3.1.9)

$$|\bar{\partial}_z \tilde{f}(x+iy)| \leq |y|^N C_N \Big(\big\|\xi^{N+1}\mathcal{F}(f)\big\|_{\mathrm{L}^1(\mathbb{R})} + \max_{0 \leq j \leq N+2} \Big\|f^{(j)}\Big\|_{\mathrm{L}^1(\mathbb{R})} \Big) \quad \forall\, y \in [-1,1], \tag{3.1.10}$$

where $C_N > 0$ is independent of f. Now we observe $(i\xi)^{N+1}\mathcal{F}(f) = \mathcal{F}\big(f^{(N+1)}\big)$, and in addition we note that for every Schwartz function f on $\mathbb{R}$ and every $j \in \{0, 1, 2, \ldots\}$

$$\Big\|\mathcal{F}\big(f^{(j)}\big)\Big\|_{\mathrm{L}^1(\mathbb{R})} \leq C_j \max_{0 \leq k \leq 2} \Big\|f^{(j+k)}\Big\|_{\mathrm{L}^1(\mathbb{R})}, \tag{3.1.11}$$

with $C_j > 0$ independent of f, see e.g. [63, Lemma 3.5]. Moreover, for a function with support of finite volume, we have the standard integral estimate

$$\Big\|f^{(l)}\Big\|_{\mathrm{L}^1(\mathbb{R})} \leq \mathrm{vol}(\mathrm{supp}\ f) \Big\|f^{(l)}\Big\|_{\infty} \quad \forall\, l \in \{0, 1, 2, \ldots\}. \tag{3.1.12}$$

Taking into account the estimates (3.1.11) and (3.1.12), the result (3.1.10) turns into

$$|\bar{\partial}_z \tilde{f}(z)| \leq C_N |\mathrm{Im}\ z|^N \mathrm{vol}(\mathrm{supp}\ f) \max_{0 \leq j \leq N+3} \Big\|f^{(j)}\Big\|_{\infty} \qquad \forall\, z \in \mathbb{C},\ |\mathrm{Im}\ z| \leq 1 \tag{3.1.13}$$

with new constants $C_N > 0$ that are independent of f. The estimate (3.1.13) is exactly the modified version of (3.1.8) that we were looking for. Setting $f = f_h$, we get for each $N \in \mathbb{N}$ and all $z \in \mathbb{C}$ with $|\mathrm{Im}\ z| \leq 1$

$$|\bar{\partial}_z \tilde{f}_h(z)| \leq C_N |\mathrm{Im}\ z|^N \mathrm{vol}(\mathrm{supp}\ f_h) \max_{0 \leq j \leq N+3} \Big\|f_h^{(j)}\Big\|_{\infty} \qquad \forall\, h \in (0,1], \tag{3.1.14}$$

[2] Note that $\chi' = 0$ in a neighborhood of 0.

with $C_N > 0$ independent of h. We will now use (3.1.14) to prove

$$\int\limits_{|\mathrm{Im}\ z|\leq h^\gamma} \bar{\partial}_z \tilde{f}_h(z)\, r_z\, dz \in S_h^{-\infty}(1_{\mathbb{R}^{2n}}) \quad \forall\, \gamma > \delta,\ f_h \in \mathcal{S}_\delta^{\mathrm{comp}} \tag{3.1.15}$$

which is slightly weaker than the f_h-version of (3.1.6), but sufficient for our purposes. Recall that the statement (3.1.15) means that for each $h \in (0, 1]$ the function on $\mathbb{R}^{2n}$ defined by the integral is smooth and one has for all $v \in \mathbb{R}^{2n},\ h \in (0, 1],\ \gamma > \delta,\ N = 0, 1, \ldots$:

$$\left|\partial_v^\alpha \int\limits_{|\mathrm{Im}\ z|\leq h^\gamma} \bar{\partial}_z \tilde{f}_h(z)\, r_z(v, h)\, dz\right| \leq C_{N,\alpha,\gamma} h^N \tag{3.1.16}$$

with constants $C_{N,\alpha,\gamma} > 0$. As $\partial_z \tilde{f}_h$ is compactly supported, and the integrand is smooth, we can interchange integration and differentiation, so that the function on $\mathbb{R}^{2n}$ defined by the integral is indeed smooth for each h. Let now $\gamma > \delta$. Then (3.1.5) and (3.1.14) imply that there is a number $h_0 \in (0, 1]$ such that for each $N = 0, 1, 2, \ldots$ and each α there is a $C_{N,\alpha} > 0$ such that for all $h \in (0, h_0]$

$$\begin{aligned}
&\left|\partial_v^\alpha \int\limits_{|\mathrm{Im}\ z|\leq h^\gamma} \bar{\partial}_z \tilde{f}_h(z)\, r_z(v, h)\, dz\right| \leq \int\limits_{|\mathrm{Im}\ z|\leq h^\gamma} \left|\bar{\partial}_z \tilde{f}_h(z)\, \partial_v^\alpha r_z(v, h)\right| dz \\
&\leq C_{N,\alpha} \mathrm{vol}(\mathrm{supp}\ f_h) \max_{0\leq j\leq N+3} \left\|f_h^{(j)}\right\|_\infty \int\limits_{\substack{|\mathrm{Im}\ z|\leq h^\gamma \\ z\in \mathrm{supp}\ \bar{\partial}_z \tilde{f}_h}} \max\Big(1, \frac{h^{1/2}}{|\mathrm{Im}\ z|}\Big)^{2n+1} |\mathrm{Im}\ z|^{-|\alpha|-1} |\mathrm{Im}\ z|^N\, dz.
\end{aligned}$$

For each $N \geq |\alpha| + 2n + 2$, we can estimate the final integral according to

$$\begin{aligned}
&\int\limits_{\substack{|\mathrm{Im}\ z|\leq h^\gamma \\ z\in \mathrm{supp}\ \bar{\partial}_z \tilde{f}_h}} \max\Big(1, \frac{h^{1/2}}{|\mathrm{Im}\ z|}\Big)^{2n+1} |\mathrm{Im}\ z|^{-|\alpha|-1} |\mathrm{Im}\ z|^N\, dz \\
&= \int\limits_{\substack{h^{1/2}\leq|\mathrm{Im}\ z|\leq h^\gamma \\ z\in \mathrm{supp}\ \bar{\partial}_z \tilde{f}_h}} |\mathrm{Im}\ z|^{-|\alpha|-1} |\mathrm{Im}\ z|^N\, dz \ +\ h^{n+1/2} \int\limits_{\substack{|\mathrm{Im}\ z|< h^{1/2} \\ z\in \mathrm{supp}\ \bar{\partial}_z \tilde{f}_h}} |\mathrm{Im}\ z|^{-|\alpha|-2n-2} |\mathrm{Im}\ z|^N\, dz \\
&\leq C'_{\alpha,N} \mathrm{vol}_{\mathbb{C}}\big(\mathrm{supp}\ \bar{\partial}_z \tilde{f}_h\big)\big(h^{\gamma(N-|\alpha|-1)} + h^{n+1/2(1+N-|\alpha|-2n-2)}\big) \\
&\leq 2C'_{\alpha,N} \mathrm{vol}_{\mathbb{C}}\big(\mathrm{supp}\ \bar{\partial}_z \tilde{f}_h\big)\big(h^{\min(\gamma,1/2)N-\max(\gamma,1/2)(|\alpha|+1)}\big).
\end{aligned}$$

Turning our attention to the term $\mathrm{vol}_{\mathbb{C}}\ \mathrm{supp}\ \bar{\partial}_z \tilde{f}_h$, note that by construction of the almost analytic extension $\tilde{f}_h$

$$\mathrm{supp}\ \bar{\partial}_z \tilde{f}_h \subset \mathrm{supp}\ \tilde{f}_h \subset (\mathrm{supp}\ \psi_{f_h}) \times [L, L]i \subset \mathbb{C}, \tag{3.1.17}$$

where $L > 0$ is some constant depending only on the cutoff function χ. Due to our assumptions on the function ψ_{f_h} in the paragraph before (3.1.9) one has

$$\mathrm{vol}\big(\mathrm{supp}\ \psi_{f_h}\big) \leq \mathrm{diam}(\mathrm{supp}\ f_h) + 2 + 2 \tag{3.1.18}$$

and we obtain $\mathrm{vol}_{\mathbb{C}}\big((\mathrm{supp}\ \psi_{f_h}) \times [L, L]i\big) \leq 2L(\mathrm{diam}(\mathrm{supp}\ f_h) + 4)$. Collecting all estimates together yields for $h \in (0, h_0]$ and $N \geq |\alpha| + 2n + 2$

$$\left|\partial_v^\alpha \int\limits_{|\mathrm{Im}\ z| \leq h^\gamma} \bar{\partial}_z \tilde{f}_h(z)\, r_z(v,h)\, dz\right| \leq C_{N,\alpha} h^{\min(\gamma,1/2)N - \max(\gamma,1/2)(|\alpha|+1)}$$
$$\mathrm{vol}(\mathrm{supp}\ f_h)(1 + \mathrm{diam}(\mathrm{supp}\ f_h)) \max_{0 \leq j \leq N+3} \left\| f_h^{(j)} \right\|_\infty, \quad (3.1.19)$$

for some new constant $C_{N,\alpha}$ which is independent of h. We now use the regularity conditions on the function $(t,h) \mapsto f_h(t)$ encoded in the assumption $f_h \in \mathcal{S}_\delta^{\mathrm{comp}}$. The condition that the function is in $S_{h;\delta}(1_{\mathbb{R}})$ yields

$$\max_{0 \leq j \leq N+3} \left\| f_h^{(j)} \right\|_\infty = \mathrm{O}\big(h^{-(N+3)\delta}\big) \qquad \text{as } h \to 0,$$

and because the diameter of the support of f_h grows at most polynomially in h^{-1} as $h \to 0$, there is a constant $r \geq 0$ such that $\mathrm{vol}(\mathrm{supp}\ f_h)(1 + \mathrm{diam}(\mathrm{supp}\ f_h)) = \mathrm{O}\big(h^{-r}\big)$ as $h \to 0$. Thus, we conclude

$$\left|\partial_v^\alpha \int\limits_{|\mathrm{Im}\ z| \leq h^\gamma} \bar{\partial}_z \tilde{f}_h(z)\, r_z(v,h)\, dz\right| \leq C_{N,\alpha}\, h^{N(\min(\gamma,1/2)-\delta) - \max(\gamma,1/2)(|\alpha|+1) - 3\delta - r} \qquad (3.1.20)$$

with a new constant $C_{N,\alpha}$. Given $N' \in \mathbb{N}$, we can set $N(N') := \left\lceil \frac{N' + \max(\gamma,1/2)(|\alpha|+1) + 3\delta + r}{\min(\gamma,1/2) - \delta} \right\rceil$ to obtain

$$\left|\partial_v^\alpha \int\limits_{|\mathrm{Im}\ z| \leq h^\gamma} \bar{\partial}_z \tilde{f}_h(z)\, r_z(v,h)\, dz\right| \leq C_{N(N'),\alpha}\, h^{N'} \quad \forall\, h \in (0, h_0]. \qquad (3.1.21)$$

Thus, after performing a global rescaling $h' := h/h_0$, we have shown (3.1.16), or equivalently (3.1.15). The next intermediate result in the proof of Theorem 2.1.4 that we want to generalize involves the integral over the whole complex plane. Namely, one easily obtains

$$\int\limits_{\mathbb{C}} \bar{\partial}_z \tilde{f}(z)\, r_z\, dz \in S_h(1_{\mathbb{R}^{2n}}) \quad \forall\, f \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}) \qquad (3.1.22)$$

by taking into account that the integrand has compact support and estimating its L^∞-norm using (3.1.5) and (3.1.8). Just as (3.1.6), (3.1.22) is a statement about uniform estimates in $h \in (0, 1]$, so it does not directly generalize to h-dependent functions. We would like to prove

$$\int\limits_{\mathbb{C}} \bar{\partial}_z \tilde{f}_h(z)\, r_z\, dz \in S_{h;\delta}(1_{\mathbb{R}^{2n}}) \quad \forall\, f_h \in \mathcal{S}_\delta^{\mathrm{comp}}. \qquad (3.1.23)$$

Let us try to prove (3.1.23) in the same way as (3.1.22) by estimating the L^∞-norm of the integrand and using that the integrand has compact support. With (3.1.16), we can write

$$\begin{aligned}
&\Big|\partial_v^\alpha \int_{\mathbb{C}} \bar\partial_z \tilde f_h(z)\, r_z(v,h)\, dz\Big| \\
&\qquad = \Big|\partial_v^\alpha \int_{|\mathrm{Im}\ z|\le h^{1/2}} \bar\partial_z \tilde f_h(z)\, r_z(v,h)\, dz + \partial_v^\alpha \int_{|\mathrm{Im}\ z|> h^{1/2}} \bar\partial_z \tilde f_h(z)\, r_z(v,h)\, dz\Big| \\
&\qquad\qquad \le \int_{\substack{|\mathrm{Im}\ z|> h^{1/2}\\ z\in \mathrm{supp}\ \bar\partial_z \tilde f_h}} \big|\bar\partial_z \tilde f_h(z)\, \partial_v^\alpha r_z(v,h)\big|\, dz + \mathrm{O}(h^\infty),
\end{aligned}$$

the $\mathrm{O}(h^\infty)$ estimate being uniform in v. Using (3.1.5), (3.1.14), (3.1.17), and (3.1.18), it follows that with $N = |\alpha| + 1$

$$\begin{aligned}
\Big|\partial_v^\alpha \int_{\mathbb{C}} \bar\partial_z \tilde f_h(z)\, r_z(v,h)\, dz\Big| &\le C_{N,\alpha}\mathrm{vol}(\mathrm{supp}\ f_h) \\
&\quad \max_{0\le j\le N+3} \Big\|f_h^{(j)}\Big\|_\infty \int_{\substack{|\mathrm{Im}\ z|> h^{1/2}\\ z\in \mathrm{supp}\ \bar\partial_z \tilde f_h}} |\mathrm{Im}\ z|^{-|\alpha|-1} |\mathrm{Im}\ z|^N\, dz + \mathrm{O}(h^\infty) \\
&\le C_{|\alpha|+1,\alpha}\mathrm{vol}(\mathrm{supp}\ f_h) \max_{0\le j\le |\alpha|+1+3} \Big\|f_h^{(j)}\Big\|_\infty \mathrm{vol}\ \mathrm{supp}\ \bar\partial_z \tilde f_h + \mathrm{O}(h^\infty) \\
&\le C_{\alpha,\delta}\mathrm{vol}(\mathrm{supp}\ f_h)(1+\mathrm{diam}(\mathrm{supp}\ f_h)) h^{-\delta(|\alpha|+4)} + \mathrm{O}(h^\infty) \\
&= \mathrm{O}\big(h^{-\delta(|\alpha|+4)-2r}\big),
\end{aligned}$$

where $r > 0$ is chosen such that the diameter of the support of f_h is of order h^{-r} as $h \to 0$. Thus, we arrive at the statement

$$\int_{\mathbb{C}} \bar\partial_z \tilde f_h(z)\, r_z\, dz \in S^{4\delta+2r}_{h;\delta}(1_{\mathbb{R}^{2n}}) \quad \forall\, f_h \in \mathcal{S}^{\mathrm{comp}}_\delta \tag{3.1.24}$$

which is considerably weaker than (3.1.23). Temporarily, (3.1.24) will be sufficient to continue with the proof, and we will deduce (3.1.23) later. As in the proof of Theorem 2.1.4, we deduce from (3.1.24)

$$\int_{\mathbb{C}} \bar\partial_z \tilde f_h(z)\, r_z\, dz \in S^{4\delta+2r}_{h;\delta}(\mathfrak{m}^{-k}) \qquad \forall\, k \in \{0,1,2,\dots\} \tag{3.1.25}$$

by writing $f_{h,k}(t) := (t+i)^k f_h(t)$ and observing that

$$f_h(\overline{\mathrm{Op}_h(s)}) = \big(\overline{\mathrm{Op}_h(s)} + i\big)^{-k} f_{h,k}\big(\overline{\mathrm{Op}_h(s)}\big),$$

see [17, Thm. 8.7]. To proceed, fix some $\gamma \in (\delta, \frac{1}{2})$. It is shown in the proof of Theorem 2.1.4 that if $|z| \ge h^\gamma$, $|z| \le$ const., the function r_z belongs to the symbol class $S^\gamma_{h;\gamma}(\mathfrak{m}^{-1})$ with estimates that are uniform in z, and r_z has an expansion in $S^\gamma_{h;\gamma}(\mathfrak{m}^{-1})$ of the form

$$r_z(y,\eta,h) \sim \sum_{j=0}^{\infty} h^j \sum_{k=0}^{2j} \frac{q_{j,k}(y,\eta,h)\, z^k}{(z - s_0(y,\eta,h))^{2j+1}}, \qquad q_{j,k} \in S_h(\mathfrak{m}^{2j-k}), \qquad q_{0,0} \equiv 1,$$

with uniform estimates on the domain $|z| \geq h^\gamma$, $|z| \leq$ const., see [17, p. 102]. Thus, by (3.1.3), (3.1.4), (3.1.16), and (3.1.25) one obtains $f_h(\overline{\mathrm{Op}_h(s)}) = \mathrm{Op}_h(a)$ with $a \in S^{4\delta+2r}_{h;\delta}(\mathfrak{m}^{-1})$ having an asymptotic expansion in $S^{4\delta+2r}_{h;\delta}(\mathfrak{m}^{-1})$

$$a \sim \sum_{j=0}^{\infty} h^j \,\widetilde{a}_j, \qquad \widetilde{a}_j(y,\eta,h) = \frac{-1}{\pi} \int\limits_{|z|\geq h^\gamma} \bar{\partial}_z \tilde{f}_h(z)\, \frac{q_j(y,\eta,z,h)}{(z - s_0(y,\eta,h))^{2j+1}}\, dz, \tag{3.1.26}$$

where we wrote $q_j(y,\eta,z,h) := \sum_{k=0}^{2j} q_{j,k}(y,\eta,h)\, z^k$. For the same reason why (3.1.15) holds, one can replace each $\widetilde{a}_j$ up to an error in $S_h^{-\infty}(\mathfrak{m}^{-1})$ by

$$\begin{aligned} a_j(y,\eta,h) &= \frac{-1}{\pi} \int\limits_{\mathbb{C}} \bar{\partial}_z \tilde{f}_h(z)\, \frac{q_j(y,\eta,z,h)}{(z - s_0(y,\eta,h))^{2j+1}}\, dz \\ &= \frac{1}{(2j)!} \Big(\frac{\partial}{\partial t}\Big)^{2j} \big(q_j(y,\eta,t,h) f_h(t)\big)_{t=s_0(y,\eta,h)}, \end{aligned}$$

where the evaluation of the complex integral between the first and the second line is as in [17, (8.16) on p. 103]. We obtain

$$a \sim \sum_{j=0}^{\infty} h^j\, a_j \quad \text{in } S^{4\delta+2r}_{h;\delta}(\mathfrak{m}^{-1}). \tag{3.1.27}$$

Now, since f_h is an element of $S_{h;\delta}(1_{\mathbb{R}})$ and only derivatives of f_h of order at most $2j$ occur in a_j, we conclude that $a_j \in S^{2j\delta}_{h;\delta}(\mathfrak{m}^{-1})$. Therefore, the expansion (3.1.27) implies that a is in fact an element of $S_{h;\delta}(\mathfrak{m}^{-1}) \subset S^{4\delta+2r}_{h;\delta}(\mathfrak{m}^{-1})$ and has the same expansion in $S_{h;\delta}(\mathfrak{m}^{-1})$. Thus, the evaluation of the complex integrals in the individual terms of the expansion (3.1.26) has finally provided a proof for (3.1.23). Just as we deduced (3.1.25) from (3.1.24), we deduce from (3.1.23) that $a \in \bigcap_{k\in\mathbb{N}} S_{h;\delta}(\mathfrak{m}^{-k})$. □

As a corollary, we get a semiclassical trace formula that generalizes Theorem 2.1.5.

Corollary 3.1.2. *Let $\mathfrak{m} : \mathbb{R}^{2n} \to (0,\infty)$ be an order function with $\mathfrak{m} \geq 1$, and $s \in S_h(\mathfrak{m})$ be a real-valued symbol function with an asymptotic expansion*

$$s \sim \sum_{j=0}^{\infty} h^j s_j \;\; \text{in } S_h(\mathfrak{m})$$

such that $s + i$ is $\mathfrak{m}$-elliptic. Let $I \subset \mathbb{R}$ be a bounded open interval with

$$\liminf_{\|v\|\to+\infty} \operatorname{dist}(s(v,h), I) \geq C \qquad \forall\, h \in (0,1]$$

for a constant $C > 0$ that is independent of h, and let $f_h \in \mathrm{C}^\infty_\mathrm{c}(I) \subset \mathrm{C}^\infty_\mathrm{c}(\mathbb{R})$ be given such that the function $(t,h) \mapsto f_h(t)$ is an element of the symbol class $S_{h;\delta}(1_{\mathbb{R}})$ for some $\delta \in [0,\frac{1}{2})$. Then, the operator $f_h\big(\overline{\mathrm{Op}_h(s)}\big) : \mathrm{L}^2(\mathbb{R}^n) \to \mathrm{L}^2(\mathbb{R}^n)$ is of trace class for small h, and as $h \to 0$ one has

$$\begin{aligned} \operatorname{tr}_{\mathrm{L}^2(\mathbb{R}^n)} f_h\Big(\overline{\mathrm{Op}_h(s)}\Big) &= \frac{1}{(2\pi h)^n} \int\limits_{\mathbb{R}^{2n}} f_h\big(s_0(y,\eta,h)\big)\, dy\, d\eta \\ &\qquad + \mathrm{O}\Big(h^{1-2\delta-n} \mathrm{vol}_{\mathbb{R}^{2n}}\big(\operatorname{supp} f_h \circ s_0(\cdot,h)\big)\Big). \end{aligned}$$

Proof. In view of Theorem 3.1.1 and Corollary 2.1.2, one can prove the statements of the theorem by complete analogy to [17, proof of Theorem 9.6]. □

Chapter 4

Results for closed Riemannian manifolds

In this chapter, we generalize Theorem 2.1.6 to functions which depend on h, and we establish explicit statements that can be used to prove trace formulas. For the whole chapter, let us fix the following setup. As introduced in Chapter 1, let M be a closed connected Riemannian manifold of dimension n. We choose a finite atlas $\{U_\alpha, \gamma_\alpha\}_{\alpha\in\mathcal{A}}$ with charts $\gamma_\alpha : U_\alpha \xrightarrow{\simeq} \mathbb{R}^n$, $U_\alpha \subset M$ open, and a subordinate partition of unity $\{\varphi_\alpha\}_{\alpha\in\mathcal{A}}$. For each $\alpha \in \mathcal{A}$, we choose in addition a compact set $K_\alpha \subset \mathbb{R}^n$ such that $\operatorname{supp} \varphi_\alpha \circ \gamma_\alpha^{-1} \subset \operatorname{Int}(K_\alpha)$, and three cutoff functions $\overline{\varphi}_\alpha, \overline{\overline{\varphi}}_\alpha, \overline{\overline{\overline{\varphi}}}_\alpha \in \mathrm{C}^\infty(M)$ with supports contained in $\gamma_\alpha^{-1}(\operatorname{Int}(K_\alpha)) \subset U_\alpha$ and with $\overline{\varphi}_\alpha \equiv 1$ on $\operatorname{supp} \varphi_\alpha$, $\overline{\overline{\varphi}}_\alpha \equiv 1$ on $\operatorname{supp} \overline{\varphi}_\alpha$, and $\overline{\overline{\overline{\varphi}}}_\alpha \equiv 1$ on $\operatorname{supp} \overline{\overline{\varphi}}_\alpha$. For a point $x \in U_\alpha$, let $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ be the coordinates of the point $\gamma_\alpha(x)$. Furthermore, we have a local metric $g_\alpha := (g_\alpha^{ij})$ with coefficients $g_\alpha^{ij} : \mathbb{R}^n \to \mathbb{R}$ and inverse matrix (g_{ij}^α), together with an associated volume density $\operatorname{Vol}_{g_\alpha}(y) := \sqrt{\det\, g_\alpha(y)}$. Note that $\lim_{\|y\|\to\infty} \operatorname{Vol}_{g_\alpha}(y) = 0$, since otherwise U_α would have infinite Riemannian volume in contradiction to the compactness of M. It follows that the positive function $y \to \operatorname{Vol}_{g_\alpha}(y)$ is bounded.

4.1 Technical preparations

The Schrödinger operator $\breve{P}(h)$ acts on a function $f \in \mathrm{C}_\mathrm{c}^\infty(U_\alpha) \subset \mathrm{C}^\infty(M)$ by the formula

$$\begin{aligned}\breve{P}(h)(f)(x) = \breve{S}_\alpha(h)(f \circ \gamma_\alpha^{-1})(y) := \frac{-h^2}{\operatorname{Vol}_{g_\alpha}(y)} \sum_{i,j=1}^{n} \frac{\partial}{\partial\, y_j} \left(g_{ij}^\alpha \operatorname{Vol}_{g_\alpha} \frac{\partial (f \circ \gamma_\alpha^{-1})}{\partial\, y_i} \right)(y) \\ + (V \circ \gamma_\alpha^{-1})(y) \cdot (f \circ \gamma_\alpha^{-1})(y) \qquad (4.1.1)\end{aligned}$$

for $x \in U_\alpha$, and $\breve{P}(h)(f)(x) = 0$ for $x \in M - U_\alpha$. The so defined operator

$$\breve{S}_\alpha(h) : \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^n) \to \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^n)$$

is a second order elliptic differential operator on $\mathbb{R}^n$, in the sense that its principal symbol is nowhere 0. However, $\breve{S}_\alpha(h)$ is not uniformly elliptic in the sense that its principal symbol is bounded away from 0, because the coefficients g_α^{ij} and g_{ij}^α can tend to zero towards infinity. To circumvent this problem, let $\tau_\alpha \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^n, [0,1])$ be a function which fulfills $\tau_\alpha \equiv 1$ in a neighborhood of K_α, and define a new differential operator $\mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^n) \to \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^n)$ by

$$\breve{P}_\alpha(h) := \tau_\alpha \breve{S}_\alpha(h) + (1-\tau_\alpha)(-h^2\Delta), \tag{4.1.2}$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial y_i^2}$. Clearly, $\breve{P}_\alpha(h)$ agrees with $\breve{S}_\alpha(h)$ on functions supported inside K_α. The reason why we introduced the new operator $\breve{P}_\alpha(h)$ is

Lemma 4.1.1. *Let $\mathfrak{m} : \mathbb{R}^2 \to (0,\infty)$ be the order function given by $(y,\eta) \mapsto \langle\eta\rangle^2$. Then, for each α, one has $\breve{P}_\alpha(h) = \mathrm{Op}_h(p_\alpha)$ for a real-valued symbol function $p_\alpha \in S_h(\mathfrak{m})$ such that $p_\alpha + i$ is $\mathfrak{m}$-elliptic. Furthermore, $\breve{P}_\alpha(h)$ has a unique self-adjoint extension $P_\alpha(h) : \mathrm{H}_h^2(\mathbb{R}^n) \to \mathrm{L}^2(\mathbb{R}^n)$, and for $z \in \mathbb{C}$, $\operatorname{Im} z \neq 0$, the resolvent $(P_\alpha(h)-z)^{-1} : \mathrm{L}^2(\mathbb{R}^n) \to \mathrm{H}_h^2(\mathbb{R}^n)$ exists as a bounded operator.*

Proof. Fix $\alpha \in \mathcal{A}$ and note that as M is compact we can assume without loss of generality that all the coefficients $g_\alpha^{ij}, g_{ij}^\alpha : \mathbb{R}^n \to \mathbb{R}$ and their derivatives are bounded. By (4.1.1) and (4.1.2),

$$\breve{P}_\alpha(h) = \mathrm{Op}_h(p_\alpha)$$

for a function $p_\alpha \in \mathrm{C}^\infty(\mathbb{R}^{2n} \times (0,1])$ of the form

$$p_\alpha(y,\eta,h) = \underbrace{\sum_{i,j=1}^n p_\alpha^{ij}(y)\eta_i\eta_j + V_\alpha(y)}_{=:p_{\alpha,0}(y,\eta)} + h\sum_{i=1}^n p_\alpha^i(y)\eta_i, \qquad V_\alpha := V \circ \gamma_\alpha^{-1}, \tag{4.1.3}$$

where

$$p_\alpha^{i,j}(y) = \tau_\alpha(y) g_\alpha^{ij}(y) + (1-\tau_\alpha(y))\delta^{ij}, \qquad p_\alpha^i \in \mathrm{C}^\infty(\mathbb{R}^n,\mathbb{R}),$$

and the functions p_α^i and V_α are bounded. Here, δ^{ij} is the *Kronecker delta*. Since all the coefficients in the polynomial p_α and all of their derivatives are bounded functions, there is for each non-negative $2n$-dimensional multiindex β a constant $C_\beta > 0$ such that $|\partial^\beta p_\alpha(y,\eta,h)| \leq C_\beta \langle\eta\rangle^2$ holds for all $(y,\eta) \in \mathbb{R}^{2n}$ and all $h \in (0,1]$. Thus, we conclude that $p_\alpha \in S_h(\mathfrak{m})$. It remains to show that there is some constant $\varepsilon_\alpha > 0$ such that $|p_\alpha + i| \geq \varepsilon_\alpha \mathfrak{m}$. Let $y \in \mathbb{R}^n$. Since $g_\alpha(y)$ is a norm-induced metric on $\mathbb{R}^n$ and all norm-induced metrics on $\mathbb{R}^n$ are equivalent, it holds $\eta^T g_\alpha(y)\eta \geq c_\alpha(y)|\eta|^2$ for some $c_\alpha(y) > 0$. Clearly, the function $\mathbb{R}^n \to (0,\infty)$, $y \mapsto c_\alpha(y)$, is smooth and thus assumes a minimum m_α on the compact support of τ_α. It follows

$$\Big|\sum_{i,j} p_\alpha^{i,j}(y)\eta_i\eta_j\Big| \geq \min(1,m_\alpha)|\eta|^2 \qquad \forall\, y \in \mathbb{R}^n. \tag{4.1.4}$$

Now, recall that p_α^i and V_α are bounded functions, which in view of (4.1.4) implies that we can find a constant $r_\alpha > 0$ such that $\big|h\sum_i p_\alpha^i(y)\eta_i + V_\alpha(y)\big| < \frac{1}{2}\big|\sum_{i,j} p_\alpha^{i,j}(y)\eta_i\eta_j\big|$ holds for all η with $|\eta| > r_\alpha$ and all $y \in \mathbb{R}^n$, $h \in (0,1]$. Thus, we conclude for $|\eta| > r_\alpha$:

$$\big|p_\alpha(y,\eta,h) + i\big|^2 \geq \Big(\frac{1}{2}\min(1,m_\alpha)|\eta|^2\Big)^2 + 1 \qquad \forall\, y \in \mathbb{R}^n,\ h \in (0,1].$$

Now, choose $R_\alpha \geq r_\alpha$ large enough and $C_\alpha > 0$ small enough such that

$$\Big(\frac{1}{2}\min(1,m_\alpha)|\eta|^2\Big)^2 + 1 \geq C_\alpha^2\big(|\eta|^2+1\big)^2$$

for all $|\eta| \geq R_\alpha$. Then $\left|p_\alpha(y,\eta,h)+i\right| \geq C_\alpha \langle\eta\rangle^2$ for all $|\eta| \geq R_\alpha$ and all $y \in \mathbb{R}^n$, $h \in (0,1]$. To obtain an analogous statement also for $|\eta| \leq R_\alpha$, note that we trivially have $|p_\alpha + i| \geq 1$, because p_α is real-valued. Assuming w.l.o.g. that $R_\alpha \geq 1$, we get

$$\left|p_\alpha(y,\eta,h)+i\right| \geq 1 = \frac{2R_\alpha^2}{2R_\alpha^2} \geq \frac{1}{2R_\alpha^2}|\eta|^2 + \frac{1}{2} \geq \frac{1}{2R_\alpha^2}|\eta|^2 + \frac{1}{2R_\alpha^2} = \frac{1}{2R_\alpha^2}\big(|\eta|^2+1\big).$$

We obtain for arbitrary (y,η,h) that $\left|p_\alpha(y,\eta,h)+i\right| \geq \widetilde{C}_\alpha \langle\eta\rangle^2$ with $\widetilde{C}_\alpha := \min\big(C_\alpha, \frac{1}{2R_\alpha^2}\big)$, so that we are done with the proof that $p_\alpha + i$ is $\mathfrak{m}$-elliptic. The remaining statements of the lemma follow from Theorem 2.1.3 and the observation (2.1.9). □

Lemma 4.1.2. *For each $\alpha \in \mathcal{A}$, define the vector space*

$$\mathrm{L}^2_{\mathrm{comp},\alpha}(M) := \big\{f \in \mathrm{L}^2(M),\ \text{ess. supp}\ f\circ\gamma_\alpha^{-1} \subset K_\alpha\big\},$$

and equip it with the norm induced from $\mathrm{L}^2(M)$. Then

$$\Gamma_\alpha^* : \mathcal{B}(\mathrm{L}^2(\mathbb{R}^n)) \to \mathcal{B}(\mathrm{L}^2_{\mathrm{comp},\alpha}(M), \mathrm{L}^2(M)), \qquad A \mapsto \big(f \mapsto \overline{A(f\circ\gamma_\alpha^{-1})\circ\gamma_\alpha}^{\,0}\big)$$

is a bounded linear operator, where $\overline{u}^0$ denotes continuation of the function u by zero outside U_α, and the vector spaces $\mathcal{B}(\mathrm{L}^2(M))$ and $\mathcal{B}(\mathrm{L}^2_{\mathrm{comp},\alpha}(M), \mathrm{L}^2(M))$ are each equipped with the operator norm.

Proof. First, note that a function $f \in \mathrm{L}^2_{\mathrm{comp},\alpha}(M)$ indeed pulls back to a function $f\circ\gamma_\alpha^{-1}$ in $\mathrm{L}^2(\mathbb{R}^n)$: As the volume density Vol_{g_α} is bounded, the only critical issue here is decay at infinity, and $f\circ\gamma_\alpha^{-1}$ has compact support. It now suffices to show that there are constants $C_\alpha, C'_\alpha > 0$ such that

$$\left\|f\circ\gamma_\alpha^{-1}\right\|_{\mathrm{L}^2(\mathbb{R}^n)} \leq C_\alpha \|f\|_{\mathrm{L}^2(M)} \qquad \forall\, f \in \mathrm{L}^2_{\mathrm{comp},\alpha}(M), \tag{4.1.5}$$

$$\|a\circ\gamma_\alpha\|_{\mathrm{L}^2(M)} \leq C'_\alpha \|a\|_{\mathrm{L}^2(\mathbb{R}^n)} \qquad \forall\, a \in \mathrm{L}^2(\mathbb{R}^n). \tag{4.1.6}$$

Then it will follow that

$$\|\Gamma_\alpha^*(A)\|_{\mathcal{B}(\mathrm{L}^2_{\mathrm{comp},\alpha}(M),\mathrm{L}^2(M))} \leq C_\alpha C'_\alpha \|A\|_{\mathcal{B}(\mathrm{L}^2(\mathbb{R}^n))} \qquad \forall\, A \in \mathcal{B}(\mathrm{L}^2(\mathbb{R}^n)). \tag{4.1.7}$$

The first relation (4.1.5) can be proved easily by observing that for a function $f \in \mathrm{L}^2_{\mathrm{comp},\alpha}(M)$ one has

$$\|f\|^2_{\mathrm{L}^2(M)} \geq \int\limits_{U_\alpha} |f|^2\, dM = \int\limits_{\mathbb{R}^n} |f(\gamma_\alpha^{-1}(y))|^2 \mathrm{Vol}_{g_\alpha}(y)\, dy = \int\limits_{K_\alpha} |f(\gamma_\alpha^{-1}(y))|^2 \mathrm{Vol}_{g_\alpha}(y)\, dy.$$

Setting

$$C_\alpha^{-2} := \min_{y\in K_\alpha} \mathrm{Vol}_{g_\alpha}(y) > 0,$$

it follows

$$\|f\|^2_{\mathrm{L}^2(M)} \geq C_\alpha^{-2} \int\limits_{K_\alpha} |f(\gamma_\alpha^{-1}(y))|^2\, dy = C_\alpha^{-2} \int\limits_{\mathbb{R}^n} |f(\gamma_\alpha^{-1}(y))|^2\, dy \equiv C_\alpha^{-2} \left\|f\circ\gamma_\alpha^{-1}\right\|^2_{\mathrm{L}^2(\mathbb{R}^n)}.$$

The assertion (4.1.6) follows from the boundedness of the function Vol_{g_α}. Namely, for $a \in \mathrm{L}^2(\mathbb{R}^n)$ we get

$$\begin{aligned}\|a\circ\gamma_\alpha\|^2_{\mathrm{L}^2(M)} \equiv \int_M |a\circ\gamma_\alpha|^2\,dM = \int_{U_\alpha} |a\circ\gamma_\alpha|^2\,dM = \int_{\mathbb{R}^n} |a(y)|^2 \mathrm{Vol}_{g_\alpha}(y)\,dy \\ \leq \underbrace{\Big(\sup_{y\in\mathbb{R}^n} \mathrm{Vol}_{g_\alpha}(y)\Big)}_{=:C'^2_\alpha} \int_{\mathbb{R}^n} |a(y)|^2\,dy \equiv C'^2_\alpha \,\|a\|^2_{\mathrm{L}^2(\mathbb{R}^n)}\,.\end{aligned}$$

□

The following resolvent estimate will be very useful.

Lemma 4.1.3. *For $z\in\mathbb{C}$, $\mathrm{Im}\ z \neq 0$, consider the resolvent $(P_\alpha(h)-z)^{-1}$ from Lemma 4.1.1 for some $\alpha\in\mathcal{A}$. Let $r,s\in \mathrm{C}^\infty_\mathrm{c}(\mathbb{R}^n)$ have disjoint supports and associated multiplication operators $\Phi_r,\Phi_s : \mathrm{L}^2(\mathbb{R}^n)\to\mathrm{L}^2(\mathbb{R}^n)$. Then, for each $N\in\mathbb{N}$ there is a constant $C_N>0$, depending on r and s, such that for $|z|\leq$ const. one has the estimate*

$$\big\|\Phi_r\circ(P_\alpha(h)-z)^{-1}\circ\Phi_s\big\|_{\mathcal{B}(\mathrm{L}^2(\mathbb{R}^n))} \leq C_N h^N |\mathrm{Im}\ z|^{-N-1}.$$

Proof. We owe the trick used in this proof to Maciej Zworski. Fix some $N\in\mathbb{N}$. Set $r_1 := r$ and choose functions $r_2\dots,r_N$ which fulfill $r_i\equiv 1$ on $\mathrm{supp}\ r_{i-1}$ and $\mathrm{supp}\ r_i\cap\mathrm{supp}\ s=\emptyset$ for $i\in\{2,\dots,N\}$. Let $\Phi_i:\mathrm{L}^2(\mathbb{R}^n)\to\mathrm{L}^2(\mathbb{R}^n)$ be the pointwise multiplication operator associated to r_i. Then we have $\Phi_1\circ\Phi_2\circ\cdots\circ\Phi_N=\Phi_r$. Next, observe that for any operator A on $\mathrm{L}^2(\mathbb{R}^n)$ the commutators $[P_\alpha(h)-z,A]$ and $[P_\alpha(h),A]$ agree, since z is just a multiple of the identity operator and hence has zero commutator. In addition, note that $\Phi_i\circ\Phi_s=0$ for all $i\in\{1,\dots,N\}$, by choice of the functions r_i, and that $\Phi_k\circ(P_\alpha(h)-z)\circ(\mathbf{1}-\Phi_{k+1})=0$, since $P_\alpha(h)-z$ is a differential operator and as such a local operator. With those observations, one verifies easily

$$\begin{aligned}(P_\alpha(h)-z)^{-1}\circ[P_\alpha(h),\Phi_1]\circ(P_\alpha(h)-z)^{-1}\circ[P_\alpha(h),\Phi_2]\circ \\ \cdots\circ(P_\alpha(h)-z)^{-1}\circ[P_\alpha(h),\Phi_N]\circ(P_\alpha(h)-z)^{-1}\circ\Phi_s \\ =\Phi_1\circ\Phi_2\circ\cdots\circ\Phi_N\circ(P_\alpha(h)-z)^{-1}\circ\Phi_s=\Phi_r\circ(P_\alpha(h)-z)^{-1}\circ\Phi_s.\end{aligned}$$

Each commutator is independent of z, and by [17, top of p. 102] we have for $|z|\leq$ const. the estimate

$$\big\|[P_\alpha(h),\Phi_i]\circ(P_\alpha(h)-z)^{-1}\big\|_{\mathcal{B}(\mathrm{L}^2(\mathbb{R}^n))} = \mathrm{O}(h|\mathrm{Im}\ z|^{-1}) \qquad \forall\, i\in\{1,\dots,N\}$$

and

$$\big\|(P_\alpha(h)-z)^{-1}\big\|_{\mathcal{B}(\mathrm{L}^2(\mathbb{R}^n))} = \mathrm{O}(|\mathrm{Im}\ z|^{-1}).$$

Therefore, we can conclude that

$$\begin{aligned}&\big\|\Phi_r\circ(P_\alpha(h)-z)^{-1}\circ\Phi_s\big\|_{\mathcal{B}(\mathrm{L}^2(\mathbb{R}^n))}\\ &\leq \big\|(P_\alpha(h)-z)^{-1}\big\|_{\mathcal{B}(\mathrm{L}^2(\mathbb{R}^n))}\big\|[P_\alpha(h),\Phi_1]\circ(P_\alpha(h)-z)^{-1}\big\|_{\mathcal{B}(\mathrm{L}^2(\mathbb{R}^n))}\\ &\qquad\cdots\big\|[P(h),\Phi_N]\circ(P_\alpha(h)-z)^{-1}\big\|_{\mathcal{B}(\mathrm{L}^2(\mathbb{R}^n))}\|\Phi_s\|_{\mathcal{B}(\mathrm{L}^2(\mathbb{R}^n))}\\ &\leq C_N h^N(|\mathrm{Im}\ z|^{-1})^{N+1}.\end{aligned}$$

□

4.2 Operator norm estimates

We can now state and prove our first theorem about the semiclassical functional calculus for h-dependent functions and the Schrödinger operator $P(h)$ on M with associated Hamiltonian p, relating the functional and symbolic calculi with operator norm remainder estimates. The theorem we will prove is actually much more explicit than Result 2 from the summary in Chapter 1, where it was stated in a condensed form. Choose $\varrho_h \in \mathcal{S}^{\mathrm{comp}}_\delta$. We then obtain for each $h \in (0,1]$ an operator $\varrho_h(P(h)) \in \mathcal{B}(\mathrm{L}^2(M))$. In addition, we introduce $B \in \Psi^0_{h;\delta}(M) \subset \mathcal{B}(\mathrm{L}^2(M))$ with principal symbol $[b]$, where $b \in S^0_{h;\delta}(M)$.

Theorem 4.2.1. *The family of operators $\{B \circ \varrho_h(P(h))\}_{h\in(0,1]} \subset \mathcal{B}(\mathrm{L}^2(M))$ has the following properties:*

- *There exists a constant $h_0 \in (0,1]$, a family of symbol functions $\{e_\alpha\}_{\alpha\in\mathcal{A}} \subset S_{h;\delta}(1_{\mathbb{R}^{2n}})$, and for each $h \in (0,h_0]$ an operator $R(h) \in \mathcal{B}(\mathrm{L}^2(M))$ such that*

$$(B \circ \varrho_h(P(h))(f) = \sum_{\alpha\in\mathcal{A}} \overline{\varphi}_\alpha \cdot \mathrm{Op}_h(e_\alpha)((\overline{\overline{\varphi}}_\alpha \cdot f) \circ \gamma_\alpha^{-1}) \circ \gamma_\alpha + R(h)(f) \tag{4.2.1}$$

  holds for all $h \in (0,h_0]$ and all $f \in \mathrm{L}^2(M)$, and

$$\|R(h)\|_{\mathcal{B}(\mathrm{L}^2(M))} = \mathrm{O}\big(h^\infty\big) \qquad \text{as } h \to 0.$$

  The operator $R(h)$ depends on B, p, ϱ_h, and the choice of the functions $\{\varphi_\alpha, \overline{\varphi}_\alpha, \overline{\overline{\varphi}}_\alpha\}_{\alpha\in\mathcal{A}}$.

- *For each $\alpha \in \mathcal{A}$, the symbol function e_α has an asymptotic expansion in $S_{h;\delta}(1_{\mathbb{R}^{2n}})$ of the form*

$$e_\alpha \sim \sum_{j=0}^{\infty} e_{\alpha,j}, \qquad e_{\alpha,j} \in S^{j(2\delta-1)}_{h;\delta}(1_{\mathbb{R}^{2n}}), \tag{4.2.2}$$

  where $e_{\alpha,j}$ is for fixed $h \in (0,h_0]$ an element of $\mathrm{C}^\infty_\mathrm{c}(\mathbb{R}^{2n})$, and

$$e_{\alpha,0} = \big((\varrho_h \circ p) \cdot b \cdot \varphi_\alpha\big) \circ (\gamma_\alpha^{-1}, (\partial\gamma_\alpha^{-1})^T). \tag{4.2.3}$$

  Moreover, for each α, j and each fixed h one has

$$\mathrm{supp}\ e_{\alpha,j} \subset \mathrm{supp}\ \big((\varrho_h \circ p) \cdot b \cdot \varphi_\alpha\big) \circ (\gamma_\alpha^{-1}, (\partial\gamma_\alpha^{-1})^T). \tag{4.2.4}$$

Proof. Let us first summarize briefly the strategy of the proof, which is divided into four steps. In Step 0, we use the Helffer-Sjöstrand formula to reduce the calculations involving $\varrho_h(P(h))$ to calculations involving the resolvent $(P(h)-z)^{-1}$ for $z \in \mathbb{C}$, Im $z \neq 0$, and estimates which are valid uniformly in z. In Step 1, we construct a parametrix which approximates $(P(h) - z)^{-1}$ up to an explicitly given remainder operator. In order to construct the parametrix, we localize the problem using the finite atlas $\{(U_\alpha, \gamma_\alpha)\}_{\alpha\in\mathcal{A}}$ for M and the partition of unity $\{\varphi_\alpha\}_{\alpha\in\mathcal{A}}$, obtaining a local parametrix for each coordinate chart, and sum up these local parametrices to a global parametrix. In Step 2, we plug the result of Step 1 into the Helffer-Sjöstrand formula which transforms the leading term in our calculations into a sum of pullbacks of operators in $\mathcal{B}(\mathrm{L}^2(\mathbb{R}^n))$. Then we can apply the semiclassical functional calculus on $\mathbb{R}^n$, and in particular Theorem 3.1.1. Finally, in Step 3, we use the concrete form of the obtained symbol functions from Step 2 to deduce the assertions (4.2.1-4.2.4).

Step 0. The operator $P(h)-z$ is invertible in $\mathcal{B}(\mathrm{L}^2(M))$ for $z \in \mathbb{C}$, Im $z \neq 0$, see [63, Lemma 14.6], and by the Helffer-Sjöstrand formula [17, Theorem 8.1] one has

$$\varrho_h(P(h)) = \frac{1}{i\pi} \int\limits_{\mathbb{C}} \bar{\partial}_z \tilde{\varrho}_h(z)(P(h)-z)^{-1}\, dz,$$

where dz denotes the Lebesgue measure, $\tilde{\varrho}_h : \mathbb{C} \to \mathbb{C}$ is the same almost analytic extension of ϱ_h as in (3.1.7), and $\bar{\partial}_z = (\partial_x + i\partial_y)/2$ when $z = x+iy$. The Helffer-Sjöstrand formula shows that $\varrho_h\,(P(h))$ will be expressed as a sum of pullbacks of operators in $\mathcal{B}(\mathrm{L}^2(\mathbb{R}^n))$ once we establish the same for the resolvent $(P(h)-z)^{-1}$. This is our strategy. Now, by Lemma 4.1.1, to the three global operators

$$\breve{P}(h) : \mathrm{C}^\infty(M) \to \mathrm{C}^\infty(M), \qquad P(h) : \mathrm{H}^2(M) \to \mathrm{L}^2(M), \qquad (P(h)-z)^{-1} : \mathrm{L}^2(M) \to \mathrm{H}^2(M)$$

there correspond three families of local operators, indexed by the finite atlas $\mathcal{A}$

$$\begin{aligned} \breve{P}_\alpha(h) : \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^n) \to \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^n), \qquad & P_\alpha(h) : \mathrm{H}^2_h(\mathbb{R}^n) \to \mathrm{L}^2(\mathbb{R}^n), \\ (P_\alpha(h)-z)^{-1} : \mathrm{L}^2(\mathbb{R}^n) \to \mathrm{H}^2_h(\mathbb{R}^n), & \end{aligned}$$

which are related to the global operators according to

$$\breve{P}(h)(f) = \breve{P}_\alpha(h)(f \circ \gamma_\alpha^{-1}) \circ \gamma_\alpha \qquad \forall\, f \in \mathrm{C}^\infty_{\mathrm{c}}(M),\ \mathrm{supp}\ f \circ \gamma_\alpha^{-1} \subset K_\alpha. \tag{4.2.5}$$

Step 1. In this step we will deduce a formula for $(P(h)-z)^{-1}$ using the family of local resolvents $\{(P_\alpha(h)-z)^{-1}\}_{\alpha\in\mathcal{A}}$. For each $\alpha \in \mathcal{A}$, denote by $\Phi_\alpha, \overline{\Phi}_\alpha, \overline{\overline{\Phi}}_\alpha$ the operators $\mathrm{L}^2(M) \to \mathrm{L}^2(M)$ given by pointwise multiplication with $\varphi_\alpha, \overline{\varphi}_\alpha, \overline{\overline{\varphi}}_\alpha$, and by $\Psi_\alpha, \overline{\Psi}_\alpha, \overline{\overline{\Psi}}_\alpha$ the operators $\mathrm{L}^2(\mathbb{R}^n) \to \mathrm{L}^2(\mathbb{R}^n)$ given by pointwise multiplication with $\varphi_\alpha\circ\gamma_\alpha^{-1}, \overline{\varphi}_\alpha\circ\gamma_\alpha^{-1}, \overline{\overline{\varphi}}_\alpha\circ\gamma_\alpha^{-1}$, respectively. We will denote (bi-)restrictions of these operators to linear subspaces of their domains by the same symbols. Furthermore, let us introduce the pullback maps

$$\begin{aligned} \gamma_\alpha^* &: \mathcal{L}(\mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^n), \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)) \to \mathcal{L}(\mathrm{C}^\infty_{\mathrm{c}}(U_\alpha), \mathrm{C}^\infty_{\mathrm{c}}(U_\alpha)), \qquad A \mapsto \big(f \mapsto A(f \circ \gamma_\alpha^{-1}) \circ \gamma_\alpha\big), \\ \gamma_\alpha^{-1*} &: \mathcal{L}(\mathrm{C}^\infty_{\mathrm{c}}(U_\alpha), \mathrm{C}^\infty_{\mathrm{c}}(U_\alpha)) \to \mathcal{L}(\mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^n), \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)), \qquad A \mapsto \big(f \mapsto A(f \circ \gamma_\alpha) \circ \gamma_\alpha^{-1}\big), \end{aligned}$$

which are each other's inverses. Elliptic regularity implies that the resolvent $(P_\alpha(h)-z)^{-1}$ induces an operator

$$(P_\alpha(h)-z)^{-1}|_{\mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)} : \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^n) \to \mathrm{C}^\infty(\mathbb{R}^n).$$

Regarding $\mathrm{C}^\infty_{\mathrm{c}}(U_\alpha)$ as a subset of $\mathrm{C}^\infty(M)$ for each α, we can define an operator $\mathrm{C}^\infty(M) \to \mathrm{C}^\infty(M)$ by

$$Y(h,z) := \sum_{\alpha\in\mathcal{A}} \gamma_\alpha^*\big(\Psi_\alpha \circ (P_\alpha(h)-z)^{-1}|_{\mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)}\big) \circ \overline{\Phi}_\alpha.$$

Using that $\breve{P}(h) - z$ is a local operator and taking into account (4.2.5), one now computes

$$\begin{aligned}
&Y(h,z) \circ (\breve{P}(h) - z) \\
&= \sum_{\alpha \in \mathcal{A}} \gamma_\alpha^* \big(\Psi_\alpha \circ (P_\alpha(h) - z)^{-1}|_{\mathrm{C}_c^\infty(\mathbb{R}^n)}\big) \circ \overline{\Phi}_\alpha \circ (\breve{P}(h) - z) \circ \overline{\overline{\Phi}}_\alpha \\
&= \sum_{\alpha \in \mathcal{A}} \Big[\gamma_\alpha^* \big(\Psi_\alpha \circ (P_\alpha(h) - z)^{-1}|_{\mathrm{C}_c^\infty(\mathbb{R}^n)}\big) \circ \gamma_\alpha^* \big(\breve{P}_\alpha(h) - z\big) \circ \overline{\overline{\Phi}}_\alpha && (4.2.6) \\
&\qquad - \underbrace{\gamma_\alpha^* \big(\Psi_\alpha \circ (P_\alpha(h) - z)^{-1}|_{\mathrm{C}_c^\infty(\mathbb{R}^n)}\big) \circ (\mathbf{1} - \overline{\Phi}_\alpha) \circ (\breve{P}(h) - z) \circ \overline{\overline{\Phi}}_\alpha}_{=:\widetilde{\mathcal{R}}_\alpha(h,z)} \Big] \\
&= \sum_{\alpha \in \mathcal{A}} \Big[\gamma_\alpha^* \Big(\Psi_\alpha \circ (P_\alpha(h) - z)^{-1}|_{\mathrm{C}_c^\infty(\mathbb{R}^n)} \circ (\breve{P}_\alpha(h) - z)\Big) \circ \overline{\overline{\Phi}}_\alpha - \widetilde{\mathcal{R}}_\alpha(h,z)\Big] \\
&= \sum_{\alpha \in \mathcal{A}} \Big[\gamma_\alpha^* \big(\Psi_\alpha\big) \circ \overline{\overline{\Phi}}_\alpha - \widetilde{\mathcal{R}}_\alpha(h,z)\Big] = \sum_{\alpha \in \mathcal{A}} \Big[\Phi_\alpha - \widetilde{\mathcal{R}}_\alpha(h,z)\Big] \\
&= \mathbf{1}_{\mathrm{C}^\infty(M)} - \sum_{\alpha \in \mathcal{A}} \widetilde{\mathcal{R}}_\alpha(h,z). && (4.2.7)
\end{aligned}$$

Note how we inserted the additional cutoff operator $\overline{\overline{\Phi}}_\alpha$ before (4.2.6) to be able to split off a remainder term which involves an operator that is composed from the left and from the right with multiplication operators by functions whose supports are disjoint. It immediately follows from (4.2.7) that

$$(P(h) - z)^{-1}|_{\mathrm{C}^\infty(M)} = Y(h,z) + \sum_{\alpha \in \mathcal{A}} \mathcal{R}_\alpha(h,z), \tag{4.2.8}$$

where

$$\mathcal{R}_\alpha(h,z) := \widetilde{\mathcal{R}}_\alpha(h,z) \circ (P(h) - z)^{-1}|_{\mathrm{C}^\infty(M)}.$$

We introduced the pullbacks γ_α^* and γ_α^{-1*} for temporary use because they are inverses of each other and they respect compositions of operators, allowing the easy construction of the parametrix $Y(h,z)$ on $\mathrm{C}^\infty(M)$. To get statements about operators in $\mathcal{B}(\mathrm{L}^2(M))$, we will work from now on with the pullback Γ_α^* from Lemma 4.1.2. Taking into account Lemma 4.1.1, we observe that the bounded operator

$$\sum_{\alpha \in \mathcal{A}} \Phi_\alpha \circ \Gamma_\alpha^* \big((P_\alpha(h) - z)^{-1}\big) \circ \overline{\Phi}_\alpha : \mathrm{L}^2(M) \to \mathrm{L}^2(M)$$

agrees with $Y(h,z)$ on $\mathrm{C}^\infty(M)$. As $\mathrm{C}^\infty(M)$ is dense in $\mathrm{L}^2(M)$, it follows from (4.2.8) that

$$(P(h) - z)^{-1} = \sum_{\alpha \in \mathcal{A}} \Big[\Phi_\alpha \circ \Gamma_\alpha^* \big((P_\alpha(h) - z)^{-1}\big) \circ \overline{\Phi}_\alpha + \mathfrak{R}_\alpha(h,z)\Big], \tag{4.2.9}$$

where

$$\mathfrak{R}_\alpha(h,z) := \Phi_\alpha \circ \Gamma_\alpha^* \big((P_\alpha(h) - z)^{-1}\big) \circ (\mathbf{1} - \overline{\Phi}_\alpha) \circ (P(h) - z) \circ \overline{\overline{\Phi}}_\alpha \circ (P(h) - z)^{-1}. \tag{4.2.10}$$

Step 2. Plugging the result of Step 1 into the Helffer-Sjöstrand formula yields

$$\begin{aligned}
\varrho_h(P(h)) &= \frac{1}{i\pi} \int_{\mathbb{C}} \bar{\partial}_z \tilde{\varrho}_h(z) \Big(\sum_{\alpha \in \mathcal{A}} \Big[\Phi_\alpha \circ \Gamma_\alpha^* \big((P_\alpha(h) - z)^{-1}\big) \circ \overline{\Phi}_\alpha + \mathfrak{R}_\alpha(h,z)\Big]\Big)\, dz \\
&= \sum_{\alpha \in \mathcal{A}} \Big[\Phi_\alpha \circ \Gamma_\alpha^* \Big(\frac{1}{i\pi} \int_{\mathbb{C}} \bar{\partial}_z \tilde{\varrho}_h(z) (P_\alpha(h) - z)^{-1}\, dz\Big) \circ \overline{\Phi}_\alpha + \mathfrak{R}_\alpha(h)\Big],
\end{aligned}$$

where
$$\mathfrak{R}_\alpha(h) = \frac{1}{i\pi} \int\limits_{\mathbb{C}} \bar{\partial}_z \tilde{\varrho}_h(z) \mathfrak{R}_\alpha(h,z)\, dz.$$

For each α, the functional calculus for the operator $P_\alpha(h)$ applies (by Lemma 4.1.1 and [17, Theorem 8.1]) and gives
$$\frac{1}{i\pi} \int\limits_{\mathbb{C}} \bar{\partial}_z \tilde{\varrho}_h(z) (P_\alpha(h) - z)^{-1}\, dz = \varrho_h\left(P_\alpha(h)\right).$$

We obtain the result
$$\varrho_h(P(h)) = \sum_{\alpha \in \mathcal{A}} \Big[\Phi_\alpha \circ \Gamma_\alpha^* \big(\varrho_h\left(P_\alpha(h)\right) \big) \circ \overline{\Phi}_\alpha + \mathfrak{R}_\alpha(h) \Big]. \tag{4.2.11}$$

This formula expresses the bounded operator $\varrho_h(P(h)) : \mathrm{L}^2(M) \to \mathrm{L}^2(M)$ in terms of the bounded operators $\varrho_h\left(P_\alpha(h)\right) : \mathrm{L}^2(\mathbb{R}^n) \to \mathrm{L}^2(\mathbb{R}^n)$, up to the remainder $\sum_{\alpha \in \mathcal{A}} \mathfrak{R}_\alpha(h)$. We proceed by estimating for fixed α the operator norm of $\mathfrak{R}_\alpha(h)$. In order to do this, we note that with (4.2.10)
$$\mathfrak{R}_\alpha(h) = \frac{1}{i\pi} \int\limits_{\mathbb{C}} \bar{\partial}_z \tilde{\varrho}_h(z) \Gamma_\alpha^* \Big(\Psi_\alpha \circ (P_\alpha(h) - z)^{-1} \circ (\mathbf{1} - \overline{\Psi}_\alpha) \Big) \circ (P(h) - z) \circ \overline{\overline{\Phi}}_\alpha \circ (P(h) - z)^{-1}\, dz.$$

Here we have replaced the operators Φ_α and $(\mathbf{1} - \overline{\Phi}_\alpha)$ by the corresponding operators inside the pullback. We now want to estimate the operator norm of the remainder operator $\mathfrak{R}_\alpha(h)$ by estimating the operator norm of the integrand, which works because the integration domain is in fact the support of $\bar{\partial}_z \tilde{\varrho}_h(z)$ which is compact and thus has finite volume. By Lemma 4.1.3 and Lemma 4.1.2, we get for each $N \in \mathbb{N}$ a constant $C_N > 0$ such that for $|z| \leq$ const.
$$\Big\| \Gamma_\alpha^* \Big(\Psi_\alpha \circ (P_\alpha(h) - z)^{-1} \circ (\mathbf{1} - \overline{\Psi}_\alpha) \Big) \Big\|_{\mathcal{B}(\mathrm{L}^2_{\mathrm{comp},\alpha}(M), \mathrm{L}^2(M))} \leq C_N h^N |\mathrm{Im}\ z|^{-N-1}.$$

This crucial estimate is precisely the reason why it was helpful to split off the remainder term the way we did in (4.2.6). Moreover, when introducing a commutator, we get for $|z| \leq$ const.
$$\begin{aligned} \Big\| (P(h) - z) \circ \overline{\overline{\Phi}}_\alpha \circ (P(h) - z)^{-1} \Big\|_{\mathcal{B}(\mathrm{L}^2(M))} &= \Big\| \big[P(h), \overline{\overline{\Phi}}_\alpha\big] \circ (P(h) - z)^{-1} + \overline{\overline{\Phi}}_\alpha \Big\|_{\mathcal{B}(\mathrm{L}^2(M))} \\ &\leq \Big\| \big[P(h), \overline{\overline{\Phi}}_\alpha\big] \circ (P(h) - z)^{-1} \Big\|_{\mathcal{B}(\mathrm{L}^2(M))} + \Big\| \overline{\overline{\Phi}}_\alpha \Big\|_{\mathcal{B}(\mathrm{L}^2(M))} \leq Ch(|\mathrm{Im}\ z|)^{-1} + 1, \end{aligned}$$

so that in total we obtain a new set of constants $\{C'_N\}$, $N = 0, 1, 2, \ldots$, such that for $|z| \leq$ const.
$$\begin{aligned} \Big\| \Gamma_\alpha^* \Big(\Psi_\alpha \circ (P_\alpha(h) - z)^{-1} \circ (\mathbf{1} - \overline{\Psi}_\alpha) \Big) \circ (P(h) - z) \circ \overline{\overline{\Phi}}_\alpha \circ (P(h) - z)^{-1} \Big\|_{\mathcal{B}(\mathrm{L}^2(M))} \\ \leq C'_N h^N |\mathrm{Im}\ z|^{-N-1}. \end{aligned} \tag{4.2.12}$$

We thus have successfully estimated the operator norm of $\mathfrak{R}_\alpha(z,h)$. In order to estimate also the supremum norm of the function $\bar{\partial}_z \tilde{\varrho}_h(z)$, recall from (3.1.14) that for $N = 0, 1, 2, \ldots$, there is a constant $C_N > 0$ such that
$$|\bar{\partial}_z \tilde{\varrho}_h(z)| \leq C_N |\mathrm{Im}\ z|^N \mathrm{vol}(\mathrm{supp}\ \varrho_h) \max_{0 \leq j \leq N+3} \Big\| \varrho_h^{(j)} \Big\|_\infty \qquad \forall\, z \in \mathbb{C},\ \forall\, h \in (0,1].$$

Together with (4.2.12), this implies that we get for each $N \in \mathbb{N}$ a new constant $C_N > 0$ such that for all $u \in \mathrm{L}^2(M)$ and for $|z| \leq$ const.

$$\begin{aligned} &|\bar{\partial}_z \tilde{\varrho}_h(z)|^2 \, \|\mathfrak{R}_\alpha(z,h)u\|^2_{\mathrm{L}^2(M)} \\ &\qquad \leq C_N h^{2N} \mathrm{vol}(\mathrm{supp}\ \varrho_h)^2 \max_{0\leq j\leq N+4} \left\| \varrho_h^{(j)} \right\|^2_\infty \|u\|^2_{\mathrm{L}^2(M)} \quad \forall\, h \in (0,1]. \end{aligned} \tag{4.2.13}$$

Thus, for all $h \in (0,1]$ and $N \in \mathbb{N}$ one has

$$\begin{aligned} \|\mathfrak{R}_\alpha(h)u\|^2_{\mathrm{L}^2(M)} &= \int_M \left| \frac{1}{i\pi} \int_{\mathbb{C}} \bar{\partial}_z \tilde{\varrho}_h(z) \mathfrak{R}_\alpha(z,h)(u)(x)\, dz \right|^2 dM(x) \\ \leq \frac{1}{\pi} \int_{\mathbb{C}} \int_M |\bar{\partial}_z \tilde{\varrho}_h(z)|^2 \big|\mathfrak{R}_\alpha(z,h)(u)(x)\big|^2 \, dM(x)\, dz &= \frac{1}{\pi} \int_{\mathbb{C}} |\bar{\partial}_z \tilde{\varrho}_h(z)|^2 \, \|\mathfrak{R}_\alpha(z,h)u\|^2_{\mathrm{L}^2(M)} \, dz \\ &\leq C_N h^{2N} \mathrm{vol}_{\mathbb{C}}(\mathrm{supp}\ \bar{\partial}_z \tilde{\varrho}_h) \mathrm{vol}(\mathrm{supp}\ \varrho_h)^2 \max_{0\leq j\leq N+4} \left\| f_h^{(j)} \right\|^2_\infty \|u\|^2_{\mathrm{L}^2(M)}. \end{aligned}$$

Note that (3.1.5) and (3.1.14) imply for each $h \in (0,1]$ that the function $M \times \mathbb{C} \to \mathbb{R}$ given by

$$(x,z) \mapsto |\bar{\partial}_z \tilde{\varrho}_h(z)|^2 \big|\mathfrak{R}_\alpha(z,h)(u)(x)\big|^2$$

has finite L^1-norm with respect to the product measure $dz\, dM$. This justifies the application of the Fubini theorem. We are now in essentially the same situation as we were in (3.1.19), so that with analogous arguments as in the lines following (3.1.19) we conclude

$$\|\mathfrak{R}_\alpha(h)u\|^2_{\mathrm{L}^2(M)} = \mathrm{O}(h^\infty)\, \|u\|^2_{\mathrm{L}^2(M)}$$

with estimates independent of u, and as $u \in \mathrm{L}^2(M)$ was arbitrary, it follows

$$\|\mathfrak{R}_\alpha(h)\|_{\mathcal{B}(\mathrm{L}^2(M))} = \mathrm{O}(h^\infty).$$

The estimation of the operator norm of the remainder is now almost complete. Namely, since $\mathcal{A}$ is finite, we can re-write (4.2.11) as

$$\varrho_h(P(h)) = \sum_{\alpha\in\mathcal{A}} \Phi_\alpha \circ \Gamma_\alpha^*\big(\varrho_h\left(P_\alpha(h)\right)\big) \circ \overline{\Phi}_\alpha + \widetilde{R}(h), \tag{4.2.14}$$

where

$$\widetilde{R}(h) := \sum_{\alpha\in\mathcal{A}} \mathfrak{R}_\alpha(h) : \mathrm{L}^2(M) \to \mathrm{L}^2(M)$$

has operator norm of order h^∞. Next, we compose with the operator B and an additional cutoff operator. That yields

$$B \circ \varrho_h(P(h)) = \sum_{\alpha\in\mathcal{A}} \overline{\Phi}_\alpha \circ B \circ \Phi_\alpha \circ \Gamma_\alpha^*\big(\varrho_h\left(P_\alpha(h)\right)\big) \circ \overline{\Phi}_\alpha + R(h), \tag{4.2.15}$$

where

$$R(h) := \sum_{\alpha\in\mathcal{A}} (\mathbf{1} - \overline{\Phi}_\alpha) \circ B \circ \Phi_\alpha \circ \Gamma_\alpha^*\big(\varrho_h\left(P_\alpha(h)\right)\big) \circ \overline{\Phi}_\alpha \;+\; B \circ \widetilde{R}(h). \tag{4.2.16}$$

As φ_α and $1-\overline{\varphi}_\alpha$ have disjoint supports, the operator norm of $(\mathbf{1}-\overline{\Phi}_\alpha)\circ B\circ\Phi_\alpha$ is of order h^∞. Moreover, by Lemma 4.1.2 and the spectral theorem, we have

$$\begin{aligned}\left\|\Gamma_\alpha^*\big(\varrho_h\left(P_\alpha(h)\right)\big)\right\|_{\mathcal{B}(\mathrm{L}^2_{\mathrm{comp},\alpha}(M),\mathrm{L}^2(M))}\\ \leq C\left\|\varrho_h\left(P_\alpha(h)\right)\right\|_{\mathcal{B}(\mathrm{L}^2(\mathbb{R}^n))}\leq C\left\|\varrho_h\right\|_\infty\leq C' \qquad \forall\, h\in(0,1]\end{aligned}$$

with constants $C,C'>0$, the last inequality being a consequence of the assumption $\varrho_h\in S_{h;\delta}(1_{\mathbb{R}})$. In addition, we know that the operator norm of $\widetilde{R}(h)$ is of order h^∞. From these observations, it follows

$$\|R(h)\|_{\mathcal{B}(\mathrm{L}^2(M))}=\mathrm{O}(h^\infty).$$

Step 3. We now express the summands in the leading term of (4.2.15) as pullbacks of semiclassical pseudodifferential operators on $\mathbb{R}^n$. Since B is a semiclassical pseudodifferential operator of order $(0,\delta)$ with principal symbol $[b]$, one has for each α

$$\overline{\Phi}_\alpha\circ B\circ\Phi_\alpha=\overline{\Phi}_\alpha\circ\Gamma_\alpha^*\big(\mathrm{Op}_h(b_\alpha)\big)\circ\Phi_\alpha,\qquad b_\alpha\in S_{h;\delta}(1_{\mathbb{R}^{2n}}),\tag{4.2.17}$$

with a symbol function b_α that has the property

$$b_\alpha=b\circ\gamma_\alpha^{-1}+h^{1-2\delta}\,\widetilde{b}_\alpha,\qquad \widetilde{b}_\alpha\in S_{h;\delta}(1_{\mathbb{R}^{2n}}).\tag{4.2.18}$$

To proceed, note that by Lemma 4.1.1, we can apply Theorem 3.1.1 to $P_\alpha(h)$ for each α, which gives us a symbol function $s_\alpha\in\bigcap_{k\in\mathbb{N}}S_{h;\delta}(\mathfrak{m}^{-k})$, where $\mathfrak{m}(y,\eta)=\langle\eta\rangle^2$, and a number $h_{0,\alpha}\in(0,1]$ such that for $h\in(0,h_{0,\alpha}]$

$$\varrho_h\left(P_\alpha(h)\right)=\mathrm{Op}_h(s_\alpha).\tag{4.2.19}$$

Each local operator $\varrho_h\left(P_\alpha(h)\right)$ is thus a semiclassical pseudodifferential operator. Moreover, Theorem 3.1.1 implies that there is an asymptotic expansion in $S_{h;\delta}(1/\mathfrak{m})$

$$s_\alpha\sim\sum_{j=0}^\infty s_{\alpha,j},\qquad s_{\alpha,j}(y,\eta,h)=\frac{1}{(2j)!}\Big(\frac{\partial}{\partial t}\Big)^{2j}\big(q_j(y,\eta,t,h)\varrho_h(t)\big)_{t=p_{\alpha,0}(y,\eta)}\tag{4.2.20}$$

for a sequence of polynomials $\{q_j(t)\}_{j=0,1,2,\dots}$ in one variable $t\in\mathbb{R}$ with coefficients being h-dependent functions in $\mathrm{C}^\infty(\mathbb{R}^{2n})$ and with $q_0\equiv 1$. In particular, one has $s_{\alpha,j}\in S^{j(2\delta-1)}_{h;\delta}(1/\mathfrak{m})$ and

$$s_{\alpha,0}(y,\eta,h)=\varrho_h(p_{\alpha,0}(y,\eta)),\tag{4.2.21}$$

where

$$p_{\alpha,0}(y,\eta)=\tau_\alpha(y)(|\eta|^2_{g_\alpha(y)}+V_\alpha(y))+(1-\tau_\alpha(y))|\eta|^2\tag{4.2.22}$$

is the h^0-coefficient in the full symbol of $\check{P}_\alpha(h)$, see (4.1.3). Since $1/\mathfrak{m}\leq 1$, it holds $S^{j(2\delta-1)}_{h;\delta}(1/\mathfrak{m})\subset S^{j(2\delta-1)}_{h;\delta}(1_{\mathbb{R}^{2n}})$, so that we can replace in the statements above $S^{j(2\delta-1)}_{h;\delta}(1/\mathfrak{m})$ with $S^{j(2\delta-1)}_{h;\delta}(1_{\mathbb{R}^{2n}})$, obtaining in particular $s_\alpha\in S_{h;\delta}(1_{\mathbb{R}^{2n}})$. Set $h_0:=\min_{\alpha\in\mathcal{A}}h_{0,\alpha}>0$. By (4.2.15), (4.2.17), and (4.2.19), we have proved that one has for all $h\in(0,h_0]$

$$B\circ\varrho_h(P(h))=\sum_{\alpha\in\mathcal{A}}\overline{\Phi}_\alpha\circ\Gamma_\alpha^*\big(\mathrm{Op}_h(b_\alpha)\circ\Psi_\alpha\circ\mathrm{Op}_h(s_\alpha)\big)\circ\overline{\Phi}_\alpha+R(h).\tag{4.2.23}$$

Let us now prove that the function $s_{\alpha,j}(y,\cdot,h):\eta\mapsto s_{\alpha,j}(y,\eta,h)$ is an element of $\mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ for each y,h,j which fulfills

$$\operatorname{supp}s_{\alpha,j}(\gamma_\alpha(x),\cdot,h)\subset(\partial\gamma_\alpha)^T\big((\operatorname{supp}\varrho_h\circ p)\cap T_x^*M\big)\quad\forall\,x\in\gamma_\alpha^{-1}(K_\alpha).\tag{4.2.24}$$

Indeed, this statement follows from formula (4.2.20). By that formula, at each point $(y,\eta,h)\in\mathbb{R}^{2n}\times(0,1]$ the number $s_{\alpha,j}(y,\eta,h)$ is a polynomial in derivatives of ϱ_h at $p_{\alpha,0}(y,\eta)$. However, each derivative of ϱ_h has compact support inside supp ϱ_h, so that for each y, the function $\eta\mapsto s_{\alpha,j}(y,\eta,h)$ is supported inside

$$\operatorname{supp}(\varrho_h\circ p_{\alpha,0})\cap\{(y,\eta):\eta\in\mathbb{R}^n\}.$$

Since $\tau_\alpha\equiv 1$ on K_α, it holds for $x\in\gamma_\alpha^{-1}(K_\alpha)$

$$\operatorname{supp}(\varrho_h\circ p_{\alpha,0})\cap\{(\gamma_\alpha(x),\eta):\eta\in\mathbb{R}^n\}=(\partial\gamma_\alpha)^T\big((\operatorname{supp}\,\varrho_h\circ p)\cap T_x^*M\big).$$

This proves (4.2.24). Now, we apply the composition formula to $\mathrm{Op}_h(b_\alpha)\circ\Psi_\alpha\circ\mathrm{Op}_h(s_\alpha)$, treating Ψ_α here as a zero order h-pseudodifferential operator. Theorem 2.1.1 then yields

$$\begin{aligned}\mathrm{Op}_h(b_\alpha)\circ\Psi_\alpha\circ\mathrm{Op}_h(s_\alpha)=\mathrm{Op}_h(e_\alpha),\qquad & e_\alpha\in S_{h;\delta}(1_{\mathbb{R}^{2n}}),\\ e_\alpha\sim\sum_{j=0}^{\infty}e_{\alpha,j},\qquad & e_{\alpha,j}\in S_{h;\delta}^{j(2\delta-1)}(1_{\mathbb{R}^{2n}}),\\ e_{\alpha,0}=\big((\varphi_\alpha\cdot b)\circ(\gamma_\alpha^{-1},(\partial\gamma_\alpha^{-1})^T)\big)\cdot(\varrho_h\circ p_{\alpha,0}).\end{aligned}\tag{4.2.25}$$

Here we took (4.2.18) and (4.2.21) into account. The function φ_α is compactly supported inside $\gamma_\alpha^{-1}(K_\alpha)$, and we have seen that s_α has the expansion (4.2.20) in terms of symbol functions which are compactly supported in the co-tangent space variable η. The summands in the expansion (2.1.3) of the composition formula are products of derivatives of the original symbol functions. Therefore, if one of the functions is compactly supported in the co-tangent space variable η, and the other one in the manifold variable y, the whole summand is compactly supported in $\mathbb{R}^{2n}$. Taking into account (4.2.24), the statement (4.2.4) follows. To finish the proof, we recall from (4.2.22) how $p_{\alpha,0}$ was defined, and that $\tau_\alpha\in\mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^n)$ is identically 1 on K_α. Since $\varphi_\alpha\circ\gamma_\alpha^{-1}$ is supported inside K_α, the claim (4.2.3) finally follows. □

4.3 Trace norm estimates

Our next goal is to deduce a refined version of Theorem 4.2.1, with a remainder operator of trace class. In order to achieve this, we need to relate the functional and symbolic calculi with trace norm remainder estimates. Suppose that we are in the situation introduced at the beginning of this chapter. For each $\alpha\in\mathcal{A}$, set

$$u_{\alpha,0}:=\big((\varrho_h\circ p)\cdot b\cdot\varphi_\alpha\big)\circ(\gamma_\alpha^{-1},(\partial\gamma_\alpha^{-1})^T),\tag{4.3.1}$$

with b as in Theorem 4.2.1. Then, one has the following result.

Theorem 4.3.1. *Suppose that $\varrho_h\in\mathcal{S}_\delta^{\mathrm{bcomp}}$. Then, for each $N\in\mathbb{N}$, there is a number $h_0\in(0,1]$, a collection of symbol functions $\{r_{\alpha,\beta,N}\}_{\alpha,\beta\in\mathcal{A}}\subset S_{h;\delta}^{2\delta-1}(1_{\mathbb{R}^{2n}})$ and an operator $\mathfrak{R}_N(h)\in\mathcal{B}(\mathrm{L}^2(M))$ such that*

- *one has for all $f\in\mathrm{L}^2(M)$, $h\in(0,h_0]$ the relation*

$$\begin{aligned}B\circ\varrho_h(P(h))(f)=&\sum_{\alpha\in\mathcal{A}}\overline{\varphi}_\alpha\cdot\mathrm{Op}_h(u_{\alpha,0})\big((f\cdot\overline{\overline{\overline{\varphi}}}_\alpha)\circ\gamma_\alpha^{-1}\big)\circ\gamma_\alpha\\ &+\sum_{\alpha,\beta\in\mathcal{A}}\overline{\varphi}_\beta\cdot\mathrm{Op}_h(r_{\alpha,\beta,N})\big((f\cdot\overline{\overline{\varphi}}_\alpha\cdot\overline{\overline{\overline{\varphi}}}_\beta)\circ\gamma_\beta^{-1}\big)\circ\gamma_\beta\;+\;\mathfrak{R}_N(h)(f);\end{aligned}$$

- *the operator* $\mathfrak{R}_N(h) \in \mathcal{B}(\mathrm{L}^2(M))$ *is of trace class and its trace norm fulfills*

$$\|\mathfrak{R}_N(h)\|_{\mathrm{tr},\mathrm{L}^2(M)} = \mathrm{O}\big(h^N\big) \quad \text{as } h \to 0; \tag{4.3.2}$$

- *for fixed* $h \in (0, h_0]$, *each symbol function* $r_{\alpha,\beta,N}$ *is an element of* $\mathrm{C}^\infty_\mathrm{c}(\mathbb{R}^{2n})$ *that fulfills*

$$\mathrm{supp}\ r_{\alpha,\beta,N} \subset \mathrm{supp}\ \big((\varrho_h \circ p) \cdot b \cdot \varphi_\alpha\big) \circ (\gamma_\alpha^{-1}, (\partial\gamma_\alpha^{-1})^T). \tag{4.3.3}$$

Proof. The proof is divided into five steps. Let the notation be as in the proof of Theorem 4.2.1.

Step 0. Consider the collection of symbol functions $\{e_{\alpha,j}\}$ with $e_{\alpha,j} \in S^{j(2\delta-1)}_{h;\delta}(1_{\mathbb{R}^{2n}})$ obtained in the proof of Theorem 4.2.1. Let $R(h) \in \mathcal{B}(\mathrm{L}^2(M))$ be the remainder operator from (4.2.16), whose operator norm is of order h^∞. The statement (4.2.2) means $e_\alpha - \sum_{j=0}^N e_{\alpha,j} \in S^{(N+1)(2\delta-1)}_{h;\delta}(1_{\mathbb{R}^{2n}})$, which by (2.1.4) implies

$$\Big\|\mathrm{Op}_h(e_\alpha) - \sum_{j=0}^N \mathrm{Op}_h(e_{\alpha,j})\Big\|_{\mathcal{B}(\mathrm{L}^2(\mathbb{R}^n))} \leq C_\alpha h^{(1-2\delta)(N+1)},$$

with a constant $C_\alpha > 0$ independent of h. Since $\mathcal{A}$ is finite, and applying analogous arguments as in the proof of Lemma 4.1.2, we obtain

$$\Big\| \sum_{\alpha\in\mathcal{A}} \overline{\varphi}_\alpha \mathrm{Op}_h(e_\alpha)((\overline{\overline{\varphi}}_\alpha f) \circ \gamma_\alpha^{-1}) \circ \gamma_\alpha - \sum_{\substack{\alpha\in\mathcal{A}\\ 0\leq j\leq N}} \overline{\varphi}_\alpha \mathrm{Op}_h(e_{\alpha,j})((\overline{\overline{\varphi}}_\alpha f) \circ \gamma_\alpha^{-1}) \circ \gamma_\alpha \Big\|_{\mathcal{B}(\mathrm{L}^2(M))} \leq C' h^{(1-2\delta)(N+1)}$$

for some constant $C' > 0$ independent of h. Thus, setting

$$R_N(h) := \sum_{\alpha\in\mathcal{A}} \overline{\varphi}_\alpha \mathrm{Op}_h(e_\alpha)((\overline{\overline{\varphi}}_\alpha f) \circ \gamma_\alpha^{-1}) \circ \gamma_\alpha - \sum_{\substack{\alpha\in\mathcal{A}\\ 0\leq j\leq N}} \overline{\varphi}_\alpha \mathrm{Op}_h(e_{\alpha,j})((\overline{\overline{\varphi}}_\alpha f) \circ \gamma_\alpha^{-1}) \circ \gamma_\alpha + R(h),$$

we have

$$\|R_N(h)\|_{\mathcal{B}(\mathrm{L}^2(M))} = \mathrm{O}(h^{(1-2\delta)(N+1)}), \tag{4.3.4}$$

and by Theorem 4.2.1 we obtain for sufficiently small h and each $f \in \mathrm{L}^2(M)$

$$B \circ \varrho_h(P(h))(f) = \sum_{\substack{\alpha\in\mathcal{A}\\ 0\leq j\leq N}} \overline{\varphi}_\alpha \mathrm{Op}_h(e_{\alpha,j})((\overline{\overline{\varphi}}_\alpha f) \circ \gamma_\alpha^{-1}) \circ \gamma_\alpha + R_N(h)(f). \tag{4.3.5}$$

This looks promising, but since we are interested in *trace norm* remainder estimates, there is still some work to do.

Step 1. To proceed, we recall that $P(h)$ has only finitely many eigenvalues

$$E(h)_1, \ldots, E(h)_{N(h)}$$

in supp ϱ_h, and the corresponding eigenspaces are all finite-dimensional. By the spectral theorem,

$$\varrho_h\left(P(h)\right) = \sum_{j=1}^{N(h)} \varrho_h\left(E_j(h)\right) \Pi_j(h), \tag{4.3.6}$$

where $\Pi_j(h)$ denotes the spectral projection onto the eigenspace of $P(h)$ corresponding to the eigenvalue $E_j(h)$. Hence, $\varrho_h(P(h))$ is a finite sum of projections onto finite-dimensional spaces and, consequently, a finite rank operator, and therefore of trace class.
Now, we get prepared to use a trick that allows us to partially estimate trace norms by operator norms. The trick has been used already by Helffer and Robert in [27, Proof of Prop. 5.3], and to implement it, we proceed as follows. For $h \in (0,1]$, choose $\overline{\varrho}_h \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R})$ such that

$$\overline{\varrho}_h = 1 \qquad \text{on the support of } \varrho_h, \tag{4.3.7}$$

and such that the function $(t,h) \mapsto \overline{\varrho}_h(t)$ is an element of the symbol class[1] $S_h(1_\mathbb{R})$ and

$$\operatorname{supp} \overline{\varrho}_h \subset \overline{I} \qquad \forall\, h \in (0,1] \tag{4.3.8}$$

for some h-independent closed interval $\overline{I} \subset \mathbb{R}$. The abstract functional calculus given by the spectral theorem fulfills $f(A) \circ g(A) = (f \cdot g)(A)$ for any self-adjoint operator A in a Hilbert space and any two bounded Borel functions f, g on $\mathbb{R}$. We therefore get

$$\varrho_h(P(h)) \circ \overline{\varrho}_h(P(h)) = (\varrho_h \cdot \overline{\varrho}_h)(P(h)) = \varrho_h(P(h)) \quad \forall\, h \in (0,1]. \tag{4.3.9}$$

Now, basic operator theory tells us that for operators $Z, S \in \mathcal{B}(\mathrm{L}^2(M))$ of which Z is of trace class, $Z \circ S$ and $S \circ Z$ are also of trace class and it holds

$$\|Z \circ S\|_{\mathrm{tr},\mathrm{L}^2(M)} \leq \|Z\|_{\mathrm{tr},\mathrm{L}^2(M)} \|S\|_{\mathcal{B}(\mathrm{L}^2(M))}, \tag{4.3.10}$$

$$\|S \circ Z\|_{\mathrm{tr},\mathrm{L}^2(M)} \leq \|Z\|_{\mathrm{tr},\mathrm{L}^2(M)} \|S\|_{\mathcal{B}(\mathrm{L}^2(M))}. \tag{4.3.11}$$

The trick is to use the latter estimates together with (4.3.9) to estimate the trace norm of remainders by operator norms. Indeed, we can apply all our predecing results, and in particular Theorem 4.2.1, also to the operator $\overline{\varrho}_h(P(h))$. From now on, choose h_0 to be the minimum of the two h_0 we obtain for ϱ_h and $\overline{\varrho}_h$ from Theorem 4.2.1. Choosing $B = \mathbf{1}_{\mathrm{L}^2(M)}$, one then has by (4.3.5) for $N = 0, 1, 2, \ldots$, $f \in \mathrm{L}^2(M)$, and $h \in (0, h_0]$

$$\overline{\varrho}_h(P(h))(f) = \sum_{\substack{\alpha \in \mathcal{A} \\ 0 \leq j \leq N}} \overline{\varphi}_\alpha \mathrm{Op}_h(\overline{e}_{\alpha,j})((\overline{\overline{\varphi}}_\alpha f) \circ \gamma_\alpha^{-1}) \circ \gamma_\alpha + \overline{R}_N(h)(f), \tag{4.3.12}$$

where $\overline{R}_N(h) \in \mathcal{B}(\mathrm{L}^2(M))$ fulfills

$$\left\|\overline{R}_N(h)\right\|_{\mathcal{B}(\mathrm{L}^2(M))} = \mathrm{O}(h^{N+1}), \tag{4.3.13}$$

and the symbols $\overline{e}_{\alpha,j} \in S_h^{-j}(1_{\mathbb{R}^{2n}})$ have analogous properties as the symbols $\{e_{\alpha,j}\}$. In particular,

$$\overline{e}_{\alpha,0} = \big((\overline{\varrho}_h \circ p) \cdot \varphi_\alpha\big) \circ (\gamma_\alpha^{-1}, (\partial\gamma_\alpha^{-1})^T) \qquad \forall\, h \in (0, h_0]. \tag{4.3.14}$$

We now use (4.3.9) and (4.3.12) to get for $N = 0, 1, 2, \ldots$ and $h \in (0, h_0]$:

$$\begin{aligned} B \circ \varrho_h(P(h)) &= B \circ \varrho_h(P(h)) \circ \overline{\varrho}_h(P(h)) \\ &= \sum_{\substack{\alpha \in \mathcal{A} \\ 0 \leq j \leq N}} B \circ \varrho_h(P(h)) \circ \overline{\Phi}_\alpha \circ \Gamma_\alpha^*\big(\mathrm{Op}_h(\overline{e}_{\alpha,j})\big) \circ \overline{\overline{\Phi}}_\alpha + B \circ \varrho_h(P(h)) \circ \overline{R}_N(h). \end{aligned}$$

[1] The larger symbol class $S_{h;\delta}(1_\mathbb{R})$ would also do. However, as the diameter of the support of $\overline{\varrho}_h$ can be assumed to be bounded away from 0, the symbol class $S_h(1_\mathbb{R})$ is more natural.

From now on, we fix N and assume h_0 to be small enough such that $\overline{R}_N(h)$ has operator norm less than $\frac{1}{2}$ for each $h \in (0, h_0]$, which implies that $\mathbf{1}_{\mathrm{L}^2(M)} - \overline{R}_N(h)$ is invertible. Note that this makes h_0 depend on N. Using also the corresponding Neumann series, one obtains for $h \in (0, h_0]$ the equality

$$\begin{aligned} B \circ \varrho_h(P(h)) &= \sum_{\substack{\alpha \in \mathcal{A} \\ 0 \le j \le N}} B \circ \varrho_h(P(h)) \circ \Gamma_\alpha^*\big(\overline{\Psi}_\alpha \circ \mathrm{Op}_h(\overline{e}_{\alpha,j})\big) \circ \overline{\overline{\Phi}}_\alpha \circ \Big(\mathbf{1}_{\mathrm{L}^2(M)} - \overline{R}_N(h)\Big)^{-1} \\ &= \sum_{\substack{\alpha \in \mathcal{A} \\ 0 \le j \le N}} B \circ \varrho_h(P(h)) \circ \Gamma_\alpha^*\big(\overline{\Psi}_\alpha \circ \mathrm{Op}_h(\overline{e}_{\alpha,j})\big) \circ \overline{\overline{\Phi}}_\alpha \circ \Big(\sum_{k=0}^{\infty} \overline{R}_N(h)^k\Big) \\ &= \sum_{\substack{\alpha \in \mathcal{A} \\ 0 \le j \le N}} B \circ \varrho_h(P(h)) \circ \Gamma_\alpha^*\big(\overline{\Psi}_\alpha \circ \mathrm{Op}_h(\overline{e}_{\alpha,j})\big) \circ \overline{\overline{\Phi}}_\alpha \\ &\quad + \overbrace{\sum_{\substack{\alpha \in \mathcal{A} \\ 0 \le j \le N}} B \circ \varrho_h(P(h)) \circ \Gamma_\alpha^*\big(\overline{\Psi}_\alpha \circ \mathrm{Op}_h(\overline{e}_{\alpha,j})\big) \circ \overline{\overline{\Phi}}_\alpha \circ \Big(\sum_{k=1}^{\infty} \overline{R}_N(h)^k\Big)}^{:=\widetilde{\mathcal{R}}_N(h)}. \end{aligned}$$

To proceed, we insert (4.3.5) into the first summand, which yields

$$\begin{aligned} B \circ \varrho_h(P(h)) = &\sum_{\substack{\alpha, \beta \in \mathcal{A} \\ 0 \le j,k \le N}} \Gamma_\beta^*\big(\overline{\Psi}_\beta \circ \mathrm{Op}_h(e_{\beta,k})\big) \circ \overline{\overline{\Phi}}_\beta \circ \Gamma_\alpha^*\big(\overline{\Psi}_\alpha \circ \mathrm{Op}_h(\overline{e}_{\alpha,j})\big) \circ \overline{\overline{\Phi}}_\alpha \\ &+ \overbrace{\sum_{\substack{\alpha \in \mathcal{A} \\ 0 \le j \le N}} R_N(h) \circ \Gamma_\alpha^*\big(\overline{\Psi}_\alpha \circ \mathrm{Op}_h(\overline{e}_{\alpha,j})\big) \circ \overline{\overline{\Phi}}_\alpha + \widetilde{\mathcal{R}}_N(h)}^{:=\mathcal{R}_N(h)}. \end{aligned} \tag{4.3.15}$$

We see that a drawback of the trick is that besides a new remainder term, we now also have a different leading term for each N.

Step 2. In this step we prove that the operator $\mathcal{R}_N(h)$ is in fact of trace class and satisfies a good trace norm estimate. Recall that the function $\overline{\varphi}_\alpha \circ \gamma_\alpha^{-1}$ is compactly supported inside the interior of the compactum $K_\alpha \subset \mathbb{R}^n$. Now, in view of (4.2.24) and (4.3.8), the results leading to (2.1.7) imply that $\overline{\Psi}_\alpha \circ \mathrm{Op}_h\big(\overline{e}_{\alpha,j}\big)$ is of trace class, and by (2.1.7) it holds for $h \in (0, h_0]$

$$\begin{aligned} &\big\|\overline{\Psi}_\alpha \circ \mathrm{Op}_h\big(\overline{e}_{\alpha,j}\big)\big\|_{\mathrm{tr}, \mathrm{L}^2(\mathbb{R}^n)} \\ &\qquad \le C_{\alpha,j} h^{-n}(1 + \mathrm{vol}_{T^*M}(\mathrm{supp}\ \varrho_h \circ p)) \sum_{|\beta| \le 2n+1} \max_{(y,\eta) \in K_\alpha \times \mathbb{R}^n} |\partial^\beta \overline{e}_{\alpha,j}(\cdot, h)| \end{aligned}$$

for some constant $C_{\alpha,j} > 0$ which is independent of h. Next, we use that $\overline{e}_{\alpha,j}$ is an element of $S_h^{-j}(1_{\mathbb{R}^{2n}})$, which implies

$$\sum_{|\beta| \le 2n+1} \max_{(y,\eta) \in K_\alpha \times \mathbb{R}^n} |\partial^\beta \overline{e}_{\alpha,j}(\cdot, h)| \le \widetilde{C}_{\alpha,n} h^j \qquad \forall\ h \in (0, h_0].$$

In summary, we obtain the estimate

$$\big\|\overline{\Psi}_\alpha \circ \mathrm{Op}_h\big(\overline{e}_{\alpha,j}\big)\big\|_{\mathrm{tr}, \mathrm{L}^2(\mathbb{R}^n)} \le C'_{\alpha,j}(1 + \mathrm{vol}_{T^*M}(\mathrm{supp}\ \varrho_h \circ p)) h^{j-n} \quad \forall\ h \in (0, h_0] \tag{4.3.16}$$

for some new constant $C'_{\alpha,j} > 0$ which is independent of h. To proceed, note that as M is compact and Vol_{g_α} is bounded and on K_α also bounded away from zero, our trace norm estimates in $\mathrm{L}^2(\mathbb{R}^n)$ carry over to trace norm estimates in $\mathrm{L}^2(M)$ by using Schwartz kernel estimates similar to [17, (9.1) on p. 112]. Combining now (4.3.10), (4.3.11), (4.3.16), and (4.3.4), we conclude

$$\begin{aligned}\Big\| R_N(h)\circ\Gamma_\alpha^*\big(\overline{\Psi}_\alpha\circ\mathrm{Op}_h(\overline{e}_{\alpha,j})\big)\circ\overline{\overline{\Phi}}_\alpha\Big\|_{\mathrm{tr},\mathrm{L}^2(M)}&\\ \leq C_{\alpha,j}(1+\mathrm{vol}_{T^*M}(\mathrm{supp}\ \varrho_h\circ p))h^{(N+1)(1-2\delta)+j-n}&\qquad\forall\, h\in(0,h_0]\end{aligned}$$

with a constant $C_{\alpha,j} > 0$ that is independent of h, and it follows from the finiteness of $\mathcal{A}$ that there is $C > 0$ such that

$$\begin{aligned}\Big\|\sum_{\substack{\alpha\in\mathcal{A}\\0\leq j\leq N}} R_N(h)\circ\Gamma_\alpha^*\big(\overline{\Psi}_\alpha\circ\mathrm{Op}_h(\overline{e}_{\alpha,j})\big)\circ\overline{\overline{\Phi}}_\alpha\Big\|_{\mathrm{tr},\mathrm{L}^2(M)}&\\ \leq C(1+\mathrm{vol}_{T^*M}(\mathrm{supp}\ \varrho_h\circ p))h^{(N+1)(1-2\delta)-n}&\qquad\forall\, h\in(0,h_0].\end{aligned}$$

Similarly, taking into account that the operator norm of $\varrho_h(P(h))$ is uniformly bounded in h by the spectral theorem and the assumption that $\varrho_h\in\mathcal{S}_\delta^{\mathrm{bcomp}}$, and writing

$$\sum_{k=1}^\infty \overline{R}_N(h)^k = \overline{R}_N(h)\circ\Big(\sum_{k=0}^\infty\overline{R}_N(h)^k\Big) = \overline{R}_N(h)\circ\Big(\mathbf{1}_{\mathrm{L}^2(M)}-\overline{R}_N(h)\Big)^{-1},$$

where $\big(\mathbf{1}_{\mathrm{L}^2(M)}-\overline{R}_N(h)\big)^{-1}$ has operator norm less than 2 for $h\in(0,h_0]$, it follows from (4.3.16) and (4.3.13) that there is $C' > 0$ such that

$$\Big\|\widetilde{\mathcal{R}}_N(h)\Big\|_{\mathrm{tr},\mathrm{L}^2(M)} \leq C'(1+\mathrm{vol}_{T^*M}(\mathrm{supp}\ \varrho_h\circ p))h^{N+1-n}\qquad\forall\, h\in(0,h_0].$$

By assumption, the diameter of the support of ϱ_h is bounded uniformly in h, and M is compact, so the number $\mathrm{vol}_{T^*M}(\mathrm{supp}\ \varrho_h\circ p)$ is also bounded uniformly in h. From the last two estimates, we therefore conclude finally

$$\|\mathcal{R}_N(h)\|_{\mathrm{tr},\mathrm{L}^2(M)} \leq Ch^{(N+1)(1-2\delta)-n}\qquad\forall\, h\in(0,h_0]\tag{4.3.17}$$

with a new constant $C > 0$ that is independent of h. Our estimation of the trace norm of the remainder operator $\mathcal{R}_N(h)$ is finished.

Step 3. We now turn our attention to the leading term in (4.3.15) with summands given by

$$\Gamma_\beta^*\big(\overline{\Psi}_\beta\circ\mathrm{Op}_h(e_{\beta,k})\big)\circ\overline{\overline{\Phi}}_\beta\circ\Gamma_\alpha^*\big(\overline{\Psi}_\alpha\circ\mathrm{Op}_h(\overline{e}_{\alpha,j})\big)\circ\overline{\overline{\Phi}}_\alpha.$$

The problem with these terms is that they do not yet have the right form as claimed in the first statement of Theorem 4.3.1, in particular they involve two pullbacks, one along the chart γ_α and one along γ_β, and we need to combine them into a single pullback. This will be done using a coordinate transformation from the α-th chart to the β-th chart. Before we can perform this transformation, we need to localize further to the intersection of both chart domains. To this end, note that since $\overline{\overline{\varphi}}_\beta$ and $1-\overline{\overline{\overline{\varphi}}}_\beta$ have disjoint supports, we have

$$\overline{\overline{\Phi}}_\beta\circ\Gamma_\alpha^*\big(\overline{\Psi}_\alpha\circ\mathrm{Op}_h(\overline{e}_{\alpha,j})\big)\circ\overline{\overline{\Phi}}_\alpha = \overline{\overline{\Phi}}_\beta\circ\Gamma_\alpha^*\big(\overline{\Psi}_\alpha\circ\mathrm{Op}_h(\overline{e}_{\alpha,j})\big)\circ\overline{\overline{\Phi}}_\alpha\circ\overline{\overline{\overline{\Phi}}}_\beta+R_{\alpha,\beta,j}(h),$$

for a remainder operator $R_{\alpha,\beta,j}(h) \in \mathcal{B}(\mathrm{L}^2(M))$ with $\|R_{\alpha,\beta,j}(h)\|_{\mathcal{B}(\mathrm{L}^2(M))} = \mathrm{O}(h^\infty)$. Similarly as in (4.3.16), it follows that the operator $\Gamma^*_\beta\big(\overline{\Psi}_\beta \circ \mathrm{Op}_h(e_{\beta,k})\big)$ is of trace class in $\mathrm{L}^2(M)$, and its trace norm is of order $(1 + \mathrm{vol}_{T^*M}(\mathrm{supp}\ \varrho_h \circ p))h^{j(1-2\delta)-(2n+1)\delta-n}$. Again, the number $\mathrm{vol}_{T^*M}(\mathrm{supp}\ \varrho_h \circ p)$ is bounded uniformly in h. Thus, the trace norm of $\Gamma^*_\beta\big(\overline{\Psi}_\beta \circ \mathrm{Op}_h(e_{\beta,k})\big)$ is bounded uniformly in h. Therefore, setting

$$\mathcal{R}_{\alpha,\beta,j,k}(h) := \Gamma^*_\beta\big(\overline{\Psi}_\beta \circ \mathrm{Op}_h(e_{\beta,k})\big) \circ R_{\alpha,\beta,j}(h)$$

we conclude

$$\begin{aligned}\|\mathcal{R}_{\alpha,\beta,j,k}(h)\|_{\mathrm{tr},\mathrm{L}^2(M)} \\ \leq \big\|\Gamma^*_\beta\big(\overline{\Psi}_\beta \circ \mathrm{Op}_h(e_{\beta,k})\big)\big\|_{\mathrm{tr},\mathrm{L}^2(M)} \|R_{\alpha,\beta,j}(h)\|_{\mathcal{B}(\mathrm{L}^2(M))} = \mathrm{O}(h^\infty). \quad (4.3.18)\end{aligned}$$

The reason why we inserted the cutoff operator $\overline{\overline{\overline{\Phi}}}_\beta$ corresponding to the function $\overline{\overline{\overline{\varphi}}}_\beta$ is that we are now prepared to perform the required coordinate transformation. Indeed, one has

$$(\overline{\overline{\varphi}}_\alpha \overline{\overline{\overline{\varphi}}}_\beta f) \circ \gamma_\alpha^{-1} = (\overline{\overline{\varphi}}_\alpha \overline{\overline{\overline{\varphi}}}_\beta f) \circ \gamma_\beta^{-1} \circ \gamma_\beta \circ \gamma_\alpha^{-1},$$

which leads to

$$\begin{aligned}&\overline{\varphi}_\beta \mathrm{Op}_h(e_{\beta,k})\Big(\Big(\overline{\overline{\varphi}}_\beta \overline{\varphi}_\alpha \mathrm{Op}_h(\overline{e}_{\alpha,j})((\overline{\overline{\varphi}}_\alpha f) \circ \gamma_\alpha^{-1}) \circ \gamma_\alpha\Big) \circ \gamma_\beta^{-1}\Big) \circ \gamma_\beta \\ &\quad = \overline{\varphi}_\beta \mathrm{Op}_h(e_{\beta,k})\Big(\big((\overline{\overline{\varphi}}_\beta \overline{\varphi}_\alpha) \circ \gamma_\beta^{-1}\big)\big[\mathrm{Op}_h(\overline{e}_{\alpha,j})((\overline{\overline{\varphi}}_\alpha \overline{\overline{\overline{\varphi}}}_\beta f) \circ \gamma_\beta^{-1} \circ \Theta_{\alpha\beta}^{-1})\big] \circ \Theta_{\alpha\beta}\Big) \circ \gamma_\beta \\ &\qquad + \mathcal{R}_{\alpha,\beta,j,k}(h)(f), \quad (4.3.19)\end{aligned}$$

where $\mathcal{R}_{\alpha,\beta,j,k}(h)$ fulfills $\|\mathcal{R}_{\alpha,\beta,j,k}(h)\|_{\mathrm{tr},\mathrm{L}^2(M)} = \mathrm{O}(h^\infty)$, as shown above, and we introduced

$$\Theta_{\alpha\beta} := \gamma_\alpha \circ \gamma_\beta^{-1} : \gamma_\beta(U_\alpha \cap U_\beta) \to \gamma_\alpha(U_\alpha \cap U_\beta).$$

By the coordinate-transformation formula [63, Theorem 9.3], it then holds

$$\big((\overline{\overline{\varphi}}_\beta \overline{\varphi}_\alpha) \circ \gamma_\beta^{-1}\big)\big[\mathrm{Op}_h(\overline{e}_{\alpha,j})((\overline{\overline{\varphi}}_\alpha \overline{\overline{\overline{\varphi}}}_\beta f) \circ \gamma_\beta^{-1} \circ \Theta_{\alpha\beta}^{-1})\big] \circ \Theta_{\alpha\beta} = \mathrm{Op}_h(\overline{u}_{\alpha,\beta,j})((\overline{\overline{\varphi}}_\alpha \overline{\overline{\overline{\varphi}}}_\beta f) \circ \gamma_\beta^{-1})$$

for a new symbol function $\overline{u}_{\alpha,\beta,j} \in S_h^{-j}(1_{\mathbb{R}^{2n}})$ which is for each fixed h a Schwartz function on $\mathbb{R}^{2n}$ and fulfills

$$\begin{aligned}\overline{u}_{\alpha,\beta,j}(y,\eta,h) = (\overline{\overline{\varphi}}_\beta \overline{\varphi}_\alpha) \circ \gamma_\beta^{-1}(y)\, \underbrace{\overline{e}_{\alpha,j}(\Theta_{\alpha\beta}(y), \partial\Theta_{\alpha\beta}(\Theta_{\alpha\beta}(y))^T\eta, h)}_{=:\Theta^*_{\alpha\beta}\overline{e}_{\alpha,j}(y,\eta,h)} \\ + h\,\overline{r}_{\alpha,\beta,j}(y,\eta,h), \quad (4.3.20)\end{aligned}$$

with a remainder symbol function $\overline{r}_{\alpha,\beta,j} \in S_h^{-j}(1_{\mathbb{R}^{2n}})$ that is for each fixed h a Schwartz function on $\mathbb{R}^{2n}$, too. We thus obtain for $f \in \mathrm{L}^2(M)$ the equality

$$\begin{aligned}&\Gamma^*_\beta\big(\overline{\Psi}_\beta \circ \mathrm{Op}_h(e_{\beta,k})\big)\Big(\overline{\overline{\varphi}}_\beta \overline{\varphi}_\alpha \mathrm{Op}_h(\overline{e}_{\alpha,j})((\overline{\overline{\varphi}}_\alpha f) \circ \gamma_\alpha^{-1}) \circ \gamma_\alpha\Big) \\ &\quad = \Gamma^*_\beta\Big(\overline{\Psi}_\beta \circ \mathrm{Op}_h(e_{\beta,k}) \circ \mathrm{Op}_h(\overline{u}_{\alpha,\beta,j}) \circ \overline{\overline{\Psi}}_{\alpha\beta}\Big) \circ \overline{\overline{\overline{\Phi}}}_\beta(f), \quad (4.3.21)\end{aligned}$$

where $\overline{\overline{\Psi}}_{\alpha\beta} : \mathrm{L}^2(\mathbb{R}^n) \to \mathrm{L}^2(\mathbb{R}^n)$ is the operator given by pointwise multiplication with the function $\overline{\overline{\varphi}}_\alpha \circ \gamma_\beta^{-1}$. Finally, we can apply the composition formula, described in Theorem 2.1.1. It tells us that

$$\mathrm{Op}_h(e_{\beta,k}) \circ \mathrm{Op}_h(\overline{u}_{\alpha,\beta,j}) = \mathrm{Op}_h(s_{\alpha,\beta,j,k}), \tag{4.3.22}$$

where $s_{\alpha,\beta,j,k} \in S^{(j+k)(2\delta-1)}_{h;\delta}(1_{\mathbb{R}^{2n}})$. Moreover, Theorem 2.1.1 says that $s_{\alpha,\beta,j,k}$ has an asymptotic expansion in $S^{(j+k)(2\delta-1)}_{h;\delta}(1_{\mathbb{R}^{2n}})$:

$$s_{\alpha,\beta,j,k} \sim \sum_{l=0}^{\infty} s_{\alpha,\beta,j,k,l}, \qquad s_{\alpha,\beta,j,k,l} \in S^{(j+k+l)(2\delta-1)}_{h;\delta}(1_{\mathbb{R}^{2n}}), \tag{4.3.23}$$

where

$$s_{\alpha,\beta,j,k,0} = e_{\beta,k} \cdot ((\overline{\overline{\varphi}}_\beta \cdot \overline{\varphi}_\alpha) \circ \gamma_\beta^{-1}) \cdot \Theta^*_{\alpha\beta}\overline{e}_{\alpha,j}. \tag{4.3.24}$$

Similarly as in our first application of the composition formula after (4.2.25), we conclude from the relations

$$\begin{aligned} \operatorname{supp} e_{\beta,k} &\subset \operatorname{supp} \big((\varrho_h \circ p) \cdot b \cdot \varphi_\beta\big) \circ (\gamma_\beta^{-1}, (\partial\beta_\beta^{-1})^T), \\ \operatorname{supp} \overline{e}_{\alpha,j} &\subset \operatorname{supp} \big((\overline{\varrho}_h \circ p) \cdot b \cdot \varphi_\alpha\big) \circ (\gamma_\alpha^{-1}, (\partial\gamma_\alpha^{-1})^T) \end{aligned}$$

that $s_{\alpha,\beta,j,k,l}$ is compactly supported inside

$$\operatorname{supp} \Big(\big((\varrho_h \circ p) \cdot b \cdot \varphi_\beta\big) \circ (\gamma_\beta^{-1}, (\partial\beta_\beta^{-1})^T)\Big) \cap \operatorname{supp} \Big(\big((\overline{\varrho}_h \circ p) \cdot b \cdot \varphi_\alpha\big) \circ (\gamma_\alpha^{-1}, (\partial\gamma_\alpha^{-1})^T)\Big) \subset \mathbb{R}^{2n}$$

for each l and each fixed $h \in (0, h_0]$, and consequently its support fulfills

$$\operatorname{vol} \operatorname{supp} s_{\alpha,\beta,j,k,l} \leq C_{\alpha,\beta,j,k,l} \operatorname{vol}_{T^*M}(\operatorname{supp} \varrho_h \circ p) \qquad \forall\, h \in (0, h_0]$$

with some constant $C_{\alpha,\beta,j,k,l} > 0$ that is independent of h. It also follows that $s_{\alpha,\beta,j,k}$ is for each fixed $h \in (0, h_0]$ a Schwartz function on $\mathbb{R}^{2n}$. By (4.3.23), we have for each $M \in \mathbb{N}$ that

$$\mathfrak{R}_{\alpha,\beta,j,k,M} := s_{\alpha,\beta,j,k} - \sum_{l=0}^{M} s_{\alpha,\beta,j,k,l} \in S^{(j+k+M+1)(2\delta-1)}_{h;\delta}(1_{\mathbb{R}^{2n}}),$$

and Corollary 2.1.2 says that $\mathrm{Op}_h(\mathfrak{R}_{\alpha,\beta,j,k,M})$ is of trace class, with a trace norm bound for $h \in (0, h_0]$

$$\|\mathrm{Op}_h(\mathfrak{R}_{\alpha,\beta,j,k,M})\|_{\mathrm{tr},\mathrm{L}^2(\mathbb{R}^n)} \leq C_{\alpha,\beta,j,k,M} h^{(j+k+M+1)(1-2\delta)-(2n+1)\delta-n}, \tag{4.3.25}$$

where $C_{\alpha,\beta,j,k,M} > 0$ is independent of h. The fact that we need Corollary 2.1.2 here, which requires the considered symbol functions to be supported inside an h-independent compactum in $\mathbb{R}^{2n}$, is the *only* reason why we need the additional assumption in this theorem that $\varrho_h \in \mathcal{S}^{\mathrm{bcomp}}_\delta$. Collecting everything together, we get from (4.3.15-4.3.25) for each $N, M \in \mathbb{N}$:

$$\begin{aligned} &B \circ \varrho_h(P(h)) \\ &= \sum_{\substack{\alpha,\beta\in\mathcal{A} \\ 0\leq j,k\leq N}} \Big[\Gamma^*_\beta\Big(\overline{\Psi}_\beta \circ \mathrm{Op}_h\Big(\sum_{0\leq l\leq M} s_{\alpha,\beta,j,k,l} + \mathfrak{R}_{\alpha,\beta,j,k,M}\Big) \circ \overline{\overline{\Psi}}_{\alpha\beta}\Big) \circ \overline{\overline{\overline{\Phi}}}_\beta + \mathcal{R}_{\alpha,\beta,j,k}(h)\Big] + \mathcal{R}_N(h). \end{aligned}$$

This is the final result of Step 3. We have transformed the leading term of (4.3.15) into a more desired form that involves only pullbacks by one chart at a time.
Step 4. We complete the proof by setting

$$u_{\alpha,\beta,0} := s_{\alpha,\beta,0,0,0}, \quad u_{\alpha,\beta,M,N} := \sum_{\substack{0\leq j,k\leq N\\ 0\leq l\leq M}} s_{\alpha,\beta,j,k,l} - s_{\alpha,\beta,0,0,0},$$

$$\mathfrak{R}_{M,N}(h) := \sum_{\substack{\alpha,\beta\in\mathcal{A}\\ 0\leq j,k\leq N}} \Big[\Gamma^*_\beta\big(\overline{\Psi}_\beta \circ \mathrm{Op}_h(\mathfrak{R}_{\alpha,\beta,j,k,M}) \circ \overline{\overline{\Psi}}_{\alpha\beta}\big) \circ \overline{\overline{\overline{\Phi}}}_\beta + \mathcal{R}_{\alpha,\beta,j,k}(h)\Big] + \mathcal{R}_N(h).$$

One then has for each $M, N \in \mathbb{N}$

$$B \circ \varrho_h(P(h)) = \sum_{\alpha,\beta\in\mathcal{A}} \Gamma^*_\beta\Big(\overline{\Psi}_\beta \circ \mathrm{Op}_h\big(u_{\alpha,\beta,0} + u_{\alpha,\beta,M,N}\big) \circ \overline{\overline{\Psi}}_{\alpha\beta}\Big) \circ \overline{\overline{\overline{\Phi}}}_\beta + \mathfrak{R}_{M,N}(h),$$

where $u_{\alpha,\beta,0} \in S_{h;\delta}(1_{\mathbb{R}^{2n}})$ and $u_{\alpha,\beta,M,N} \in S^{2\delta-1}_{h;\delta}(1_{\mathbb{R}^{2n}})$ are elements of $\mathrm{C}^\infty_{\mathrm{c}}(K_\alpha \cap K_\beta \times \mathbb{R}^n) \subset \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^{2n})$ for each fixed $h \in (0, h_0]$, and

$$\|\mathfrak{R}_{M,N}(h)\|_{\mathrm{tr},\mathrm{L}^2(M)} = \mathrm{O}\Big(h^{(\min(N,M)+1)(1-2\delta)-n-(2n+1)\delta}\Big) \quad \text{as } h \to 0.$$

Let $\widetilde{N} \in \mathbb{N}$. Since $1 - 2\delta > 0$, we can find numbers $N(\widetilde{N}), M(\widetilde{N}) \in \mathbb{N}$ large enough such that

$$\Big\|\mathfrak{R}_{M(\widetilde{N}),N(\widetilde{N})}(h)\Big\|_{\mathrm{tr},\mathrm{L}^2(M)} = \mathrm{O}\big(h^{\widetilde{N}}\big) \quad \text{as } h \to 0.$$

Defining

$$\mathfrak{R}_{\widetilde{N}}(h) := \mathfrak{R}_{M(\widetilde{N}),N(\widetilde{N})}(h), \qquad r_{\alpha,\beta,\widetilde{N}} := u_{\alpha,\beta,M(\widetilde{N}),N(\widetilde{N})} \in S^{2\delta-1}_{h;\delta}(1_{\mathbb{R}^{2n}}),$$

we arrive for arbitrary $\widetilde{N} \in \mathbb{N}$ at the equality

$$B \circ \varrho_h(P(h)) = \sum_{\alpha,\beta\in\mathcal{A}} \Gamma^*_\beta\Big(\overline{\Psi}_\beta \circ \mathrm{Op}_h(u_{\alpha,\beta,0} + r_{\alpha,\beta,\widetilde{N}}) \circ \overline{\overline{\Psi}}_{\alpha\beta}\Big) \circ \overline{\overline{\overline{\Phi}}}_\beta + \mathfrak{R}_{\widetilde{N}}(h).$$

To finish the proof, recall the identities

$$e_{\beta,0} = \big((\varrho_h \circ p) \cdot \varphi_\beta\big) \circ (\gamma_\beta^{-1}, (\partial\gamma_\beta^{-1})^T), \qquad \overline{e}_{\alpha,0} = \big((\overline{\varrho}_h \circ p) \cdot \varphi_\alpha\big) \circ (\gamma_\alpha^{-1}, (\partial\gamma_\alpha^{-1})^T).$$

With these identities and the definition of the pullback by the function $\Theta_{\alpha\beta} \equiv \gamma_\alpha \circ \gamma_\beta^{-1}$, one computes

$$\begin{aligned}
&e_{\beta,0} \cdot \Theta^*_{\alpha\beta}\overline{e}_{\alpha,0} = e_{\beta,0} \cdot \Big(\overline{e}_{\alpha,0} \circ (\Theta_{\alpha\beta}, \partial\Theta^T_{\alpha\beta})\Big)\\
&= \Big(\big((\varrho_h \circ p) \cdot \varphi_\beta\big) \circ \big(\gamma_\beta^{-1}, (\partial\gamma_\beta^{-1})^T\big)\Big) \cdot \Big(\big((\overline{\varrho}_h \circ p) \cdot \varphi_\alpha\big) \circ (\gamma_\alpha^{-1}, (\partial\gamma_\alpha^{-1})^T) \circ (\Theta_{\alpha\beta}, \partial\Theta^T_{\alpha\beta})\Big)\\
&= \Big(\big((\varrho_h \circ p) \cdot \varphi_\beta\big) \circ \big(\gamma_\beta^{-1}, (\partial\gamma_\beta^{-1})^T\big)\Big)\\
&\qquad\qquad \cdot \Big(\big((\overline{\varrho}_h \circ p) \cdot \varphi_\alpha\big) \circ (\gamma_\alpha^{-1}, (\partial\gamma_\alpha^{-1})^T) \circ \big(\gamma_\alpha \circ \gamma_\beta^{-1}, (\partial\gamma_\alpha)^T \circ (\partial\gamma_\beta^{-1})^T\big)\Big)\\
&= \big((\varrho_h \circ p) \cdot (\overline{\varrho}_h \circ p) \cdot \varphi_\alpha \cdot \varphi_\beta\big) \circ \big(\gamma_\beta^{-1}, (\partial\gamma_\beta^{-1})^T\big).
\end{aligned}$$

Taking finally into account that the functions decorated with a bar are identically 1 on the supports of the corresponding functions without bar, it holds

$$u_{\alpha,\beta,0} \equiv s_{\alpha,\beta,0,0,0} = e_{\beta,0} \cdot ((\overline{\overline{\varphi}}_\beta \cdot \overline{\varphi}_\alpha) \circ \gamma_\beta^{-1}) \cdot \Theta^*_{\alpha\beta} \overline{e}_{\alpha,0} = \big((\varrho_h \circ p) \cdot \varphi_\alpha \cdot \varphi_\beta\big) \circ \big(\gamma_\beta^{-1}, (\partial \gamma_\beta^{-1})^T\big).$$

In particular, since $\sum_{\alpha \in \mathcal{A}} \varphi_\alpha = 1_M$, we can set

$$u_{\beta,0} := \sum_{\alpha \in \mathcal{A}} u_{\alpha,\beta,0} = \big((\varrho_h \circ p) \cdot \varphi_\beta\big) \circ \big(\gamma_\beta^{-1}, (\partial \gamma_\beta^{-1})^T\big)$$

which finally yields

$$\sum_{\alpha,\beta \in \mathcal{A}} \Gamma^*_\beta \Big(\overline{\Psi}_\beta \circ \mathrm{Op}_h(u_{\alpha,\beta,0}) \circ \overline{\overline{\Psi}}_{\alpha\beta} \Big) \circ \overline{\overline{\overline{\Phi}}}_\beta = \sum_{\beta \in \mathcal{A}} \Gamma^*_\beta \Big(\overline{\Psi}_\beta \circ \mathrm{Op}_h(u_{\beta,0}) \Big) \circ \overline{\overline{\overline{\Phi}}}_\beta.$$

□

From the previous theorem one immediately deduces

Corollary 4.3.2 (Semiclassical trace formula for Schrödinger operators)**.** *In the situation of the previous theorem, one has in the semiclassical limit $h \to 0$*

$$\begin{aligned} &\mathrm{tr}_{\,\mathrm{L}^2(M)} \big[B \circ \varrho_h(P(h)) \big] \\ &\quad = \frac{1}{(2\pi h)^n} \int\limits_{T^*M} b \cdot (\varrho_h \circ p) \, d(T^*M) \; + \; \mathrm{O}\Big(h^{1-2\delta-n} \mathrm{vol}_{\,T^*M} \big[\operatorname{supp} \big(b \cdot (\varrho_h \circ p) \big) \big] \Big). \end{aligned} \tag{4.3.26}$$

Remark 4.3.3. If $\mathrm{vol}_{\,T^*M}\big[\operatorname{supp}\big(b \cdot (\varrho_h \circ p)\big)\big] \neq 0$, i.e. in all non-trivial cases, we can divide both sides of (4.3.26) by $\mathrm{vol}_{\,T^*M}\big[\operatorname{supp}\big(b \cdot (\varrho_h \circ p)\big)\big]$ to obtain the equivalent statement

$$(2\pi h)^n \frac{\mathrm{tr}_{\,\mathrm{L}^2(M)} \big[B \circ \varrho_h(P(h)) \big]}{\mathrm{vol}_{\,T^*M} \big[\operatorname{supp} \big(b \cdot (\varrho_h \circ p) \big) \big]} = \fint\limits_{\operatorname{supp}\, b \cdot (\varrho_h \circ p)} b \cdot (\varrho_h \circ p) \, d(T^*M) + \mathrm{O}\big(h^{1-2\delta}\big) \qquad \text{as } h \to 0$$

in which the distinction between the leading term and the remainder term is emphasized more.

Proof. For convenience of the reader, we give the short proof which involves only standard arguments. By Theorem 4.3.1, there is a number $h_0 \in (0,1]$ and for each $N \in \mathbb{N}$ a collection of symbol functions $\{r_{\alpha,\beta,N}\}_{\alpha,\beta \in \mathcal{A}} \subset S^{2\delta-1}_{h;\delta}(1_{\mathbb{R}^{2n}})$ and an operator $\mathfrak{R}_N(h) \in \mathcal{B}(\mathrm{L}^2(M))$ such that for $h \in (0, h_0]$

$$\begin{aligned} B \circ \varrho_h(P(h))(f) &= \sum_{\alpha \in \mathcal{A}} \overline{\varphi}_\alpha \cdot \mathrm{Op}_h(u_{\alpha,0}) \big((f \cdot \overline{\overline{\varphi}}_\alpha) \circ \gamma_\alpha^{-1} \big) \circ \gamma_\alpha \\ &\quad + \sum_{\alpha,\beta \in \mathcal{A}} \overline{\varphi}_\beta \cdot \mathrm{Op}_h(r_{\alpha,\beta,N}) \big((f \cdot \overline{\overline{\varphi}}_\alpha \cdot \overline{\overline{\overline{\varphi}}}_\beta) \circ \gamma_\beta^{-1} \big) \circ \gamma_\beta + \mathfrak{R}_N(h)(f) \qquad \forall\, f \in \mathrm{L}^2(M), \end{aligned}$$

where $u_{\alpha,0} = \big((\varrho_h \circ p) \cdot b \cdot \varphi_\alpha\big) \circ (\gamma_\alpha^{-1}, (\partial \gamma_\alpha^{-1})^T)$. Moreover, the operator $\mathfrak{R}_N(h) \in \mathcal{B}(\mathrm{L}^2(M))$ is of trace class and its trace norm is of order h^N as $h \to 0$, while for fixed $h \in (0, h_0]$, each symbol function $r_{\alpha,\beta,N}$ is an element of $\mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^{2n})$ that fulfills

$$\operatorname{vol} \operatorname{supp} r_{\alpha,\beta,N} \leq C_{\alpha,\beta,N} \operatorname{vol} \operatorname{supp} \big((\varrho_h \circ p) \cdot b \big) \tag{4.3.27}$$

with a constant $C_{\alpha,\beta,N} > 0$ that is independent of h. In particular, each of the operators

$$A_\alpha : f \mapsto \overline{\varphi}_\alpha \cdot \mathrm{Op}_h(u_{\alpha,0})\big((f \cdot \overline{\overline{\overline{\varphi}}}_\alpha)\circ\gamma_\alpha^{-1}\big), \quad A_{\alpha,\beta,N} : f \mapsto \overline{\varphi}_\beta \cdot \mathrm{Op}_h(r_{\alpha,\beta,N})\big((f \cdot \overline{\overline{\varphi}}_\alpha \cdot \overline{\overline{\overline{\varphi}}}_\beta)\circ\gamma_\beta^{-1}\big)\circ\gamma_\beta$$

has a smooth, compactly supported Schwartz kernel, given by

$$\begin{aligned}
&K_{A_\alpha}(x_1,x_2)\\
&= \frac{1}{(2\pi h)^n}\overline{\varphi}_\alpha(x_1)\int\limits_{\mathbb{R}^n} e^{\frac{i}{h}(y_1-y_2)\cdot\eta} u_{\alpha,0}\Big(\frac{y_1+y_2}{2},\eta,h\Big)\overline{\overline{\overline{\varphi}}}_\alpha(\gamma_\alpha^{-1}(y_2))\,d\eta\big(\mathrm{Vol}_{g_\alpha}(y)\big)^{-1}\\
&K_{A_{\alpha,\beta,N}}(x_1,x_2)\\
&= \frac{1}{(2\pi h)^n}\overline{\varphi}_\beta(x_1)\int\limits_{\mathbb{R}^n} e^{\frac{i}{h}(y_1-y_2)\cdot\eta} r_{\alpha,\beta,N}\Big(\frac{y_1+y_2}{2},\eta,h\Big)(\overline{\overline{\varphi}}_\alpha\cdot\overline{\overline{\overline{\varphi}}}_\beta)(\gamma_\beta^{-1}(y_2))\,d\eta\big(\mathrm{Vol}_{g_\beta}(y)\big)^{-1},
\end{aligned}$$

where $x_1, x_2 \in M$ and y_i denotes $\gamma_\alpha(x_i)$ and $\gamma_\beta(x_i)$ in the first line and the second line, respectively. We obtain for arbitrary $N \in \mathbb{N}$

$$\operatorname{tr}_{\mathrm{L}^2(M)}\big[B\circ\varrho_h(P(h))\big] = \sum_{\alpha\in\mathcal{A}}\int\limits_M K_{A_\alpha}(x,x)\,dM(x) + \sum_{\alpha,\beta\in\mathcal{A}}\int\limits_M K_{A_{\alpha,\beta,N}}(x,x)\,dM(x) + \mathrm{O}(h^N).$$

Let us consider first the integrals in the second summand. Using (4.3.27) we obtain that there is a constant $C_{\alpha,\beta,N} > 0$, independent of h, such that

$$\begin{aligned}
&\Big|\int\limits_M K_{A_{\alpha,\beta,N}}(x,x)\,dM(x)\Big|\\
&\qquad\qquad \le C_{\alpha,\beta,N}\frac{1}{(2\pi h)^n}\,\big\|\overline{\varphi}_\beta\big\|_\infty\,\|r_{\alpha,\beta,N}\|_\infty\,\Big\|\overline{\overline{\varphi}}_\alpha\cdot\overline{\overline{\overline{\varphi}}}_\beta\Big\|_\infty \text{vol supp }\big((\varrho_h\circ p)\cdot b\big).
\end{aligned}$$

As $r_{\alpha,\beta,N}$ is an element of $S^{2\delta-1}_{h;\delta}(1_{\mathbb{R}^{2n}})$, one has $\|r_{\alpha,\beta,N}\|_\infty = \mathrm{O}(h^{1-2\delta})$, and so we conclude

$$\Big|\int\limits_M K_{A_{\alpha,\beta,N}}(x,x)\,dM(x)\Big| = \mathrm{O}\Big(h^{1-2\delta-n}\text{vol supp }\big((\varrho_h\circ p)\cdot b\big)\Big) \quad \text{as } h\to 0.$$

Since $\mathcal{A}$ is finite, it follows

$$\operatorname{tr}_{\mathrm{L}^2(M)}\big[B\circ\varrho_h(P(h))\big] = \sum_{\alpha\in\mathcal{A}}\int\limits_M K_{A_\alpha}(x,x)\,dM(x) + \mathrm{O}\big(h^{1-2\delta-n}\text{vol supp }((\varrho_h\circ p)\cdot b\big).$$

To finish the proof, we calculate the leading term to be given by

$$\begin{aligned}
\sum_{\alpha\in\mathcal{A}}\int\limits_M K_{A_\alpha}(x,x)\,dM(x) &= \sum_{\alpha\in\mathcal{A}}\frac{1}{(2\pi h)^n}\int\limits_{\mathbb{R}^{2n}}\overline{\varphi}_\alpha(\gamma_\alpha^{-1}(y))u_{\alpha,0}(y,\eta,h)\overline{\overline{\overline{\varphi}}}_\alpha(\gamma_\alpha^{-1}(y))\,d\eta\,dy\\
&= \frac{1}{(2\pi h)^n}\sum_{\alpha\in\mathcal{A}}\int\limits_{\mathbb{R}^{2n}}\overline{\varphi}_\alpha(\gamma_\alpha^{-1}(y))\big((\varrho_h\circ p)\cdot b\cdot\varphi_\alpha\big)(\gamma_\alpha^{-1}(y),(\partial\gamma_\alpha^{-1})^T\eta,h)\overline{\overline{\overline{\varphi}}}_\alpha(\gamma_\alpha^{-1}(y))\,d\eta\,dy\\
&= \frac{1}{(2\pi h)^n}\sum_{\alpha\in\mathcal{A}}\int\limits_{T^*M}((\varrho_h\circ p)\cdot b)(x,\xi)\cdot\varphi_\alpha(x)\,d(T^*M)(x,\xi)\\
&= \frac{1}{(2\pi h)^n}\int\limits_{T^*M}((\varrho_h\circ p)\cdot b)(x,\xi)\,d(T^*M)(x,\xi).
\end{aligned}$$

$\square$

Part II

Semiclassical analysis and symmetry reduction

Chapter 5

Overview

5.1 Motivation and setup

Let M be a closed connected Riemannian manifold of dimension n with Riemannian volume density dM, and denote by Δ the Laplace-Beltrami operator on M. One of the central problems in spectral geometry consists in studying the properties of eigenvalues and eigenfunctions of $-\Delta$ in the limit of large eigenvalues. Concretely, let $\{u_j\}$ be an orthonormal basis of $\mathrm{L}^2(M)$ of eigenfunctions of $-\Delta$ with respective eigenvalues $\{E_j\}$, repeated according to their multiplicity. As $E_j \to \infty$, one is interested among other things in the asymptotic distribution of eigenvalues, the pointwise convergence of the u_j, bounds of the L^p-norms of the u_j for $1 \leq p \leq \infty$, and the weak convergence of the measures $|u_j|^2 dM$.

This thesis addresses these problems for Schrödinger operators in case that the underlying classical system possesses certain symmetries.

In Chapter 7, we shall concentrate on the distribution of eigenvalues. The question is then how the symmetries of the underlying Hamiltonian system determine the fine structure of the spectrum. In Chapter 8, we shall concentrate on the ergodic properties of eigenfunctions. In both chapters, the guiding idea behind is the correspondence principle of semiclassical physics. To explain this in more detail, consider the unit co-sphere bundle S^*M, which corresponds to the phase space of a classical free particle moving with constant energy. Each point in S^*M represents a state of the classical system, its motion being given by the geodesic flow in S^*M, and classical observables correspond to functions $a \in \mathrm{C}^\infty(S^*M)$. On the other hand, by the Kopenhagen interpretation of quantum mechanics, quantum observables correspond to self-adjoint operators A in the Hilbert space $\mathrm{L}^2(M)$. The elements $\psi \in \mathrm{L}^2(M)$ are interpreted as states of the quantum mechanical system, and the expectation value for measuring the property A while the system is in the state ψ is given by $\langle A\psi, \psi\rangle_{\mathrm{L}^2(M)}$. The transition between the classical and the quantum-mechanical picture is given by a *quantization map*

$$S^k_h(M) \ni a \quad \longmapsto \quad \mathrm{Op}_h(a), \qquad k \in \mathbb{R},$$

where $\mathrm{Op}_h(a)$ is a pseudodifferential operator in $\mathrm{L}^2(M)$ depending on Planck's constant h and the particular choice of the map Op_h, and $S^k_h(M) \subset \mathrm{C}^\infty(T^*M)$ denotes a suitable space of symbol functions. The *correspondence principle* then says that, in the limit of high energies, the quantum mechanical system should behave more and more like the corresponding classical system. The study of the asymptotic distribution of eigenvalues has a history of more than a hundred years that goes back to work of Weyl [59], Levitan [39], Avacumovič [2], and

Hörmander [32], the central result being *Weyl's law*, which says, in the setup above, that

$$\#\{j : \sqrt{E_j} \leq E\} = \frac{\nu_n}{(2\pi)^n} \mathrm{vol}\,(M) E^n + \mathrm{O}\big(E^{n-1}\big) \qquad \text{as } E \to \infty, \tag{5.1.1}$$

where ν_n denotes the euclidean volume of the unit ball in $\mathbb{R}^n$ and $\mathrm{vol}\,(M)$ is the Riemannian volume of M. In comparison, the behavior of eigenfunctions has been examined more intensively during the last decades. One of the major results in this direction is the *quantum ergodicity theorem* for chaotic systems, due to Shnirelman [52], Zelditch [61], and Colin de Verdière [15]. To explain it, consider the distributions[1]

$$\mu_j : \mathrm{C}^\infty(S^*M) \longrightarrow \mathbb{C}, \quad a \longmapsto \langle \mathrm{Op}_h(a) u_j, u_j \rangle_{\mathrm{L}^2(M)}\,.$$

If it exists, the distribution limit $\mu = \lim_{j\to\infty} \mu_j$ constitutes a so-called *quantum limit* for the eigenfunction sequence $\{u_j\}$. Furthermore, the probability measure on S^*M defined by a quantum limit is invariant under the geodesic flow and independent of the choice of Op_h. Since the measure μ projects to a weak limit $\bar\mu$ of the measures $\bar\mu_j = |u_j|^2 dM$, it is called a *microlocal lift* of $\bar\mu$, and one can reduce the study of the measures $\bar\mu$ to the classification of quantum limits. The quantum ergodicity theorem then says that if the geodesic flow on S^*M is ergodic with respect to the Liouville measure $d(S^*M)$, then there exists a subsequence $\{u_{j_k}\}_{k\in\mathbb{N}}$ of density 1 such that the μ_{j_k} converge to $(\mathrm{vol}\, S^*M)^{-1} d(S^*M)$ as distributions, and consequently the measures $\bar\mu_{j_k}$ converge weakly to $(\mathrm{vol}\, M)^{-1} dM$. Intuitively, the geodesic flow being ergodic means that the geodesics are distributed on S^*M in a sufficiently chaotic way, and this equidistribution of trajectories in the classical system implies asymptotic equidistribution for a density 1 subsequence of states of the corresponding quantum system.

A large class of manifolds whose geodesic flow is ergodic are closed manifolds with strictly negative sectional curvature [31, 7], and one of the main conjectures in the field is the Rudnick-Sarnak conjecture on quantum unique ergodicity (QUE) [49] which says that if M has strictly negative sectional curvature, the whole sequence $|u_j|^2 dM$ converges weakly to the normalized Riemannian measure $(\mathrm{vol}\, M)^{-1} dM$ as $j \to \infty$. It has been verified in certain arithmetic situations by Lindenstrauss [40], but in general, the conjecture is still very open. Sequences of eigenfunctions with a quantum limit different from the Liouville measure are called *exceptional subsequences*, and it has been shown by Jacobson and Zelditch [35] that any flow-invariant measure on the unit co-sphere bundle of a standard n-sphere occurs as a quantum limit for the Laplacian, showing that the family of exceptional subsequences for the Laplacian can be quite large if the geodesic flow fails to be ergodic. However, it was shown by Faure, Nonnenmacher, and de Bièvre [22] that ergodicity of the geodesic flow alone is not sufficient to rule out the existence of exceptional subsequences for particular elliptic operators. Examples of ergodic billiard systems that admit exceptional subsequences of eigenfunctions were recently found by Hassel [26].

Chapter 8 addresses the problem of determining quantum limits for sequences of eigenfunctions of Schrödinger operators in case that the underlying classical system possesses certain symmetries. Due to the presence of conserved quantitites, the corresponding Hamiltonian flow will in parts be integrable, and not totally chaotic, in contrast to the hitherto examined chaotic systems. The question is then how the partially chaotic behavior of the Hamiltonian flow is reflected in the ergodic properties of the eigenfunctions.

[1] Here one regards $s \in \mathrm{C}^\infty(S^*M)$ as an element in $S^0_1(M) \subset \mathrm{C}^\infty(T^*M)$ by extending it 0-homogeneously to T^*M with the zero-section removed, and then cutting off that extension smoothly near the zero section.

5.1.1 Setup

To explain things more precisely, let G be a compact connected Lie group that acts effectively and isometrically on M. Note that there might be orbits of different dimensions, and that the orbit space $\widetilde{M} := M/G$ won't be a manifold in general, but a topological quotient space. If G acts on M with finite isotropy groups, $\widetilde{M}$ is a compact orbifold, and its singularities are not too severe. Consider now a Schrödinger operator on M given by

$$\breve{P}(h) = -h^2\breve{\Delta} + V, \qquad \breve{P}(h) : \mathrm{C}^\infty(M) \to \mathrm{C}^\infty(M), \qquad h \in (0,1],$$

where $\breve{\Delta}$ denotes the Laplace-Beltrami operator as differential operator on M with domain $\mathrm{C}^\infty(M)$ and $V \in \mathrm{C}^\infty(M,\mathbb{R})$ a G-invariant potential. As mentioned already in Part I, $\breve{P}(h)$ has a unique self-adjoint extension

$$P(h) : \mathrm{H}^2(M) \to \mathrm{L}^2(M) \tag{5.1.2}$$

as an unbounded operator in $\mathrm{L}^2(M)$, where $\mathrm{H}^2(M) \subset \mathrm{L}^2(M)$ denotes the second Sobolev space, and one calls $P(h)$ a Schrödinger operator, too. For each $h \in (0,1]$, the spectrum of $P(h)$ is discrete, consisting of eigenvalues $\{E_j(h)\}_{j\in\mathbb{N}}$ which we repeat according to their multiplicity and which form a non-decreasing sequence unbounded towards $+\infty$. Thus, the spectrum of $P(h)$ is bounded from below and its eigenspaces are finite-dimensional. The associated sequence of eigenfunctions $\{u_j(h)\}_{j\in\mathbb{N}}$ constitutes a Hilbert basis in $\mathrm{L}^2(M)$, and each eigenfunction $u_j(h)$ is smooth. When studying the spectral asymptotics of Schrödinger operators, one often uses the *semiclassical method*, as in Part I of this thesis. Instead of examining the spectral properties of $P(h)$ for fixed h and high energies, h being Planck's constant, one considers fixed energy intervals and allows $h \in (0,1]$ to become small, regarding it no longer as a physical constant. The two methods are essentially equivalent. In the special case $V \equiv 0$, the Schrödinger operator is just a rescaled version of $-\Delta$ so that the semiclassical method can be used to study the spectral asymptotics of the Laplace-Beltrami operator. Now, since $P(h)$ commutes with the isometric G-action, one can use representation theory to describe the spectrum and the eigenspaces of $P(h)$ in a more refined way. Indeed, by the Peter-Weyl theorem, the unitary left-regular representation of G

$$G \times \mathrm{L}^2(M) \to \mathrm{L}^2(M), \qquad (g,f) \mapsto \left(L_g f : x \mapsto f(g^{-1}\cdot x)\right),$$

has an orthogonal decomposition into isotypic components according to

$$\mathrm{L}^2(M) = \bigoplus_{\chi\in\widehat{G}} \mathrm{L}^2_\chi(M), \qquad \mathrm{L}^2_\chi(M) = T_\chi\,\mathrm{L}^2(M), \tag{5.1.3}$$

where we wrote $\widehat{G}$ for the set of equivalence classes of irreducible unitary G-representations, and $T_\chi : \mathrm{L}^2(M) \to \mathrm{L}^2_\chi(M)$ for the associated orthogonal projections. The *character* belonging to an element $\chi \in \widehat{G}$ is given by $\chi(g) := \mathrm{tr}_{\mathrm{L}^2(M)}\,\pi_\chi(g)$, where π_χ denotes a representation of class χ. It is also denoted by χ, and the projection operators T_χ are given by the explicit formula

$$T_\chi : f \mapsto \Big(x \mapsto d_\chi \int_G \overline{\chi(g)} f(g^{-1}\cdot x)\,dg\Big), \tag{5.1.4}$$

where dg is the normalized Haar measure on G, and d_χ the dimension of an irreducible representation of class χ. Since each eigenspace of the Schrödinger operator $P(h)$ constitutes a unitary G-module, it has an analogous decomposition into a direct sum of irreducible

G-representations, which represents the so-called *fine structure* of the spectrum of $P(h)$. To study this fine structure and the eigenfunctions of $P(h)$ asymptotically, consider for a fixed $\chi \in \widehat{G}$ and any operator $A : D \to \mathrm{L}^2(M)$ on a G-invariant subset $D \subset \mathrm{L}^2(M)$ the corresponding *reduced operator*

$$A_\chi := T_\chi \circ A \circ T_\chi|_D. \tag{5.1.5}$$

Since $P(h)$ commutes with T_χ, the reduced operator $P(h)_\chi$ corresponds to the bi-restriction $P(h)|_\chi : \mathrm{L}^2_\chi(M) \cap \mathrm{H}^2(M) \to \mathrm{L}^2_\chi(M)$. More generally, one can consider the bi-restriction of $P(h)$ to h-dependent sums of isotypic components of the form

$$\mathrm{L}^2_{\mathcal{W}_h}(M) = \bigoplus_{\chi \in \mathcal{W}_h} \mathrm{L}^2_\chi(M), \tag{5.1.6}$$

choosing for each $h \in (0,1]$ an appropriate finite subset $\mathcal{W}_h \subset \widehat{G}$ whose cardinality is allowed to grow in a controlled way as $h \to 0$. A natural problem is then to examine the spectral asymptotics and the eigenfunctions of $P(h)$ bi-restricted to $\mathrm{L}^2_{\mathcal{W}_h}(M)$. The study of a single isotypic component corresponds to choosing $\mathcal{W}_h = \{\chi\}$ for all h and a fixed $\chi \in \widehat{G}$. Note that, so far, it is irrelevant whether the group action has various different orbit types or not.

On the other hand, the principal symbol of the Schrödinger operator (5.1.2) is represented by the G-invariant Hamiltonian

$$p : T^*M \to \mathbb{R}, \qquad (x,\xi) \mapsto \|\xi\|_x^2 + V(x), \tag{5.1.7}$$

on the co-tangent bundle T^*M of M with canonical symplectic form ω. It describes the classical mechanical properties of the underlying Hamiltonian system, and defines a Hamiltonian flow $\varphi_t : T^*M \to T^*M$, which in the special case $V \equiv 0$ corresponds to the geodesic flow on $T^*M \cong TM$. Consider now for a regular value c of p the hypersurface $\Sigma_c := p^{-1}(\{c\}) \subset T^*M$. It is invariant under the Hamiltonian flow φ_t, and carries a canonical Liouville measure $d\Sigma_c$ induced by ω. The measure space $(\Sigma_c, d\Sigma_c)$ is related to the asymptotic distribution of the eigenvalues of $P(h)$ that are close to c by its volume which occurs in the leading term of the semiclassical Weyl law, and in case that φ_t is ergodic on $(\Sigma_c, d\Sigma_c)$, the measure space is also known to be related to the asymptotic equidistribution of the eigenfunctions $\{u_j(h)\}$ by the semiclassical quantum ergodicity theorem [20, Appendix D, Theorem 5]. Now, as we are interested in the asymptotic distribution of eigenvalues close to c along subspaces of $\mathrm{L}^2(M)$ of the form (5.1.6) and properties of the corresponding eigenfunctions, we can expect these issues to be related to measure spaces modeled on subsets of Σ_c or quotients of such subsets by the G-action. Indeed, if G is non-trivial, φ_t cannot be ergodic on $(\Sigma_c, d\Sigma_c)$ due to the presence of additional conserved quantities besides the total energy c. To describe the classical dynamical properties of the system, it is therefore convenient to divide out the symmetries, which can be done by performing a procedure called *symplectic reduction*. The latter is based on the fundamental fact that the presence of conserved quantities or first integrals of motion leads to elimination of variables, and reduces the given configuration space with its symmetries to a lower-dimensional one, in which the degeneracies and the conserved quantitites have been eliminated. Namely, let $\mathbb{J} : T^*M \to \mathfrak{g}^*$ denote the momentum map of the induced Hamiltonian G-action on T^*M, which represents the conserved quantitites of the dynamical system, and consider the topological quotient space

$$\widetilde{\Omega} := \Omega/G, \qquad \Omega := \mathbb{J}^{-1}(\{0\}).$$

In contrast to the situation encountered in the Peter-Weyl theorem, the orbit structure of the underlying G-action on M is not at all irrelevant to the symplectic reduction. Namely, if the

G-action is not free the spaces Ω and $\widetilde{\Omega}$ need not be manifolds. Nevertheless, they are stratified spaces, where each stratum is a smooth manifold that consists of orbits of one particular type. In particular, Ω and $\widetilde{\Omega}$ each have a *principal stratum* Ω_{reg} and $\widetilde{\Omega}_{\mathrm{reg}}$, respectively, which is the smooth manifold consisting of (the union of) all orbits whose isotropy type is the minimal of M. Moreover, $\widetilde{\Omega}_{\mathrm{reg}}$ carries a canonical symplectic structure, and the Hamiltonian flow on T^*M induces a flow $\widetilde{\varphi}_t : \widetilde{\Omega}_{\mathrm{reg}} \to \widetilde{\Omega}_{\mathrm{reg}}$, which is the Hamiltonian flow associated to the reduced Hamiltonian $\widetilde{p} : \widetilde{\Omega}_{\mathrm{reg}} \to \mathbb{R}$ induced by p. One calls $\widetilde{\varphi}_t$ the *reduced Hamiltonian flow*. Since the orbit projection $\Omega_{\mathrm{reg}} \to \widetilde{\Omega}_{\mathrm{reg}}$ is a submersion, c is also a regular value of $\widetilde{p}$, and we define $\widetilde{\Sigma}_c := \widetilde{p}^{-1}(\{c\}) \subset \widetilde{\Omega}_{\mathrm{reg}}$. Similarly to $(\Sigma_c, d\Sigma_c)$, the smooth hypersurface $\widetilde{\Sigma}_c = (\Omega_{\mathrm{reg}} \cap \Sigma_c)/G \subset \widetilde{\Omega}_{\mathrm{reg}}$ carries a Liouville measure $d\widetilde{\Sigma}_c$ induced by the symplectic form on $\widetilde{\Omega}_{\mathrm{reg}}$, and one can interpret the measure space $(\widetilde{\Sigma}_c, d\widetilde{\Sigma}_c)$ as the symplectic reduction of $(\Sigma_c, d\Sigma_c)$. If one prefers a symplectically reduced measure space modeled on a subset of Σ_c rather than a quotient space, one can use that $(\widetilde{\Sigma}_c, d\widetilde{\Sigma}_c)$ corresponds to the measure space $\big(\Omega_{\mathrm{reg}} \cap \Sigma_c, \frac{d\mu_c}{\mathrm{vol}\,\mathcal{O}}\big)$, where $d\mu_c$ denotes the induced volume density on the smooth hypersurface $\Omega_{\mathrm{reg}} \cap \Sigma_c \subset \Omega_{\mathrm{reg}}$, and the function $\mathrm{vol}\,_{\mathcal{O}} : \Sigma_c \cap \Omega_{\mathrm{reg}} \to (0, \infty)$, $x \mapsto \mathrm{vol}\,(G \cdot x)$ assigns to an orbit its Riemannian volume. For a detailed exposition of these facts, we refer the reader to Sections 6.2 and 6.3.

5.1.2 Problem and methods

Let us now come back to our initial questions. Suppose that we have chosen for each $h \in (0, 1]$ an appropriate finite set $\mathcal{W}_h \subset \widehat{G}$ whose cardinality does not grow too fast as $h \to 0$, see Definition 5.2.1 below. Then, in Chapter 7, we shall study the distribution of the eigenvalues of $P(h)$ along the h-dependent family of isotypic components $\mathrm{L}^2_{\mathcal{W}_h}(M)$ in the Peter-Weyl decomposition of $\mathrm{L}^2(M)$ as $h \to 0$, and the way their distribution in a spectral window $[c, c + h^\delta]$ around a regular value c of p is related to the symplectic reduction $(\widetilde{\Sigma}_c, d\widetilde{\Sigma}_c)$ of the corresponding Hamiltonian system, $\delta > 0$ being a suitable small number. Similar problems were studied for h-pseudodifferential operators in $\mathbb{R}^n$ in [21], [13], [58], and within the classical high-energy approach in [18], [10], [28, 29], and [46]. In our approach, we shall combine well-known methods from semiclassical analysis and symplectic reduction with results on singular equivariant asymptotics recently developed in [46, 47]. We will also use Theorem 4.3.1 from the first part of this thesis. In case of the Laplacian, it would also be possible to study the problem via the original classical approach of Shnirelman, Zelditch and Colin de Verdière.

In Chapter 8, we assume that the reduced Hamiltonian flow $\widetilde{\varphi}_t$ is ergodic on $(\widetilde{\Sigma}_c, d\widetilde{\Sigma}_c)$, and we then ask whether there is a non-trivial family of index sets $\{\Lambda(h)\}_{h \in (0,1]}$, $\Lambda(h) \subset \mathbb{N}$, such that for $j \in \Lambda(h)$ we have $u_j(h) \in \mathrm{L}^2_\chi(M)$ for some $\chi \in \mathcal{W}_h$, the associated eigenvalue $E_j(h)$ is close to c, and the distributions

$$\mu_j(h) : \mathrm{C}^\infty_{\mathrm{c}}(\Sigma_c) \longrightarrow \mathbb{C}, \quad a \longmapsto \langle \mathrm{Op}_h(a) u_j(h), u_j(h) \rangle_{\mathrm{L}^2(M)}$$

converge for $j \in \Lambda(h)$ and $h \to 0$ to a distribution limit with density 1, which would answer the corresponding question for the measures $|u_j(h)|^2 dM$. In particular, in the special case $V \equiv 0$, $c = 1$, the problem is equivalent to finding quantum limits for sequences of eigenfunctions of the Laplace-Beltrami operator. In case that $\widetilde{M}$ is an orbifold and $\mathcal{W}_h = \{\chi_0\}$ for all h, where χ_0 corresponds to the trivial representation, this problem has been dealt with recently by Kordyukov [38] using classical techniques.

The general idea behind the approach to equivariant quantum ergodicity in this thesis can be summarized as follows. The existence of symmetries of a classical Hamiltonian system implies the existence of conserved quantitites and partial integrability of the Hamiltonian flow, forcing the system to behave less chaotically. Symplectic reduction divides out the symmetries, and hence, order, and allows to study the symmetry-reduced spectral and ergodic properties of the corresponding quantum system. In particular, eigenfunctions should reflect the partially chaotic behavior of the classical system.

5.2 Summary of main results

To formulate our results, we first need to introduce some additional notation. As explained in Section 6.2, the G-action on M has a principal isotropy type (H), represented by a principal isotropy group H, as well as a principal orbit type. We denote by κ the dimension of the principal orbits, which agrees with the maximal dimension of a G-orbit in M, and we assume throughout the whole second part of this thesis that $\kappa < n = \dim M$. Furthermore, we denote by Λ_M^G the maximal number of elements of a totally ordered subset of isotropy types of the G-action on M. For an equivalence class $\chi \in \widehat{G}$ with corresponding irreducible G-representation π_χ, we write $[\pi_\chi|_H : \mathbb{1}]$ for the multiplicity of the trivial representation in the restriction of π_χ to H. Let $\widehat{G}' \subset \widehat{G}$ denote the subset consisting of those classes of representations that appear in the decomposition (5.1.3) of $\mathrm{L}^2(M)$. In order to consider a growing number of isotypic components of $\mathrm{L}^2(M)$ in the semiclassical limit, we make the following

Definition 5.2.1. *A family $\{\mathcal{W}_h\}_{h\in(0,1]}$ of finite sets $\mathcal{W}_h \subset \widehat{G}'$ is called* semiclassical character family *if there exists a $\vartheta \geq 0$ such that for each $N \in \{0,1,2,\dots\}$ and each differential operator D on G of order N, there is a constant $C > 0$ independent of h with*

$$\frac{1}{\#\mathcal{W}_h} \sum_{\chi\in\mathcal{W}_h} \frac{\|D\,\overline{\chi}\|_\infty}{[\pi_\chi|_H : \mathbb{1}]} \leq C\, h^{-\vartheta N} \qquad \forall\, h \in (0,1].$$

We call the smallest possible ϑ the growth rate *of the semiclassical character family.*

Remark 5.2.1. Note that $[\pi_\chi|_H : \mathbb{1}] \geq 1$ for $\chi \in \widehat{G}'$, since the irreducible G-representations appearing in the decomposition (5.1.3) of $\mathrm{L}^2(M)$ are precisely those representations appearing in $\mathrm{L}^2(G/H)$, and by the Frobenius reciprocity theorem one has $[\pi_\chi|_H : \mathbb{1}] = [\mathrm{L}^2(G/H) : \pi_\chi]$, compare [18, Section 2].

Example 5.2.2. For $G = \mathrm{SO}(2) \cong S^1 \subset \mathbb{C}$, one has $\widehat{G} \equiv \{\chi_k : k \in \mathbb{Z}\}$, where the k-th character $\chi_k : G \to \mathbb{C}$ is given by $\chi_k\big(e^{i\varphi}\big) = e^{ik\varphi}$. One then obtains a semiclassical character family with growth rate less or equal to ϑ by setting $\mathcal{W}_h := \{\chi_k : |k| \leq h^{-\vartheta}\}$.

Example 5.2.3. More generally, let G be a connected compact semi-simple Lie group with Lie algebra $\mathfrak{g}$ and $T \subset G$ a maximal torus with Lie algebra $\mathfrak{t}$. By the Cartan-Weyl classification of irreducible finite dimensional representations of reductive Lie algebras over $\mathbb{C}$, $\widehat{G}$ can be identified with the set of dominant integral and T-integral linear forms Λ on the complexification $\mathfrak{t}_\mathbb{C}$ of the Lie algebra $\mathfrak{t}$ of T. Let therefore denote $\Lambda_\chi \in \mathfrak{t}_\mathbb{C}^*$ the element associated with a class $\chi \in \widehat{G}$, and put $\mathcal{W}_h := \big\{\chi \in \widehat{G} : |\Lambda_\chi| \leq h^{-\vartheta}\big\}$, where $\vartheta \geq 0$, $h \in (0,1]$. Then $\{\mathcal{W}_h\}_{h\in(0,1]}$ constitutes a semiclassical character family with growth rate less or equal to ϑ in the sense of Definition 5.2.1, see [47, Section 3.2] for details.

Denote by $\Psi^m_{h;\delta}(M)$, $m \in \mathbb{R} \cup \{-\infty\}$, $\delta \in [0, \frac{1}{2})$, the set of semiclassical pseudodifferential operators on M of order (m,δ). The principal symbols of these operators are represented by symbol functions in the classes $S^m_{h;\delta}(M)$, where the index δ describes the growth properties of the symbol functions as $h \to 0$, see Section 2.1 for the precise definitions. An important point to note is that elements of $S^0_{h;\delta}(M)$ define operators on $\mathrm{L}^2(M)$ with operator norm bounded uniformly in h. Finally, for any measurable function f with domain D a G-invariant subset of M or T^*M, we write

$$\langle f\rangle_G(x) := \int_G f(g\cdot x)\,dg, \tag{5.2.1}$$

and denote by $\widetilde{\langle f\rangle}_G$ the function induced on the orbit space D/G by the G-invariant function $\langle f\rangle_G$. In order to begin with the statements of the results, let

$$\delta \in \Big(0, \frac{1}{2\kappa+4}\Big)$$

and choose a semiclassical character family $\{\mathcal{W}_h\}_{h\in(0,1]}$ for the G-action on M with growth rate

$$\vartheta < \frac{1-(2\kappa+4)\delta}{2\kappa+3}.$$

Furthermore, write

$$J(h) := \big\{j\in\mathbb{N} : E_j(h)\in[c,c+h^\delta],\ \chi_j(h)\in\mathcal{W}_h\big\},$$

where $\chi_j(h)\in\widehat{G}$ is defined by $u_j(h)\in \mathrm{L}^2_{\chi_j(h)}(M)$. We can now state the main result of Chapter 7.

Result 4 (**Generalized equivariant semiclassical Weyl law**, Theorem 7.2.1)**.** *For an operator $B\in\Psi^0_{h;\delta}(M)\subset\mathcal{B}(\mathrm{L}^2(M))$ with principal symbol represented by $b\in S^0_{h;\delta}(M)$, one has in the semiclassical limit $h\to 0$*

$$\begin{aligned}\frac{(2\pi)^{n-\kappa}h^{n-\kappa-\delta}}{\#\mathcal{W}_h}\sum_{J(h)}\frac{\langle Bu_j(h),u_j(h)\rangle_{\mathrm{L}^2(M)}}{d_{\chi_j(h)}\,[\pi_{\chi_j(h)}|_H:\mathbb{1}]} &= \int_{\Sigma_c\cap\Omega_{\mathrm{reg}}} b\,\frac{d\mu_c}{\mathrm{vol}\,\mathcal{O}}\\ &\quad+\ \mathrm{O}\Big(h^\delta + h^{\frac{1-(2\kappa+3)\vartheta}{2\kappa+4}-\delta}\left(\log h^{-1}\right)^{\Lambda^G_M-1}\Big).\end{aligned}\tag{5.2.2}$$

When considering only a fixed isotypic component $\mathrm{L}^2_\chi(M)$ the statement becomes simpler, yielding the asymptotic formula

$$\begin{aligned}(2\pi)^{n-\kappa}h^{n-\kappa-\delta}\sum_{\substack{j\in\mathbb{N}:\,u_j(h)\in\mathrm{L}^2_\chi(M),\\ E_j(h)\in[c,c+h^\delta]}}\langle Bu_j(h),u_j(h)\rangle_{\mathrm{L}^2(M)} &= d_\chi\,[\pi_\chi|_H:\mathbb{1}]\int_{\widetilde{\Sigma}_c}\widetilde{\langle b\rangle}_G\,d\widetilde{\Sigma}_c\\ &\quad+\ \mathrm{O}\Big(h^\delta + h^{\frac{1}{2\kappa+4}-\delta}\left(\log h^{-1}\right)^{\Lambda^G_M-1}\Big),\end{aligned}$$

see Theorem 7.2.3. Via co-tangent bundle reduction, the integral in the leading term of (5.2.2) can actually be viewed as an integral over the smooth bundle

$$S^*_{\widetilde{p},c}(\widetilde{M}_{\mathrm{reg}}) := \big\{(x,\xi)\in T^*(\widetilde{M}_{\mathrm{reg}}) : \widetilde{p}(x,\xi)=c\big\},$$

where $\widetilde{M}_{\mathrm{reg}}$ is the space of principal orbits in M and $\widetilde{p}$ is the function induced by p on $T^*(\widetilde{M}_{\mathrm{reg}})$, compare Lemma 6.2.3. In case that $\widetilde{M}$ is an orbifold, the mentioned integral is given by an integral over the orbifold bundle $S^*_{\widetilde{p},c}(\widetilde{M}) := \big\{(x,\xi)\in T^*\widetilde{M} : \widetilde{p}(x,\xi)=c\big\}$, compare Remark 7.1.2.

We prove Theorem 7.2.1 by applying a semiclassical trace formula for h-dependent test functions, which is the content of Theorem 7.1.1. The trace formula established here is a generalization of [58, Theorem 3.1] for Schrödinger operators to closed G-manifolds, h-dependent test functions, and h-dependent families of isotypic components. In particular, the fact that we consider h-dependent test functions is crucial for the applications in Chapter 8 and implicitly involves the technical difficulties whose treatment was outsourced to the first part of this thesis. Ultimately, the proof of Theorem 7.1.1 reduces to the asymptotic description of certain oscillatory integrals that have recently been studied in [46, 47] by Ramacher using resolution of singularities. The involved phase functions are given in terms of the underlying G-action on M, and if singular orbits occur, the corresponding critical sets are no longer smooth, so that a partial desingularization process has to be implemented in order to obtain asymptotics with remainder estimates via the stationary phase principle. The explicit remainder estimates obtained in [47] do not only account for the quantitative form of the remainder in (5.2.2). They also provide the qualitative basis for our study of h-dependent families of isotypic components and the localization to the hypersurface $\widetilde{\Sigma}_c$. Without the remainder estimates from [47], only a fixed single isotypic component $\mathrm{L}^2_\chi(M)$ could be studied, and only eigenvalues $E_j(h)$ lying in a non-shrinking energy strip of the form $[c, c+\varepsilon]$ with a fixed $\varepsilon > 0$ could be considered, compare Remark 7.2.4. Proceeding with our summary, the main result of Chapter 8 is

Result 5 (**Equivariant quantum ergodicity for Schrödinger operators**, Theorem 8.2.6)**.** *Suppose that the reduced Hamiltonian flow $\widetilde{\varphi}_t$ is ergodic on $\widetilde{\Sigma}_c$. Then, there is a number $h_0 \in (0,1]$ such that for each $h \in (0, h_0]$ we have two subsets $\Lambda^1(h),\ \Lambda^2(h) \subset J(h)$ satisfying*

$$\lim_{h\to 0} \frac{\#\Lambda^1(h)}{\#J(h)} = 1, \qquad \lim_{h\to 0} \frac{\sum_{j\in\Lambda^2(h)} \frac{1}{d_{\chi_j(h)}[\pi_{\chi_j(h)}|_H:\mathbb{1}]}}{\sum_{j\in J(h)} \frac{1}{d_{\chi_j(h)}[\pi_{\chi_j(h)}|_H:\mathbb{1}]}} = 1,$$

such that for each semiclassical pseudodifferential operator $A \in \Psi^0_h(M)$ with principal symbol $\sigma(A) = [a]$, where a is h-independent, the following holds. For all $\varepsilon > 0$ there is a $h_\varepsilon \in (0, h_0]$ such that for all $h \in (0, h_\varepsilon]$ one has

$$\frac{1}{\sqrt{d_{\chi_j(h)}[\pi_{\chi_j(h)}|_H:\mathbb{1}]}}\Big|\langle Au_j(h), u_j(h)\rangle_{\mathrm{L}^2(M)} - \fint_{\Sigma_c\cap\Omega_{\mathrm{reg}}} a\,\frac{d\mu_c}{\operatorname{vol}\mathcal{O}}\Big| < \varepsilon \quad \forall\, j \in \Lambda^1(h), \tag{5.2.3}$$

$$\Big|\langle Au_j(h), u_j(h)\rangle_{\mathrm{L}^2(M)} - \fint_{\Sigma_c\cap\Omega_{\mathrm{reg}}} a\,\frac{d\mu_c}{\operatorname{vol}\mathcal{O}}\Big| < \varepsilon \quad \forall\, j \in \Lambda^2(h), \tag{5.2.4}$$

where the integrals in (5.2.3) and (5.2.4) equal $\fint_{\widetilde{\Sigma}_c} \widetilde{\langle a\rangle_G}\, d\widetilde{\Sigma}_c$.

Result 5 shows that when considering general semiclassical character families, one can choose between two versions of the equivariant quantum ergodicity theorem, corresponding to the two families of subsets $\Lambda^1(h)$ and $\Lambda^2(h)$: Either, one prefers the usual notion of a density 1 family of subsets, then one has to include the factor $1/\sqrt{d_{\chi_j(h)}[\pi_{\chi_j(h)}|_H:\mathbb{1}]}$ in the convergence statement; or one accepts a new representation theoretic definition of a density 1 family of subsets of $J(h)$ with respect to the limit $h \to 0$, and then one can avoid introducing the seemingly inelegant additional factor in the convergence statement. If $\mathcal{W}_h$ consists of just a single character, the two options agree and the statement of Result 5 is simpler, see Theorem 8.2.7. Result 5 will be deduced from Result 4. Again, it should be emphasized that the remainder estimate for the equivariant semiclassical Weyl law proved in Chapter 7, and consequently the desingularization process implemented in [46], are crucial for studying shrinking spectral windows and growing families of representations in Result 5. The obtained quantum limits

$$(\mathrm{vol}_{\frac{\mu_c}{\mathrm{vol}\,\mathcal{O}}}(\Sigma_c \cap \Omega_{\mathrm{reg}}))^{-1} \frac{d\mu_c}{\mathrm{vol}\,\mathcal{O}}$$

describe the ergodic properties of the eigenfunctions in the presence of symmetries. They are singular measures since they are supported on $\Sigma_c \cap \Omega_{\mathrm{reg}}$, which is a submanifold of Σ_c of codimension κ. In fact, they correspond to Liouville measures on the smooth bundles

$$S^*_{\widetilde{p},c}(\widetilde{M}_{\mathrm{reg}}) := \big\{(x,\xi) \in T^*(\widetilde{M}_{\mathrm{reg}}) : \widetilde{p}(x,\xi) = c\big\}$$

over the space of principal orbits in M; if $\widetilde{M}$ is an orbifold, they are given by integrals over the orbifold bundles $S^*_{\widetilde{p},c}(\widetilde{M}) := \big\{(x,\xi) \in T^*\widetilde{M} : \widetilde{p}(x,\xi) = c\big\}$, see Remark 7.2.2. In the latter case, the ergodicity of the reduced flow $\widetilde{\varphi}_t$ on $\widetilde{\Sigma}_c$ is equivalent to the ergodicity of the corresponding Hamiltonian flow on the orbifold bundle $S^*_{\widetilde{p},c}(\widetilde{M})$ with respect to the canonical Liouville measures.

In the special case of the Laplacian, Result 5 becomes an equivariant version of the classical quantum ergodicity theorem of Shnirelman [52], Zelditch [61], and Colin de Verdière [15]. To state it in the version corresponding to the $\Lambda^1(h)$-statement of Result 5, let $\{u_j\}_{j\in\mathbb{N}}$ be an orthonormal basis in $\mathrm{L}^2(M)$ of eigenfunctions of $-\Delta$ with associated eigenvalues $\{E_j\}_{j\in\mathbb{N}}$.

Result 6 (**Equivariant quantum limits for the Laplacian**, Theorem 8.3.2)**.** *Assume that the reduced geodesic flow is ergodic. Choose a semiclassical character family $\{\mathcal{W}_h\}_{h\in(0,1]}$ of growth rate $\vartheta < \frac{1}{2\kappa+3}$ and a partition $\mathcal{P}$ of the set $\{E_j\}_{j\in\mathbb{N}}$ of order $\delta \in \big(0, \frac{1-(2\kappa+3)\vartheta}{2\kappa+4}\big)$ in the sense of Definition 8.3.1. Define the set of eigenfunctions*

$$\big\{u_i^{\mathcal{W},\mathcal{P}}\big\}_{i\in\mathbb{N}} := \big\{u_j : \chi_j \in \mathcal{W}_{E_{\mathcal{P}(j)}^{-1/2}}\big\},$$

where χ_j is defined by $u_j \in \mathrm{L}^2_{\chi_j}(M)$. Define $\chi_i^{\mathcal{W},\mathcal{P}}$ by $u_i^{\mathcal{W},\mathcal{P}} \in \mathrm{L}^2_{\chi_i^{\mathcal{W},\mathcal{P}}}(M)$. Then, there is a subsequence $\big\{u_{i_k}^{\mathcal{W},\mathcal{P}}\big\}_{k\in\mathbb{N}}$ of density 1 in $\big\{u_i^{\mathcal{W},\mathcal{P}}\big\}_{i\in\mathbb{N}}$ such that for all $s \in \mathrm{C}^\infty(S^*M)$ one has

$$\frac{1}{\sqrt{d_{\chi_{i_k}^{\mathcal{W},\mathcal{P}}}[\pi_{\chi_{i_k}^{\mathcal{W},\mathcal{P}}}|_H : \mathbb{1}]}} \left| \left\langle \mathrm{Op}(s) u_{i_k}^{\mathcal{W},\mathcal{P}}, u_{i_k}^{\mathcal{W},\mathcal{P}} \right\rangle_{\mathrm{L}^2(M)} - \fint_{S^*M\cap\Omega_{\mathrm{reg}}} s \frac{d\mu}{\mathrm{vol}\,\mathcal{O}} \right| \longrightarrow 0 \qquad \text{as } k\to\infty,$$

where we wrote μ for μ_1 and Op *for* Op_1*, which is the ordinary non-semiclassical quantization, see Chapter 6.*

Of course, there is also a second version of Result 6 corresponding to the $\Lambda^2(h)$-statement of Result 5, involving a subsequence with a more complicated density property but a more elegant convergence statement of the form

$$\left\langle \mathrm{Op}(s) u_{i_k}^{\mathcal{W},\mathcal{P}}, u_{i_k}^{\mathcal{W},\mathcal{P}} \right\rangle_{\mathrm{L}^2(M)} \longrightarrow \fint_{S^*M\cap\Omega_{\mathrm{reg}}} s \frac{d\mu}{\mathrm{vol}\,\mathcal{O}} \qquad \text{as } k\to\infty.$$

In the special case of a single isotypic component, Result 6 simplifies to the following statement. Let $\{u_j^\chi\}_{j\in\mathbb{N}}$ be an orthonormal basis of $\mathrm{L}^2_\chi(M)$ consisting of eigenfunctions of $-\Delta$. Then, by Theorem 8.3.9, there is a subsequence $\{u_{j_k}^\chi\}_{k\in\mathbb{N}}$ of density 1 in $\{u_j^\chi\}_{j\in\mathbb{N}}$ such that for all $s \in \mathrm{C}^\infty(S^*M)$ one has

$$\left\langle \mathrm{Op}(s) u_{j_k}^\chi, u_{j_k}^\chi \right\rangle_{\mathrm{L}^2(M)} \longrightarrow \fint_{S^*M\cap\Omega_{\mathrm{reg}}} s \frac{d\mu}{\mathrm{vol}\,\mathcal{O}} \qquad \text{as } k\to\infty.$$

Projecting from $S^*M \cap \Omega_{\mathrm{reg}}$ onto M we immediately deduce from Result 6 for any $f \in \mathrm{C}(M)$

$$\frac{1}{\sqrt{d_{\chi^{\mathcal{W},\mathcal{P}}_{i_k}}[\pi_{\chi^{\mathcal{W},\mathcal{P}}_{i_k}}|_H : \mathbb{1}]}} \left| \int_M f |u^{\mathcal{W},\mathcal{P}}_{i_k}|^2 dM - \fint_M f \, \frac{dM}{\operatorname{vol}_{\mathcal{O}}} \right| \longrightarrow 0 \qquad \text{as } k \to \infty,$$

which describes the asymptotic equidistribution of the eigenfunctions in the presence of symmetries, see Corollary 8.3.5. If we consider instead the variant of Result 6 corresponding to the $\Lambda^2(h)$-statement of Result 5, where the density property of the obtained subsequence is more complicated, we arrive at the more elegant assertion that there is a weak convergence of measures

$$|u^{\mathcal{W},\mathcal{P}}_{i_k}|^2 \, dM \longrightarrow \left(\operatorname{vol}_{\frac{dM}{\operatorname{vol}_{\mathcal{O}}}} M\right)^{-1} \frac{dM}{\operatorname{vol}_{\mathcal{O}}} \qquad \text{as } k \to \infty.$$

Again, for a single isotypic component $\mathrm{L}^2_\chi(M)$, the two versions agree and we get a simpler result, compare Corollary 8.3.10. The fact that the reduced and the non-reduced flow cannot be simultaneously ergodic is consistent with the QUE conjecture, since otherwise the results of Chapter 8 would, in principle, imply the existence of exceptional subsequences for ergodic geodesic flows. In this sense, our results can be understood as complementary to the previously known results. Applying some elementary representation theory, one can deduce from Corollary 8.3.5 a statement on convergence of measures on the topological Hausdorff space $\widetilde{M}$ associated to irreducible G-representations. For this, choose an orthogonal decomposition of $\mathrm{L}^2(M)$ into a direct sum $\bigoplus_{i \in \mathbb{N}} V_i$ of irreducible unitary G-modules such that each V_i is contained in an eigenspace of the Laplace-Beltrami operator corresponding to some eigenvalue $E_{j(i)}$. Denote by $\chi_i \in \widehat{G}$ the class of V_i.

Result 7 (**Representation-theoretic equidistribution theorem**, Theorem 8.3.8)**.** *Assume that the reduced geodesic flow is ergodic. Choose a semiclassical character family $\{\mathcal{W}_h\}_{h\in(0,1]}$ of growth rate $\vartheta < \frac{1}{2\kappa+3}$ and a partition $\mathcal{P}$ of $\{E_j\}_{j\in\mathbb{N}}$ of order $\delta \in (0, \frac{1-(2\kappa+3)\vartheta}{2\kappa+4})$. Define the set of irreducible G-modules*

$$\{V_l^{\mathcal{W},\mathcal{P}}\}_{l\in\mathbb{N}} := \{V_i : \chi_i \in \mathcal{W}_{E^{-1/2}_{\mathcal{P}(j(i))}}\}.$$

As in Lemma 8.3.7, assign to each $V_l^{\mathcal{W},\mathcal{P}}$ the G-invariant function $\Theta_l := \Theta_{V_l^{\mathcal{W},\mathcal{P}}} : M \to [0,\infty)$, regard it as a function on $M/G = \widetilde{M}$, and write $\chi_l^{\mathcal{W},\mathcal{P}}$ for the class of $V_l^{\mathcal{W},\mathcal{P}}$. Then, there is a subsequence $\{V_{l_m}^{\mathcal{W},\mathcal{P}}\}_{m\in\mathbb{N}}$ with

$$\lim_{N\to\infty} \frac{\sum_{l_m \le N} d_{\chi^{\mathcal{W},\mathcal{P}}_{l_m}}}{\sum_{l\le N} d_{\chi^{\mathcal{W},\mathcal{P}}_{l}}} = 1$$

for which one has in the limit $m \to \infty$

$$\frac{1}{\sqrt{d_{\chi^{\mathcal{W},\mathcal{P}}_{l_m}}[\pi_{\chi^{\mathcal{W},\mathcal{P}}_{l_m}}|_H : \mathbb{1}]}} \left| \int_{\widetilde{M}} f \, \Theta_{l_m} \, d\widetilde{M} - \fint_{\widetilde{M}} f \frac{d\widetilde{M}}{\operatorname{vol}} \right| \longrightarrow 0 \qquad \forall\, f \in \mathrm{C}(\widetilde{M}),$$

where $d\widetilde{M} = \pi_* dM$ is the pushforward measure defined by the orbit projection $\pi : M \to \widetilde{M}$ and $\operatorname{vol} : \widetilde{M} \to (0,\infty)$ *assigns to an orbit its Riemannian volume.*

For a single isotypic component, one obtains a simpler statement by considering an orthogonal decomposition of $\mathrm{L}^2_\chi(M)$ into a sum $\bigoplus_{i\in\mathbb{N}} V_i^\chi$ of irreducible unitary G-modules of class χ such that each V_i^χ is contained in some eigenspace of the Laplace-Beltrami operator. Then, we have the weak convergence of measures

$$\Theta_{i_k}^\chi \, d\widetilde{M} \stackrel{k\to\infty}{\longrightarrow} \left(\mathrm{vol}_{\frac{d\widetilde{M}}{\mathrm{vol}}} \widetilde{M}\right)^{-1} \frac{d\widetilde{M}}{\mathrm{vol}}$$

for a subsequence $\{V_{i_k}^\chi\}_{k\in\mathbb{N}}$ of density 1 in $\{V_i^\chi\}_{i\in\mathbb{N}}$, see Theorem 8.3.11. Note that Result 7 is a statement about limits of representations, or multiplicities, and not eigenfunctions, since it assigns to each unitary irreducible G-module in $\mathrm{L}^2(M)$ a measure on the space $\widetilde{M}$, and then considers the weak convergence of those measures. In essence, it can therefore be regarded as a representation-theoretic statement in which the spectral theory for the Laplacian only enters in *choosing a concrete decomposition* of each isotypic component. In the case of the trivial group $G = \{e\}$, there is only one isotypic component in $\mathrm{L}^2(M)$, associated to the trivial representation, and choosing a family of irreducible G-modules is equivalent to choosing a Hilbert basis of $\mathrm{L}^2(M)$ of eigenfunctions of the Laplace-Beltrami operator. Result 7 then reduces to the classical equidistribution theorem for the Laplacian.

5.3 Discussion

5.3.1 Applications

In Section 8.4 we consider some concrete examples to illustrate the results of Chapter 8. They include

- compact locally symmetric spaces $\mathbb{Y} := \Gamma\backslash\mathcal{G}/K$, where $\mathcal{G}$ is a connected semisimple Lie group of rank 1 with finite center, Γ a discrete co-compact subgroup, and K a maximal compact subgroup;
- all surfaces of revolution diffeomorphic to the standard 2-sphere;
- S^3-invariant metrics on the 4-sphere.

In the first case, K acts with finite isotropy groups on $\mathbb{X} := \Gamma\backslash\mathcal{G}$, so that $\mathbb{Y}$ is an orbifold. Furthermore, the orbit volume is constant. The reduced geodesic flow on $M = \mathbb{X} := \Gamma\backslash\mathcal{G}$ coincides with the geodesic flow on $\mathbb{Y}$ and is ergodic, since $\mathbb{Y}$ has strictly negative sectional curvature. Our results recover the Shnirelman-Zelditch-Colin-de-Verdière theorem for $\mathrm{L}^2(\mathbb{Y}) \simeq \mathrm{L}^2(\mathbb{X})^K$, and generalize it to non-trivial isotypic components of $\mathrm{L}^2(\mathbb{X})$. In the examples of the 2- and 4-dimensional spheres, the considered actions have two fixed points, and the reduced geodesic flow is ergodic for topological reasons, regardless of the choice of invariant Riemannian metric and in spite of the fact that the geodesic flow can be totally integrable. Since the eigenfunctions of the Laplacian on the standard 2-sphere – the spherical harmonics – are well understood, we can independently verify Result 7 for single isotypic components in this case.

5.3.2 Previous results

Let us first collect the previously known results comparable to those of Chapter 7. For general isometric and effective group actions the asymptotic distribution of the spectrum of an invariant operator along single isotypic components of $\mathrm{L}^2(M)$ was studied within the classical framework via heat kernel methods by Donnelly [18] and Brüning–Heintze [10]. These methods allow to determine the leading term in the expansion, while remainder estimates or growing families of isotypic components are not accessible via this approach. On the other hand, using Fourier integral operator techniques, remainder estimates were obtained for actions with orbits of the same dimension by Donnelly [18], Brüning–Heintze [10], Brüning [9], Helffer–Robert [28, 29], and Guillemin–Uribe [23] in the classical setting, and El-Houakmi–Helffer [21] and Cassanas [13] in the semiclassical setting. For general effective group actions, remainder estimates were derived by Cassanas–Ramacher [14] and Ramacher [46] in the classical, and by Weich [58] in the semiclassical framework using resolution of singularites. The idea of considering families of isotypic components that vary with the asymptotic parameter has been known since the 1980s. Thus, for the Laplacian and free isometric group actions, so-called *ladder subspaces* of $\mathrm{L}^2(M)$ and *fuzzy ladders* have been considered in [50, 23] and [62], respectively, see also below.

Let us now collect the previously known results which are comparable to those of Chapter 8. In case that G acts on M with only one orbit type, $\widetilde{M}$ is a compact smooth manifold with Riemannian metric induced by the G-invariant Riemannian metric on M. By co-tangent bundle reduction, $T^*\widetilde{M}$ is symplectomorphic to $\mathbb{J}^{-1}(\{0\})/G$, so the ergodicity of the reduced geodesic flow on M and that of the geodesic flow on $\widetilde{M}$ are equivalent. Under these circumstances, one can apply the classical Shnirelman-Zelditch-Colin-de-Verdière equidistribution theorem to $\widetilde{M}$, yielding an equidistribution statement for the eigenfunctions of the Laplacian $\Delta_{\widetilde{M}}$ on $\widetilde{M}$ in terms of weak convergence of measures on $\widetilde{M}$. On the other hand, one could as well apply Corollary 8.3.5 and Theorem 8.3.8 to M, yielding also a statement about weak convergence of measures on $\widetilde{M}$, but this time with measures related to eigenfunctions of the Laplacian Δ_M on M in families of isotypic components of $\mathrm{L}^2(M)$. It is then an obvious question how these two results are related. The answer is rather difficult in general, since – in spite of the presence of the isometric group action – the geometry of M may be much more complicated than that of $\widetilde{M}$. Consequently, the eigenfunctions of Δ_M, even those in the trivial isotypic component, that is, those that are G-invariant, may be much harder to understand than the eigenfunctions of $\Delta_{\widetilde{M}}$. Only in case that all orbits are totally geodesic or minimal submanifolds, or, more generally, do all have the same volume, one can show that an eigenfunction of $\Delta_{\widetilde{M}}$ lifts to a unique G-invariant eigenfunction of Δ_M [57, 8, 3]. In this particular situation, it is easy to see that the application of the Shnirelman-Zelditch-Colin-de-Verdière equidistribution theorem implies the special case of 8.3.8 for the single trivial isotypic component. The case of a compact locally symmetric space treated in Section 8.4.1 is an example of this in the torsion-free case. In cases where the orbit volume is not constant, no significant results about the relation between the eigenfunctions of $\Delta_{\widetilde{M}}$ and Δ_M are known to the author.

An explicitly studied case is that of a general free G-action, when the projection $M \to M/G = \widetilde{M}$ is a Riemannian principal G-bundle. Extending work of Schrader and Taylor [50], Zelditch [62] obtained quantum limits for sequences of eigenfunctions of Δ_M in so-called *fuzzy ladders*. These are subsets of $\mathrm{L}^2(M)$ associated to a so-called *ray of representations* originating from some chosen $\chi \in \widehat{G}$. The obtained quantums limit are directly related to the symplectic orbit reduction $\mathbb{J}^{-1}(\mathcal{O}_\chi)/G \simeq T^*\widetilde{M}$, where $\mathcal{O}_\chi \subset \mathfrak{g}^*$ is the co-adjoint orbit associated to χ by the Borel-Weil theorem. They are given by Liouville measures on hypersurfaces in $\mathbb{J}^{-1}(\mathcal{O}_\chi)/G$, and their projections onto the base manifold agree with those obtained in Chapter 8.

Further, significant efforts were recently made towards the understanding of quantum (unique) ergodicity for locally symmetric spaces, which are particular manifolds of negative sectional curvature. As before, let $\mathcal{G}$ be a connected, semisimple Lie group with finite center, $\mathcal{G} = KAN$ an Iwasawa decomposition of $\mathcal{G}$, and Γ a torsion-free, discrete subgroup in $\mathcal{G}$. Following earlier work of Zelditch and Lindenstrauss, Silberman and Venkatesh introduced in [54] certain representation theoretic lifts from $\mathbb{Y} = \Gamma\backslash\mathcal{G}/K$ to $\mathbb{X} = \Gamma\backslash\mathcal{G}$ that substitute the previously considered microlocal lifts and take into account the additional structure of locally symmetric spaces. These representation theoretic lifts should play an important role in solving the QUE conjecture, already settled by Lindenstrauss in particular cases, also for higher rank symmetric spaces. In case that Γ is co-compact, their results were generalized by Bunke and Olbrich [12] to homogeneous vector bundles $\mathbb{X} \times_K V_\chi$ over $\mathbb{Y}$ associated to equivalence classes of irreducible representations $\chi \in \widehat{K}$ of the maximal compact subgroup K. The constructed representation theoretic lifts are invariant with respect to the action of A, which corresponds to the invariance of the microlocal lifts under the geodesic flow. Since Γ has no torsion, K acts on $\mathbb{X}$ only with one orbit type.

To close the summary of previously known results comparable to those of Chapter 8, it might be appropriate to mention that Marklof and O'Keefe [41] obtained quantum limits in situations where the geodesic flow is ergodic only in certain regions of phase space. Conceptually, this is both similar and contrary to the approach of this thesis, since in this case the geodesic flow is partially ergodic as well, but not due to symmetries.

Finally, comparable to both Chapters 7 and 8, there has been much work in recent times concerning the spectral theory of elliptic operators on orbifolds. Such spaces are locally homeomorphic to a quotient of euclidean space by a finite group while, globally, any (reduced) orbifold is a quotient of a smooth manifold by a compact Lie group action with finite isotropy groups, that is, in particular, with no singular isotropy types [1, 43]. As it turns out, the theory of elliptic operators on orbifolds is essentially equivalent to the theory of invariant elliptic operators on manifolds carrying the action of a compact Lie group with finite isotropy groups [11, 19, 56]. In particular, Kordyukov [38] obtained the Shnirelman-Zelditch-Colin-de-Verdière theorem for elliptic operators on compact orbifolds, using their original high-energy approach. Result 6 recovers his result for the Laplacian, and generalizes it to singular group actions and growing families of isotypic components.

Thus, in all the previously examined cases, no singular orbits occur. Actually, this thesis can be viewed as part of an attempt to develop an equivariant spectral theory of elliptic operators on general singular G-spaces.

5.3.3 Comments and outlook

Weaker versions of Results 5 and 6 can be proved in the case of a single isotypic component by the same methods employed here with a less sharp energy localization in a fixed interval $[c, c+\varepsilon]$ instead of a shrinking interval $[c, c+h^\delta]$. The point is that for these weaker statements no remainder estimate in the semiclassical Weyl law is necessary, see Remark 8.2.5. Thus, at least the weaker version of Result 6 could have also been obtained within the classical framework in the late 1970's using heat kernel methods as in [18] or [10]. In contrast, for the stronger versions of equivariant quantum ergodicity proved in Results 5 and 6, remainder estimates in the equivariant Weyl law, and in particular the results obtained in [46] for general group actions via resolution of singularities, are necessary. However, the weaker versions would still be strong enough to imply Result 7 for a single isotypic component. Therefore, in principle, Theorem 8.3.11 could have been proved already when Shnirelman formulated his theorem more than 40 years ago.

As mentioned above, the idea of considering families of representations that vary with the asymptotic parameter has been known since the end of the 1980's, compare [50, 23, 62], and it is a natural problem to determine what kind of families can be considered in the context of quantum ergodicity, and study them from a more conceptional point of view. To illustrate this, consider the standard 2-sphere $S^2 \subset \mathbb{R}^3$, acted upon by the group SO(2) of rotations around the z-axis in $\mathbb{R}^3$. This action has exactly two fixed points given by the north pole and the south pole of S^2. The orbits of all other points are circles, so in this case we have $\kappa = 1$ and $H = \{e\}$, the trivial group. The eigenvalues of $-\Delta$ on S^2 are given by the numbers $l(l+1)$, $l = 0, 1, 2, 3\ldots$, and the corresponding eigenspaces $\mathcal{E}_l$ are of dimension $2l + 1$. They are spanned by the spherical harmonics, given in spherical polar coordinates by

$$Y_{l,m}(\phi, \theta) = \sqrt{\tfrac{2l+1}{4\pi}\tfrac{(l-m)!}{(l+m)!}}P_{l,m}(\cos\theta)e^{im\phi}, \qquad 0 \leq \phi < 2\pi,\ 0 \leq \theta < \pi, \tag{5.3.1}$$

where $m \in \mathbb{Z}$, $|m| \leq l$, and $P_{l,m}$ are the associated Legendre polynomials. Each subspace $\mathbb{C} \cdot Y_{lm}$ corresponds to an irreducible representation of SO(2), and each irreducible representation with character χ_k and $|k| \leq l$ occurs in the eigenspace $\mathcal{E}_l$ with multiplicity 1, where χ_k is as in Example 5.2.2. Considering the limit $l \to \infty$ and the Laplacian $-\Delta$ is equivalent to

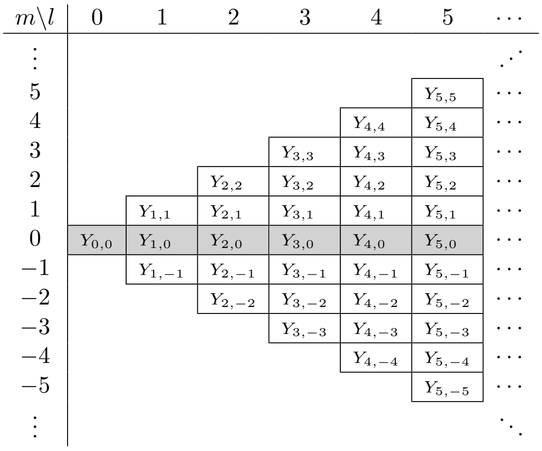

Figure 5.3.1: The single trivial isotypic component in the case $M = S^2$, $G = \mathrm{SO}(2)$, described by $m = 0$. The k-th row spans the isotypic component $\mathrm{L}^2_{\chi_k}(S^2)$ and the l-th column represents the l-th eigenspace of $-\Delta$.

studying the limit $h \to 0$ and the semiclassical Laplacian $-h^2\Delta$. Figure 5.3.1 depicts a single isotypic component, corresponding to a constant semiclassical character family. It means that

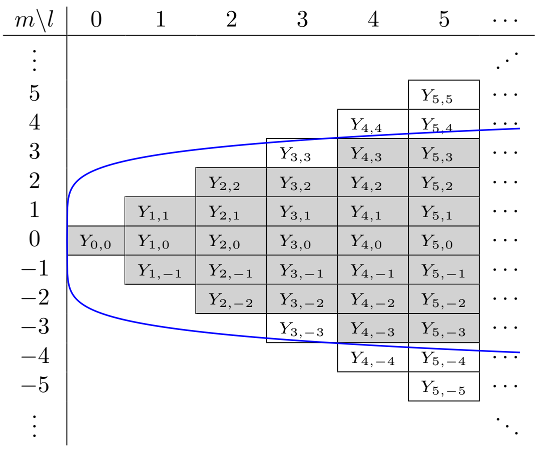

Figure 5.3.2: Spherical harmonics on S^2 in isotypic components corresponding to a semiclassical character family with growth rate $\frac{1}{6}$.

one keeps m fixed and studies the limit $l \to \infty$. In contrast, the semiclassical character family from Example 5.2.2 for $\vartheta = \frac{1}{6}$ corresponds to the gray region in Figure 5.3.2. As opposed to the results of Chapter 8, Figure 5.3.3 illustrates a cone-like family of representations that would correspond to subsequences of eigenfunctions of density larger than zero, while Figure 5.3.4 depicts the sequence of zonal spherical harmonics $Y_{l,l}$, which are known to localize at the equator of S^2 as $l \to \infty$, and therefore yield a different limit measure than the one implied by Result 7, see Section 8.4.2 and in particular Remark 8.4.5. Hence, different kinds of families of representations give rise to qualitatively different quantum limits, and it would be illuminating to understand this interrelation in a deeper way.

$m\backslash l$	0	1	2	3	4	5	$\cdots$
$\vdots$							$\iddots$
5						$Y_{5,5}$	$\cdots$
4					$Y_{4,4}$	$Y_{5,4}$	$\cdots$
3				$Y_{3,3}$	$Y_{4,3}$	$Y_{5,3}$	$\cdots$
2			$Y_{2,2}$	$Y_{3,2}$	$Y_{4,2}$	$Y_{5,2}$	$\cdots$
1		$Y_{1,1}$	$Y_{2,1}$	$Y_{3,1}$	$Y_{4,1}$	$Y_{5,1}$	$\cdots$
0	$Y_{0,0}$	$Y_{1,0}$	$Y_{2,0}$	$Y_{3,0}$	$Y_{4,0}$	$Y_{5,0}$	$\cdots$
-1		$Y_{1,-1}$	$Y_{2,-1}$	$Y_{3,-1}$	$Y_{4,-1}$	$Y_{5,-1}$	$\cdots$
-2			$Y_{2,-2}$	$Y_{3,-2}$	$Y_{4,-2}$	$Y_{5,-2}$	$\cdots$
-3				$Y_{3,-3}$	$Y_{4,-3}$	$Y_{5,-3}$	$\cdots$
-4					$Y_{4,-4}$	$Y_{5,-4}$	$\cdots$
-5						$Y_{5,-5}$	$\cdots$
$\vdots$							$\ddots$

Figure 5.3.3: Spherical harmonics on S^2 in cone-like families of representations.

$m\backslash l$	0	1	2	3	4	5	$\cdots$
$\vdots$							$\iddots$
5						$Y_{5,5}$	$\cdots$
4					$Y_{4,4}$	$Y_{5,4}$	$\cdots$
3				$Y_{3,3}$	$Y_{4,3}$	$Y_{5,3}$	$\cdots$
2			$Y_{2,2}$	$Y_{3,2}$	$Y_{4,2}$	$Y_{5,2}$	$\cdots$
1		$Y_{1,1}$	$Y_{2,1}$	$Y_{3,1}$	$Y_{4,1}$	$Y_{5,1}$	$\cdots$
0	$Y_{0,0}$	$Y_{1,0}$	$Y_{2,0}$	$Y_{3,0}$	$Y_{4,0}$	$Y_{5,0}$	$\cdots$
-1		$Y_{1,-1}$	$Y_{2,-1}$	$Y_{3,-1}$	$Y_{4,-1}$	$Y_{5,-1}$	$\cdots$
-2			$Y_{2,-2}$	$Y_{3,-2}$	$Y_{4,-2}$	$Y_{5,-2}$	$\cdots$
-3				$Y_{3,-3}$	$Y_{4,-3}$	$Y_{5,-3}$	$\cdots$
-4					$Y_{4,-4}$	$Y_{5,-4}$	$\cdots$
-5						$Y_{5,-5}$	$\cdots$
$\vdots$							$\ddots$

Figure 5.3.4: Zonal spherical harmonics on S^2.

As further lines of research, it would be interesting to see whether the results of Chapters 7 and 8 can be generalized to G-vector bundles, as well as manifolds with boundary and non-compact situations. Also, in view of Result 7, it might be possible to deepen our understanding of equivariant quantum ergodicity via representation theory. Finally, one can ask what could be a suitable symmetry-reduced version of the QUE conjecture. In the particular case of the SO(2)-action on the standard 2-sphere studied in Section 8.4, we actually see that in each fixed isotypic component the representation-theoretic equidistribution theorem for the Laplacian applies to the full sequence of spherical harmonics, so that equivariant QUE holds in this case. However, even for this simple example it is unclear whether equivariant QUE holds for growing families of isotypic components.

Chapter 6

Background II

Here, we explain in more detail the concepts, definitions and techniques that are relevant to the setup of this second part of the thesis, in addition to those from Chapter 2 in Part I.

Classical pseudodifferential operators and symbols While most chapters of this thesis use the language of semiclassical analysis introduced in Chapter 2 of Part I, there is the exception of the Sections 8.3 and 8.4, where we specialize from general Schrödinger operators to the Laplace-Beltrami operator and maintaining the semiclassical notation would be artificial. Therefore, we mention here briefly that the corresponding classical quantities are obtained from the introduced semiclassical definitions by setting $h = 1$. In order to simplify the notation when $h = 1$, we define for $m \in \mathbb{R} \cup \{-\infty\}$

$$\begin{aligned}\mathrm{Op} := \mathrm{Op}_1, \qquad \Psi^m_\delta(M) := \Psi^m_{1;\delta}(M), \quad \Psi^m(M) := \Psi^m_{1;0}(M), \quad \Psi(M) := \Psi^0_{1;0}(M),\\ S^m_\delta(M) := S^m_{1;\delta}(M), \quad S^m(M) := S^m_{1;0}(M), \qquad S(M) := S^0_{1;0}(M).\end{aligned}$$

6.1 Smooth actions of compact Lie groups

In what follows, we recall some essential facts from the general theory of compact Lie group actions on smooth manifolds. We will actually need only a small number of results. For a detailed introduction, we refer the reader to [6, Chapters I, IV, VI]. Let $\mathbf{X}$ be a smooth manifold of dimension n and G a Lie group acting locally smoothly on $\mathbf{X}$. For $x \in \mathbf{X}$, denote by G_x the isotropy group and by $G \cdot x = \mathcal{O}_x$ the G-orbit through x so that

$$G_x = \{g \in G,\ g \cdot x = x\}, \qquad \mathcal{O}_x = G \cdot x = \{g \cdot x \in \mathbf{X},\ g \in G\}.$$

Note that $G \cdot x$ and G/G_x are homeomorphic. The equivalence class of an orbit $\mathcal{O}_x$ under equivariant homeomorphisms, written $[\mathcal{O}_x]$, is called its *orbit type*. The conjugacy class of a stabilizer group G_x is called its *isotropy type*, and written (G_x). If K_1 and K_2 are closed subgroups of G, a partial ordering of orbit and isotropy types is given by

$$[G/K_1] \leq [G/K_2] \iff (K_2) \leq (K_1) \iff K_2 \text{ is conjugate to a subgroup of } K_1.$$

For any closed subgroup $K \subset G$, one denotes by $\mathbf{X}(K) := \{x \in \mathbf{X} : G_x \sim K\}$ the subset of points of isotropy type (K).

Assume now that G is compact. The set of all orbits is denoted by $\mathbf{X}/G$, and equipped with the quotient topology it becomes a topological Hausdorff space [6, Theorem I.3.1]. In the following we shall assume that it is connected. One of the central results in the theory of compact group actions is the *principal orbit theorem* [6, Theorem IV.3.1], which states that there exists a maximum orbit type $[\mathcal{O}_{\max}]$ with associated minimal isotropy type (H). Furthermore, $\mathbf{X}(H)$ is open and dense in $\mathbf{X}$, and its image in $\mathbf{X}/G$ is connected. We call $[\mathcal{O}_{\max}]$ the *principal orbit type* of the G-action on $\mathbf{X}$ and a representing orbit a *principal orbit.* Similarly, we call the isotropy type (H) and an isotropy group $G_x \sim H$ *principal.* Casually, we will identify orbit types with isotropy types and say *an orbit of type* (H) or even *an orbit of type* H, making no distinction between equivalence classes and their representants. The reduced space $\mathbf{X}(H)/G$ is a smooth manifold of dimension $n-\kappa$, where κ is the dimension of $\mathcal{O}_{\max}$, since G acts with only one orbit type on $\mathbf{X}(H)$.

6.2 Symplectic reduction

Let us now review in some detail the theory of symplectic reduction of Marsden and Weinstein, Sjamaar, Lerman and Bates. The theory emerged out of classical mechanics, and is based on the fundamental fact that the presence of conserved quantities or integrals of motion leads to the elimination of variables. A thorough exposition can be found in [44].

Let $(\mathbf{X},\omega)$ be a connected symplectic manifold, and assume that $(\mathbf{X},\omega)$ carries a global Hamiltonian action of a Lie group G. In particular, we will be interested in the case where $\mathbf{X}=T^*M$ is the co-tangent bundle of our closed Riemannian manifold M. Let

$$\mathbb{J}:\mathbf{X}\to\mathfrak{g}^*, \qquad \mathbb{J}(\eta)(X)=\mathbb{J}_X(\eta),$$

be the corresponding *momentum map*, where $\mathbb{J}_X:\mathbf{X}\to\mathbb{R}$ is a C^∞-function depending linearly on $X\in\mathfrak{g}$ such that the fundamental vector field $\widetilde{X}$ on $\mathbf{X}$ associated to X is given by the Hamiltonian vector field of $\mathbb{J}_X$. It is clear from the definition that $\mathrm{Ad}^*(g^{-1})\circ\mathbb{J}=\mathbb{J}\circ g$. Furthermore, for each $X\in\mathfrak{g}$ the function $\mathbb{J}_X$ is a *conserved quantity* or *integral of motion* for any G-invariant function $p\in\mathrm{C}^\infty(\mathbf{X})$ since in this case

$$\{\mathbb{J}_X,p\}=\omega(\text{s-grad}\,\mathbb{J}_X,\text{s-grad}\,p)=-\omega(\widetilde{X},\text{s-grad}\,p)=dp(\widetilde{X})=\widetilde{X}(p)=0,$$

where $\{\cdot,\cdot\}$ is the Poisson-bracket on $\mathbf{X}$ given by ω. Now, define

$$\Omega:=\mathbb{J}^{-1}(\{0\}), \qquad \widetilde{\Omega}:=\Omega/G.$$

Unless the G-action on $\mathbf{X}$ is free, the reduced space $\widetilde{\Omega}$ will in general not be a smooth manifold, but a topological quotient space. Nevertheless, one can show that $\widetilde{\Omega}$ constitutes a stratified symplectic space in the following sense. A function $\widetilde{f}:\widetilde{\Omega}\to\mathbb{R}$ is defined to be *smooth*, if there exists a G-invariant function $f\in\mathrm{C}^\infty(\mathbf{X})^G$ such that $f|_\Omega=\pi^*\widetilde{f}$, where $\pi:\Omega\to\widetilde{\Omega}$ denotes the orbit map. One can then show that $\mathrm{C}^\infty(\widetilde{\Omega})$ inherits a Poisson algebra structure from $\mathrm{C}^\infty(\mathbf{X})$ which is compatible with a stratification of the reduced space into symplectic manifolds. Moreover, the Hamiltonian flow φ_t corresponding to f is G-invariant and leaves Ω invariant, and consequently descends to a flow $\widetilde{\varphi}_t$ on $\widetilde{\Omega}$ [55].

More precisely, let μ be a value of $\mathbb{J}$, and G_μ the isotropy group of μ with respect to the co-adjoint action on $\mathfrak{g}^*$. Consider further an isotropy group $K\subset G$ of the G-action on $\mathbf{X}$, let $\eta\in\mathbb{J}^{-1}(\{\mu\})$ be such that $G_\eta=K$, and $\mathbf{X}_K^\eta$ be the connected component of $\mathbf{X}_K:=\{\zeta\in\mathbf{X}:G_\zeta=K\}$ containing η. Then [44, Theorem 8.1.1] the set $\mathbb{J}^{-1}(\{\mu\})\cap G_\mu\cdot\mathbf{X}_K^\eta$ is a smooth submanifold of $\mathbf{X}$, and the quotient

$$\widetilde{\Omega}_\mu^{(K)}:=\big(\mathbb{J}^{-1}(\{\mu\})\cap G_\mu\cdot\mathbf{X}_K^\eta\big)/G_\mu$$

possesses a differentiable structure such that the projection $\pi_\mu^{(K)} : \mathbb{J}^{-1}(\{\mu\}) \cap G_\mu \cdot \mathbf{X}_K^\eta \to \widetilde{\Omega}_\mu^{(K)}$ is a surjective submersion. Furthermore, there exists a unique symplectic form $\widetilde{\omega}_\mu^{(K)}$ on $\widetilde{\Omega}_\mu^{(K)}$ such that $(\iota_\mu^{(K)})^*\omega = (\pi_\mu^{(K)})^*(\widetilde{\omega}_\mu^{(K)})$, where $\iota_\mu^{(K)} : \mathbb{J}^{-1}(\{\mu\}) \cap G_\mu \cdot \mathbf{X}_K^\eta \hookrightarrow \mathbf{X}$ denotes the inclusion. Finally, if $p \in \mathrm{C}^\infty(\mathbf{X})$ is a G-invariant function, $H_p :=$ s-grad p its Hamiltonian vector field, and φ_t the corresponding flow, then φ_t leaves the connected components of the space $\mathbb{J}^{-1}(\{\mu\}) \cap G_\mu \cdot \mathbf{X}_K^\eta$ invariant and commutes with the G_μ-action, yielding a reduced flow $\widetilde{\varphi}_t^\mu$ on $\widetilde{\Omega}_\mu^{(K)}$ given by

$$\pi_\mu^{(K)} \circ \varphi_t \circ \iota_\mu^{(K)} = \widetilde{\varphi}_t^\mu \circ \pi_\mu^{(K)}. \tag{6.2.1}$$

This reduced flow $\widetilde{\varphi}_t^\mu$ on $\widetilde{\Omega}_\mu^{(K)}$ turns out to be Hamiltonian, and its Hamiltonian $\widetilde{p}_\mu^{(K)} : \widetilde{\Omega}_\mu^{(K)} \to \mathbb{R}$ satisfies $\widetilde{p}_\mu^{(K)} \circ \pi_\mu^{(K)} = p \circ \iota_\mu^{(K)}$.

Remark 6.2.1. With the notation above we have $G \cdot \mathbf{X}_K = \mathbf{X}(K)$. Indeed, for $x \in \mathbf{X}_K$, the isotropy group of x is K. If $g'g \cdot x = g \cdot x$ for some $g, g' \in G$, then $g^{-1}g'g \cdot x = x$, hence $g^{-1}g'g \in K$, that is $g' \in (K)$. That shows $G \cdot \mathbf{X}_K \subset \mathbf{X}(K)$. On the other hand, if $x \in \mathbf{X}(K)$, then $(G_x) = (K)$, hence for every $g' \in G_x$, there is a $k \in K$ and a $g \in G$ such that $g' = gkg^{-1}$. But then $kg^{-1} \cdot x = g^{-1} \cdot x$, so that $g^{-1} \cdot x \in \mathbf{X}_K$, and in particular $x \in G \cdot \mathbf{X}_K$.

Example 6.2.2. An important class of examples of Hamiltonian G-actions of a given Lie group G is given by induced actions on co-tangent bundles of G-manifolds. Thus, let $\Psi : G \times M \to M, (g, x) \to \Psi_g(x) := g \cdot x$ be a smooth G-action on a smooth manifold M. The induced action on T^*M is given by

$$(g \cdot \eta_x)(v) = ((\Psi_{g^{-1}})^*_{g\cdot x} \cdot \eta_x)(v) = \eta_x((\Psi_{g^{-1}})_{*,g\cdot x} \cdot v), \qquad \eta_x \in T^*_x M,\ v \in T_{g\cdot x} M,$$

where $(\Psi_g)_{*,x} : T_x M \to T_{g\cdot x} M$ denotes the derivative of the map $g : M \to M, x \mapsto g \cdot x$. Now, if $\tau : \mathbf{X} = T^*M \to M$ denotes the co-tangent bundle with standard symplectic form $\omega = -d\theta$, where θ is the *tautological* or *Liouville* one-form on T^*M, then

$$\mathbb{J} : T^*M \ni \eta \mapsto \mathbb{J}(\eta)(X) := \eta\big(\widetilde{X}_{\tau(\eta)}\big), \qquad X \in \mathfrak{g}, \tag{6.2.2}$$

defines a momentum map, meaning that the G-action on T^*M is Hamiltonian. Here $\widetilde{X}_{\tau(\eta)}$ denotes the fundamental vector field on M corresponding to X evaluated at the point $\tau(\eta)$. In the particular case when $M = G$ is itself a Lie group, and $L : G \times G \to G$ denotes the left action of G onto itself, there exists a vector bundle isomorphism

$$T^*G \xrightarrow{\simeq} G \times \mathfrak{g}^*, \qquad \eta_g \mapsto (g, (L_g)^*_e \cdot \eta_g), \tag{6.2.3}$$

called the *left trivialization* of T^*G, and the induced left action takes the form

$$g \cdot (h, \mu) = (gh, \mu), \qquad g, h \in G,\ \mu \in \mathfrak{g}^*.$$

Consequently, the decomposition of T^*G into orbit types of this action is given by the one of G and

$$(T^*G)(H) = T^*(G(H)),$$

H being an arbitrary closed subgroup of G. On the other hand, the momentum map reads $\mathbb{J}(g, \mu) = \mathrm{Ad}^*_{g^{-1}}\mu$, since with $\mu = (L_g)^*_e \cdot \eta_g$ one computes for $X \in \mathfrak{g}$

$$\begin{aligned} \mathbb{J}(g,\mu)(X) = \mathbb{J}(\eta_g)(X) &= (L_{g^{-1}})^*_g\, \mu(\widetilde{X}_g) \\ &= \mu((L_{g^{-1}})_{*,g}\widetilde{X}_g) = \mu\Big(\frac{d}{dt}(g^{-1}\, \mathrm{e}^{tX}\, g)_{|t=0}\Big) = \mu(\mathrm{Ad}(g^{-1})X), \end{aligned}$$

compare [44, Example 4.5.5].

Let us now apply these general results to the situation of this second part of the thesis. Thus, let $\mathbf{X} = T^*M$, where M is a closed connected Riemannian manifold of dimension n, carrying an isometric effective action of a compact connected Lie group G. In all what follows, the principal isotropy type of the action will be denoted by (H), H being a closed subgroup of G, and the dimension of the principal orbits in M by κ. Furthermore, we shall always assume that $\kappa < n$. T^*M constitutes a Hamiltonian G-space when endowed with the canonical symplectic structure and the G-action induced from the smooth action on M, and one has

$$\Omega = \mathbb{J}^{-1}(\{0\}) = \bigsqcup_{x\in M} \operatorname{Ann} T_x(G\cdot x), \tag{6.2.4}$$

where $\operatorname{Ann} V_x \subset T_x^*M$ denotes the annihilator of a subspace $V_x \subset T_xM$. Further, let

$$M_{\text{reg}} := M(H), \qquad \Omega_{\text{reg}} := \Omega \cap (T^*M)(H),$$

where $M(H)$ and $(T^*M)(H)$ denote the union of orbits of type (H) in M and T^*M, respectively. By the principal orbit theorem, M_{reg} is open in M, hence M_{reg} is a smooth submanifold. We then define

$$\widetilde{M}_{\text{reg}} := M_{\text{reg}}/G.$$

$\widetilde{M}_{\text{reg}}$ is a smooth boundaryless manifold, since G acts on M_{reg} with only one orbit type and M_{reg} is open in M. Moreover, because the Riemannian metric on M is G-invariant, it induces a Riemannian metric on $\widetilde{M}_{\text{reg}}$. On the other hand, by symplectic reduction Ω_{reg} is a smooth submanifold of T^*M, and the quotient

$$\widetilde{\Omega}_{\text{reg}} := \Omega_{\text{reg}}/G$$

possesses a unique differentiable structure such that the projection $\pi : \Omega_{\text{reg}} \to \widetilde{\Omega}_{\text{reg}}$ is a surjective submersion. Furthermore, there exists a unique symplectic form $\widetilde{\omega}$ on $\widetilde{\Omega}_{\text{reg}}$ such that $\iota^*\omega = \pi^*\widetilde{\omega}$, where $\iota : \Omega_{\text{reg}} \hookrightarrow T^*M$ denotes the inclusion and ω the canonical symplectic form on T^*M. Consider now the inclusion $j : (T^*M_{\text{reg}} \cap \Omega)/G \hookrightarrow \widetilde{\Omega}_{\text{reg}}$. The symplectic form $\widetilde{\omega}$ on $\widetilde{\Omega}_{\text{reg}}$ induces a symplectic form $j^*\widetilde{\omega}$ on $(T^*M_{\text{reg}} \cap \Omega)/G$. We then have the following

Lemma 6.2.3 (Singular co-tangent bundle reduction). *Let $\widehat{\omega}$ denote the canonical symplectic form on the co-tangent bundle $T^*\widetilde{M}_{\text{reg}}$. Then the two $2(n-\kappa)$-dimensional symplectic manifolds*

$$((T^*M_{\text{reg}} \cap \Omega)/G, j^*\widetilde{\omega}) \simeq (T^*\widetilde{M}_{\text{reg}}, \widehat{\omega})$$

are canonically symplectomorphic.

Remark 6.2.4. If $M = M_{\text{reg}}$, the previous lemma simply asserts that

$$T^*\widetilde{M} \simeq \widetilde{\Omega} = \Omega/G \tag{6.2.5}$$

as symplectic manifolds. In case that G acts on M only with finite isotropy groups, $\widetilde{M}$ is an orbifold, and the relation (6.2.5) holds as well, being the quotient presentation of the co-tangent bundle of $\widetilde{M}$ as an orbifold [38].

Proof. First, we apply [44, Theorem 8.1.1] once to the manifold T^*M and once to the manifold T^*M_{reg}. Noting that the momentum map of the G-action on T^*M_{reg} agrees with the restriction of the momentum map of the G-action on T^*M to T^*M_{reg}, we get that $j^*\widetilde{\omega}$ is the unique symplectic form on $(T^*M_{\text{reg}} \cap \Omega_{\text{reg}})/G$ which fulfills

$$i^*\omega = \Pi^* j^*\widetilde{\omega}, \tag{6.2.6}$$

where $\Pi : T^*M_{\text{reg}} \to T^*M_{\text{reg}}/G$ is the orbit projection, $i : T^*M_{\text{reg}} \cap \Omega_{\text{reg}} \hookrightarrow T^*M_{\text{reg}}$ is the inclusion, and ω is the canonical symplectic form on T^*M_{reg}.

What follows now is essentially a proof of the *co-tangent bundle reduction theorem* [42, Theorem 2.2.2] for the manifold M_{reg}. The proof given in [42, Theorem 2.2.2] works in principle also for our situation. However, in [42, Theorem 2.2.2] the authors assume a free G-action, which is not necessarily the case here, so that the situation is not completely identical. For convenience, we therefore provide a detailed proof of our theorem.
We are going to construct in a canonical way a diffeomorphism $\Phi : (T^*M_{\text{reg}} \cap \Omega_{\text{reg}})/G \to T^*\widetilde{M}_{\text{reg}}$ such that $\Phi^*\widehat{\omega}$ also fulfills (6.2.6), that is, such that $i^*\omega = \Pi^*\Phi^*\widehat{\omega}$ holds. We will then be able to conclude from the uniqueness statement associated to (6.2.6) that $j^*\widetilde{\omega} = \Phi^*\widehat{\omega}$, so that Φ is a symplectomorphism. Denote by $\pi : M_{\text{reg}} \to \widetilde{M}_{\text{reg}} \equiv M_{\text{reg}}/G$ the orbit projection. Together with the pointwise derivatives, π induces a morphism of smooth vector bundles

$$\partial\pi : TM_{\text{reg}} \to T\widetilde{M}_{\text{reg}},$$

which has the fiberwise kernel $\ker \partial\pi|_x = T_x(G \cdot x)$. The Riemannian metrics on M_{reg} and $\widetilde{M}_{\text{reg}}$ provide us with the usual isomorphisms of smooth vector bundles $\alpha : TM_{\text{reg}} \simeq T^*M_{\text{reg}}$ and $\beta : T\widetilde{M}_{\text{reg}} \simeq T^*\widetilde{M}_{\text{reg}}$. For each $x \in M_{\text{reg}}$, we have

$$\alpha^{-1}(\text{Ann } T_x(G \cdot x)) = (T_x(G \cdot x))^\perp,$$

so that $\partial\pi$ is fiberwise injective when restricted to $\alpha^{-1}(T^*M_{\text{reg}} \cap \Omega_{\text{reg}})$. However, the dimensions of $(T_x(G \cdot x))^\perp$ and $T_{\pi(x)}\widetilde{M}_{\text{reg}}$ are both $n - \kappa$. So denoting the restriction of $\partial\pi$ to $\alpha^{-1}(T^*M_{\text{reg}} \cap \Omega_{\text{reg}})$ by $\overline{\partial\pi}$ it follows that

$$\overline{\partial\pi} : \alpha^{-1}(T^*M_{\text{reg}} \cap \Omega_{\text{reg}}) \to T\widetilde{M}_{\text{reg}}$$

together with the map $\pi : M_{\text{reg}} \to \widetilde{M}_{\text{reg}}$ is a morphism of smooth vector bundles which is fiberwise an isomorphism. For $x \in M_{\text{reg}}$, denote the restriction of $\overline{\partial\pi}$ to $T_x(G \cdot x)^\perp$ by γ_x. That defines an isomorphism $\gamma_x : (T_x(G \cdot x))^\perp \xrightarrow{\simeq} T_{\pi(x)}\widetilde{M}_{\text{reg}}$. Looking at the definition of $\overline{\partial\pi}$, we see that $\overline{\partial\pi}$ is G-invariant and that for each $\xi \in T_{\pi(x)}\widetilde{M}_{\text{reg}}$, $\overline{\partial\pi}^{-1}(\{\xi\}) = G \cdot \gamma_x^{-1}(\xi)$, the orbit of $\gamma_x^{-1}(\xi)$ under the G-action on TM_{reg} induced from the G-action on M_{reg}. We conclude that $\overline{\partial\pi}$ induces a diffeomorphism

$$\widetilde{\partial\pi} : (\alpha^{-1}(T^*M_{\text{reg}} \cap \Omega_{\text{reg}}))/G \to T\widetilde{M}_{\text{reg}}.$$

Because the G-action on M is isometric, α is G-invariant. Therefore, $\overline{\partial\pi} \circ \alpha^{-1}$ is G-invariant and induces a diffeomorphism

$$\widetilde{\partial\pi \circ \alpha^{-1}} : (T^*M_{\text{reg}} \cap \Omega_{\text{reg}})/G \to T\widetilde{M}_{\text{reg}}.$$

Composition with β yields the map

$$\Phi := \beta \circ \widetilde{\partial\pi \circ \alpha^{-1}} : (T^*M_{\text{reg}} \cap \Omega_{\text{reg}})/G \to T^*\widetilde{M}_{\text{reg}}. \tag{6.2.7}$$

As a composition of diffeomorphisms, Φ is a diffeomorphism. Moreover, the Riemannian metric on $\widetilde{M}_{\text{reg}}$ that is used to define β is defined as the induced Riemannian metric obtained from the G-invariant Riemannian metric on M_{reg}. Therefore, one can check that the map Φ is actually independent from the choice of G-invariant metric on M_{reg}. That makes the definition of Φ canonical. Now, in view of (6.2.6), we want to show that $\Phi^*\widehat{\omega}$ fulfills

$$i^*\omega = \Pi^*\Phi^*\widehat{\omega}, \tag{6.2.8}$$

where $\widehat{\omega}$ is the canonical symplectic form on $T^*\widetilde{M}_{\mathrm{reg}}$. Since $\widehat{\omega} = -d\widehat{\varrho}$, with $\widehat{\varrho}$ the tautological one-form on $T^*\widetilde{M}_{\mathrm{reg}}$, the desired equation (6.2.8) would follow, by applying the exterior derivative d to both sides and using compatibility of pullbacks with d, from the more basic relation

$$i^*\varrho = \Pi^*\Phi^*\widehat{\varrho}, \tag{6.2.9}$$

where ϱ is the tautological one-form on T^*M_{reg}. We will now show that (6.2.9) is in fact true. Since by definition $\widetilde{\partial\pi \circ \alpha^{-1}} \circ \Pi = \partial\pi \circ \alpha^{-1}$, we have

$$\Phi \circ \Pi = \beta \circ \partial\pi \circ \alpha^{-1} : T^*M_{\mathrm{reg}} \cap \Omega_{\mathrm{reg}} \to T^*\widetilde{M}_{\mathrm{reg}}.$$

Taking into account that $\Pi^*\Phi^* = (\Phi\Pi)^*$, what we have to show is

$$\varrho|_\eta(v) = ((\beta \circ \partial\pi \circ \alpha^{-1})^*\widehat{\varrho})|_\eta(v) \quad \forall\, \eta \in T^*M_{\mathrm{reg}} \cap \Omega_{\mathrm{reg}},\ v \in T_\eta(T^*M_{\mathrm{reg}} \cap \Omega_{\mathrm{reg}}). \tag{6.2.10}$$

Let $\tau : T^*M_{\mathrm{reg}} \to M_{\mathrm{reg}}$ and $\widehat{\tau} : T^*\widetilde{M}_{\mathrm{reg}} \to \widetilde{M}_{\mathrm{reg}}$ be the co-tangent bundles. By definition of the tautological one-forms on the two bundles we have

$$\varrho|_\eta(v) = \eta(\partial\tau|_\eta(v)), \quad \eta \in T^*M_{\mathrm{reg}},\ v \in T_\eta T^*M_{\mathrm{reg}}, \tag{6.2.11}$$

$$\widehat{\varrho}|_{\widetilde{\eta}}(\widetilde{v}) = \widetilde{\eta}(\partial\widehat{\tau}|_{\widetilde{\eta}}(\widetilde{v})), \quad \widetilde{\eta} \in T^*\widetilde{M}_{\mathrm{reg}},\ \widetilde{v} \in T_{\widetilde{\eta}}T^*\widetilde{M}_{\mathrm{reg}}. \tag{6.2.12}$$

Using the definition of the pullback of differential forms and inserting definition (6.2.12) into the right hand side of (6.2.10) we obtain for $\eta \in T^*M_{\mathrm{reg}},\ v \in T_\eta(T^*M_{\mathrm{reg}} \cap \Omega_{\mathrm{reg}})$

$$\begin{aligned} ((\beta \circ \partial\pi \circ \alpha^{-1})^*\widehat{\varrho})|_\eta(v) &= \widehat{\varrho}|_{\beta\circ\partial\pi\circ\alpha^{-1}(\eta)}(\partial(\beta \circ \partial\pi \circ \alpha^{-1})v) \\ &= (\beta \circ \partial\pi \circ \alpha^{-1})(\eta)(\partial\widehat{\tau}|_{\beta\circ\partial\pi\circ\alpha^{-1}(\eta)}\partial(\beta \circ \partial\pi \circ \alpha^{-1})v). \end{aligned} \tag{6.2.13}$$

Denote the Riemannian metrics on M_{reg} and $\widetilde{M}_{\mathrm{reg}}$ by g and $\widetilde{g}$, respectively, so that $\alpha(v) = g|_x(v,\cdot)$ for $v \in T_xM_{\mathrm{reg}}$, $x \in M_{\mathrm{reg}}$ and $\beta(\widetilde{v}) = \widetilde{g}|_{\widetilde{x}}(\widetilde{v},\cdot)$ for $\widetilde{v} \in T_{\widetilde{x}}\widetilde{M}_{\mathrm{reg}}$, $\widetilde{x} \in \widetilde{M}_{\mathrm{reg}}$. Inserting the definition of β into (6.2.13), we obtain for $\eta \in T^*M_{\mathrm{reg}},\ v \in T_\eta(T^*M_{\mathrm{reg}} \cap \Omega_{\mathrm{reg}})$

$$\begin{aligned} &((\beta \circ \partial\pi \circ \alpha^{-1})^*\widehat{\varrho})|_\eta(v) \\ &\qquad = \widetilde{g}|_{\widehat{\sigma}\circ\partial\pi\circ\alpha^{-1}(\eta)}\left((\partial\pi \circ \alpha^{-1})(\eta), \partial\widehat{\tau}|_{\beta\circ\partial\pi\circ\alpha^{-1}(\eta)}\partial(\beta \circ \partial\pi \circ \alpha^{-1})v)\right), \end{aligned} \tag{6.2.14}$$

where we wrote $\widehat{\sigma} : T\widetilde{M}_{\mathrm{reg}} \to \widetilde{M}_{\mathrm{reg}}$ for the tangent bundle. Consider now the commutative diagram

$$
\begin{tikzcd}
& TTM_{\mathrm{reg}} \arrow[r, "\partial(\partial\pi)"] \arrow[dl, "\partial\alpha"'] \arrow[d, "\partial\sigma = \text{can.}"] & TT\widetilde{M}_{\mathrm{reg}} \arrow[d, "\partial\widehat{\sigma} = \text{can.}"] \arrow[dr, "\partial\beta"] & \\
TT^*M_{\mathrm{reg}} \arrow[r, "\partial\tau = \text{can.}"] \arrow[d, "\text{can.}"] & TM_{\mathrm{reg}} \arrow[r, "\partial\pi"] \arrow[dl, "\alpha"'] \arrow[d, "\sigma = \text{can.}"] & T\widetilde{M}_{\mathrm{reg}} \arrow[d, "\widehat{\sigma} = \text{can.}"] \arrow[dr, "\beta"] & TT^*\widetilde{M}_{\mathrm{reg}} \arrow[l, "\partial\widehat{\tau} = \text{can.}"] \arrow[d, "\text{can.}"] \\
T^*M_{\mathrm{reg}} \arrow[r, "\tau = \text{can.}"] & M_{\mathrm{reg}} \arrow[r, "\pi"] & \widetilde{M}_{\mathrm{reg}} & T^*\widetilde{M}_{\mathrm{reg}} \arrow[l, "\widehat{\tau} = \text{can.}"]
\end{tikzcd}
$$

In the diagram, “can.” denotes the canonical projection of a tangent or co-tangent bundle. The other maps have been introduced in the paragraph above. That the triangles and the square in the bottom row of the diagram commute is a consequence of the relevant pairs of maps being morphisms of vector bundles. The triangles and the square in the top row of the diagram are just the derivatives of the corresponding triangles and the square in the bottom row. Therefore, they commute. From the commutativity of the top row of the diagram we get

$$\partial\widehat{\tau} \circ \partial\beta \circ \partial(\partial\pi) \circ \partial\alpha^{-1} = \partial\pi \circ \partial\tau. \tag{6.2.15}$$

We can simplify (6.2.14) using (6.2.15), which leads to

$$((\beta \circ \partial\pi \circ \alpha^{-1})^* \widehat{\varrho})|_\eta(v) = \widetilde{g}|_{\widehat{\sigma}\circ\partial\pi\circ\alpha^{-1}(\eta)} \left(\partial\pi \circ \alpha^{-1}(\eta), \partial\pi \circ \partial\tau(v)\right).$$

By definition of the metric $\widetilde{g}$, this can be rewritten as

$$((\beta \circ \partial\pi \circ \alpha^{-1})^* \widehat{\varrho})|_\eta(v) = g_{\sigma\circ\alpha^{-1}(\eta)} \left(\alpha^{-1}(\eta), \partial\tau(v)\right).$$

Moreover, by definition of α^{-1}, this simplifies into

$$((\beta \circ \partial\pi \circ \alpha^{-1})^* \widehat{\varrho})|_\eta(v) = \eta\left(\partial\tau(v)\right). \tag{6.2.16}$$

In view of (6.2.11), we have shown (6.2.10), and so we have proved (6.2.9). By the discussion at the beginning of the proof and in the lines above (6.2.9), we are done. □

6.3 Relevant measure spaces

This section contains a listing of the spaces and measures that will be relevant in the upcoming sections, followed by a technical subsection in which some basic properties and interrelations of the introduced measure spaces are studied. If not stated otherwise, measures are not assumed to be normalized. As before, let M be a closed connected Riemannian manifold of dimension n with Riemannian volume density dM, carrying an isometric effective action of a compact connected Lie group G with Haar measure dg. Let κ be the dimension of an orbit of principal type. Note that if $\dim G > 0$, dg is equivalent to the normalized Riemannian volume density on G associated to any choice of left-invariant Riemannian metric on G. If $\dim G = 0$, which in our case implies that G is trivial, dg is the normalized counting measure. Consider further T^*M with its canonical symplectic form ω, endowed with the natural Sasaki metric. Then the Riemannian volume density $d(T^*M)$ given by the Sasaki metric coincides with the symplectic volume form $\omega^n/n!$, see [36, page 537]. Next, if $\Omega := \mathbb{J}^{-1}(\{0\})$ denotes the zero level of the momentum map $\mathbb{J} : T^*M \to \mathfrak{g}^*$ of the underlying Hamiltonian action, we regard $\Omega_{\text{reg}} \subset T^*M$ as a Riemannian submanifold with Riemannian metric induced by the Sasaki metric on T^*M, and denote the associated Riemannian volume density by $d\Omega_{\text{reg}}$. Similarly, let

$$\mathcal{C} := \{(\eta, g) \in \Omega \times G : \; g \cdot \eta = \eta\}. \tag{6.3.1}$$

As Ω, the space $\mathcal{C}$ is not a manifold in general. We consider therefore the space $\operatorname{Reg} \mathcal{C}$ of all regular points in $\mathcal{C}$, that is, all points that have a neighborhood which is a smooth manifold. $\operatorname{Reg} \mathcal{C}$ is a smooth, non-compact submanifold of $T^*M \times G$ of co-dimension 2κ, and it is not difficult to see that

$$\operatorname{Reg} \mathcal{C} = \{(\eta, g) \in \Omega \times G, \; g \cdot \eta = \eta, \; \eta \in \Omega_{\text{reg}}\},$$

see e.g. [46, (17)]. We then regard $\operatorname{Reg} \mathcal{C} \subset T^*M \times G$ as a Riemannian submanifold with Riemannian metric induced by the product metric of the Sasaki metric on T^*M and some left-invariant Riemannian metric on G, and denote the corresponding Riemannian volume density by $d(\operatorname{Reg} \mathcal{C})$. In the same way, if $x \in M$ and $\eta \in T^*M$ are points, the orbits $G \cdot x$ and $G \cdot \eta$ are smooth submanifolds of M and T^*M, respectively, and if they have dimension greater than 0, we endow them with the corresponding Riemannian orbit measures, denoted by $d\mu_{G\cdot x}$ and $d\mu_{G\cdot \eta}$, respectively. If the dimension of an orbit is 0, it is a finite collection of isolated points, since G is compact, and we define $d\mu_{G\cdot x}$ and $d\mu_{G\cdot \eta}$ to be the counting measures. Further, for any Riemannian G-space $\mathbf{X}$, we define the orbit volume functions

$$\operatorname{vol}_{\mathcal{O}} : \mathbf{X} \to (0, \infty), \quad x \mapsto \operatorname{vol} \mathcal{O}_x = \operatorname{vol} (G \cdot x), \qquad \operatorname{vol} : \mathbf{X}/G \to (0, \infty), \quad \mathcal{O} \mapsto \operatorname{vol} \mathcal{O}.$$

Note that by definition we have $\operatorname{vol} > 0$, $\operatorname{vol}_{\mathcal{O}} > 0$ for all orbits, singular or not, since the orbit volume is defined using the induced Riemannian measure for smooth orbits and the counting measure for finite orbits. An important property of the orbit measures is their relation to the normalized Haar measure on G. Namely, for any orbit $G \cdot x$ and any continuous function $f : G \cdot x \to \mathbb{C}$, we have

$$\int_{G\cdot x} f(x')\, d\mu_{G\cdot x}(x') = \operatorname{vol} (G \cdot x) \int_G f(g \cdot x)\, dg. \tag{6.3.2}$$

To see why (6.3.2) holds, recall that there is a G-equivariant diffeomorphism $\Phi : G{\cdot}x \to G/G_x$. Then $\Phi_*(d\mu_{G\cdot x})$ is a G-invariant finite measure on G/G_x. Similarly, if $\Pi : G \to G/G_x$ denotes the canonical projection, $\Pi_*(dg)$ is also a G-invariant finite measure on G/G_x. Hence, $\Phi_*(d\mu_{G\cdot x}) = C{\cdot}\Pi_*(dg)$ for some constant C which is precisely given by $\operatorname{vol}(G{\cdot}x)$, since $\Pi_*(dg)$ is normalized. Observing that $\int_G f(gx)\, dg = \int_{G/G_x} f(gG_x x)\Pi_*(dg)$, (6.3.2) follows.

We describe now the quotient spaces and measures on them that will be relevant to us. Let $d\widetilde{M}_{\text{reg}}$ be the Riemannian volume density on $\widetilde{M}_{\text{reg}}$ associated to the Riemannian metric on $\widetilde{M}_{\text{reg}}$ induced by the G-invariant metric on M. Regarding the co-tangent bundle $T^*\widetilde{M}_{\text{reg}}$, we endow it with the canonical symplectic structure and let $d(T^*\widetilde{M}_{\text{reg}})$ be the corresponding symplectic volume form. Again, it coincides with the Riemannian volume form given by the natural Sasaki metric on $T^*\widetilde{M}_{\text{reg}}$. Similarly, the symplectic stratum $\widetilde{\Omega}_{\text{reg}}$ carries a canonical symplectic form $\widetilde{\omega}$ from [44, Theorem 8.1.1], and $d\widetilde{\Omega}_{\text{reg}} = \widetilde{\omega}^{(n-\kappa)}/(n-\kappa)!$ denotes the corresponding symplectic volume form. One can then show that $d\widetilde{\Omega}_{\text{reg}}$ agrees with the Riemannian volume density associated to the Riemannian metric on $\widetilde{\Omega}_{\text{reg}}$ induced by the Riemannian metric on Ω_{reg}, see Lemma 6.3.1. Since orbit projections on principal strata define fiber bundles [6, Theorem IV.3.3], Lemma 6.3.1 implies that $d\mu_{G\cdot x}$ and $d\mu_{G\cdot\eta}$ are the unique measures on the orbits in M_{reg} and Ω_{reg} such that

$$\int\limits_{M_{\text{reg}}} f(x)\, dM(x) = \int\limits_{\widetilde{M}_{\text{reg}}} \int\limits_{G\cdot x} f(x')\, d\mu_{G\cdot x}(x')\, d\widetilde{M}_{\text{reg}}(G\cdot x) \quad \forall\, f \in \mathrm{C}(M_{\text{reg}}), \tag{6.3.3}$$

$$\int\limits_{\Omega_{\text{reg}}} f(\eta)\, d(\Omega_{\text{reg}})(\eta) = \int\limits_{\widetilde{\Omega}_{\text{reg}}} \int\limits_{G\cdot\eta} f(\eta')\, d\mu_{G\cdot\eta}(\eta')\, d\widetilde{\Omega}_{\text{reg}}(G\cdot\eta) \quad \forall\, f \in \mathrm{C}(\Omega_{\text{reg}}). \tag{6.3.4}$$

Next, hypersurfaces given by the inverse image of a regular value c of (maps induced by) our Hamiltonian function $p : T^*M \to \mathbb{R}$ will be endowed with the Liouville measure induced by the measure on the ambient manifold, see Lemma 6.3.8 and Corollary 6.3.9. Thus, there is a canonical measure $d\Sigma_c$ on the hypersurface $\Sigma_c = p^{-1}(\{c\})$, induced by the symplectic volume form on T^*M, or equivalently, by the Riemannian volume density associated to the Sasaki metric. In the case $\Sigma_c = S^*M$ it is denoted by $d(S^*M)$. Similarly, for $S^*\widetilde{M}_{\text{reg}}$, the unit co-sphere bundle over $\widetilde{M}_{\text{reg}}$, the induced Liouville measure is denoted by $d(S^*\widetilde{M}_{\text{reg}})$, and for the hypersurface $\widetilde{\Sigma}_c := \widetilde{p}^{-1}(\{c\})$, where $\widetilde{p}$ is induced by p and $\widetilde{\Omega}_{\text{reg}}$ is endowed with the measure $d\widetilde{\Omega}_{\text{reg}}$, we denote the induced Liouville measure by $d\widetilde{\Sigma}_c$. Furthermore, since the intersection is transversal, $\Sigma_c \cap \Omega_{\text{reg}} = p|_{\Omega_{\text{reg}}}^{-1}(\{c\})$ is a smooth hypersurface of Ω_{reg}, and carries a Liouville measure μ_c induced by $d\Omega_{\text{reg}}$. As the orbit projection $\Sigma_c \cap \Omega_{\text{reg}} \to \widetilde{\Sigma}_c$ is a fiber bundle, μ_c fulfills

$$\int\limits_{\Sigma_c \cap \Omega_{\text{reg}}} f(\eta)\, d\mu_c(\eta) = \int\limits_{\widetilde{\Sigma}_c} \int\limits_{G\cdot\eta} f(\eta')\, d\mu_{G\cdot\eta}(\eta')\, d\widetilde{\Sigma}_c(G\cdot\eta) \quad \forall\, f \in \mathrm{C}(\Sigma_c \cap \Omega_{\text{reg}}). \tag{6.3.5}$$

Finally, let $d\widetilde{M} := \pi_* dM$ be the pushforward of dM along the canonical projection $\pi : M \to \widetilde{M} := M/G$. This means that, for $f \in \mathrm{C}(\widetilde{M})$, we have

$$\int\limits_{\widetilde{M}} f(\mathcal{O})\, d\widetilde{M}(\mathcal{O}) = \int\limits_{M} f \circ \pi(x)\, dM(x).$$

In what follows, we will use the orbit volume functions vol and $\mathrm{vol}_{\mathcal{O}}$ together with the previously defined measures to form new measures. This way we obtain on $\widetilde{M}$ the measure $\frac{d\widetilde{M}}{\mathrm{vol}}$, and on $\Sigma_c \cap \Omega_{\text{reg}}$ the measure $\frac{\mu_c}{\mathrm{vol}_{\mathcal{O}}}$. These measures are of fundamental importance in the following chapters. Finally, for a measure space $(\mathbf{X}, \mu)$ with $0 < \mu(\mathbf{X}) < \infty$ and a measurable function f on $\mathbf{X}$, we shall use the common notation

$$⨍\limits_{\mathbf{X}} f\, d\mu := \frac{1}{\mu(\mathbf{X})} \int\limits_{\mathbf{X}} f\, d\mu.$$

6.3.1 Interrelations and properties of the measure spaces

In this subsection, we shall collect a few useful technical facts related to the spaces and measures introduced above. As before, we are not assuming that the considered measures are normalized, unless otherwise stated. With the notation as above, we have

Lemma 6.3.1. *The measure $d\widetilde{\Omega}_{\mathrm{reg}}$ agrees with the Riemannian volume density defined by the Riemannian metric on $\widetilde{\Omega}_{\mathrm{reg}}$ that is induced by the Sasaki metric on T^*M.*

Proof. By [4, Theorem 4.6] all metrics on $\widetilde{\Omega}_{\mathrm{reg}}$ which are associated to the symplectic form $\widetilde{\omega}$ by an almost complex structure define the same Riemannian volume density, and that density agrees with the one defined by the symplectic form $\widetilde{\omega}$. Hence, it suffices to show that the Riemannian metric on $\widetilde{\Omega}_{\mathrm{reg}}$ induced by the G-invariant Sasaki metric on T^*M is associated to $\widetilde{\omega}$ by an almost complex structure. Now, the Sasaki metric g_S on T^*M is associated to the canonical symplectic form ω on T^*M by an almost complex structure $\mathcal{J} : TT^*M \to TT^*M$. Consequently, the Riemannian metric ι^*g_S on Ω_{reg} is associated to the symplectic form $\iota^*\omega$ by the almost complex structure $\iota^*\mathcal{J}$, where $\iota : \Omega_{\mathrm{reg}} \to T^*M$ is the inclusion. Since both ι^*g_S and $\iota^*\omega$ are G-invariant, $\iota^*\mathcal{J} : T\Omega_{\mathrm{reg}} \to T\Omega_{\mathrm{reg}}$ is G-equivariant, and therefore induces an almost complex structure $\widetilde{\iota^*\mathcal{J}} : T\widetilde{\Omega}_{\mathrm{reg}} \to T\widetilde{\Omega}_{\mathrm{reg}}$ which associates the metric induced by ι^*g_S on $\widetilde{\Omega}_{\mathrm{reg}}$ with $\widetilde{\omega}$. □

Lemma 6.3.2. *$M - M_{\mathrm{reg}}$ is a null set in (M, dM), and $\Omega_{\mathrm{reg}} - (T^*M_{\mathrm{reg}} \cap \Omega)$ is a null set in $(\Omega_{\mathrm{reg}}, d\Omega_{\mathrm{reg}})$.*

Proof. The proof is completely analogous to the proof of [14, Lemma 3]. □

Similarly, on $\widetilde{M} = M/G$ we have

Corollary 6.3.3. *$\widetilde{M} - \widetilde{M}_{\mathrm{reg}}$ is a null set in $(\widetilde{M}, d\widetilde{M})$, and $\widetilde{\Omega}_{\mathrm{reg}} - (T^*M_{\mathrm{reg}} \cap \Omega)/G$ is a null set in $(\widetilde{\Omega}_{\mathrm{reg}}, d\widetilde{\Omega}_{\mathrm{reg}})$.*

Proof. The first claim is true by definition of the measure $d\widetilde{M}$ and Lemma 6.3.2. Concerning the second claim, note that

$$(\Omega_{\mathrm{reg}} - (T^*M_{\mathrm{reg}} \cap \Omega_{\mathrm{reg}}))/G = \widetilde{\Omega}_{\mathrm{reg}} - (T^*M_{\mathrm{reg}} \cap \Omega_{\mathrm{reg}})/G.$$

Consequently, (6.3.4) and Lemma 6.3.2 together yield

$$\mathrm{vol}\,\left(\widetilde{\Omega}_{\mathrm{reg}} - (T^*M_{\mathrm{reg}} \cap \Omega_{\mathrm{reg}})/G\right) = \int\limits_{\Omega_{\mathrm{reg}} - (T^*M_{\mathrm{reg}} \cap \Omega_{\mathrm{reg}})} \frac{1}{\mathrm{vol}\,(G\cdot\eta)} d\Omega_{\mathrm{reg}}(\eta) = 0.$$

□

Lemma 6.3.4. *The orbit volume function $\mathrm{vol}_{\mathcal{O}}|_{M_{\mathrm{reg}}} : M_{\mathrm{reg}} \to \mathbb{R}$, $x \mapsto \mathrm{vol}\,(G\cdot x)$, is smooth. Moreover, if the dimension of the principal orbits is at least 1, the function $\mathrm{vol}_{\mathcal{O}}|_{M_{\mathrm{reg}}}$ can be extended by zero to a continuous function $\overline{\mathrm{vol}}_{\mathcal{O}} : M \to \mathbb{R}$.*

Proof. See [45, Proposition 1]. □

Remark 6.3.5. The function $\overline{\mathrm{vol}}_{\mathcal{O}} : M \to \mathbb{R}$ from the previous lemma is in general different from the original orbit volume function $\mathrm{vol}_{\mathcal{O}} : M \to \mathbb{R}$, $x \mapsto \mathrm{vol}(G\cdot x)$. The latter function is by definition nowhere zero and not continuous if there are some orbits of dimension 0 and some of dimension > 0.

Lemma 6.3.6. *On $\widetilde{M} = M/G$ we have*

$$\frac{d\widetilde{M}}{\mathrm{vol}}\Big|_{\widetilde{M}_{\mathrm{reg}}} = d\widetilde{M}_{\mathrm{reg}}, \qquad \frac{d\widetilde{M}}{\mathrm{vol}}\Big|_{\widetilde{M}-\widetilde{M}_{\mathrm{reg}}} \equiv 0.$$

Proof. Considering (6.3.3), (6.3.2), and Corollary 6.3.3, the claimed relations are obvious. □

Corollary 6.3.7. *The following two measures on $(T^*M_{\mathrm{reg}} \cap \Omega_{\mathrm{reg}})/G$ agree:*

1. *the measure $j^* d\widetilde{\Omega}_{\mathrm{reg}}$, where j is the inclusion $j : (T^*M_{\mathrm{reg}} \cap \Omega_{\mathrm{reg}})/G \hookrightarrow \widetilde{\Omega}_{\mathrm{reg}}$ and $d\widetilde{\Omega}_{\mathrm{reg}}$ the symplectic volume form on $\widetilde{\Omega}_{\mathrm{reg}}$;*
2. *the measure $\Phi^* d(T^*\widetilde{M}_{\mathrm{reg}})$, where $\Phi : (T^*M_{\mathrm{reg}} \cap \Omega_{\mathrm{reg}})/G \to T^*\widetilde{M}_{\mathrm{reg}}$ is the canonical symplectomorphism from Lemma 6.2.3 and $d(T^*\widetilde{M}_{\mathrm{reg}})$ the symplectic volume form on $T^*\widetilde{M}_{\mathrm{reg}}$.*

Proof. The measures $d\widetilde{\Omega}_{\mathrm{reg}}$ and $d(T^*\widetilde{M}_{\mathrm{reg}})$ are defined by the volume forms $\widetilde{\omega}^{n-\kappa}/(n-\kappa)!$ and $\widehat{\omega}^{n-\kappa}/(n-\kappa)!$, respectively, which implies that the measures $j^* d\widetilde{\Omega}_{\mathrm{reg}}$ and $\Phi^* d(T^*\widetilde{M}_{\mathrm{reg}})$ are defined by the volume forms $j^*\widetilde{\omega}^{n-\kappa}/(n-\kappa)!$ and $\Phi^*\widehat{\omega}^{n-\kappa}/(n-\kappa)!$, respectively. Using compatibility of the wedge product with pullbacks and Lemma 6.2.3 we obtain

$$j^*\widetilde{\omega}^{n-\kappa} = (j^*\widetilde{\omega})^{n-\kappa} = (\Phi^*\widehat{\omega})^{n-\kappa} = \Phi^*(\widehat{\omega}^{n-\kappa}).$$

□

The next lemma describes the Liouville measures on hypersurfaces that we use frequently.

Lemma 6.3.8. *Let $c \in \mathbb{R}$ be a regular value of our Hamiltonian function p. For each $\delta > 0$, let $I_\delta \subset [c-\delta, c+\delta]$ be a non-empty interval. Then, for all $f \in C(T^*M)$ the limit*

$$\lim_{\delta\to 0} \frac{1}{\mathrm{vol}_{\mathbb{R}}(I_\delta)} \int\limits_{p^{-1}(I_\delta)} f\, d(T^*M) =: \int\limits_{p^{-1}(\{c\})} f\, d\Sigma_c \tag{6.3.6}$$

exists, and defines a finite measure $d\Sigma_c$ on the hypersurface $\Sigma_c := p^{-1}(\{c\})$. Furthermore, for each $f \in C(T^*M)$, one has in the limit $\delta \to 0$ the estimate

$$\frac{1}{\mathrm{vol}_{\mathbb{R}}(I_\delta)} \int\limits_{p^{-1}(I_\delta)} f\, d(T^*M) - \int\limits_{p^{-1}(\{c\})} f\, d\Sigma_c = \mathrm{O}(\delta). \tag{6.3.7}$$

Proof. Since M is compact, $p^{-1}([c-r, c+r]) \subset T^*M$ is compact for every $r > 0$. Thus, we can find $\varepsilon > 0$ such that each $t \in [c-\varepsilon, c+\varepsilon]$ is a regular value of p. This implies that there is an atlas for T^*M such that the intersection of any chart with $p^{-1}(\{t\})$ is given by the points whose last coordinate is equal to $t-c$ for each $t \in [c-\varepsilon, c+\varepsilon]$. As $p^{-1}([c-\varepsilon, c+\varepsilon])$ is compact, we can reduce such an atlas to a finite collection of charts that still cover $p^{-1}([c-\varepsilon, c+\varepsilon])$. Denote the so obtained finite collection of charts by $\{U_\alpha, \gamma_\alpha\}_{\alpha\in\mathcal{A}}$, $\gamma_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^{2n}$, $U_\alpha \subset T^*M$. W.l.o.g. we can assume that $V_\alpha \subset \mathbb{R}^n$ is bounded. Let $\{\varphi_\alpha\}_{\alpha\in\mathcal{A}}$ be a partition of unity subordinated to the family $\{U_\alpha\}_{\alpha\in\mathcal{A}}$. Then, by definition, it holds for $\delta < \varepsilon$ and an interval $I_\delta \subset [c-\delta, c+\delta]$:

$$\int_{p^{-1}(I_\delta)} f\, d(T^*M) = \sum_{\alpha\in\mathcal{A}} \int_{\gamma_\alpha(U_\alpha \cap p^{-1}(I_\delta))} (f\cdot\varphi_\alpha)\big(\gamma_\alpha^{-1}(y)\big) D_\alpha(y)\, dy \qquad \forall\, f \in C(T^*M),$$

where $D_\alpha : U_\alpha \to \mathbb{R}$ is the smooth function defining the restriction of the volume density $d(T^*M)$ to the chart U_α in our local coordinates. (In fact, here $D_\alpha = \sqrt{\det(g_\alpha(y))}$, where g_α is the local matrix defined by the Sasaki metric on T^*M). Due to our special choice of coordinates in the chart U_α, we get for $f \in C(T^*M)$:

$$\begin{aligned}\int_{p^{-1}(I_\delta)} f\, d(T^*M) &= \sum_{\alpha\in\mathcal{A}} \int_{V_\alpha} (f\cdot\varphi_\alpha)\big(\gamma_\alpha^{-1}(y_1,\dots,y_{2n})\big) D_\alpha(y_1,\dots,y_{2n})\, dy_1\dots dy_{2n-1}\, dy_{2n}\\ &= \sum_{\alpha\in\mathcal{A}} \int_{I_\delta}\int_{V_\alpha\cap\mathbb{R}^{2n-1}} (f\cdot\varphi_\alpha)\big(\gamma_\alpha^{-1}(y_1,\dots,y_{2n-1},t)\big) D_\alpha(y_1,\dots,y_{2n-1},t)\, dy_1\dots dy_{2n-1}\, dt,\end{aligned}$$

where we used the Fubini theorem for the Lebesgue integral. Since $p^{-1}([c-\varepsilon, c+\varepsilon])$ is compact, we can assume w.l.o.g. that the function D_α is uniformly continuous. Furthermore, we know that the functions f and φ_α are uniformly continuous on $p^{-1}([c-\varepsilon, c+\varepsilon])$. This implies for each $\alpha \in \mathcal{A}$ and $y \in V_\alpha$:

$$\begin{aligned}\Big|&(f\cdot\varphi_\alpha)\big(\gamma_\alpha^{-1}(y_1,\dots,y_{2n-1},t)\big) D_\alpha(y_1,\dots,y_{2n-1},t)\\ &\quad - (f\cdot\varphi_\alpha)\big(\gamma_\alpha^{-1}(y_1,\dots,y_{2n-1},c)\big) D_\alpha(y_1,\dots,y_{2n-1},c)\Big| \leq C_\alpha(|t-c|) \qquad \forall\, t\in I_\delta,\end{aligned}$$

with some constant $C_\alpha > 0$ that is independent of y. To shorten the notation, set

$$\mathcal{I}(c,\alpha) := \int\limits_{V_\alpha\cap\mathbb{R}^{2n-1}} (f\cdot\varphi_\alpha)\big(\gamma_\alpha^{-1}(y_1,\dots,y_{2n-1},c)\big) D_\alpha(y_1,\dots,y_{2n-1},c)\, dy_1\dots dy_{2n-1}$$

With $|t-c| \leq \delta$ we then get

$$\begin{aligned}&\int_{I_\delta}\int_{V_\alpha\cap\mathbb{R}^{2n-1}} (f\cdot\varphi_\alpha)\big(\gamma_\alpha^{-1}(y_1,\dots,y_{2n-1},t)\big) D_\alpha(y_1,\dots,y_{2n-1},t)\, dy_1\dots dy_{2n-1}\, dt\\ &= \int\limits_{I_\delta} \big(\mathcal{I}(c,\alpha) + \mathrm{O}_\alpha(|t-c|)|\big)\, dt = \mathrm{vol}_{\mathbb{R}}(I_\delta)\mathcal{I}(c,\alpha) + \int_{I_\delta} \mathrm{O}_\alpha(|t-c|)|\, dt = \mathrm{vol}_{\mathbb{R}}(I_\delta)\big(\mathcal{I}(c,\alpha) + \mathrm{O}_\alpha(\delta)\big).\end{aligned}$$

Since $\mathcal{A}$ is finite, we conclude

$$\frac{1}{\mathrm{vol}_{\mathbb{R}}(I_\delta)} \int_{p^{-1}(I_\delta)} f\, d(T^*M) = \sum_{\alpha\in\mathcal{A}} \mathcal{I}(c,\alpha) + \mathrm{O}_\alpha(\delta) =: \int\limits_{p^{-1}(\{c\})} f\, d\mu_c + \mathrm{O}(\delta).$$

As the left hand side of the equation above is a global expression independent of any choice of coordinates, the definition of $\int_{p^{-1}(\{c\})} f\, d\mu_c$ is independent of the chosen coordinates. $\square$

From the previous lemma, one immediately obtains a symmetry-reduced version, defining Liouville measures on the reduced hypersurfaces occuring in this thesis.

Corollary 6.3.9. *Let $c \in \mathbb{R}$ be a regular value of our Hamiltonian function $p : T^*M \to \mathbb{R}$, and consider the induced maps*

$$p|_{\Omega_{reg}} : \Omega_{reg} \to \mathbb{R}, \qquad \widetilde{p} : \widetilde{\Omega}_{reg} \to \mathbb{R}, \qquad {}'\widetilde{p} : T^*(\widetilde{M}_{reg}) \to \mathbb{R},$$

for which c is also a regular value, as well as the smooth hypersurfaces

$$\Sigma_c \cap \Omega_{reg} = p|_{\Omega_{reg}}^{-1}(\{c\}) \subset \Omega_{reg}, \qquad \widetilde{\Sigma}_c := \widetilde{p}^{-1}(\{c\}) \subset \widetilde{\Omega}_{reg}, \qquad {}'\widetilde{\Sigma}_c :={}' \widetilde{p}^{-1}(\{c\}) \subset T^*(\widetilde{M}_{reg}).$$

For each $\delta > 0$, let $I_\delta \subset [c-\delta, c+\delta]$ be a non-empty interval. Then, for all G-invariant $f \in C(T^*M)$, inducing

$$f|_{\Omega_{reg}} \in C(\Omega_{reg}), \qquad \widetilde{f} \in C(\widetilde{\Omega}_{reg}), \qquad {}'\widetilde{f} \in C(T^*(\widetilde{M}_{reg})),$$

the limits

$$\begin{aligned}
\lim_{\delta\to 0} \frac{1}{\operatorname{vol}_{\mathbb{R}}(I_\delta)} \int_{p|_{\Omega_{reg}}^{-1}(I_\delta)} f|_{\Omega_{reg}}\, d\widetilde{\Omega}_{reg} &=: \int_{\Sigma_c\cap\Omega_{reg}} f|_{\Omega_{reg}}\, d\mu_c\\
\lim_{\delta\to 0} \frac{1}{\operatorname{vol}_{\mathbb{R}}(I_\delta)} \int_{\widetilde{p}^{-1}(I_\delta)} \widetilde{f}\, d\widetilde{\Omega}_{reg} &=: \int_{\widetilde{\Sigma}_c} \widetilde{f}\, d\widetilde{\Sigma}_c\\
\lim_{\delta\to 0} \frac{1}{\operatorname{vol}_{\mathbb{R}}(I_\delta)} \int_{'\widetilde{p}^{-1}(I_\delta)} {}'\widetilde{f}\, d(T^*(\widetilde{M}_{reg})) &=: \int_{'\widetilde{\Sigma}_c} {}'\widetilde{f}\, d\,{}'\widetilde{\Sigma}_c
\end{aligned}$$

exist, and define finite measures on the corresponding hypersurfaces. Furthermore, for each G-invariant $f \in C(T^*M)$, one has in the limit $\delta \to 0$ the estimates

$$\begin{aligned}
\frac{1}{\operatorname{vol}_{\mathbb{R}}(I_\delta)} \int_{p|_{\Omega_{reg}}^{-1}(I_\delta)} f|_{\Omega_{reg}}\, d\widetilde{\Omega}_{reg} - \int_{\Sigma_c\cap\Omega_{reg}} f|_{\Omega_{reg}}\, d\mu_c &= \mathrm{O}(\delta),\\
\frac{1}{\operatorname{vol}_{\mathbb{R}}(I_\delta)} \int_{\widetilde{p}^{-1}(I_\delta)} \widetilde{f}\, d\Omega_{reg} - \int_{\widetilde{\Sigma}_c} \widetilde{f}\, d\widetilde{\Sigma}_c &= \mathrm{O}(\delta),\\
\frac{1}{\operatorname{vol}_{\mathbb{R}}(I_\delta)} \int_{'\widetilde{p}^{-1}(I_\delta)} {}'\widetilde{f}\, d(T^*(\widetilde{M}_{reg})) - \int_{'\widetilde{\Sigma}_c} {}'\widetilde{f}\, d\,{}'\widetilde{\Sigma}_c &= \mathrm{O}(\delta).
\end{aligned}$$

Proof. Since the functions $p|_{\Omega_{\mathrm{reg}}}$, $\widetilde{p}$, $'\widetilde{p}$, $f|_{\Omega_{\mathrm{reg}}}$, $\widetilde{f}$, $'\widetilde{f}$, and the volume densities on Ω_{reg}, $\widetilde{\Omega}_{\mathrm{reg}}$, and $T^*(\widetilde{M}_{\mathrm{reg}})$ are all induced by G-invariance from their counterparts on T^*M, we can proceed by analogy to the proof of Lemma 6.3.8, even though $\Sigma_c \cap \Omega_{\mathrm{reg}}$, $\widetilde{\Sigma}_c$, and $'\widetilde{\Sigma}_c$ are not compact. In particular, we can find $\varepsilon > 0$ such that each $t \in [c-\varepsilon, c+\varepsilon]$ is a regular value of $p|_{\Omega_{\mathrm{reg}}}$, $\widetilde{p}$, and $'\widetilde{p}$, we can cover $p|_{\Omega_{\mathrm{reg}}}^{-1}([c-\varepsilon, c+\varepsilon])$, $\widetilde{p}^{-1}([c-\varepsilon, c+\varepsilon])$, and $'\widetilde{p}^{-1}([c-\varepsilon, c+\varepsilon])$ by finitely many charts, and on these "ε-thickened hypersurfaces", the maps $p|_{\Omega_{\mathrm{reg}}}$, $\widetilde{p}$, $'\widetilde{p}$, $f|_{\Omega_{\mathrm{reg}}}$, $\widetilde{f}$, and $'\widetilde{f}$, as well as the corresponding volume densities are uniformly continuous. Thus, we can perform analogous calculations and estimates as in the proof of Lemma 6.3.8. □

Remark 6.3.10. When $V \equiv 0$ and $c = 1$, the Liouville measure on $'\widetilde{p}^{-1}(\{1\}) = S^*\widetilde{M}_{\mathrm{reg}}$ obtained in Corollary 6.3.9 agrees with the Liouville measure $d(S^*\widetilde{M}_{\mathrm{reg}})$ induced by the canonical symplectic form on $T^*(\widetilde{M}_{\mathrm{reg}})$.

Remark 6.3.11. Suppose we regard the hypersurfaces Σ_c, $\Sigma_c \cap \Omega_{\mathrm{reg}}$, and $'\widetilde{\Sigma}_c$ as Riemannian submanifolds of T^*M, Ω_{reg}, resp. $T^*\widetilde{M}_{\mathrm{reg}}$, where the latter spaces are equipped with the (induced) Sasaki metrics. Then the induced Riemannian measures $d\Sigma_c^R$, $d\mu_c^R$, $d'\widetilde{\Sigma}_c^R$ are related to the Liouville measures defined above by the norm of the gradient of the defining functions:

$$d\Sigma_c = \frac{1}{\|\nabla p\|} d\Sigma_c^R, \qquad d\mu_c = \frac{1}{\big\|\nabla p|_{\Omega_{\mathrm{reg}}}\big\|} d\mu_c^R, \qquad d'\widetilde{\Sigma}_c = \frac{1}{\|\nabla' \widetilde{p}\|} d'\widetilde{\Sigma}_c^R.$$

The following simple yet important lemma tells us that taking averages over certain subsets of the co-tangent bundle of functions which actually depend only on the base manifold variable is equivalent to averaging the functions over the corresponding base manifold.

Lemma 6.3.12. *Denote by $\tau : T^*M \to M$ the co-tangent bundle projection, and by $\overline{\tau} : (T^*M_{\mathrm{reg}} \cap \Omega_{\mathrm{reg}})/G \to \widetilde{M}_{\mathrm{reg}}$ the smooth map induced by the G-equivariant map $\tau|_{T^*M_{\mathrm{reg}}\cap\Omega_{\mathrm{reg}}}$. Let $f \in \mathrm{C}(M)$ be G-invariant, inducing $\widetilde{f} \in \mathrm{C}(M/G)$, and let $p : T^*M \to \mathbb{R}$ be the G-invariant map given by $p(x,\xi) = \|\xi\|_x^2$, inducing $\widetilde{p} \in \mathrm{C}^\infty(\widetilde{\Omega}_{\mathrm{reg}})$. Then we have for all $a, b \in \mathbb{R}$ with $0 < a < b$*

$$\fint_{\widetilde{p}^{-1}([a,b])\cap((T^*M_{\mathrm{reg}}\cap\Omega_{\mathrm{reg}})/G)} \widetilde{f}(\overline{\tau}(G\cdot\eta))\, d\widetilde{\Omega}_{\mathrm{reg}}(G\cdot\eta) = \fint_{\widetilde{M}_{\mathrm{reg}}} \widetilde{f}(G\cdot x)\, d\widetilde{M}_{\mathrm{reg}}(G\cdot x). \tag{6.3.8}$$

Proof. For convenience, and because the resulting Corollary 6.3.13 is important for the applications in Section 8.3, we provide a detailed proof. Using the canonical symplectomorphism $\Phi : (T^*M_{\mathrm{reg}} \cap \Omega_{\mathrm{reg}})/G \to T^*\widetilde{M}_{\mathrm{reg}}$ from Lemma 6.2.3 and Corollary 6.3.7, we obtain

$$\begin{aligned}\int_{\widetilde{p}^{-1}([a,b])\cap((T^*M_{\mathrm{reg}}\cap\Omega_{\mathrm{reg}})/G)} &\widetilde{f}(\overline{\tau}(G\cdot\eta))\, d\widetilde{\Omega}_{\mathrm{reg}}(G\cdot\eta)\\ &= \int_{\Phi(\widetilde{p}^{-1}([a,b])\cap((T^*M_{\mathrm{reg}}\cap\Omega_{\mathrm{reg}})/G))} \widetilde{f}\circ\overline{\tau}\circ\Phi^{-1}\, d(T^*\widetilde{M}_{\mathrm{reg}}).\end{aligned} \tag{6.3.9}$$

Now that we can work with the co-tangent bundle $T^*\widetilde{M}_{\mathrm{reg}}$ together with its natural symplectic volume form as our ambient measure space, the situation is simpler and the rest of the proof could in principle be written down for an arbitrary Riemannian manifold instead of $\widetilde{M}_{\mathrm{reg}}$.

Let $\psi : U \overset{\cong}{\to} V$ be a chart, where $U \subset \widetilde{M}_{\mathrm{reg}}$, $V \subset \mathbb{R}^{n-\kappa}$ are open. Together with its derivative $\partial\psi$, the chart defines an isomorphism of vector bundles $TU \cong V \times \mathbb{R}^{n-\kappa}$. Let $(e_1, \dots, e_{n-\kappa})$ be the standard orthonormal basis of $\mathbb{R}^{n-\kappa}$. Then we obtain for each $x \in U$ a basis $\{(\partial\psi)|_x^{-1}e_1, \dots, (\partial\psi)|_x^{-1}e_{n-\kappa}\}$ of T_xU, and an associated dual basis $\{dq_1|_x, \dots, dq_{n-\kappa}|_x\}$ for T_x^*U, where $dq_i|_x$ is defined on the chosen basis vectors of T_xU by

$$dq_i|_x\big((\partial\psi)|_x^{-1}e_j\big) := \begin{cases} 1, & j = i,\\ 0, & j \neq i.\end{cases} \tag{6.3.10}$$

Fixing this choice of frame for T^*U yields an isomorphism of vector bundles $\Psi : T^*U \cong V \times \mathbb{R}^{n-\kappa}$ which is at the same time a chart on $T^*\widetilde{M}_{\mathrm{reg}}$ with domain T^*U. For a co-vector $\eta \in T_x^*U$, let $(q_1, \dots, q_{n-\kappa}, p_1, \dots, p_{n-\kappa}) \in V \times \mathbb{R}^{n-\kappa}$ be its coordinates with respect to the chart Ψ. In these coordinates, the pullback of the canonical volume form $d(T^*\widetilde{M}_{\mathrm{reg}})$ restricted to T^*U along Ψ^{-1} is given by

$$(\Psi^{-1})^*d(T^*\widetilde{M}_{\mathrm{reg}})(q,p) \equiv \frac{1}{(n-\kappa)!}dq_1\wedge\ldots\wedge dq_{n-\kappa}\wedge dp_1\wedge\ldots\wedge dp_{n-\kappa}, \qquad (q,p) \in V\times\mathbb{R}^{n-\kappa},$$

and the local Riemannian volume density on U reads

$$(\psi^{-1})^*d\widetilde{M}_{\mathrm{reg}}(q) = \sqrt{\det m(q)}\,|dq_1\wedge\ldots\wedge dq_{n-\kappa}|, \qquad q \in V,$$

where $m(q)$ is the matrix representing the Riemannian metric on T_xU, where $x = \psi^{-1}(q)$. The choice (6.3.10) of basis for T_x^*U is the natural one for considering the local pullback of the volume form $d(T^*\widetilde{M}_{\text{reg}})$, since the obtained expression for $(\Psi^{-1})^* d(T^*\widetilde{M}_{\text{reg}})(q,p)$ is invariant under a change of coordinates on U and hence independent of the initial choice of the chart ψ. However, in order to identify the co-tangent bundle globally with the tangent bundle, we need to use the Riemannian metric to relate each tangent space with its dual space, as explained in the *Conventions and notation* at the beginning of this thesis. This means in our context that we choose a different basis $\{b_1^x, \dots, b_{n-\kappa}^x\}$ for T_x^*U given by

$$b_i^x\big((\partial\psi)|_x^{-1} e_j\big) := m(q)_{i,j} = m(q)_{j,i}.$$

Denote the coordinates of $\eta \in T_x^*U$ with respect to this new choice of basis by

$$(q_1, \dots, q_{n-\kappa}, \zeta_1, \dots, \zeta_{n-\kappa}) \in V \times \mathbb{R}^{n-\kappa}.$$

Clearly, they are related to the other coordinates according to

$$\begin{pmatrix} p_1 \\ \vdots \\ p_{n-\kappa} \end{pmatrix} = m(q) \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_{n-\kappa} \end{pmatrix}. \tag{6.3.11}$$

Let $A \subset T^*U$ be measurable and $\Xi : A \to \mathbb{C}$ be an integrable function. Then, one has

$$\begin{aligned}
&\int_A \Xi \, d(T^*\widetilde{M}_{\text{reg}}) \equiv \int_{\Psi(A)} (\Psi^{-1})^*\big(\Xi \, d(T^*\widetilde{M}_{\text{reg}})\big) \\
&= \frac{1}{(n-\kappa)!} \int_{\Psi(A)} \Xi \circ \Psi^{-1}(q,p) \, dq_1 \wedge \ldots \wedge dq_{n-\kappa} \wedge dp_1 \wedge \ldots \wedge dp_{n-\kappa} \\
&= \frac{1}{(n-\kappa)!} \int_{\widehat{\tau}(\Psi(A))} \frac{1}{\sqrt{\det m(q)}} \int_{\{q\}\times\mathbb{R}^{n-\kappa}\cap\Psi(A)} \Xi \circ \Psi^{-1}(q,p) \, |dp_1 \wedge \ldots \wedge dp_{n-\kappa}| (\psi^{-1})^* d\widetilde{M}_{\text{reg}}(q),
\end{aligned}$$

where $\widehat{\tau} : V \times \mathbb{R}^{n-\kappa} \to V$ denotes the projection onto V. In order to switch from the p-coordinates to the ζ-coordinates, we perform now for each $q \in V$ the linear substitution (6.3.11). The transformation formula for the Lebesgue integral then gives us

$$\begin{aligned}
&\int_{\{q\}\times\mathbb{R}^{n-\kappa}\cap\Psi(A)} \Xi \circ \Psi^{-1}(q,p) \, |dp_1 \wedge \ldots \wedge dp_{n-\kappa}| \\
&\qquad = \int_{m(q)^{-1}\left(\{q\}\times\mathbb{R}^{n-\kappa}\cap\Psi(A)\right)} \Xi \circ \Psi^{-1}(q, m(q)\zeta) |\det m(q)| \, |d\zeta_1 \wedge \ldots \wedge d\zeta_{n-\kappa}|,
\end{aligned}$$

which combined with the previous calculations leads to

$$\begin{aligned}
\int_A \Xi \, d(T^*\widetilde{M}_{\text{reg}}) = \frac{1}{(n-\kappa)!} &\int_{\widehat{\tau}(\Psi(A))} \sqrt{\det m(q)} \\
&\int_{m(q)^{-1}\left(\{q\}\times\mathbb{R}^{n-\kappa}\cap\Psi(A)\right)} \Xi \circ \Psi^{-1}(q, m(q)\zeta) \, d\zeta \, (\psi^{-1})^* d\widetilde{M}_{\text{reg}}(q),
\end{aligned} \tag{6.3.12}$$

where we introduced the short hand notation $d\zeta := |d\zeta_1 \wedge \ldots \wedge d\zeta_{n-\kappa}|$, and the notation $m(q)^{-1}\big(\{q\} \times \mathbb{R}^{n-\kappa} \cap \Psi(A)\big)$ means that the vectors in $\mathbb{R}^{n-\kappa} \cap \Psi(A)$ are multiplied with the inverse matrix. Now, we apply the general result (6.3.12) to the concrete integral (6.3.9). By the definition of Φ we have $\psi \circ \overline{\tau} \circ \Phi^{-1} \circ \Psi^{-1} = \widehat{\tau}$, obtaining

$$\begin{aligned}(n-\kappa)! & \int\limits_{T^*U\cap\Phi(\widetilde{p}^{-1}([a,b])\cap((T^*M_{\mathrm{reg}}\cap\Omega_{\mathrm{reg}})/G))} \widetilde{f}\circ\overline{\tau}\circ\Phi^{-1}\, d(T^*\widetilde{M}_{\mathrm{reg}}) \\ &= \int\limits_{\psi\circ\overline{\tau}\left(\Phi^{-1}(T^*U)\cap\widetilde{p}^{-1}([a,b])\cap((T^*M_{\mathrm{reg}}\cap\Omega_{\mathrm{reg}})/G)\right)} \\ & \int\limits_{m(q)^{-1}\left[\{q\}\times\mathbb{R}^{n-\kappa}\cap\Psi\circ\Phi\left(\widetilde{p}^{-1}([a,b])\cap((T^*M_{\mathrm{reg}}\cap\Omega_{\mathrm{reg}})/G)\right)\right]} \widetilde{f}\big(\psi^{-1}(q)\big)\, d\zeta\, (\psi^{-1})^* d\widetilde{M}_{\mathrm{reg}}(q). \quad (6.3.13)\end{aligned}$$

Similarly, taking into account the definitions of Φ and $\widetilde{p}$, one easily sees using (6.2.4) that

$$\overline{\tau}\big(\Phi^{-1}(T^*U) \cap \widetilde{p}^{-1}([a,b]) \cap ((T^*M_{\mathrm{reg}} \cap \Omega_{\mathrm{reg}})/G)\big) = U$$

holds as well as

$$m(q)^{-1}\big[\{q\} \times \mathbb{R}^{n-\kappa} \cap \Psi\circ\Phi\big(\widetilde{p}^{-1}([a,b]) \cap ((T^*M_{\mathrm{reg}} \cap \Omega_{\mathrm{reg}})/G)\big)\big] = \big\{(q,\zeta) : a \leq \zeta^T m(q)\zeta \leq b\big\},$$

compare (0.0.1). With these observations, and considering that the integrand does not depend on ζ, the right hand side of (6.3.13) simplifies tremendously to

$$\int\limits_{\psi(U)} \widetilde{f}\big(\psi^{-1}(q)\big)\sqrt{\det m(q)}\,\mathrm{vol}_{d\zeta}\big(\{\zeta : a \leq \zeta^T m(q)\zeta \leq b\}\big)(\psi^{-1})^* d\widetilde{M}_{\mathrm{reg}}(q). \qquad (6.3.14)$$

We now calculate $\mathrm{vol}_{d\zeta}\big(\{\zeta : a \leq \zeta^T m(q)\zeta \leq b\}\big)$ to be given by

$$\begin{aligned}\mathrm{vol}_{d\zeta}\{\zeta : a \leq \zeta^T m(q)\zeta \leq b\} &= \int\limits_{\{\zeta : a\leq\zeta^T m(q)\zeta\leq b\}} d\zeta \\ &= \int\limits_{\left\{\sqrt{m(q)}^{-1}\zeta' : a\leq\zeta'^T\zeta'\leq b\right\}} d\zeta = \int\limits_{\{\zeta' : a\leq\zeta'^T\zeta'\leq b\}} \big|\det \sqrt{m(q)}^{-1}\big|\, d\zeta' \\ &= \sqrt{\det m(q)}^{-1} V_{n-\kappa}(a,b), \qquad (6.3.15)\end{aligned}$$

where $\sqrt{m(q)}^{-1}$ is the inverse of the square root of the positive definite matrix $m(q)$ which fulfills

$$\det\left(\sqrt{m(q)}^{-1}\right) = \sqrt{\det m(q)}^{-1},$$

and we wrote $V_{n-\kappa}(a,b)$ for the euclidean volume of $B^{n-\kappa}_{\sqrt{b}} - B^{n-\kappa}_{\sqrt{a}}$, $B^{n-\kappa}_r$ denoting the standard ball of radius r around 0 in $\mathbb{R}^{n-\kappa}$. By (6.3.14) and (6.3.15), line (6.3.13) becomes

$$\begin{aligned}&\int\limits_{T^*U\cap\Phi(\widetilde{p}^{-1}([a,b])\cap((T^*M_{\mathrm{reg}}\cap\Omega_{\mathrm{reg}})/G))} \widetilde{f}\circ\overline{\tau}\circ\Phi^{-1}\, d(T^*\widetilde{M}_{\mathrm{reg}}) \\ &= \frac{V_{n-\kappa}(a,b)}{(n-\kappa)!}\int\limits_{\psi(U)} \widetilde{f}\big(\psi^{-1}(q)\big)(\psi^{-1})^* d\widetilde{M}_{\mathrm{reg}}(q) \equiv \frac{V_{n-\kappa}(a,b)}{(n-\kappa)!}\int\limits_U \widetilde{f}\, d\widetilde{M}_{\mathrm{reg}}. \quad (6.3.16)\end{aligned}$$

The result above holds for an arbitrary chart domain $U \subset \widetilde{M}_{\text{reg}}$. Using a locally finite partition of unity, we can establish the global result

$$\int_{\Phi(\widetilde{p}^{-1}([a,b])\cap((T^*M_{\text{reg}}\cap\Omega_{\text{reg}})/G))} \widetilde{f}\circ\overline{\tau}\circ\Phi^{-1}\, d(T^*\widetilde{M}_{\text{reg}}) = \frac{V_{n-\kappa}(a,b)}{(n-\kappa)!}\int_{\widetilde{M}_{\text{reg}}} \widetilde{f}\, d(\widetilde{M}_{\text{reg}}). \quad (6.3.17)$$

In view of (6.3.9) we have shown

$$\int_{\widetilde{p}^{-1}([a,b])\cap((T^*M_{\text{reg}}\cap\Omega_{\text{reg}})/G)} \widetilde{f}(\overline{\tau}(G\cdot\eta))\, d\widetilde{\Omega}_{\text{reg}}(G\cdot\eta) = \frac{V_{n-\kappa}(a,b)}{(n-\kappa)!}\int_{\widetilde{M}_{\text{reg}}} \widetilde{f}(G\cdot x)\, d(\widetilde{M}_{\text{reg}})(G\cdot x).$$

Taking averages finally yields the assertion (6.3.8). □

Corollary 6.3.13. *Let $\pi : S^*M \to M$ be the canonical projection of the co-sphere bundle. Then, for every $f \in \mathrm{C}(M)$, one has*

$$\fint_{S^*M\cap\Omega_{\text{reg}}} f\circ\pi\,\frac{d\mu}{\operatorname{vol}\mathcal{O}} = \fint_M f\,\frac{dM}{\operatorname{vol}\mathcal{O}}.$$

Proof. A consequence of (6.3.5), Corollaries 6.3.9, 6.3.7, 6.3.3 and Lemmas 6.3.6, 6.3.12. □

6.4 Singular equivariant asymptotics

As it will become apparent in the next chapter, the results proved in this second part of the thesis rely on the description of the asymptotic behavior of certain oscillatory integrals that were already examined in [46, 47] while studying the spectrum of an invariant elliptic operator. Thus, let M be a Riemannian manifold of dimension n carrying a smooth effective action of a connected compact Lie group G. Consider a chart $\gamma : M \supset U \xrightarrow{\simeq} V \subset \mathbb{R}^n$ on M, and write (x,ξ) for an element in $T^*U \simeq U \times \mathbb{R}^n$ with respect to the canonical trivialization of the co-tangent bundle over the chart domain. Let $a_\mu \in \mathrm{C}^\infty_{\mathrm{c}}(T^*U \times G)$ be an amplitude that might depend on a parameter $\mu \in \mathbb{R}_{>0}$ such that $(x,\xi,g) \in \operatorname{supp} a_\mu$ implies $g \cdot x \in U$, and assume that there is a compact μ-independent set $\mathcal{K} \subset T^*U \times G$ such that $\operatorname{supp} a_\mu \subset \mathcal{K}$ for each μ. Further, consider the phase function

$$\Phi(x,\xi,g) := \langle \gamma(x) - \gamma(g\cdot x), \xi\rangle, \qquad (x,\xi,g) \in \operatorname{supp} a_\mu, \quad (6.4.1)$$

where $\langle\cdot,\cdot\rangle$ denotes the euclidean scalar product on $\mathbb{R}^n$. It represents a global analogue of the momentum map, and oscillatory integrals with phase function given by the latter have been examined in [48] in the context of equivariant cohomology. The phase function Φ has the critical set [46, equation following (3.3)]

$$\begin{aligned}\operatorname{Crit}(\Phi) &= \left\{(x,\xi,g) \in T^*U \times G : (\Phi_*)_{(x,\xi,g)} = 0\right\}\\ &= \{(x,\xi,g) \in (\Omega \cap T^*U) \times G :\ g\cdot(x,\xi) = (x,\xi)\} = \mathcal{C} \cap T^*U,\end{aligned}$$

with $\mathcal{C}$ as in (6.3.1), and the central question is to describe the asymptotic behavior as $\mu \to +\infty$ of oscillatory integrals of the form

$$I(\mu) = \int_{T^*U}\int_G e^{i\mu\Phi(x,\xi,g)} a_\mu(x,\xi,g)\, dg\, d\,(T^*U)\,(x,\xi). \quad (6.4.2)$$

The major difficulty here resides in the fact that, unless the G-action on T^*M is free, the considered momentum map is not a submersion, so that the zero set Ω of the momentum map and the critical set of the phase function Φ are not smooth manifolds. The stationary phase theorem can therefore not immediately be applied to the integrals $I(\mu)$. Nevertheless, it was shown in [46] that by constructing a strong resolution of the set

$$\mathcal{N} = \{(p, g) \in M \times G : g \cdot p = p\}$$

a partial desingularization $\mathcal{Z} : \widetilde{\mathbf{X}} \to \mathbf{X} := T^*M \times G$ of the critical set $\mathcal{C}$ can be achieved, and by applying the stationary phase theorem in the resolution space $\widetilde{\mathbf{X}}$, an asymptotic description of $I(\mu)$ can be obtained. Indeed, the map $\mathcal{Z}$ yields a partial monomialization of the local ideal $I_\Phi = (\Phi)$ generated by the phase function (6.4.1) according to

$$\mathcal{Z}^*(I_\Phi) \cdot \mathcal{E}_{\tilde{x},\widetilde{\mathbf{X}}} = \prod_j \sigma_j^{l_j} \cdot \mathcal{Z}_*^{-1}(I_\Phi) \cdot \mathcal{E}_{\tilde{x},\widetilde{\mathbf{X}}},$$

where $\mathcal{E}_{\widetilde{\mathbf{X}}}$ denotes the structure sheaf of rings of $\widetilde{\mathbf{X}}$, $\mathcal{Z}^*(I_\Phi)$ the total transform, and $\mathcal{Z}_*^{-1}(I_\Phi)$ the weak transform of I_Φ, while the σ_j are local coordinate functions near each $\tilde{x} \in \widetilde{\mathbf{X}}$ and the l_j natural numbers. As a consequence, the phase function factorizes locally according to $\Phi \circ \mathcal{Z} \equiv \prod \sigma_j^{l_j} \cdot \tilde{\Phi}^{wk}$, and one shows that the weak transforms $\tilde{\Phi}^{wk}$ have clean critical sets. Asymptotics for the integrals $I(\mu)$ are then obtained by pulling them back to the resolution space $\widetilde{\mathbf{X}}$, and applying the stationary phase theorem to the $\tilde{\Phi}^{wk}$ with the variables σ_j as parameters. Thus, with κ and Λ_M^G as in Section 5.2 one has

Theorem 6.4.1 ([47, Theorem 2.1])**.** *In the limit $\mu \to +\infty$ one has*

$$\begin{aligned} &\left| I(\mu) - \left(\frac{2\pi}{\mu}\right)^\kappa \int_{\operatorname{Reg}\mathcal{C}} \frac{a_\mu(x,\xi,g)}{\left|\det \Phi''(x,\xi,g)_{|N_{(x,\xi,g)}Reg\,\mathcal{C}}\right|^{1/2}} \, d(\operatorname{Reg}\mathcal{C})(x,\xi,g) \right| \\ &\qquad \leq C \sup_{l \leq 2\kappa+3} \left\| D^l a_\mu \right\|_\infty \mu^{-\kappa-1} (\log \mu)^{\Lambda_M^G - 1}, \end{aligned} \tag{6.4.3}$$

where D^l is a differential operator of order l independent of μ and a_μ and $C > 0$ a constant independent of μ and a_μ, too. The expression $\Phi''(x,\xi,g)_{N_{(x,\xi,g)}Reg\,\mathcal{C}}$ denotes the restriction of the Hessian of Φ to the normal space of $Reg\,\mathcal{C}$ inside $T^*U \times G$ at the point (x,ξ,g). In particular, the integral in (6.4.3) exists.

The precise form of the remainder estimate in the previous theorem will allow us to give remainder estimates also in the case when the amplitude depends on μ. To conclude, let us note the following

Lemma 6.4.2. *Let $b \in C_c^\infty(\Omega \cap T^*U)$ and $\chi \in \widehat{G}$. Then*

$$\int_{\operatorname{Reg}\mathcal{C}} \frac{\overline{\chi}(g) b(x,\xi)}{\left|\det \Phi''(x,\xi,g)_{|N_{(x,\xi,g)}Reg\,\mathcal{C}}\right|^{1/2}} \, d(\operatorname{Reg}\mathcal{C})(x,\xi,g) = [\pi_\chi|_H : \mathbb{1}] \int_{\Omega_{\text{reg}}} b(x,\xi) \frac{d\Omega_{\text{reg}}(x,\xi)}{\operatorname{vol}_{\mathcal{O}}(G \cdot (x,\xi))}.$$

Proof. By using a partition of unity, the proof essentially reduces to the one of [14, Lemma 7], which is based on a result of [13, Section 3.3.2], and involves only local calculations. Furthermore, $b \in C_c^\infty(\Omega_{\text{reg}})$ is required there. However, similarly as in [46, Lemma 9.3], one can use Fatou's Lemma to show that it suffices to require only $b \in C_c^\infty(\Omega)$. $\square$

Chapter 7

Generalized equivariant semiclassical Weyl law

7.1 An equivariant semiclassical trace formula

In this section, we generalize the semiclassical trace formula (4.3.26) to an equivariant semiclassical trace formula, which will be crucial for proving the generalized equivariant Weyl law in the next section. As before, let M be a closed connected Riemannian manifold of dimension n, carrying an isometric effective action of a compact connected Lie group G such that the dimension κ of the principal orbits is strictly smaller than n. Recall the Peter-Weyl decomposition (5.1.3) of the left-regular G-representation in $\mathrm{L}^2(M)$, as well as the function class $\mathcal{S}^{\mathrm{bcomp}}_\delta$, see Section 1.1, semiclassical character families, see Definition 5.2.1, and the Notation (5.1.5). Consider then a Schrödinger operator (5.1.2) with G-invariant potential and Hamiltonian (5.1.7).

Theorem 7.1.1 (Equivariant semiclassical trace formula for Schrödinger operators). *Let $\delta \in \big[0, \frac{1}{2\kappa+3}\big)$, $\varrho_h \in \mathcal{S}^{\mathrm{bcomp}}_\delta$, and choose an operator $B \in \Psi^0_{h;\delta}(M) \subset \mathcal{B}(\mathrm{L}^2(M))$ with principal symbol $[b]$ represented by $b \in S^0_{h;\delta}(M)$. Consider further for each $h \in (0,1]$ the trace-class operator*

$$\varrho_h(P(h)) \circ B : \mathrm{L}^2(M) \to \mathrm{L}^2(M).$$

Then, for each semiclassical character family $\{\mathcal{W}_h\}_{h\in(0,1]}$ with growth rate $\vartheta < \frac{1}{2\kappa+3} - \delta$ one has in the semiclassical limit $h \to 0$ the asymptotic formula

$$\frac{(2\pi h)^{n-\kappa}}{\#\mathcal{W}_h} \sum_{\chi \in \mathcal{W}_h} \frac{\mathrm{tr}_{\mathrm{L}^2(M)} \big(\varrho_h(P(h)) \circ B\big)_\chi}{d_\chi\,[\pi_\chi|_H : \mathbb{1}]}$$
$$= \int\limits_{\Omega_{\mathrm{reg}}} b \cdot (\varrho_h \circ p)\, \frac{d\Omega_{\mathrm{reg}}}{\mathrm{vol}\,\mathcal{O}} \;+\; \mathrm{O}\Big(h^{1-(2\kappa+3)(\delta+\vartheta)} \big(\log h^{-1}\big)^{\Lambda^G_M - 1}\Big). \tag{7.1.1}$$

Remark 7.1.2. The integral in the leading term can also be written as

$$\int\limits_{T^*\widetilde{M}_{\mathrm{reg}}} (\varrho_h \circ \widetilde{p})\, \widetilde{\langle b \rangle}_G \, d(T^*\widetilde{M}_{\mathrm{reg}}),$$

with notation as in (5.2.1). To see this, one has to take into account (6.3.2), (6.3.4) and the G-invariance of p, and apply Lemma 6.2.3 and Corollary 6.3.3. In case that $\widetilde{M} = M/G$ is an orbifold, the mentioned integral is given by an integral over the orbifold co-tangent bundle $T^*\widetilde{M}$, see Remark 6.2.4.

Remark 7.1.3. If G is trivial, the result agrees almost completely with (4.3.26), the only difference being that the remainder estimate in (4.3.26) is of order $h^{1-2\delta}$, while (7.1.1) yields for trivial G (i.e. $\kappa = \vartheta = 0$, $\Lambda^G_M = 1$) only the weaker order $h^{1-3\delta}$.

Proof. Let us consider first a fixed $\chi \in \widehat{G}$. Introduce a finite atlas $\{U_\alpha, \gamma_\alpha\}_{\alpha\in\mathcal{A}}$, $\gamma_\alpha : U_\alpha \xrightarrow{\simeq} \mathbb{R}^n$, with a subordinate compactly supported partition of unity $\{\varphi_\alpha\}_{\alpha\in\mathcal{A}}$ on M and a family of functions $\{\overline{\varphi}_\alpha, \overline{\overline{\varphi}}_\alpha, \overline{\overline{\overline{\varphi}}}_\alpha\}_{\alpha\in\mathcal{A}} \subset \mathrm{C}^\infty_\mathrm{c}(M)$ such that supp $\overline{\varphi}_\alpha$, supp $\overline{\overline{\varphi}}_\alpha$, supp $\overline{\overline{\overline{\varphi}}}_\alpha \subset U_\alpha$ and $\overline{\varphi}_\alpha \equiv 1$ on supp φ_α, $\overline{\overline{\varphi}}_\alpha \equiv 1$ on supp $\overline{\varphi}_\alpha$, and $\overline{\overline{\overline{\varphi}}}_\alpha \equiv 1$ on supp $\overline{\overline{\varphi}}_\alpha$. For each $\alpha \in \mathcal{A}$, set

$$u_{\alpha,0} := \big((\varrho_h \circ p)\cdot b \cdot \varphi_\alpha\big) \circ (\gamma_\alpha^{-1}, (\partial\gamma_\alpha^{-1})^T).$$

Clearly, $u_{\alpha,0} \in S^0_{h;\delta}(\mathbb{R}^n)$. Then, by Theorem 4.2.1, there is for each $N \in \mathbb{N}$ a number $h_0 > 0$ and a collection of symbol functions $\{r_{\alpha,\beta,N}\}_{\alpha,\beta\in\mathcal{A}} \subset h^{1-2\delta} S^0_{h;\delta}(\mathbb{R}^n)$ and an operator $\mathfrak{R}_N(h) \in \mathcal{B}(\mathrm{L}^2(M))$ such that for $h \in (0, h_0]$

$$\big[T_\chi \circ B \circ \varrho_h(P(h))\big](f) = \sum_{\alpha\in\mathcal{A}} T_\chi\Big(\overline{\varphi}_\alpha \cdot \mathrm{Op}_h(u_{\alpha,0})\big((f\cdot \overline{\overline{\overline{\varphi}}}_\alpha)\circ\gamma_\alpha^{-1}\big)\circ\gamma_\alpha\Big)$$
$$+ \sum_{\alpha,\beta\in\mathcal{A}} T_\chi\Big(\overline{\varphi}_\beta \cdot \mathrm{Op}_h(r_{\alpha,\beta,N})\big((f\cdot\overline{\overline{\varphi}}_\alpha\cdot\overline{\overline{\overline{\varphi}}}_\beta)\circ\gamma_\beta^{-1}\big)\circ\gamma_\beta\Big) + T_\chi\circ\mathfrak{R}_N(h)(f)$$

for all $f \in \mathrm{L}^2(M)$. Moreover, the operator $\mathfrak{R}_N(h) \in \mathcal{B}(\mathrm{L}^2(M))$ is of trace class, its trace norm fulfills

$$\|\mathfrak{R}_N(h)\|_{\mathrm{tr},\mathrm{L}^2(M)} = \mathrm{O}\big(h^N\big) \quad \text{as } h \to 0,$$

and for fixed $h \in (0, h_0]$ each symbol function $r_{\alpha,\beta,N}$ is an element of $\mathrm{C}^\infty_\mathrm{c}(\mathbb{R}^{2n})$ satisfying

$$\mathrm{supp}\, r_{\alpha,\beta,N} \subset \mathrm{supp}\, \big((\varrho_h\circ p)\cdot\varphi_\alpha\big)\circ(\gamma_\alpha^{-1}, (\partial\gamma_\alpha^{-1})^T). \tag{7.1.2}$$

Inserting the definition (5.1.4) of the projection T_χ one sees with (2.1.2) that each of the operators

$$A^\chi_\alpha : f \mapsto T_\chi\Big(\overline{\varphi}_\alpha \cdot \mathrm{Op}_h(u_{\alpha,0})\big((f\cdot\overline{\overline{\overline{\varphi}}}_\alpha)\circ\gamma_\alpha^{-1}\big)\circ\gamma_\alpha\Big),$$
$$A^\chi_{\alpha,\beta,N} : f \mapsto T_\chi\Big(\overline{\varphi}_\beta \cdot \mathrm{Op}_h(r_{\alpha,\beta,N})\big((f\cdot\overline{\overline{\varphi}}_\alpha\cdot\overline{\overline{\overline{\varphi}}}_\beta)\circ\gamma_\beta^{-1}\big)\circ\gamma_\beta\Big)$$

has a smooth, compactly supported Schwartz kernel given respectively by

$$K_{A^\chi_\alpha}(x_1,x_2) = \frac{d_\chi}{(2\pi h)^n}\int_G\int_{\mathbb{R}^n} \overline{\chi(g)}\,\overline{\varphi}_\alpha(g^{-1}\cdot x_1)\, e^{\frac{i}{h}\langle\gamma_\alpha(g^{-1}\cdot x_1)-\gamma_\alpha(x_2),\eta\rangle}$$
$$u_{\alpha,0}\Big(\frac{\gamma_\alpha(g^{-1}\cdot x_1)+\gamma_\alpha(x_2)}{2},\eta,h\Big)\overline{\overline{\overline{\varphi}}}_\alpha(x_2)\,d\eta\,dg\big(\mathrm{Vol}_{g_\alpha}(\gamma_\alpha(x_2))\big)^{-1},$$
$$K_{A^\chi_{\alpha,\beta,N}}(x_1,x_2) = \frac{d_\chi}{(2\pi h)^n}\int_G\int_{\mathbb{R}^n} \overline{\chi(g)}\,\overline{\varphi}_\beta(g^{-1}\cdot x_1)\, e^{\frac{i}{h}\langle\gamma_\beta(g^{-1}\cdot x_1)-\gamma_\beta(x_2),\eta\rangle}$$
$$r_{\alpha,\beta,N}\Big(\frac{\gamma_\beta(g^{-1}\cdot x_1)+\gamma_\beta(x_2)}{2},\eta,h\Big)(\overline{\overline{\varphi}}_\alpha\cdot\overline{\overline{\overline{\varphi}}}_\beta)(x_2)\,d\eta\,dg\big(\mathrm{Vol}_{g_\beta}(\gamma_\beta(x_2))\big)^{-1},$$

where $x_1, x_2 \in M$, and $\mathrm{Vol}_{g_\alpha} : \mathbb{R}^n \to (0,\infty)$ denotes the Riemannian volume density function in local coordinates, given by

$$\mathrm{Vol}_{g_\alpha}(y) = \sqrt{\det g_\alpha(y)},$$

g_α being the matrix representing the Riemannian metric on M over the chart U_α. Consequently, we obtain for arbitrary $N \in \mathbb{N}$ that

$$\begin{aligned} &\mathrm{tr}_{\mathrm{L}^2(M)} \big(\varrho_h(P(h)) \circ B\big)_\chi = \mathrm{tr}_{\mathrm{L}^2(M)} \big[T_\chi \circ B \circ \varrho_h(P(h))\big] \\ &= \sum_{\alpha\in\mathcal{A}} \int\limits_M K_{A^\chi_\alpha}(x,x)\, dM(x) + \sum_{\alpha,\beta\in\mathcal{A}} \int\limits_M K_{A^\chi_{\alpha,\beta,N}}(x,x)\, dM(x) + \mathrm{O}(h^N), \end{aligned} \tag{7.1.3}$$

where we took into account that the trace is invariant under cyclic permutations and T_χ commutes with $\varrho_h(P(h))$. Furthermore, $|\,\mathrm{tr}_{\mathrm{L}^2(M)} Q| \leq \|Q\|_{\mathrm{tr},\mathrm{L}^2(M)}$ for any trace class operator Q, and because T_χ is a projection one has

$$\|T_\chi \circ \mathfrak{R}_N(h)\|_{\mathrm{tr},\mathrm{L}^2(M)} \leq \|\mathfrak{R}_N(h)\|_{\mathrm{tr},\mathrm{L}^2(M)} \|T_\chi\|_{\mathcal{B}(\mathrm{L}^2(M))} \leq \|\mathfrak{R}_N(h)\|_{\mathrm{tr},\mathrm{L}^2(M)}$$

so that the $\mathrm{O}(h^N)$-estimate in (7.1.3) is independent of χ. Let us consider first the integrals in the second summand

$$\int\limits_M K_{A^\chi_{\alpha,\beta,N}}(x,x)\, dM(x) = \frac{d_\chi}{(2\pi h)^n} \int\limits_G \int\limits_{T^*U_\beta} e^{\frac{i}{h}\langle \gamma_\beta(x) - \gamma_\beta(g\cdot x), \xi\rangle} u^{\chi,N}_{\alpha,\beta}(x,\xi,g,h)\, dg\, d(T^*U_\beta)(x,\xi),$$

where $u^{\chi,N}_{\alpha,\beta}(\cdot,h) \in \mathrm{C}^\infty_{\mathrm{c}}(T^*U_\beta \times G)$ is given by

$$u^{\chi,N}_{\alpha,\beta}(x,\xi,g,h) = \overline{\chi(g)}\overline{\varphi}_\beta(x) r_{\alpha,\beta,N}\Big(\frac{\gamma_\beta(x) + \gamma_\beta(g\cdot x)}{2}, \xi, h\Big)(\overline{\overline{\varphi}}_\alpha \cdot \overline{\overline{\overline{\varphi}}}_\beta)(g\cdot x) J(x,g), \tag{7.1.4}$$

$J(x,g)$ being the Jacobian of the substitution $x = g\cdot x'$. By definition of the class $\mathcal{S}^{\mathrm{bcomp}}_\delta$ there is a compact interval $I \subset \mathbb{R}$ with $\operatorname{supp} \varrho_h \subset I$ for all $h \in (0,1]$. Taking into account (7.1.2) and the definition (5.1.7) of p we see that the function $u^{\chi,N}_{\alpha,\beta}(\cdot,h)$ is supported inside a compact h-independent subset of $T^*U_\beta \times G$. Theorem 6.4.1 now implies for each $N \in \mathbb{N}$ the estimate

$$\begin{aligned} \Bigg| (2\pi h)^n &\int\limits_M K_{A^\chi_{\alpha,\beta,N}}(x,x)\, dM(x) \\ &- d_\chi\, (2\pi h)^\kappa \int\limits_{\mathrm{Reg}\, \mathcal{C}_\beta} \frac{u^{\chi,N}_{\alpha,\beta}(x,\xi,g,h)}{\left|\det \Phi''(x,\xi,g)_{|N_{(x,\xi,g)} Reg\, \mathcal{C}_\beta}\right|^{1/2}}\, d(\mathrm{Reg}\, \mathcal{C}_\beta)(x,\xi,g)\Bigg| \\ &\leq C_{\alpha,\beta,N}\, d_\chi \sup_{l\leq 2\kappa+3} \big\|D^l u^{\chi,N}_{\alpha,\beta}\big\|_\infty h^{\kappa+1} \big(\log h^{-1}\big)^{\Lambda^G_M - 1}, \end{aligned} \tag{7.1.5}$$

where

$$\mathrm{Reg}\, \mathcal{C}_\beta = \{(x,\xi,g) \in (\Omega \cap T^*U_\beta) \times G,\ g\cdot(x,\xi) = (x,\xi),\ x \in M(H)\},$$

D^l is a differential operator of order l, and $\Phi''(x,\xi,g)_{|N_{(x,\xi,g)}\mathrm{Reg}\, \mathcal{C}_\beta}$ denotes the restriction of the Hessian of the function

$$\Phi(x,\xi,g) = \langle \gamma_\beta(x) - \gamma_\beta(g\cdot x), \xi\rangle$$

to the normal space of $\operatorname{Reg}\mathcal{C}_\beta$ inside $T^*U_\beta \times G$ at the point (x,ξ,g). Note that the domain of integration of the integral

$$\mathfrak{A}^\chi_{\alpha,\beta,N}(h) := \int\limits_{\operatorname{Reg}\mathcal{C}_\beta} \frac{u^{\chi,N}_{\alpha,\beta}(x,\xi,g,h)}{\left|\det \Phi''(x,\xi,g)_{|N_{(x,\xi,g)}Reg\,\mathcal{C}_\beta}\right|^{1/2}}\, d(\operatorname{Reg}\mathcal{C}_\beta)(x,\xi,g)$$

contains only such g and x for which $g\cdot x = x$, so that it simplifies to

$$\mathfrak{A}^\chi_{\alpha,\beta,N}(h) = \int\limits_{\operatorname{Reg}\mathcal{C}_\beta} \frac{\overline{\chi(g)} r_{\alpha,\beta,N}(\gamma_\beta(x),\xi,h)(\overline{\overline{\varphi}}_\alpha \cdot \overline{\varphi}_\beta)(x)}{\left|\det \Phi''(x,\xi,g)_{|N_{(x,\xi,g)}Reg\,\mathcal{C}_\beta}\right|^{1/2}}\, d(\operatorname{Reg}\mathcal{C}_\beta)(x,\xi,g). \tag{7.1.6}$$

Here we used that $J(x,g)=1$ in the domain of integration, since the substitution $x' = g\cdot x$ is the identity when $g\cdot x = x$, and that $\overline{\overline{\overline{\varphi}}}_\beta \equiv 1$ on $\operatorname{supp}\overline{\varphi}_\beta$. By Lemma 6.4.2 this simplifies further to

$$\mathfrak{A}^\chi_{\alpha,\beta,N}(h) = [\pi_\chi|_H : \mathbb{1}] \int\limits_{\Omega_{\mathrm{reg}}} r_{\alpha,\beta,N}(\gamma_\beta(x),\xi,h)(\overline{\overline{\varphi}}_\alpha \cdot \overline{\varphi}_\beta)(x) \frac{d\Omega_{\mathrm{reg}}(x,\xi)}{\mathrm{vol}\,(G\cdot(x,\xi))}.$$

We obtain that there is a constant $C_{\alpha,\beta,N}>0$, independent of h and χ, such that

$$\left|\mathfrak{A}^\chi_{\alpha,\beta,N}(h)\right| \leq C_{\alpha,\beta,N}\,[\pi_\chi|_H : \mathbb{1}]\, \|r_{\alpha,\beta,N}\|_\infty \left\|(\overline{\overline{\varphi}}_\alpha \cdot \overline{\varphi}_\beta)\right\|_\infty .$$

As $r_{\alpha,\beta,N}$ is an element of $h^{1-2\delta}S^0_{h;\delta}(\mathbb{R}^n)$ we have that

$$\|r_{\alpha,\beta,N}\|_\infty = \mathrm{O}(h^{1-2\delta}),$$

so that we conclude

$$\frac{\left|\mathfrak{A}^\chi_{\alpha,\beta,N}(h)\right|}{[\pi_\chi|_H : \mathbb{1}]} = \mathrm{O}\big(h^{1-2\delta}\big) \quad \text{as } h\to 0,$$

the estimate being independent of χ. Now, we combine this result with (7.1.5) and the relation $r_{\alpha,\beta,N} \in h^{1-2\delta}S^0_{h;\delta}(\mathbb{R}^n)$ to obtain the estimate

$$\begin{aligned}\Big|\int\limits_M K_{A^\chi_{\alpha,\beta,N}}(x,x)\,dM(x)\Big| = \mathrm{O}\Big(& d_\chi\,[\pi_\chi|_H : \mathbb{1}]\, h^{1-2\delta-n+\kappa} + \\ & d_\chi \sup_{l\leq 2\kappa+3} \big\|\widetilde{D}^l\overline{\chi}\big\|_\infty h^{-n+\kappa+1-\delta(2\kappa+3)+1-2\delta}\left(\log h^{-1}\right)^{\Lambda^G_M-1}\Big),\end{aligned} \tag{7.1.7}$$

where $\widetilde{D}^l$ is a differential operator of order l on G and the constant in the estimate is independent of χ. Since $\mathcal{A}$ is finite, we conclude from (7.1.3) and (7.1.7) that

$$\begin{aligned}&\mathrm{tr}_{\mathrm{L}^2(M)}\left(\varrho_h(P(h))\circ B\right)_\chi = \sum_{\alpha\in\mathcal{A}} \int\limits_M K_{A^\chi_\alpha}(x,x)\,dM(x) \\ &+ \mathrm{O}\Big(h^{1-2\delta-n+\kappa} d_\chi\Big[[\pi_\chi|_H : \mathbb{1}] + \sup_{l\leq 2\kappa+3}\big\|\widetilde{D}^l\overline{\chi}\big\|_\infty h^{1-\delta(2\kappa+3)}\left(\log h^{-1}\right)^{\Lambda^G_M-1}\Big]\Big),\end{aligned} \tag{7.1.8}$$

with the constant in the estimate being independent of χ. Let us now calculate the integrals in the leading term. Again, we can apply Theorem 6.4.1, and by steps analogous to those above we arrive at

$$\left|(2\pi h)^n \int\limits_M K_{A_\alpha^\chi}(x,x)\, dM(x) - d_\chi\,(2\pi h)^\kappa\,[\pi_\chi|_H : \mathbb{1}] \int\limits_{\Omega_{\mathrm{reg}}} u_{\alpha,0}(\gamma_\alpha(x),\xi,h)\overline{\varphi}_\alpha(x) \frac{d\Omega_{\mathrm{reg}}(x,\xi)}{\mathrm{vol}\,(G\cdot(x,\xi))}\right|$$

$$\leq C_\alpha d_\chi \sup_{l'\leq 2\kappa+3} \left\|\widetilde{D}^{l'}\overline{\chi}\right\|_\infty \sup_{l\leq 2\kappa+3}\left\|\hat{D}^l u_{\alpha,0}\right\|_\infty h^{\kappa+1}\left(\log h^{-1}\right)^{\Lambda_M^G-1},$$

where C_α is independent of h and χ, and $\hat{D}^l$ is a differential operator on $\mathbb{R}^{2n}$ of order l. Since ϱ_h is an element of $\mathcal{S}_\delta^{\mathrm{bcomp}}$ one has

$$\sup_{l\leq 2\kappa+3}\left\|\hat{D}^l u_{\alpha,0}\right\|_\infty = \mathrm{O}(h^{-(2\kappa+3)\delta}),$$

yielding the estimate

$$\left|(2\pi h)^n \int\limits_M K_{A_\alpha^\chi}(x,x)\, dM(x) - d_\chi\,(2\pi h)^\kappa\,[\pi_\chi|_H : \mathbb{1}] \int\limits_{\Omega_{\mathrm{reg}}} \big((\varrho_h\circ p)\cdot b\cdot\varphi_\alpha\big)(x,\xi) \frac{d\Omega_{\mathrm{reg}}(x,\xi)}{\mathrm{vol}\,(G\cdot(x,\xi))}\right|$$

$$= \mathrm{O}\Big(h^{1+\kappa-(2\kappa+3)\delta}\, d_\chi \sup_{l\leq 2\kappa+3}\left\|\widetilde{D}^l\overline{\chi}\right\|_\infty \left(\log h^{-1}\right)^{\Lambda_M^G-1}\Big)$$

as $h\to 0$. Summing over the finite set $\mathcal{A}$, and using (7.1.8) together with $\overline{\varphi}_\alpha\equiv 1$ on supp φ_α and $\sum_{\alpha\in\mathcal{A}}\varphi_\alpha = 1$ we finally obtain

$$(2\pi h)^{n-\kappa}\,\frac{\mathrm{tr}_{\mathrm{L}^2(M)}\big(\varrho_h(P(h))\circ B\big)_\chi}{d_\chi\,[\pi_\chi|_H : \mathbb{1}]} = \int\limits_{\Omega_{\mathrm{reg}}} \big((\varrho_h\circ p)\cdot b\big)(x,\xi)\frac{d\Omega_{\mathrm{reg}}(x,\xi)}{\mathrm{vol}\,(G\cdot(x,\xi))}$$

$$+\,\mathrm{O}\Big(h^{1-2\delta} + W_\kappa(\chi) h^{1-\delta(2\kappa+3)}\left(\log h^{-1}\right)^{\Lambda_M^G-1}\Big), \quad (7.1.9)$$

where the constant in the estimate is independent of χ and we introduced the notation

$$W_\kappa(\chi) := \frac{\sup_{l\leq 2\kappa+3}\left\|\widetilde{D}^l\overline{\chi}\right\|_\infty}{[\pi_\chi|_H : \mathbb{1}]}. \quad (7.1.10)$$

Now, having established the result (7.1.9) for a fixed χ, we know precisely how the remainder estimate depends on χ and we see that the leading term is independent of χ. Thus, we can average for each $h\in(0,1]$ each summand in (7.1.9) over the finite set $\mathcal{W}_h$ to obtain the result

$$\frac{(2\pi h)^{n-\kappa}}{\#\mathcal{W}_h}\sum_{\chi\in\mathcal{W}_h}\frac{\mathrm{tr}_{\mathrm{L}^2(M)}\big(\varrho_h(P(h))\circ B\big)_\chi}{d_\chi\,[\pi_\chi|_H : \mathbb{1}]} = \int\limits_{\Omega_{\mathrm{reg}}} \big((\varrho_h\circ p)\cdot b\big)(x,\xi)\frac{d\Omega_{\mathrm{reg}}(x,\xi)}{\mathrm{vol}\,(G\cdot(x,\xi))}$$

$$+\,\mathrm{O}\Big(h^{1-2\delta} + \Big[\frac{1}{\#\mathcal{W}_h}\sum_{\chi\in\mathcal{W}_h} W_\kappa(\chi)\Big] h^{1-\delta(2\kappa+3)}\left(\log h^{-1}\right)^{\Lambda_M^G-1}\Big).$$

To finish the proof, it suffices to observe that since the growth rate of the family $\{\mathcal{W}_h\}_{h\in(0,1]}$ is ϑ we have $\frac{1}{\#\mathcal{W}_h}\sum_{\chi\in\mathcal{W}_h} W_\kappa(\chi) = \mathrm{O}\big(h^{-(2\kappa+3)\vartheta}\big)$ as $h\to 0$, and the assertion (7.1.1) follows. $\square$

7.2 A generalized equivariant Weyl law

Let the notation be as in the previous sections and the *Overview* in Chapter 5. We are now in the position to state and prove the main result of this chapter.

Theorem 7.2.1 (Generalized equivariant semiclassical Weyl law). *Let $\delta \in \big(0, \frac{1}{2\kappa+4}\big)$ and choose an operator $B \in \Psi^0_{h;\delta}(M) \subset \mathcal{B}(\mathrm{L}^2(M))$ with principal symbol represented by $b \in S^0_{h;\delta}(M)$ and a semiclassical character family $\{\mathcal{W}_h\}_{h\in(0,1]}$ with growth rate $\vartheta < \frac{1-(2\kappa+4)\delta}{2\kappa+3}$. Write*

$$J(h) := \big\{j \in \mathbb{N} : E_j(h) \in [c, c+h^\delta],\ \chi_j(h) \in \mathcal{W}_h\big\},$$

where $\chi_j(h) \in \widehat{G}$ is defined by $u_j(h) \in \mathrm{L}^2_{\chi_j(h)}(M)$. Then, one has in the semiclassical limit $h \to 0$

$$\frac{(2\pi)^{n-\kappa} h^{n-\kappa-\delta}}{\#\mathcal{W}_h} \sum_{J(h)} \frac{\langle Bu_j(h), u_j(h)\rangle_{\mathrm{L}^2(M)}}{d_{\chi_j(h)}\,[\pi_{\chi_j(h)}|_H : \mathbb{1}]}$$
$$= \int_{\Sigma_c \cap\, \Omega_{\mathrm{reg}}} b\, \frac{d\mu_c}{\mathrm{vol}\,\mathcal{O}} + \mathrm{O}\Big(h^\delta + h^{\frac{1-(2\kappa+3)\vartheta}{2\kappa+4}-\delta}\big(\log h^{-1}\big)^{\Lambda^G_M-1}\Big).$$

Remark 7.2.2. Note that the second summand in the remainder dominates the estimate if and only if $\delta \geq \frac{1-(2\kappa+3)\vartheta}{4\kappa+8}$. Further, the integral in the leading term equals $\int_{\widetilde{\Sigma}_c} \widetilde{\langle b\rangle_G}\, d\widetilde{\Sigma}_c$, compare Section 6.3, and it can actually be viewed as an integral over the smooth bundle

$$S^*_{\widetilde{p},c}(\widetilde{M}_{\mathrm{reg}}) := \Big\{(x,\xi) \in T^*(\widetilde{M}_{\mathrm{reg}}) : \widetilde{p}(x,\xi) = c\Big\},$$

where $\widetilde{p}$ is the function on $T^*(\widetilde{M}_{\mathrm{reg}})$ induced by p via Lemma 6.2.3. In case that $\widetilde{M}$ is an orbifold, the mentioned integral is given by an integral over the orbifold bundle $S^*_{\widetilde{p},c}(\widetilde{M}) := \big\{(x,\xi) \in T^*\widetilde{M} : \widetilde{p}(x,\xi) = c\big\}$, compare Remark 7.1.2.

In the special case of a constant semiclassical character family, corresponding to the study of a single fixed isotypic component, we obtain as a direct corollary

Theorem 7.2.3. *Choose a fixed $\chi \in \widehat{G}$. Then for each $\delta \in (0, \frac{1}{2\kappa+4})$ one has the asymptotic formula*

$$(2\pi)^{n-\kappa} h^{n-\kappa-\delta} \sum_{\substack{j\in\mathbb{N}:\, u_j(h)\in \mathrm{L}^2_\chi(M),\\ E_j(h)\in[c,c+h^\delta]}} \langle Bu_j(h), u_j(h)\rangle_{\mathrm{L}^2(M)} = d_\chi\,[\pi_\chi|_H : \mathbb{1}] \int_{\Sigma_c \cap\, \Omega_{\mathrm{reg}}} b\, \frac{d\mu_c}{\mathrm{vol}\,\mathcal{O}}$$
$$+\ \mathrm{O}\Big(h^\delta + h^{\frac{1}{2\kappa+4}-\delta}\big(\log h^{-1}\big)^{\Lambda^G_M-1}\Big), \qquad h \to 0. \tag{7.2.1}$$

Remark 7.2.4. A weaker version of Theorem 7.2.3 can be proved if instead of the spectral window $[c, c+h^\delta]$ one considers a fixed interval $[r,s]$, the numbers r, s being regular values of p. One can then show that

$$(2\pi h)^{n-\kappa} \sum_{\substack{j\in\mathbb{N}:\, u_j(h)\in \mathrm{L}^2_\chi(M),\\ E_j(h)\in[r,s]}} \langle Bu_j(h), u_j(h)\rangle_{\mathrm{L}^2(M)} = d_\chi\,[\pi_\chi|_H : \mathbb{1}] \int_{p^{-1}([r,s])\cap\Omega_{\mathrm{reg}}} b\, \frac{d\Omega_{\mathrm{reg}}}{\mathrm{vol}\,\mathcal{O}}$$
$$+\,\mathrm{O}\left(h^{\frac{1}{2\kappa+4}}(\log h^{-1})^{\Lambda^G_M-1}\right),$$

which is proven by complete analogy. The even weaker statement

$$\lim_{h\to 0}(2\pi h)^{n-\kappa}\sum_{\substack{j\in\mathbb{N}:\, u_j(h)\in \mathrm{L}^2_\chi(M),\\ E_j(h)\in[r,s]}}\langle Bu_j(h),u_j(h)\rangle_{\mathrm{L}^2(M)}=d_\chi\,[\pi_\chi|_H:\mathbb{1}]\int_{p^{-1}([r,s])\cap\Omega_{\mathrm{reg}}} b\,\frac{d\Omega_{\mathrm{reg}}}{\mathrm{vol}\,\mathcal{O}}$$

could in principle also be obtained without the remainder estimates from [47] using heat kernel methods as in [18] or [10], adapted to the semiclassical setting. Nevertheless, for the study of growing families of isotypic components and shrinking spectral windows as in Theorem 7.2.1 remainder estimates are necessary due to the lower rate of convergence.

Proof of Theorem 7.2.1. The proof is an adaptation of the proof of [63, Theorem 15.3] to our situation, but with a sharper energy localization. Again, we consider first a single character $\chi\in\widehat{G}$. Let $h\in(0,1]$ and fix a positive number $\lambda<\frac{1}{2\kappa+3}-\delta$. Choose $f_{\lambda,h},g_{\lambda,h}\in \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R},[0,1])$ such that $\operatorname{supp} f_{\lambda,h}\subset[-\frac12+h^\lambda,\frac12-h^\lambda]$, $f_{\lambda,h}\equiv 1$ on $[-\frac12+3h^\lambda,\frac12-3h^\lambda]$, $\operatorname{supp} g_{\lambda,h}\subset[-\frac12-3h^\lambda,\frac12+3h^\lambda]$, $g_{\lambda,h}\equiv 1$ on $[-\frac12-h^\lambda,\frac12+h^\lambda]$, and

$$|\,\partial_y^j f_{\lambda,h}(y)|\leq C_j\,h^{-\lambda j},\qquad |\,\partial_y^j g_{\lambda,h}(y)|\leq C_j\,h^{-\lambda j},\tag{7.2.2}$$

compare [33, Theorem 1.4.1 and (1.4.2)]. Put $c(h):=ch^{-\delta}+\frac12$, so that $x\mapsto h^{-\delta}x-c(h)$ defines a diffeomorphism from $[c,c+h^\delta]$ to $[-1/2,1/2]$, and set $f_{\lambda,\delta,h}(x):=f_{\lambda,h}(h^{-\delta}x-c(h))$, $g_{\lambda,\delta,h}(x):=g_{\lambda,h}(h^{-\delta}x-c(h))$. Let Π_χ be the projection onto the span of $\{u_j(h)\in \mathrm{L}^2_\chi(M): E_j(h)\in[c,c+h^\delta]\}$. Then

$$\begin{aligned} f_{\lambda,\delta,h}(P(h))_\chi\circ\Pi_\chi&=\Pi_\chi\circ f_{\lambda,\delta,h}(P(h))_\chi=f_{\lambda,\delta,h}(P(h))_\chi,\\ g_{\lambda,\delta,h}(P(h))_\chi\circ\Pi_\chi&=\Pi_\chi\circ g_{\lambda,\delta,h}(P(h))_\chi=\Pi_\chi.\end{aligned}\tag{7.2.3}$$

Note that $f_{\lambda,\delta,h}(P(h))$, $g_{\lambda,\delta,h}(P(h))$, Π_χ are finite rank operators. For that elementary reason, all operators we consider in the following are trace class. In particular, by (7.2.3) we have

$$\begin{aligned}\sum_{\substack{j\in\mathbb{N}:\, u_j(h)\in \mathrm{L}^2_\chi(M)\\ E_j(h)\in[c,c+h^\delta]}}\langle Bu_j(h),u_j(h)\rangle_{\mathrm{L}^2(M)}&=\operatorname{tr}_{\mathrm{L}^2(M)}\Pi_\chi\circ B\circ\Pi_\chi\\ &=\operatorname{tr}_{\mathrm{L}^2(M)} f_{\lambda,\delta,h}(P(h))_\chi\circ B_\chi+\operatorname{tr}_{\mathrm{L}^2(M)}\Pi_\chi\circ g_{\lambda,\delta,h}(P(h))_\chi\circ(1-f_{\lambda,\delta,h}(P(h))_\chi)\circ B_\chi\circ\Pi_\chi\\ &=\operatorname{tr}_{\mathrm{L}^2(M)}\big(f_{\lambda,\delta,h}(P(h))\circ B\big)_\chi\\ &\qquad+\underbrace{\operatorname{tr}_{\mathrm{L}^2(M)}\Pi_\chi\circ\big(g_{\lambda,\delta,h}(P(h))\circ(1-f_{\lambda,\delta,h}(P(h)))\circ B\big)_\chi\circ\Pi_\chi}_{=:\mathfrak{R}_{\lambda,\delta,h}}.\end{aligned}\tag{7.2.4}$$

In what follows, we shall show that the first summand in (7.2.4) represents the main contribution, while $\mathfrak{R}_{\lambda,\delta,h}$ becomes small as h goes to zero. For this, we estimate $\mathfrak{R}_{\lambda,\delta,h}$ using the trace norm. Recall that if $L\in\mathcal{B}(\mathrm{L}^2(M))$ is of trace class and $M\in\mathcal{B}(\mathrm{L}^2(M))$, then $\|LM\|_{\mathrm{tr},\mathrm{L}^2(M)}\leq\|L\|_{\mathrm{tr},\mathrm{L}^2(M)}\|M\|_{\mathcal{B}(\mathrm{L}^2(M))}$, see e.g. [63, p. 337]. By the spectral functional calculus this implies

$$\begin{aligned}|\mathfrak{R}_{\lambda,\delta,h}|&\leq\left\|\Pi_\chi\circ\big(g_{\lambda,\delta,h}(P(h))\circ(1-f_{\lambda,\delta,h}(P(h)))\circ B\big)_\chi\circ\Pi_\chi\right\|_{\mathrm{tr},\mathrm{L}^2(M)}\\ &\leq\left\|\big(g_{\lambda,\delta,h}(P(h))\circ(1-f_{\lambda,\delta,h}(P(h)))\big)_\chi\right\|_{\mathrm{tr},\mathrm{L}^2(M)}\|B\|_{B(\mathrm{L}^2(M))}\\ &=\|v_{\lambda,\delta,h}(P(h))_\chi\|_{\mathrm{tr},\mathrm{L}^2(M)}\|B\|_{B(\mathrm{L}^2(M))},\end{aligned}\tag{7.2.5}$$

where we set

$$v_{\lambda,\delta,h} := g_{\lambda,\delta,h}(1 - f_{\lambda,\delta,h}) \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}, [0,1]).$$

In particular, $v_{\lambda,\delta,h}$ is non-negative. By the spectral functional calculus, $v_{\lambda,\delta,h}(P(h))$ is a positive operator. T_χ is a projection, hence positive as well. Moreover, by the spectral functional calculus, $v_{\lambda,\delta,h}(P(h))_\chi$ commutes with T_χ, as $P(h)$ does. It follows that $v_{\lambda,\delta,h}(P(h))_\chi$ is positive as the composition of positive commuting operators. For a positive operator, the trace norm is identical to the trace. Therefore (7.2.5) implies

$$|\mathfrak{R}_{\lambda,\delta,h}| \leq \|B\|_{B(\mathrm{L}^2(M))} \operatorname{tr}_{\mathrm{L}^2(M)} v_{\lambda,\delta,h}(P(h))_\chi. \tag{7.2.6}$$

By construction of the supports of $f_{\lambda,h}$ and $g_{\lambda,h}$, we have

$$\operatorname{supp} v_{\lambda,\delta,h} \subset [c - 3h^\lambda h^\delta, c + 3h^\lambda h^\delta] \cup [c + h^\delta - 3h^\lambda h^\delta, c + h^\delta + 3h^\lambda h^\delta]. \tag{7.2.7}$$

Now, note that the functions $f_{\lambda,\delta,h}$, $g_{\lambda,\delta,h}$, $v_{\lambda,\delta,h}$ are elements of $\mathcal{S}^{\mathrm{bcomp}}_{\lambda+\delta}$. Since we chose λ such that $\lambda + \delta < \frac{1}{2\kappa+3}$, we can apply (7.1.9) with $B = \mathbf{1}_{\mathrm{L}^2(M)}$ to conclude

$$\begin{aligned}\Bigg| (2\pi h)^{n-\kappa} \frac{\operatorname{tr}_{\mathrm{L}^2(M)} v_{\lambda,\delta,h}\left(P(h)\right)_\chi}{d_\chi \left[\pi_\chi|_H : \mathbb{1}\right]} - \int\limits_{\Omega_{\mathrm{reg}}} (v_{\lambda,\delta,h} \circ p) \frac{d\Omega_{\mathrm{reg}}}{\operatorname{vol}_{\mathcal{O}}} \Bigg| \\ \leq C\Big(h^{1-2(\lambda+\delta)} + W_\kappa(\chi) h^{1-(\lambda+\delta)(2\kappa+3)} \left(\log h^{-1}\right)^{\Lambda^G_M - 1}\Big),\end{aligned} \tag{7.2.8}$$

where C is independent of h and χ, and $W_\kappa(\chi)$ was defined in (7.1.10). On the other hand, applying (7.1.9) to the first summand on the right hand side of (7.2.4) yields

$$\begin{aligned}\Bigg| (2\pi h)^{n-\kappa} \frac{\operatorname{tr}_{\mathrm{L}^2(M)} \left(f_{\lambda,\delta,h}\left(P(h)\right) \circ B\right)_\chi}{d_\chi \left[\pi_\chi|_H : \mathbb{1}\right]} - \int\limits_{\Omega_{\mathrm{reg}}} (f_{\lambda,\delta,h} \circ p)\, b \frac{d\Omega_{\mathrm{reg}}}{\operatorname{vol}_{\mathcal{O}}} \Bigg| \\ \leq C\Big(h^{1-2(\lambda+\delta)} + W_\kappa(\chi) h^{1-(\lambda+\delta)(2\kappa+3)} \left(\log h^{-1}\right)^{\Lambda^G_M - 1}\Big),\end{aligned} \tag{7.2.9}$$

where C is a new constant independent of h and χ. Next, we use that Corollary 6.3.9 implies

$$\Big| \int_{\Omega_{\mathrm{reg}}} (v_{\lambda,\delta,h} \circ p) \frac{d\Omega_{\mathrm{reg}}}{\operatorname{vol}_{\mathcal{O}}} \Big| = \Big| \int_{\widetilde{\Omega}_{\mathrm{reg}}} (v_{\lambda,\delta,h} \circ \widetilde{p})\, d(\widetilde{\Omega}_{\mathrm{reg}}) \Big| = \mathrm{O}(h^{\delta+\lambda}) \qquad \text{as } h \to 0. \tag{7.2.10}$$

Combining (7.2.4)-(7.2.10) leads to

$$\begin{aligned}(2\pi h)^{n-\kappa} \sum_{\substack{j \in \mathbb{N}:\, u_j(h) \in \mathrm{L}^2_\chi(M) \\ E_j(h) \in [c, c+h^\delta]}} \frac{\langle B u_j(h), u_j(h)\rangle_{\mathrm{L}^2(M)}}{d_\chi \left[\pi_\chi|_H : \mathbb{1}\right]} = \int\limits_{\Omega_{\mathrm{reg}}} (f_{\lambda,\delta,h} \circ p)\, b \frac{d\Omega_{\mathrm{reg}}}{\operatorname{vol}_{\mathcal{O}}} \\ +\ \mathrm{O}\Big(h^{\delta+\lambda} + h^{1-2(\lambda+\delta)} + W_\kappa(\chi) h^{1-(\lambda+\delta)(2\kappa+3)} \left(\log h^{-1}\right)^{\Lambda^G_M - 1}\Big),\end{aligned} \tag{7.2.11}$$

the constant in the estimate being independent of χ. We proceed by observing

$$\begin{aligned}\Bigg| \int\limits_{\Omega_{\mathrm{reg}}} (f_{\lambda,\delta,h} \circ p)\, b \frac{d\Omega_{\mathrm{reg}}}{\operatorname{vol}_{\mathcal{O}}} - \int\limits_{\Omega_{\mathrm{reg}} \cap p^{-1}([c, c+h^\delta])} b \frac{d\Omega_{\mathrm{reg}}}{\operatorname{vol}_{\mathcal{O}}} \Bigg| \\ \leq \Bigg| \int\limits_{T^*\widetilde{M}_{\mathrm{reg}}} (v_{\lambda,\delta,h} \circ \widetilde{p})\, b\, d(T^*\widetilde{M}_{\mathrm{reg}}) \Bigg| = \mathrm{O}(h^{\lambda+\delta}).\end{aligned} \tag{7.2.12}$$

Furthermore, with

$$\Sigma_c = p^{-1}(\{c\}), \qquad \widetilde{\Sigma}_c = \widetilde{p}^{-1}(\{c\}),$$

and the notation from (5.2.1) one computes

$$\begin{aligned}\frac{1}{h^\delta}\int_{\Omega_{\mathrm{reg}}\cap p^{-1}([c,c+h^\delta])} b\,\frac{d\Omega_{\mathrm{reg}}}{\operatorname{vol}\mathcal{O}} &= \frac{1}{h^\delta}\int_{\widetilde{p}^{-1}([c,c+h^\delta])} \widetilde{\langle b\rangle}_G\, d\widetilde{\Omega}_{\mathrm{reg}} \\ &= \int_{\widetilde{\Sigma}_c} \widetilde{\langle b\rangle}_G\, d\widetilde{\Sigma}_c + \mathrm{O}(h^\delta) = \int\limits_{\Sigma_c\cap\Omega_{\mathrm{reg}}} b\,\frac{d\mu_c}{\operatorname{vol}\mathcal{O}} + \mathrm{O}(h^\delta),\end{aligned} \tag{7.2.13}$$

where we took again Corollary 6.3.9 into account. Combining (7.2.11)-(7.2.13) then yields for a fixed $\chi \in \widehat{G}$

$$\begin{aligned}&(2\pi)^{n-\kappa}h^{n-\kappa-\delta} \sum_{\substack{j\in\mathbb{N}:\, u_j(h)\in \mathrm{L}^2_\chi(M)\\ E_j(h)\in[c,c+h^\delta]}} \frac{\langle Bu_j(h), u_j(h)\rangle_{\mathrm{L}^2(M)}}{d_\chi\,[\pi_\chi|_H : \mathbb{1}]} - \int\limits_{\Sigma_c\cap\Omega_{\mathrm{reg}}} b\,\frac{d\mu_c}{\operatorname{vol}\mathcal{O}} \\ &= \mathrm{O}\Big(h^\delta + h^\lambda + W_\kappa(\chi)h^{1-(\lambda+\delta)(2\kappa+3)-\delta}\left(\log h^{-1}\right)^{\Lambda^G_M-1}\Big).\end{aligned} \tag{7.2.14}$$

Here, the constant in the estimate is independent of χ. Just as at the end of the proof of Theorem 7.1.1, we can now take for each $h \in (0,1]$ the average over the finite set $\mathcal{W}_h$, and knowing that $\mathcal{W}_h$ has growth rate ϑ, we get

$$\begin{aligned}&\frac{(2\pi)^{n-\kappa}h^{n-\kappa-\delta}}{\#\mathcal{W}_h} \sum_{J(h)} \frac{\langle Bu_j(h), u_j(h)\rangle_{\mathrm{L}^2(M)}}{d_{\chi_j(h)}\left[\pi_{\chi_j(h)}|_H : \mathbb{1}\right]} - \int\limits_{\Sigma_c\cap\Omega_{\mathrm{reg}}} b\,\frac{d\mu_c}{\operatorname{vol}\mathcal{O}} \\ &\qquad = \mathrm{O}\Big(h^\delta + h^\lambda + h^{1-(\lambda+\delta+\vartheta)(2\kappa+3)-\delta}\left(\log h^{-1}\right)^{\Lambda^G_M-1}\Big).\end{aligned} \tag{7.2.15}$$

Finally, we choose λ such that the remainder estimate is optimal for the given constants δ and ϑ. This is the case iff $\lambda = 1-(\lambda+\delta+\vartheta)(2\kappa+3)-\delta$, which is equivalent to

$$\lambda = \frac{1-(2\kappa+3)\vartheta}{2\kappa+4} - \delta.$$

This choice for λ is compatible with the general technical requirement that $\lambda < \frac{1}{2\kappa+3} - \delta$, and the assertion of Theorem 7.2.1 follows. $\square$

As consequence of the previous theorem we obtain in particular

Theorem 7.2.5 (Equivariant Weyl law for semiclassical character families). *For each $\chi \in \widehat{G}$, denote by $\mathrm{mult}_\chi(E_j(h))$ the multiplicity of the irreducible representation π_χ in the eigenspace $\mathcal{E}_j(h)$ corresponding to the eigenvalue $E_j(h)$. Then one has in the limit $h \to 0$ the asymptotic formula*

$$\frac{(2\pi)^{n-\kappa}h^{n-\kappa-\delta}}{\#\mathcal{W}_h} \sum_{\substack{\chi\in\mathcal{W}_h,\, j\in\mathbb{N}:\\ E_j(h)\in[c,c+h^\delta]}} \frac{\mathrm{mult}_\chi(E_j(h))}{\dim\mathcal{E}_j(h)\cdot[\pi_\chi|_H:\mathbb{1}]}$$
$$= \mathrm{vol}_{d\widetilde{\Sigma}_c}\widetilde{\Sigma}_c \;+\; \mathrm{O}\Big(h^\delta + h^{\frac{1-(2\kappa+3)\vartheta}{2\kappa+4}-\delta}\left(\log h^{-1}\right)^{\Lambda_M^G-1}\Big). \quad (7.2.16)$$

$\square$

Again, in the special case that $\mathcal{W}_h = \{\chi\}$ for all $h \in (0,1]$ and some fixed $\chi \in \widehat{G}$ we obtain

Theorem 7.2.6 (Equivariant Weyl law for single isotypic components). *Choose a fixed $\chi \in \widehat{G}$. Then one has in the limit $h \to 0$*

$$(2\pi)^{n-\kappa}h^{n-\kappa-\delta} \sum_{\substack{j\in\mathbb{N}:\\ E_j(h)\in[c,c+h^\delta]}} \frac{\mathrm{mult}_\chi(E_j(h))}{\dim\mathcal{E}_j(h)}$$
$$= [\pi_\chi|_H:\mathbb{1}]\mathrm{vol}_{d\widetilde{\Sigma}_c}\widetilde{\Sigma}_c \;+\; \mathrm{O}\Big(h^\delta + h^{\frac{1}{2\kappa+4}-\delta}\left(\log h^{-1}\right)^{\Lambda_M^G-1}\Big). \quad (7.2.17)$$

$\square$

Remark 7.2.7. Note that the leading terms in the formulas above are non-zero. Indeed, if c is a regular value of p and consequently of $\widetilde{p}$, both Σ_c and $\widetilde{\Sigma}_c$ are non-degenerate hypersurfaces, which implies that their volumes are non-zero.

As mentioned before, the proof of the generalized equivariant semiclassical Weyl law in Theorem 7.2.1 relies on the singular equivariant asymptotics which are the content of Theorem 6.4.1. Hereby one cannot assume that the considered integrands are supported away from the singular part of Ω, in particular when localizing to $\Sigma_c \cap \Omega_{\mathrm{reg}}$ in (7.2.13). This means that for general group actions a desingularization process is indeed necessary, as the following examples illustrate.

Examples 7.2.8.

1. Let G be a compact Lie group of dimension at least 1, acting effectively and locally smoothly on the n-sphere S^n with precisely one orbit type. Then G either acts transitively or freely on S^n [6, Theorem IV.6.2]. In the latter case, G is either S^1, S^3, or the normalizer of S^1 in S^3. Consequently, if M is an arbitrary compact G-manifold, S^*M will contain non-principal isotropy types in general. As a simple example, consider the linear action of $G = S^1$ on the standard 3-sphere $M = S^3 = \{x \in \mathbb{R}^4 : \|x\| = 1\}$ given by

$$S^1 = \left\{z = e^{i\phi},\, \phi \in [0,2\pi)\right\} \ni z \longmapsto R(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & \sin\phi \\ 0 & 0 & -\sin\phi & \cos\phi \end{pmatrix} \in \mathrm{SO}(4)$$

with isotropy types $(\{e\})$ and (S^1). The induced action on the tangent bundle

$$TS^3 = \bigsqcup_x T_xS^3 = \bigsqcup_x \left\{(x,v) \in \mathbb{R}^8 : x \in S^3,\ v \perp x\right\},$$

which we identify with T^*S^3 via the induced metric, is given by

$$z \cdot (x,v) \mapsto (R(z)x, R(z)v),$$

and has the same isotropy types. Let now $x \in S^3(S^1)$ be of singular orbit type. Then $x_3 = x_4 = 0$ and S^1 acts on

$$S_xS^3 = \left\{(x,v) \in S^3 \times S^3 : v_1x_1 + v_2x_2 = 0\right\}$$

with isotropy types $(\{e\})$ and (S^1). In particular, the S^1-action on $S_xS^3 \simeq S_x^*S^3$ is neither transitive nor free.

2. Let $M = G$ be a Lie group with a left-invariant Riemannian metric and $K \subset G$ a compact subgroup. Consider the left action of G on itself and the decomposition of T^*G into isotropy types with respect to the induced left K-action. Taking into account the left trivialization $T^*G \simeq G \times \mathfrak{g}^*$ explained in [44, Example 4.5.5] one has

$$(S^*G)(H) = S^*(G(H))$$

for an arbitrary closed subgroup $H \subset K$. Thus, in general, the co-sphere bundle of G will contain non-principal isotropy types. Assume now that G is compact and consider a Schrödinger operator $P(h)$ on G with K-invariant symbol function p. Let $c \in \mathbb{R}$ be a regular value of p and $\Sigma_c = p^{-1}(\{c\})$. Then the results of Theorem 7.2.1 apply. By the previous considerations,

$$S^*G \cap \Omega_{\text{reg}} = \Omega \cap S^*(G(H)), \qquad \Omega = \mathbb{J}_K^{-1}(0),$$

where $\mathbb{J}_K : T^*G \to \mathfrak{k}^*$ is the momentum map of the K-action, and H a principal isotropy group. Consequently, the closure of $S^*G \cap \Omega_{\text{reg}}$, and more generally of $\Sigma_c \cap \Omega_{\text{reg}}$, will contain non-principal isotropy types in general.

In case that G acts on M with finite isotropy groups, G-invariant pseudodifferential operators on M correspond to pseudodifferential operators on the orbifold $\widetilde{M}$, and vice versa. In fact, the spectral theory of elliptic operators on compact orbifolds has attracted quite much attention recently [19, 56, 38] and, as mentioned earlier, the work presented here can be viewed as part of an attempt to develop a spectral theory of elliptic operators on general singular G-spaces.

Chapter 8

Equivariant quantum ergodicity

8.1 Symmetry-reduced classical ergodicity

We begin now with our study of ergodicity, and first turn to the examination of classical ergodicity in the presence of symmetries within the framework of symplectic reduction. As we already mentioned, the latter is based on the fundamental fact that the presence of conserved quantities or first integrals of motion leads to the elimination of variables, and reduces the given configuration space with its symmetries to a lower-dimensional one, in which the degeneracies and the conserved quantitites have been eliminated. In particular, the Hamiltonian flows associated to G-invariant Hamiltonians give rise to corresponding reduced Hamiltonian flows on the different symplectic strata of the reduction. Therefore, the concept of ergodicity can be studied naturally in the context of symplectic reduction, leading to a symmetry-reduced notion of ergodicity.

Recall that, in general, a measure-preserving transformation $T : \mathbf{X} \to \mathbf{X}$ on a finite measure space $(\mathbf{X}, \nu)$ is called *ergodic* if $T^{-1}(A) = A$ implies $\nu(A) \in \{0, \nu(\mathbf{X})\}$ for every measurable set $A \subset \mathbf{X}$. Consider now a connected, symplectic manifold $(\mathbf{X}, \omega)$ with a global Hamiltonian action of a Lie group G, and let $\mathbb{J} : \mathbf{X} \to \mathfrak{g}^*$, $\mathbb{J}(\eta)(X) = \mathbb{J}_X(\eta)$ be the corresponding momentum map. As already noted in Section 6.2, for each $X \in \mathfrak{g}$ the function $\mathbb{J}_X$ is a conserved quantity for any G-invariant function $p \in \mathrm{C}^\infty(\mathbf{X}, \mathbb{R})$, so that $\{\mathbb{J}_X, p\} = 0$. This implies that for any value μ of $\mathbb{J}$, the fiber $\mathbb{J}^{-1}(\{\mu\})$ is invariant under the Hamiltonian flow of p, which means that $\mathbb{J}$ fulfills *Noether's condition.* In particular, if $c \in \mathbb{R}$ is a regular value of p and $\Sigma_c := p^{-1}(\{c\})$, the pre-image under $\mathbb{J}$ of any open proper subset in $\mathbb{J}(\Sigma_c)$ will be an open proper subset in Σ_c that is invariant under the Hamiltonian flow of p, so the latter cannot be ergodic with respect to the induced Liouville measure on Σ_c, unless G is trivial.

Let now p and μ be fixed, $K \subset G$ an isotropy group of the G-action on $\mathbf{X}$, and $\eta \in \mathbb{J}^{-1}(\{\mu\})$. With the notation as in Section 6.2, let $c \in \mathbb{R}$, and put $\widetilde{\Sigma}^{(K)}_{\mu,c} := (\widetilde{p}^{(K)}_\mu)^{-1}(\{c\})$. Let $\widetilde{g}$ be a Riemannian metric on $\widetilde{\Omega}^{(K)}_\mu$ and $\mathcal{J} : T\widetilde{\Omega}^{(K)}_\mu \to T\widetilde{\Omega}^{(K)}_\mu$ the almost complex structure determined by $\widetilde{\omega}^{(K)}_\mu$ and $\widetilde{g}$, so that $(\widetilde{\Omega}^{(K)}_\mu, \mathcal{J}, \widetilde{g})$ becomes an almost Hermitian manifold. We then make the following

Assumption 1. *c is a regular value of $\widetilde{p}^{(K)}_\mu$.*

Note that this assumption is implied by the condition that for all $\xi \in \mathbb{J}^{-1}(\{\mu\}) \cap G_\mu \cdot \mathbf{X}^\eta_K \cap \Sigma_c$ one has

$$H_p(\xi) \notin \mathfrak{g}_\mu \cdot \xi,$$

where $\mathfrak{g}_\mu$ denotes the Lie algebra of G_μ. Indeed, assume that there exists some $[\xi] \in \widetilde{\Sigma}_{\mu,c}^{(K)}$ such that $\operatorname{grad} \widetilde{p}_\mu^{(K)}([\xi]) = 0$. Since

$$\widetilde{\omega}_\mu^{(K)}(\text{s-grad}\, \widetilde{p}_\mu^{(K)}, \mathfrak{X}) = d\widetilde{p}_\mu^{(K)}(\mathfrak{X}) = \widetilde{g}(\operatorname{grad} \widetilde{p}_\mu^{(K)}, \mathfrak{X}) \qquad \forall\, \mathfrak{X} \in T\widetilde{\Omega}_\mu^{(K)},$$

we infer that $H_{\widetilde{p}_\mu^{(K)}}([\xi]) = \text{s-grad}\, \widetilde{p}_\mu^{(K)}([\xi]) = 0$, which means that $[\xi] \in \widetilde{\Sigma}_{\mu,c}^{(K)}$ is a stationary point for the reduced flow, so that $\widetilde{\varphi}_t^\mu([\xi]) = [\xi]$ for all $t \in \mathbb{R}$. By (6.2.1), this is equivalent to

$$\pi_\mu^{(K)} \circ \varphi_t \circ \iota_\mu^{(K)}(\xi') = \widetilde{\varphi}_t^\mu([\xi]) \qquad \forall\, t \in \mathbb{R},\ \xi' \in G_\mu \cdot \xi,$$

which in turn is equivalent to $\varphi_t \circ \iota_\mu^{(K)}(\xi') \in G_\mu \cdot \xi'$. Thus, there exists a G_μ-orbit in $\mathbb{J}^{-1}(\{\mu\}) \cap G_\mu \cdot \mathbf{X}_K^\eta \cap \Sigma_c$ which is invariant under φ_t. In particular one has $H_p(\xi') \in \mathfrak{g}_\mu \cdot \xi'$ for all $\xi' \in G_\mu \cdot \xi$.

Assumption 1 ensures that $\widetilde{\Sigma}_{\mu,c}^{(K)}$ is a smooth submanifold of $\widetilde{\Omega}_\mu^{(K)}$. Equipping $\widetilde{\Omega}_\mu^{(K)}$ with the symplectic volume form defined by the unique symplectic form on $\widetilde{\Omega}_\mu^{(K)}$ described in Section 6.2, there is a unique induced Liouville measure $\nu_{\mu,c}^{(K)}$ on $\widetilde{\Sigma}_{\mu,c}^{(K)}$, compare Corollary 6.3.9. Moreover, $\nu_{\mu,c}^{(K)}$ is invariant under the reduced flow $\widetilde{\varphi}_t^\mu$, since the latter constitutes a symplectomorphism due to Cartan's homotopy formula. Suppose now that the hypersurface $\widetilde{\Sigma}_{\mu,c}^{(K)}$ has finite volume with respect to the measure $\nu_{\mu,c}^{(K)}$. It is then natural to make the following

Definition 8.1.1. *The reduced flow $\widetilde{\varphi}_t^\mu$ is called* ergodic *on $\widetilde{\Sigma}_{\mu,c}^{(K)}$ if for any measurable subset $A \subset \widetilde{\Sigma}_{\mu,c}^{(K)}$ with $\widetilde{\varphi}_t^\mu(A) = A$ one has*

$$\nu_{\mu,c}^{(K)}(A) = 0 \qquad \text{or} \qquad \nu_{\mu,c}^{(K)}(A) = \nu_{\mu,c}^{(K)}(\widetilde{\Sigma}_{\mu,c}^{(K)}).$$

We can now formulate

Theorem 8.1.1 (Symmetry-reduced mean ergodic theorem). *Let Assumption 1 above be fulfilled, and suppose that $\widetilde{\Sigma}_{\mu,c}^{(K)}$ has finite, non-zero volume with respect to its hypersurface measure $\nu_{\mu,c}^{(K)}$, and that the reduced flow $\widetilde{\varphi}_t^\mu$ is ergodic on $\widetilde{\Sigma}_{\mu,c}^{(K)}$. Then, for each $f \in \mathrm{L}^2(\widetilde{\Sigma}_{\mu,c}^{(K)}, d\nu_{\mu,c}^{(K)})$ we have*

$$\langle f \rangle_T \quad \stackrel{T\to\infty}{\longrightarrow} \quad \frac{1}{\nu_{\mu,c}^{(K)}(\widetilde{\Sigma}_{\mu,c}^{(K)})} \int\limits_{\widetilde{\Sigma}_{\mu,c}^{(K)}} f \, d\nu_{\mu,c}^{(K)}$$

with respect to the norm topology of $\mathrm{L}^2(\widetilde{\Sigma}_{\mu,c}^{(K)}, d\nu_{\mu,c}^{(K)})$, where

$$\langle f \rangle_T ([\mu]) := \frac{1}{T} \int\limits_0^T f\left(\widetilde{\varphi}_t^\mu([\mu])\right) dt, \quad [\mu] \in \widetilde{\Sigma}_{\mu,c}^{(K)}.$$

Proof. The proof is completely analogous to the existing proofs of the classical mean ergodic theorem, compare e.g. [63, Theorem 15.1]. □

In what follows, we shall apply the general results outlined above to the case where $\mathbf{X} = T^*M$ with M and G as in Chapter 5, $\mu = 0$, $K = H$ is given by a principal isotropy group, and p is the Hamiltonian function (5.1.7). We shall then use the simpler notation

$$\widetilde{\Omega}_{\text{reg}} = \widetilde{\Omega}_0^{(H)}, \quad \widetilde{\varphi}_t = \widetilde{\varphi}_t^0, \quad \widetilde{\Sigma}_c = \widetilde{\Sigma}_{0,c}^{(H)}, \quad d\widetilde{\Sigma}_c = d\nu_{0,c}^{(H)}, \quad \widetilde{p} = \widetilde{p}_0^{(H)}.$$

As a special case of Theorem 8.1.1 we get the following

Theorem 8.1.2. *Suppose that the reduced flow $\widetilde{\varphi}_t$ is ergodic on $\big(\widetilde{\Sigma}_c, d\widetilde{\Sigma}_c\big)$. Then for each $f \in \mathrm{L}^2\big(\widetilde{\Sigma}_c, d\widetilde{\Sigma}_c\big)$,*

$$\lim_{T\to\infty} \int_{\widetilde{\Sigma}_c} \Big(\langle f \rangle_T - \fint_{\widetilde{\Sigma}_c} f \, d\widetilde{\Sigma}_c \Big)^2 d\widetilde{\Sigma}_c = 0.$$

Remark 8.1.3. Note that if $\widetilde{M}$ is an orbifold, the ergodicity of the reduced flow $\widetilde{\varphi}_t$ on $\big(\widetilde{\Sigma}_c, d\widetilde{\Sigma}_c\big)$ is equivalent to the ergodicity of the corresponding Hamiltonian flow on the orbifold bundle $S^*_{\widetilde{p},c}(\widetilde{M}) = \big\{(x,\xi) \in T^*(\widetilde{M}) : \widetilde{p}(x,\xi) = c\big\}$ with respect to Liouville measure.

Next, we examine the relation between classical time evolution and symmetry reduction. Let $a \in \mathrm{C}^\infty(T^*M)$. For a G-equivariant diffeomorphism $\Phi : T^*M \to T^*M$, we have

$$\langle a \circ \Phi \rangle_G (\eta) = \int_G a(\Phi(g\cdot\eta))\, dg = \int_G a(g \cdot \Phi(\eta))\, dg = \langle a \rangle_G (\Phi(\eta)),$$

so that $\langle a\circ\Phi\rangle_G = \langle a\rangle_G \circ \Phi$ and consequently $(\langle a\circ\Phi\rangle_G)\widetilde{\ } = (\langle a\rangle_G\circ\Phi)\widetilde{\ }$ holds. Now, we apply this result to the case $\Phi = \varphi_t$, where φ_t is the Hamiltonian flow associated to the symbol function p of the Schrödinger operator. If $i : \Omega_{\mathrm{reg}} \hookrightarrow T^*M$ denotes the inclusion and $\pi : \Omega_{\mathrm{reg}} \to \widetilde{\Omega}_{\mathrm{reg}}$ the projection onto the G-orbit space, we have $\pi \circ \varphi_t \circ i = \widetilde{\varphi}_t \circ \pi$. Since

$$\langle a\rangle_G \circ \varphi_t \circ i = (\langle a\rangle_G\circ\varphi_t)\widetilde{\ } \circ \pi, \qquad \langle a\rangle_G \circ i = \widetilde{\langle a\rangle}_G \circ \pi,$$

we get

$$\widetilde{\langle a\rangle}_G \circ \widetilde{\varphi}_t \circ \pi = \widetilde{\langle a\rangle}_G \circ \pi \circ \varphi_t \circ i = \langle a\rangle_G \circ i \circ \varphi_t \circ i = \langle a\rangle_G \circ \varphi_t \circ i = (\langle a\rangle_G \circ \varphi_t)\widetilde{\ } \circ \pi,$$

where we used that $i \circ \varphi_t \circ i = \varphi_t \circ i$. Since π is surjective, we have shown

Lemma 8.1.4. *Let $a \in \mathrm{C}^\infty(T^*M)$ and φ_t be the flow on T^*M associated to the Hamiltonian p. Let $\widetilde{\varphi}_t$ be the reduced flow on $\widetilde{\Omega}_{\mathrm{reg}}$ associated to $\widetilde{p}$. Then time evolution and reduction commute:*

$$(\langle a\rangle_G \circ \varphi_t)\widetilde{\ } = \widetilde{\langle a\rangle}_G \circ \widetilde{\varphi}_t.$$

8.2 Equivariant quantum ergodicity

We are now ready to formulate our first quantum ergodic theorem in a symmetry-reduced context. Let the notation be as in the previous sections and Chapter 5.

Theorem 8.2.1 (Integrated equivariant quantum ergodicity). *Suppose that the reduced flow $\widetilde{\varphi}_t$ corresponding to the reduced Hamiltonian function $\widetilde{p}$ is ergodic on $\widetilde{\Sigma}_c = \widetilde{p}^{-1}(\{c\})$. Let $A \in \Psi^0_h(M)$ be a semiclassical pseudodifferential operator with principal symbol $\sigma(A) = [a]$, where $a \in S^0_h(M)$ is independent of h. For a number $\delta \in \big(0, \frac{1}{2\kappa+4}\big)$ and a semiclassical character family $\{\mathcal{W}_h\}_{h\in(0,1]}$ with growth rate $\vartheta < \frac{1-(2\kappa+4)\delta}{2\kappa+3}$ set*

$$J(h) := \{j \in \mathbb{N} : E_j(h) \in [c, c+h^\delta],\ \chi_j(h) \in \mathcal{W}_h\},$$

where $\chi_j(h)$ is defined by $u_j(h) \in \mathrm{L}^2_{\chi_j(h)}(M)$. Then, one has

$$\lim_{h\to 0} \frac{h^{n-\kappa-\delta}}{\#\mathcal{W}_h} \sum_{J(h)} \frac{1}{d_{\chi_j(h)}[\pi_{\chi_j(h)}|_H : \mathbb{1}]} \left| \langle A u_j(h), u_j(h)\rangle_{\mathrm{L}^2(M)} - \fint_{\Sigma_c\cap\Omega_{\mathrm{reg}}} a \frac{d\mu_c}{\mathrm{vol}\,\mathcal{O}} \right|^2 = 0. \quad (8.2.1)$$

Remark 8.2.2. Again, the integral in (8.2.1) can also we written as $⨍_{\widetilde{\Sigma}_c} \widetilde{\langle a \rangle}_G \, d\widetilde{\Sigma}_c$, and if $\widetilde{M}$ is an orbifold, it can be written as an integral over $S^*_{\widetilde{p},c}(\widetilde{M})$, compare Remark 7.2.2.

Proof. We shall adapt the existing proofs of quantum ergodicity to the equivariant situation, following mainly [63, Theorem 15.4]. Let us write $u_j(h) = u_j$ and $E_j(h) = E_j$, and choose a function $\varrho \in \mathrm{C^\infty_c}(\mathbb{R}, [0,1])$ with $\varrho \equiv 1$ in a neighborhood of c. Without loss of generality we may assume for the rest of the proof that h is small enough so that $\varrho \equiv 1$ on $[c, c + h^\delta]$. Set

$$B := \varrho(P(h)) \circ \big(A - \alpha \, \mathbf{1}_{\mathrm{L}^2(M)}\big), \qquad \alpha := ⨍_{\Sigma_c \cap \Omega_{\mathrm{reg}}} a \, \frac{d\mu_c}{\mathrm{vol}\, \mathcal{O}} = ⨍_{\widetilde{\Sigma}_c} \widetilde{\langle a \rangle}_G \, d\widetilde{\Sigma}_c, \tag{8.2.2}$$

where $\widetilde{\langle a \rangle}_G$ was defined in (5.2.1). Note that by the semiclassical calculus we have $B \in \Psi_h^{-\infty}(M)$. Furthermore,

$$\sigma(B) = (\varrho \circ \sigma(P(h))) \, \sigma\left(A - \alpha \, \mathbf{1}_{\mathrm{L}^2(M)}\right) = [(\varrho \circ p) \, (a - \alpha \, 1_{T^*M})] \in S_h^{-\infty}(M)/hS_h^{-\infty}(M).$$

Let us write $b := (\varrho \circ p) \, (a - \alpha \, 1_{T^*M})$, so that $\sigma(B) = [b]$. Clearly,

$$\begin{aligned} \widetilde{\langle b \rangle}_G &= ((\varrho \circ p) \, (\langle a \rangle_G - \alpha \, 1_{T^*M}))^{\widetilde{}} \\ &= (\varrho \circ \widetilde{p}) \, (\langle a \rangle_G - \alpha \, 1_{T^*M})^{\widetilde{}} = (\varrho \circ \widetilde{p}) \left(\widetilde{\langle a \rangle}_G - \alpha \, 1_{\widetilde{\Omega}_{\mathrm{reg}}}\right). \end{aligned} \tag{8.2.3}$$

Next, we define

$$\mathcal{L}(h) := \frac{(2\pi)^{n-\kappa} h^{n-\kappa-\delta}}{\# \mathcal{W}_h} \sum_{J(h)} \frac{1}{d_{\chi_j(h)} [\pi_{\chi_j(h)}|_H : \mathbb{1}]} \left| \langle B u_j, u_j \rangle_{\mathrm{L}^2(M)} \right|^2. \tag{8.2.4}$$

By the spectral theorem, $\varrho(P(h)) u_j = u_j$ for $E_j \in [c, c + h^\delta]$, since $\varrho \equiv 1$ on $[c, c + h^\delta]$. Taking into account the self-adjointness of $\varrho(P(h))$ one sees that for $E_j \in [c, c + h^\delta]$

$$\langle B u_j, u_j \rangle_{\mathrm{L}^2(M)} = \langle A u_j, u_j \rangle_{\mathrm{L}^2(M)} - \alpha. \tag{8.2.5}$$

Consequently, we will be done with the proof if we can show that

$$\lim_{h \to 0} \mathcal{L}(h) = 0. \tag{8.2.6}$$

In order to do so, one considers the time evolution operator

$$F^h(t) : \mathrm{L}^2(M) \to \mathrm{L}^2(M), \quad F^h(t) := e^{-itP(h)/h}, \qquad t \in \mathbb{R},$$

which by Stone's theorem [60, Section XI.13] is a well-defined bounded operator. One then sets

$$B(t) := F^h(t)^{-1} B F^h(t).$$

In order to make use of classical ergodicity, one notes that the expectation value

$$\begin{aligned} \langle B u_j, u_j \rangle_{\mathrm{L}^2(M)} &= \left\langle B e^{-itE_j/h} u_j, e^{-itE_j/h} u_j \right\rangle_{\mathrm{L}^2(M)} = \left\langle B e^{-itP(h)/h} u_j, e^{-itP(h)/h} u_j \right\rangle_{\mathrm{L}^2(M)} \\ &= \langle B(t) u_j, u_j \rangle_{\mathrm{L}^2(M)}, \qquad t \in [0, T], \end{aligned}$$

is actually time-independent. This implies for each $T > 0$

$$\langle Bu_j, u_j\rangle_{L^2(M)} = \langle \langle B\rangle_T u_j, u_j\rangle_{L^2(M)},$$

where we set $\langle B\rangle_T = \frac{1}{T}\int_0^T B(t)dt \in \Psi_h^{-\infty}(M)$. Taking into account $\|u_j\|^2_{L^2(M)} = 1$ and the Cauchy-Schwarz inequality one arrives at

$$\left|\langle Bu_j, u_j\rangle_{L^2(M)}\right|^2 \leq \|\langle B\rangle_T u_j\|^2_{L^2(M)}.$$

We therefore conclude from (8.2.4) for each $T > 0$ that

$$\left|\mathcal{L}(h)\right| \leq \frac{(2\pi)^{n-\kappa}h^{n-\kappa-\delta}}{\#\mathcal{W}_h}\sum_{J(h)}\frac{1}{d_{\chi_j(h)}[\pi_{\chi_j(h)}|_H : \mathbb{1}]}\;\langle\langle B^*\rangle_T\langle B\rangle_T u_j, u_j\rangle_{L^2(M)}. \tag{8.2.7}$$

Next, let $\overline{B}(t)$ be an element in $\Psi_h^{-\infty}(M)$ with principal symbol $\sigma(B)\circ\varphi_t$. By the weak Egorov theorem [63, Theorem 15.2] one has

$$\left\|B(t) - \overline{B}(t)\right\|_{\mathcal{B}(L^2(M))} = \mathrm{O}(h) \quad \text{uniformly for } t \in [0,T],$$

which implies

$$\langle B\rangle_T = \langle\overline{B}\rangle_T + \mathrm{O}^T_{\mathcal{B}(L^2(M))}(h). \tag{8.2.8}$$

From the definition of $\overline{B}$ we get

$$\sigma\left(\langle\overline{B}\rangle_T\right) = \left[\frac{1}{T}\int_0^T b\circ\varphi_t\,dt\right].$$

Furthermore, the symbol map is a $*$-algebra homomorphism from $\Psi_h^{-\infty}(M)$ with involution given by the adjoint operation to $S_h^{-\infty}(M)/hS_h^{-\infty}(M)$ with involution given by pointwise complex conjugation. That leads to

$$\sigma\Big(\langle\overline{B}^*\rangle_T\langle\overline{B}\rangle_T\Big) = \left[\Big|\frac{1}{T}\int_0^T b\circ\varphi_t\,dt\Big|^2\right].$$

Now, note that by Lemma 8.1.4

$$\left\langle\frac{1}{T}\int_0^T b\circ\varphi_t\,dt\right\rangle_G^{\sim} = \frac{1}{T}\int_0^T\left(\langle b\rangle_G\circ\varphi_t\right)^{\sim}\,dt = \frac{1}{T}\int_0^T\widetilde{\langle b\rangle}_G\circ\widetilde{\varphi}_t\,dt = \langle\widetilde{\langle b\rangle}_G\rangle_T, \tag{8.2.9}$$

which is where the transition from the flow φ_t to the reduced flow $\widetilde{\varphi}_t$ takes place. We can then apply the generalized equivariant Weyl law, Theorem 7.2.1, which together with (8.2.9) yields

$$\begin{aligned}\frac{(2\pi)^{n-\kappa}h^{n-\kappa-\delta}}{\#\mathcal{W}_h}&\sum_{J(h)}\frac{1}{d_{\chi_j(h)}[\pi_{\chi_j(h)}|_H : \mathbb{1}]}\;\langle\langle\overline{B}^*\rangle_T\langle\overline{B}\rangle_T u_j, u_j\rangle_{L^2(M)}\\ &= \int_{\widetilde{\Sigma}_c}|\langle\widetilde{\langle b\rangle}_G\rangle_T|^2\,d\widetilde{\Sigma}_c + \mathrm{O}_T\Big(h^\delta + h^{\frac{1-(2\kappa+3)\vartheta}{2\kappa+4}-\delta}(\log h^{-1})^{\Lambda^G_M-1}\Big).\end{aligned} \tag{8.2.10}$$

From (8.2.3) we see that over $\widetilde{\Sigma}_c = \widetilde{p}^{-1}(\{c\})$ we have $\widetilde{\langle b\rangle}_G|_{\widetilde{\Sigma}_c} = \widetilde{\langle a\rangle}_G|_{\widetilde{\Sigma}_c} - \alpha\cdot 1_{\widetilde{\Sigma}_c} =: \widetilde{b}_c$. With (8.2.7), (8.2.8) and (8.2.10) we deduce for each $T>0$

$$\begin{aligned}\big|\mathcal{L}(h)\big| \leq \int\limits_{\widetilde{\Sigma}_c} |\langle\widetilde{b}_c\rangle_T|^2\, d\widetilde{\Sigma}_c + \mathrm{O}_T\Big(h^\delta + h^{\frac{1-(2\kappa+3)\vartheta}{2\kappa+4}-\delta}(\log h^{-1})^{\Lambda_M^G - 1}\Big)\\ + \left[\frac{h^{n-\kappa-\delta}}{\#\mathcal{W}_h}\sum_{J(h)}\frac{1}{d_{\chi_j(h)}[\pi_{\chi_j(h)}|_H : \mathbb{1}]}\right]\cdot \mathrm{O}_T(h).\end{aligned}$$

By Theorem 7.2.1, the factor in front of the $\mathrm{O}_T(h)$-remainder is convergent and therefore bounded as $h\to 0$. Moreover, the number $\int_{\widetilde{\Sigma}_c}|\langle\widetilde{b}_c\rangle_T|^2\,d\widetilde{\Sigma}_c$ is independent of h, as we assume that a is independent of h. Thus,

$$\limsup_{h\to 0}\big|\mathcal{L}(h)\big| \leq \int\limits_{\widetilde{\Sigma}_c}|\langle\widetilde{b}_c\rangle_T|^2\,d\widetilde{\Sigma}_c \qquad \forall\, T>0. \tag{8.2.11}$$

This is now the point where symmetry-reduced classical ergodicity is used. Since $\widetilde{b}_c$ fulfills $\fint_{\widetilde{\Sigma}_c}\widetilde{b}_c\,d\widetilde{\Sigma}_c = 0$, Theorem 8.1.2 yields $\lim_{T\to\infty}\int_{\widetilde{\Sigma}_c}|\langle\widetilde{b}_c\rangle_T|^2 d\widetilde{\Sigma}_c = 0$. Because the left hand side of (8.2.11) is independent of T, it follows that it must be zero, yielding (8.2.6). □

Remark 8.2.3. Note that one could have still exhibited the Weyl law remainder estimate in (8.2.11). But since the rate of convergence in Theorem 8.1.2 is unknown in general, it is not possible to give a remainder estimate in Theorem 8.2.1 with the methods employed here. Nevertheless, in certain dynamical situations, the rate could probably be made explicit.

In the special case of a constant semiclassical character family, corresponding to the study of a single fixed isotypic component, we obtain as a direct consequence

Theorem 8.2.4 (Integrated equivariant quantum ergodicity for single isotypic components). *Suppose that the reduced flow $\widetilde{\varphi}_t$ corresponding to the reduced Hamiltonian function $\widetilde{p}$ is ergodic on $\widetilde{\Sigma}_c := \widetilde{p}^{-1}(\{c\})$. Let $A\in\Psi_h^0(M)$ be a semiclassical pseudodifferential operator with principal symbol $\sigma(A) = [a]$, where $a\in S_h^0(M)$ is independent of h. Choose $\delta\in\big(0,\frac{1}{2\kappa+4}\big)$ and $\chi\in\widehat{G}$. Then, one has*

$$\lim_{h\to 0} h^{n-\kappa-\delta}\sum_{J^\chi(h)}\Big|\langle Au_j(h),u_j(h)\rangle_{\mathrm{L}^2(M)} - \fint\limits_{\Sigma_c\cap\Omega_{\mathrm{reg}}} a\,\frac{d\mu_c}{\mathrm{vol}\,\mathcal{O}}\Big|^2 = 0, \tag{8.2.12}$$

where

$$J^\chi(h) := \big\{j\in\mathbb{N} : E_j(h)\in[c,c+h^\delta],\ u_j(h)\in \mathrm{L}^2_\chi(M)\big\}. \tag{8.2.13}$$

Remark 8.2.5. A weaker version of Theorem 8.2.4 can be proved with a less sharp energy localization in an interval $[r,s]$ with $r<s$ by the same methods employed here. In fact, under the additional assumption that the mean value α introduced in (8.2.2) is the same for all $c\in[r,s]$ and all considered c are regular values of p, the reduced flow being ergodic on each of the contemplated hypersurfaces $\widetilde{\Sigma}_c$, one can show that

$$\lim_{h\to 0} h^{n-\kappa}\sum_{\substack{j\in\mathbb{N}:\, u_j(h)\in \mathrm{L}^2_\chi(M),\\ E_j(h)\in[r,s]}}\Big|\langle Au_j(h),u_j(h)\rangle_{\mathrm{L}^2(M)} - \fint\limits_{p^{-1}([r,s])\cap\Omega_{\mathrm{reg}}} a\,\frac{d\Omega_{\mathrm{reg}}}{\mathrm{vol}\,\mathcal{O}}\Big|^2 = 0. \tag{8.2.14}$$

The proof of this relies on a corresponding semiclassical Weyl law for the interval $[r, s]$ and single isotypic components, see Remark 7.2.4. The point is that for the weaker statement (8.2.14) a remainder estimate of order $\mathrm{o}(h^{n-\kappa})$ is sufficient in Weyl's law, since the rate of convergence in (8.2.14) is the one of the leading term. Thus, in principle, this weaker result could have also been obtained using heat kernel methods as in [18] or [10] adapted to the semiclassical setting, at least for the Laplacian. Nevertheless, for the stronger version proved in Theorem 8.2.4, remainder estimates of order $\mathrm{O}(h^{n-\kappa-\delta})$ in Weyl's law and in particular the results of [46] are necessary.

In what follows, we shall use our previous results to prove the main result of this chapter, a symmetry-reduced quantum ergodicity theorem for Schrödinger operators.

Theorem 8.2.6 (Equivariant quantum ergodicity for Schrödinger operators). *With the notation and assumptions as in Theorem 8.2.1, there is a $h_0 \in (0,1]$ such that for each $h \in (0, h_0]$ we have two subsets $\Lambda^1(h)$, $\Lambda^2(h) \subset J(h)$ satisfying*

$$\lim_{h\to 0} \frac{\#\Lambda^1(h)}{\#J(h)} = 1, \qquad \lim_{h\to 0} \frac{\sum_{j\in\Lambda^2(h)} \frac{1}{d_{\chi_j(h)}[\pi_{\chi_j(h)}|_H:\mathbb{1}]}}{\sum_{j\in J(h)} \frac{1}{d_{\chi_j(h)}[\pi_{\chi_j(h)}|_H:\mathbb{1}]}} = 1, \tag{8.2.15}$$

such that for each semiclassical pseudodifferential operator $A \in \Psi_h^0(M)$ with principal symbol $\sigma(A) = [a]$, where a is h-independent, the following holds. For all $\varepsilon > 0$ there is a $h_\varepsilon \in (0, h_0]$ such that for all $h \in (0, h_\varepsilon]$ one has

$$\frac{1}{\sqrt{d_{\chi_j(h)}[\pi_{\chi_j(h)}|_H : \mathbb{1}]}} \Big| \langle A u_j(h), u_j(h)\rangle_{\mathrm{L}^2(M)} - \fint_{\Sigma_c\cap\Omega_{\mathrm{reg}}} a \, \frac{d\mu_c}{\operatorname{vol}\mathcal{O}} \Big| < \varepsilon \quad \forall\, j \in \Lambda^1(h), \tag{8.2.16}$$

$$\Big| \langle A u_j(h), u_j(h)\rangle_{\mathrm{L}^2(M)} - \fint_{\Sigma_c\cap\Omega_{\mathrm{reg}}} a \, \frac{d\mu_c}{\operatorname{vol}\mathcal{O}} \Big| < \varepsilon \quad \forall\, j \in \Lambda^2(h), \tag{8.2.17}$$

where the integrals in (8.2.16) and (8.2.17) equal $\fint_{\widetilde{\Sigma}_c} \widetilde{\langle a\rangle}_G \, d\widetilde{\Sigma}_c$.

Proof. Again, this proof is an adaptation of existing proofs like [63, Theorem 15.5] to the equivariant setting, only that we do not need the technical condition that the value of the integral $\fint_{\widetilde{\Sigma}_c} \widetilde{a}\, d\widetilde{\Sigma}_c$ must stay the same when varying c in some interval, which slightly simplifies the proof. On the other hand, the consideration of semiclassical character families adds a few subtleties.

Write $u_j(h) = u_j$ and $E_j(h) = E_j$. By Theorem 7.2.1 we can choose a $h_0 \in (0,1]$ such that $J(h) \neq \emptyset$ for all $h \in (0, h_0]$, and suppose that $h \in (0, h_0]$. With the notation as in (5.2.1), we set for any smooth function s on T^*M

$$\alpha(s) := \fint_{\widetilde{\Sigma}_c} \widetilde{\langle s\rangle}_G \, d\widetilde{\Sigma}_c.$$

Let $\tau \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}, [0,1])$ be such that $\tau \equiv 1$ in a neighborhood of c. Without loss of generality, we assume for the rest of the proof that h_0 is small enough so that $\tau \equiv 1$ on $[c, c + h_0^\delta]$. Now, for any operator A as in the statement of the theorem set

$$B := A - \alpha(a)\, \tau\left(P(h)\right).$$

By the semiclassical calculus we know that the principal symbol of B is given by $\sigma(B) = [b]$ with $b := a - \alpha(a)\, \tau \circ p$. Clearly, $\alpha(b) = 0$, since $\tau \circ \widetilde{p} \equiv 1$ on $\widetilde{\Sigma}_c$. Let us now assume that the statement of the theorem holds for all operators A with $\alpha(a) = 0$. Then, there are subsets $\Lambda^1(h)$, $\Lambda^2(h)$ fulfilling (8.2.15) such that for all $\varepsilon > 0$ there is a $h_\varepsilon \in (0, h_0]$ with

$$\sqrt{\Xi^i(j,h)}\,\left|\langle Bu_j, u_j\rangle_{\mathrm{L}^2(M)}\right| < \varepsilon \qquad \forall\, h \in (0, h_\varepsilon], \quad \forall\, j \in \Lambda^i(h),\ i \in \{1,2\}, \tag{8.2.18}$$

where we introduced the notation

$$\Xi^i(j,h) := \begin{cases} \frac{1}{d_{\chi_j(h)}[\pi_{\chi_j(h)}|_H:\mathbb{1}]}, & i = 1, \\ 1, & i = 2. \end{cases}$$

Due to the choice of the function τ we have $\tau\left(P(h)\right)(u_j) = u_j$ for all u_j with $E_j \in [c, c+h^\delta]$. Consequently, (8.2.18) implies for each $i \in \{1,2\}$ that for all $\varepsilon > 0$ there is $h_\varepsilon \in (0, h_0]$ such that

$$\sqrt{\Xi^i(j,h)}\,\left|\langle Au_j, u_j\rangle_{\mathrm{L}^2(M)} - \alpha(a)\right| < \varepsilon \qquad \forall\, h \in (0, h_\varepsilon], \quad \forall\, j \in \Lambda^i(h),$$

and we obtain the statement of the theorem for general A. We are therefore left with the task of proving (8.2.18) for arbitrary operators B with $\alpha(b) = 0$, and shall proceed in a similar fashion to parts 1 - 5 of the proof of [63, Theorem 15.5], pointing out only the main arguments. By Theorem 8.2.1 we have for fixed B

$$\frac{h^{n-\kappa-\delta}}{\#\mathcal{W}_h} \sum_{j \in J(h)} \frac{\left|\langle Bu_j, u_j\rangle_{\mathrm{L}^2(M)}\right|^2}{d_{\chi_j(h)}[\pi_{\chi_j(h)}|_H : \mathbb{1}]} =: r(h) \to 0$$

as $h \to 0$. We then define for $h \in (0, h_0]$ and $i \in \{1,2\}$ the following B-dependent subsets:

$$\Lambda^i(h) := J(h) - \Gamma^i(h), \qquad \Gamma^i(h) := \left\{ j \in J(h) : \Xi^i(j,h) |\langle Bu_j, u_j\rangle_{\mathrm{L}^2(M)}|^2 \geq \sqrt{r(h)} \right\}.$$

Clearly, with the notation

$$\widehat{i} := \begin{cases} 2, & i = 1, \\ 1, & i = 2, \end{cases}$$

one has for each $i \in \{1,2\}$

$$\sum_{j \in \Gamma^i(h)} \Xi_{\widehat{i}}(j,h) \leq \sum_{j \in J(h)} \frac{|\langle Bu_j, u_j\rangle_{\mathrm{L}^2(M)}|^2}{d_{\chi_j(h)}[\pi_{\chi_j(h)}|_H : \mathbb{1}]\,\sqrt{r(h)}} = \frac{\#\mathcal{W}_h\,\sqrt{r(h)}}{h^{n-\kappa-\delta}},$$

and taking $B = \mathbf{1}_{\mathrm{L}^2(M)}$ in Theorem 7.2.1 one computes

$$\begin{aligned} \frac{\#\mathcal{W}_h\,\sqrt{r(h)}}{h^{n-\kappa-\delta}\#J(h)} &\leq \frac{\#\mathcal{W}_h\sqrt{r(h)}}{h^{n-\kappa-\delta}\sum\limits_{j \in J(h)} \frac{1}{d_{\chi_j(h)}[\pi_{\chi_j(h)}|_H:\mathbb{1}]}} \\ &= \frac{\sqrt{r(h)}}{(2\pi)^{\kappa-n}\mathrm{vol}_{d\widetilde{\Sigma}_c}\widetilde{\Sigma}_c + \mathrm{O}\big(h^\delta + h^{\frac{1-(2\kappa+3)\vartheta}{2\kappa+4}-\delta}\left(\log h^{-1}\right)^{\Lambda^G_M-1}\big)} \longrightarrow 0. \end{aligned}$$

On the other hand,

$$\sqrt{\Xi^i(j,h)}\,\left|\langle Bu_j, u_j\rangle_{\mathrm{L}^2(M)}\right| < r(h)^{1/4} \qquad \forall\, j \in \Lambda^i(h),\ i \in \{1,2\},$$

yielding (8.2.18) for these particular $\Lambda^1(h), \Lambda^2(h)$ and B.

Consider now a family $\{A_k\}_{k\in\mathbb{N}}$ of semiclassical pseudodifferential operators in $\Psi_h^0(M)$ with principal symbols represented by h-independent symbol functions. By our previous considerations, for $i \in \{1,2\}$ and each k there are subsets $\Lambda_k^i(h) \subset J(h)$ such that (8.2.15-8.2.17) hold for each particular A_k and $\Lambda_k^i(h)$. One then shows that for sufficiently small h there are subsets $\Lambda_\infty^i(h) \subset J(h)$ fulfilling (8.2.15) such that $\Lambda_k^i(h) \subset \Lambda_\infty^i(h)$ for each k. Hence, the theorem is true for countable families of operators. To obtain it for all operators in $\Psi_h^{-\infty}(M)$, it suffices to find a sequence of operators $\{A_k\}_{k\in\mathbb{N}}$ which is dense in the set of operators in $\Psi_h^{-\infty}(M)$ whose principal symbols are represented by an h-independent symbol function, in the sense that for any given $A \in \Psi_h^{-\infty}(M)$ of the mentioned form and any $\varepsilon > 0$ there exists a k such that

$$\|A - A_k\|_{\mathcal{B}(\mathrm{L}^2(M))} < \varepsilon, \qquad \fint_{\widetilde{\Sigma}_c} \widetilde{\langle a - a_k\rangle_G}\, d\widetilde{\Sigma}_c < \varepsilon$$

for sufficiently small h. To find such a sequence $\{A_k\}_{k\in\mathbb{N}} \subset \Psi_h^{-\infty}(M)$, note that for two symbol functions a and b and the corresponding semiclassical quantizations A and B, one has

$$\|A - B\|_{\mathcal{B}(\mathrm{L}^2(M))} \leq \|a - b\|_{\mathrm{L}^\infty(T^*M)} + C\sqrt{h}, \qquad \fint_{\widetilde{\Sigma}_c} \widetilde{\langle a - b\rangle_G}\, d\widetilde{\Sigma}_c \leq C\|a - b\|_{\mathrm{L}^\infty(T^*M)}.$$

Consequently, we only need to find a sequence of h-independent symbol functions that is dense in $S_h^{-\infty}(M)$ equipped with the L^∞-norm. That such a sequence exists follows directly from the facts that $\mathrm{C}_\mathrm{c}^\infty(T^*M)$ is L^∞-norm dense in the Banach space $\mathrm{C}_0(T^*M) \supset S_h^{-\infty}(M)$ of continuous functions vanishing at infinity, and that $\mathrm{C}_\mathrm{c}^\infty(T^*M)$ is separable. This proves the theorem for operators A in $\Psi_h^{-\infty}(M)$ with principal symbol represented by an h-independent symbol function. Finally, if $A \in \Psi_h^0(M)$ is a general operator with principal symbol represented by an h-independent symbol function, one composes A with the smoothing operator $\varrho(P(h))$, where $\varrho \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R})$ equals 1 near c. This completes the proof of the theorem. □

Again, in the special case that $\mathcal{W}_h = \{\chi\}$ for all $h \in (0,1]$ and some fixed $\chi \in \widehat{G}$, we obtain a simpler statement:

Theorem 8.2.7 (Equivariant quantum ergodicity for Schrödinger operators and single isotypic components). *With the notation and assumptions as in Theorem 8.2.4, let $\chi \in \widehat{G}$, $\delta \in \left(0, \frac{1}{2\kappa+4}\right)$ be fixed, and let $J^\chi(h)$ be as in (8.2.13). Then, there is a $h_0 \in (0,1]$ such that for each $h \in (0, h_0]$ we have a subset $\Lambda^\chi(h) \subset J^\chi(h)$ satisfying*

$$\lim_{h\to 0} \frac{\#\Lambda^\chi(h)}{\#J^\chi(h)} = 1$$

such that for each semiclassical pseudodifferential operator $A \in \Psi_h^0(M)$ with principal symbol $\sigma(A) = [a]$, a being h-independent, the following holds. For all $\varepsilon > 0$ there is a $h_\varepsilon \in (0, h_0]$ such that

$$\left| \langle Au_j(h), u_j(h)\rangle_{\mathrm{L}^2(M)} - \fint_{\Sigma_c\cap\Omega_{\mathrm{reg}}} a \frac{d\mu_c}{\mathrm{vol}\,\mathcal{O}} \right| < \varepsilon \qquad \forall\, j \in \Lambda^\chi(h),\ \forall\, h \in (0, h_\varepsilon].$$

8.3 Equivariant quantum limits for the Laplacian

We shall now apply the semiclassical results from the previous section to study the distribution of eigenfunctions of the Laplace-Beltrami operator on a closed connected Riemannian G-manifold M in the limit of large eigenvalues, G being a compact connected Lie group acting isometrically and effectively on M, with principal orbits of dimension $\kappa < n = \dim M$. For simplicity of presentation, we will apply in this chapter only the version of Theorem 8.2.6 corresponding to the statement for the family $\Lambda^1(h)$, which means that we keep the classical notion of a density 1 family of subsets and introduce a representation theoretic correction factor in the convergence statements.

8.3.1 Eigenfunctions of the Laplace-Beltrami operator

Let Δ be the unique self-adjoint extension of the Laplace-Beltrami operator $\breve{\Delta}$ on M, and choose an orthonormal basis $\{u_j\}_{j\in\mathbb{N}}$ of $\mathrm{L}^2(M)$ of eigenfunctions of $-\Delta$ with corresponding eigenvalues $\{E_j\}_{j\in\mathbb{N}}$, repeated according to their multiplicity. Consider further the Schrödinger operator $P(h)$ given by (5.1.2) with $V \equiv 0$ and principal symbol defined by the symbol function $p = \|\cdot\|^2_{T^*M}$. Clearly, $P(h) = -h^2\Delta$, and each u_j is an eigenfunction of $P(h)$ with eigenvalue $E_j(h) = h^2 E_j$. Furthermore, under the identification $T^*M \simeq TM$ given by the Riemannian metric, the Hamiltonian flow φ_t induced by p corresponds to the geodesic flow of M. Each $c > 0$ is a regular value of p, and since $V \equiv 0$ the dynamics of the reduced geodesic flow $\widetilde{\varphi}_t$ are equivalent on any two hypersurfaces $\widetilde{\Sigma}_c$ and $\widetilde{\Sigma}_{c'}$. In the following, we shall therefore choose $c = 1$ without loss of generality, and call the reduced geodesic flow *ergodic* if it is ergodic on $\widetilde{\Sigma}_1 = \widetilde{p}^{-1}(\{1\})$. The following construction will allow a simpler formulation of the subsequent theorems.

Definition 8.3.1. *Let $\{a_j\}_{j\in\mathbb{N}}$ be a non-decreasing unbounded sequence of real numbers and $j_0 \in \mathbb{N}$ be an index such that $a_{j_0} > 0$. For $\delta > 0$, the* partition of $\{a_j\}_{j\in\mathbb{N}}$ of order δ *with* initial index j_0 *is the non-decreasing sequence $\mathcal{P} = \{\mathcal{P}(j)\}_{j\in\mathbb{N}} \subset \mathbb{N}$ defined as follows. Consider the subsequence $\{j_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ of indices given by the inductive rule*

$$j_1 = j_0, \qquad j_{k+1} := \min\big\{j \in \mathbb{N} : a_{j_k}(1 + a_{j_k}^{-\delta/2}) < a_j\big\}.$$

Then, $\mathcal{P}(j) := j_k$, where j_k is uniquely defined by $a_{j_k} \le a_j < a_{j_{k+1}}$.

Example 8.3.1. If $a_j = E_j = j(j+1)$, the j-th eigenvalue of the Laplacian on the standard 2-sphere S^2, then the partition of $\{E_j\}_{j\in\mathbb{N}}$ of order $\frac{1}{6}$ with initial index 1 is given by

$$\{j_k\}_{k\in\mathbb{N}} = \{1,2,3,5,7,10,14,\dots\}, \quad \{\mathcal{P}(j)\}_{j\in\mathbb{N}} = \{1,2,3,3,5,5,7,7,7,10,10,10,10,14,\dots\}.$$

We are now prepared to state and prove an equivariant version of the classical Shnirelman-Zelditch-Colin-de-Verdière quantum ergodicity theorem [52, 61, 15]. In the special case that $\widetilde{M} = M/G$ is an orbifold, a similar statement has been proved by Kordyukov [38] for the trivial isotypic component.

Theorem 8.3.2 (Equivariant quantum limits for the Laplacian). *With the notation as above, assume that the reduced geodesic flow is ergodic. Choose a semiclassical character family $\{\mathcal{W}_h\}_{h\in(0,1]}$ of growth rate $\vartheta < \frac{1}{2\kappa+3}$ and a partition $\mathcal{P}$ of $\{E_j\}_{j\in\mathbb{N}}$ of order $\delta \in \big(0, \frac{1-(2\kappa+3)\vartheta}{2\kappa+4}\big)$. Define the set of eigenfunctions*

$$\{u_i^{\mathcal{W},\mathcal{P}}\}_{i\in\mathbb{N}} := \{u_j : \chi_j \in \mathcal{W}_{E_{\mathcal{P}(j)}^{-1/2}}\},$$

where χ_j is defined by $u_j \in \mathrm{L}^2_{\chi_j}(M)$. Define $\chi_i^{\mathcal{W},\mathcal{P}}$ by $u_i^{\mathcal{W},\mathcal{P}} \in \mathrm{L}^2_{\chi_i^{\mathcal{W},\mathcal{P}}}(M)$. Then, there is a subsequence $\{u_{i_k}^{\mathcal{W},\mathcal{P}}\}_{k\in\mathbb{N}}$ of[1] *density 1 in $\{u_i^{\mathcal{W},\mathcal{P}}\}_{i\in\mathbb{N}}$ such that for all $s \in \mathrm{C}^\infty(S^*M)$ one has in the limit $k \to \infty$*

$$\frac{1}{\sqrt{d_{\chi_{i_k}^{\mathcal{W},\mathcal{P}}}[\pi_{\chi_{i_k}^{\mathcal{W},\mathcal{P}}}|_H : \mathbb{1}]}} \Big| \Big\langle \mathrm{Op}(s)u_{i_k}^{\mathcal{W},\mathcal{P}}, u_{i_k}^{\mathcal{W},\mathcal{P}} \Big\rangle_{\mathrm{L}^2(M)} - \fint_{S^*M\cap\Omega_{\mathrm{reg}}} s \,\frac{d\mu}{\mathrm{vol}\,\mathcal{O}} \Big| \longrightarrow 0, \tag{8.3.1}$$

where we wrote μ for μ_1 and Op *for* Op_1*, which is the ordinary non-semiclassical quantization, see Chapter 6.*

Remark 8.3.3. While the maximal order of the partition $\mathcal{P}$ is restricted in the theorem, the initial index is arbitrary. In fact, the need to partition the eigenfunction sequence to apply our results is just an artefact from the semiclassical formulation of the original general theorems.

Remark 8.3.4. The integral in (8.3.1) can also be written as $\fint_{S^*\widetilde{M}_{\mathrm{reg}}} s'\, d(S^*\widetilde{M}_{\mathrm{reg}})$, where $s' \in \mathrm{C}^\infty(S^*\widetilde{M}_{\mathrm{reg}})$ is the function corresponding to $\widetilde{\langle s\rangle}_G$ under the diffeomorphism $\widetilde{\Sigma}_1 \simeq S^*\widetilde{M}_{\mathrm{reg}}$ up to a null set, and $d(S^*\widetilde{M}_{\mathrm{reg}})$ is the Liouville measure on the unit co-sphere bundle, see Lemma 6.2.3, Corollary 6.3.3, and Remark 6.3.10. In the orbifold case, this integral is given by an integral over the orbifold co-sphere bundle $S^*\widetilde{M}$.

Proof. First, we extend s to a function $\overline{s} \in S^0_h(M) \subset \mathrm{C}^\infty(T^*M)$ with $\overline{s}|_{S^*M} = s$ as follows. Set $\widehat{\overline{s}}(x,\xi) := s(x, \xi/\|\xi\|_x)$ for $x \in M$, $\xi \in T^*_xM - \{0\}$. Choose a small $\delta > 0$ and a smooth cut-off function $\varphi : T^*M \to [0,1]$ with

$$\begin{aligned} \varphi(x,\xi) &= 1 \qquad \forall\, x \in M, \qquad \forall\, \xi \in T^*_xM \text{ with } \|\xi\|_x \geq 1-\delta, \\ \varphi(x,\xi) &= 0 \qquad \forall\, x \in M, \qquad \forall\, \xi \in T^*_xM \text{ with } \|\xi\|_x \leq \delta. \end{aligned}$$

Now set $\overline{s}(x,\xi) := \varphi(x,\xi)\cdot\widehat{\overline{s}}(x,\xi)$ for $\xi \in T^*_xM - \{0\}$ and $\overline{s}(x,0) := 0$. Then $\mathrm{Op}(\overline{s})$ is a pseudodifferential operator in $\Psi^0(M)$. Because $\overline{s}$ is polyhomogenous of degree 0 and therefore independent of $|\xi|$ for large ξ, the ordinary non-semiclassical quantization $\mathrm{Op}(\overline{s})$ differs only by an operator in $h^\infty\Psi_h^{-\infty}(M)$ from the semiclassical pseudodifferential operator $\mathrm{Op}_h(\overline{s}) \in \Psi^0_h(M)$ with principal symbol $\sigma(\mathrm{Op}_h(\overline{s})) = [\overline{s}]$. Thus, when applying Theorem 8.2.6 to $P(h) = -h^2\Delta$, we are allowed to replace $\mathrm{Op}_h(\overline{s})$ by $\mathrm{Op}(\overline{s})$ in the results. Fix some $\delta \in \big(0, \frac{1-(2\kappa+3)\vartheta}{2\kappa+4}\big)$. With $c = 1$ and $E_j(h) = h^2E_j$ one has

$$\begin{aligned} J(h) &= \{j \in \mathbb{N} : E_j(h) \in [c, c+h^\delta],\ \chi_j(h) \in \mathcal{W}_h\} \\ &= \left\{ j \in \mathbb{N} : E_j \in \left[\frac{1}{h^2}, \frac{1}{h^2} + \frac{1}{h^{2-\delta}}\right],\ \chi_j \in \mathcal{W}_h \right\}. \end{aligned}$$

Now, by Theorem 8.2.6, there is a number $h_0 \in (0,1]$ together with subsets $\Lambda(h) := \Lambda^1(h) \subset J(h)$, $h \in (0, h_0]$, satisfying

$$\lim_{h\to 0} \frac{\#\Lambda(h)}{\#J(h)} = 1, \tag{8.3.2}$$

and for each $s \in \mathrm{C}^\infty(S^*M)$ and arbitrary $\varepsilon > 0$ there is a $h_\varepsilon \in (0, h_0]$ such that

$$\frac{1}{\sqrt{d_{\chi_j}[\pi_{\chi_j}|_H : \mathbb{1}]}} \Big| \langle \mathrm{Op}(\overline{s})u_j, u_j\rangle_{\mathrm{L}^2(M)} - \fint_{S^*M\cap\Omega_{\mathrm{reg}}} s\,\frac{d\mu_1}{\mathrm{vol}\,\mathcal{O}} \Big| < \varepsilon \qquad \forall\, j \in \Lambda(h),\ \forall\, h \in (0, h_\varepsilon]. \tag{8.3.3}$$

[1] The expression *of density 1* means that $\lim_{m\to\infty} \#\{k : i_k \leq m\}/m = 1$.

Next, consider the partition $\mathcal{P}$ of $\{E_j\}_{j\in\mathbb{N}}$ of order δ with j_0, j_k, $\mathcal{P}(j)$ as in Definition 8.3.1. Since there are only finitely many eigenvalues E_j with $j < j_0$ and $h_0 < \frac{1}{\sqrt{E_j}}$, there is a $k_0 \in \mathbb{N}$ such that $h_k := \frac{1}{\sqrt{E_{j_k}}} \leq h_0$ for all $k \geq k_0$. Let us apply the results above to the sequence $\{h_k\}_{k\geq k_0}$. By construction, $k \neq k'$ implies $J(h_k) \cap J(h_{k'}) = \emptyset$ since

$$\begin{aligned} J(h_k) &= \left\{ j \in \mathbb{N} : E_j \in \left[E_{j_k}, E_{j_k}\big(1 + E_{j_k}^{-\delta/2}\big)\right],\ \chi_j \in \mathcal{W}_{E_{j_k}^{-1/2}} \right\} \\ &= \left\{ j \in \mathbb{N} : E_j \in \big[E_{j_k}, E_{j_{k+1}}\big),\ \chi_j \in \mathcal{W}_{E_{\mathcal{P}(j)}^{-1/2}} \right\}. \end{aligned}$$

Now, if $(a_q)_{q\in\mathbb{N}}$ and $(b_q)_{q\in\mathbb{N}}$ are sequences of real numbers such that $0 < a_q \leq b_q$ for all q, and $\liminf_{q\to\infty} b_q > 0$, $\lim_{q\to\infty} \frac{a_q}{b_q} = 1$, the Stolz-Cesaro lemma implies that

$$\lim_{N\to\infty} \frac{\sum_{q=1}^N a_q}{\sum_{q=1}^N b_q} = 1.$$

Applied to our situation and taking into account that $J(h_k) \cap J(h_{k'}) = \emptyset$ when $k \neq k'$ we deduce from (8.3.2) that

$$\lim_{N\to\infty} \frac{\#\bigcup_{k=k_0}^N \Lambda(h_k)}{\#\bigcup_{k=k_0}^N J(h_k)} = \lim_{N\to\infty} \frac{\sum_{k=k_0}^N \#\Lambda(h_k)}{\sum_{k=k_0}^N \#J(h_k)} = 1. \tag{8.3.4}$$

If we therefore set

$$J := \bigcup_{k\geq k_0} J(h_k) = \left\{ j \in \mathbb{N} : j \geq j_0,\ \frac{1}{\sqrt{E_j}} \leq h_0,\ \chi_j \in \mathcal{W}_{E_{\mathcal{P}(j)}^{-1/2}} \right\}, \qquad \Lambda := \bigcup_{k\geq k_0} \Lambda(h_k),$$

we obtain from (8.3.4)

$$\lim_{N\to\infty} \frac{\#\{\lambda \in \Lambda : \lambda \leq N\}}{\#\{j \in J : j \leq N\}} = 1.$$

Consequently, $\{i_k\}_{k\in\mathbb{N}} := \Lambda$ is a density 1 subsequence of $\{j \in \mathbb{N} : \chi_j \in \mathcal{W}_{E_{\mathcal{P}(j)}^{-1/2}}\}$, since the latter set differs from J by only finitely many elements. From (8.3.3) we conclude that the sequence $\{u_{i_k}^{\mathcal{W},\mathcal{P}}\}_{k\in\mathbb{N}}$ fulfills (8.3.1), completing the proof. $\square$

Projecting from $S^*M \cap \Omega_{\text{reg}}$ onto M we obtain

Corollary 8.3.5 (Equidistribution of eigenfunctions of the Laplacian). *In the situation of Theorem 8.3.2, we have for any $f \in \mathrm{C}(M)$*

$$\frac{1}{\sqrt{d_{\chi_{i_k}^{\mathcal{W},\mathcal{P}}}[\pi_{\chi_{i_k}^{\mathcal{W},\mathcal{P}}}|_H : \mathbb{1}]}} \left| \int_M f|u_{i_k}^{\mathcal{W},\mathcal{P}}|^2 dM - \fint_M f\, \frac{dM}{\operatorname{vol}\mathcal{O}} \right| \longrightarrow 0 \qquad \text{as } k \to \infty.$$

Proof. Let $\pi : T^*M \to M$ be the co-tangent bundle projection and consider for $f \in \mathrm{C}^\infty(M)$ the pseudodifferential operator $\operatorname{Op}(f\circ\pi)$, which corresponds to pointwise multiplication with f up to lower order terms. Since the Sasaki metric on T^*M projects onto the Riemannian metric on M and is fiber-wise just the euclidean metric, and the Sasaki metric induces $d\mu$, we have

$$\fint_{S^*M\cap\Omega_{\text{reg}}} f\circ\pi\, \frac{d\mu}{\operatorname{vol}\mathcal{O}} = \fint_M f\, \frac{dM}{\operatorname{vol}\mathcal{O}},$$

see Corollary 6.3.13. Consequently, the assertion follows directly from Theorem 8.3.2 by approximating continuous functions on M by smooth functions. $\square$

8.3.2 Limits of representations

Corollary 8.3.5 immediately leads to a statement about measures on the topological Hausdorff space $\widetilde{M} = M/G$ and to a representation-theoretic formulation of our results.

Corollary 8.3.6. *In the situation of Theorem 8.3.2, we have for any $f \in \mathrm{C}(\widetilde{M})$*

$$\frac{1}{\sqrt{d_{\chi^{\mathcal{W},\mathcal{P}}_{i_k}}[\pi_{\chi^{\mathcal{W},\mathcal{P}}_{i_k}}|_H : \mathbb{1}]}} \left| \int_{\widetilde{M}} f \widetilde{\left\langle |u^{\mathcal{W},\mathcal{P}}_{i_k}|^2 \right\rangle_G} d\widetilde{M} - \fint_{\widetilde{M}} f \frac{d\widetilde{M}}{\mathrm{vol}} \right| \longrightarrow 0 \qquad \text{as } k \to \infty.$$

Proof. Let $f \in \mathrm{C}(\widetilde{M})$, $\pi : M \to \widetilde{M}$ be the canonical projection, and denote by $\overline{f} := f \circ \pi \in \mathrm{C}(M)$ the lift of f to a G-invariant function. With (6.3.3) and Corollary 6.3.3 one deduces for any $u \in \mathrm{C}^\infty(M)$

$$\begin{aligned}
\int_M \overline{f}(x)|u(x)|^2\, dM(x) &= \int_{M_{\mathrm{reg}}} \overline{f}(x)|u(x)|^2\, dM(x) = \int_{\widetilde{M}_{\mathrm{reg}}} \int_{G\cdot x} \overline{f}(x')|u(x')|^2\, d\mu_{G\cdot x}(x')\, d\widetilde{M}_{\mathrm{reg}}(G\cdot x)\\
&= \int_{\widetilde{M}_{\mathrm{reg}}} f(G\cdot x) \int_{G\cdot x} |u(x')|^2\, d\mu_{G\cdot x}(x')\, d\widetilde{M}_{\mathrm{reg}}(G\cdot x)\\
&= \int_{\widetilde{M}_{\mathrm{reg}}} f(G\cdot x)\mathrm{vol}\,(G\cdot x) \int_G |u(g\cdot x)|^2\, dg\, d\widetilde{M}_{\mathrm{reg}}(G\cdot x)\\
&= \int_{\widetilde{M}} f(G\cdot x)\widetilde{\langle |u|^2\rangle_G}(G\cdot x)\, d\widetilde{M}(G\cdot x),
\end{aligned}$$

as well as $\fint_M \overline{f}\, \frac{dM}{\mathrm{vol}\,\mathcal{O}} = \fint_{\widetilde{M}_{\mathrm{reg}}} f\, d\widetilde{M}_{\mathrm{reg}} = \fint_{\widetilde{M}} f\, \frac{d\widetilde{M}}{\mathrm{vol}}$. The claim now follows from Corollary 8.3.5. □

Next, let us state a simple fact from elementary representation theory.

Lemma 8.3.7. *Let $V \subset \mathrm{L}^2(M)$ be an irreducible G-module of class $\chi \in \widehat{G}$. Let further $\{v_1, \dots, v_{d_\chi}\}$ denote an L^2-orthonormal basis of V, and $a \in V \cap \mathrm{C}^\infty(M)$ have L^2-norm equal to 1. Then, for any $x \in M$,*

$$\left\langle |a|^2 \right\rangle_G (x) = d_\chi^{-1} \sum_{k=1}^{d_\chi} |v_k(x)|^2. \tag{8.3.5}$$

In particular, the function

$$\Theta_V : M \to \mathbb{R}, \quad x \mapsto d_\chi^{-1} \sum_{k=1}^{d_\chi} |v_k(x)|^2,$$

is a G-invariant element of $\mathrm{C}^\infty(M)$ that is independent of the choice of orthonormal basis, and the left hand side of (8.3.5) is independent of the choice of a.

Proof. Since the left hand side of (8.3.5) is clearly G-invariant, smooth, and independent of the choice of orthonormal basis, it suffices to prove (8.3.5). Now, one has $a = \sum_{j=1}^{d_\chi} a_j v_j$ with

$a_j \in \mathbb{C}$, $\sum_{j=1}^{d_\chi} |a_j|^2 = 1$, and

$$(L_g a)(x) = a(g^{-1} \cdot x) = \sum_{j=1}^{d_\chi} a_j v_j(g^{-1} \cdot x) = \sum_{j,k=1}^{d_\chi} a_j c_{jk}(g) v_k(x), \qquad g \in G,\, x \in M,$$

where $\{c_{jk}\}_{1 \leq j,k \leq d_\chi}$ denote the matrix coefficients of the G-representation on V. This yields

$$\begin{aligned}\int_G |a(g^{-1} \cdot x)|^2 \, dg &= \int_G a(g^{-1} \cdot x)\overline{a}(g^{-1} \cdot x) \, dg \\ &= \int_G \left(\sum_{j,k=1}^{d_\chi} a_j c_{jk}(g) v_k(x) \right) \left(\sum_{l,m=1}^{d_\chi} \overline{a}_l \overline{c}_{lm}(g) \overline{v}_m(x) \right) dg,\end{aligned}$$

and we obtain (8.3.5) by taking into account the Schur orthogonality relations [37, Corollary 1.10]

$$\int_G c_{jk}(g) \overline{c}_{lm}(g) \, dg = d_\chi^{-1} \delta_{jl} \delta_{km},$$

and the fact that the substitution $g \mapsto g^{-1}$ leaves the Haar measure invariant. $\square$

We can now restate Corollary 8.3.5 in representation-theoretic terms.

Theorem 8.3.8 (Representation-theoretic equidistribution theorem). *Assume that the reduced geodesic flow is ergodic. By the spectral theorem, choose an orthogonal decomposition* $\mathrm{L}^2(M) = \bigoplus_{i \in \mathbb{N}} V_i$ *into irreducible unitary G-modules such that each V_i is contained in an eigenspace of the Laplace-Beltrami operator corresponding to some eigenvalue $E_{j(i)}$. Denote by $\chi_i \in \widehat{G}$ the class of V_i. Choose a semiclassical character family $\{\mathcal{W}_h\}_{h \in (0,1]}$ of growth rate $\vartheta < \frac{1}{2\kappa+3}$ and a partition $\mathcal{P}$ of $\{E_j\}_{j \in \mathbb{N}}$ of order $\delta \in \big(0, \frac{1-(2\kappa+3)\vartheta}{2\kappa+4}\big)$. Define the set of irreducible G-modules*

$$\big\{V_l^{\mathcal{W},\mathcal{P}}\big\}_{l \in \mathbb{N}} := \big\{V_i : \chi_i \in \mathcal{W}_{E^{-1/2}_{\mathcal{P}(j(i))}}\big\}.$$

As in Lemma 8.3.7, assign to each $V_l^{\mathcal{W},\mathcal{P}}$ the G-invariant function $\Theta_l := \Theta_{V_l^{\mathcal{W},\mathcal{P}}} : M \to [0,\infty)$, regard it as a function on $M/G = \widetilde{M}$, and write $\chi_l^{\mathcal{W},\mathcal{P}}$ for the class of $V_l^{\mathcal{W},\mathcal{P}}$. Then, there is a subsequence $\big\{V_{l_m}^{\mathcal{W},\mathcal{P}}\big\}_{m \in \mathbb{N}}$ with

$$\lim_{N \to \infty} \frac{\sum_{l_m \leq N} d_{\chi_{l_m}^{\mathcal{W},\mathcal{P}}}}{\sum_{l \leq N} d_{\chi_l^{\mathcal{W},\mathcal{P}}}} = 1$$

for which one has in the limit $m \to \infty$

$$\frac{1}{\sqrt{d_{\chi_{l_m}^{\mathcal{W},\mathcal{P}}} [\pi_{\chi_{l_m}^{\mathcal{W},\mathcal{P}}}|_H : \mathbb{1}]}} \left| \int_{\widetilde{M}} f \, \Theta_{l_m} \, d\widetilde{M} - \fint_{\widetilde{M}} f \frac{d\widetilde{M}}{\mathrm{vol}} \right| \longrightarrow 0 \qquad \forall\, f \in \mathrm{C}(\widetilde{M}),$$

where $d\widetilde{M} = \pi_* dM$ is the pushforward measure defined by the orbit projection $\pi : M \to \widetilde{M}$ and vol $: \widetilde{M} \to (0,\infty)$ *assigns to an orbit its Riemannian volume.*

Proof. Consider the set of eigenfunctions

$$\{u_i^{\mathcal{W},\mathcal{P}}\}_{i\in\mathbb{N}} = \{u_j : \chi(u_j) \in \mathcal{W}_{E_{\mathcal{P}(j)}^{-1/2}}\}$$

from Theorem 8.3.2, where we have replaced the notation χ_j by $\chi(u_j)$ to avoid confusion. For each $l \in \mathbb{N}$ one has $V_l^{\mathcal{W},\mathcal{P}} = \mathrm{span}\,\{u_i^{\mathcal{W},\mathcal{P}} : i \in J_l\}$ for a unique index set J_l with $\#J_l = d_{\chi_l^{\mathcal{W},\mathcal{P}}}$. Without loss of generality, we can assume $\min(J_1) = 1$ and $\min(J_{l+1}) = \max(J_l) + 1$ for each $l \in \mathbb{N}$. By Corollary 8.3.6, there is a subsequence $\{u_{i_k}^{\mathcal{W},\mathcal{P}}\}_{k\in\mathbb{N}}$ of density 1 in $\{u_i^{\mathcal{W},\mathcal{P}}\}_{i\in\mathbb{N}}$ such that we have for any $f \in \mathrm{C}(\widetilde{M})$

$$\frac{1}{\sqrt{d_{\chi(u_{i_k}^{\mathcal{W},\mathcal{P}})}[\pi_{\chi(u_{i_k}^{\mathcal{W},\mathcal{P}})}|_H : \mathbb{1}]}} \Big| \int_{\widetilde{M}} f \Big\langle \widetilde{|u_{i_k}^{\mathcal{W},\mathcal{P}}|^2} \Big\rangle_G d\widetilde{M} - \fint_{\widetilde{M}} f \, \frac{d\widetilde{M}}{\mathrm{vol}} \Big| \longrightarrow 0 \qquad \text{as } k \to \infty,$$

and by Lemma 8.3.7,

$$\Big\langle \widetilde{|u_{i_k}^{\mathcal{W},\mathcal{P}}|^2} \Big\rangle_G = \Theta_l \qquad \text{if } i_k \in J_l. \tag{8.3.6}$$

Let now $\{l_m\}_{m\in\mathbb{N}}$ be the sequence of those indices l occurring in (8.3.6) when k varies over all of $\mathbb{N}$. Then, due to the way how we indexed our sets J_l, we have for each $N \in \mathbb{N}$

$$\sum_{l_m \le N} d_{\chi_{l_m}^{\mathcal{W},\mathcal{P}}} \ge \sum_{l_m \le N} \#\{k : i_k \in J_{l_m}\} = \#\Big\{k : i_k \le \sum_{l \le N} d_{\chi_l^{\mathcal{W},\mathcal{P}}}\Big\}$$

Passing to the limit $N \to \infty$ we obtain

$$1 \ge \limsup_{N\to\infty} \frac{\sum_{l_m \le N} d_{\chi_{l_m}^{\mathcal{W},\mathcal{P}}}}{\sum_{l \le N} d_{\chi_l^{\mathcal{W},\mathcal{P}}}} \ge \liminf_{N\to\infty} \frac{\sum_{l_m \le N} d_{\chi_{l_m}^{\mathcal{W},\mathcal{P}}}}{\sum_{l \le N} d_{\chi_l^{\mathcal{W},\mathcal{P}}}} \ge \lim_{N\to\infty} \frac{\#\Big\{k : i_k \le \sum_{l \le N} d_{\chi_l^{\mathcal{W},\mathcal{P}}}\Big\}}{\sum_{l \le N} d_{\chi_l^{\mathcal{W},\mathcal{P}}}} = 1,$$

where the final equality holds because $\{u_{i_k}^{\mathcal{W},\mathcal{P}}\}_{k\in\mathbb{N}}$ has density 1 in $\{u_i^{\mathcal{W},\mathcal{P}}\}_{i\in\mathbb{N}}$. This concludes the proof of the theorem. □

Note that Theorem 8.3.8 is a statement about limits of representations, or multiplicities, in the sense that it assigns to each irreducible G-module in the character family a measure on $\widetilde{M}$, and then considers the limit measure.

To conclude this section, let us notice that in the special case that $\mathcal{W}_h = \{\chi\}$ for all $h \in (0, 1]$ and some fixed $\chi \in \widehat{G}$, the partitioning of the eigenfunction sequence $\{E_j\}$ is not necessary, and the statements proved in this section become much simpler. Thus, as a direct consequence of Theorem 8.3.2 we obtain

Theorem 8.3.9 (Equivariant quantum limits for the Laplacian and single isotypic components). *Assume that the reduced geodesic flow is ergodic, and choose $\chi \in \widehat{G}$. Let $\{u_j^\chi\}_{j\in\mathbb{N}}$ be an orthonormal basis of $\mathrm{L}^2_\chi(M)$ consisting of eigenfunctions of $-\Delta$. Then, there is a subsequence $\{u_{j_k}^\chi\}_{k\in\mathbb{N}}$ of density 1 in $\{u_j^\chi\}_{j\in\mathbb{N}}$ such that for all $s \in \mathrm{C}^\infty(S^*M)$ one has*

$$\langle \mathrm{Op}(s) u_{j_k}^\chi, u_{j_k}^\chi \rangle_{\mathrm{L}^2(M)} \longrightarrow \frac{1}{\mathrm{vol}_{\frac{\mu}{\mathrm{vol}\,\mathcal{O}}}(S^*M \cap \Omega_{\mathrm{reg}})} \int_{S^*M\cap\Omega_{\mathrm{reg}}} s \, \frac{d\mu}{\mathrm{vol}\,\mathcal{O}} \qquad \textit{as } k \to \infty. \tag{8.3.7}$$

Next, recall that a sequence of measures μ_j on a metric space $\mathbf{X}$ is said to *converge weakly* to a measure μ, if for all bounded and continuous functions f on $\mathbf{X}$ one has

$$\int_{\mathbf{X}} f \, d\mu_j \longrightarrow \int_{\mathbf{X}} f \, d\mu \qquad \text{as } j \to \infty.$$

We immediately deduce from Corollary 8.3.5

Corollary 8.3.10 (Equidistribution of eigenfunctions of the Laplacian for single isotypic components). *In the situation of Theorem 8.3.9 we have the weak convergence of measures*

$$|u^{\chi}_{j_k}|^2 \, dM \longrightarrow \big(\mathrm{vol}_{\frac{dM}{\mathrm{vol}\,\mathcal{O}}} M\big)^{-1} \frac{dM}{\mathrm{vol}\,\mathcal{O}} \qquad \textit{as } k \to \infty.$$

On the other hand, Theorem 8.3.8 directly implies

Theorem 8.3.11 (Representation-theoretic equidistribution theorem for single isotypic components). *Assume that the reduced geodesic flow is ergodic, and let $\chi \in \widehat{G}$. By the spectral theorem, choose an orthogonal decomposition $\mathrm{L}^2_\chi(M) = \bigoplus_{i\in\mathbb{N}} V_i^\chi$ into irreducible unitary G-modules of class χ such that each V_i^χ is contained in some eigenspace of the Laplace-Beltrami operator. As in Lemma 8.3.7, assign to each V_i^χ the G-invariant function $\Theta_i := \Theta_{V_i^\chi} : M \to [0,\infty)$, and regard it as a function on $M/G = \widetilde{M}$. Then, there is a subsequence $\{V^\chi_{i_k}\}_{k\in\mathbb{N}}$ of density 1 in $\{V_i^\chi\}_{i\in\mathbb{N}}$ such that we have the weak convergence*

$$\Theta^\chi_{i_k} \, d\widetilde{M} \overset{k\to\infty}{\longrightarrow} \Big(\mathrm{vol}_{\frac{d\widetilde{M}}{\mathrm{vol}}} \widetilde{M}\Big)^{-1} \frac{d\widetilde{M}}{\mathrm{vol}}.$$

8.4 Applications

In what follows, we apply our results for the Laplacian to some concrete situations where a closed connected Riemannian manifold carries an effective isometric action of a compact connected Lie group such that the principal orbits are of lower dimension than the manifold, and the reduced geodesic flow is ergodic.

8.4.1 Compact locally symmetric spaces

Let G be a connected semisimple Lie group with finite center and Lie algebra $\mathfrak{g}$, and Γ a discrete co-compact subgroup. Consider a Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

of $\mathfrak{g}$, and denote the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$ by K. Choose a left-invariant metric on G given by an $\mathrm{Ad}(K)$-invariant bilinear form on $\mathfrak{g}$. The quotient $M = \mathbb{X} := \Gamma\backslash G$ is a closed smooth manifold, and by requiring that the projection $G \to \mathbb{X}$ is a Riemannian submersion, we obtain a Riemannian structure on $\mathbb{X}$. K acts on G and on $\mathbb{X}$ from the right in an isometric and effective way, and the isotropy group of a point $\Gamma g \in \mathbb{X}$ is conjugate to the finite group $gKg^{-1} \cap \Gamma$. Hence, all K-orbits in $\mathbb{X}$ are either principal or exceptional. Since the maximal compact subgroups of G are precisely the conjugates of K, exceptional K-orbits arise from elements in Γ of finite order. Now, let $\mathbb{J}_G : T^*\mathbb{X} \to \mathfrak{g}^*$ be the momentum map of the right G-action on $\mathbb{X}$ and $\mathrm{res} : \mathfrak{g}^* \to \mathfrak{k}^*$ the natural restriction map. Then $\mathbb{J}_K = \mathrm{res} \circ \mathbb{J}_G$ is the momentum map of the right K-action on $\mathbb{X}$. As usual,

$$
\begin{tikzcd}
 & & & \widetilde{\Omega} = \Omega/K \simeq T^*\mathbb{Y} \simeq \mathbb{Y} \times \mathfrak{p}^* \arrow[ddl] \\
\mathbb{X} \times \mathfrak{g}^* \simeq T^*\mathbb{X} \arrow[ddr] & \Omega \arrow[l, "\iota"'] \arrow[urr, dashed] & & \\
 & & \mathbb{Y} = \Gamma\backslash G/K & \\
 & \mathbb{X} = \Gamma\backslash G \arrow[ur, dashed] & & G/K \arrow[ul, dashed] \\
 & & G \arrow[ul, dashed] \arrow[ur, dashed] &
\end{tikzcd}
$$

Figure 8.4.1: Co-tangent bundle reduction for a locally symmetric space

let $\Omega := \mathbb{J}_K^{-1}(\{0\})$. Let us consider first the case when Γ has no torsion, meaning that no non-trivial element $\gamma \in \Gamma$ is conjugate in G to an element of K. In this case, there are no exceptional orbits, the action of Γ on G/K is free, and $\mathbb{Y} := \Gamma\backslash G/K$ becomes a closed manifold of dimension $n - d$, where $n = \dim \mathbb{X}$ and $d = \dim K$. Furthermore, by co-tangent bundle reduction,

$$T^*\mathbb{Y} \simeq \Omega/K =: \widetilde{\Omega} \tag{8.4.1}$$

as symplectic manifolds, compare Lemma 6.2.3 and Figure 8.4.1. In what follows, we give a more intrinsic description of this symplectomorphism. The left trivialization $T^*G \simeq G \times \mathfrak{g}^*$ described in (6.2.3) induces the trivialization

$$T^*\mathbb{X} \xrightarrow{\simeq} \mathbb{X} \times \mathfrak{g}^*, \qquad \xi_{\Gamma g} \longmapsto (\Gamma g, (L_g)^*_e \cdot \eta_g), \quad \mathrm{pr}^*(\xi_{\Gamma g}) = \eta_g,\, \eta_g \in T^*_g G,$$

$\mathrm{pr} : G \to \Gamma\backslash G$ being a submersion. The right G-action on $T^*\mathbb{X}$ then takes the form

$$T^*_{\Gamma gh}(\mathbb{X}) \ni \xi_{\Gamma g} \cdot h = (R_{h^{-1}})^*_{\Gamma gh}\, \xi_{\Gamma g} \longmapsto (\Gamma gh, (L_{gh})^*_e \circ (R_{h^{-1}})^*_{gh}\, \eta_g),$$

so that with $\mu = (L_g)^*_e \cdot \eta_g$ we have

$$(\Gamma g, \mu) \cdot h = (\Gamma gh, \mathrm{Ad}^*(h)\mu), \qquad h \in G. \tag{8.4.2}$$

Now, for $X \in \mathfrak{g}$ one computes

$$\begin{aligned}
\mathbb{J}_G(\Gamma g, \mu)(X) &= \mathbb{J}_G(\xi_{\Gamma g})(X) = (L_{g^{-1}})^*_g\, \mu(\widetilde{X}^{\mathrm{R}}_g) \\
&= \mu((L_{g^{-1}})_{*,g}\widetilde{X}^{\mathrm{R}}_g) = \mu\Big(\frac{d}{dt}(g^{-1} g\, \mathrm{e}^{tX}\,)_{|t=0}\Big) = \mu(X),
\end{aligned}$$

where $\widetilde{X}^{\mathrm{R}}_g$ denotes the vector field generated by the right action of X, compare Example 6.2.2, so that the momentum map reads

$$\mathbb{J}_G(\Gamma g, \mu) = \mu, \qquad (\Gamma g, \mu) \in T^*\mathbb{X}.$$

If $\eta = (\Gamma g, \mu) \in \mathbb{J}_K^{-1}(\{0\}) \subset T^*\mathbb{X}$, the last equality implies $\mu = \mathbb{J}_G(\Gamma g, \mu) \in \mathfrak{p}^*$. Furthermore, in view of the Cartan decomposition $G = PK$, where P is the parabolic subgroup with Lie algebra $\mathfrak{p}$, one has the diffeomorphism $G/K \simeq P$. Consequently, we can choose as representant of the class $[\eta] \in \widetilde{\Omega}$ an element $\eta = (\Gamma g, \mu)$ with $g \in P$ and $\mu \in \mathfrak{p}^*$, yielding the identification

$$\widetilde{\Omega} \simeq (\Gamma\backslash P) \times \mathfrak{p}^* \simeq \mathbb{Y} \times \mathfrak{p}^*. \tag{8.4.3}$$

On the other hand, the left trivialization $T^*P \simeq P \times \mathfrak{p}^*$ and the previous arguments imply the trivialization

$$T^*\mathbb{Y} \simeq \mathbb{Y} \times \mathfrak{p}^*. \tag{8.4.4}$$

Comparing (8.4.3) and (8.4.4) then yields the desired intrinsic realization of the symplectomorphism (8.4.1).

Let us now assume that G has real rank 1. In this case, the orbit space $\mathbb{Y}$ has strictly negative sectional curvature inherited from G/K. Consequently, its geodesic flow ψ_t is ergodic. Since the measures on the spaces $T^*\mathbb{Y} \simeq \widetilde{\Omega}$ are given by the corresponding symplectic forms, this implies that the reduced geodesic flow $\widetilde{\varphi}_t$ on $\widetilde{\Omega}$, which corresponds to ψ_t under the symplectomorphism (8.4.1), is ergodic, and the results from Section 8.3 apply.

Next, let us consider a discrete co-compact subgroup Γ_1 with torsion. In this case K acts on $\mathbb{X}_1 := \Gamma_1\backslash G$ with non-conjugated finite isotropy groups, so that $\mathbb{Y}_1 := \Gamma_1\backslash G/K$ is no longer a manifold, but an orbifold. Now, by a theorem of Selberg [51], any finitely generated linear group contains a torsion-free subgroup of finite index. More generally, Borel [5] showed that every finitely generated group of isometries of a simply connected Riemannian symmetric manifold has a normal torsion-free subgroup of finite index. Let therefore $\Gamma \subset \Gamma_1$ be a normal torsion-free co-compact subgroup of finite index [10]. In this case, $\mathbb{Y} = \Gamma\backslash G/K$ is a smooth manifold and a finite covering of $\mathbb{Y}_1$, and

$$\mathbb{X}_1 \simeq F\backslash\mathbb{X}, \qquad \mathbb{Y}_1 \simeq F\backslash\mathbb{Y},$$

where F denotes the finite group $F := \Gamma_1/\Gamma$. Next, let $\mathbb{J}^1_G : T^*\mathbb{X}_1 \to \mathfrak{g}^*$ be the momentum map of the right G-action on $\mathbb{X}_1$, $\mathbb{J}^1_K := \mathrm{res} \circ \mathbb{J}^1_G$, and $\Omega_1 := (\mathbb{J}^1_K)^{-1}(\{0\})$. As in the torsion-free case we have the left trivialization $T^*\mathbb{X}_1 \simeq \mathbb{X}_1 \times \mathfrak{g}^*$ as smooth manifolds, and by analogy to (8.4.1) one shows that as symplectic orbifolds

$$T^*\mathbb{Y}_1 \simeq \widetilde{\Omega}_1, \tag{8.4.5}$$

which represents the quotient presentation of the co-tangent bundle of $\mathbb{Y}_1$. Furthermore, with (8.4.4) we obtain

$$T^*\mathbb{Y}_1 \simeq \widetilde{\Omega}_1 \simeq F\backslash\widetilde{\Omega} \simeq F\backslash(T^*\mathbb{Y}) \simeq \mathbb{Y}_1 \times \mathfrak{p}^*.$$

Consequently, we have a diagram analogous to Figure 8.4.1 with Γ being replaced by Γ_1. Besides, if $\mathbb{X}_1'$ denotes the stratum of orbits of principal type of $\mathbb{X}_1$, notice that singular co-tangent bundle reduction (Lemma 6.2.3) implies

$$T^*\mathbb{Y}_1 \supset T^*(\mathbb{X}_1'/K) \simeq \big((\mathbb{J}^1_K)^{-1}(\{0\}) \cap T^*\mathbb{X}_1'\big)/K \subset (\widetilde{\Omega}_1)_{\mathrm{reg}},$$

the measures on these spaces being given by the corresponding symplectic forms, and the complements of the inclusions having measure zero. Consider now the commutative diagram in Figure 8.4.2, where π_K and π_F denote the projections of the K- and F-actions, respectively.

$$\begin{tikzcd}
T^*\mathbb{X} \supset \Omega \arrow[r, "\pi_K"] \arrow[d, "\pi_F"] & \widetilde{\Omega} \simeq T^*\mathbb{Y} \arrow[d, "\pi_F"] \\
T^*\mathbb{X}_1 \supset \Omega_1 \arrow[r, "\pi_K"] & \widetilde{\Omega}_1 \simeq T^*\mathbb{Y}_1
\end{tikzcd}$$

Figure 8.4.2: Orbit projections and symplectic quotients

To relate the dynamics on the symplectic quotients $\widetilde{\Omega}$ and $\widetilde{\Omega}_1$, let $\widetilde{p}_1 \in \mathrm{C}^\infty(\widetilde{\Omega}_1)$ be a smooth function. By definition, there exists a function $p_1 \in \mathrm{C}^\infty(T^*\mathbb{X}_1)^K$ such that $p_1|_{\Omega_1} = \pi_K^* \widetilde{p}_1$. The Hamiltonian flow φ_t^1 of p_1 then induces a Hamiltonian flow $\widetilde{\varphi}_t^1$ on $\widetilde{\Omega}_1$, compare Section 6.2. On the other hand, $\widetilde{p}_1$ yields a function $\widetilde{p} \in \mathrm{C}^\infty(\widetilde{\Omega})^F$ with Hamiltonian flow $\widetilde{\varphi}_t$ induced by the corresponding flow φ_t on $T^*\mathbb{X}$. Since φ_t induces the flow φ_t^1, it is clear that $\widetilde{\varphi}_t$ induces a flow on $\widetilde{\Omega}_1$ given precisely by $\widetilde{\varphi}_t^1$. Indeed, for $\widetilde{f}_1 \in \mathrm{C}^\infty(\widetilde{\Omega}_1)$ and $\widetilde{\eta}_1 = \pi_K(\eta_1) = \pi_F \circ \pi_K(\eta) = \pi_F(\widetilde{\eta}) \in \widetilde{\Omega}_1$ one computes for $\widetilde{f}_1(\widetilde{\varphi}_t^1(\widetilde{\eta}_1))$

$$\pi_K^* \widetilde{f}_1(\varphi_t^1(\eta_1)) = (\pi_F^* \circ \pi_K^* \widetilde{f}_1)(\varphi_t(\eta)) = (\pi_K^* \circ \pi_F^* \widetilde{f}_1)(\varphi_t(\eta)) = \pi_F^* \widetilde{f}_1(\widetilde{\varphi}_t(\widetilde{\eta})).$$

Furthermore, in view of (8.4.5), $\widetilde{\varphi}_t^1$ yields a flow ψ_t^1 on $\mathbb{Y}_1$.

Let now ψ_t be the geodesic flow on $\mathbb{Y}$, and assume that the rank of G is 1, so that ψ_t is ergodic. Then the induced flow ψ_t^1 on $\mathbb{Y}_1$ is ergodic, too, with respect to the orbifold symplectic measure on $T^*\mathbb{Y}_1$. More precisely, by our previous considerations the ergodicity of the flow $\widetilde{\varphi}_t$ on $\widetilde{\Omega}$ implies that

$$(\widetilde{\varphi}_t^1)|_{(\widetilde{\Omega}_1)_{\mathrm{reg}}},$$

which is precisely the reduced geodesic flow on the symplectic stratum $(\widetilde{\Omega}_1)_{\mathrm{reg}}$ given by (6.2.1), must be ergodic with respect to the symplectic measure $d((\widetilde{\Omega}_1)_{\mathrm{reg}})$. Summing up, our results from Section 8.3 apply. For simplicity, let us state here only the results for single isotypic components. Then, Theorem 8.3.9 and Corollary 8.3.10 yield

Proposition 8.4.1. *Let G be a connected semisimple Lie group of rank 1 with finite center, K a maximal compact subgroup, and Γ a discrete co-compact subgroup, possibly with torsion. Let Δ be the Laplace–Beltrami operator on $\mathbb{X} = \Gamma\backslash G$, $\chi \in \widehat{K}$, and let $\{u_j^\chi\}_{j\in\mathbb{N}}$ be an orthonormal basis of $\mathrm{L}^2_\chi(\mathbb{X})$ of eigenfunctions of $-\Delta$. Then, there is a subsequence $\{u_{j_k}^\chi\}_{k\in\mathbb{N}}$ of density 1 in $\{u_j^\chi\}_{j\in\mathbb{N}}$ such that for all $s \in \mathrm{C}^\infty(S^*\mathbb{X})$ one has*

$$\left\langle \mathrm{Op}(s) u_{j_k}^\chi, u_{j_k}^\chi \right\rangle_{\mathrm{L}^2(\mathbb{X})} \stackrel{k\to\infty}{\longrightarrow} \fint_{S^*\mathbb{X}\cap\Omega_{\mathrm{reg}}} s \, \frac{d\mu}{\mathrm{vol}\,\mathcal{O}}, \tag{8.4.6}$$

as well as

$$|u_{j_k}^\chi|^2 \, d\mathbb{X} \stackrel{k\to\infty}{\longrightarrow} \left(\mathrm{vol}_{\frac{d\mathbb{X}}{\mathrm{vol}\,\mathcal{O}}} \mathbb{X}\right)^{-1} \frac{d\mathbb{X}}{\mathrm{vol}\,\mathcal{O}}, \qquad \left\langle \widetilde{|u_{j_k}^\chi|^2} \right\rangle_G d\mathbb{Y} \stackrel{k\to\infty}{\longrightarrow} \left(\mathrm{vol}_{\frac{d\mathbb{Y}}{\mathrm{vol}}} \mathbb{Y}\right)^{-1} \frac{d\mathbb{Y}}{\mathrm{vol}}, \tag{8.4.7}$$

where $\mathbb{Y} = \Gamma\backslash G/K$ is in general an orbifold, and $d\mathbb{Y}$ is the pushforward of the measure $d\mathbb{X}$ along the orbit projection $\mathbb{X} \to \mathbb{Y}$, see Section 6.3.

Notice that the limit integral in (8.4.6) represents an integral over the orbifold co-sphere bundle $S^*\mathbb{Y}$. Since the orbit volume function is constant in this case, eigenfunctions of the Laplacian $\Delta_\mathbb{Y}$ on $\mathbb{Y}$ correspond to K-invariant eigenfunctions of Δ on $\mathbb{X}$, compare Section 5.3.2. Furthermore, up to the constant given by the orbit volume, the pushforward measure $d\mathbb{Y}$ agrees in the orbifold case with the orbifold volume form. Consequently, in the special case that χ corresponds to the trivial representation, Proposition 8.4.1 yields the following result already implied by the work of Kordyukov [38].

Corollary 8.4.2 (Shnirelman-Zelditch-Colin-de-Verdière equidistribution theorem for $\mathbb{Y}$). *With the assumptions of Proposition 8.4.1, let $\{v_j\}_{j\in\mathbb{N}}$ be an orthonormal basis of $\mathrm{L}^2(\mathbb{Y})$ of eigenfunctions of $-\Delta_\mathbb{Y}$. Then, there is a subsequence $\{v_{j_k}\}_{k\in\mathbb{N}}$ of density 1 in $\{v_j\}_{j\in\mathbb{N}}$ such that we have the weak convergence of measures*

$$|v_{j_k}|^2 \, d\mathbb{Y} \stackrel{k\to\infty}{\longrightarrow} (\mathrm{vol}_{d\mathbb{Y}}\mathbb{Y})^{-1} \, d\mathbb{Y}.$$

Notice that in view of the left trivialization $T^*\mathbb{X} \simeq \mathbb{X}\times\mathfrak{g}^*$ and (8.4.2), K acts on $\Omega \subset \mathbb{X}\times\mathfrak{p}^*$ by right multiplication according to

$$\Omega \ni (\Gamma g, \mu) \cdot k = (\Gamma g k, \mathrm{Ad}^*(k)\mu), \qquad k \in K,$$

$\mathfrak{p}$ being $\mathrm{Ad}(K)$-invariant. In particular, regarding the decomposition of $T^*\mathbb{X}$ into isotropy types with respect to the right K-action, whenever Γ contains non-trivial elliptic elements, the closure of $S^*\mathbb{X} \cap \Omega_{\mathrm{reg}}$ in Ω will contain exceptional isotropy types, which means that in the proofs of Theorems 8.2.1 and 8.3.2 one cannot assume that one can stay away from the singular points of Ω, compare also Examples 7.2.8.

8.4.2 Invariant metrics on spheres in dimensions 2 and 4

In contrast to genuinely chaotic cases, it can happen that the reduced geodesic flow is ergodic simply for topological reasons. Namely, when the singular symplectic reduction of the co-sphere-bundle is just 1-dimensional, a single closed orbit of the reduced flow can have full measure. Although non-generic, this situation is topologically invariant, so that if it occurs for some particular G-space, it occurs for any choice of G-invariant Riemannian metric on that space, leading to a whole class of examples which might well be complicated geometrically.

In what follows, we will show that the spheres in dimensions 2 and 4, with appropriate group actions and invariant Riemannian metrics, are examples of the form just described. The reason why we consider only the dimensions 2 and 4 is that, in general, the n-sphere is topologically the suspension of the $(n-1)$-sphere, but only for $n \in \{2,4\}$, the $(n-1)$-sphere has the structure of a compact connected Lie group. Thus, let G be a compact connected Lie group. The *suspension* of G is the topological quotient space

$$SG := \big([-1,1] \times G\big)/\sim,$$

where the equivalence relation is defined by

$$(t,g) \sim (t',g') \iff \begin{cases} (t,g) = (t',g') & \text{or} \\ t = t' = 1 & \text{or} \\ t = t' = -1. \end{cases}$$

SG is a compact connected Hausdorff space that carries an effective G-action induced by the G-action on G by left-multiplication and the trivial action on $[-1,1]$. We will call this induced action the *suspension of the G-action*. It has exactly two fixed points $N := [\{1\}\times G]$ and $S := [\{-1\}\times G]$ which we may call *north pole* and *south pole*. Now, in general, SG does not possess a differentiable structure. However, if G is an n-sphere, then SG is homeomorphic to the $(n+1)$-sphere, and consequently carries a canonical smooth structure making it diffeomorphic to the standard $(n+1)$-sphere. As is well-known, the only connected Lie groups that are spheres are $\mathrm{SO}(2) \cong S^1$ and $\mathrm{SU}(2) \cong S^3$.

Note that S^2, with the S^1-action given by the suspension of left-multiplication on S^1 and equipped with an S^1-invariant Riemannian metric, is just a surface of revolution diffeomorphic to the standard 2-sphere. Similarly, for $G = S^3$, we equip the suspension $S^4 \cong SS^3$ with the S^3-action given by the suspension of left-multiplication on S^3 and an S^3-invariant Riemannian metric, obtaining a class of 4-dimensional examples. We now have the following

Proposition 8.4.3. *For $n \in \{2,4\}$, equip the n-sphere $S^n \cong SS^{n-1}$ with the S^{n-1}-action given by the suspension of left-multiplication on S^{n-1}. Then the reduced geodesic flow with respect to any S^{n-1}-invariant Riemannian metric on S^n is ergodic.*

Proof. First, we prove the result for S^2. It will then become clear that the situation is entirely analogous for S^4. Thus, let $G = S^1 \cong \mathrm{SO}(2)$. Then, for any choice of SO(2)-invariant metric on $M := SS^1$, we can identify M with a surface of revolution in $\mathbb{R}^3$ diffeomorphic to the 2-sphere and endowed with the induced metric from $\mathbb{R}^3$. We assume that the poles are given by the points $N = (0,0,1)$ and $S = (0,0,-1)$. The corresponding meridians are orthogonal to the SO(2)-orbits, and since the metric is SO(2)-invariant, each meridian is a closed geodesic. Now, for $(x,\xi) \in T^*M$, set $p(x,\xi) := \|\xi\|_x^2$. Let $c > 0$ and put $\Sigma_c := p^{-1}(\{c\})$ and $\widetilde{\Sigma}_c := \widetilde{p}^{-1}(\{c\})$, where $\widetilde{p} \in \mathrm{C}^\infty(\widetilde{\Omega}_{\mathrm{reg}})$ is the function induced by $p|_{\Omega_{\mathrm{reg}}}$. Clearly, c is a regular value of p. To examine whether the reduced geodesic flow is ergodic on $\widetilde{\Sigma}_c$, note that with the identification $T^*M \simeq TM$ given by the Riemannian metric one has

$$\Omega = \mathbb{J}^{-1}(\{0\}) \simeq \bigsqcup_{x\in M} T_x(G\cdot x)^\perp, \tag{8.4.8}$$

compare (6.2.4), so that

$$\Omega_{\mathrm{reg}} \simeq \Big(\bigcup_{x\in M_{\mathrm{reg}}} \{x\}\times T_x(G\cdot x)^\perp\Big) \cup \Big(\{N\}\times(T_NM\backslash\{0\})\Big) \cup \Big(\{S\}\times(T_SM\backslash\{0\})\Big),$$

$$\widetilde{\Omega}_{\mathrm{reg}} \simeq \big((-1,1)\times\mathbb{R}\big) \cup \big(\{1\}\times(0,\infty)\big) \cup \big(\{-1\}\times(0,\infty)\big) \simeq \mathbb{R}^2\backslash\{(0,1),(0,-1)\},$$

where $M_{\mathrm{reg}} = M\backslash\{N,S\}$, $M_{\mathrm{reg}}/G \simeq (-1,1)$. The diffeomorphism $\widetilde{\Omega}_{\mathrm{reg}} \simeq \mathbb{R}^2\backslash\{(0,1),(0,-1)\}$ is illustrated in Figures 8.4.3 and 8.4.4 on the next page for S^2 with the round metric, which is the generic case since M is SO(2)-equivariantly diffeomorphic to it. Under the diffeomorphism $\widetilde{\Omega}_{\mathrm{reg}} \simeq \mathbb{R}^2\backslash\{(0,1),(0,-1)\}$, the hypersurface $\widetilde{\Sigma}_c$ corresponds to an ellipse with radii determined by c, as illustrated in Figure 8.4.4. Let now $G\cdot(x,\xi) \in \widetilde{\Sigma}_c$. Since $\xi \in T_x(G\cdot x)^\perp$, the geodesic flow φ_t transports (x,ξ) around curves in T^*M that project onto meridians through N and S, so that the reduced geodesic flow $\widetilde{\varphi}_t(G\cdot(x,\xi)) \equiv G\cdot\varphi_t(x,\xi)$ through $G\cdot(x,\xi)$ corresponds to a periodic flow around the ellipse $\widetilde{\Sigma}_c$. Consequently, the only subsets of $\widetilde{\Sigma}_c$ which are invariant under $\widetilde{\varphi}_t$ are the whole ellipse and the empty set, implying that the reduced flow $\widetilde{\varphi}_t$ on $\widetilde{\Sigma}_c$ is ergodic for arbitrary $c > 0$. Besides, note that the points on the segment between $(0,1)$ and $(0,-1)$ are stationary under $\widetilde{\varphi}_t$.

Next, let us check what happens for a general compact connected Lie group G. Due to the definition of SG and its G-action, it is clear that SG/G is homeomorphic to $[-1,1]$ and, due to (8.4.8), that $\widetilde{\Omega}_{\mathrm{reg}}$ is diffeomorphic to $\mathbb{R}^2\backslash\{(0,1),(0,-1)\}$ whenever SG is a smooth manifold, so that we always obtain not only an analogous but essentially the same picture as in Figure 8.4.4. Hence, for $G = S^3$, the reduced geodesic flow is given by a periodic flow around an ellipse, and therefore ergodic. □

We shall now apply some of our results from Section 8.3 to a surface of revolution diffeomorphic to the standard 2-sphere. Thus, let $M \subset \mathbb{R}^3$ be given by rotating a suitable smooth curve $\gamma : [0,L] \to \mathbb{R}^2_{x\geq 0}$ in the xz-half plane around the z-axis in $\mathbb{R}^3$. In particular, $\gamma'(t)$ has to be perpendicular to the z-axis at $\gamma(0)$ and $\gamma(L)$. We assume that $\gamma(0) = (0,-1)$ and $\gamma(L) = (0,1)$ and that γ is parametrized by arc length, so that $\gamma : [0,L] \ni \theta \mapsto (R(\theta), z(\theta))$, where $R : [0,L] \to [0,\infty)$, $R(0) = R(L) = 0$, $R(\theta) > 0$ for $\theta \in (0,L)$ corresponds to the distance to the z-axis, and $z : [0,L] \to \mathbb{R}$ is smooth. This leads to a parametrization of M according to

$$M = \big\{(R(\theta)\cos\phi, R(\theta)\sin\phi, z(\theta)),\ \theta \in [0,L],\ \phi \in [0,2\pi)\big\}.$$

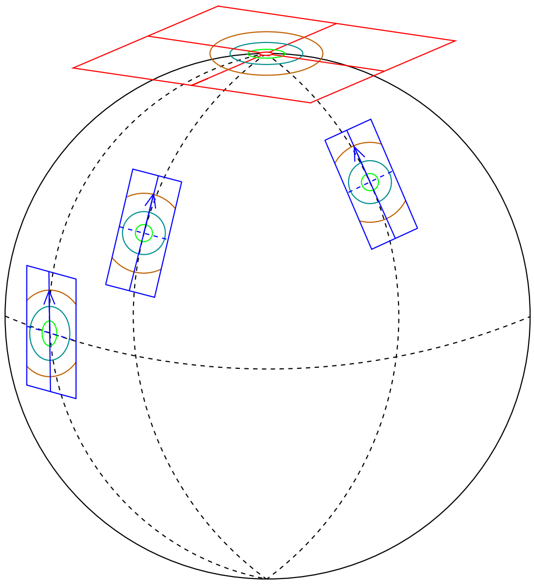

Figure 8.4.3: The space $T^*_N S^2\backslash\{0\}$ (red) and three co-tangent spaces (blue) with arrows that represent elements of Ω_{reg}. The three circles in each plane (brown, teal, green) correspond to the intersection of the plane with Σ_c for three different values of c.

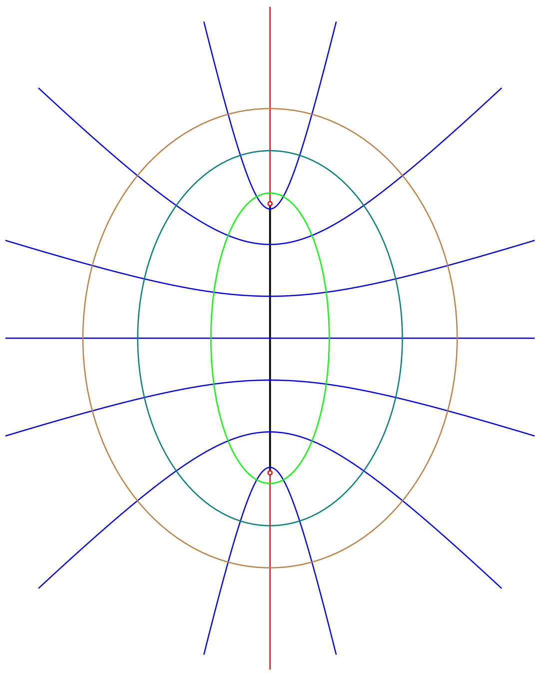

Figure 8.4.4: Under the projection $\Omega_{\mathrm{reg}} \to \widetilde{\Omega}_{\mathrm{reg}}$, $T^*_N S^2\backslash\{0\}$ and $T^*_S S^2\backslash\{0\}$ collapse to open half-lines (red) and for every $x \in S^2\backslash\{N, S\}$, $T^*_x S^2 \cap \Omega_{\mathrm{reg}}$ collapses to a line (blue). The ellipses (brown, teal, green) depict $\widetilde{\Sigma}_c$ for three different values of c.

Now, let M be endowed with the induced metric from $\mathbb{R}^3$. The Laplace-Beltrami operator Δ on M commutes with ∂_ϕ, so that separation of variables leads to a Hilbert basis of $\mathrm{L}^2(M)$ of joint eigenfunctions of both operators of the form

$$e_{l,m}(\phi,\theta) = f_{l,m}(\theta)e^{im\phi}, \qquad (l,m) \in \mathcal{I} \subset \mathbb{Z}\times\mathbb{Z}. \tag{8.4.9}$$

The irreducible representations of $\mathrm{SO}(2) \simeq S^1 = \{e^{i\varphi},\ \varphi \in [0,2\pi)\} \subset \mathbb{C}$ are all 1-dimensional, and given by the characters $\chi_k(e^{i\phi}) = e^{-ik\phi}$, $k \in \mathbb{Z}$. Thus, each subspace $\mathbb{C}\cdot e_{l,m}$ corresponds to an irreducible representation of SO(2), and $\{e_{l,m}\}_{l:\,(l,m)\in\mathcal{I}}$ is a Hilbert basis of $\mathrm{L}^2_{\chi_m}(M)$. Furthermore, $|e_{l,m}|^2$ is manifestly SO(2) invariant. Theorem 8.3.11 then yields for each $m \in \mathbb{Z} \simeq \widehat{\mathrm{SO}(2)}$ a subsequence $\{e_{l_k,m}\}_{k\in\mathbb{N}}$ of density 1 in $\{e_{l,m}\}_{l:\,(l,m)\in\mathcal{I}}$ such that for all $a \in \mathrm{C}(\widetilde{M})$

$$\int\limits_{\widetilde{M}} a|e_{l_k,m}|^2\, d\widetilde{M} \overset{k\to\infty}{\longrightarrow} \left(\int_{\widetilde{M}} \frac{d\widetilde{M}}{\mathrm{vol}}\right)^{-1} \int\limits_{\widetilde{M}} a\frac{d\widetilde{M}}{\mathrm{vol}}, \tag{8.4.10}$$

where as before $\widetilde{M} = M/\mathrm{SO}(2)$. Let us write (8.4.10) more explicitly. An SO(2)-orbit of a point $x \in M$ with coordinates (ϕ,θ) is of the form $\{(\phi',\theta):\ 0 < \phi' < 2\pi\}$, up to a set of measure zero with respect to the induced orbit measure $d\mu_{\mathrm{SO}(2)\cdot x} \equiv R(\theta)\,d\phi$, and we obtain $\mathrm{vol}(\mathrm{SO}(2)\cdot x) = \int_0^{2\pi} R(\theta)\,d\phi = 2\pi R(\theta)$. Furthermore, $\widetilde{M}$ is homeomorphic to the closed interval $[0,L] \subset \mathbb{R}$, and the pushforward measure on $\widetilde{M}$ is given by $d\widetilde{M}(\theta) \equiv 2\pi R(\theta)\,d\theta$, where we identified $\mathrm{SO}(2)\cdot x$ and θ. Summing up, (8.4.10) yields

$$2\pi \int\limits_0^L a(\theta)|f_{l_k,m}|^2(\theta)R(\theta)\,d\theta \overset{k\to\infty}{\longrightarrow} \frac{1}{L}\int\limits_0^L a(\theta)\,d\theta, \qquad a \in \mathrm{C}([0,L]), \tag{8.4.11}$$

which is a result about weak convergence of measures on $\widetilde{M} \cong [0,L]$. Formulated on M, Corollary 8.3.10 yields that for each m there is a subsequence $\{f_{l_k,m}\}_{k\in\mathbb{N}}$ of density 1 in $\{f_{l,m}\}_{l:\,(l,m)\in\mathcal{I}}$ such that one has the weak convergence of measures

$$|f_{l_k,m}|^2\,dM \quad \overset{k\to\infty}{\longrightarrow} \quad \frac{1}{2\pi L}\frac{dM}{R}. \tag{8.4.12}$$

Here, $\frac{dM}{R}$ is to be understood as the extension by zero of the smooth measure $dM(\phi,\theta)/R(\phi,\theta)$ from $\{(\phi,\theta),\ \theta \in (0,L)\}$ to $\{(\phi,\theta),\ \theta \in [0,L]\}$, and we used that $\mathrm{vol}_{\frac{dM}{R}} M = 2\pi L$. In particular, the obtained quantum limit on M is, up to a constant, related to the Riemannian volume density on M by the reciprocal of the distance function R, which tends to infinity towards the poles. This is illustrated in Figure 8.4.5 on the next page, where the function $1/R$ is plotted on a surface of revolution.

So far, for simplicity of presentation, we have restricted ourselves to the special case of considering a single fixed isotypic component, which means keeping the index m fixed. Even in this case, we do not know whether the results (8.4.11) and (8.4.12) are known for general surfaces of revolution diffeomorphic to the standard 2-sphere. Having actually the more general Theorem 8.3.8 at hand, the results (8.4.11) and (8.4.12) directly generalize to the situation of a semiclassical character family of growth rate $\vartheta < \frac{1}{5}$ as illustrated in Figure 5.3.2 since the dimensions of the irreducible representations are all 1 in this case, and all principal isotropy groups are trivial, so that $[\pi_\chi|_H : \mathbb{1}] = d_\chi = 1$ for all $\chi \in \widehat{\mathrm{SO}(2)}$.

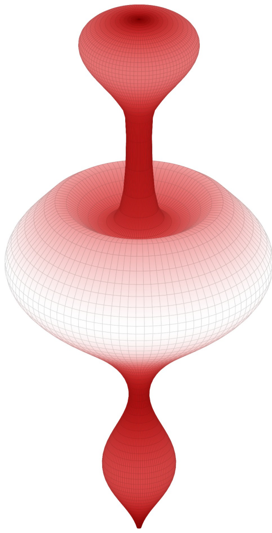

Figure 8.4.5: A quantum limit on a surface of revolution.

Physically, one can interprete these results as follows. For each family of symmetry types that does not grow too fast in the high-energy limit, there is a sequence of quantum states such that the corresponding sequence of probability densities on M converges weakly and with density 1 in the high-energy limit to the probability density of finding within a certain surface element of M a classical particle with known energy and zero angular momentum with respect to the z-axis, but unknown momentum.

In the simplest case of the standard 2-sphere $M = S^2$ with the round metric, the eigenfunctions are explicitly known, and we show in the following that at least our simplest result (8.4.11) for fixed isotypic components is implied by the classical theory of spherical harmonics. In fact, we will see that one does not need to pass to a subsequence of density 1. Recall from Section 5.3.3 that the eigenvalues of $-\Delta$ on S^2 are given by the numbers $l(l+1)$, $l = 0, 1, 2, 3\dots$, and the corresponding eigenspaces E_l are of dimension $2l+1$ and spanned by the spherical harmonics

$$Y_{l,m}(\phi,\theta) = \sqrt{\tfrac{2l+1}{4\pi}\tfrac{(l-m)!}{(l+m)!}}P_{l,m}(\cos\theta)e^{im\phi}, \qquad 0 \leq \phi < 2\pi,\ 0 \leq \theta < \pi, \tag{8.4.13}$$

where $m \in \mathbb{Z}$, $|m| \leq l$, and $P_{l,m}$ are the associated Legendre polynomials

$$P_{l,m}(x) = \frac{(-1)^m}{2^l l!}\left(1-x^2\right)^{\frac{m}{2}}\frac{d^{l+m}}{dx^{l+m}}\left(x^2-1\right)^l, \tag{8.4.14}$$

compare (8.4.9). Each subspace $\mathbb{C}\cdot Y_{l,m}$ corresponds to an irreducible representation of SO(2), and each irreducible representation χ_k with $|k| \leq l$ occurs in the eigenspace E_l with multiplicity 1. The situation is illustrated in Figure 5.3.1. For each m, the result (8.4.11) now turns into the statement about Legendre polynomials that for all $f \in \mathrm{C}([0,\pi])$ one has

$$\frac{2l_k+1}{2}\frac{(l_k-m)!}{(l_k+m)!}\int_0^\pi f(\theta)\sin(\theta)|P_{l_k,m}(\cos\theta)|^2\,d\theta \xrightarrow{k\to\infty} \frac{1}{\pi}\int_0^\pi f(\theta)\,d\theta. \tag{8.4.15}$$

We conclude this section by proving

Proposition 8.4.4. *For fixed m, (8.4.15) holds for the full sequence of Legendre polynomials, that is, if l_k is replaced by l and "$k \to \infty$" is replaced by "$l \to \infty$".*

Proof. Let us begin by recalling the following classical result about the asymptotic behavior of Legendre polynomials [30, page 303]. For fixed $m \in \mathbb{Z}$ and each small $\varepsilon > 0$ one has

$$\frac{1}{l^m} P_{l,m}(\cos\theta) = \left(\frac{2}{l\pi\sin\theta}\right)^{1/2} \cos\left(\left(l+\frac{1}{2}\right)\theta - \frac{\pi}{4} + \frac{m\pi}{2}\right) + \mathrm{O}\big(l^{-3/2}\big) \tag{8.4.16}$$

as $l \to \infty$ uniformly for $\theta \in (\varepsilon, \pi - \varepsilon)$. From (8.4.13) and (8.4.16) we therefore obtain

$$\begin{aligned}
\widetilde{|Y_{l,m}|}(\theta)^2 &= \left|\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_{l,m}(\cos\theta)\right|^2 = \frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!} l^{2m} \left|\frac{1}{l^m} P_{l,m}(\cos\theta)\right|^2 \\
&= \frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!} l^{2m} \left|\left(\frac{2}{l\pi\sin\theta}\right)^{1/2} \cos\left(\left(l+\frac{1}{2}\right)\theta - \frac{\pi}{4} + \frac{m\pi}{2}\right) + \mathrm{O}\big(l^{-3/2}\big)\right|^2 \\
&= \frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!} l^{2m} \left(\frac{2}{l\pi\sin\theta} \cos^2\left(\left(l+\frac{1}{2}\right)\theta - \frac{\pi}{4} + \frac{m\pi}{2}\right) + \mathrm{O}\big(l^{-2}\big)\right).
\end{aligned}$$

The asymptotic relation

$$(l-m)!/(l+m)! \sim l^{-2m} \quad \text{as } l \to \infty \tag{8.4.17}$$

implies that $\frac{(l-m)!}{(l+m)!} l^{2m}$ is bounded in l, so we can use the simple relation $\frac{2l+1}{l} = 2 + \mathrm{O}(l^{-1})$ to obtain

$$\widetilde{|Y_{l,m}|}(\theta)^2 = \frac{(l-m)!}{(l+m)!} l^{2m} \frac{1}{\pi^2\sin\theta} \cos^2\left(\left(l+\frac{1}{2}\right)\theta - \frac{\pi}{4} + \frac{m\pi}{2}\right) + \mathrm{O}\big(l^{-1}\big), \tag{8.4.18}$$

uniformly for $\theta \in (\varepsilon, \pi - \varepsilon)$ and each small $\varepsilon > 0$. Now let $f \in \mathrm{C}([0,\pi], \mathbb{R})$ and choose $\varepsilon > 0$. Due to the uniform estimate (8.4.18) and boundedness of the integration domain we get

$$\begin{aligned}
&2\pi \int_\varepsilon^{\pi-\varepsilon} f(\theta) \widetilde{|Y_{l,m}|}(\theta)^2 \sin\theta \, d\theta \\
&\quad = 2\pi \int_\varepsilon^{\pi-\varepsilon} f(\theta) \frac{(l-m)!}{(l+m)!} l^{2m} \frac{1}{\pi^2\sin(\theta)} \cos^2\left(\left(l+\frac{1}{2}\right)\theta - \frac{\pi}{4} + \frac{m\pi}{2}\right) \sin(\theta)\, d\theta + \mathrm{O}\big(l^{-1}\big) \\
&\quad = \frac{2}{\pi}\frac{(l-m)!}{(l+m)!} l^{2m} \int_\varepsilon^{\pi-\varepsilon} f(\theta) \cos^2\left(\left(l+\frac{1}{2}\right)\theta - \frac{\pi}{4} + \frac{m\pi}{2}\right) d\theta + \mathrm{O}\big(l^{-1}\big).
\end{aligned} \tag{8.4.19}$$

The oscillatory integral in (8.4.19) has the limit

$$\begin{aligned}
&\lim_{l\to\infty} \int_\varepsilon^{\pi-\varepsilon} f(\theta) \cos^2\left(\left(l+\frac{1}{2}\right)\theta - \frac{\pi}{4} + \frac{m\pi}{2}\right) d\theta \\
&\qquad = \lim_{l\to\infty} \int_\varepsilon^{\pi-\varepsilon} f(\theta) \cos^2(l\theta)\, d\theta = \frac{1}{2} \int_\varepsilon^{\pi-\varepsilon} f(\theta)\, d\theta,
\end{aligned} \tag{8.4.20}$$

where the final equality is true because

$$\lim_{l\to\infty}\int\limits_{\varepsilon}^{\pi-\varepsilon} f(\theta)\cos^2(l\theta)\,d\theta = \lim_{l\to\infty}\int\limits_{\varepsilon}^{\pi-\varepsilon} f(\theta)\sin^2(l\theta)\,d\theta$$

and $\sin^2 + \cos^2 = 1$. Using (8.4.20) and (8.4.17) we conclude from (8.4.19) for each small $\varepsilon > 0$ that

$$\lim_{l\to\infty} 2\pi\int\limits_{\varepsilon}^{\pi-\varepsilon} f(\theta)\widetilde{|Y_{l,m}|}(\theta)^2\sin(\theta)\,d\theta = \frac{1}{\pi}\int\limits_{\varepsilon}^{\pi-\varepsilon} f(\theta)\,d\theta. \tag{8.4.21}$$

On the other hand, (8.4.17) and (8.4.18) imply the pointwise asymptotic property

$$\begin{aligned}\limsup_{l\to\infty}\widetilde{|Y_{l,m}|}(\theta)^2 &= \limsup_{l\to\infty}\frac{(l-m)!}{(l+m)!}l^{2m}\frac{1}{\pi^2\sin\theta}\cos^2\left(\left(l+\frac{1}{2}\right)\theta-\frac{\pi}{4}+\frac{m\pi}{2}\right)\\ &= \frac{1}{\pi^2\sin\theta}\lim_{l\to\infty}\left(\frac{(l-m)!}{(l+m)!}l^{2m}\right)\limsup_{l\to\infty}\cos^2\left(\left(l+\frac{1}{2}\right)\theta-\frac{\pi}{4}+\frac{m\pi}{2}\right)\\ &\le \frac{1}{\pi^2\sin\theta}\quad\forall\,\theta\in(0,\pi).\end{aligned}$$

Now, Fatou's Lemma implies that for each small $\varepsilon > 0$ one has the estimate

$$\begin{aligned}\limsup_{l\to\infty} 2\pi\int\limits_0^{\varepsilon} f(\theta)\widetilde{|Y_{l,m}|}(\theta)^2\sin\theta\,d\theta &\le 2\pi\int\limits_0^{\varepsilon}\limsup_{l\to\infty}|f(\theta)|\widetilde{|Y_{l,m}|}(\theta)^2\sin\theta\,d\theta\\ &\le 2\pi\int\limits_0^{\varepsilon}|f(\theta)|\frac{1}{\pi^2\sin\theta}\sin\theta\,d\theta \le \frac{2\varepsilon}{\pi}\|f\|_\infty\,,\end{aligned}$$

and analogously for the integral over $(\pi-\varepsilon,\pi)$. In a similar way one computes

$$\begin{aligned}\liminf_{l\to\infty} 2\pi\int\limits_0^{\varepsilon} f(\theta)\widetilde{|Y_{l,m}|}(\theta)^2\sin\theta\,d\theta &\ge 2\pi\int\limits_0^{\varepsilon}\liminf_{l\to\infty} f(\theta)\widetilde{|Y_{l,m}|}(\theta)^2\sin\theta\,d\theta\\ &\ge -2\pi\int\limits_0^{\varepsilon}|f(\theta)|\limsup_{l\to\infty}\widetilde{|Y_{l,m}|}(\theta)^2\sin\theta\,d\theta \ge -2\pi\int\limits_0^{\varepsilon}|f(\theta)|\frac{1}{\pi^2\sin\theta}\sin\theta\,d\theta\\ &\ge \frac{-2\varepsilon}{\pi}\|f\|_\infty\end{aligned}$$

and again analogously for the integral over $(\pi-\varepsilon,\pi)$. Together with (8.4.21) this allows us to conclude that

$$\begin{aligned}\limsup_{l\to\infty}\ 2\pi\int\limits_0^{\pi} f(\theta)\widetilde{|Y_{l,m}|}(\theta)^2\sin\theta\,d\theta &\le \limsup_{l\to\infty}\ 2\pi\int\limits_0^{\varepsilon} f(\theta)\widetilde{|Y_{l,m}|}(\theta)^2\sin\theta\,d\theta\\ &+\lim_{l\to\infty}\ 2\pi\int\limits_{\varepsilon}^{\pi-\varepsilon} f(\theta)\widetilde{|Y_{l,m}|}(\theta)^2\sin\theta\,d\theta + \limsup_{l\to\infty}\ 2\pi\int\limits_{\pi-\varepsilon}^{\pi} f(\theta)\widetilde{|Y_{l,m}|}(\theta)^2\sin\theta\,d\theta\\ &\le \frac{2\varepsilon}{\pi}\|f\|_\infty + \frac{1}{\pi}\int\limits_{\varepsilon}^{\pi-\varepsilon} f(\theta)\,d\theta + \frac{2\varepsilon}{\pi}\|f\|_\infty = \frac{1}{\pi}\int\limits_{\varepsilon}^{\pi-\varepsilon} f(\theta)\,d\theta + \frac{4\varepsilon}{\pi}\|f\|_\infty\end{aligned} \tag{8.4.22}$$

and similarly

$$\liminf_{l\to\infty} \; 2\pi \int_0^{\pi} f(\theta)\widetilde{|Y_{l,m}|}(\theta)^2 \sin\theta \, d\theta \geq \frac{1}{\pi} \int_{\varepsilon}^{\pi-\varepsilon} f(\theta)\, d\theta - \frac{4\varepsilon}{\pi} \|f\|_{\infty} . \tag{8.4.23}$$

The left hand sides of (8.4.22) and (8.4.23) are independent of ε, so that passing to the limit $\varepsilon \to 0$ we obtain

$$\lim_{l\to\infty} \; 2\pi \int_0^{\pi} f(\theta)\widetilde{|Y_{l,m}|}(\theta)^2 \sin(\theta) \, d\theta = \frac{1}{\pi} \int_0^{\pi} f(\theta)\, d\theta. \tag{8.4.24}$$

□

Remark 8.4.5. We do not know whether for the standard 2-sphere Theorem 8.3.8 is directly implied by the classical theory of Legendre polynomials. Moreover, it is crucial that m grows slower than l as $l \to \infty$. Indeed, if one considers the diagonal sequence $Y_{l,l}$ of zonal spherical harmonics, it is not difficult to see that, contrasting with our results, they concentrate along the equator in S^2 as $l \to \infty$ in the sense that for a given $\varepsilon > 0$ there is a constant $c(\varepsilon) > 0$ such that

$$\int_{S^2 - B_\varepsilon} |Y_{l,l}|^2 \, dS^2 = \mathrm{O}\big(e^{-c(\varepsilon) l}\big),$$

where B_ε denotes the tubular neighborhood of the equator of width ε, compare [15] and Figure 5.3.4, yielding qualitatively quite different limit measures.

Glossary

χ Isomorphism class of irreducible complex G-representations, identified with the associated character $G \to \mathbb{C}$.

$D_x \equiv \frac{1}{i}(\partial_{x_1}, \dots, \partial_{x_m}), \quad x \in \mathbb{R}^m,\ m \in \mathbb{N}$.

d_χ Dimension of the irreducible G-representations of class χ; in character notation one has $d_\chi = \chi(e)$, $e \in G$ denoting the neutral element.

$\breve{\Delta}$ Laplace-Beltrami operator on M, mapping $\mathrm{C}^\infty(M) \to \mathrm{C}^\infty(M) \subset \mathrm{L}^2(M)$.

Δ Unique self-adjoint extension of $\breve{\Delta}$, mapping $\mathrm{H}^2(M) \to \mathrm{L}^2(M)$, also called Laplace-Beltrami operator or just *Laplacian*.

G Compact connected Lie group.

$\widehat{G}$ Set of isomorphism classes of irreducible complex G-representations, identified with the character ring of G.

H Principal isotropy group of the G-action on M.

κ Dimension of the principal orbits of the G-action on M.

Λ_M^G Maximal number of elements in a totally ordered set of isotropy types of the G-action on M.

M Compact connected Riemannian manifold without boundary.

$M_{\mathrm{reg}} \equiv M(H) \subset M$, the principal stratum of M.

$\widetilde{M}_{\mathrm{reg}} \equiv M_{\mathrm{reg}}/G$, a smooth manifold.

$\widetilde{M} \equiv M/G$, in general not a manifold.

n Dimension of the manifold M.

$\Omega \equiv \mathbb{J}^{-1}(\{0\})$, the zero set of the momentum map $\mathbb{J} : T^*M \to \mathfrak{g}^*$. In general not a manifold.

$\Omega_{\mathrm{reg}} \equiv \Omega \cap T^*M(H)$, where $T^*M(H)$ denotes the union of all orbits of type H in T^*M. A smooth manifold.

$\widetilde{\Omega}_{\mathrm{reg}}$ $\equiv \Omega_{\mathrm{reg}}/G$, a smooth manifold.

p Given by $p(x,\xi) = \|\xi\|_x^2 + V(x)$, the Hamiltonian function $T^*M \to \mathbb{R}$ with the potential $V \in \mathrm{C}^\infty(M,\mathbb{R})$.

$\breve{P}(h)$ Schrödinger operator $\mathrm{C}^\infty(M) \to \mathrm{C}^\infty(M) \subset \mathrm{L}^2(M)$ with principal symbol represented by p.

$P(h)$ Unique self-adjoint extension of $\breve{P}(h)$, mapping $\mathrm{H}^2(M) \to \mathrm{L}^2(M)$, also called Schrödinger operator.

$[\pi_\chi|_H : \mathbb{1}]$ Frobenius factor: the multiplicity of the trivial representation in the H-representation $\pi_\chi|_H$.

$\mathcal{S}_\delta^{\mathrm{comp}}$, $\mathcal{S}_\delta^{\mathrm{bcomp}}$ See Section 1.1.

Σ_c $\equiv p^{-1}(\{c\})$, the smooth hypersurface associated to a regular value c of the Hamiltonian function p.

$\widetilde{\Sigma}_c$ $\equiv \big(\Sigma_c \cap \Omega_{\mathrm{reg}}\big)/G$, a smooth manifold.

T_χ Projection operator in $\mathrm{L}^2(M)$ onto the isotypic component $\mathrm{L}^2_\chi(M)$, see (5.1.3) and (5.1.4).

$\mathcal{W}_h$ Semiclassical character family, see Definition 5.2.1.

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