Equilibrium Asset Pricing

under Incomplete Information on Regimes

Dissertation

zur Erlangung des akademischen Grades
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Marburg, März 2015

Danksagung

Mein besonderer Dank gilt meinem Doktorvater und Erstgutachter, Prof. Dr. Bernhard Nietert. Seine Anregungen und Bereitschaft zur konstruktiven Diskussion haben diese Arbeit ermöglicht und mitgeprägt.

Auch Prof. Dr. Marc Steffen Rapp schulde ich Dank für die Übernahme des Zweitgutachtens und wertvolle Hinweise.

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Glossary of Commonly Used Symbols

$\longrightarrow$	symbol that denotes the whole history of a variable; $\overrightarrow{X_t}$ is defined as $\overrightarrow{X_t} = (X_0, .., X_t)$
$\forall$	for all
$*$	indicates optimal values of decision variables
$\underline{0}_k$	k-dimensional vector with only zero components
$^{(1)}$	superscript referring to assets that pay cash flows in every period
$AfR(.)$	adjustment for risk; $AfR(.) = \frac{\frac{\partial}{\partial W_{t+1}} J(.)}{E\left(\frac{\partial}{\partial W_{t+1}} J(.) \middle| .\right)}$
$a_{ij,t}$	loading of factor j in the affine linear model for asset i
$b_{i,t}$	loading of residual in the affine linear model for asset i
$^{\text{cI}}$	superscript denoting complete information
$corr_{d_1,d_2}$:	correlation in dividend growth of two assets conditional on the regime
$cov(.,.)$	covariance between two variables
$cov(.,.\|.)$	conditional covariance between two variables
C_t	consumption at time t
$d_i(.)$	functional form of growth rate of asset i's cash flows
${d_{t+1}}^{market}$	aggregate dividend growth
$D_i(.)$	functional form of risky cash flows
D_t	n-dimensional vector of risky cash flows at time t
${D_t}^{market}$	aggregate cash flows paid by all risky assets in the market portfolio at time t
$E_t(.)$	expected value (conditional on information at time t)
$E_{rn}(.)$	expectation taken with respect to the risk-neutralized conditional probability measure (density)
e_{t+1}	n-dimensional vector of residuals
$f(.\|.)$	conditional density function
fe_{t+1}	summarizes factors and residual in one symbol (used in cases where economic analysis does not need to distinguish between factors and residuals)
f_{t+1}	m-dimensional vector of factors
$f_{z,\tau}$	functional form weighting arguments of $z_{\tau+1} = f_{z,\tau}(z_\tau, \xi_{\tau+1})$
${f_{z,\tau}}^d$	functional form weighting arguments of ${z_{\tau+1}}^d = {f_{z,\tau}}^d\left({z_\tau}^d, {\xi_{\tau+1}}^d\right)$
${f_{z,\tau}}^p(.)$	functional form weighting arguments of ${z_{\tau+1}}^p = {f_{z,\tau}}^p({z_\tau}^p, {\xi_{\tau+1}}^p)$

H_t	wealth invested in the one-period riskless asset at time t
$\overline{H}_t^{eq}$	equilibrium aggregated wealth invested in the one-period riskless asset at time t
I_0	information at time 0 (which may be either complete or incomplete)
I_t^{cI}	complete information structure
I_t^{iI}	incomplete information structure
iI	superscript denoting incomplete information
$J(W,.,t)$	value function at time t that is assumed to be differentiable with $\frac{\partial}{\partial W_t}\bar{J} > 0$
$\bar{J}(.)$	value function without the erm containing the time precerence rate
K	finite number of regimes
m_{t+1}	function that takes strictly positive values for all possible realizations of the variable $z_{t+1}{}^p$ in the context of pricing with CARA utility
n_I	number of identical investors
N_t	vector of portfolio holdings of the risky assets at time t
$\overline{N}$	portfolio holding in market equilibrium
P_t	price vector of risky asset prices at time t
$P(S_{t+1} = j \| S_t = k, S_{t-}$ $= P(S_{t+1} = j \| S_t = k)$ $= p_{kj}$	probability of transiting form regime k at time t to regime j at time $t + 1$
p_{Draw}	probability that a regime is drawn
$p_{mean,Draw}$	probability of a draw of regimes in expected dividend growth rates
$p_{Stddev,Draw}$	probability of a draw of regimes in standard deviations of dividend growth rates
$p_{Mean,s}$	conditional transition probabilities, i.e., that a regime s in expected dividend growth rates becomes the new regime
$p_{\mu,s}$	conditional transition probabilities, i.e., that a regime s in expected dividend growth rates becomes the new regime in the discretized (Veronesi, 2000) model
$p_{Stddev,s}$	conditional transition probabilities, i.e., that a regime s in standard deviation of dividend growth rates becomes the new regime
$q_{t,t+\tau}$	stochastic discount factors that prices cash flow at time $t+\tau$ from the perspective of time t
$RP_t(.)$	vector of risk premia at time t
r_t	locally riskless interest rate at time t
Sig_{t+1}	signals at time $t+1$
S_t	regime; represents the changing economic conditions over time
$S_{t,t'}$	path of regimes from time t to t'

t	arbitrary future point in time
$t_{(k)}$	cash flows times
^{T}	superscript indicating transposition
T	(finite) planning horizon
$U(.)$	utility function that is assumed to be twice continuously differentiable with $U' > 0$ and $U'' < 0$
$Var_t(.)$	variance (conditional on information at time t)
$V_{t+1}{}^{market}$	aggregate value of the market portfolio of risky assets at time $t + 1$
W_τ	wealth at time τ
$\overline{W}_{t+1}{}^{eq}$	equilibrium aggregate wealth at time $t + 1$
z_t	sufficient statistic for the joint distribution of asset price and cash flow processes; $z_t \equiv (z_t{}^p, z_t{}^d)$
$z_t{}^d$	describes the state of the cash flow process at time t as perceived by each of the individual investors, i.e., relevant information from the perspective of the individual investor
$z_t{}^{d,0}$	current cash flow part of $z_t{}^d$
$z_t{}^{d,+}$	sufficient statistic part of $z_t{}^d$
$z_t{}^p$	information relevant to pricing
α	risk preference parameter of the exponential utility function (constant absolute risk aversion)
α_{t+1}	positive constant in the context of pricing with CARA utility; $\alpha_T = \alpha$
δ_t	relative dividend contribution (of assets' dividends to aggregate dividend)
$^{(\Delta_C)}$	superscript referring to assets that pay cash flows every Δ_C periods
Δ_C	number of time periods per cash flow period
Δ_{t+1}	indicates the change of a position at time $t + 1$
$\Delta\sigma_D{}^{max}$	maximum difference in standard deviations of dividend growth across regimes
η_{t+1}	signal noise
γ	risk preference parameter of the power utility function (constant relative risk aversion of the utility function $U(C) = \frac{C^{1-\gamma}}{1-\gamma}$)
$\lambda(i,j)$	Eigenvalues of matrix $M_{i,j}$
$\mu_{i,t}$	expected value of asset i's cash flows in the affine linear model
$\pi_{.,t} \equiv P(S_t = . \mid .)$	abbreviated notation for conditional regime probabilities
$\pi_0 \equiv \begin{pmatrix} \pi_{1,0} \\ \cdots \\ \pi_{K,0} \end{pmatrix}$	vector of initial regime probabilities

ρ	time preference rate
σ_D	standard deviation of dividend growth conditional on the regime
θ	time index
$\theta_t(.)$	risk-neutralized regime probabilities
$\{\xi_\tau{}^p\}$	noise term

List of Abbreviations

CARA	constant absolute risk aversion
CRRA	constant relative risk aversion
e.g.	for example (abbreviation for *exempli gratia*)
etc.	and so on (abbreviation for *et cetera*)
GDP	Gross Domestic Product
i.e.	that is (abbreviation for *id est*)
i.i.d.	independent and identically distributed
p.	page
pp.	pages

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1 Introduction

1.1 Introduction to the Problem

Modern financial theory acknowledges that decision makers do not perfectly know the stochastic process of financial figures, in particular corporate cash flows[1]. In other words, there is imperfect information in the sense of incomplete information. Incomplete information in the context of this thesis is modeled with the help of an unobservable underlying regime model, i.e., corporate dividends can assume several regimes. For example normally distributed dividend growth can switch between a regime with parameters μ_{high} and σ_{high} and another regime with parameters μ_{low} and σ_{low}. Regime switching models are regarded as parsimonious yet powerful model to capture abrupt changes in the behavior of financial time series including ARCH-effects, skewness, fat tails, non-linear dynamics, and time-varying correlations as Ang/Timmermann (2011), pp. 1, 5, and 6, point out. Moreover, regime switches have been detected empirically in financial time series (e.g., Guidolin/Timmermann (2006) and Whitelaw (2001)).

It is clear that incomplete information regarding dividends should be reflected in companies' stock prices. It is not so clear how incomplete information translates into risk premia. Does incomplete information as a second source of risk in addition to "normal" stock price fluctuations (first source of risk) (i) automatically increase overall risk and, hence, call for an adequate compensation, or (ii) can incomplete information and "normal" stock price risk interact in a way that overall risk is reduced and risk premia decrease? At least the classical intertemporal CAPM of Merton (1973) seems to indicate that a second source of risk might increase or decrease stocks' risk premia.

For that reason, it is the objective of this thesis to analyze whether incomplete information risk premia are greater or less than complete information risk premia. To achieve a certain degree of robustness of findings, the effects of various model assumptions on prices and risk premia are analyzed:

(i) Utility functions: in a first step, results are derived for general types of utility function. The implications of special utility functions (CARA and CRRA) are examined in a second step.

(ii) Types of regimes: a rich class of regime processes is considered; in particular, regimes in standard deviations are analyzed in addition to the regimes in expectations that are found in the literature.

[1] See, e.g., Chen/Epstein (2002), Epstein/Wang (1994), Hansen/Sargent/Tallarini (1999), and Maenhout (2004).

(iii) Form of the cash flow model: both fairly general cash flow functions and special functional forms are analyzed.
(iv) Number of risky assets: models with one risky asset as well as models with several risky assets are considered.

These goals are primarily tackled by means of theoretical analysis in discrete time. In addition, Chapter 5 contains numerical analysis for illustrative purposes.

The results of this thesis can be summarized as follows.

Incomplete information exerts a substantial influence on risk premia for all models considered in this thesis - CARA and CRRA utility functions, richer class of regime processes, various forms of cash flow model, and more than one risky asset - as the analytical analyses demonstrate. Core of all pricing approaches is the covariance between stochastic discount factor and asset return. Incomplete information fundamentally alters this covariance. The numerical analyses illustrate that the theoretical pricing results are also relevant from an economic point of view: incomplete information risk premia are significantly different from complete information risk premia and the different model versions also translate into significantly different risk premia.

1.2 Review of and Contribution Compared to the Literature

This thesis contributes to incomplete information asset pricing with regimes. Even though regime switching models are regarded as parsimonious and powerful model to capture empirical features of financial time series (e.g., Ang/Timmermann (2011), pp. 1, 5, and 6, Guidolin/Timmermann (2006), and Whitelaw (2001)), pricing models under regime switching are rare. One strand of the literature deals with incomplete information asset pricing without regime, i.e., does not model abrupt changes in the behavior of time series. The classical papers in this field are Detemple (1986) and Detemple/Murthy (1994), the most important more recent papers are Bansal/Yaron (2004), Ai (2010). A second strand of the literature comprises regimes, but assumes complete information, e.g., Abel (1988), Cecchetti/Lam/Mark (1990), Whitelaw (2001), and Elliot/Miao/Yu (2008).

The combination of incomplete information and regimes is only dealt with in the following few papers: David (1997), Veronesi (2000), Brandt/Zeng/Zhang (2004), Lettau/Ludvigson/Wachter (2008).

David (1997) considers optimal portfolio selection with two assets and two regimes and explores with the help of optimal portfolio weights implications for market risk premia. From that perspective, he discusses asset pricing implications only as a by-product. Moreover, he uses very special regimes: two assets with regimens in means where the regimes are inversely related, i.e., if the mean of one asset is high, the mean of the other asset is low and vice versa.

The most important paper in the field of incomplete information asset pricing with regimes, Veronesi (2000), argues within a very narrow framework: the stock market consists of only one stock whose dividend growth rate is normally distributed and its expected value is subject to regimes where decision makers possess power utility. This thesis extends Veronesi (2000) by allowing for CARA utility function (besides CRRA), non-normally distributed cash flows, regimes in means and standard deviations as well as more than one risky asset. Using more than one risky asset is often decisive because then the dividend of the risky asset no longer fully determines the stochastic discount factor; the fact that dividends determine stochastic discount factors substantially drive Veronesi (2000)'s results.

Brandt/Zeng/Zhang (2004) employ a model with Epstein/Zin (1989) preferences that is otherwise similar to Veronesi (2000); their focus is on different learning behaviors such as rational Bayesian updating, behavioral updating, etc. and whether incomplete information risk premia under regimes lead to fluctuations of the (conditional) equity risk premium that is comparable to the empirically observed of the equity risk premium. Since their main interest is the fluctuation of risk premia over time and not the size of incomplete information risk premium, in particular not the comparison of complete and incomplete risk premia, their emphasis is different from my research question.

Lettau/Ludvigson/Wachter (2008) consider Epstein/Zin (1989) preferences, one stock, and two regimes in means and standard deviations. While Epstein/Zin (1989) preferences is more general than CRRA utility, they do not consider CARA. Moreover, this thesis adds more general regimes and several risky assets. – Again, considering just one risky asset might substantially impact results.

1.3 Organization of the Thesis

The rest of the thesis is organized as follows. In Chapter 2 the regime switching model under incomplete information is outlined. Chapter 3 derives pricing results for CARA utility, Chapter 4 for CRRA utility. Chapter 5 contains numerical analyses that illustrate the economic significance of Chapter 4's extensions of the Veronesi (2000) model. The thesis ends with a conclusion (Chapter 6) and an ample appendix.

2 A Framework for the Analysis of Asset Prices under Incomplete Information

In the absence of arbitrage, asset prices can be expressed as the expected value of stochastically discounted future cash flows (see, e.g., Cochrane (2005), p. 61). Hence, asset pricing requires, first, to determine the cash flow stream, second to determine the stochastic discount factor and, third, to compute the expected value). Chapter 2 deals with the determination of the cash flow stream (first step, Section 2.1) and preparatory works (Sections 2.2 and 2.3) for the second and third step (which are then conducted in Chapters 3 and 4)

2.1 Cash Flows

2.1.1 Nature of Cash Flows

Cash flows are assumed to be risky and stem from equity positions. As such, they have no explicit maturity date. There is no default in the model and, therefore, cash flows of corporate debt are riskless and indistinguishable from the risk free asset. To capture institutional features of companies, two special models of cash-flows are used. In one version, cash flows are paid by companies with limited liability to their owners; hence these cash flows can be interpreted as dividends and will always be non-negative. In a second version, owners without limited liability are considered. In this case, cash flows may at times be negative.

2.1.2 Exogenous Cash Flows

The perspective of an investor is taken who is unable to influence cash flows but can describe them by a stochastic model. Cash flows will then throughout be exogenous rather than the results of corporate decisions. Production and dividend policies are not modeled explicitly.

2.1.3 Cash flows at Discrete Points of Time

In contrast to a substantial body of literature on asset pricing under incomplete information, cash flows are paid at discrete time intervals rather than continuously (as, e.g., in Detemple/Murthy (1994), Veronesi (2000)). There are several reasons to justify this difference. First, real-world cash flows are always discrete. This is especially the case for dividends which are paid quarterly, biannually or annually and, thus, deviate significantly from cash flows that are available from trading stocks. Second, assuming cash flows to follow a diffusion process of the type used in the related literature on incomplete information eliminates uncertainty about second moments of the cash flow processes. To see this, refer to, e.g., Merton (1980). He analyzes the problem of estimating the instantaneous expected growth rate and standard deviation of an Itô process of the form

2-1

$$\frac{dM_t}{M_t} = \alpha(t)dt + \sigma(t)dZ_t$$

under the additional assumption that the instantaneous expected growth rate $\alpha(t)$ and the instantaneous standard deviation $\sigma(t)$ are piecewise constant at least over short finite time intervals. Any such time interval contains an infinite number of realizations of the stochastic process. Merton then demonstrates how an estimator for the instantaneous standard deviation can be constructed whose variance goes to zero as the number of observations goes to infinity. By contrast, the properties of the estimator for the instantaneous expected growth rate are fundamentally different. Its variance depends on the length of the finite time interval over which the process is observed instead of the number of observations used to compute the estimator (see Merton (1980), p. 356, for the estimators and their variances). From a practical point of view, this means that observing the process over an arbitrarily short time interval, say a minute, suffices to compute the instantaneous standard deviation with any desired precision, whereas the accuracy of the estimate for the instantaneous growth rate can only be improved by observing the process over a longer time interval, say a day instead of a minute. It also implies that infrequent switches in the value of the standard deviation will immediately be detected, in contrast to switches in the instantaneous expected growth rate. Although Merton developed these results in the context of asset returns, it is clear that they formally also hold in the context of continuous dividends of the form (2-1). As a consequence, models such as Veronesi (2000) only include unobservable means. It is, however, obvious that standard deviations and correlations are also subject to incomplete information in the real world. If dividends are paid biannually, one has twenty observations over ten years and, in the face of changing market conditions, it is unclear how many of them still contain valuable information about future dividends. In conclusion, continuous time cash flow streams assume away important sources of incomplete information by supposing that second order moments can be estimated with arbitrary precision, i.e., by relying on a mathematical artifact. Third, using a discrete time model for cash flows allows for a wide variety of other information sources which are available at shorter time intervals than cash flows. Most movements in stock prices are due to information sources other than dividend announcements, for example the release of macroeconomic data, the release of sales figures for a particular industry, cuts in sovereign ratings, rumors about mergers and acquisitions etc. It is then desirable to include points of time into the model with such signals but without cash flows, and this is logically impossible in a framework with a continuous-time dividend stream.

2.1.4 Cash Flow Dynamics over Short and Long Time Horizons: Motivation and Overview

To motivate the formal cash-flow model given below, Figure 2-1 shows dividends per share for German car manufacturers from 1998 to 2013

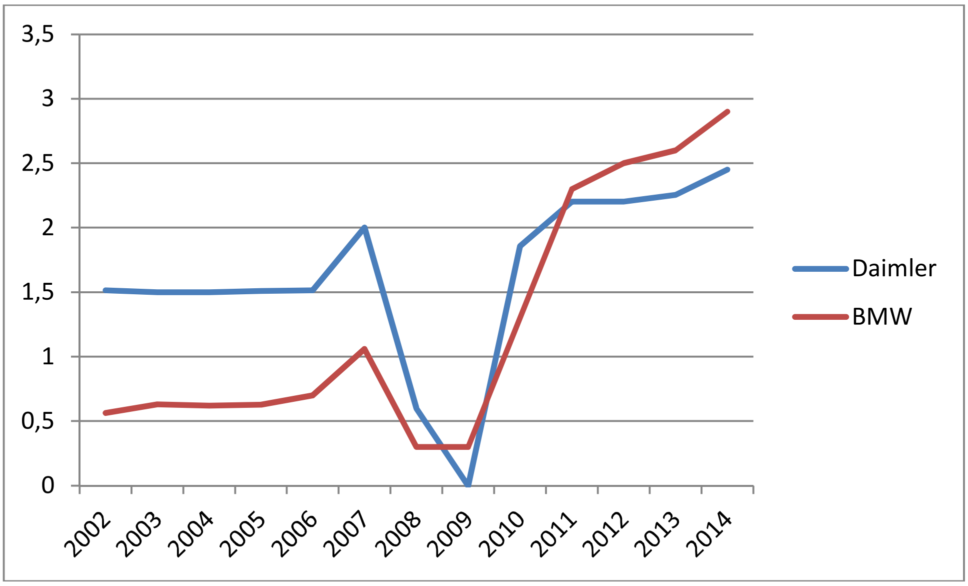

Figure 2-1: Dividends per Share (stock split-adjusted) for BMW AG and Daimler AG for the time period from 2002 to 2014. Source: OSIRIS database -> Stock Data -> Annual Stock Data

These examples demonstrate two important aspects of dividend behavior. (i) Dividends are influenced by common factors. In addition, there seems to be some firm-specific component. The effect of the economic crisis in 2008/2009 on dividends illustrates this: In Figure 2-1, a deep dent appears in the dividends of BMW and Daimler. (ii) Dividends are characterized by regimes. Dividend behavior changes substantially over time, often in the form of abrupt and unexpected breaks. For example Daimler had a stable dividend in the years 2002 to 2006 and 2011 to 2013, with levels changing in each phase. Note that such abrupt changes in regimes can be firm-specific, industry-wide, or economy wide. This suggests that changes in regimes are priced only insofar as they are "systematic risks" that affect aggregate dividends.

This motivational example suggests the following cash flow model: there is a short-term cash flow model and a long-term cash flow model. The short term cash flow model describes the behavior of cash flows over one period. It captures cash flow fluctuations given one economic environment and is modeled through factors and firm-specific influences referred to as residuals. Returning to the example, it is clear that a dividend model of Daimler would have to be substantially different in, say, 2008, compared to 2002. The long-term cash flow model portrays the change in the economic environment.

2.1.5 Short-Term Cash Flow Model: Factor Model

2.1.5.1 Factors and Residuals

If t and $t+1$ are two consecutive times of cash flow payments, a factor over the cash flow period $[t, t+1]$ is a random variable which makes cash flows at time $t+1$ stochastic from the point of view of market participants at time t. The factors are denoted by $f_{j,t+1}, j = 1, \dots, m$. In vector notation, f_{t+1} is the m-dimensional vector of factors. The number of factors then is m and assumed to be identical in all cash flow periods.

Factors can be broadly categorized into economy-wide, industry-wide and firm-specific, depending on the number of risky cash flows they affect. Economy-wide factors then affect all cash flows in some way. Typical examples are GDP growth or other macroeconomic variables. In the example from Section 2.1.4, the economic crisis in 2008/2009 will have influenced all firms in the economy to some extent. Industry-wide factors are limited to firms acting in a particular market, either directly or indirectly through intermediate goods. Again in the example from Section 2.1.4, automobile demand in China would be a factor relevant to the automobile industry. Firm-specific factors, as the name suggests, influence only cash flows of one particular firm. In Figure 2-1 there are several examples where dividends of one firm behave differently from those of other firms. For example, there is no analogue to the low dividend paid by DaimlerChrysler in 2001, and Volkswagen dividends still rise in 2008 when BMW and Daimler had already substantially cut theirs.

Residuals are similar to firm-specific factors, being linked to one particular risky asset, but differ in two ways. First, residuals have no precise economic interpretation. Instead, they capture all unspecified sources of cash flow variation which are not caused by factors. Second, it will always be assumed that there is a residual for each risky cash flow, whereas there might theoretically be assets without any factors. To motivate both differences, assume that a particular problem of asset pricing under incomplete information is to be analyzed. The definition of appropriate factors would be one of the first steps. For example, one might want to study the case where there is one single common factor for all risky assets, say GDP growth. It is, however, implausible that cash flows of all assets can be entirely attributed to this factor, i.e., there will remain a variety of firm-specific influences for all risky cash flows which will be lumped together into “residuals“. Mathematically, the residual of asset i over the period $[t, t+1]$ will in the following be denoted by $e_{i,t+1}$. If the number of risky cash flows is denoted by n, e_{t+1} is the n-dimensional vector of residuals.

Implicit in the notion of firm-specific factors and residuals is the absence of interrelations. Residuals and firm-specific factors should be unrelated to common factors. Moreover, neither the residual nor the specific factors of one cash flow should be related to residuals or specific factors of other cash flows. The question arises as to the correct formalization of these ideas; stochastic independence and the weaker assumption of uncorrelatedness are the two possible choices. As the probability distributions for factors and residuals and the functional form of the cash flow models will be allowed to be fairly general in the more theoretical part of the analysis, stochastic independence is assumed. For example, covariances are a measure for the linear relationship between two variables, and thus it would be less suited for other forms of relations. In conclusion, it is assumed that the vector of residuals e_{t+1} consists of n stochastically independent random variables, conditional on the information at t.

Although the particular distribution for factors and residuals is deliberately left open, the restriction that all variances and expectations be finite is imposed. This excludes some forms of distributions, such as the Cauchy distribution. Moreover, it is assumed that all factors and residuals at all times have a non-zero variance, i.e., are truly stochastic. The motivation of factors and residuals is to have stochastic sources in the model. All (conditionally) non-stochastic elements of cash flow will be parameters of the cash flow function. Therefore, the assumption is not restrictive. Through simple rescaling arguments, it can then be assumed that all factors and residuals have zero expectations and unit variances (in the context of a linear model (see Ingersoll (1987), p. 166):

2-2

$$E_t(e_{t+1}) = \underline{0}_n$$

2-3

$$Var_t(e_{i,t+1}) = 1$$
$$i = 1, \ldots, n$$

2-4

$$E_t(f_{t+1}) = \underline{0}_m$$

2-5

$$Var_t(f_{j,t+1}) = 1, j = 1, \ldots, m$$

where $\underline{0}_k, k \epsilon \mathbb{N}$, denotes the k-dimensional vector with only zero components.

Correlations between factors are explicitly allowed, insofar as the economic interpretation does not require independence, for example in the case of a single firm-specific factor for some risky cash flow.

Whenever the economic interpretation of results does not need to distinguish between factors and residuals, I simply combine factors and residuals into one i.i.d. random variable denoted by fe_t.

2.1.5.2 Functional Forms for the Relation between Cash Flows, Factors and Residuals

2.1.5.2.1 General Functional Form

The general relation between cash flows, factors and residuals considered in the analysis is

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$$D_{i,t+1} = D_i(D_t, t, f_{t+1}, e_{i,t+1})$$
$$i = 1, \dots, n$$

where D_t denotes the n-dimensional vector of risky cash flows at time t, $D_{i,t+1}$ is the i^{th} component of D_{t+1} and $D_i(.)$ is some known functional form.

(2-6) precludes a direct dependency on past cash flows with a time lag greater than one. Apart from that, it merely restates the ideas outlined above in mathematical form. Note in particular the time index as an argument on the right-hand side which represents dependence on conditions prevailing at time t.

Note that (2-6) is able to include stylized facts regarding companies' dividend mentioned in the seminal, but still valid paper of Lintner (1956): (i) dividends depend on past dividends. Changes in dividends rather than the dividend level are the main decision variable (p. 99); (ii) In addition to past dividends, earnings (captured in my model with the help of time, factors, and residuals) is the second major determinant of dividends (p. 101).

2.1.5.2.2 Cash Flows without Lags in Levels

While some results can be obtained for the general model, others depend on the interaction between cash flow model and assumptions about incomplete information, and thus some special cases of (2-6) are analyzed in detail. As a first special class of cash flow functions, models of the form

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$$D_{i,t+1} = D_i(t, fe_{t+1}), i = 1, \dots, n$$

are considered. In contrast to the general form (2-6), cash flows do not exhibit lags in levels, i.e., do not depend on D_t. An example would be an affine linear factor model for cash flows, i.e., cash flows are given by an additive relation between factors and residuals:

2-8

$$D_{i,t+1} = \mu_{i,t} + \sum_{j=1}^{m} a_{ij,t} \cdot f_{j,t+1} + b_{i,t} \cdot e_{i,t+1}, i = 1, \dots, n$$

The dependence on the time index in (2-6) here takes the form of time-dependent coefficients $\mu_{i,t}, a_{ij,t}$ and b_i

The model (2-8) implies the possibility of negative cash flows (unless additional assumptions are imposed on the distributions of factors and residuals) and is thus suited for the unlimited liability case.

2.1.5.2.3 Cash Flows without Lags in Growth Rates

As a second special class of cash flow functions, models of the form

2-9

$$D_{i,t+1} = D_{i,t} \cdot [1 + d_i(t, fe_{t+1})]$$
$$i = 1, \dots, n$$

with

$$d_i(t, fe_{t+1}) \geq -1$$
$$D_{i,0} > 0$$
$$i = 1, \dots, n$$

will be considered. Although cash flows exhibit lags in this special class of cash flow functions, the growth rate $d_i(.) = \frac{D_{i,t+1}}{D_{i,t}}$ does not depend on past cash flows or cash flow growth rates. For brevity, this class of cash flow functions is referred to as "cash flows without lags in growth rates". The assumption

$$d_i(t, fe_{t+1}) \geq -1$$

jointly with the assumption $D_{i,0} > 0$ implies that cash flows are either positive or zero. Moreover, if cash flows are zero at one point of time, they will remain zero at all future points of time. Clearly, the assumption of non-negative cash flows makes this special class of cash flow models a particularly suitable model for dividends. An example for this special class of models would be an affine-linear model for cash flow growth rates,

2-10

$$ln\big(1 + d_i(t, fe_{t+1})\big) = \mu_{i,t} + \sum_{j=1}^{m} a_{ij,t} \cdot f_{j,t+1} + b_{i,t} \cdot e_{i,t+1}, i = 1, \dots, n$$

Model (2-10) has a very close relation to dividend models in the related literature, which is desirable for comparing results, e.g., Veronesi (2000) and Brandt/Zeng/Zhang (2004). These dividend models are of the following form which obviously is a special case of (2-10) with zero factors:

2-11

$$ln\left(\frac{D_{i,t+1}}{D_{i,t}}\right) = \mu_{i,t} + b_{i,t} \cdot e_{i,t+1}$$
$$i = 1, \dots, n$$

2.1.6 Long-term Cash Flow Model: Regime Switching

The definition of the model is completed by specifying how the conditioning information in the factor model changes over time. In the case of the models based on an affine linear function, this takes the form of stochastic processes for both the coefficients, $\mu_{i,t}, a_{ij,t}$ and $b_{i,t}$, and factors and residuals. Here a Markov chain is used as it introduces regimes into cash flows, i.e., the second of the two empirical facts (after factors) described in Section 2.1.4. Factors and residuals will be assumed to be i.i.d.

2.1.6.1 Definition of the Regime Process

A finite time-homogeneous Markov Chain, denoted by S_t and referred to as the “regime process“, represents the changing economic conditions over time

2-12

$$S_t \epsilon \{1, \dots, K\}$$

2-13

$$P(S_{t+1} = j | S_t = k, S_{t-1} = l, \dots) = P(S_{t+1} = j | S_t = k) = p_{kj}$$
$$j, k \epsilon \{1, \dots, K\}$$

where p_{kj} denotes the probability of transiting form regime k at time t to regime j at time $t + 1$ and K is the finite number of regimes. Initial probabilities are denoted by

2-14

$$\pi_{k,0} \equiv P(S_0 = k)$$
$$kj \epsilon \{1, \dots, K\}$$

2-15

$$\pi_0 \equiv \begin{pmatrix} \pi_{1,0} \\ \cdots \\ \pi_{K,0} \end{pmatrix}$$

2.1.6.2 Definition of the Process of Factors and Residuals

It is assumed that factors and residuals are both i.i.d. and stochastically independent of the process of regimes:

2-16

$$\{fe_t\}\ i.i.d.$$

2-17

$$\{fe_t\} \text{ and } \{S_t\} \text{ stochastically independent}$$

This will imply that all intertemporal relations between cash flows are entirely attributable to the regime process, whereas factors and residuals acquire the properties of one-period disturbance terms. Because this structure is comparatively simple, results will not be blurred by the interaction of intertemporal dependencies in factors and regimes, for example.

2.1.6.3 Definition of the Models

The intertemporal version of the general model (2-6) becomes:

2-18

$$D_{i,t+1} = D_i\left(D_t, S_t, f_{t+1}, e_{i,t+1}\right)$$
$$i = 1, \dots, n$$

The affine linear models (2-8) and (2-10) are both defined by:

2-19

$$\mu_{i,t} = \mu_i(S_t)$$
$$i = 1, \dots, n$$

2-20

$$a_{ij,t} = a_{ij}(S_t)$$
$$i = 1, \dots, n, j = 1, \dots, m$$

2-21

$$b_{i,t} = b_i(S_t)$$

Note that (2-18) is again able to include additional elements of the Lintner (1956) model: (i) Firms do not adjust dividends immediately to meet the target payout level. If there is a permanent increase or decrease in earnings, dividends will be adjusted upwards or downwards over the course of several dividend periods through a process of “partial adjustment“ (see Lintner (1956), p. 100). (ii) Firms are reluctant to cut dividends, but will do so if there is a permanent decline in earning (see Lintner (1956), p. 101).

2.1.7 Time Periods without Cash Flows

In the models defined in Section 2.1.6, cash flows are paid at every point of time t. It is, however, desirable to have points of time without cash flows to facilitate the inclusion of information sources into the model that arrive at shorter time intervals than cash flows (see Section 2.1.3). This requires an adjustment of the time structure, and a subsequent definition of cash flow models with regimes and factors relative to the new time structure.

2.1.7.1 Time Structure

The points of time will again be discrete and denoted by $t \epsilon \mathbb{N}_0$. Δ_C denotes the number of time periods per cash flow period. If, for instance, time is measured in months and cash flows are paid biannually, then $[t, t+1]$is one month and Δ_C= 6. Cash flows would be paid at times 0,6,12, … etc. More generally, the times of cash flow payments are

2-22

$$t_{(k)} = k \cdot \Delta_C, k\epsilon\mathbb{N}_0$$

There are then no cash flows at times $t_{(k)} + 1, \dots, t_{(k+1)} - 1$. If Δ_C= 1, the model with cash flows at every period results as a special case.

2.1.7.2 Process of Regimes

Regimes are again given by the Markov chain (2-12) to (2-15). This implies that there may be regime switches at any point of time t.

2.1.7.3 Cash Flow Models

Cash flows now may depend on the entire history of regimes over $\left[t_{(k)}, t_{(k+1)}\right]$, i.e., $S_{t_{(k)}}, \dots, S_{t_{(k+1)}-1}$. In addition, cash flows are influenced by factors and residuals $fe_{t_{(k+1)}}$ at the time of payment:

2-23

$$D_{i,t_{(k+1)}} = D_i\left(D_{t_{(k)}}, S_{t_{(k)}}, \dots, S_{t_{(k+1)}-1}, fe_{t_{(k+1)}}\right)$$

2.1.8 Extensions and Limitations of the Cash Flow Model

2.1.8.1 Finite Number of Regimes

There could be circumstances where an infinite number of regimes would be a realistic choice, relevance, e.g., dividend growth rates could be normally distributed and its mean and standard deviation could be regime-dependent. This implies that mean and standard deviation could assume arbitrary values within a given interval, i.e., there is an infinite number of regimes (see, e.g., Veronesi, (2000), p. 810 for such a procedure).

A sufficiently fine partition of these intervals, however, could serve as a finite approximation.

2.1.8.2 Markov Property for Transition Probabilities

The definition of a Markov chain entails that transition probabilities may only depend on the current regime, but not on the past history of regimes (Assumption (2-17)). Nevertheless, an adequate redefinition of the regime process makes it possible to include scenarios into the analysis with transition probabilities also depending on past regimes (see, e.g., Cox/Miller (1977), p. 133): for convenience, assume that there are only two possible regimes, i.e., $S_t \epsilon \{0; 1\}$, and that transition probabilities at time t should depend on S_{t-1} and S_t:

2-24

$$P(S_{t+1} = 1 | S_t = 1, S_{t-1} = 1) \equiv p_{11,1}$$
$$P(S_{t+1} = 1 | S_t = 1, S_{t-1} = 0) \equiv p_{01,1}$$
$$P(S_{t+1} = 1 | S_t = 0, S_{t-1} = 1) \equiv p_{10,1}$$
$$P(S_{t+1} = 1 | S_t = 0, S_{t-1} = 0) \equiv p_{00,1}$$

Corresponding to each pair (S_{t-1}, S_t), one defines new regimes $S_t{}'$ (see again, Cox/Miller (1977), p. 133). The new regimes $S_t{}'$ are given through a (bijective) function:

2-25

$$S_t{}' : \{0; 1\}^2 \rightarrow \{0; 1; 2; 3\}$$
$$S_t{}'(0,0) = 0, \qquad S_t{}'(0,1) = 1, \qquad S_t{}'(1,0) = 2, \qquad S_t{}'(1,1) = 3$$

If regimes in the original formulation are $S_{t-1} = 0, S_t = 1, S_{t+1} = 1$, then this corresponds to a transition from $S_t{}'(0,1) = 1$ to $S_{t+1}{}'(1,1) = 3$, with $P(S_{t+1}{}' = 3 | S_t{}' = 1) \equiv p_{01,1}$. Some transition in the redefined system must have a zero transition probability. For example, a transition from $S_t{}'(0,0) = 0$ to $S_{t+1}{}'(1,0) = 2$ is not possible: $S_t{}'(0,0) = 0$ corresponds to $(S_{t-1} = 0, S_t = 0)$, and $S_t{}'(1,0) = 2$ corresponds to $(S_t = 1, S_{t+1} = 0)$, i.e., S_t would have to take both the values zero and one. Hence, it is necessary to define $P(S_{t+1}{}' = 2 | S_t{}' = 0) \equiv 0$.

As the example demonstrates, it is possible to define models where, motivated through economic considerations, transition probabilities depend on a certain number of past regimes. The problem can then be reformulated and solved under Assumption (2-17), before the economic interpretation of results takes place in the original formulation.

The obvious drawback of this method is the quickly expanding number of regimes in the redefined model. For K regimes and dependence of transition probabilities on k_{order} past regimes, $K^{k_{order}}$ regimes in the redefined model will be needed.

2.1.8.3 Time-Homogeneity of Transition Probabilities

By definition, the transition probabilities of a time-homogeneous Markov chain do not depend on the time index (Assumption (2-17)). On the other hand, it appears plausible that transition probabilities change over time. An increase in the probability of a transition of a severe economic crisis, for example, could be expected to lower prices of assets that react especially sensitive to such crises, as appears to be the case for the automobile industry. Nevertheless, such constellations can still be analyzed within the model framework with Assumption (2-17).

Again, a suitable redefinition of the Markov chain can be used to integrate changing transition probabilities into the model. For simplicity, take again the case where there are only two regimes, one corresponding to a "boom" state, $S_t = 1$, and one corresponding to a "recession" state, $S_t = 0$. The probability for a transition from a "boom" regime to a "recession" regime will now be allowed to take one of two possible values, $0 \leq {p_{10}}^l \leq {p_{10}}^h \leq 1$. To keep the example simple, the probability of remaining in a recession state, p_{00}, is taken to be constant. Finally, let λ denote the probability for a change in the recession probability. Now define a new regime model ${S_t}'$ with four possible regimes:

$${S_t}' \epsilon \{0, \overline{0}, 1, \overline{1}\}$$

Each of the pairs $0, 0'$ and $1, 1'$ consists of regimes which are indistinguishable in terms of cash flow, i.e.,

$$D_i(D_t, f_{t+1}, e_{i,t+1}, 0) = D_i(D_t, f_{t+1}, e_{i,t+1}, \overline{0})$$
$$D_i(D_t, f_{t+1}, e_{i,t+1}, 1) = D_i(D_t, f_{t+1}, e_{i,t+1}, \overline{1})$$

for all risky assets, all times and all possible realizations of factors and residuals. Let $\overline{0}$ and $\overline{1}$ correspond to the case with a high recession probability ${p_{10}}^h$, and let 0 and 1 represent the opposite case of a low recession probability, ${p_{10}}^l$. If transitions between the recession and boom regime groups, i.e., $\{0, \overline{0}\}$ and $\{1, \overline{1}\}$, and between the high and low transition probability groups, i.e., $\{0,1\}$ and $\{\overline{0}, \overline{1}\}$, occur independent, then it is easy to verify that the following transition probabilities solve the problem:

${S_t}' \backslash {S_{t+1}}'$	0	1	$\overline{0}$	$\overline{1}$
0	$(1-\lambda) \cdot p_{00}$	$(1-\lambda) \cdot (1-p_{00})$	$\lambda \cdot p_{00}$	$\lambda \cdot (1-p_{00})$
1	$(1-\lambda) \cdot {p_{10}}^l$	$(1-\lambda) \cdot (1-{p_{10}}^l)$	$\lambda \cdot {p_{10}}^l$	$\lambda \cdot (1-{p_{10}}^l)$
$\overline{0}$	$\lambda \cdot p_{00}$	$\lambda \cdot (1-p_{00})$	$(1-\lambda) \cdot p_{00}$	$(1-\lambda) \cdot (1-p_{00})$
$\overline{1}$	$\lambda \cdot {p_{10}}^h$	$\lambda \cdot (1-{p_{10}}^h)$	$(1-\lambda) \cdot {p_{10}}^h$	$(1-\lambda) \cdot (1-{p_{10}}^h)$

Table 2-1: Inclusion of Time-Dependent Transitions Probabilities into the Markov Chain Model.

2.1.8.4 i.i.d. Assumption on Factors and Residuals

Factors and residuals are assumed to be i.i.d. and independent of regimes. However, there is a wide variety of other plausible modeling choices for the processes of factors and residuals. For example, factors could exhibit mean reversion or auto-regression.

The general model (2-18), however, has the core feature that short- and long-term effects are separated. For that reason, these intertemporal dependencies are captures by means of regimes only and, thus, are categorizes as long-term effects. The advantage of this separation is a better economic interpretation of short- and long-term effects.

2.2 Decision Makers

There are assumed to be n_I identical investors that maximize additively separable expected utility over a finite time horizon T,

2-26

$$E\left(\sum_{\tau=0}^{T}\left\{\frac{1}{1+\rho}\right\}^{\tau} U(C_\tau)\middle| I_0\right)$$

where the utility function is assumed to be twice continuously differentiable with $U' > 0$ and $U'' < 0$. In addition to this general form of utility function $U(.)$, two special utility functions are considered:

<u>Constant absolute risk aversion</u>

2-27

$$U(C) = -exp(-\alpha \cdot C)$$
$$\alpha > 0$$

<u>Constant absolute risk aversion</u>

2-28

$$U(C) = \frac{C^{1-\gamma}}{1-\gamma}$$
$$\gamma > 0, \gamma \neq 1$$

For the case $\gamma = 1$ (where (2-28) is not defined), the utility function is defined as $ln(C)$,[2] in accordance with the literature.

The cases of constant and absolute relative risk aversion are of special interest because they are widely used in the literature on dynamic consumption and portfolio selection. In addition, assuming a special form of utility function often either admits an explicit solution where none can be found for general utility functions (see Section 3.1.4), or simplifies existing solutions. Moreover, a specification of the utility function is necessary for numerical computations.

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2 Since adding a constant to a utility function does not affect decisions, the utility function $\frac{C^{1-\gamma}-1}{1-\gamma}$ can be considered instead of $\frac{C^{1-\gamma}}{1-\gamma}$, and the limit $lim_{\gamma\to 1}\frac{C^{1-\gamma}-1}{1-\gamma} = ln(C)$, by l'Hôpital's rule, extends (2-28) to the case $\gamma = 1$.

2.3 Information Structures

Since this thesis wishes to analyze how incomplete information affects asset prices, it is necessary to describe what decision makers know and what is unknown to them. Although incomplete information is the direct object of interest, it must first be clarified what is meant by its logical opposite, complete information, which both defines incomplete information negatively and provides a benchmark case against which asset pricing results will be compared. Once incomplete information has been defined, it will be discussed how various information sources can be used to form conditional probabilities for the unobservable parts of the cash flow process, making it possible to treat the problem of asset pricing in an expected utility maximization framework.

2.3.1 Definition of Complete Information

There are two possible definitions of complete information at some time t: (i) a straightforward definition is knowledge of the model structure, combined with knowledge of all realizations of random processes up to time t, for example all regimes from time zero to time t. However, this definition will prove impracticable and too restrictive: most of this information would be discarded as irrelevant by decision-makers since it is not needed for taking optimal decisions (see Bertsekas (2005), pp. 251, for this argument in a closely related context). As a consequence, information can alternatively be defined as (ii) complete if decision-makers know everything that is truly relevant to them.

2.3.1.1 Complete Information as all Possible Past and Current Information

To apply the definition of complete information model structure, combined with knowledge of all realizations of random processes up to time, it is necessary to identify random processes and the elements of the model structure.

Random processes are the processes $\{S_t\}, \{fe_t\}$ underlying cash flows, cash flows themselves, i.e., $\{D_t\}$, the processes of prices of risky assets and interest rates, denoted by $\{P_t\}$ and $\{r_t\}$, respectively, and finally the processes of consumption, $\{C_t\}$, portfolios of risky assets, $\{N_t\}$, and wealth invested in the one-period riskless asset, $\{H_t\}$.

Realizations of these processes up to time t can be categorized as asset-related,

2-29

$$\left(\overrightarrow{S_t}, \overrightarrow{fe_t}, \overrightarrow{D_t}, \overrightarrow{P_t}, \overrightarrow{r_t}\right)$$

and investor-related,

2-30

$$\left(\overrightarrow{C_{t-1}}, \overrightarrow{N_{t-1}}, \overrightarrow{H_{t-1}}\right)$$

where, for any sequence $\{X_t\}, t\epsilon\mathbb{N}_0$, the symbol $\overrightarrow{X_t}$ is defined as

2-31

$$\overrightarrow{X_t} = (X_0, .., X_t)$$

Note that the index in (2-30) is $t-1$ instead of t. The reason for this is that the information derived from observing (2-29) and (2-30) defines the situation at time t immediately before new decisions are taken. In the case where cash flows are paid at times $t_k = k \cdot \Delta_C, k\epsilon\mathbb{N}_0$, the symbols $\overrightarrow{D_t}$ and $\overrightarrow{fe_t}$ are defined differently as

2-32

$$\overrightarrow{D_t} \equiv \left\{D_0, D_{\Delta_C}, \dots, D_{k(t)\cdot\Delta_C}\right\}$$

2-33

$$\overrightarrow{(fe)_t} \equiv \left\{(fe)_0, (fe)_{\Delta_C}, \dots, (fe)_{k(t)\cdot\Delta_C}\right\}$$

with $k(t)$ referring to the maximum natural number such that $k(t) \cdot \Delta_C \leq t$, and, hence, with $k(t) \cdot \Delta_C$ referring to the latest cash flow payment before or at time t. This convention ensures that, for example, $\overrightarrow{D_t}$ can always be interpreted as the history of cash flow payments up to time t.

Complete information about the model structure, combined with knowledge of all realizations of random processes up to time t then requires that decision-makers observe the processes (2-29) and (2-30) at all points of time. The information derived from (2-29) and (2-30) is denoted by ${I_t}^{cI}$.

Complete information further means: (i) decision-makers know the cash flow model; (ii) they understand both the institutional details of the asset markets; (iii) information on other market participants including their preferences, information, wealth, and past and present decisions is known.

It should be briefly mentioned what complete information does not mean: It should not be confused with perfect foresight. Information is complete at time t if all events up to time t are known, but it does not entail knowledge of future events. For example, complete information means that $S_0, S_1, \dots, S_t$ are known, not that S_{t+1} is already known at time t.

2.3.1.2 Complete Information as all Relevant Past and Current Information

The information that is truly relevant to decision-makers (I_t) will be substantially less than information I_t^{cI}. The general idea is that if decision-makers have access to information I_t which is not complete information in the sense of I_t^{cI}, i.e., [3]

2-34
$$I_t^{cI} \subseteq I_t$$

and if the optimal decision based on I_t is always the same as the optimal decision based on I_t^{cI}, then knowing I_t suffices (see Bertsekas (2005), p. 252). Similarly, it is possible that decision-makers do not know certain aspects of the model structure, but this lack of information may turn out to be irrelevant for taking decisions.

What precisely constitutes complete information in the sense of relevant past and current information within our framework is an important part of the analysis and cannot be determined at this stage. It is, therefore, perhaps helpful to choose an example that is entirely unrelated to the model. Assume that a decision-maker with preferences defined over the mean and variance of final wealth solves a static portfolio selection problem of the type that is commonly discussed in Markowitz portfolio selection Markowitz (1952). He or she may then not know the precise common distribution of asset returns. However, if means, variances, and covariances of returns are known, this would be entirely sufficient since the decision problem can be entirely characterized in terms of these moments.

2.3.2 Definition of Incomplete Information

It follows immediately that there are two possible definitions of incomplete information, depending on which definition of complete information is chosen. Corresponding to the definition of complete information as known model structure, combined with knowledge of all realizations of random processes up to time t, incomplete information would mean that there is at least some time t where decision-makers cannot observe I_t^{cI}, and/or they do not know some aspects of the model structure. Corresponding to the definition of complete information as relevant past and current information, information is incomplete (i) if there is at least some time t where decision-makers do only have less precise information than I_t^{cI}, and where this further results in decisions that deviate from those that would be taken based on I_t^{cI}, (ii) and/or decision-makers do not know a crucial part of the model.

Since asset prices will be derived from decisions, the definition of complete information as relevant past and current information is adequate here. In what follows, incomplete information will

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[3] Illustration of (2-34): Note that $I_t^{cI} \subset I_t$ means that I_t^{cI} contains more information than I_t because more states of the world are ruled out by I_t^{cI} than by I_t.

then always mean absence of complete information in the sense of relevant past and current information.

Note that incomplete information should not be confused with asymmetric information in the literature on rational expectation equilibria and game theory (for an overview of asset pricing under asymmetric information, see the textbook of Brunnermeier (2001)).

2.3.3 Description of an Information Structure with Unobservable Regimes, Factors and Residuals

2.3.3.1 Motivational Background

The information structure ${I_t}^{cI}$ contains all information that investors could theoretically know at time t. Since asset pricing under incomplete information is the topic of this thesis, incomplete information structures with less information than ${I_t}^{cI}$ must be defined. This section aims at specifying what “less information” means.

Caveat: Incomplete information structures are defined (see Section 2.3.2) as absence of complete information in the sense of relevant past and current information. This means that information structures with less information than ${I_t}^{cI}$ can still be complete information, and, thus, are not incomplete information. To illustrate this, it will turn out in Chapter 3 that investors at time t are not interested in realizations of factors and residuals up to time t as long as they know the current regime. Therefore, an information structure where the history of regimes, but not the one of factors and residuals, is observable, would still be complete information.

2.3.3.2 Overview of Information Structures

While it appears reasonable to assume that decision-makers can observe the past history of risky asset prices, interest rates and cash flows, it is unrealistic that decision-makers also can perfectly observe realizations of regimes, factors and residuals. Returning to the example of Figure 2-1, it appears likely from dividend data and other information sources that there may be a new regime from 2011 on. However, it is unclear what this regime might look like, i.e., the regime is not precisely observable. Factors may not be perfectly observable either. In the example of the automobile industry, the existence of common factors can be suspected from realized dividend data and information sources such as newspapers, but this does not mean that factors are precisely observable. Finally, residuals by definition refer to a sum of unspecified influences and are not observable.

This motivates the definition of the following information structure ${I_t}^{alt,asset}$ as containing potentially less information than ${I_t}^{cI}$. Investors observe the realizations of the processes

2-35

$$\overrightarrow{Sig_t}, \overrightarrow{D_t}, \overrightarrow{P_t}, \overrightarrow{r_t}$$

where the arrow above variables denotes the whole history of these variables, see (2-31), and

2-36

$$\pi_0, P_0, r_0$$

where Sig_t is a vector of signals[4] and π_0 is the vector of initial probabilities for regimes.

General form of signals

Signals in the context of incomplete information capture information sources on regimes, factors and residuals other than cash flows such as earnings reports, macroeconomic forecasts etc. From that perspective, it is plausible to add signals to information structures because they provide information on unobservable regimes, factors, and residuals.

The following fairly general structure of signals is assumed:

$\Delta_C = 1$ (cash flows paid in every period)

$$Sig_{t+1} = Sig_{t+1}(S_t, S_{t+1}, fe_{t+1}, \eta_{t+1})$$

Signals at time $t+1$ are allowed to be a function of the current regime S_{t+1}, the previous regime S_t, and current factors and residuals fe_{t+1}. Making signals a function of these unobservable elements of the cash flow process makes signals a useful source of information. In addition, there is an element of noise in signals, denoted by η_{t+1} and referred to as "signal noise" (which is independent of S_t, S_{t+1}, and fe_{t+1}).

$\Delta_C > 1$ (cash flows paid every Δ_C periods)

$$Sig_{t+1} = Sig_{t+1}\left(S_{t_{(k)},t}, S_{t+1}, fe_{t+1}, \eta_{t+1}\right)$$

The structure of signals is similar to the case $\Delta_C = 1$, with one exception. Signals are allowed to depend on the path of regimes from time $t_{(k)}$ (i.e., last cash flow payment date from the point of view of time t) instead of S_t. The idea is that cash flows at the next payment date $t_{(k+1)}$ will depend on the entire path of regimes $S_{t_{(k)},t_{(k+1)}-1}$ rather than a single regime. Note that if no cash flows are paid at time $t+1$, signals cannot depend on factors and residuals and read

$$Sig_{t+1} = Sig_{t+1}\left(S_{t_{(k)},t}, S_{t+1}, \eta_{t+1}\right)$$

---

[4] It should be noted that signals are entirely exogenous and should, thus, be distinguished from signals that are sent by a better informed party in the context of asymmetric information (see, e.g., Spence (1973)).

Illustrative examples

As a first example, take the case where regimes become observable with some time delay: although decision-makers may not know the current regime, past regimes may become known over time as new information arrives. For example, GDP data on one quarter is usually only available in the subsequent quarter and provides information on the past regime representing the state of the business cycle. This idea can be formalized by assuming that there are as many possible signal realizations as there are regimes and by setting

$$Sig_{t+1} = S_t$$

A second example makes the signal dependent on both S_{t+1} and S_t. This signal structure models that decision-makers may be sure that there has been a change in regime, but are unsure about what the new regime is. An example would be the change in central banks' monetary policy from standard monetary policy to quantitative easing in the wake of the financial crisis. Such a change in policy will almost certainly affect financial markets, i.e., there is a regime switch. If, however, the type of assets included in the purchase program of the central bank is not known, the exact form of the new regime is unclear.

A very stylized formalization of such a situation is given by the following signal model with only two possible signal realizations

2-37

$$Sig_{t+1} = \begin{cases} 1 & S_{t+1} = S_t \\ 2 & S_{t+1} \neq S_t \end{cases}$$

Here, signal 2 would be observed if and only if there is a regime switch, without providing further information on the new regime.

A third example covers signals that also depend on factors and residuals. Consider the model without lags in growth rates and a single regime in means (as in Veronesi (2000)),

$$1 + d(t, fe_{t+1}) = exp(\mu(S_t) + b \cdot e_{t+1})$$

A signal then might take the form of "true mean plus noise", i.e., Veronesi (2000), p. 810

$$Sig_{t+1} = \mu(S_t) + \eta_{t+1}$$

No signals

An information structure without signals means that all information comes from the history of cash flows; investors, therefore, merely observe

2-38

$$\overrightarrow{D_t}, \overrightarrow{P_t}, \overrightarrow{r_t}$$

instead of (2-35).

2.3.4 Conditional Regime Probabilities

Since investors do not know the regime, they can only assign probabilities to regimes conditional on past cash flows and signal observations. In this section, the problem of computing conditional regime probabilities is discussed in detail.

Note that conditional regime probabilities do not include any of the endogenous variables such as asset prices, interest rates or individual decisions. At this stage of the analysis, nothing is known about these variables, and conditional probabilities are derived exclusively from observations of exogenous variables, i.e., cash flows and signals. In contrast to asymmetric information (see, e.g., Grossman (1978)), asset prices under symmetric (but incomplete) information cannot convey private information from other investors.

More formally, conditional regime probabilities

2-39

$$P\left(S_t = s \middle| \overrightarrow{D_t}, \overrightarrow{Sig_{,t}},\right)$$

need to be determined, including the cases where cash flows are not paid every period ($\Delta_C > 1$).

2.3.4.1 Information from Cash Flows Only

In the case information from cash flows only, it is sufficient to discuss the case with cash flows paid in every period ($\Delta_C = 1$): the case $\Delta_C > 1$ is motivated by the idea that signals can arrive at shorter time intervals than cash flow payments, thus it is not interesting in the case without signals.

The problem then is the determination of conditional regime probabilities:

2-40

$$P\left(S = s \middle| \overrightarrow{D_t}\right)$$

This problem of determining conditional regime probabilities can be solved recursively similar to Hamilton (1994), p. 693, and the mathematical details can be found in Appendix A1.1.

<u>General cash flow model</u>

For general cash flow models

2-18

$$D_{i,t} = D_i\left(D_{t-1}, S_{t-1}, f_t, e_{i,t}\right)$$
$$i = 1, \dots, n$$

conditional regime probabilities are given by the following recursion:

2-41

$$P\left(S_t = \theta_0 \middle| \overrightarrow{D_t}\right) = \frac{\sum_{\theta_1=1}^{K} f(D_t | S_{t-1} = \theta_1, D_{t-1}) \cdot p_{\theta_1\theta_0} \cdot P\left(S_{t-1} = \theta_1 \middle| \overrightarrow{D_{t-1}}\right)}{\sum_{\nu_1=1}^{K} f(D_t | S_{t-1} = \nu_1, D_{t-1}) \cdot P\left(S_{t-1} = \nu_1 \middle| \overrightarrow{D_{t-1}}\right)}$$
$$\theta_0 \epsilon \{1, \dots, K\}$$

where the recursion is started with the initial probabilities

2-42

$$P\left(S_0 = \theta_0 \middle| \overrightarrow{D_{-1}}\right) \equiv P(S_0 = k)$$
$$\theta_0 \epsilon \{1, \dots, K\}$$

with conditional density function[5] of cash flows at time $t + 1$

2-43

$$f(D_t | D_{t-1}, S_{t-1})$$

If, for example, the affine linear factor model for cash flows, $D_t = \mu(S_{t-1}) + A(S_{t-1}) fe_t$, is chosen and fe_t is assumed to be multivariate standard normal, then $f(D_t | S_{t-1}, D_{t-1})$ is the density of a multivariate normal distribution with mean vector $\mu_{t-1} = \mu(S_{t-1})$ and covariance matrix $\Sigma_{t-1} = A(S_{t-1}) A(S_{t-1})^T$.

Cash flows without lags in levels

For cash flows without lags in levels

2-7

$$D_{i,t} = D_i\left(S_{t-1}, f_t, e_{i,t}\right)$$
$$i = 1, \dots, n$$

the conditional regime probability will not depend on D_{t-1} and (2-41) simplifies to:

2-44

$$P\left(S_t = \theta_0 \middle| \overrightarrow{D_t}\right) = \frac{\sum_{j=1}^{K} f(D_t | S_{t-1} = \theta_1) \cdot p_{jk} \cdot P\left(S_{t-1} = \theta_1 \middle| \overrightarrow{D_{t-1}}\right)}{\sum_{\nu_1=1}^{K} f(D_t | S_{t-1} = \nu_1) \cdot P\left(S_{t-1} = \nu_1 \middle| \overrightarrow{D_{t-1}}\right)}$$
$$\theta_0 \epsilon \{1, \dots, K\}$$

Cash flows without lags in growth rates

For cash flows without lags in growth rates

2-9

$$D_{i,t+1} = D_{i,t} \cdot [1 + d_i(t, fe_{t+1})]$$

conditional regime probabilities can be expressed in terms of dividend growth rates,

$$P\left(S_t = \theta_0 \middle| \overrightarrow{d_t}\right) = \frac{\sum_{j=1}^{K} f(d_t | S_{t-1} = \theta_1) \cdot p_{jk} \cdot P\left(S_{t-1} = \theta_1 \middle| \overrightarrow{d_{t-1}}\right)}{\sum_{\nu_1=1}^{K} f(d_t | S_{t-1} = \nu_1) \cdot P\left(S_{t-1} = \nu_1 \middle| \overrightarrow{d_{t-1}}\right)}$$

with

2-45

$$\overrightarrow{d_t} \equiv \left(\frac{D_1}{D_0} - 1, \dots, \frac{D_t}{D_{t-1}} - 1\right)$$

---

[5] D_t can have either a continuous or discrete distribution. If the distribution of D_t is discrete, $f(.|.)$ is to be interpreted as a conditional probability.

Notational simplification

For most of the analysis, the exact functional for of conditional regime probabilities is not relevant for the argument and an abstract notation that ignores the details is desirable. To this end, first use a simplified notation for conditional probabilities

2-46

$$\pi_{k,t} \equiv P\left(S_t = k \middle| \overrightarrow{D_t}\right)$$
$$k\epsilon\{1, \dots, K\}$$

2-47

$$\pi_t \equiv \begin{pmatrix} \pi_{1,t} \\ \cdots \\ \pi_{K,t} \end{pmatrix}$$

and recall the definition of initial probabilities

2-14

$$\pi_{k,0} \equiv P(S_0 = k)$$
$$k, j\epsilon\{1, \dots, K\}$$

and

2-15

$$\pi_0 \equiv \begin{pmatrix} \pi_{1,0} \\ \cdots \\ \pi_{K,0} \end{pmatrix}$$

Then conditional regime probabilities can be written as follows:

General cash flow model

2-48

$$\pi_t = \Pi(\pi_{t-1}, D_{t-1}, D_t)$$

where Π is an appropriate function.

Cash flow models with lags in levels

2-49

$$\pi_t = \Pi(\pi_{t-1}, D_t)$$

Cash flow model with lags in growth rates

2-50

$$\pi_t = \Pi(\pi_{t-1}, d_t)$$

2.3.4.2 Information from Cash Flows and Signals on Regimes

In the case with signals, both the cases $\Delta_C = 1$ (cash flows in every period) and $\Delta_C > 1$ (cash flows every Δ_C periods) must be considered. Since conditional regime probabilities for the case $\Delta_C = 1$ can be thought of as a special case of $\Delta_C \geq 1$, I discuss conditional regime probabilities for the general case $\Delta_C \geq 1$ and obtain conditional regime probabilitiesfor $\Delta_C = 1$ as a by-product.

$\Delta_C \geq 1$

Generally, if $\Delta_C \geq 1$, cash flows at a cash flow payment date $t_{(k+1)}$ depend on the path of regimes since the last payment date $t_{(k)}$ and up to the point of time prior to $t_{(k+1)}$, i.e., on the path $S_{t_{(k)},t_{(k+1)}-1}$. For that reason, conditional probabilities for regime paths (rather than single regimes) must be computed (see Appendix A1.2).

General cash flow model

Consider the general cash flow model

2-23

$$D_{i,t_{(k+1)}} = D_i\left(D_{t_{(k)}}, S_{t_{(k)}}, \dots, S_{t_{(k+1)}-1}, fe_{t_{(k+1)}}\right)$$
$$i = 1, \dots, n, k \epsilon \mathbb{N}$$

Conditional regime path probabilities can be found recursively where two cases have to be distinguished:

Case $t+1 = t_{(k+1)}$: (Δ_C)-periodic cash flows are paid at $t+1$

In this case, the regime path is degenerate because it consists of one single regime $S_{t_{(k+1)}}$. Hence, the recursion for regime path probabilities reads

2-51

$$P\left(\hat{S}_{t_{(k+1)}} \middle| \overrightarrow{\widehat{D}_{t_{(k+1)}}^{(\Delta_C)}}, \overrightarrow{\widehat{D}_{t_{(k+1)}}^{(1)}}, \overrightarrow{\widehat{Sig}_{t_{(k+1)}}}\right)$$
$$= \frac{\sum_{s_{t_{(k)},t}} \left\{ \begin{array}{c} P\left(\widehat{D}_{t_{(k+1)}}^{(\Delta_C)}, \widehat{D}_{t_{(k+1)}}^{(1)}, \widehat{Sig}_{t_{(k+1)}} \middle| \hat{S}_{t_{(k+1)}}, S_{t_{(k)},t} = s_{t_{(k)},t}, \widehat{D}_{t_{(k)}}^{(\Delta_C)}, \widehat{D}_t^{(1)}\right) \\ \cdot p_{\hat{S}_{t_{(k+1)}-1}, \hat{S}_{t_{(k+1)}}} \cdot P\left(S_{t_{(k)},t} = s_{t_{(k)},t} \middle| \overrightarrow{\widehat{D}_{t_{(k)}}^{(\Delta_C)}}, \overrightarrow{\widehat{D}_t^{(1)}}, \overrightarrow{\widehat{Sig}_t}\right) \end{array} \right\}}{\sum_{s_{t_{(k+1)}}}^{K} \sum_{s_{t_{(k)},t}} \left\{ \begin{array}{c} P\left(\widehat{D}_{t_{(k+1)}}^{(\Delta_C)}, \widehat{D}_{t_{(k+1)}}^{(1)}, \widehat{Sig}_{t_{(k+1)}} \middle| \begin{array}{c} S_{t_{(k+1)}} = s_{t_{(k+1)}}, S_{t_{(k)},t} = s_{t_{(k)},t} \\ , \widehat{D}_{t_{(k)}}^{(\Delta_C)}, \widehat{D}_t^{(1)} \end{array} \right) \\ \cdot p_{\hat{S}_{t_{(k+1)}-1}, \hat{S}_{t_{(k+1)}}} \cdot P\left(S_{t_{(k)},t} = s_{t_{(k)},t} \middle| \overrightarrow{\widehat{D}_{t_{(k)}}^{(\Delta_C)}}, \overrightarrow{\widehat{D}_t^{(1)}}, \overrightarrow{\widehat{Sig}_t}\right) \end{array} \right\}}$$

where $\hat{S}_{t_{(k+1)}}$ is a realization of the regime at time $t_{(k+1)}$, $t = t_{(k+1)} - 1$ is the time immediately before (Δ_C)-periodic cash flows are paid, $\widehat{D}_t^{(1)}$ are cash flows of assets that pay in every period (if such assets exist), and $p_{\hat{S}_{t_{(k+1)}-1}, \hat{S}_{t_{(k+1)}}}$ is the transition probability from regime $\hat{S}_{t_{(k+1)}-1}$ to regime $\hat{S}_{t_{(k+1)}}$.

$P\left(\widehat{D}_{t_{(k+1)}}^{(\Delta_C)}, \widehat{D}_{t_{(k+1)}}^{(1)}, \widehat{Sig}_{t_{(k+1)}} \middle| \hat{S}_{t_{(k+1)}}, S_{t_{(k)},t} = s_{t_{(k)},t}, \widehat{D}_{t_{(k)}}^{(\Delta_C)}, \widehat{D}_t^{(1)}\right)$ is a form of "likelihood" function that yields the probability or density of the observable quantities $\widehat{D}_{t_{(k+1)}}^{(\Delta_C)}, \widehat{D}_{t_{(k+1)}}^{(1)}, \widehat{Sig}_{t_{(k+1)}}$ conditional on the unobservable regimes $S_{t_{(k)},t}$ and $S_{t_{(k+1)}}$ and the previous cash flow payments $\widehat{D}_{t_{(k)}}^{(\Delta_C)}, \widehat{D}_t^{(1)}$; the precise form of this probability/density depends on the particular signal model.

Case $t+1 < t_{(k+1)}$: no (Δ_C)-periodic cash flows are paid at $t+1$

The recursion for regime path probabilities reads

2-52

$$P\left(\hat{S}_{t_{(k)},t+1}\middle|\overrightarrow{\widehat{D}_{t_{(k_t)}}}^{(\Delta_C)},\overrightarrow{\widehat{D}_t}^{(1)},\overrightarrow{\widehat{Sig}_t}\right)$$

$$=\frac{P\left(\widehat{D}_{t+1}{}^{(1)},\widehat{Sig}_{t+1}\middle|\hat{S}_{t_{(k_t)},t+1},\widehat{D}_t{}^{(1)}\right)\cdot p_{\hat{S}_t,\hat{S}_{t+1}}\cdot P\left(\hat{S}_{t_{(k)},t}\middle|\overrightarrow{\widehat{D}_{t_{(k_t)}}}^{(\Delta_C)},\overrightarrow{\widehat{D}_t}^{(1)},\overrightarrow{\widehat{Sig}_t}\right)}{\sum_{\bar{\bar{S}}_{t_{(k)},t+1}}P\left(\widehat{D}_{t+1}{}^{(1)},\widehat{Sig}_{t+1}\middle|\bar{\bar{S}}_{t_{(k)},t+1},\widehat{D}_t{}^{(1)}\right)\cdot p_{\bar{\bar{S}}_t,\bar{\bar{S}}_{t+1}}\cdot P\left(\bar{\bar{S}}_{t_{(k)},t}\middle|\overrightarrow{\widehat{D}_{t_{(k_t)}}}^{(\Delta_C)},\overrightarrow{\widehat{D}_t}^{(1)},\overrightarrow{\widehat{Sig}_t}\right)}$$

The structure of (2-52) is similar to the case $t+1 = t_{(k+1)}$ (i.e., (2-51)). Obviously, there are no (Δ_C)-periodic cash flows, and all new information comes from signals $\widehat{Sig}_{t+1}$ and, if existing, cash flows of assets that pay in every period, $\widehat{D}_{t+1}{}^{(1)}$.

Notational simplification

Both (2-51) and (2-52) can be notationally simplified and combined to one single recursion:

2-53

$$\pi_{t_{(k)},t+1}=\begin{cases}\Pi_0\left(\pi_{t_{(k)},t},D_t{}^{(1)},D_{t+1}{}^{(1)},Sig_{t+1}\right) & t+1<t_{(k+1)}\\ \Pi_1\begin{pmatrix}\pi_{t_{(k)},t},D_{t_{(k)}}{}^{(\Delta_C)},D_{,t}{}^{(1)}\\ ,D_{t_{(k+1)}}{}^{(\Delta_C)},D_{t_{(k+1)}}{}^{(1)},Sig_{t_{(k+1)}}\end{pmatrix} & t+1=t_{(k+1)}\end{cases}$$

where $\pi_{t_{(k)},t+1}$ is the vector of probabilities for all possible regime paths from time $t_{(k)}$ to time $t+1$.

$\Delta_C = 1$

If cash flows are paid in every period, the problem can be solved in one single step (see Appendix A1.1):

2-54

$$\pi_{j,t+1}=\frac{\sum_{k=1}^{K}P\left(D_{t+1},Sig_{S,t+1}\middle|D_t,S_{t+1}=j,S_t=k\right)\cdot p_{kj}\cdot\pi_{k,t}}{\sum_{\nu_0=1}^{K}\sum_{\nu_1=1}^{K}P\left(D_{t+1},Sig_{S,t+1}\middle|D_t,S_{t+1}=\nu_0,S_t=\nu_1\right)\cdot p_{\nu_1\nu_0}\cdot\pi_{\nu_1.t}}$$
$$j\epsilon\{1,\dots,K\}$$

where $\pi_{j,t+1}$ is the conditional probability of regime j at time $t+1$ (conditional on the history of signals and cash flows up to time $t+1$).

The simplified notation for the recursion is

2-55

$$\pi_t=\Pi_S\left(\pi_{t-1},Sig_{S,t},D_{t-1},D_t\right)$$

Cash flows without lags in levels

$$\pi_{j,t+1} = \frac{\sum_{k=1}^{K} P\left(D_{t+1}, Sig_{S,t+1} \middle| S_{t+1} = j, S_t = k\right) \cdot p_{kj} \cdot \pi_{k,t}}{\sum_{\nu_0=1}^{K} \sum_{\nu_1=1}^{K} P\left(D_{t+1}, Sig_{S,t+1} \middle| S_{t+1} = \nu_0, S_t = \nu_1\right) \cdot p_{\nu_1 \nu_0} \cdot \pi_{\nu_1 . t}}$$

$$j \epsilon \{1, \dots, K\}$$

The simplified notation for the recursion is

2-56

$$\pi_t = \Pi_S\left(\pi_{t-1}, Sig_{S,t}, D_t\right)$$

Cash flows without lags in growth rates

$$\pi_{j,t+1} = \frac{\sum_{k=1}^{K} P\left(d_{t+1}, Sig_{S,t+1} \middle| S_{t+1} = j, S_t = k\right) \cdot p_{kj} \cdot \pi_{k,t}}{\sum_{\nu_0=1}^{K} \sum_{\nu_1=1}^{K} P\left(d_{t+1}, Sig_{S,t+1} \middle| S_{t+1} = \nu_0, S_t = \nu_1\right) \cdot p_{\nu_1 \nu_0} \cdot \pi_{\nu_1 . t}}$$

$$j \epsilon \{1, \dots, K\}$$

The simplified notation for the recursion is

2-57

$$\pi_t = \Pi_S\left(\pi_{t-1}, Sig_{S,t}, d_t\right)$$

3 Partial Equilibrium Asset Pricing

Chapter 3 comprises definitions and general results (Section 3.1), results derived in the context of constant relative risk aversion (CRRA) (Section 3.2), and results that hold for constant absolute risk aversion (CARA) (Section 3.3).

Within this general structure, Section 3.1 serves two purposes. First, it analyzes what asset pricing results and economic interpretations can be obtained without having to rely on concrete specifications of utility functions (such as CARA or CRRA) or particular information scenarios (complete or incomplete information). Second, it allows Section 3.2 and 3.3 to analyze exclusively the particular insights from utility functions, cash flow models, and information scenarios instead of repeating all pricing results and interpretations of the general section for all combinations of utility functions, cash flow models, and information scenarios. To illustrate this structuring idea, consider one example. The derivation of equilibrium asset prices under complete and incomplete information involves technical steps that are virtually identical. Hence, Section 3.1 develops a general procedure instead of repeating all these steps in the sections on complete and incomplete information in Sections 3.2 and 3.3.

3.1 Definitions and General Results

3.1.1 Partial Equilibrium and investors' decision problem

In a dynamic consumption and portfolio selection problem the market for risky assets is in partial equilibrium if (i) all investors behave optimally at all points in time within the planning horizon and if (ii) demand for risky assets is equal to the exogenous supply of risky assets. The riskless interest rate is exogenous and constant at some positive value r; investors may borrow or lend any desired amount at this rate. In particular, the riskless asset is not assumed to be in zero net supply. Cash holdings are not regarded as an investment alternative. Moreover, cash flows are specified exogenously and not derived endogenously from optimal production decisions making this economy an exchange economy in the sense of Lucas (1978). Since an equilibrium relation is applied to the risky, but not the exogenous riskless asset, a partial equilibrium as in, e.g., Merton (1973) is obtained.

The time horizon is finite and denoted by T. Cash flows are paid at times $t = 0, \dots, T$; trading takes place at times $t = 0, \dots, T-1$. Investors consume at all points of time $t = 0, \dots, T$. The model ends at time T with the consumption of final wealth.

There is a large number of investors with identical initial wealth, preferences, and information. In principle, these investors can be aggregated into a single representative investor (this approach is taken, for example, in Lucas (1978)); however, certain economic aspects of asset pricing in competitive markets are easier to understand if it is explicitly assumed that there are many investors, a fact that outweighs the notational and mathematical simplifications resulting from the assumption of a representative investor.

Formally, a partial equilibrium consists of (i) an exogenously specified riskless interest rate r, (ii) a price process $\{P_t\}_{0\leq t\leq T-1}$, and (iii) an optimal strategy for each of the identical investors that determines consumption, portfolio holdings of risky assets and riskless investment, denoted by $\{C_t(I_t)\}_{0\leq t\leq T}$, $\{N_t(I_t)\}_{0\leq t\leq T-1}$,$\{H_t(I_t)\}_{0\leq t\leq T-1}$ respectively, such that the demand for risky assets by all investors is equal to the exogenous market supply of risky assets. Equilibrium, in other words, consists of a price process and an optimal market-clearing portfolio and consumption strategy.[6]

Optimality then means that the strategy of each of the individual investors must solve the problem[7]

3-1

$$max_{\{N_\tau,H_\tau,C_\tau\},0\leq\tau\leq T-1,C_T} E\left(\sum_{t=0}^{T}\frac{1}{(1+\rho)^t}\cdot U(C_t)\middle|I_0\right)$$

subject to the budget constraints

3-2

$$N_\tau{}^T P_\tau + H_\tau + C_\tau = W_\tau, 0\leq\tau\leq T-1$$

3-3

$$C_T = W_T$$

where I_0 is information at time 0 (which may be either complete or incomplete) and ρ denotes a time preference rate

Wealth at time τ is defined as

3-4

$$W_\tau \equiv \begin{cases} W_0{}^{initial} & \tau = 0 \\ N_{\tau-1}{}^T\{P_\tau + D_\tau\} + H_{\tau-1}\cdot(1+r) & 1\leq\tau\leq T-1 \\ N_{T-1}{}^T D_T + H_{T-1}\cdot(1+r) & \tau = T \end{cases}$$

where initial wealth is denoted by $W_0{}^{initial}$.

Note that there is no price vector P_T at time T since the model ends at time T. By defining

3-5

$$P_T = \underline{0}_n$$

however, the price process can formally be extended to time T allowing a unified definition of wealth for times $1\leq\tau\leq T-1$ and time $\tau = T$:

---

6 The formal definition of equilibrium as consisting of a price process and optimal behavior is standard in the literature of equilibrium asset pricing, see, e.g, Cox/Ingersoll/Ross (1985), p. 371, or Lucas (1978), p. 1432.

7 Indices referring to investors can be omitted from the description of the optimization problem because all investors are assumed to be identical.

3-6

$$W_\tau \equiv \begin{cases} W_0{}^{initial} & \tau = 0 \\ N_{\tau-1}{}^T\{P_\tau + D_\tau\} + H_{\tau-1} \cdot (1+r) & 1 \le \tau \le T \end{cases}$$

This simplifies the notation because the final period (from time $\tau = T - 1$ to time T) does not have to be treated separately (unless explicitly desired).

Market clearing means that the demand for risky assets must always be equal to the exogenous market portfolio (denoted by the vector $\overline{N}$),

3-7

$$\sum_{\nu=1}^{n_I} N_t{}^{(\nu)}(I_t) = \overline{N}\ \forall I_t$$

The assumption of identical investors implies that each investor must hold

3-8

$$N_t{}^{(\nu)}(I_t) = N_t(I_t) = \frac{1}{n_I} \cdot \overline{N}\ \forall I_t$$

$$\nu = 1, \ldots, n_I$$

Although the problem of any investor is completely described by (3-1) subject to the constraints (3-2) and (3-3), it is convenient (and standard in dynamic consumption and portfolio selection problems) to transform the problem into an equivalent unconstrained problem where investors choose risky portfolio holdings and consumption, but determine the riskless investment H_τ by means of the budget constraint (3-2). Then the problem of investors reads[8]

3-9

$$max_{\{N_\tau, C_\tau\}, 0 \le \tau \le T-1, C_T} E\left(\sum_{t=0}^{T} \frac{1}{(1+\rho)^t} \cdot U(C_t) \middle| I_0\right)$$

with wealth now given recursively by

3-10

$$W_{\tau+1} = [W_\tau - C_\tau] \cdot (1+r)$$
$$+ N_\tau{}^T\{P_{\tau+1} + D_{\tau+1} - (1+r) \cdot P_\tau\}$$
$$0 \le \tau \le T-1$$

3.1.2 Outline of the Derivation of Partial Equilibrium Asset Prices

3.1.2.1 Approaches to the Determination of Equilibrium Prices

In principle, there are two approaches of determining equilibrium asset prices. The first one postulates a price process and verifies that the postulated price process corresponds to partial equilibrium by solving the associated problem of investors (i.e., (3-9) and (3-10)). The second one derives as-

8 One should bear in mind that there is an implicit decision on the riskless investment – the riskless investment is chosen to balance the budget.

set prices recursively (rather than postulating and verifying them). Although the first approach has the advantage of mathematical simplicity, it does not reveal much economic intuition about why a given price process should correspond to partial equilibrium. It will, therefore, be more helpful for the understanding of asset pricing if asset prices are derived rather than postulated and verified. For that reason, the verification approach is only used in the appendix to formally ensure the validity of the argument. In the text part, however, the economic intuition behind the derivation of equilibrium pricing and its economic interpretation are in the center of attention.

The derivation of equilibrium asset prices is confronted with the following problem. On the one hand, equilibrium prices result from aggregated optimal individual decisions. On the other hand, individual investors make optimal consumption and portfolio decisions based on market prices. This means, the asset prices to be determined are at the same time input to and outcome of individual decisions. Based on the solution procedures found in the literature (e.g., Merton (1973) and Breeden (1979)), the following four-step procedure is used to cope with this difficulty.

(i) A class of models for the dynamics of asset prices and cash flows is exogenously specified. Thereby, it is implied that the equilibrium asset price dynamics that is yet to be determined belongs to this class of models.

(ii) Investors take optimal decisions based on the model specified in step i and derive that way the demand function.

(iii) Equilibrium asset prices are derived by aggregating individual optimal decisions in equilibrium.

(iv) A consistency check is performed to prove that the class of models specified in step i is indeed consistent with the actual equilibrium asset prices obtained in step iii.

To illustrate this four-step procedure by means of an example, consider the well-known model of Merton (1973). In this paper he assumes (Section 3) that stocks follow a diffusion process with stochastic drift and volatility parameters (= class of models, step i of the four-step procedure). He then (Section 5) derives investors' optimal decisions thereby generating the demand function (step ii). Aggregating demand in equilibrium (Section 8) leads to an equilibrium drift of the stocks to be valued (step iii) and the verification that the equilibrium asset price dynamics is indeed consistent with the assumed class of stock price models (step iv).

3.1.2.2 Step 1: Class of Models for Equilibrium Asset Prices and Cash Flows

3.1.2.2.1 Class of Models for Equilibrium Asset Prices

No-arbitrage ensures that asset prices are discounted future cash flows,

3-11

$$P_t = \sum_{\tau=1}^{T-t} \frac{1}{(1+\rho)^\tau} \cdot E\left(q_{t,t+\tau} \cdot D_{t+\tau} \middle| I_t\right)$$

Any reasonable class of models for equilibrium asset prices must, therefore, be compatible with the discounting approach (3-11). Within this general structure of equilibrium asset prices, the translation of information I_t into equilibrium asset prices is of particular importance. The central idea here is that not all available information is truly relevant for pricing as can be inferred from Grossman (1976) and Feldman (2007).[9] In other words, the desired class of models consists of restrictions on the dynamics of information relevant to pricing within the general structure of discounting future cash flows.

To be able to specify the class of models, it must be clarified what is meant by information relevant to pricing. Before attempting a general characterization, the idea of information relevant to pricing is illustrated by means of an example: consider the case of complete information where information frequency is equal to cash flow frequency. The current regime will certainly be information relevant to pricing because it captures the conditional distribution of future cash flows and asset prices are discounted future cash flows. By contrast, past regimes will be irrelevant information because regimes follow a Markov chain and all investors need to know to characterize the conditional distribution of cash flows is the current regime.

On a more general level, the problem of finding information relevant to pricing amounts to identifying a limited number of variables which describe asset prices at time t. Since this section concentrates on information relevant to pricing, the fact that asset prices are discounted cash flows is suppressed and the original asset pricing relation (3-11) is rewritten in the abbreviated form

3-12

$$P_t = {P_t}^I(I_t)$$

where ${P_t}^I(.)$ is an appropriate function that relates asset prices to information.

Note that the function ${P_t}^I(.)$ comprises future prices, cash flows, discount factors, and information.

[9] The idea that not all information is relevant to pricing, was developed first by Grossman (1976) In an asymmetric information setting. Feldman (2007) offers an overview and clarification of this idea in an incomplete information setting. In Section 1.2, he discusses the re-representation of incomplete information economies through a finite number of moments which capture relevant information.

Identifying information relevant to pricing then is tantamount to finding variables ${z_t}^p$ which are based on less information than I_t,

3-13

$${z_t}^p = {z_t}^p(I_t)$$

This information relevant to pricing can be categorized into (i) information relevant to pricing all assets and (ii) information relevant to pricing one asset i. (i) requires more information than (ii) because information relevant to price all assets must include information relevant to price one asset i.

Information relevant to pricing is, of course, endogenously determined as the result of equilibrium processes and, therefore, cannot be specified arbitrarily. Nevertheless, it is possible to formulate three requirements that ${z_t}^p$ must meet:

Sufficiency

Asset prices must be sufficiently described by ${z_t}^p$. Formally,[10] sufficiency means that there must be a function $P_t(.)$ that assigns equilibrium asset prices to ${z_t}^p$ at all points of time and for all possible values of I_t:

3-14

$$P_t\left({z_t}^p(I_t)\right) = {P_t}^I(I_t)$$
$$\forall I_t \forall t = 0, \dots, T-1$$

In particular, if asset prices at time $t+1$ are described by ${z_{t+1}}^p$, i.e., $P_{t+1} = P_{t+1}({z_{t+1}}^p)$, then equilibrium asset prices at time t must likewise satisfy $P_t = P_t({z_t}^p)$.

Irreducibility

All elements of ${z_t}^p$ must be essential in the sense that the sufficiency condition cannot be met by a new variable $({z_t}^p)'$ which is obtained by omitting one or several elements from ${z_t}^p$. The irreducibility property formalizes the idea that ${z_t}^p$ should only contain information that is relevant to the pricing of assets.

In addition to these two requirements, ${z_t}^p$ should possess the following property for conceptual simplicity:

Time-Independent Composition and Markov property of ${z_t}^p$

${z_t}^p$ should at all points of time consist of the same elements. For example, if ${z_t}^p$ consists of the current regime and current cash flows (S_t, D_t), then ${z_{t+1}}^p$ should likewise consist of (S_{t+1}, D_{t+1}).

[10] This formalization is based on Bertsekas (2005), p. 252.

In formalizing these ideas, I assume that ${z_t}^p$ follows a (possibly time-dependent) Markov process

3-15

$$ {z_{\tau+1}}^p = {f_{z,\tau}}^p({z_\tau}^p, {\xi_{\tau+1}}^p) $$
$$ \tau = 0, \dots, T-1 $$

where the noise terms $\{{\xi_\tau}^p\}$ are a vector-valued process which is assumed to consist of independent random variables.

The idea behind the independence assumption is that ${z_\tau}^p$ should capture all information relevant to pricing on future realizations ${z_{\tau+1}}^p$,..., ${z_T}^p$. If $\{\xi_\tau\}$ exhibited some form of dependence, then knowledge about the history of noise term realizations could possibly help to better predict these future values ${z_{\tau+1}}^p$,..., ${z_T}^p$ relative to a prediction based on ${z_\tau}^p$ alone.

3.1.2.2.2 Class of Cash Flow Models

In addition to the class of models for equilibrium asset prices, a class cash flow model is needed: asset prices are discounted future cash flows, and without adequate restrictions on cash flows little can be learned about asset prices. Moreover, investors will formulate their decision problem in terms of the joint distribution of asset prices and cash flows. Hence, an abstract asset price model alone will not suffice to derive the demand for assets and, therefore, asset prices.

The specification of a class of cash flow models is developed along the lines of the one for equilibrium asset prices: it is assumed that there is a set of variables ${z_t}^d$ that describes the state of the cash flow process at time t as perceived by each of the individual investors, i.e., relevant information from the perspective of the individual investor. The state of the cash flow process at time t perceived by each of the individual investors is defined as comprising (i) current cash flows (D_t) and (ii) a sufficient statistic for the distribution of future cash flows ($D_{t+1}, .., D_T$) conditional on information at time t. Denote by ${z_t}^{d,0}$ the current cash flow part and by ${z_t}^{d,+}$ the sufficient statistic part. Then ${z_t}^d$ can be written as

3-16

$$ {z_t}^d = \left({z_t}^{d,0}, {z_t}^{d,+}\right) $$

In contrast to ${z_t}^p$, ${z_t}^d$ is not the result of an equilibrium process but exogenous, i.e., input of the model.

To be able to capture the idea of relevant information regarding cash flows, ${z_t}^d$ must possess the following five properties:

Sufficiency

$z_t{}^d$ must be sufficient in the sense that it condenses all relevant information available at time t concerning the state of the cash flow process. Formally,

for the current cash flow part

3-17

$$D_t = D_t\left(z_t{}^{d,0}\right)$$
$$\forall t = 0, \ldots, T$$

for the sufficient statistic part

3-18

$$Prob_{D_{t+\tau}|I_t} = Prob_{D_{t+\tau}|I_t}\left(z_t{}^{d,+}, t, \tau\right)$$
$$\forall t = 0, \ldots, T-1$$
$$\forall \tau = 1, \ldots, T-t$$

where $Prob_{D_{t+\tau}|I_t}$ denotes the probability measure of cash flows $D_{t+\tau}$ conditional on information available at time t from the perspective of each of the individual investors.

Irreducibility

$z_t{}^d$ must not contain any components that can be omitted.

Compatibility with cash flow models defined in (2-18), (2-7), (2-9), and (2-23)

The cash flow model described by $z_t{}^d$ must be general enough to include the various versions of the cash flow processes defined in the previous chapter, both under complete and incomplete information, and for information frequencies equal to or higher than cash flow frequencies.

Perspective of Investors

The variables $z_t{}^d$ must describe the state of the cash flow process from the investors' point of view rather than an omniscient observer. If, for example, information is incomplete, then $z_t{}^d$ cannot contain the non-observable cash flow regime. Instead, it can only contain information derived from past signals and cash flows.

Time-Independent Composition and Markov property of $z_t{}^d$

$z_t{}^d$ should at all points of time consist of the same elements. It is assumed to follow a (possibly time-dependent) Markov process

3-19

$$z_{\tau+1}{}^d = f_{z,\tau}{}^d\left(z_\tau{}^d, \xi_{\tau+1}{}^d\right)$$
$$\tau = 0, \ldots, T-1$$

where the noise terms $\left\{\xi_\tau{}^d\right\}$ are a vector-valued process which is assumed to consist of independent random variables.

The rationale for the independence assumption is again that ${z_\tau}^d$ should summarize all information available at time τ.

3.1.2.2.3 Joint Dynamics of ${z_t}^p$ and ${z_t}^d$

So far ${z_t}^p$ and ${z_t}^d$ have been modeled isolated of each other. However, since asset prices are discounted future cash flows, there must be a close relation between ${z_t}^p$ and ${z_t}^d$. Hence, the last missing piece before the desired class of models is completely specified is a model of this relation, i.e., of the joint dynamics of the information relevant to pricing, ${z_t}^p$, and the state of the cash flow process, ${z_t}^d$.

The relation between ${z_t}^p$ and ${z_t}^d$ is captured by means of a vector-valued process of independent variables $\{\xi_\tau\}$.ξ_τ contains all components of ${\xi_\tau}^p$ and ${\xi_\tau}^d$. Hence, some components of ξ_t will affect both ${z_t}^p$ and ${z_t}^d$, whereas others only affect either ${z_t}^p$ or ${z_t}^d$.

If ${z_t}^p$ and ${z_t}^d$ are combined into the "state variable" z_t,

3-20

$$z_t \equiv \left({z_t}^p, {z_t}^d\right)$$

then it follows that the dynamics of z_t is described by a Markov process

3-21

$$z_{\tau+1} = f_{Z,\tau}(z_\tau, \xi_{\tau+1})$$
$$\tau = 0, \dots, T-1$$

z_t is a sufficient statistic for the joint distribution of asset price and cash flow processes. In particular, z_t completely describes, but is not identical to investors' investment opportunity set at time t; investor's opportunity set, a term introduced by Merton (1973), p. 870, is defined as the conditional distribution of one-period returns.

3.1.2.3 Step 2: The Optimization Problem of an Individual Investor

Pricing always refers to optimal decisions of each of the identical investors. To find these optimal decisions, the technique of dynamic programming is applied. In its most abstract form within a consumption and portfolio selection context, the problem of each of the identical investors can be formulated as (see, for example, Bertsekas (2005), Section 5.1)

3-22

$$J(I_t, t) = sup_{N,C}\left\{\frac{1}{(1+\rho)^t} \cdot U(C) + E(J(I_{t+1}, t+1)|I_t)\right\}$$

with wealth dynamics (based on (3-10))

$$W_{t+1}\left(z_{t+1}{}^{p}, z_t{}^{p}, z_{t+1}{}^{d}, W_t; C_t, N_t\right)$$
$$= [W_t - C_t] \cdot (1+r) + N_t{}^{T}\left\{P_{t+1}(z_{t+1}{}^{p}) + D_{t+1}\left(z_{t+1}{}^{d}\right) - (1+r) \cdot P_t(z_t{}^{p})\right\}$$

At the individual level, a problem arises which is similar (and closely related) to the problem of identifying information relevant to pricing in market equilibrium: information relevant to solve (3-22) will not include all information I_t but only a subset. Moreover, the information relevant to solve (3-22) will not completely coincide with information relevant to pricing in market equilibrium. For example, investors need to know their individual wealth to solve (3-22); individual wealth, however, is negligible at an aggregate level because there aggregate wealth of all investors is needed.

Finding the information relevant to solve (3-22) means that the arguments of the value function must be found. These arguments can be heuristically identified for time $t+1$ (and then inductively confirmed for time t):

The value function at time $t+1$, $J(.,t+1)$

The value function $J(.,t+1)$ will depend on the time index, investor wealth, and the state variable z_{t+1}:

3-23

$$J(I_{t+1}, t+1) = J(W_{t+1}, z_{t+1}, t+1)$$

To see this, note that $J(.,t+1)$ is the utility derived from pursuing an optimal policy from time $t+1$ to T based on the starting conditions at time $t+1$. Hence investors wealth must clearly be an argument of the value function because W_{t+1} determines how much can be consumed and invested at times $t+1$ to T, directly at time $t+1$ and indirectly as the basis for reinvestment. The success of reinvestment depends on the conditions at which investment can occur, i.e., the investment opportunity set, which is completely described by the state variable z_{t+1}. For that reason, the value function must be a function of z_{t+1}. Finally, the dependence of the value function on the time index arises from the finite time horizon of the decision problem.

Bellman Equation

Putting the results regarding information relevant to solve (3-22) together, the Bellman equation (3-22) reads

3-24

$$J(W_t, z_t, t) = \frac{1}{(1+\rho)^t}$$
$$\cdot sup_{N,C}\left\{U(C) + \frac{1}{1+\rho} \cdot E\left(\bar{J}\left(W_{t+1}\left(z_{t+1}{}^{p}, z_t{}^{p}, z_{t+1}{}^{d}, W_t; C, N\right), z_{t+1}, t+1\right) \middle| z_t, W_t\right)\right\}$$

where the wealth dynamics is given by

$$W_{t+1}\left(z_{t+1}{}^{p}, z_t{}^{p}, z_{t+1}{}^{d}, W_t; C_t, N_t\right)$$
$$= [W_t - C_t] \cdot (1+r) + N_t{}^{T}\left\{P_{t+1}(z_{t+1}{}^{p}) + D_{t+1}\left(z_{t+1}{}^{d}\right) - (1+r) \cdot P_t(z_t{}^{p})\right\}$$

Note that the value function for time $t+1$ can without loss of generality be written in the form

3-25

$$J(W_{t+1}, z_{t+1}, t+1) \equiv \frac{1}{(1+\rho)^{t+1}} \cdot \bar{J}(W_{t+1}, z_{t+1}, t+1)$$

Since the value function $\bar{J}(W_{t+1}, z_{t+1}, t+1)$ is the derived utility of a risk averse decision maker, it must meet the usual requirements of such a utility function, i.e., must possess positive marginal utility of wealth and concavity in wealth. Formally,

3-26

$$\frac{\partial}{\partial W_{t+1}} \bar{J}(W_{t+1}, z_{t+1}, t+1) > 0$$

3-27

$$\frac{\partial^2}{\partial {W_{t+1}}^2} \bar{J}(W_{t+1}, z_{t+1}, t+1) < 0$$

Consequently, only first order conditions are needed to find a maximum of (3-24); second order conditions are redundant because the maximand in the Bellman equation is concave (see Appendix A2.1).

First order conditions

Optimal values for portfolio holdings and consumption are found by differentiating the Bellman equation (3-24) with respect to N and C.

First-order condition for optimal portfolio holdings

3-28

$$E\left(\left. \begin{array}{c} \frac{\partial}{\partial W_{t+1}} \bar{J}\left(W_{t+1}\left({z_{t+1}}^p, {z_t}^p, {z_{t+1}}^d, {W_t}^{(\nu)}; {C_t}^{*(\nu)}, {N_t}^{*(\nu)}\right), z_{t+1}, t+1\right) \\ \cdot \left\{P_{t+1}({z_{t+1}}^p) + D_{t+1}\left({z_{t+1}}^d\right) - (1+r) \cdot P_t({z_t}^p)\right\} \end{array} \right| z_t, {W_t}^{(\nu)} \right) = \underline{0}_n$$

$$\nu = 1, \dots, n_I$$

First-order condition for optimal consumption

3-29

$$U'\left({C_t}^{*(\nu)}\right) - \frac{1+r}{1-\rho} \cdot E\left(\frac{\partial}{\partial W_{t+1}} \bar{J}\left(W_{t+1}\left({z_{t+1}}^p, {z_t}^p, {z_{t+1}}^d, {W_t}^{(\nu)}; {C_t}^{*(\nu)}, {N_t}^{*(\nu)}\right), z_{t+1}, t+1\right) \middle| z_t, {W_t}^{(\nu)} \right) = 0$$

3.1.2.4 Step 3: Equilibrium Asset Prices

To obtain equilibrium asset prices, three steps must be undertaken. First, prices for each of the individual investors must be obtained. Second, these prices must be transferred to market equilibrium. Third, a final pricing formula must be derived taking equilibrium consumption into account.

Individual pricing equation

Solving the first-order portfolio conditions of the identical investors (3-28) with respect to asset prices yields

$$P_t = \frac{1}{1+r} \cdot E\left(\left. \frac{\frac{\partial}{\partial W_{t+1}} \bar{J}\left(W_{t+1}\left(z_{t+1}{}^p, z_t{}^p, z_{t+1}{}^d, W_t{}^{(\nu)}; C_t{}^{*(\nu)}, N_t{}^{*(\nu)}\right), z_{t+1}, t+1\right)}{E\left(\left.\frac{\partial}{\partial W_{t+1}} \bar{J}\left(W_{t+1}\left(z_{t+1}{}^p, z_t{}^p, z_{t+1}{}^d, W_t{}^{(\nu)}; C_t{}^{*(\nu)}, N_t{}^{*(\nu)}\right), z_{t+1}, t+1\right) \right| z_t, W_t{}^{(\nu)}\right)} \cdot \left\{P_{t+1}(z_{t+1}{}^p) + D_{t+1}(z_{t+1}{}^d)\right\} \right| z_t, W_t{}^{(\nu)} \right)$$

$$\nu = 1, \dots, n_I$$

Prices in market equilibrium

Since equilibrium asset prices are of interest in this study and not individual asset prices, the individual demand must be aggregated in market equilibrium.

In equilibrium, wealth of each of the identical investors at time $t+1$ must be equal to $\frac{1}{n_I}$-th of equilibrium aggregate wealth $\overline{W}_{t+1}{}^{eq}$, yielding a relation between equilibrium asset prices at time t and equilibrium aggregate wealth at time $t+1$:

3-30

$$P_t = \frac{1}{1+r} \cdot E\left(\left. \frac{\frac{\partial}{\partial W_{t+1}} \bar{J}\left(\frac{1}{n_I} \cdot \overline{W}_{t+1}{}^{eq}, z_{t+1}, t+1\right)}{E\left(\left.\frac{\partial}{\partial W_{t+1}} \bar{J}\left(\frac{1}{n_I} \cdot \overline{W}_{t+1}{}^{eq}, z_{t+1}, t+1\right) \right| z_t, \overline{W}_t{}^{eq}\right)} \cdot \left\{P_{t+1}(z_{t+1}{}^p) + D_{t+1}(z_{t+1}{}^d)\right\} \right| z_t, \overline{W}_t{}^{eq} \right)$$

If the process of equilibrium aggregate wealth was observable, i.e., exogenously given, asset prices could readily be derived as, e.g., in the classical CAPM. However, here equilibrium aggregate wealth $\overline{W}_{t+1}{}^{eq}$ is endogenous.[11]

Final pricing formula – no closed-form solution

This endogeneity of $\overline{W}_{t+1}{}^{eq}$ will become clear if the individual wealth dynamics (3-10) is aggregated at equilibrium prices to obtain

3-31

$$\overline{W}_{t+1}{}^{eq} = \left[\overline{W}_t{}^{eq} - n_I \cdot C_t{}^*\right] \cdot (1+r) + \overline{N}^T \left\{P_{t+1}(z_{t+1}{}^p) + D_{t+1}(z_{t+1}{}^d) - (1+r) \cdot P_t(z_t{}^p)\right\}$$

where $C_t{}^*$ is the equilibrium level of consumption chosen by the identical investors.

[11] Exogenous here is to be understood as either not a decision variable at time t or not influenced by a decision variable. On the flipside, endogenous means either decision variable or influenced by a decision variable.

It is immediately evident that equilibrium aggregate wealth $\overline{W}_{t+1}{}^{eq}$ depends on equilibrium asset prices P_t and equilibrium consumption of each of the identical investors, $C_t{}^*$. Moreover, $\overline{W}_{t+1}{}^{eq}$ depends on equilibrium riskless investment $\overline{H}_{t-1}{}^{eq}$. To see this last dependence, aggregate individual wealth (3-4) at time t to equilibrium wealth at time t, $\overline{W}_t{}^{eq}$:

3-32

$$\overline{W}_t{}^{eq} = \overline{H}_{t-1}{}^{eq} \cdot (1+r) + \overline{N}^T \{P_t(z_t{}^p) + D_t(z_t{}^d)\}$$

– The dependence on equilibrium aggregate riskless investment $\overline{H}_{t-1}{}^{eq}$ becomes visible.

Finally, note that equilibrium consumption of each investor $C_t{}^*$ is endogenous as well: start with the first-order condition for optimal consumption (3-29) and observe that all investors in equilibrium choose the same level of consumption, $C_t{}^{*(v)} = C_t{}^*$ and possess wealth $W_{t+1}{}^{(v)} = \frac{1}{n_I} \cdot \overline{W}_{t+1}{}^{eq}$. Thus equilibrium consumption must satisfy

3-33

$$U'(C_t{}^*) = \frac{1+r}{1-\rho} \cdot E\left(\frac{\partial}{\partial W_{t+1}} J\left(\frac{1}{n_I} \cdot \overline{W}_{t+1}{}^{eq}, z_{t+1}, t+1\right) \middle| z_t, \overline{W}_t{}^{eq}\right)$$

By substituting equilibrium aggregate wealth at time t (3-32) and $t+1$ (3-31) into the pricing equation (3-30) and the condition for equilibrium consumption (3-33), a system of equations is obtained in which P_t and $C_t{}^*$ are implicit.

3.1.2.5 Step 4: Consistency Conditions

The fourth step identifies conditions under which the class of models for equilibrium asset prices specified in step i is indeed consistent with the actual equilibrium asset prices obtained in step iii (Equation (3-32)). This implies checking $z_t{}^p$ with respect to sufficiency, irreducibility, and time-independent composition as well as Markov property (see Section 3.1.2.2.1). I discuss these conditions simultaneously because they are interrelated: finding a set of variables $z_t{}^p$ which is sufficient obviously leads to the question of which elements cannot be omitted and, therefore, the problem of irreducibility; it will also reveal the composition of $z_t{}^p$ which, in turn, makes it possible to examine the dynamics of $z_t{}^p$ and, that way, to check time-independent composition as well as Markov property.

3.1.2.5.1 Sufficiency, irreducibility, and time-independent composition

From the system of equations (3-30), (3-31), (3-32), and (3-33) two requirements regarding sufficiency, irreducibility, and time-independent composition of $z_t{}^p$ can be deduced. In particular it is shown that information relevant to pricing ${z_t}^p$ must contain an equilibrium component (first consistency requirement) and a cash flow-based component (second consistency requirement).

Sufficiency: first consistency requirement

Based on the system of equations (3-30),(3-31), (3-32), and (3-33) that determines in principle asset prices, it becomes clear: asset prices depend on ${z_t}^p$, ${D_t}^{market}$ and ${\overline{H}_{t-1}}^{eq}$, i.e., $P_t = P_t\left({z_t}^p, {D_t}^{market}, {\overline{H}_{t-1}}^{eq} \cdot (1+r)\right)$ where where ${D_{t+1}}^{market}$ is the aggregate cash flows paid by all risky assets in the market portfolio. However, $P_t = P_t\left({z_t}^p, {D_t}^{market}, {\overline{H}_{t-1}}^{eq} \cdot (1+r)\right)$, would imply that $P_t \neq P_t({z_t}^p)$, i.e., ${z_t}^p$ was not sufficient and (3.14) violated. Hence, the first consistency requirements demands that the exogenous parts of equilibrium aggregate wealth ${\overline{W}_t}^{eq}$, ${D_t}^{market}$ and ${\overline{H}_{t-1}}^{eq} \cdot (1+r)$, must be captured by ${z_t}^p$, i.e., ${D_t}^{market} = {D_t}^{market}({z_t}^p)$ and ${\overline{H}_{t-1}}^{eq} \cdot (1+r) = {\overline{H}_{t-1}}^{eq}({z_t}^p) \cdot (1+r)$ so that $P_t = P_t({z_t}^p)$ (3-14) still holds.

The economic intuition behind this formal argument can be seen from

3-30

$$P_t = \frac{1}{(1+r)} \cdot E\left(\left.\begin{array}{c}\dfrac{\dfrac{\partial}{\partial W_{t+1}}\bar{J}\left(\dfrac{1}{n_I}\cdot {\overline{W}_{t+1}}^{eq}, z_{t+1}, t+1\right)}{E\left(\dfrac{\partial}{\partial W_{t+1}}\bar{J}\left(\dfrac{1}{n_I}\cdot {\overline{W}_{t+1}}^{eq}, z_{t+1}, t+1\right)\middle| z_t, {\overline{W}_t}^{eq}\right)} \\ \cdot\left\{P_{t+1}({z_{t+1}}^p) + D_{t+1}({z_{t+1}}^d)\right\}\end{array}\right| z_t, {\overline{W}_t}^{eq}\right)$$

(3-30) expresses prices as present values of future cash flows where marginal utility of wealth is used as (stochastic) discount factor. Hence, the dependence of the discount factor and prices on equilibrium wealth becomes evident.

Applying the fact that ${\overline{W}_t}^{eq}$, ${D_t}^{market}$ and ${\overline{H}_{t-1}}^{eq} \cdot exp(r)$ are already contained in ${z_t}^p$, simplifies the pricing equation (3-30) to

3-34

$$P_t(z_t) = \frac{1}{(1+r)} \cdot E\left(\left.\begin{array}{c}\dfrac{\dfrac{\partial}{\partial W_{t+1}}\bar{J}\left(\dfrac{1}{n_I}\cdot {\overline{W}_{t+1}}^{eq}, z_{t+1}, t+1\right)}{E\left(\dfrac{\partial}{\partial W_{t+1}}\bar{J}\left(\dfrac{1}{n_I}\cdot {\overline{W}_{t+1}}^{eq}, z_{t+1}, t+1\right)\middle| z_t\right)} \\ \cdot\left\{P_{t+1}({z_{t+1}}^p) + D_{t+1}({z_{t+1}}^d)\right\}\end{array}\right| z_t\right)$$

Sufficiency: second consistency requirement

According to pricing equation (3-34), information relevant to pricing reads z_t. However, sufficiency requires that ${z_t}^p$ is information relevant to pricing and not $z_t = \left({z_t}^p, {z_t}^d\right)$.

The difference between ${z_t}^p$ and z_t obviously is ${z_t}^d$, i.e., ${z_t}^d = \left({z_t}^{d,0}, {z_t}^{d,+}\right)$ (see (3-16)). Hence, the solution to this problem technically is simply: integrate elements of ${z_t}^d$ into ${z_t}^p$. The question, however, is: what elements of ${z_t}^d$ must be integrated into ${z_t}^p$, i.e., are information relevant to pricing? Since P_t is the present value of prices and cash flows at time t + 1, a sufficient statistic of the conditional cash flow distribution must be information relevant to pricing. In other words, ${z_t}^{d,+}$ must be part of ${z_t}^{p}$. On the other hand, the current cash flow part ${z_t}^{d,0}$ does not provide any information relevant to pricing in addition to ${z_t}^{d,+}$ and, hence, is not contained in ${z_t}^{p}$.

The economic intuition behind this formal argument is straight forward. Prices are expected discounted future cash flows. Hence, information regarding future cash flows must be information relevant to pricing.

One clarification regarding the role of current cash flows is in order. If current cash flows influence the conditional distribution of future cash flows, e.g., due to an autoregressive structure of cash flows like $D_{t+1} = D(D_t, S_t, fe_{t+1})$, current cash flow will be part of both ${z_t}^{d,+}$ (influence on D_{t+1}) and ${z_t}^{d,0}$ (cash flow at t). On the other hand, if current cash flows do not influence the conditional distribution of future cash flows, e.g., due to independent structure of cash flows like $D_{t+1} = D(S_t, fe_{t+1})$, current cash flow will not be contained in ${z_t}^{d,+}$ (no influence on D_{t+1}) but in ${z_t}^{d,0}$ (cash flow at t).

Irreducibility and time-independent composition

The discussion regarding sufficiency has shown that ${z_t}^{p}$ consists of the exogenous parts of equilibrium aggregate wealth, ${D_t}^{market}$ and ${\overline{H}_{t-1}}^{eq} \cdot (1+r)$, as well as ${z_t}^{d,+}$:

3-35

$${z_t}^{p} = \left({D_t}^{market}, {\overline{H}_{t-1}}^{eq} \cdot (1+r), {z_t}^{d,+}\right)$$

Hence, it is clear that none of these three components can be omitted in general because they are needed for sufficiency.

Moreover, all three elements will be needed for sufficiency at all points in time. For that reason, time-independent composition is given.

3.1.2.5.2 Markov property

In order to ensure that the state variable z_t follows a Markov process, it must be shown that z_{t+1} can be obtained from z_t and independent noise terms. The results concerning the composition of z_t and z_{t+1} provide a first step to the solution of this problem:

3-36

$$z_t = \left({D_t}^{market}, {\overline{H}_{t-1}}^{eq} \cdot (1+r), {z_t}^{d,+}, {z_t}^{d,0}\right)$$

$$z_{t+1} = \left({D_{t+1}}^{market}({z_{t+1}}^{p}), {\overline{H}_t}^{eq}({z_{t+1}}^{p}) \cdot (1+r), {z_{t+1}}^{d,+}, {z_{t+1}}^{d,0}\right)$$

${z_t}^d = \left({z_t}^{d,0}, {z_t}^{d,+}\right)$ is model input and Markov for the models chosen in this analysis. Nor does ${D_t}^{market}$ introduce any difficulties: it is a function of ${z_t}^d$, ${D_t}^{market} = {D_t}^{market}\left({z_t}^d\right)$, thus no additional stochastic elements (which might be non-Markovian) are needed for ${D_t}^{market}$.

Therefore, the only element which remains to be checked with respect to the Markov property is equilibrium aggregate riskless investment ${\overline{H}_{t-1}}^{eq}$. To that end, equilibrium aggregate riskless investment ${\overline{H}_{t-1}}^{eq}$ must be made explicit – So far it is only implicitly contained in the budget equation.

In equilibrium, wealth satisfies both

3-37

$${\overline{W}_t}^{eq}({z_t}^p) = {\overline{H}_{t-1}}^{eq}({z_t}^p) \cdot (1+r) + \overline{N}^T P_t({z_t}^p) + {D_t}^{market}({z_t}^p)$$

and

3-38

$${\overline{W}_t}^{eq}({z_t}^p) = {\overline{H}_t}^{eq} + n_I \cdot {C_t}^* + \overline{N}^T P_t({z_t}^p)$$

By combining these equations, equilibrium consumption can be expressed through equilibrium riskless investment:

3-39

$${C_t}^* = \frac{1}{n_I} \cdot \left\{ {\overline{H}_{t-1}}^{eq}({z_t}^p) \cdot (1+r) + {D_t}^{market}({z_t}^p) - {\overline{H}_t}^{eq} \right\}$$

In a next step plug (3-37), (3-38), and (3-39) into the system that determined equilibrium prices and consumption (3-30), (3-31), (3-32), and (3-33) to obtain:

3-40

$$P_t({z_t}^p) = \frac{1}{(1+r)} \cdot E\left(\left. \frac{\frac{\partial}{\partial W_{t+1}} J\left(\frac{1}{n_I} \cdot \left[{\overline{H}_t}^{eq}({z_t}^p) \cdot (1+r) + \overline{N}^T \left\{P_{t+1}({z_{t+1}}^p) + D_{t+1}\left({z_{t+1}}^d\right)\right\}\right], z_{t+1}, t+1\right)}{E\left(\left. \frac{\partial}{\partial W_{t+1}} J\left(\frac{1}{n_I} \cdot \left[{\overline{H}_t}^{eq}({z_t}^p) \cdot (1+r) + \overline{N}^T \left\{P_{t+1}({z_{t+1}}^p) + D_{t+1}({z_{t+1}}^d)\right\}\right], z_{t+1}, t+1 \right) \right| {z_t}^p \right)} \cdot \left\{P_{t+1}({z_{t+1}}^p) + D_{t+1}\left({z_{t+1}}^d\right)\right\} \right| {z_t}^p \right)$$

3-41

$$U'\left(\frac{1}{n_I} \cdot \left\{ {\overline{H}_{t-1}}^{eq}({z_t}^p) \cdot exp(r) + {D_t}^{market}\left({z_t}^d\right) - {\overline{H}_t}^{eq}({z_t}^p) \right\}\right) = \frac{1+r}{1-\rho} \cdot E\left(\left. \frac{\partial}{\partial W_{t+1}} J\left(\frac{1}{n_I} \cdot \left[{\overline{H}_t}^{eq}({z_t}^p) \cdot (1+r) + \overline{N}^T \left\{P_{t+1}({z_{t+1}}^p) + D_{t+1}\left({z_{t+1}}^d\right)\right\}\right], z_{t+1}, t+1\right) \right| {z_t}^p \right)$$

The equation system (3-40) and (3-41) does not permit a closed-form solution for riskless investment and, hence, a direct analysis of its Markov property. Nevertheless, it provides an abstract characterization of equilibrium riskless investment and its dynamics that is precise enough to check the Markov property.

First, observe that $\overline{H}_t^{eq}$ is a function of $z_t{}^p$ since the equation system (3-40) and (3-41) is parameterized by $z_t{}^p$ only. Second, (compounded) $\overline{H}_t^{eq}$ is an element of $z_{t+1}{}^p$ (see (3-36). Combining both steps, it can be concluded that $\overline{H}_t^{eq}(z_t{}^p) = function_t(z_t{}^p)$. This is a special case – no noise term – of

$$z_{t+1}{}^p = f_{z,t}{}^p(z_t{}^p, \xi_{t+1}{}^p)$$

Hence, $\overline{H}_t^{eq}$ as a deterministic quantity is a (degenerated) Markov process.

3.1.2.6 Dependency of Equilibrium Consumption, of the Value Function, and of Reinvestment Opportunities on $z_t{}^p$

To prepare for the economic interpretation of asset pricing results, the dependency of equilibrium consumption and the value function J on relevant information must be made explicit and more specific. So far, however, we know $P_t(z_t{}^p)$ (see end of Section 3.1.2.3), but only $J(W_{t+1}, z_{t+1}, t+1)$ (see (3-23)) and no statement regarding the dependency of equilibrium consumption on z_t.

Using the additional details regarding the composition of z_{t+1} elaborated in Section 3.1.2.5, allows improving the specification of equilibrium consumption and the value function.

3.1.2.6.1 Dependency of Equilibrium Consumption on $z_t{}^p$

Equilibrium consumption of each of the identical investors is a function of $z_t{}^p$: to see this, recall that equilibrium aggregate wealth at time t and equilibrium aggregate riskless investment at time t are both functions of $z_t{}^p$ ((3-37) and Markov property, p. 45, respectively). Solving the equilibrium aggregate budget equation (3-38) for equilibrium consumption yields

3-42

$$C_t{}^*(z_t{}^p) = \frac{1}{n_I} \cdot \left\{\overline{W}_t^{eq}(z_t{}^p) - \overline{H}_t^{eq}(z_t{}^p) - \overline{N}^T P_t(z_t{}^p)\right\}$$

The economic intuition behind this result is as follows. The difference between $z_t{}^p$ and z_t is the current cash flow part $z_t{}^{d,0}$. However, current cash flow is integrated into current equilibrium wealth and, hence, already taken into consideration.

3.1.2.6.2 Dependency of the Value Function on $z_t{}^p$

To prove that the second argument of the value function is $z_t{}^p$ rather than z_t, inductively assume that the value function at time $t+1$ is $\bar{J}(W_{t+1}, z_{t+1}{}^p, t+1)$. Then the Bellman (3-24) equation reads

$$J(W_t, z_t, t) = \frac{1}{(1+\rho)^t} \cdot U\left(C_t{}^*(z_t{}^p, W_t)\right) + \frac{1}{1+\rho} \cdot E\left(\bar{J}\left(W_{t+1}\left(z_{t+1}{}^p, z_t{}^p, z_{t+1}{}^d, W_t; C_t{}^*(z_t{}^p, W), N_t{}^*(z_t{}^p, W_t)\right), z_{t+1}{}^p, t+1\right) \middle| z_t, W_t\right)$$ [12]

where $C_t{}^*(z_t{}^p, W)$ and $N_t{}^*(z_t{}^p, W)$ are optimal consumption and portfolio holdings of risky assets if the individual investor possesses wealth W_t and if the relevant pricing information is $z_t{}^p$.

Observe, first, that all random elements in the expectation arise from $z_{t+1}{}^p$ and $z_{t+1}{}^d$. Second, since information relevant ot pricing is Markov, $z_t{}^p$ is all that needs to be known about $z_{t+1}{}^p$. Third, $z_{t+1}{}^d$ contains cash flows at time $t+1$ and sufficient statistic for the distribution of future cash flows $(D_{t+2}, .., D_T)$ conditional on information at time $t+1$. Obviously, current cash flow at time t cannot influence $z_{t+1}{}^d$, only $z_t{}^{d,+}$ can do it. However, $z_t{}^{d,+}$ is already contained in $z_t{}^p$ as the second consistency requirement (see p. 42) has shown. In other words, $z_t{}^p$ is also all that needs to be known about $z_{t+1}{}^d$.

Taking these three aspects together, it is obtained for the left-hand side

3-43

$$J(W_t, z_t{}^p, t) = \frac{1}{(1+\rho)^t} \cdot U\left(C_t{}^*(z_t{}^p, W_t)\right) + \frac{1}{1+\rho} \cdot E\left(\bar{J}\left(W_{t+1}\left(z_{t+1}{}^p, z_t{}^p, z_{t+1}{}^d, W_t; C_t{}^*(z_t{}^p, W), N_t{}^*(z_t{}^p, W_t)\right), z_{t+1}{}^p, t+1\right) \middle| z_t{}^p, W_t\right)$$

To see the economic intuition behind the formal argument, proceed as follows. The value function consists of utility from optimal current consumption and the expected utility of optimal future consumption and reinvestment opportunities. Current cash flows do not provide any additional information over wealth concerning optimal consumption at time t. Nor do current cash flows convey information about future consumption and future reinvestment opportunities. Hence, the value function cannot depend on current cash flow.

3.1.2.6.3 Dependency of Reinvestment Opportunities on $z_t{}^p$

Reinvestment opportunities at time t consist of prices of risky assets at time t, the riskless interest rate at time t, and the conditional distributions of future asset prices and cash flows from the per-

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[12] The base case of the assumption is given by the boundary condition $J(W_T, z_T{}^p, T) = J(W_T, T) = \frac{1}{(1+\rho)^T} \cdot U(W_T)$.

spective of time t. Prices of risky assets at time t must be a function of ${z_t}^p$ (see the definition of ${z_t}^p$ in (3-14)). The riskless interest rate is exogenous. The conditional distribution of future asset prices is captured by means of ${z_t}^p$ because future prices of risky assets at some time $t+\tau$ are functions of ${z_{t+\tau}}^p$ (sufficiency) and the process of information relevant to pricing is Markov. Finally, the second consistency requirement (see p. 42) demands that the conditional distribution of future cash flows is described by ${z_t}^p$.

In other words, ${z_t}^p$ can be interpreted not only as information relevant to pricing but also as reinvestment opportunities at time t.

3.1.3 Economic Interpretation of Asset Prices

The influence of regimes on asset prices under complete and incomplete information and, hence, the economic interpretation of asset prices, is analyzed with the help of two questions:

(i) How are risky assets priced, both under complete and incomplete information?

(ii) How are risky assets priced relative to the riskless asset, both under complete and incomplete information?

3.1.3.1 First Question: Economic Interpretation of Stochastic Discount Factors

The reasoning of the preceding sections leads to the following pricing equations for risky assets

3-44

$$P_t({z_t}^p) = \frac{1}{1+r} \cdot E\left(\left. \frac{\frac{\partial}{\partial W_{t+1}} \bar{J}\left(\frac{1}{n_I} \cdot {\overline{W}_{t+1}}^{eq}({z_{t+1}}^p), {z_{t+1}}^p, t+1\right)}{E\left(\frac{\partial}{\partial W_{t+1}} \bar{J}\left(\frac{1}{n_I} \cdot {\overline{W}_{t+1}}^{eq}({z_{t+1}}^p), {z_{t+1}}^p, t+1\right) \middle| {z_t}^p\right)} \cdot \left\{P_{t+1}({z_{t+1}}^p) + D_{t+1}\left({z_{t+1}}^d\right)\right\} \right| {z_t}^p \right)$$

The structure of this pricing equation is identical to the stochastic discount factor approach of asset pricing. Therefore, the stochastic discount factor approach will be the basis of the economic analysis of asset prices and be preferred over the alternative (but equivalent) approaches of pricing by means of risk-neutralized probabilities or pricing by means of certainty equivalents.

Making the stochastic discount factor explicit, the pricing equation reads

3-45

$$P_t({z_t}^p) = E\left(q_{t,t+1}({z_{t+1}}^p, {z_t}^p) \cdot \left\{P_{t+1}({z_{t+1}}^p) + D_{t+1}\left({z_{t+1}}^d\right)\right\} \middle| {z_t}^p\right)$$

with stochastic discount factor

$$q_{t,t+1}({z_{t+1}}^p, {z_t}^p) \equiv \frac{1}{1+r} \cdot \frac{\frac{\partial}{\partial W_{t+1}} \bar{J}\left(\frac{1}{n_I} \cdot {\overline{W}_{t+1}}^{eq}({z_{t+1}}^p), {z_{t+1}}^p, t+1\right)}{E\left(\frac{\partial}{\partial W_{t+1}} \bar{J}\left(\frac{1}{n_I} \cdot {\overline{W}_{t+1}}^{eq}({z_{t+1}}^p), {z_{t+1}}^p, t+1\right) \middle| {z_t}^p\right)}$$

Principally, the analysis of the stochastic discount factor in (3-45) can be approached from two different economic angles. The first angle focuses on the risk adjustment nature of the discount factor by comparing marginal utility to expected marginal utility. The second angle stresses the underlying intertemporal consumption choice by relating marginal utility of consumption at time $t+1$, adjusted for time preference, to marginal utility of consumption at time t.

3.1.3.1.1 The Stochastic Discount Factor Expressed Through an Adjustment for Risk

Adjustment for Risk Based on the Value Function

From (3-45), it immediately follows that the stochastic discount factor between t and $t+1$ consists of two parts: (i) the riskless discounting, $exp(-r)$, and (ii) the adjustment for risk:

3-46

$$AfR_{t,t+1}({z_{t+1}}^p, {z_t}^p) \equiv \frac{\frac{\partial}{\partial W_{t+1}} \bar{J}\left(\frac{1}{n_I} \cdot {\overline{W}_{t+1}}^{eq}({z_{t+1}}^p), {z_{t+1}}^p, t+1\right)}{E\left(\frac{\partial}{\partial W_{t+1}} \bar{J}\left(\frac{1}{n_I} \cdot {\overline{W}_{t+1}}^{eq}({z_{t+1}}^p), {z_{t+1}}^p, t+1\right) \middle| {z_t}^p\right)}$$

The adjustment for risk is based on the value function of each of the individual) investors that in turn depends on equilibrium wealth of each of the individual investors at time $t+1$ and on the reinvestment opportunities at time $t+1$ (through ${z_{t+1}}^p$ as the second argument of the value function). The adjustment for risk can be interpreted as a scarcity indicator (see Wilhelm (1985), p. 16-17 and p. 82). To understand the interpretation of the adjustment for risk as scarcity indicator and the two channels through which it is affected, consider the effect of equilibrium wealth and the effect of reinvestment opportunities first separated – i.e., while one channel is varied, the other effect is hypothetically held fixed – and then combined:

For given reinvestment opportunities (i.e., by fixing the second argument ${z_{t+1}}^p$ of the value function), the effect of equilibrium wealth on the discount factor reads as follows. The concavity of the value function in W_{t+1} (see (3-27)) assures that $\frac{\partial}{\partial W_{t+1}} \bar{J}$ will be high if wealth is low. If wealth is low enough so that $\frac{\partial}{\partial W_{t+1}} \bar{J} > E\left(\frac{\partial}{\partial W_{t+1}} \bar{J} \middle| {z_t}^p\right)$, the adjustment for risk will exceed unity: risk-averse investors highly value contributions to wealth W_{t+1} in such states of the world resulting in a discount rate that is lower than the riskless rate.

For given wealth ${\overline{W}_{t+1}}^{eq}$, the effect of reinvestment opportunities (through ${z_{t+1}}^p$) on the discount factor becomes accessible. If $\frac{\partial}{\partial W_{t+1}} \bar{J}$ is higher than $E\left(\frac{\partial}{\partial W_{t+1}} \bar{J} \middle| {z_t}^p\right)$ for some ${z_{t+1}}^p$, the discount rate is lower than the riskless interest rate suggesting that reinvestment opportunities will be bad at time $t+1$. By contrast, those states of the world where $\frac{\partial}{\partial W_{t+1}} \bar{J}$ is below $E\left(\frac{\partial}{\partial W_{t+1}} \bar{J} \middle| {z_t}^p\right)$ and where the discount rate is above the riskless rate provide good reinvestment opportunities at time $t + 1$.

Now the combined effect of wealth and reinvestment opportunities on the adjustment for risk can be easily discussed: wealth $\overline{W}_{t+1}{}^{eq}$ and $z_{t+1}{}^{p}$ together constitute a scarce state of the world if $\frac{\partial}{\partial W_{t+1}}\bar{J}\left(\overline{W}_{t+1}{}^{eq}(z_{t+1}{}^{p}), z_{t+1}{}^{p}, t+1\right)$ is above $E\left(\frac{\partial}{\partial W_{t+1}}\bar{J}\middle|z_t{}^{p}\right)$, implying that wealth is comparatively low and/or $z_{t+1}{}^{p}$ leads to poor reinvestment opportunities. By contrast, a state of the world is abundant if $\frac{\partial}{\partial W_{t+1}}\bar{J}\left(\overline{W}_{t+1}{}^{eq}(z_{t+1}{}^{p}), z_{t+1}{}^{p}, t+1\right)$ is below $E\left(\frac{\partial}{\partial W_{t+1}}\bar{J}\middle|z_t{}^{p}\right)$, i.e., wealth is comparatively high or $z_{t+1}{}^{p}$ leads to favorable reinvestment opportunities.

An example might further illustrate the combined effect of wealth and reinvestment opportunities on the adjustment for risk: consider a state of the world with a secular decline in economic growth (relative to the status quo at time t) where, as a consequence, forecasted growth rates of cash flows paid by the market portfolio of risky assets will also be low. Even if a somewhat above average realization of wealth $\overline{W}_{t+1}{}^{eq}$ is considered, an unfavorable change in reinvestment opportunities due to a bad realization of $z_{t+1}{}^{p}$ may considerably overcompensate the, in principle, positive wealth situation: the multi-period utility as measured by the value function $\bar{J}$ decreases more due to $z_{t+1}{}^{p}$ than it increases due to $\overline{W}_{t+1}{}^{eq}$. As a consequence, such a state of the world will be scarce; contributions to future wealth will be welcomed by the decision maker and, hence, be discounted at a rate which is lower than the riskless rate.

<u>Adjustment for Risk Based on the Marginal Utility of Consumption</u>

The stochastic discount factor in the form (3-45) has the disadvantage that the precise form of the function $\bar{J}$ is not known. Hence, it might be promising from an economic perspective to express discounting in terms of the known direct utility function.

This task can be achieved as follows: if the envelope condition is valid for each of the individual investors,[13]

$$\frac{\partial}{\partial W_{t+1}}\bar{J}\left(W_{t+1}{}^{(\nu)}, z_{t+1}{}^{p}, t+1\right) = U'\left(C_{t+1}{}^{*(\nu)}\right)$$

$$\nu = 1, \ldots, n_I$$

then an equilibrium version of the envelope condition holds,

3-47

$$\frac{\partial}{\partial W_{t+1}}\bar{J}\left(\frac{1}{n_I}\cdot\overline{W}_{t+1}{}^{eq}(z_{t+1}{}^{p}), z_{t+1}{}^{p}, t+1\right) = U'\left(C_{t+1}{}^{*}(z_{t+1}{}^{p})\right)$$

13 It is sufficient for the envelope condition to hold that the optimal controls, consumption and the portfolio holdings of risky assets, are differentiable with respect to wealth, see Ingersoll (1987), p. 236.

and an expression of the stochastic discount factor in terms of the known (direct utility) function U is obtained:

3-48

$$q_{t,t+1}(z_{t+1}{}^{p}, z_t{}^{p}) = \frac{1}{1+r} \cdot \frac{U'\left(C_{t+1}{}^{*}(z_{t+1}{}^{p})\right)}{E\left(U'\left(C_{t+1}{}^{*}(z_{t+1}{}^{p})\right)\middle|z_t{}^{p}\right)}$$

This means that a state of the world can be categorized as scarce or abundant, depending on whether marginal consumption $U'\left(C_{t+1}{}^{*}(z_{t+1}{}^{p})\right)$ is below or above $E\left(U'\left(C_{t+1}{}^{*}(z_{t+1}{}^{p})\right)\middle|z_t{}^{p}\right)$ in that state.

It should, however, be emphasized that consumption is not exogenously given but endogenously determined in the optimum of the decision maker. Hence the practical use of the stochastic discount factor in the form (3-48) is limited and its simplicity deceptive. – It is subject to similar problems as (3-45): there the value function could not be observed and had to be computed. Here the direct utility function is known, but its argument consumption cannot be observed and, thus, must be computed.

3.1.3.1.2 The Stochastic Discount Factor Expressed Through Time Preference and Relative Marginal Utility of Consumption

Both the version of the stochastic discount factor formulated in terms of $\bar{J}$ (3-45) and the direct utility function U (3-48) focus on the pricing of risky cash flows at time $t+1$. However, the underlying nature of intertemporal decisions can be emphasized more clearly if marginal utility of consumption at time t is compared with marginal utility of consumption at time $t+1$. Formally, this comparison can be achieved when the aggregate versions of the first order consumption condition

3-33'

$$U'\left(C_t{}^{*}(z_t{}^{p})\right) = \frac{1+r}{1+\rho} \cdot E\left(\frac{\partial}{\partial W_{t+1}} \bar{J}\left(\frac{1}{n_I} \cdot \overline{W}_{t+1}{}^{eq}(z_{t+1}{}^{p}), z_{t+1}{}^{p}, t+1\right)\middle| z_t{}^{p}\right)$$

and the envelope condition

3-47

$$\frac{\partial}{\partial W_{t+1}} \bar{J}\left(\frac{1}{n_I} \cdot \overline{W}_{t+1}{}^{eq}(z_{t+1}{}^{p}), z_{t+1}{}^{p}, t+1\right) = U'\left(C_{t+1}{}^{*}(z_{t+1}{}^{p})\right)$$

are combined:

$$U'\left(C_t{}^{*}(z_t{}^{p})\right) = \frac{1+r}{1+\rho} \cdot E\left(U'\left(C_{t+1}{}^{*}(z_{t+1}{}^{p})\right)\middle|z_t{}^{p}\right)$$

Substituting this result into the stochastic discount factor (3-45) leads to

3-49

$$q_{t,t+1}(z_{t+1}{}^{p}, z_t{}^{p}) = \frac{1}{1+\rho} \cdot \frac{U'\left(C_{t+1}{}^{*}(z_{t+1}{}^{p})\right)}{U'\left(C_t{}^{*}(z_t{}^{p})\right)}$$

(3-49) illustrates that cash flows are especially valuable in those states of the world where marginal utility of consumption at time $t+1$ is high relative to marginal utility of consumption at time t.

The intertemporal nature of the optimal decisions will become even more pronounced if a multi-period stochastic discount factor is defined. Since the multi-period stochastic discount factor $q_{t,t+\tau}$ is simply the product of neighboring one-period stochastic discount factors,

$$q_{t,t+\tau} = \prod_{\nu=1}^{\tau} q_{t+\nu-1,t+\nu}$$

the stochastic discount factor in the form (3-49) also easily reveals the desired structure of multi-period stochastic discount factors,

3-50

$$q_{t,t+\tau}(z_{t+\tau}{}^p, z_t{}^p) = \frac{1}{(1+\rho)^\tau} \cdot \frac{U'\left(C_{t+\tau}{}^*(z_{t+\tau}{}^p)\right)}{U'\left(C_t{}^*(z_t{}^p)\right)}$$

However, both the single- (3-49) and the multi-period stochastic discount factor (3-50) are subject to the same limitations as the stochastic discount factor in the form (3-48): optimal consumption is not exogenously given but endogenously determined as the result of optimal decisions.

3.1.3.2 Second Question: Equilibrium Risk Premia

To analyze how risky assets are priced relative to the riskless asset, the concept of a one-period risk premium is used. It relates expected values of assets to their compounded prices:

3-51

$$RP_t(z_t{}^p) \equiv E\left(P_{t+1}(z_{t+1}{}^p) + D_{t+1}\left(z_{t+1}{}^d\right)\middle| z_t{}^p\right) - (1+r) \cdot P_t(z_t{}^p)$$

In the context of risk premia, two different aspects are of interest:

(i) How does the co-movement of payoffs and adjustment for risk influence risk premia as a whole?

(ii) Are there particular states that contribute exceptionally to risk premia?

3.1.3.2.1 Risk Premia Expressed Through Covariances with the Adjustment for Risk

To address the first aspect, risk premia are expressed through covariances with the adjustment for risk. Using the identity

$$cov(X,Y) = E(X \cdot Y) - E(X) \cdot E(Y)$$

in combination with the pricing equation (3-45), yields as risk premium formulation

3-52

$$RP_t(z_t{}^p)$$
$$= -cov\left(\frac{\frac{\partial}{\partial W_{t+1}} \bar{J}_{t+1}\left(\overline{W}_{t+1}{}^{eq}(z_{t+1}{}^p), z_{t+1}{}^p, t+1\right)}{E\left(\frac{\partial}{\partial W_{t+1}} \bar{J}_{t+1}\left(\overline{W}_{t+1}{}^{eq}(z_{t+1}{}^p), z_{t+1}{}^p, t+1\right)\middle| z_t{}^p\right)}, P_{t+1}(z_{t+1}{}^p) + D_{t+1}\left(z_{t+1}{}^d\right)\middle| z_t{}^p\right)$$

Three different signs of the risk premium (3-52) can occur for any asset i, depending on the sign of the covariance of $P_{i,t+1} + D_{i,t+1}$ with the adjustment for risk (3-46)

3-53

$$AfR_t(z_{t+1}{}^p, z_t{}^p) \equiv \frac{\frac{\partial}{\partial W_{t+1}} \bar{J}_{t+1}\left(\overline{W}_{t+1}{}^{eq}(z_{t+1}{}^p), z_{t+1}{}^p, t+1\right)}{E\left(\frac{\partial}{\partial W_{t+1}} \bar{J}_{t+1}\left(\overline{W}_{t+1}{}^{eq}(z_{t+1}{}^p), z_{t+1}{}^p, t+1\right) \middle| z_t{}^p\right)}$$

The risk premium of asset i will be positive if the covariance between $P_{i,t+1} + D_{i,t+1}$ and adjustment for risk is negative, zero if the covariance between $P_{i,t+1} + D_{i,t+1}$ and adjustment for risk is zero, and negative if the covariance between $P_{i,t+1} + D_{i,t+1}$ and adjustment for risk is positive (see, for example, Cochrane (2005), p. 13). To understand the economic logic behind this mathematical result, consider each of these three cases in more detail: (i) a negative correlation between $P_{i,t+1} + D_{i,t+1}$ and adjustment for risk implies that realizations of cash flow and asset prices are mostly high when needed least, i.e., when each of the individual investors is already well off. For that reason, $P_{i,t+1} + D_{i,t+1}$ is, loosely speaking, discounted heavily in most states, resulting in a low asset price $P_{i,t}$ and, as a consequence, a positive risk premium. In other words, such an asset is regarded as worse than the riskless asset calling for a compensation in the form of a positive risk premium. (ii) if $P_{i,t+1} + D_{i,t+1}$ is uncorrelated with the adjustment for risk, cash flows and asset prices are discounted at the riskless rate, implying a zero risk premium. Although $P_{i,t+1} + D_{i,t+1}$ is stochastic, there is no systematic relationship with the adjustment for risk implying that the stochastic discount rate is sometimes higher and sometimes lower than the riskless rate, but on average the same. Hence such an asset is regarded as equally good as the riskless asset and no compensation is required. (iii) if $P_{i,t+1} + D_{i,t+1}$ is positively correlated with the adjustment for risk, asset i provides high values of $P_{i,t+1} + D_{i,t+1}$ when needed most, i.e., in scarce states of the world. The discount rate is in most states lower than the riskless rate, resulting in a relatively high price $P_{i,t}$ and, consequently, a negative risk premium. Consequently, such an asset is regarded as better than the riskless asset and a negative compensation arises – investors are willing to pay a premium to acquire such an asset.

3.1.3.2.2 The Contributions of Individual States to Risk Premia

To better understand why it is desirable from an economic perspective to know whether there are particular states that contribute exceptionally to risk premia, consider an example. Assume that a recession might occur with low equilibrium aggregate wealth and bad reinvestment opportunities. It is interesting to know which assets are hit most by this recession and, hence, demand a high risk premium.

Addressing a such question calls for a representation of risk premia that, similar to the stochastic discount factor representation of asset prices, shows how asset prices and cash flows in a particular state z_{t+1} (in the sense of 3-20) contribute to risk premia. To achieve this desired interpretation, proceed as follows.

After plugging the price equation (3-45) into the definition of the risk premium (3-52), it is obtained

3-54
$$RP_t(z_t{}^p) = E\left(P_{t+1}(z_{t+1}{}^p) + D_{t+1}\left(z_{t+1}{}^d\right)\middle|z_t{}^p\right)$$
$$-E\left((1+r)\cdot q_{t,t+1}(z_{t+1}{}^p, z_t{}^p)\cdot\left\{P_{t+1}(z_{t+1}{}^p) + D_{t+1}\left(z_{t+1}{}^d\right)\right\}\middle|z_t{}^p\right)$$

As the risk-neutralized conditional probability (or density) equals the product of empirical conditional probability (or density) and adjustment for risk (see Cochrane (2005), p. 51) the risk premia can be written as the difference of expectations of payoffs with respect to the empirical and the risk-neutralized measures:

3-55
$$RP_t(z_t{}^p) = E\left(P_{t+1}(z_{t+1}{}^p) + D_{t+1}\left(z_{t+1}{}^d\right)\middle|z_t{}^p\right) - E_{rn}\left(P_{t+1}(z_{t+1}{}^p) + D_{t+1}\left(z_{t+1}{}^d\right)\middle|z_t{}^p\right)$$

where E_{rn} is the expectation taken with respect to the risk-neutralized conditional probability (density).

The risk-neutralized conditional probability (density) will be compared to its empirical counterpart (i) larger if a bad state occurs, i.e., $\frac{\partial}{\partial W_{t+1}}\bar{J}_{t+1}$ is above $E\left(\frac{\partial}{\partial W_{t+1}}\bar{J}_{t+1}\middle|z_t{}^p\right)$, (ii) identical to if a neutral state occurs, i.e., $\frac{\partial}{\partial W_{t+1}}\bar{J}_{t+1}$ is equal to $E\left(\frac{\partial}{\partial W_{t+1}}\bar{J}_{t+1}\middle|z_t{}^p\right)$, (iii) smaller if a good state occurs, i.e., $\frac{\partial}{\partial W_{t+1}}\bar{J}_{t+1}$ is below $E\left(\frac{\partial}{\partial W_{t+1}}\bar{J}_{t+1}\middle|z_t{}^p\right)$. Therefore, positive payoffs $P_{t+1}(z_{t+1}{}^p) + D_{t+1}\left(z_{t+1}{}^d\right)$ (i) in bad states decrease the risk premium the more, the higher the probability of this state and the higher the payoff is; (ii) have no effect on the risk premium in neutral states; (iii) in good states increase the risk premium the more, the higher the probability of this state and the higher the payoff is.

Coming back to our motivating example, it becomes clear: assets that are subject to a recession, i.e., a bad state, will demand a high risk premium if they offer low payoffs in these states and theses states have high conditional probability (density).

3.1.4 Impossibility of a Closed-Form Solution

Although the discussion in the preceding sub-section sheds some light on asset prices and risk premia on a general and abstract level, it does not allow to derive asset prices in closed form, i.e., to finalize the pricing task: clearly, equilibrium asset prices are only implicitly given by (3-30), (3-31), (3-32), and (3-33). This motivates the analysis of particular combinations of utility functions (first specification) and state variables (second specification). State variables comprise combinations of cash flow models and information scenarios, for example, general state variable (no restriction on cash flow models and information scenarios), general cash flow model under complete information, or cash flows without lags in levels under incomplete information.

3.2 Constant Relative Risk Aversion

As a first specification to the general problem consider the case of constant relative risk aversion and general state variable. Each of the individual investors then solves the optimization problem

3-9

$$max_{\{N_\tau,C_\tau\},0\leq\tau\leq T-1,C_T} E\left(\sum\nolimits_{t=0}^{T} \frac{1}{(1+\rho)^t}\cdot U(C_t)\,\middle|I_0\right)$$

with utility function

$$U(C_t)=\frac{{C_t}^{1-\gamma}}{1-\gamma}$$

$$0<\gamma,\gamma\neq 1$$

and with wealth dynamics (3-10),

$$W_{\tau+1}\left({z_{\tau+1}}^p,{z_\tau}^p,{z_{\tau+1}}^d,W_\tau;C_\tau,N_\tau\right)$$
$$=[W_\tau-C_\tau]\cdot(1+r)+{N_\tau}^T\left\{P_{\tau+1}({z_{\tau+1}}^p)+D_{\tau+1}\left({z_{\tau+1}}^d\right)-(1+r)\cdot P_\tau({z_\tau}^p)\right\}$$

To find a solution to the optimization problem of the CRRA-investor, start with the period between time $T-1$ and T, i.e., a one-period model. There, the boundary condition implies that the value function at time T coincides with the direct utility function and takes the form

$$\bar{J}(W_T,{z_T}^p,T)=\frac{{W_T}^{1-\gamma}}{1-\gamma}$$

The system of equations to determine equilibrium asset prices and consumption reads:

3-56

$$P_{T-1}=\frac{1}{1+r}\cdot E\left(\frac{\left\{\frac{1}{n_I}\cdot{\overline{W_T}}^{eq}\right\}^{-\gamma}}{E\left(\left\{\frac{1}{n_I}\cdot{\overline{W_T}}^{eq}\right\}^{-\gamma}\middle|{z_{T-1}}^p\right)}\cdot D_T\left({z_T}^d\right)\middle|{z_{T-1}}^p\right)$$

3-57

$$\{{C_{T-1}}^*\}^{-\gamma}-\frac{1+r}{1+\rho}\cdot E\left(\left\{\frac{1}{n_I}\cdot{\overline{W_T}}^{eq}\right\}^{-\gamma}\middle|{z_{T-1}}^p\right)=0$$

with

$${\overline{W_T}}^{eq}=\left[{\overline{W_{T-1}}}^{eq}-n_I\cdot{C_{T-1}}^*\right]\cdot(1+r)+\overline{N}^T\left\{D_T\left({z_T}^d\right)-(1+r)\cdot P_{T-1}\right\}$$
$${\overline{W_{T-1}}}^{eq}=\overline{N}^T P_{T-1}+{D_{T-1}}^{market}({z_{T-1}}^p)+{\overline{H}_{T-2}}^{eq}({z_{T-1}}^p)\cdot(1+r)$$

The system of equation defined by (3-56) and (3-57) does not even have a closed-form solution for time $t=T-1$ because, e.g., ${C_{T-1}}^*$ appears on both sides of (3-57) and on the right-hand side of (3-57) (as a part of ${\overline{W_T}}^{eq}$) in a non-linear way. These non-linear equations might be solvable by numerical methods if cash flow model and utility function are specified. However, this does not generate as much insight as a closed-form solution. As a consequence, CRRA preferences are not further analyzed in the partial equilibrium framework.

3.3 Constant Absolute Risk Aversion

3.3.1 Solution for General Cash flow Models and Information Scenarios

3.3.1.1 The Optimization Problem of Investors with CARA Preferences

As an alternative first specification to the general problem consider the case of constant absolute (CARA) risk aversion and general state variable. Then each of the individual investors solves the problem

3-9

$$max_{\{N_\tau,C_\tau\},0\le\tau\le T-1,C_T} E\left(\sum_{t=0}^{T} \frac{1}{(1+\rho)^t} \cdot U(C_t) \middle| I_0\right)$$

with utility function

$$U(C_t) = -exp(-\alpha \cdot C_t)$$
$$\alpha > 0$$

and with wealth dynamics (3-10),

$$W_{\tau+1}\left(z_{\tau+1}{}^p, z_\tau{}^p, z_{\tau+1}{}^d, W_\tau; C_\tau, N_\tau\right)$$
$$= [W_\tau - C_\tau] \cdot (1+r) + N_\tau{}^T\left\{P_{\tau+1}(z_{\tau+1}{}^p) + D_{\tau+1}\left(z_{\tau+1}{}^d\right) - (1+r) \cdot P_\tau(z_\tau{}^p)\right\}$$

The system of equations to determine equilibrium asset prices and consumption (3-30), (3-31), (3-32), and (3-33) now admits an explicit solution: it is shown in Appendix A2.2 that the value function of each of the identical investors who solves the problem (3-9) is of the form

$$J(W_{t+1}, z_{t+1}{}^p, t+1) \equiv \frac{1}{(1+\rho)^{t+1}} \cdot \bar{J}(W_{t+1}, z_{t+1}{}^p, t+1)$$

3-58

$$\bar{J}(W_{t+1}, z_{t+1}{}^p, t+1) = -exp(-\alpha_{t+1} \cdot W_{t+1}) \cdot m_{t+1}(z_{t+1}{}^p)$$

where α_{t+1} is a positive constant and m_{t+1} is a function that takes strictly positive values for all possible realizations of the variable $z_{t+1}{}^p$ (for the details regarding the formalization of α_{t+1} and m_{t+1}, see again Appendix A2.2):

3-59

$$\alpha_{t+1} > 0$$

3-60

$$m_{t+1}(z_{t+1}{}^p) > 0 \; \forall z_{t+1}$$

In particular, at time $t+1 = T$, it is obtained

3-61

$$\alpha_T = \alpha$$

3-62

$$m_T{}^{cI} = 1$$

To see how m_{t+1} captures the influence of reinvestment opportunities ${z_{t+1}}^p$ on the value function $\bar{J}$, observe that a low value of m_{t+1} corresponds to realizations of ${z_{t+1}}^p$ that lead to favorable reinvestment opportunities. The reason for this is as follows: a high value of $\bar{J}$ is tantamount to high derived utility over the remaining time horizon. But the negative sign of $\bar{J}$ (3-58) and the positive sign of m_{t+1} (3-60) together imply that the value function will take high values whenever m_{t+1} is low. On the flipside, realizations of ${z_{t+1}}^p$ that lead to poor reinvestment opportunities result in high values of m_{t+1}. Note that at time T, there is no reinvestment and, therefore, m_T simplifies to one (see (3-62)), i.e., is neutral.

3.3.1.2 Equilibrium Asset Prices

Plugging the form of the value function under CARA preferences into the abstract pricing equation 3-44

$$P_t({z_t}^p) = \frac{1}{1+r} \cdot E\left(\frac{\frac{\partial}{\partial W_{t+1}} \bar{J}\left(\frac{1}{n_I} \cdot {\overline{W}_{t+1}}^{eq}({z_{t+1}}^p), {z_{t+1}}^p, t+1\right)}{E\left(\frac{\partial}{\partial W_{t+1}} \bar{J}\left(\frac{1}{n_I} \cdot {\overline{W}_{t+1}}^{eq}({z_{t+1}}^p), {z_{t+1}}^p, t+1\right) \middle| {z_t}^p\right)} \cdot \left\{P_{t+1}({z_{t+1}}^p) + D_{t+1}\left({z_{t+1}}^d\right)\right\} \middle| {z_t}^p \right)$$

with

$$ {\overline{W}_{t+1}}^{eq}({z_{t+1}}^p) = \left[{\overline{W}_t}^{eq}({z_t}^p) - n_I \cdot {C_t}^*({z_t}^p)\right] \cdot (1+r) + \overline{N}^T \left\{P_{t+1}({z_{t+1}}^p) + D_{t+1}\left({z_{t+1}}^d\right) - (1+r) \cdot P_t({z_t}^p)\right\} $$

yields for equilibrium asset prices:

3-63

$$P_t({z_t}^p) = E\left(q_{t,t+1}({z_{t+1}}^p, {z_t}^p) \cdot \left\{P_{t+1}({z_{t+1}}^p) + D_{t+1}\left({z_{t+1}}^d\right)\right\} \middle| {z_t}^p\right)$$

with

$$q_{t,t+1}({z_{t+1}}^p, {z_t}^p) = \frac{1}{1+r} \cdot \frac{exp\left(-\alpha_{t+1} \cdot \frac{1}{n_I} \cdot \left\{{D_{t+1}}^{market}({z_{t+1}}^p) + {V_{t+1}}^{market}({z_{t+1}}^p)\right\}\right) \cdot m_{t+1}({z_{t+1}}^p)}{E\left(exp\left(-\alpha_{t+1} \cdot \frac{1}{n_I} \cdot \left\{{D_{t+1}}^{market}({z_{t+1}}^p) + {V_{t+1}}^{market}({z_{t+1}}^p)\right\}\right) \cdot m_{t+1}({z_{t+1}}^p) \middle| {z_t}^p\right)}$$

where ${D_{t+1}}^{market}$ is the aggregate cash flows paid by all risky assets in the market portfolio

$${D_{t+1}}^{market}({z_{t+1}}^p) \equiv \overline{N}^T D_{t+1}\left({z_{t+1}}^d\right)$$

and where ${V_{t+1}}^{market}$ is the aggregate value of the market portfolio of risky assets

$${V_{t+1}}^{market}({z_{t+1}}^p) \equiv \overline{N}^T P_{t+1}({z_{t+1}}^p)$$

Compared to the general pricing formula (3-44), asset prices under CARA preferences possess two important additional properties: a closed-form solution becomes possible and information relevant to pricing ${z_t}^p$ is simplified in that it consists of ${z_t}^{d,+}$ only.

To understand this result, observe, first, that the stochastic discount factor at time t (3-63) does no longer depend on any of the quantities $D_t{}^{market}$, $\overline{H}_{t-1}{}^{eq}$, and $\overline{H}_t{}^{eq}$. The stochastic discount factor at time t (3-63) consists of two components: a reinvestment opportunities-related and a wealth-related part. The reinvestment opportunities-related part $m_{t+1}(z_{t+1}{}^p)$[14] is not a function of $D_t{}^{market}$, $\overline{H}_{t-1}{}^{eq}$, and $\overline{H}_t{}^{eq}$ as is shown in Appendix A2.3. The wealth-related part only depends on the contribution made at time $t+1$ by risky assets as inspection of (3-63) shows, i.e., $\frac{1}{n_I} \cdot D_{t+1}{}^{market}(z_{t+1}{}^p)$ and $\frac{1}{n_I} \cdot V_{t+1}{}^{market}(z_{t+1}{}^p)$ are the sole arguments of the wealth-related part.

As a direct consequence of this independence, the closed-form solution becomes possible. Since $\overline{H}_t{}^{eq}(z_t{}^p) \cdot (1+r) = \left[\overline{W}_t{}^{eq}(z_t{}^p) - n_I \cdot C_t{}^*(z_t{}^p) - P_t(z_t{}^p)\right] \cdot exp(r)$, P_t entered the right hand side of the pricing equation and caused the circularity problem in the general case (3-44). However, this problem disappear with the independence of $\overline{H}_t{}^{eq}$.

Second, it still remains to be shown that information relevant to pricing $z_t{}^p$ simplifies to $z_t{}^{d,+}$. Recall that in the general model

3-64

$$z_t{}^p = \left(D_t{}^{market}, \overline{H}_{t-1}{}^{eq} \cdot exp(r), z_t{}^{d,+}\right)$$

holds.

The problem behind this is as follows. Since prices are not only functions of the stochastic discount factor but also of next-period asset prices and cash flows, a characterization of the stochastic discount factor alone does not suffice. However, simple inductive reasoning (see Appendix A2.3) shows that if prices at time $t+1$ depend on relevant information $z_{t+1}{}^{d,+}$ only, then prices at time t will likewise only depend on $z_t{}^{d,+}$, thus establishing the assertion.

The economic intuition behind this result lies in the fact that investors with CARA preferences choose risky investment independently of their respective total wealth and just based on the assets' prices and cash flows (see, e.g., Hakansson (1970), p. 602)). Hence, demand for risky assets is independent of total wealth; moreover, supply is exogenous. For that reason, market clearing asset prices cannot depend on total wealth, but just on the assets' prices and cash flows in equilibrium.

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[14] $m_{t+1}(z_{t+1})$ is given by a recursion which can be found in the Appendix A2.3.

3.3.1.3 Equilibrium Risk Premia

Plugging the now known form of the value function J into the alternative risk premium formalization (3-52) yields,

3-65

$$RP_t({z_t}^p) = -cov\left(\left.\frac{exp\left(-\alpha_{t+1}\cdot\frac{1}{n_I}\cdot\left\{{D_{t+1}}^{market}({z_{t+1}}^p)+{V_{t+1}}^{market}({z_{t+1}}^p)\right\}\right)\cdot m_{t+1}({z_{t+1}}^p)}{E\left(exp\left(-\alpha_{t+1}\cdot\frac{1}{n_I}\cdot\left\{{D_{t+1}}^{market}({z_{t+1}}^p)+{V_{t+1}}^{market}({z_{t+1}}^p)\right\}\right)\cdot m_{t+1}({z_{t+1}}^p)\middle|{z_t}^p\right)}, P_{t+1}({z_{t+1}}^p)+D_{t+1}({z_{t+1}}^d)\right|{z_t}^p\right)$$

Compared to the general risk premia formula (3-52), a closed-form solution for risk premia under CARA preferences can be obtained. The simplification of information relevant to pricing ${z_t}^p$ to ${z_t}^{d,+}$ as identified in the context of the price equation (3-63) still holds in the risk premia context.

3.3.2 Equilibrium Asset Prices and Risk Premia for the General Cash Flow Model under Complete and Incomplete Information

Within the framework of CARA preferences (first specification) the state variable (combinations of cash flow models and information scenarios, second specification) is specified to address the main topic of this thesis, namely pricing implications of the information structure (complete or incomplete information) of the decision maker.

To that end, four constellations are considered: information may either be complete or incomplete, and the information frequency may either be equal to or higher than the cash flow frequency (i.e., $\Delta_C = 1$ or $\Delta_C > 1$).

3.3.2.1 All Assets Pay Cash Flows in Every Period: Information Frequency = Cash Flow Frequency

3.3.2.1.1 Cash Flow Model

If information frequency equals cash flow frequency all assets pay cash flows at all point of time t. Since the cash flow follows the general model, it reads

2-18

$$D_{t+1} = D(D_t, S_t, fe_{t+1})$$

3.3.2.1.2 Complete Information

3.3.2.1.2.1 Information Relevant to Pricing ${z_t}^p$

Since information relevant to pricing ${z_t}^p$ under CARA utility function coincides with ${z_t}^{d,+}$ (see (3-63) and the following paragraph), ${z_t}^{d,+}$ must be specified in the context of the general cash flow model under complete information.

${z_t}^{d,+}$ reads

3-66

$$ {z_t}^{cI,d,+} = (S_t, D_t) $$

where ${z_t}^{cI,d,+}$ is Markov.

${z_t}^{d,+}$ has been defined as sufficient statistic for the distribution of future cash flows $(D_{t+1}, .., D_T)$ conditional on information at time t (see, 3-16). Hence, elements must be identified to describe the conditional distribution of future cash flows. By the form of the general cash flow model under complete information, these future cash flows are functionally related to both current cash flows and the (observable) current regime. It follows that ${z_t}^{cI,d,+}$ must at least include S_t and D_t. However, it can be argued that ${z_t}^{cI,d,+}$ needs no additional components: the cash flow model does not allow any time lags greater than one. Moreover, regimes are Markov. Therefore, no cash flows or regimes prior to time t are of relevance. Finally, factors and residuals fe_{t+1} are i.i.d. and, therefore, do not convey information regarding future cash flows.

3.3.2.1.2.2 Equilibrium Asset Prices

In the special case of complete information, the pricing results for general cash flow models and information scenarios (3-63) specialize to

3-67

$$ {P_t}^{cI}(S_t, D_t) = E\left({q_{t,t+1}}^{cI}(S_{t+1}, fe_{t+1}, S_t, D_t) \cdot \left\{{P_{t+1}}^{cI}(S_{t+1}, D_{t+1}) + D_{t+1}\right\} \middle| S_t, D_t\right) $$

with stochastic discount factor

3-68

$$q_{t,t+1}{}^{cI}(S_{t+1}, fe_{t+1}, S_t, D_t) = \frac{1}{1+r} \cdot \frac{\left\{ exp\left(-\alpha_{t+1} \cdot \frac{1}{n_I} \cdot \left\{D_{t+1}{}^{market}(D_{t+1}) + V_{t+1}{}^{market}(S_{t+1}, D_{t+1})\right\}\right) \cdot m_{t+1}{}^{cI}(S_{t+1}, D_{t+1}) \right\}}{E\left(\left\{ exp\left(-\alpha_{t+1} \cdot \frac{1}{n_I} \cdot \left\{D_{t+1}{}^{market}(D_{t+1}) + V_{t+1}{}^{market}(S_{t+1}, D_{t+1})\right\}\right) \cdot m_{t+1}{}^{cI}(S_{t+1}, D_{t+1}) \right\} \middle| S_t, D_t\right)}$$

with

$$D_{t+1} = D(D_t, S_t, fe_{t+1})$$

In the case of complete information, the abstract pricing results for general cash flow models and information scenarios (3-63) take the following form. There are two sources of risk, regimes S_{t+1} as well as factors and residuals fe_{t+1}. Although regimes as well as factors and residuals are modeled stochastically independent (see (2-17)), they do jointly determine the adjustment for risk through the reinvestment opportunities-related part $m_{t+1}{}^{cI}(S_{t+1}, D_{t+1})$ and through the value of the market portfolio of risky assets $V_{t+1}{}^{market}(S_{t+1}, D_{t+1})$. Hence, it can be concluded that both regimes S_{t+1} as well as factors and residuals fe_{t+1} will, through their effect on $D_{t+1}{}^{market}$, $V_{t+1}{}^{market}$, and $m_{t+1}{}^{cI}$ give rise to correlations between asset prices, cash flows and the stochastic discount factor and, thus, are priced in a risk-adjusted way.

3.3.2.1.2.3 Equilibrium Risk Premia

In the special case of complete information, the result on risk premia for general cash flow models and information scenarios (3-65) specializes to

3-69

$$RP_t{}^{cI}(S_t, D_t) = -cov\left(\begin{matrix} AfR^{cI}(S_{t+1}, fe_{t+1}, S_t, D_t) \\ , P_{t+1}{}^{cI}(S_{t+1}, D_{t+1}) + D_{t+1} \end{matrix} \middle| S_t, D_t \right)$$

with

$$AfR^{cI}(S_{t+1}, fe_{t+1}, S_t, D_t) \equiv \frac{\left\{ exp\left(-\alpha_{t+1} \cdot \frac{1}{n_I} \cdot \left\{D_{t+1}{}^{market}(D_{t+1}) + V_{t+1}{}^{market}(S_{t+1}, D_{t+1})\right\}\right) \cdot m_{t+1}{}^{cI}(S_{t+1}, D_{t+1}) \right\}}{E\left(\left\{ exp\left(-\alpha_{t+1} \cdot \frac{1}{n_I} \cdot \left\{D_{t+1}{}^{market}(D_{t+1}) + V_{t+1}{}^{market}(S_{t+1}, D_{t+1})\right\}\right) \cdot m_{t+1}{}^{cI}(S_{t+1}, D_{t+1}) \right\} \middle| S_t, D_t\right)}$$

$$D_{t+1} = D(D_t, S_t, fe_{t+1})$$

Similar to the price equation, risk premia arise from the two sources of risk, new regime S_{t+1} as well as factors and residuals fe_{t+1}, where the pricing effects of both sources of risk cannot be separated.

3.3.2.1.3 Incomplete Information

3.3.2.1.3.1 Information Relevant to Pricing ${z_t}^p$

Under incomplete information, CARA preference, and general cash flow model, ${z_t}^p$ still coincides with ${z_t}^{d,+}$ and reads

3-70

$$ {z_t}^{iI,d,+} = (\pi_t, D_t) $$

where ${z_t}^{iI,d,+}$ is Markov and $\pi_{s,t} \equiv P\left(S_t = s \middle| \overrightarrow{Sig_t}, \overrightarrow{D_t}\right), s = 1, \dots, K$ denotes the conditional regime probability.

As in the complete information case, elements must be identified to describe the conditional distribution of future cash flows and, thus, to specify ${z_t}^{iI,d,+}$. In principle, the conditional distribution of future cash flows depends on current cash flows D_t and the current regime S_t as in the complete information case. However, S_t is unobservable in the complete information case and, hence, is replaced by current regime probabilities.

3.3.2.1.3.2 Equilibrium Asset Prices

In the special case of incomplete information and CARA utility function, the pricing result for general cash flow models and information scenarios (3-63) specializes to

3-71

$$ {P_t}^{iI}(\pi_t, D_t) = \sum_{s=1}^{K} \pi_{s,t} \cdot E\left({q_{t,t+1}}^{iI}(\eta_{t+1}, S_{t+1}, fe_{t+1}, S_t, \pi_t, D_t) \cdot \left\{{P_{t+1}}^{iI}(\pi_{t+1}, D_{t+1}) + D_{t+1}\right\} \middle| S_t = s, \pi_t, D_t\right) $$

with stochastic discount factor

3-72

$$q_{t,t+1}{}^{iI}(\eta_{t+1},S_{t+1},fe_{t+1},S_t,\pi_t,D_t)$$

$$=\frac{1}{1+r}$$

$$\cdot\frac{\left\{\begin{matrix}exp\left(-\alpha_{t+1}\cdot\frac{1}{n_I}\cdot\left\{D_{t+1}{}^{market}(D_{t+1})+V_{t+1}{}^{market}(\pi_{t+1},D_{t+1})\right\}\right)\\ \cdot m_{t+1}{}^{iI}(\pi_{t+1},D_{t+1})\end{matrix}\right\}}{\sum_{s=1}^{K}\pi_{s,t}\cdot E\left(\left\{\begin{matrix}exp\left(-\alpha_{t+1}\cdot\frac{1}{n_I}\cdot\left\{D_{t+1}{}^{market}(D_{t+1})+V_{t+1}{}^{market}(\pi_{t+1},D_{t+1})\right\}\right)\\ \cdot m_{t+1}{}^{iI}(\pi_{t+1},D_{t+1})\end{matrix}\right\}\middle|S_t=s,\pi_t,D_t\right)}$$

with

$$D_{t+1}=D(D_t,S_t,fe_{t+1})$$

$$\pi_{t+1}=\Pi(\pi_t,D_t,D_{t+1},Sig_{t+1})$$

$$Sig_{t+1}=Sig_{t+1}(S_t,S_{t+1},fe_{t+1},\eta_{t+1})$$

In the case of incomplete information, the abstract pricing results for general cash flow models and information scenarios (3-63) take the following form. There are four sources of risk, namely current regime S_t, factors and residuals fe_{t+1}, the regime at time $t+1$ S_{t+1}, and signal noise η_{t+1} where S_t and S_{t+1} are dependent; fe_{t+1}, S_{t+1}, and η_{t+1} are independent (see (2-17) and p. 21). Nevertheless, these sources of risk jointly determine the adjustment for risk and, hence, are priced in a risk-adjusted way: S_t and fe_{t+1} influence the adjustment for risk through $D_{t+1}{}^{market}(D_{t+1})$, $V_{t+1}{}^{market}(\pi_{t+1},D_{t+1})$, and $m_{t+1}{}^{iI}(\pi_{t+1},D_{t+1})$ by determining D_{t+1} (first channel) and, in addition, through signals and their effect on regime probabilities π_{t+1} (second channel). By contrast, S_{t+1} and η_{t+1} only affect the adjustment for risk through signals (second channel). This implies that in the special case without signals (i.e., all information comes from cash flows) the adjustment for risk is a function of S_t and fe_{t+1} only.

3.3.2.1.3.3 Equilibrium Risk Premia

Under incomplete information, the result on equilibrium risk premia for general cash flow models and information scenarios (3-65) reads:

3-73

$$RP_t^{iI}(\pi_t, D_t) = -cov\left(\begin{matrix} AfR_{t,t+1}^{iI}(\eta_{t+1}, S_{t+1}, fe_{t+1}, S_t, \pi_t, D_t) \\ , P_{t+1}^{iI}(\pi_{t+1}, D_{t+1}) + D_{t+1} \end{matrix}\middle| \pi_t, D_t\right)$$

with

$$AfR_{t,t+1}^{iI}(\eta_{t+1}, S_{t+1}, fe_{t+1}, S_t, \pi_t, D_t)$$

$$\equiv \frac{\left\{\begin{matrix} exp\left(-\alpha_{t+1} \cdot \frac{1}{n_I} \cdot \left\{\begin{matrix} D_{t+1}^{market}(D_{t+1}) \\ +V_{t+1}^{market}(\pi_{t+1}, D_{t+1}) \end{matrix}\right\}\right) \\ \cdot m_{t+1}^{iI}(\pi_{t+1}, D_{t+1}) \end{matrix}\right\}}{\sum_{s=1}^{K} \pi_{st} E\left(\left\{\begin{matrix} exp\left(-\alpha_{t+1} \cdot \frac{1}{n_I} \cdot \left\{\begin{matrix} D_{t+1}^{market}(D_{t+1}) \\ +V_{t+1}^{market}(\pi_{t+1}, D_{t+1}) \end{matrix}\right\}\right) \\ \cdot m_{t+1}^{iI}(\pi_{t+1}, D_{t+1}) \end{matrix}\right\}\middle| S_t = s, \pi_t, D_t\right)}$$

$$D_{t+1} = D(D_t, S_t, fe_{t+1})$$

$$\pi_{t+1} = \Pi(\pi_t, D_t, D_{t+1}, Sig_{t+1})$$

$$Sig_{t+1} = Sig_{t+1}(S_t, S_{t+1}, fe_{t+1}, \eta_{t+1})$$

Similar to the price equation, risk premia are determined by the same sources of risk as asset prices where analogous interaction results hold.

3.3.2.1.3.4 Risk Decomposition and Consequences to Prices and Risk Premia

3.3.2.1.3.4.1 Decomposition of Risky Asset Prices and Cash Flows

So far prices under incomplete information have been derived as a function of all sources of risk identified in Chapter 2, namely current regime S_t, factors and residuals fe_{t+1}, the regime at time $t+1$ S_{t+1}, and signal noise η_{t+1}. The purpose of this section is to identify the price influence of these sources of risk where special emphasis is laid on the analysis of the current regime S_t.

A closer look at these four sources of risk reveals their different reference to time. Current regime S_t refers to time t in the sense that under incomplete information the true probability law at time t governing future cash flows and prices is not known. Hence, this type of risk can be called –in a sense of a rule of thumb – “inter-distribution risk”. For example, $t+1$ cash flows are logarithmic normally distributed with parameters μ_1 and σ_1 in regime$_1$ and beta-distributed in regime$_2$.

The other three sources of risk refer to time $t+1$ in that they determine the realizations of cash flows and prices at time $t+1$ given the regime at time t. This fact can be characterized by the term "intra-distribution risk". For example, if the "true" cash flow regime at time t is the logarithmic normal distribution with parameters μ_1 and σ_1, the other sources of risk draw a realization of future cash flows based on this particular logarithmic normal distribution.

Based on this distinction between "inter-distribution risk" and "intra-distribution risk", I separate the pricing influence of these risks as follows.

As point of departure, I use an approach analogous to the standard literature on factor models (see, e.g., Ingersoll (1987), p. 166) and separate risky asset prices and cash flows ${P_{t+1}}^{iI} + D_{t+1}$ additively into an expectation component, $E\left({P_{t+1}}^{iI} + D_{t+1} \middle| \pi_t, D_t\right)$, and a risk component with zero expectation, ${P_{t+1}}^{iI} + D_{t+1} - E\left({P_{t+1}}^{iI} + D_{t+1} \middle| \pi_t, D_t\right)$.

It seems to be natural to decompose the risk component additively into "inter- and intra-distribution risk" as well, i.e.,

${P_{t+1}}^{iI} + D_{t+1} - E\left({P_{t+1}}^{iI} + D_{t+1} \middle| \pi_t, D_t\right)$ equals "intra-distribution risk" plus "inter-distribution risk".

"Intra-distribution risk" in a given regime $S_t = s$ can be identified with

$$ {P_{t+1}}^{iI} + D_{t+1} - E\left({P_{t+1}}^{iI} + D_{t+1} \middle| S_t = s, \pi_t, D_t\right) $$

Using this formalization of "intra-distribution risk" in the decomposition of the risk component leads to

$$ \begin{gathered} {P_{t+1}}^{iI} + D_{t+1} - E\left({P_{t+1}}^{iI} + D_{t+1} \middle| \pi_t, D_t\right) \\ = \left\{{P_{t+1}}^{iI} + D_{t+1} - E\left({P_{t+1}}^{iI} + D_{t+1} \middle| S_t = s, \pi_t, D_t\right)\right\} \\ + \left\{E\left({P_{t+1}}^{iI} + D_{t+1} \middle| S_t = s, \pi_t, D_t\right) - E\left({P_{t+1}}^{iI} + D_{t+1} \middle| \pi_t, D_t\right)\right\} \end{gathered} $$

The second term in brackets must then be "inter-distribution risk". However, both "inter- and intra-distribution risk" depend on a known regime $S_t = s$ and, therefore, more information than the left-hand side. This in turn means that the desired risk decomposition cannot be implemented.

However, if a similar decomposition where the known regime $S_t = s$ is replaced by the random variable S_t is applied instead of this failed approach, a successful (non-additive) risk decomposition becomes possible:

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$$ \begin{gathered} {P_{t+1}}^{iI} + D_{t+1} - E\left({P_{t+1}}^{iI} + D_{t+1} \middle| \pi_t, D_t\right) \\ = \underbrace{\left\{{P_{t+1}}^{iI} + D_{t+1} - E\left({P_{t+1}}^{iI} + D_{t+1} \middle| S_t, \pi_t, D_t\right)\right\}}_{\Delta^{comb.risk}} \\ \underbrace{+\left\{E\left({P_{t+1}}^{iI} + D_{t+1} \middle| S_t, \pi_t, D_t\right) - E\left({P_{t+1}}^{iI} + D_{t+1} \middle| \pi_t, D_t\right)\right\}}_{\Delta^{exp.risk}} \end{gathered} $$

The second term in brackets on the right-hand side captures the fact that the expectation of asset prices and cash flows conditional on the regime is unknown if the regime is unobservable. I call this form of risk "expectation risk". Continuing the above example of the logarithmic normally distributed regimes, “expectation risk” results from the fact that it is not known whether the true expectation parameter is μ_1 or the expected value of the beta distribution. It does not, however, include the risk of the unknown standard deviation, σ_1 or the standard deviation of the beta distribution, which clearly is a part of “inter-distribution risk”.

The first term in brackets on the right-hand side comprises two types of risk: first, even if the true regime was known, asset prices and cash flows would still be stochastic and deviate from their conditional expectation. Second, the true regime is not known. Therefore, I call this form of risk “combined risk”. In the example of the logarithmic normally distributed regimes, “combined risk” consists of the fact that the standard deviation parameter can be σ_1 or the standard deviation of the beta distribution and, in addition, asset prices and cash flows are still random variables.

Consequently, “inter-distribution risk” manifests itself in the form of “expectation risk” as well as the second component of “combined risk”. “Intra-distribution risk” consists of the first component of “combined risk”.

3.3.2.1.3.4.2 Pricing of the Parts of the Decomposition

Applying the stochastic discount factor to each of the three summands on the right-hand side of

$$P_{t+1}{}^{iI} + D_{t+1} = E\left(P_{t+1}{}^{iI} + D_{t+1} \middle| \pi_t, D_t\right) + \Delta^{exp.risk} + \Delta^{comb.risk}$$

leads to a decomposition of asset prices into the prices of three components.

The expectations of asset prices and cash flows conditional on the information relevant to pricing π_t, D_t are simply priced through riskless discounting, i.e.,

3-75

$$E\left(q_{t,t+1}{}^{iI} \cdot E\left(P_{t+1}{}^{iI} + D_{t+1} \middle| \pi_t, D_t\right) \middle| \pi_t, D_t\right) = \frac{1}{1+r} \cdot E\left(P_{t+1}{}^{iI} + D_{t+1} \middle| \pi_t, D_t\right)$$

Pricing “expectation risk“ yields (see Appendix A2.4.1.2.1)

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$$E\left(q_{t,t+1}{}^{iI} \cdot \Delta^{exp.risk} \middle| \pi_t, D_t\right) = -\frac{1}{1+r} \cdot \sum_{s=1}^{K} \{\pi_{s,t} - \theta_{s,t}(\pi_t, D_t)\} \cdot E\left(P_{t+1}{}^{iI} + D_{t+1} \middle| S_t = s, \pi_t, D_t\right)$$

where $\theta_{s,t}(\pi_t, D_t)$ is a risk-neutralized probability[15] of regime s,

[15] To see the interpretation of $\theta_{s,t}(\pi_t, D_t)$ as risk-neutralized regime probability, first observe that the K values $\theta_{s,t}(\pi_t, D_t), s = 1, \dots, K$ are non-negative and add to one, and, therefore, can formally be interpreted as regime probabilities. Also observe that $\theta_{s,t}(\pi_t, D_t)$ is obtained from the empirical regime probability $\pi_{s,t}$ through an adjustment for regime risk; the adjustment for risk equals the compounded stochastic discount factor.

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$$\theta_{s,t}(\pi_t, D_t) \equiv \frac{\pi_{s,t} \cdot E\left(\begin{matrix} exp\left(-\alpha_{t+1} \cdot \frac{1}{n_I} \cdot \{D_{t+1}{}^{market}(D_{t+1}) + V_{t+1}{}^{market}(\pi_{t+1}, D_{t+1})\}\right) \\ \cdot m_{t+1}{}^{iI}(\pi_{t+1}, D_{t+1}) \end{matrix} \middle| S_t = s, D_t \right)}{\sum_{s'=1}^{K} \pi_{s',t} \cdot E\left(\begin{matrix} exp\left(-\alpha_{t+1} \cdot \frac{1}{n_I} \cdot \{D_{t+1}{}^{market}(D_{t+1}) + V_{t+1}{}^{market}(\pi_{t+1}, D_{t+1})\}\right) \\ \cdot m_{t+1}{}^{iI}(\pi_{t+1}, D_{t+1}) \end{matrix} \middle| S_t = s', D_t \right)}$$

$$s = 1, \dots, K$$

With this interpretation of $\theta_{s,t}(\pi_t, D_t)$ in mind, it becomes clear that the sign of the price of "expectation risk" depends on whether the expectation of $E\left(P_{t+1}{}^{iI} + D_{t+1} \middle| S_t = s, \pi_t, D_t\right)$ with respect to empirical probabilities is higher or lower than its expectation with respect to the risk-neutralized probabilities.[16] Assets are especially valuable if they provide high conditional expected asset prices and high conditional expected cash flows in bad regimes (i.e., where $\frac{\partial}{\partial W_{t+1}} \bar{J}$ is above $E\left(\frac{\partial}{\partial W_{t+1}} \bar{J} \middle| z_t{}^p\right)$); such assets will provide a hedge against "expectation risk", and investors will be willing to pay for such a hedge. Similarly, "expectation risk" lowers the value of those assets that offer low expected asset prices and cash flows in bad regimes.

Pricing "combined risk", yields (see the Appendix A2.4.1.2.2)

3-78

$$E\left(q_{t,t+1}{}^{iI} \cdot \Delta^{comb.risk} \middle| \pi_t, D_t\right) = \frac{1}{1+r} \cdot \sum_{s=1}^{K} \theta_{s,t}(\pi_t, D_t) \cdot E\left(AfR_t{}^{conditional}(s; \eta_{t+1}, S_{t+1}, fe_{t+1}, S_t, \pi_t, D_t) \cdot \Delta^{comb.risk} \middle| S_t = s, D_t\right)$$

where the "conditional adjustment for risk" relative to regime s is defined by

$$AfR_t{}^{conditional}(s; \eta_{t+1}, S_{t+1}, fe_{t+1}, S_t, \pi_t, D_t) \equiv \frac{exp\left(-\alpha_{t+1} \cdot \frac{1}{n_I} \cdot \left\{ \begin{matrix} D_{t+1}{}^{market}(D_{t+1}) \\ +V_{t+1}{}^{market}(\pi_{t+1}, D_{t+1}) \end{matrix} \right\}\right) \cdot m_{t+1}{}^{iI}(\pi_{t+1}, D_{t+1})}{E\left(exp\left(-\alpha_{t+1} \cdot \frac{1}{n_I} \cdot \left\{ \begin{matrix} D_{t+1}{}^{market}(D_{t+1}) \\ +V_{t+1}{}^{market}(\pi_{t+1}, D_{t+1}) \end{matrix} \right\}\right) \cdot m_{t+1}{}^{iI}(\pi_{t+1}, D_{t+1}) \middle| S_t = s, D_t\right)}$$

with

$$D_{t+1} = D(D_t, S_t, fe_{t+1})$$
$$\pi_{t+1} = \Pi(\pi_t, D_t, D_{t+1}, Sig_{t+1})$$
$$Sig_{t+1} = Sig_{t+1}(S_t, S_{t+1}, fe_{t+1}, \eta_{t+1})$$

To answer the initial question of this section, namely how "inter-distribution risk" and "intra- distribution risk" are priced and whether their price effects can be separated, it can be stated: "inter-

[16] Note, that this structure is similar (and closely related) to the result on risk premia (3-55).

distribution risk” is priced via risk-neutralized regime probabilities $\theta_t(\pi_t, D_t)$ that are elements in both the prices of “expectation risk” and “combined risk”. “Intra-distribution risk” is priced via a conditional adjustment for risk and only appears in the price of “combined risk”.

3.3.2.1.3.4.3 Decomposition of Risk Premia

The risk premium $RP_t^{iI}(\pi_t, D_t)$ can likewise be decomposed into two components attributable to “expectation risk”[17] (first term) and “combined risk” (second term):

3-79

$$RP_t^{iI}(\pi_t, D_t) = \sum_{s=1}^{K} \{\pi_{s,t} - \theta_{s,t}(\pi_t, D_t)\} \cdot E\left(P_{t+1}^{iI} + D_{t+1} \middle| S_t = s, \pi_t, D_t\right) + \sum_{s=1}^{K} \theta_{s,t}(\pi_t, D_t) \cdot RP_t^{conditional}(s; \pi_t, D_t)$$

with

$$RP_t^{conditional}(s; \pi_t, D_t) \equiv -cov\left(AfR_t^{conditional}(s; \eta_{t+1}, S_{t+1}, fe_{t+1}, S_t, \pi_t, D_t), \Delta^{comb.risk} \middle| S_t = s, D_t\right)$$

Analogous to the price formulation, “inter-distribution risk” is priced via risk-neutralized regime probabilities $\theta_t(\pi_t, D_t)$ that are elements in both the prices of “expectation risk” and “combined risk”. “Intra-distribution risk” is priced via a conditional risk premium and only appears in the part of the risk premium attributable to “combined risk”.

3.3.2.2 Not All Assets Pay Cash Flows in Every Period: Information Frequency ≥ Cash Flow Frequency

3.3.2.2.1 Cash Flow Model

In the more general case where information frequency is higher than or equal to cash flow frequency, there are two groups of assets: assets where information frequency is equal to cash flow frequency, so-called (1)-periodic assets, and assets where information frequency is greater than cash flow frequency, so-called (Δ_C)-periodic assets. Denote by n_1(where n_1 can be zero) and n_{Δ_C} the number of the respective assets. In addition, the points of time where (Δ_C)-periodic assets pay cash flows read $t_{(k)} \equiv k \cdot \Delta_C, k \epsilon \mathbb{N}_0$. Although (Δ_C)-periodic assets pay cash flows only every Δ_C periods,

[17] Recall from the discussion of general asset pricing results that risk premia are differences of expectations under empirical and risk-neutralized probability measures.

trading and information arrival occurs at every point in time $t = 0, \ldots, T-1$. For (1)-periodic assets, cash flow, information, and trading times coincide.

Since cash flows for both groups of assets follow the general model, they read

For (1)-periodic assets

2-18

$$D_{t+1}{}^{(1)} = D^{(1)}\left(D_t{}^{(1)}, S_t, fe_{t+1}\right)$$

For (Δ_C)-periodic assets

2-23

$$D_{t_{(k+1)}}{}^{(\Delta_C)} = D^{(\Delta_C)}\left(D_{t_{(k)}}{}^{(\Delta_C)}, S_{t_{(k)},t_{(k+1)}-1}, fe_{t_{(k+1)}}\right)$$

3.3.2.2.2 Complete Information

3.3.2.2.2.1 Information Relevant to Pricing $z_t{}^p$

Under complete information, CARA preferences, and general cash flow model where information frequency is higher than or equal to cash flow frequency, $z_t{}^p$ coincides with $z_t{}^{d,+}$ and reads

3-80

$$z_t{}^{cI,(\Delta_C),d,+} = \left(S_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)$$

where $S_{t_{(k_t)},t}$ is the path of regimes since the last payment date for (Δ_C)-periodic cash flows and where $z_t{}^{cI,(\Delta_C),d,+}$ is Markov. $t_{(k_t)}$ denotes the most recent cash flow payment date of (Δ_C)-periodic cash flows from the perspective of time t:

3-81

$$k_t \equiv max\{k\epsilon\mathbb{N}_0 : k \cdot \Delta_C \leq t\}$$

As in the simpler case of information frequency equals cash flow frequency, elements must be identified to describe the conditional distribution of future cash flows and, thus, to specify $z_t{}^{cI,(\Delta_C),d,+}$. The functional form for (Δ_C)-periodic cash flows implies that future cash flows of this type depend on the path of regimes $S_{t_{(k_\tau)},\tau}$ as well as the most recent (Δ_C)-periodic cash flows $D_{t_{(k_t)}}{}^{(\Delta_C)}$. Future cash flows of the (1)-periodic type depend on the most recent regime S_t (which is included in the path of regimes $S_{t_{(k_t)},t}$) and current cash flows $D_t{}^{(1)}$.

3.3.2.2.2.2 Equilibrium Asset Prices

In the case of complete information where information frequency is higher than or equal to cash flow frequency, the pricing result for general cash flow models and information scenarios (3-63) specializes to two pricing equations, one relating to each group of risky assets:

Prices of (Δ_C)-periodic risky assets

3-82

$$P_t^{cI,(\Delta_C)}\left(S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}\right) = E\left(\begin{matrix} q_{t,t+1}^{cI,(\Delta_C)}\left(S_{t+1}, fe_{t+1}, S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}\right) \\ \cdot\left\{P_{t+1}^{cI,(\Delta_C)} + D_{t+1}^{(\Delta_C)}\right\}\end{matrix}\middle| S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}\right)$$

with

$$k = k_t$$

$$D_{t+1}^{(\Delta_C)} = \begin{cases} D^{(\Delta_C)}\left(D_{t_{(k)}}^{(\Delta_C)}, S_{t_{(k)},t}, fe_{t+1}\right) & t+1 = t_{(k+1)} \\ 0 & t+1 \neq t_{(k+1)} \end{cases}$$

$$D_{t+1}^{(1)} = D^{(1)}\left(D_t^{(1)}, S_t, fe_{t+1}\right)$$

$$P_{t+1}^{cI,(\Delta_C)} = \begin{cases} P_{t+1}^{cI,(\Delta_C)}\left(S_{t+1,t+1}, D_{t+1}^{(\Delta_C)}, D_{t+1}^{(1)}\right) & t+1 = t_{(k+1)} \\ P_{t+1}^{cI,(\Delta_C)}\left(S_{t_{(k)},t+1}, D_{t_{(k)}}^{(\Delta_C)}, D_{t+1}^{(1)}\right) & t+1 \neq t_{(k+1)} \end{cases}$$

Prices of (1)-periodic risky assets

3-83

$$P_t^{cI,(1)}\left(S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}\right) = E\left(\begin{matrix} q_{t,t+1}^{cI,(\Delta_C)}\left(S_{t+1}, fe_{t+1}, S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}\right) \\ \cdot\left\{P_{t+1}^{cI,(1)} + D_{t+1}^{(1)}\right\}\end{matrix}\middle| S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}\right)$$

with

$$P_{t+1}^{cI,(1)} = \begin{cases} P_{t+1}^{cI,(1)}\left(S_{t+1,t+1}, D_{t+1}^{(\Delta_C)}, D_{t+1}^{(1)}\right) & t+1 = t_{(k+1)} \\ P_{t+1}^{cI,(1)}\left(S_{t_{(k)},t+1}, D_{t_{(k)}}^{(\Delta_C)}, D_{t+1}^{(1)}\right) & t+1 \neq t_{(k+1)} \end{cases}$$

with stochastic discount factor

$$q_{t,t+1}^{cI,(\Delta_C)}\left(S_{t+1}, fe_{t+1}, S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}\right) = \frac{1}{1+r} \cdot \frac{exp\left(-\alpha_{t+1}\cdot\frac{1}{n_I}\cdot\left\{D_{t+1}^{market} + V_{t+1}^{market}\right\}\right)\cdot m_{t+1}^{cI,(\Delta_C)}}{E\left(\left\{\begin{matrix} exp\left(-\alpha_{t+1}\cdot\frac{1}{n_I}\cdot\left\{D_{t+1}^{market} + V_{t+1}^{market}\right\}\right) \\ \cdot m_{t+1}^{cI,(\Delta_C)}\end{matrix}\right\}\middle| S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}\right)}$$

with

$$D_{t+1}{}^{market} = \begin{cases} D_{t+1}{}^{market}\left(D_{t+1}{}^{(\Delta_C)}, D_{t+1}{}^{(1)}\right) & t+1 = t_{(k+1)} \\ D_{t+1}{}^{market}\left(D_{t+1}{}^{(1)}\right) & t+1 \neq t_{(k+1)} \end{cases}$$

$$V_{t+1}{}^{market} = \begin{cases} V_{t+1}{}^{market}\left(S_{t+1,t+1}, D_{t+1}{}^{(\Delta_C)}, D_{t+1}{}^{(1)}\right) & t+1 = t_{(k+1)} \\ V_{t+1}{}^{market}\left(S_{t_{(k)},t+1}, D_{t_{(k)}}{}^{(\Delta_C)}, D_{t+1}{}^{(1)}\right) & t+1 \neq t_{(k+1)} \end{cases}$$

$$m_{t+1}{}^{cI,(\Delta_C)} = \begin{cases} m_{t+1}{}^{cI,(\Delta_C)}\left(S_{t+1,t+1}, D_{t+1}{}^{(\Delta_C)}, D_{t+1}{}^{(1)}\right) & t+1 = t_{(k+1)} \\ m_{t+1}{}^{cI,(\Delta_C)}\left(S_{t_{(k)},t+1}, D_{t_{(k)}}{}^{(\Delta_C)}, D_{t+1}{}^{(1)}\right) & t+1 \neq t_{(k+1)} \end{cases}$$

(1)-periodic and (Δ_C)-periodic assets possess two different cash flow structures in that (Δ_C)-periodic assets do not pay cash flows in every point in time. This difference translates into two different price formulas: prices of (Δ_C)-periodic assets are present values of next-period prices only[18] whereas prices of (1)-periodic assets are present values of next-period prices and cash flows.

Both prices, however, are not isolated from each other but are interdependent. In particular, information relevant to pricing for both groups of assets not only includes the conditional distribution of cash flows of that particular group: regime and cash flow for the (1)-periodic assets; path of regime and cash flow for the (Δ_C)-periodic assets. Instead, prices of each group also depend on the conditional distribution of cash flows of the other group. The underlying reason is that aggregates of cash flows paid by both groups of assets are one major component of the stochastic discount factor that prices both groups of assets.

In addition, the adjustment for risk for both prices depends on two sources of risk, regimes S_{t+1} as well as factors and residuals fe_{t+1}. In the generalized case where information frequency is higher than cash flow frequency S_{t+1} can now be interpreted as the stochastic part of the new path $S_{t_{(k)},t+1}$ or, if $t+1 = t_{(k+1)}$, $S_{t_{(k+1)},t_{(k+1)}}$. In other words, the source of risk at time $t+1$ is not the entire path of regimes but only the next addition to the regime path, i.e., S_{t+1} which means that both S_{t+1} and fe_{t+1} are priced in a risk-adjusted way.

Given these identified sources of risk in the case where information frequency is higher than or equal to cash flow frequency, analogous interaction results hold to the case where information frequency equals cash flow frequency.

3.3.2.2.2.3 Equilibrium Risk Premia

In the case of complete information where information frequency is higher than or equal to cash flow frequency, the result on equilibrium risk premia for general cash flow models and information scenarios (3-65) takes the following form:

[18] With the exception of time $t = t_{(k+1)} - 1$.

Risk premia of (Δ_C)-periodic risky assets

3-84

$$RP_t^{cI,(\Delta_C)}\left(S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}\right) = -cov\left(\begin{matrix} AfR_t^{cI,(\Delta_C)}\left(S_{t+1}, fe_{t+1}, S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}\right) \\ , P_{t+1}^{cI,(\Delta_C)} + D_{t+1}^{(\Delta_C)} \end{matrix}\middle| S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}\right)$$

with

$$k = k_t$$

$$D_{t+1}^{(\Delta_C)} = \begin{cases} D^{(\Delta_C)}\left(D_{t_{(k)}}^{(\Delta_C)}, S_{t_{(k)},t}, fe_{t+1}\right) & t+1 = t_{(k+1)} \\ 0 & t+1 \neq t_{(k+1)} \end{cases}$$

$$D_{t+1}^{(1)} = D^{(1)}\left(D_t^{(1)}, S_t, fe_{t+1}\right)$$

$$P_{t+1}^{cI,(\Delta_C)} = \begin{cases} P_{t+1}^{cI,(\Delta_C)}\left(S_{t_{(k+1)},t_{(k+1)}}, D_{t+1}^{(\Delta_C)}, D_{t+1}^{(1)}\right) & t+1 = t_{(k+1)} \\ P_{t+1}^{cI,(\Delta_C)}\left(S_{t_{(k)},t+1}, D_{t_{(k)}}^{(\Delta_C)}, D_{t+1}^{(1)}\right) & t+1 \neq t_{(k+1)} \end{cases}$$

Risk premia of (1)-periodic risky assets

3-85

$$RP_t^{cI,(1)}\left(S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}\right) = -cov\left(\begin{matrix} AfR_t^{cI,(\Delta_C)}\left(S_{t+1}, fe_{t+1}, S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}\right) \\ , P_{t+1}^{cI,(1)} + D_{t+1}^{(1)} \end{matrix}\middle| S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}\right)$$

with

$$P_{t+1}^{cI,(1)} = \begin{cases} P_{t+1}^{cI,(1)}\left(S_{t+1,t+1}, D_{t+1}^{(\Delta_C)}, D_{t+1}^{(1)}\right) & t+1 = t_{(k+1)} \\ P_{t+1}^{cI,(1)}\left(S_{t_{(k)},t+1}, D_{t_{(k)}}^{(\Delta_C)}, D_{t+1}^{(1)}\right) & t+1 \neq t_{(k+1)} \end{cases}$$

with adjustment for risk

$$AfR_t^{cI,(\Delta_C)}\left(S_{t+1}, fe_{t+1}, S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}\right) \equiv \frac{\left\{exp\left(-\alpha_{t+1}\cdot\frac{1}{n_I}\cdot\left\{D_{t+1}^{market} + V_{t+1}^{market}\right\}\right)\cdot m_{t+1}^{cI,(\Delta_C)}\right\}}{E\left(\left\{exp\left(-\alpha_{t+1}\cdot\frac{1}{n_I}\cdot\left\{D_{t+1}^{market} + V_{t+1}^{market}\right\}\right)\cdot m_{t+1}^{cI,(\Delta_C)}\right\}\middle| S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}\right)}$$

with

$$D_{t+1}^{market} = \begin{cases} D_{t+1}^{market}\left(D_{t+1}^{(\Delta_C)}, D_{t+1}^{(1)}\right) & t+1 = t_{(k+1)} \\ D_{t+1}^{market}\left(D_{t+1}^{(1)}\right) & t+1 \neq t_{(k+1)} \end{cases}$$

$$V_{t+1}^{market} = \begin{cases} V_{t+1}^{market}\left(S_{t+1,t+1}, D_{t+1}^{(\Delta_C)}, D_{t+1}^{(1)}\right) & t+1 = t_{(k+1)} \\ V_{t+1}^{market}\left(S_{t_{(k)},t+1}, D_{t_{(k)}}^{(\Delta_C)}, D_{t+1}^{(1)}\right) & t+1 \neq t_{(k+1)} \end{cases}$$

$$m_{t+1}^{cI,(\Delta_C)} = \begin{cases} m_{t+1}^{cI,(\Delta_C)}\left(S_{t+1,t+1}, D_{t+1}^{(\Delta_C)}, D_{t+1}^{(1)}\right) & t+1 = t_{(k+1)} \\ m_{t+1}^{cI,(\Delta_C)}\left(S_{t_{(k)},t+1}, D_{t_{(k)}}^{(\Delta_C)}, D_{t+1}^{(1)}\right) & t+1 \neq t_{(k+1)} \end{cases}$$

Similar to the price equation, risk premia are determined by the same sources of risk as asset prices where analogous interaction results hold.

3.3.2.2.3 Incomplete Information

3.3.2.2.3.1 Information Relevant to Pricing $z_t{}^p$

Under complete information, CARA preferences, and general cash flow model where information frequency is higher than or equal to cash flow frequency, $z_t{}^p$ still coincides with $z_t{}^{d,+}$ and reads

3-86
$$z_t{}^{iI,(\Delta_C),d,+} = \left(\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)$$

where $z_t{}^{iI,(\Delta_C),d,+}$ is Markov and where

$$\pi_{t_{(k_t)},t}\left(S_{t_{(k_t)},t}\right) \equiv P\left(S_{t_{(k_t)},t} = s_{t_{(k_t)},t} \middle| \overrightarrow{Sig_t}, \overrightarrow{D_{t_{(k_t)}}{}^{(\Delta_C)}}, \overrightarrow{D_t{}^{(1)}}\right)$$

denotes conditional regime path probabilities.

As in the complete information case, elements must be identified to describe the conditional distribution of future cash flows and, thus to specify $z_t{}^{iI,(\Delta_C),d,+}$. In principle, $z_t{}^{iI,(\Delta_C),d,+}$ depends on the path of regimes and the most recent cash flows of both types. However, the path of regimes is not observable in the complete information case and hence replaced by conditional regime path probabilities.

3.3.2.2.3.2 Equilibrium Asset Prices

In the case of incomplete information where information frequency is higher than or equal to cash flow frequency, the pricing results for general cash flow models and information scenarios (3-63) specialize to two pricing equations, one relating to each group of risky assets:

<u>Prices of (Δ_C)-periodic risky assets</u>

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$$\begin{aligned} &P_t{}^{iI,(\Delta_C)}\left(\pi_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) \\ &= \sum_{s_{t_{(k)},t}} \pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right) \\ &\cdot E\left(\begin{matrix} q_{t,t+1}{}^{iI,(\Delta_C)}\left(\eta_{t+1}, S_{t+1}, fe_{t+1}, S_{t_{(k)},t}, \pi_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) \\ \cdot\left\{P_{t+1}{}^{iI,(\Delta_C)} + D_{t+1}{}^{(\Delta_C)}\right\} \end{matrix} \middle| S_{t_{(k)},t} = s_{t_{(k)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) \end{aligned}$$

with

$$k = k_t$$

$$D_{t+1}{}^{(\Delta c)} = \begin{cases} D^{(\Delta c)}\left(D_{t_{(k)}}{}^{(\Delta c)}, \pi_{t_{(k)},t}, fe_{t+1}\right) & t+1 = t_{(k+1)} \\ 0 & t+1 \neq t_{(k+1)} \end{cases}$$

$$D_{t+1}{}^{(1)} = D^{(1)}\left(D_t{}^{(1)}, S_t, fe_{t+1}\right)$$

$$P_{t+1}{}^{iI,(\Delta c)} = \begin{cases} P_{t+1}{}^{iI,(\Delta c)}\left(\pi_{t+1,t+1}, D_{t+1}{}^{(\Delta c)}, D_{t+1}{}^{(1)}\right) & t+1 = t_{(k+1)} \\ P_{t+1}{}^{iI,(\Delta c)}\left(\pi_{t_{(k)},t+1}, D_{t_{(k)}}{}^{(\Delta c)}, D_{t+1}{}^{(1)}\right) & t+1 \neq t_{(k+1)} \end{cases}$$

$$\pi_{t_{(k)},t+1} = \Pi_0\left(\pi_{t_{(k)},t+1}, D_t{}^{(1)}, D_{t+1}{}^{(1)}, Sig_{t+1}\right), t+1 \neq t_{(k+1)}$$

$$\pi_{t_{(k+1)},t_{(k+1)}} = \Pi_1\begin{pmatrix} \pi_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta c)}, D_{,t}{}^{(1)} \\ , D_{t+1}{}^{(\Delta c)}, D_{t+1}{}^{(1)}, Sig_{t+1} \end{pmatrix}$$

$$Sig_{t+1} = Sig_{t+1}\left(S_{t_{(k)},t}, S_{t+1}, fe_{t+1}, \eta_{t+1}\right)$$

Prices of (1)-periodic risky assets

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$$P_t{}^{iI,(1)}\left(\pi_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta c)}, D_t{}^{(1)}\right)$$
$$= \sum_{s_{t_{(k)},t}} \pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right)$$
$$\cdot E\left(\begin{matrix} q_{t,t+1}{}^{iI,(\Delta c)}\left(\eta_{t+1}, S_{t+1}, fe_{t+1}, S_{t_{(k)},t}, \pi_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta c)}, D_t{}^{(1)}\right) \\ \cdot\left\{P_{t+1}{}^{iI,(1)} + D_{t+1}{}^{(1)}\right\} \end{matrix}\middle| S_{t_{(k)},t} = s_{t_{(k)},t}, D_{t_{(k_t)}}{}^{(\Delta c)}, D_t{}^{(1)}\right)$$

with

$$P_{t+1}{}^{iI,(1)} = \begin{cases} P_{t+1}{}^{iI,(1)}\left(\pi_{t+1,t+1}, D_{t+1}{}^{(\Delta c)}, D_{t+1}{}^{(1)}\right) & t+1 = t_{(k+1)} \\ P_{t+1}{}^{iI,(1)}\left(\pi_{t_{(k)},t+1}, D_{t_{(k)}}{}^{(\Delta c)}, D_{t+1}{}^{(1)}\right) & t+1 \neq t_{(k+1)} \end{cases}$$

with stochastic discount factor

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$$q_{t,t+1}{}^{iI,(\Delta c)}\left(\eta_{t+1}, S_{t+1}, fe_{t+1}, S_{t_{(k)},t}, \pi_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta c)}, D_t{}^{(1)}\right)$$
$$= \frac{1}{1+r}$$
$$\cdot \frac{\left\{exp\left(-\alpha_{t+1}\cdot\frac{1}{n_I}\cdot\left\{D_{t+1}{}^{market} + V_{t+1}{}^{market}\right\}\right)\cdot m_{t+1}{}^{iI,(\Delta c)}\right\}}{\sum_{s_{t_{(k)},t}} \pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right)\cdot E\left(\begin{Bmatrix} exp\left(-\alpha_{t+1}\cdot\frac{1}{n_I}\cdot\left\{D_{t+1}{}^{market} + V_{t+1}{}^{market}\right\}\right) \\ \cdot m_{t+1}{}^{iI,(\Delta c)} \end{Bmatrix}\middle| \begin{matrix} S_{t_{(k)},t} = s_{t_{(k)},t} \\ , D_{t_{(k)}}{}^{(\Delta c)}, D_t{}^{(1)} \end{matrix}\right)}$$

with

$$D_{t+1}{}^{market} = \begin{cases} D_{t+1}{}^{market}\left(D_{t+1}{}^{(\Delta c)}, D_{t+1}{}^{(1)}\right) & t+1 = t_{(k+1)} \\ D_{t+1}{}^{market}\left(D_{t+1}{}^{(1)}\right) & t+1 \neq t_{(k+1)} \end{cases}$$

$$V_{t+1}{}^{market} = \begin{cases} V_{t+1}{}^{market}\left(\pi_{t+1t+1}, D_{t+1}{}^{(\Delta c)}, D_{t+1}{}^{(1)}\right) & t+1 = t_{(k+1)} \\ V_{t+1}{}^{market}\left(\pi_{t_{(k)},t+1}, D_{t_{(k)}}{}^{(\Delta c)}, D_{t+1}{}^{(1)}\right) & t+1 \neq t_{(k+1)} \end{cases}$$

$$m_{t+1}{}^{iI,(\Delta_C)} = \begin{cases} m_{t+1}{}^{iI,(\Delta_C)}\left(\pi_{t+1,t+1}, D_{t+1}{}^{(\Delta_C)}, D_{t+1}{}^{(1)}\right) & t+1 = t_{(k+1)} \\ m_{t+1}{}^{iI,(\Delta_C)}\left(\pi_{t_{(k)},t+1}, D_{t_{(k)}}{}^{(\Delta_C)}, D_{t+1}{}^{(1)}\right) & t+1 \neq t_{(k+1)} \end{cases}$$

As in the complete information case with information frequency higher than or equal to cash flow frequency, there are two groups of prices that are interdependent. There are two underlying reasons for this interdependence: first, and similar to the complete information case, the stochastic discount factor; second, in addition to the complete information case, an information argument: (1)-periodic cash flows, together with signals, provide information on the true path of regimes that is relevant to pricing (Δ_C)- periodic assets.

There are four sources of risk, namely the current regime path $S_{t_{(k)},t}$, factors and residuals fe_{t+1}, the regime at time $t+1$ S_{t+1} as the next addition to the regime path, and signal noise η_{t+1} where $S_{t_{(k)},t}$ and S_{t+1} are dependent; fe_{t+1}, $S_{t_{(k)},t+1}$ (or, if $t+1 = t_{(k+1)}$, $S_{t_{(k+1)},t_{(k+1)}}$), and η_{t+1} are independent (see (2-17) and p. 21). Nevertheless, these sources of risk jointly determine the adjustment for risk and, hence, are priced in a risk-adjusted way: As in the simpler case where information frequency equals cash flow frequency, these sources of risk fall into two groups: $S_{t_{(k)},t}$ and fe_{t+1} influence the adjustment for risk both through $D_{t+1}{}^{(1)}$ (or, if $t+1 = t_{(k+1)}$, $D_{t_{(k+1)}}{}^{(\Delta_C)}$ and $D_{t_{(k+1)}}{}^{(1)}$) and through signals; S_{t+1} and η_{t+1} affect the adjustment for risk only through signals.

3.3.2.2.3.3 Equilibrium Risk Premia

In the case of incomplete information where information frequency is higher than or equal to cash flow frequency, result on equilibrium risk premia for general cash flow models and information scenarios (3-65) takes the following form:

<u>Risk premia of (Δ_C)-periodic risky assets</u>

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$$RP_t{}^{iI,(\Delta_C)}\left(\pi_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) = -cov\left(\begin{matrix} AfR_t{}^{iI,(\Delta_C)}\left(\eta_{t+1}, S_{t+1}, fe_{t+1}, S_{t_{(k)},t}, \pi_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) \\ , P_{t+1}{}^{iI,(\Delta_C)} + D_{t+1}{}^{(\Delta_C)} \end{matrix} \middle| \pi_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)$$

with

$$k = k_t$$

$$D_{t+1}{}^{(\Delta_C)} = \begin{cases} D^{(\Delta_C)}\left(D_{t_{(k)}}{}^{(\Delta_C)}, S_{t_{(k)},t}, fe_{t+1}\right) & t+1 = t_{(k+1)} \\ 0 & t+1 \neq t_{(k+1)} \end{cases}$$

$$D_{t+1}{}^{(1)} = D^{(1)}\left(D_t{}^{(1)}, S_t, fe_{t+1}\right)$$

$$P_{t+1}{}^{iI,(\Delta_C)} = \begin{cases} P_{t+1}{}^{iI,(\Delta_C)}\left(\pi_{t+1,t+1}, D_{t+1}{}^{(\Delta_C)}, D_{t+1}{}^{(1)}\right) & t+1 = t_{(k+1)} \\ P_{t+1}{}^{iI,(\Delta_C)}\left(\pi_{t_{(k)},t+1}, D_{t_{(k)}}{}^{(\Delta_C)}, D_{t+1}{}^{(1)}\right) & t+1 \neq t_{(k+1)} \end{cases}$$

Risk premia of (1)-periodic risky assets

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$$RP_t{}^{iI,(1)}\left(\pi_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) = -cov\left(\begin{matrix} AfR_t{}^{iI,(\Delta_C)}\left(\eta_{t+1}, S_{t+1}, fe_{t+1}, S_{t_{(k)},t}, \pi_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) \\ , P_{t+1}{}^{iI,(1)}\left(z_{t+1}{}^{iI,(\Delta_C)}\right) + D_{t+1}{}^{(1)}\left(z_{t+1}{}^{iI,(\Delta_C),d}\right) \end{matrix} \middle| \pi_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)} \right)$$

with

$$P_{t+1}{}^{iI,(1)} = \begin{cases} P_{t+1}{}^{iI,(1)}\left(\pi_{t+1,t+1}, D_{t+1}{}^{(\Delta_C)}, D_{t+1}{}^{(1)}\right) & t+1 = t_{(k+1)} \\ P_{t+1}{}^{iI,(1)}\left(\pi_{t_{(k)},t+1}, D_{t_{(k)}}{}^{(\Delta_C)}, D_{t+1}{}^{(1)}\right) & t+1 \neq t_{(k+1)} \end{cases}$$

with adjustment for risk

$$AfR_t{}^{iI,(\Delta_C)}\left(\eta_{t+1}, S_{t+1}, fe_{t+1}, S_{t_{(k)},t}, \pi_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) \equiv \frac{\left\{ \begin{matrix} exp\left(-\alpha_{t+1} \cdot \frac{1}{n_I} \cdot \left\{D_{t+1}{}^{market} + V_{t+1}{}^{market}\right\}\right) \\ \cdot m_{t+1}{}^{iI,(\Delta_C)} \end{matrix} \right\}}{\sum_{s_{t_{(k)},t}} \pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right) \cdot E\left(\left\{ \begin{matrix} exp\left(-\alpha_{t+1} \cdot \frac{1}{n_I} \cdot \left\{D_{t+1}{}^{market} + V_{t+1}{}^{market}\right\}\right) \\ \cdot m_{t+1}{}^{iI,(\Delta_C)} \end{matrix} \right\} \middle| \begin{matrix} S_{t_{(k)},t} = s_{t_{(k)},t} \\ , D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)} \end{matrix} \right)}$$

with

$$D_{t+1}{}^{market} = \begin{cases} D_{t+1}{}^{market}\left(D_{t+1}{}^{(\Delta_C)}, D_{t+1}{}^{(1)}\right) & t+1 = t_{(k+1)} \\ D_{t+1}{}^{market}\left(D_{t+1}{}^{(1)}\right) & t+1 \neq t_{(k+1)} \end{cases}$$

$$V_{t+1}{}^{market} = \begin{cases} V_{t+1}{}^{market}\left(\pi_{t+1,t+1}, D_{t+1}{}^{(\Delta_C)}, D_{t+1}{}^{(1)}\right) & t+1 = t_{(k+1)} \\ V_{t+1}{}^{market}\left(\pi_{t_{(k)},t+1}, D_{t_{(k)}}{}^{(\Delta_C)}, D_{t+1}{}^{(1)}\right) & t+1 \neq t_{(k+1)} \end{cases}$$

$$m_{t+1}{}^{iI,(\Delta_C)} = \begin{cases} m_{t+1}{}^{cI,(\Delta_C)}\left(\pi_{t+1,t+1}, D_{t+1}{}^{(\Delta_C)}, D_{t+1}{}^{(1)}\right) & t+1 = t_{(k+1)} \\ m_{t+1}{}^{cI,(\Delta_C)}\left(\pi_{t_{(k)},t+1}, D_{t_{(k)}}{}^{(\Delta_C)}, D_{t+1}{}^{(1)}\right) & t+1 \neq t_{(k+1)} \end{cases}$$

Similar to the price equation, risk premia are determined by the same sources of risk as asset prices where analogous interaction results hold.

3.3.2.2.3.4 Risk Decomposition and Consequences to Prices and Risk Premia

The results on the pricing of “inter-distribution risk” and “intra-distribution risk” generalize without major change to the case where information frequency is higher than or equal to cash flow frequency: regimes merely have to be replaced by paths of regimes. Since the results otherwise parallel those obtained for the special case of identical information and cash flow frequencies, the details are only stated in Appendix A2.4.2.

3.3.2.3 Comparison of Asset Prices and Risk Premia across Information Structures

What can be said about the relationship between asset prices across differing information structures in a CARA partial equilibrium setting? Specifically two questions are addressed.

(i) Comparison of complete and incomplete information asset prices and risk premia. It could be argued naively that risk premia under incomplete information are always higher and asset prices always lower than their complete information counterparts because investors under incomplete information will demand a compensation for more sources of risk: regimes S_{t+1} as well as factors and residuals fe_{t+1} in the case of complete information versus four sources of risk in the incomplete information case (current regime S_t, factors and residuals fe_{t+1}, the regime at time $t+1$ S_{t+1}, and signal noise η_{t+1})

(ii) Effect of signal quality on asset prices and risk premia. To illustrate this question, consider two scenarios where investors learn much or little from signals about unobservable regimes. How do these different signals affect asset prices and risk premia?

Both the case where information frequency is equal to cash flow frequency as well as the more general case where information frequency is higher than or equal to cash flow frequency are discussed in this section.

3.3.2.3.1 Comparison of Asset Prices and Risk Premia under Complete and Incomplete Information

3.3.2.3.1.1 Asset Prices under Complete and Incomplete Information

To see if incomplete information asset prices are lower than the corresponding complete information prices and, more generally, to characterize the relationship between complete and incomplete information asset prices, it is instructive to consider time $t = T - 1$ and information frequency equal to cash flow frequency first. Then the model takes on a static character since there is no price vector at time T: Stochastic discount factors under complete (3-67) (p. 60) and incomplete (3-72) (p. 63) information simplify in the one-period model to[19]

$$q_{T-1,T}{}^{cI}(D_T, S_{T-1}, D_{T-1}) = \frac{1}{1+r} \cdot \frac{exp\left(-\alpha \cdot \frac{1}{n_I} \cdot D_T{}^{market}(D_T)\right)}{E\left(exp\left(-\alpha \cdot \frac{1}{n_I} \cdot D_T{}^{market}(D_T)\right) \middle| S_{T-1}, D_{T-1}\right)}$$

[19] Observe that the value of the market portfolio of risky assets at time T is zero, $V_T{}^{market} = 0$. Moreover, under both complete and incomplete information one has $\alpha_T = \alpha$ and $m_T{}^{cI} = m_T{}^{iI} = 1$.

$$q_{T-1,T}{}^{iI}(D_T,\pi_{T-1},D_{T-1})=\frac{1}{1+r}\cdot\frac{exp\left(-\alpha\cdot\frac{1}{n_I}\cdot D_T{}^{market}(D_T)\right)}{\sum_{s=1}^{K}\pi_{s,T-1}\cdot E\left(exp\left(-\alpha\cdot\frac{1}{n_I}\cdot D_T{}^{market}(D_T)\right)\middle|S_{T-1}=s,D_{T-1}\right)}$$

and incomplete information asset prices are expectations of complete information asset prices with respect to the risk-neutralized probabilities defined in (3-77) (p. 67):

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$$P_{T-1}{}^{iI}(\pi_{T-1},D_{T-1})=\sum_{s=1}^{K}\theta_{s,T-1}(\pi_{T-1},D_{T-1})\cdot P_{T-1}{}^{cI}(s,D_{T-1})$$

It follows from (3-92) that incomplete information asset prices can be higher, equal to, or lower than complete information asset prices: mathematically, this is a simple consequence of the fact that incomplete information asset prices at time $t=T-1$ are convex combinations of complete information asset prices (with weights $\theta_{s,T-1}(\pi_{T-1},D_{T-1}), s=1,\dots,K$).

To give an economic intuition behind this result, take any asset i and consider a simple two-regime example with a good and a bad regime in the sense that $P_{i,T-1}{}^{cI}(good,D_{T-1})>P_{i,T-1}{}^{cI}(bad,D_{T-1})$. Assume that in the complete information case the good regime occurs. In the incomplete information case, however, the bad regime has a positive probability. Hence, $P_{i,T-1}{}^{cI}(good,D_{T-1})$ is too high to induce market clearing under incomplete information and a price discount from $P_{i,T-1}{}^{cI}(good,D_{T-1})$ becomes necessary. On the flipside, if the bad regime holds in the complete information case, $P_{i,T-1}{}^{cI}(bad,D_{T-1})$ is too low to induce market clearing: the good regime might occur and each of the individual investors is willing to pay more than $P_{i,T-1}{}^{cI}(bad,D_{T-1})$.

If an arbitrary point in time t is considered and information frequency still equals cash flow frequency, incomplete information asset prices no longer are necessarily expectations of complete information asset prices. Instead, the results for time $T-1$ can be generalized along a slightly different line as expectations of "conditional asset prices" with respect to risk-neutralized probabilities:

$$P_t{}^{iI}(\pi_t,D_t)=\sum_{s=1}^{K}\theta_{s,t}(\pi_t,D_t)\cdot P_t{}^{conditional}(s,D_t)$$

with

$$P_t{}^{conditional}(s,D_t)$$
$$=E\left(exp(-r)\cdot AfR_t{}^{conditional}(S_t;\pi_{t+1},D_{t+1},\pi_t,D_t)\cdot\{P_{t+1}{}^{iI}(\pi_{t+1},D_{t+1})+D_{t+1}\}\middle|S_t=s,D_t\right)$$

Here $P_t{}^{conditional}(s,D_t)$ prices "intra-distribution risk" and is based on the conditional adjustment for risk defined in (3-78) (p. 67)

There are two reasons why incomplete information asset prices no longer are risk-neutralized expectations of complete information prices. First, prices at time $t+1$ are defined on a different set of possible outcomes than under complete information: under complete information, there are only K

possible next-period prices; under incomplete information, there are as many possible next-period prices as there are possible conditional regime probability distributions. Second, $AfR_t^{conditional}$ under incomplete information differs from its complete information counterpart AfR_t^{cI} (3-68) (p. 61). The value functions under complete and incomplete information are not the same, leading to different marginal contributions to wealth which are at the core of the adjustment for risk.[20]

Finally, in the generalized case where information frequency is higher than or equal to cash flow frequency, there cannot be a simple relationship between complete and incomplete information asset prices if it does not even hold in the special case of information frequency equal o cash flow frequency.

3.3.2.3.1.2 Risk Premia under Complete and Incomplete Information

To compare risk premia under complete and incomplete information, start again at time $T-1$, i.e., adapt the risk premia formula

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$$RP_t{}^{iI}(\pi_t, D_t) = \sum_{s=1}^{K} \{\pi_{s,t} - \theta_{s,t}(\pi_t, D_t)\} \cdot E\left(P_{t+1}{}^{iI} + D_{t+1} \middle| S_t = s, \pi_t, D_t\right) + \sum_{s=1}^{K} \theta_{s,t}(\pi_t, D_t) \cdot RP_t^{conditional}(s; \pi_t, D_t)$$

to the one-period case where information frequency equals cash flow frequency. First observe that $E\left(P_{t+1}{}^{iI} + D_{t+1} \middle| S_t = s, \pi_t, D_t\right)$ simplifies to $E(D_T|S_{T-1} = s, D_{T-1})$ because prices at time T are equal to zero. Second, conditional risk premia $RP_t^{conditional}(s; \pi_t, D_t)$ coincide with complete information risk premia $RP_{T-1}{}^{cI}(s, D_{T-1})$. The reason is that at time T asset prices are zero and complete and incomplete information value functions are identical and equal to the (direct) utility function. Hence, it is obtained for the one-period risk premia

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$$RP_{T-1}{}^{iI}(\pi_{T-1}, D_{T-1}) = \sum_{s=1}^{K} \{\pi_{s,T-1} - \theta_{s,T-1}(\pi_{T-1}, D_{T-1})\} \cdot E(D_T|S_{T-1} = s, D_{T-1}) + \sum_{s=1}^{K} \theta_{s,T-1}(\pi_{T-1}, D_{T-1}) \cdot RP_{T-1}{}^{cI}(s, D_{T-1})$$

It follows from (3-93) that incomplete information risk premia can be higher, equal to, or lower than complete information risk premia: mathematically, this is a simple consequence of the fact that

[20] Note that both differences vanish at time $T-1$. The first difference disappears because prices at time T equal zero and, hence, cannot have a different distribution in the complete and incomplete information case. The second difference does no longer exist since the boundary conditions at time T guarantees that the value function coincides with the direct utility function for both complete and incomplete information.

incomplete information risk premia at time $t = T - 1$ consist of two components. A convex combination of complete information risk premia (with weights $\theta_{s,T-1}(\pi_{T-1}, D_{T-1}), s = 1, \dots, K$, second term of (3-93)) and another component that is completely unrelated to complete information risk premia and can be positive, negative, or zero.

To give an economic intuition behind this result, observe that the first term equals the risk premium due to expectation risk". Hence, this risk premium is positive if expected cash flows are high when needed least and negative when needed most (see, p. 67). The economic intuition behind the second term is similar to the discussion of prices. Consider a simple two-regime example with a good and a bad regime in the sense that ${RP_{i,T-1}}^{cI}(good, D_{T-1}) < {RP_{i,T-1}}^{cI}(bad, D_{T-1})$. Assume that in the complete information case the good regime occurs. In the incomplete information case, however, the bad regime has a positive probability. Hence, ${RP_{i,T-1}}^{cI}(good, D_{T-1})$ is too low and ${RP_{i,T-1}}^{cI}(bad, D_{T-1})$ too high to cover the incomplete information case.

For that reason, risk premia at an arbitrary point in time t where information frequency equals cash flow frequency as well as information frequency is higher than or equal to cash flow frequency do not possess a clear-cut relation.

3.3.2.3.2 Asset Prices and Risk Premia under Differing Signal Qualities

Analyzing how changes in signal quality affect asset prices and risk premia means, it must be clarified what is meant by signal quality. Note that there are two limiting cases regarding signals: A useless signal is one that does not alter the conditional regime probabilities, i.e., all information is derived from the history of past cash flows alone. A perfect signal is one that assigns probability one to the true regime (and zero probability to all other regimes), i.e., a complete information scenario. Based on these limiting cases, high signal quality is – intuitively - defined as moving closer to the perfect signal case and low signal quality as approaching the useless signal case.

Signal quality can influence asset prices and risk premia via two channels, namely (i) dynamics of the information relevant to pricing, ${z_\tau}^{iI}$ or ${z_\tau}^{iI,(\Delta_C)}$, and (ii) the functional relation which transforms this information into prices, the price function, ${P_t}^{iI}(.)$ or ${P_t}^{iI,(\nu)}(.), \nu \epsilon \{1, \Delta_C\}$.

<u>Dynamics of the information relevant to pricing under different signal qualities</u>

The first insight concerning information relevant to pricing is that its composition does not itself depend on the signal quality: it is either given by ${z_t}^{iI,p}$ (3-70) (p. 62) or ${z_t}^{iI,(\Delta_C),p}$ (3-80) (p. 69) and the crucial part is either conditional regime probabilities or conditional regime path probabilities.

However, the dynamics of these conditional probabilities depends on signal quality. To see this point, the relation between signal quality and the dynamics of information relevant to pricing must be analyzed. The easiest way of analyzing this relation is to consider limiting cases first and then the particular information scenario at hand (intermediate case).

In the complete information case where information frequency equals cash flow frequency, the dynamics of information relevant to pricing can be identified with the transition from (S_t, D_t) to (S_{t+1}, D_{t+1}). Under complete information the dynamics of its two components regime and cash flows are conditionally independent: conditional on (S_t, D_t), the randomness of cash flows D_{t+1} is entirely due to factors and residuals, as the cash flow model

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$$D_{t+1} = D(D_t, S_t, fe_{t+1})$$

reveals. Moreover, the regime process and the process of factors and residuals have been assumed to be independent (see (2-17)). In other words, the components regime and cash flow can fluctuate totally independently of each other, i.e., in particular, a bad regime can coincide with a good cash flow.

Under the second limiting case incomplete information without signals where information frequency equals cash flow frequency, by contrast, the situation is different: the transition between times t and $t + 1$ of information relevant to pricing is entirely due to cash flows D_{t+1} which now have a double function: cash flows D_{t+1} are a component of information relevant to pricing at time $t + 1$ (direct effect) and, in addition, provide new information on the regime process (indirect effect). Thus, good cash flow must also increase the conditional probability of a good regime. In contrast to the complete information case, good cash flows and increasing probabilities of bad regimes cannot occur.

In the intermediate cases where signals are neither perfect nor useless and information frequency equals cash flow frequency, conditional probabilities depend on both signals and cash flows; the situation, therefore, is similar to complete information in that there are now two stochastic sources in the transition of information relevant to pricing. However, conditional probabilities and new cash flows are stochastically dependent, similar to incomplete information without signals. Continuing the above example, high cash flows can now coincide with a signal indicating a bad regime. Cash flows and the signal will have a conflicting effect on conditional regime probabilities: good cash flows will tend to raise the conditional probabilities of good regimes (or regime paths) whereas the bad signal will have the opposite effect. Which effect dominates will depend on how high signal quality : if signal quality is high, updated conditional regime probabilities are mostly determined by the signal; if signal quality is low, updated conditional regime probabilities mostly depend on cash flows.

These results transfer to the generalized case where information frequency is higher than or equal to cash flow frequency. Regimes and regime probabilities merely have to be replaced by regime paths and probabilities of regime paths.

Price and risk premium functions under different signal qualities

Recall that the price function translates given information relevant to pricing into concrete asset prices by discounting future cash flows. Similarly, a risk premium function can be defined by assigning the associated risk premium to information relevant to pricing. This means, even if information relevant to pricing is held constant, the price function will, in general, depend on signal quality. Since the argument for price functions and risk premium functions is exactly the same, I will focus on price functions and do not explicitly mention risk premia in the following paragraph.

To understand this connection between price function and signal quality intuitively, note that the price function not only depends on parameters like risk aversion, market portfolio, time preference rate etc., but also on the parameter signal quality. If one of these parameters is modified, it is clear that the price function can change. However, a more precise statement regarding the effects of signal quality on the price function cannot be made. Signal quality enters the price function via distribution of next period prices and stochastic discount factors. The complex interaction between both channels cannot be specified further.

More formally, let $\mathfrak{s}$ and $\mathfrak{s}'$ denote two different signal qualities. The dependence of asset prices on signal quality can be expressed by writing $P_t(\pi_t, D_t; \mathfrak{s})$ and $q_{t,t+1}(\pi_{t+1}, D_{t+1}, \pi_t, D_t; \mathfrak{s})$ instead of the less precise ${P_t}^{iI}(\pi_t, D_t)$ and ${q_{t,t+1}}^{iI}(\pi_{t+1}, D_{t+1}, \pi_t, D_t)$. In this notation the asset price under incomplete information and CARA utility function for general cash flow models (3-71) reads

$$P_t(\pi_t, D_t; \mathfrak{s}) = \sum_{s=1}^{K} \pi_{s,t} \cdot E\big(q_{t,t+1}(\pi_{t+1}, D_{t+1}, \pi_t, D_t; \mathfrak{s}) \cdot \{P_{t+1}(\pi_{t+1}, D_{t+1}; \mathfrak{s}) + D_{t+1}\} \big| S_t = s, D_t, \pi_t\big)$$

To compare the effect of signal qualities $\mathfrak{s}$ and $\mathfrak{s}'$ on asset prices, information relevant to pricing (π_t, D_t) is held constant. But although information relevant to pricing is the same, $P_{t+1}(\pi_{t+1}, D_{t+1}; \mathfrak{s})$ is a different random variable than $P_{t+1}(\pi_{t+1}, D_{t+1}; \mathfrak{s}')$. For example, conditional regime probabilities can be much more volatile under $\mathfrak{s}'$ than under $\mathfrak{s}$. Moreover, the stochastic discount factor $q_{t,t+1}(\pi_{t+1}, D_{t+1}, \pi_t, D_t; \mathfrak{s}')$ should also differ from $q_{t,t+1}(\pi_{t+1}, D_{t+1}, \pi_t, D_t; \mathfrak{s})$ because (i) the stochastic discount factor depends on the signal structure via ${V_{t+1}}^{market}(\pi_{t+1}, D_{t+1}; \mathfrak{s})$ and (ii) via $m_{t+1}(\pi_{t+1}, D_{t+1}; \mathfrak{s})$.

Finally, in the generalized case where information frequency is higher than or equal to cash flow frequency, price functions can still depend on signal quality via distribution of next period prices and stochastic discount factors. The exact functional form is even less tractable.

Signal quality and its effect on asset prices and risk premia

In conclusion, it is not possible to give a general answer to the question of this section, namely the effect of signal quality on prices and risk premia. The effect of signal quality on the dynamics of information can be determined analytically, the effect of signal qualities on price and risk premium functions, however, has proven to be analytically intractable. As a consequence, the total effect of signal qualities on asset prices and risk premia remains unclear.

3.3.3 Equilibrium Asset Prices and Risk Premia for a Special Cash Flow Model under Complete and Incomplete Information

3.3.3.1 Motivation for the Case to Analyze

In Chapter 2 several special cash flow models, which were from an economic perspective ex ante interesting, were introduced: cash flows without lags in levels and cash flows without lags in growth rates. Principally, asset pricing implications of all of these special cases must be derived. However, not all of these special cases contain insights into pricing that go beyond the pricing results that are already known for general cash flow models (Section 3.3.2).

For that reason, the pricing implications of only selected special cash flow models are examined. The selection criterion in this connection is that information relevant to pricing must be simpler than in the general cash flow model. In other words, merely substituting the special cash flow model into the pricing formula does not justify its analysis in Section 3.3.3.

Information relevant to pricing reads in the complete information case

3-66

$$z_t^{cI,d,+} = (S_t, D_t)$$

and in the incomplete information case

3-70

$$z_t^{iI,d,+} = (\pi_t, D_t)$$

In the case of cash flow models without lags in levels, information relevant to pricing for all assets simplifies considerably because the regime alone captures all inter-temporal dependencies between cash flows. By contrast, in the case of cash flows without lags in growth rates, future cash flows still depend on current cash flows. Therefore, information relevant to pricing for both all assets and one asset i is still given by (3-66) and (3-70) and cash flows without lags in growth rates do not lead to a simplification of information relevant to pricing.

The case where information frequency is higher than cash flow frequency is not considered for a different reason. The discussion for general cash flow models (Section 3.3.2) has already shown that the single current regime is replaced by a path of regimes in the more general case where information frequency is higher than cash flow frequency. Obviously, this will still hold for special cash flow cases. Thus, it suffices to consider the case with equal information and cash flow frequencies.

In sum, Section 3.3.3 will consider only cash flow models without lags in levels where information frequency equals cash flow frequency.

3.3.3.2 Cash Flow Models without Lags in Levels

Cash flow models without lags in levels read by definition

2-7

$$D_{t+1} = D(S_t, fe_{t+1})$$

In addition, a special functional form of $D(.)$, an affine linear factor model, is specified

2-8

$$D_{i,t+1} = \mu_{i,t} + \sum_{j=1}^{m} a_{ij,t} \cdot f_{j,t+1} + b_{i,t} \cdot e_{i,t+1}, i = 1, \dots, n$$

The problem is to analyze how the specific economic properties of cash flow models (2-7) and (2-8) translate into asset prices and risk premia. To that end, the asset pricing implications of the more general model $D(S_t, fe_{t+1})$ are discussed in a first step before the implications of the affine linear factor model are analyzed in a second step. This procedure has the double advantage of being not as restrictive as assuming the affine linear cash flow model from the outset and, at the same time, of making clear which asset pricing implications hinge on the assumption of the affine linear factor model and which results hold more generally for models of the type $D(S_t, fe_{t+1})$.

3.3.3.2.1 Complete Information

3.3.3.2.1.1 Information Relevant to Pricing ${z_t}^p$

Under complete information, CARA preferences, and cash flow model without lags ${z_t}^p$ still coincides with ${z_t}^{d,+}$. Furthermore, ${z_t}^{d,+}$ simplifies to

3-94

$${z_t}^{cI,p} = {z_t}^{cI,d,+} = (S_t)$$

The reason is that future cash flows are no longer functionally related to current cash flows, implying that the conditional distribution of future cash flows is completely described by the current regime.

3.3.3.2.1.2 Equilibrium Asset Prices

Cash flows without lags in levels: no additional restrictions on the functional form of $D(.)$

If cash flow models belong to the subclass of this section, CARA equilibrium asset prices under complete information and general cash flow model (3-63) simplify to

3-95

$$P_t{}^{cI}(S_t) = E\left(q_{t,t+1}{}^{cI}(S_{t+1}, fe_{t+1}, S_t) \cdot \{P_{t+1}{}^{cI}(S_{t+1}) + D_{t+1}\} \middle| S_t\right)$$

with stochastic discount factor

$$q_{t,t+1}{}^{cI}(S_{t+1}, fe_{t+1}, S_t) = \frac{1}{1+r} \cdot \frac{\left\{ \begin{array}{c} exp\left(-\alpha_{t+1} \cdot \frac{1}{n_I} \cdot \{D_{t+1}{}^{market}(D_{t+1}) + V_{t+1}{}^{market}(S_{t+1})\}\right) \\ \cdot m_{t+1}{}^{cI}(S_{t+1}) \end{array} \right\}}{E\left(\left\{ \begin{array}{c} exp\left(-\alpha_{t+1} \cdot \frac{1}{n_I} \cdot \{D_{t+1}{}^{market}(D_{t+1}) + V_{t+1}{}^{market}(S_{t+1})\}\right) \\ \cdot m_{t+1}{}^{cI}(S_{t+1}) \end{array} \right\} \middle| S_t\right)}$$

with

$$D_{t+1} = D(S_t, fe_{t+1})$$

For cash flow models without lags in levels, the adjustment for risk takes on a special multiplicative form as opposed to the general cash flow model: risks stemming from the two independent stochastic sources, factors and residuals fe_{t+1} and regime S_{t+1}, are taken into account separately by an adjustment for factor and residual risk on the one hand and an adjustment for regime risk on the other hand:

3-96

$$q_{t,t+1}{}^{cI}(S_{t+1}, fe_{t+1}, S_t) = \frac{1}{1+r} \cdot AfR_{t,t+1,fe_{t+1}}{}^{cI}(fe_{t+1}, S_t) \cdot AfR_{t,S_{t+1}}{}^{cI}(S_{t+1}, S_t)$$

with

$$AfR_{t,t+1,fe_{t+1}}{}^{cI}(fe_{t+1}, S_t) \equiv \frac{exp\left(-\alpha_{t+1} \cdot \frac{1}{n_I} \cdot \{D_{t+1}{}^{market}(D_{t+1})\}\right)}{E\left(exp\left(-\alpha_{t+1} \cdot \frac{1}{n_I} \cdot \{D_{t+1}{}^{market}(D_{t+1})\}\right) \middle| S_t\right)}$$

with

$$D_{t+1} = D(S_t, fe_{t+1})$$

and

$$AfR_{t,t+1,S_{t+1}}{}^{cI}(S_{t+1}, S_t) \equiv \frac{\left\{ \begin{array}{c} exp\left(-\alpha_{t+1} \cdot \frac{1}{n_I} \cdot V_{t+1}{}^{market}(S_{t+1})\right) \\ \cdot m_{t+1}{}^{cI}(S_{t+1}) \end{array} \right\}}{E\left(\left\{ \begin{array}{c} exp\left(-\alpha_{t+1} \cdot \frac{1}{n_I} \cdot V_{t+1}{}^{market}(S_{t+1})\right) \\ \cdot m_{t+1}{}^{cI}(S_{t+1}) \end{array} \right\} \middle| S_t\right)}$$

The economic reason for this multiplicative separation is that future cash flows over the time horizon from times $t+2$ through T are entirely described by the new regime S_{t+1} but do not depend on cash flows D_{t+1} under cash flow models without lags in levels. Therefore, the reinvestment opportunities-related part ${m_{t+1}}^{cI}(S_{t+1})$ and the value of the market portfolio of risky assets ${V_{t+1}}^{market}(S_{t+1})$ are not influenced by factors and residuals fe_{t+1}. By contrast, new cash flows D_{t+1} do not depend on the new regime S_{t+1}. This admits the separation result. Note that despite this separation both S_{t+1} and fe_{t+1} are still priced in a risk-adjusted way.

This separation of the effects of factors and residuals fe_{t+1} from the effect of the regime S_{t+1} translates from stochastic discount factors to asset prices: first observe that new prices ${P_{t+1}}^{cI} = {P_{t+1}}^{cI}(S_{t+1})$ depend on the new regime only while cash flows are functions of factors and residuals only, $D_{t+1} = D(S_t, fe_{t+1})$. Together with the structure of the stochastic discount factor, this implies that the risk in next-period asset prices is taken into account by the adjustment for risk stemming from the source S_{t+1}, whereas the rate at which next-period cash flows are discounted is determined by the adjustment for risk stemming from the source factors and residuals fe_{t+1}:

3-97

$$\begin{aligned}{P_t}^{cI}(S_t) &= \frac{1}{1+r} \cdot E\left({AfR_{t,t+1,S_{t+1}}}^{cI}(S_{t+1}, S_t) \cdot {P_{t+1}}^{cI}(S_{t+1}) \middle| S_t\right) \\ &+ \frac{1}{1+r} \cdot E\left({AfR_{t,t+1,fe_{t+1}}}^{cI}(fe_{t+1}, S_t) \cdot D_{t+1} \middle| S_t\right)\end{aligned}$$

with

$$D_{t+1} = D(S_t, fe_{t+1})$$

<u>Cash flows without lags in levels: affine linear factor model</u>

If it is further assumed that cash flows follow the affine linear factor,

$$D_{i,t+1} = \mu_{i,t} + \sum_{j=1}^{m} a_{ij,t} \cdot f_{j,t+1} + b_{i,t} \cdot e_{i,t+1}, i = 1, \dots, n$$

CARA equilibrium asset prices under complete information and general cash flow model (3-63) simplify to

3-98

$$\begin{aligned}{P_t}^{cI}(S_t) &= \frac{1}{1+r} \cdot E\left({AfR_{t,t+1,S_{t+1}}}^{cI}(S_{t+1}, S_t) \cdot {P_{t+1}}^{cI}(S_{t+1}) \middle| S_t\right) \\ &+ \frac{1}{1+r} \cdot E\left({AfR_{t,t+1,fe_{t+1}}}^{cI}(fe_{t+1}, S_t) \cdot D_{t+1} \middle| S_t\right)\end{aligned}$$

with

$$D_{t+1} = D(S_t, fe_{t+1})$$

where

$$\frac{1}{1+r}\cdot E\left(AfR_{t,t+1,fe_{t+1}}{}^{cI}(fe_{t+1},S_t)\cdot D_{i,t+1}\middle|S_t\right)=\frac{1}{1+r}\cdot\mu_i(S_t)$$
$$+\sum_{j=1}^{m}a_{ij}(S_t)\cdot\frac{1}{1+r}\cdot E\left(AfR_{t,t+1,fe_{t+1}}{}^{cI}(fe_{t+1},S_t)\cdot f_{j,t+1}\middle|S_t\right)$$
$$+b_i(S_t)\cdot\frac{1}{1+r}\cdot E\left(AfR_{t,t+1,fe_{t+1}}{}^{cI}(fe_{t+1},S_t)\cdot e_{i,t+1}\middle|S_t\right)$$
$$i=1,\dots,n$$

The first term of (3-98) prices $P_{t+1}{}^{cI}(S_{t+1})$ and cannot be significantly simplified compared to (3-97). The second term of (3-98) prices cash flows D_{t+1}. It consists of expected cash flows discounted at the riskless rate (i.e., $\frac{1}{1+r}\cdot\mu_i(S_t)$) and compensations for factor risk (second row) and residual risk (third row). - A compensation for risk consists of the quantity of risk multiplied by the price of risk.

Compensation for factor risk

The quantity of factor j risk of asset i equals $a_{ij}(s)$, the price of factor j risk is $\lambda_j{}^f(s)$

3-99

$$\lambda_j{}^f(s)\equiv\frac{1}{1+r}\cdot E\left(AfR_{t,t+1,fe_{t+1}}{}^{cI}(fe_{t+1},S_t)\cdot f_{j,t+1}\middle|S_t=s\right)$$
$$j=1,\dots,m$$

While the quantity component $a_{ij}(s)$ is a known model input, the price per unit of factor j risk, $\lambda_j{}^f(s)$, is endogenously determined and deserves further analysis. It consists of the interaction between the adjustment for risk based on factors and residuals on the one hand with factors on the other hand, i.e., exhibits the usual "covariance structure". In particular, $\lambda_j{}^f(s)$ can be positive, zero or negative: factors which, other things equal, move in the same direction as the adjustment for risk based on factors and residuals have a negative price $\lambda_j{}^f(s)$.[21] Accordingly, factors that are uncorrelated with the adjustment for risk based on factors and residuals have a price of zero and factors which tend to move in the opposite direction of adjustment for risk based on factors and residuals have a positive price.

Note that the adjustment for risk based on factors and residuals is completely determined by aggregate cash flows paid by the market portfolio of risky assets (3-96) (p. 85). Hence, these results can be alternatively restated through co-movements of factors with aggregate cash flows.

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[21] To see this, recall that factors have been assumed to possess zero expectations, hence $E\left(AfR_{t,t+1,fe_{t+1}}{}^{cI}(fe_{t+1},S_t)\cdot f_{j,t+1}\middle|S_t\right)=cov\left(AfR_{t,t+1,fe_{t+1}}{}^{cI}(fe_{t+1},S_t),f_{j,t+1}\middle|S_t\right)$.

Compensation for residual risk

The quantity of residual risk of asset i equals $b_i(s)$, the price of residual risk is $\lambda_i{}^e(s)$

3-100

$$\lambda_i{}^e(s) \equiv \frac{1}{1+r} \cdot E\left(AfR_{t,t+1,fe_{t+1}}{}^{cI}(fe_{t+1}, S_t) \cdot e_{i,t+1} \middle| S_t = s\right)$$
$$i = 1, \dots, n$$

The structure of compensation for residual risk is the same as for the compensation for factor risk. However, its economic significance is different. By their very nature, residuals are uncorrelated with other factors and affect cash flows of one single asset. If it is assumed that every asset makes only a small contribution to aggregate cash flows, residuals should be roughly uncorrelated with aggregate cash flows. In other words, residuals should approximately have a price of zero.

In sum, the linear price structure in general and, in particular, the negligible price influence of residual risk when cash flows follow an affine linear factor model parallels the results of linear factor models of asset prices (see, e.g., Ingersoll (1987), pp. 172). In other words, an affine linear factor-model of cash flows translates into an affine linear factor model of prices.

3.3.3.2.1.3 Equilibrium Risk Premia

Cash flows without lags in levels: no additional restrictions on the functional form of $D(.)$

If cash flow models belong to the subclass of this section, CARA complete information risk premia (3-69) specialize to

3-101

$$RP_t{}^{cI}(S_t) = -cov\left(\begin{matrix} AfR_{t,t+1,fe_{t+1}}{}^{cI}(fe_{t+1}, S_t) \\ , P_{t+1}{}^{cI}(S_{t+1}) + D_{t+1} \end{matrix} \middle| S_t\right)$$

with

$$AfR_{t,t+1,fe_{t+1}}{}^{cI}(fe_{t+1}, S_t) = \frac{\left\{\begin{matrix} exp\left(-\alpha_{t+1} \cdot \frac{1}{n_I} \cdot \left\{D_{t+1}{}^{market}(D_{t+1}) + V_{t+1}{}^{market}(S_{t+1})\right\}\right) \\ \cdot m_{t+1}{}^{cI}(S_{t+1}) \end{matrix}\right\}}{E\left(\left\{\begin{matrix} exp\left(-\alpha_{t+1} \cdot \frac{1}{n_I} \cdot \left\{D_{t+1}{}^{market}(D_{t+1}) + V_{t+1}{}^{market}(S_{t+1})\right\}\right) \\ \cdot m_{t+1}{}^{cI}(S_{t+1}) \end{matrix}\right\} \middle| S_t\right)}$$
$$D_{t+1} = D(S_t, fe_{t+1})$$

Given the multiplicative structure of the adjustment for risk, risk premia read as follows paralleling the results for asset prices (3-97):

3-102

$$\begin{aligned} RP_t{}^{cI}(S_t) = &-cov\left(AfR_{t,t+1,S_{t+1}}{}^{cI}(S_{t+1}, S_t), P_{t+1}{}^{cI}(S_{t+1}) \middle| S_t\right) \\ &- cov\left(AfR_{t,t+1,fe_{t+1}}{}^{cI}(fe_{t+1}, S_t), D_{t+1} \middle| S_t\right) \end{aligned}$$

with

$$D_{t+1} = D(S_t, fe_{t+1})$$

Risk premia consist of a covariance between adjustment for risk stemming from the source S_{t+1} and ${P_{t+1}}^{cI}(S_{t+1})$ (first term of (3-102)) and a covariance between the adjustment for risk stemming from the source factors and residuals fe_{t+1} and D_{t+1} (second term of (3-102)).

<u>Cash flows without lags in levels: affine linear factor model</u>

If it is further assumed that cash flows follow the affine linear factor,

$$D_{t+1} = \mu(S_t) + A(S_t) \cdot fe_{t+1}$$

CARA equilibrium risk premia under complete information and general cash flow model (3-102) simplify to

3-103

$$RP_t^{cI}(S_t) = -cov\left({AfR_{t,t+1,S_{t+1}}}^{cI}(S_{t+1}, S_t), {P_{t+1}}^{cI}(S_{t+1}) \middle| S_t \right)$$
$$- cov\left({AfR_{t,t+1,fe_{t+1}}}^{cI}(fe_{t+1}, S_t), D_{t+1} \middle| S_t \right)$$

with

$$D_{t+1} = D(S_t, fe_{t+1})$$

where

$$-cov\left({AfR_{t,t+1,fe_{t+1}}}^{cI}(fe_{t+1}, S_t), D_{i,t+1} \middle| S_t \right)$$
$$= -\frac{1}{1+r} \cdot \sum_{j=1}^{m} a_{ij}(S_t) \cdot {\lambda_j}^f(S_t)$$
$$-\frac{1}{1+r} \cdot b_i(S_t) \cdot {\lambda_i}^e(S_t)$$
$$i = 1, \ldots, n$$

The first term of (3-103) is the risk premium resulting from ${P_{t+1}}^{cI}(S_{t+1})$ and cannot be significantly simplified compared to (3-102). The second term of (3-103) is the risk premium resulting from cash flows D_{t+1} and consists of compensations for factor risk (second row) and residual risk(third row).

3.3.3.2.2 Incomplete Information

3.3.3.2.2.1 Information Relevant to Pricing ${z_t}^p$

Under incomplete information, CARA preferences, and cash flow model without lags ${z_t}^p$ still coincides with ${z_t}^{d,+}$. Furthermore, ${z_t}^{d,+}$ simplifies to:

3-104

$${z_t}^{iI,p} = {z_t}^{iI,d,+} = (\pi_t)$$

The reason is that future cash flows are no longer functionally related to current cash flows, implying that the conditional distribution of future cash flows is completely described by current regime probabilities.

3.3.3.2.2.2 Equilibrium Asset Prices

Cash flows without lags in levels: no additional restrictions on the functional form of $D(.)$

If cash flow models belong to the subclass of this section, CARA equilibrium asset prices under incomplete information and general cash flow model (3-71) simplify to

3-105

$$P_t{}^{iI}(\pi_t) = \sum_{s=1}^{K} \pi_{s,t} \cdot E\left(q_{t,t+1}{}^{iI}(\eta_{t+1}, S_{t+1}, fe_{t+1}, S_t, \pi_t) \cdot \{P_{t+1}{}^{iI}(\pi_{t+1}, D_{t+1}) + D_{t+1}\} \middle| S_t = s\right)$$

with stochastic discount factor

$$q_{t,t+1}{}^{iI}(\eta_{t+1}, S_{t+1}, fe_{t+1}, S_t, \pi_t) = \frac{1}{1+r} \cdot \frac{\left\{ exp\left(-\alpha_{t+1} \cdot \frac{1}{n_I} \cdot \{D_{t+1}{}^{market}(D_{t+1}) + V_{t+1}{}^{market}(\pi_{t+1})\}\right) \cdot m_{t+1}{}^{iI}(\pi_{t+1}) \right\}}{\sum_{s=1}^{K} \pi_{s,t} \cdot E\left(\left\{ exp\left(-\alpha_{t+1} \cdot \frac{1}{n_I} \cdot \{D_{t+1}{}^{market}(D_{t+1}) + V_{t+1}{}^{market}(\pi_{t+1})\}\right) \cdot m_{t+1}{}^{iI}(\pi_{t+1}) \right\} \middle| S_t = s\right)}$$

with

$$D_{t+1} = D(S_t, fe_{t+1})$$
$$\pi_{t+1} = \Pi(\pi_t, D_{t+1}, Sig_{t+1})^{22}$$
$$Sig_{t+1} = Sig_{t+1}(S_t, S_{t+1}, fe_{t+1}, \eta_{t+1})$$

For cash flow models without lags in levels, the adjustment for risk is very similar to the general cash flow model (3-71). In particular and opposed to the complete information case (3-96), the adjustment for risk does not possess a multiplicative structure. The reason is that conditional regime probabilities π_{t+1} are a function of cash flows D_{t+1}. Therefore, $D_{t+1}{}^{market}$, $V_{t+1}{}^{market}$ and $m_{t+1}{}^{iI}$ are all functions of D_{t+1} and stochastically dependent, making a separation impossible. To see the economic intuition behind this result, recall cash flows D_{t+1} are an important information source and contribute to conditional regime probabilities π_{t+1}. This leads to stochastic dependence of D_{t+1} and π_{t+1} (compare intermediate case where signals are neither perfect nor useless, p. 81). In other words, it is clear that all four sources of risk are priced in a risk-adjusted way.

Cash flows without lags in levels: affine linear factor model

[22] Note that the updating of conditional regime probabilities does not depend on current cash flows D_t for the subclass of models under consideration.

If it is further assumed that cash flows follow the affine linear factor,

$$D_{t+1} = \mu(S_t) + A(S_t) \cdot fe_{t+1}$$

CARA equilibrium asset prices under incomplete information and general cash flow model (3-71) take the specific form

3-106

$$P_t{}^{iI}(\pi_t) = \sum_{s=1}^{K} \pi_{s,t} \cdot E\left(q_{t,t+1}{}^{iI}(\eta_{t+1}, S_{t+1}, fe_{t+1}, S_t, \pi_t) \cdot P_{t+1}{}^{iI}(\pi_{t+1}, D_{t+1}) \middle| S_t = s\right)$$
$$+ \sum_{s=1}^{K} \pi_{s,t} \cdot E\left(q_{t,t+1}{}^{iI}(\eta_{t+1}, S_{t+1}, fe_{t+1}, S_t, \pi_t) \cdot D_{t+1} \middle| S_t = s\right)$$

where

$$\sum_{s=1}^{K} \pi_{s,t} \cdot E\left(q_{t,t+1}{}^{iI}(\eta_{t+1}, S_{t+1}, fe_{t+1}, S_t, \pi_t) \cdot D_{i,t+1} \middle| S_t = s\right) = \frac{1}{1+r} \cdot \sum_{s=1}^{K} \pi_{s,t} \cdot \mu_i(s)$$
$$+ \frac{1}{1+r} \cdot \sum_{s=1}^{K} \left[\theta_{s,t}(\pi_t) - \pi_{s,t}\right] \cdot \mu_i(s)$$
$$+ \sum_{j=1}^{m} \sum_{s=1}^{K} \theta_{s,t}(\pi_t) \cdot a_{ij}(s)$$
$$\cdot E\left(AfR_t{}^{conditional}(s; \eta_{t+1}, S_{t+1}, fe_{t+1},, \pi_t) \cdot f_{j,t+1} \middle| S_t = s\right)$$
$$+ \sum_{s=1}^{K} \theta_{s,t}(\pi_t) \cdot b_i(s) \cdot E\left(AfR_t{}^{conditional}(s; \eta_{t+1}, S_{t+1}, fe_{t+1},, \pi_t) \cdot e_{i,t+1} \middle| S_t = s\right)$$
$$i = 1, \dots, n$$

with the conditional adjustment for risk

$$AfR_t{}^{conditional}(s; \eta_{t+1}, S_{t+1}, fe_{t+1},, \pi_t)$$
$$\equiv \frac{\left\{ exp\left(-\alpha_{t+1} \cdot \frac{1}{n_I} \cdot \left\{D_{t+1}{}^{market}(D_{t+1}) + V_{t+1}{}^{market}(\pi_{t+1})\right\}\right) \cdot m_{t+1}{}^{iI}(\pi_{t+1}) \right\}}{E\left(exp\left(-\alpha_{t+1} \cdot \frac{1}{n_I} \cdot \left\{D_{t+1}{}^{market}(D_{t+1}) + V_{t+1}{}^{market}(\pi_{t+1})\right\}\right) \cdot m_{t+1}{}^{iI}(\pi_{t+1}) \middle| S_t = s \right)}$$

with

$$D_{t+1} = D(S_t, fe_{t+1})$$
$$\pi_{t+1} = \Pi(\pi_t, D_{t+1}, Sig_{t+1})$$
$$Sig_{t+1} = Sig_{t+1}(S_t, S_{t+1}, fe_{t+1}, \eta_{t+1})$$

and with risk-neutralized regime probabilities

$$\theta_{s,t}(\pi_t) \equiv \frac{\pi_{s,t} \cdot E\left(exp\left(-\alpha_{t+1} \cdot \frac{1}{n_I} \cdot \left\{D_{t+1}{}^{market}(D_{t+1}) + V_{t+1}{}^{market}(\pi_{t+1})\right\}\right) \cdot m_{t+1}{}^{iI}(\pi_{t+1}) \middle| S_t = s \right)}{\sum_{s'=1}^{K} \pi_{s',t} \cdot E\left(exp\left(-\alpha_{t+1} \cdot \frac{1}{n_I} \cdot \left\{D_{t+1}{}^{market}(D_{t+1}) + V_{t+1}{}^{market}(\pi_{t+1})\right\}\right) \cdot m_{t+1}{}^{iI}(\pi_{t+1}) \middle| S_t = s' \right)}$$

The first term of (3-106) prices $P_{t+1}{}^{iI}(\pi_{t+1})$ and cannot be significantly simplified compared to (3-105). The second term of (3-106) prices cash flows D_{t+1}. It consists of expected cash flows discounted at the riskless rate (i.e., $\frac{1}{1+r} \cdot \sum_{s=1}^{K} \pi_{s,t} \cdot \mu_i(s)$) and compensations for "expectation risk" (second row) and combined risk (third and fourth row), where "combined risk" contains factor risk (third row) and residual risk (fourth row).

Compensation for "expectation risk"

Recall that "expectation" risk captures the fact that the expectation of cash flows conditional on the regime is unknown if the regime is unobservable. Under the affine linear factor model, the conditional expectation of cash flows is given by $\mu(S_t)$. Since it is unobservable, it gives rise to "expectation risk" and a compensation is demanded by each of the individual investors. This compensation reads $\frac{1}{1+r} \cdot \sum_{s=1}^{K} \left[\theta_{s,t}(\pi_t) - \pi_{s,t}\right] \cdot \mu_i(s)$. It is a special case of the general result that the price of "expectation risk" stems from the difference between risk-neutralized and empirical regime probabilities. Clearly, it does not have a complete information counterpart.

Compensation for "combined risk"

Recall that "combined risk" captures the fact that (i) even if the true regime was known, cash flows would still be stochastic and deviate from their conditional expectation and (ii) the true regime is not known. (i) is a form of intra-distribution whereas (ii) is a form of inter-distribution risk.

Compensation for factor risk

Generally, factors have zero expectations and, hence, factor risk falls into the category of "combined risk" and not "expectation risk". Under incomplete information, factor risk contains both an intra-distribution and an inter-distribution component.

The intra-distribution component within regime s possesses the structure quantity of risk multiplied by the price of risk:

$$a_{ij}(s) \cdot \lambda_j^{f,conditional}(s, \pi_t)$$

with

3-107

$$\lambda_j^{f,conditional}(s, \pi_t) \equiv \frac{1}{1+r} \cdot E\left(AfR^{iI,conditional}(s; \pi_{t+1}, \pi_t) \cdot f_{j,t+1} \middle| S_t = s\right)$$

where $a_{ij}(s)$ denotes the quantity component and $\lambda_j^{f,conditional}(s, \pi_t)$ the price component of factor j conditional on regime s.

The intra-distribution component of factor risk is, thus, analogous to the complete information case.

The inter-distribution component stems from the fact that both the quantity and the price components are regime-dependent and, therefore, not known. Hence, additional compensation is required so that the total compensation for factor risk reads:

$$\sum\nolimits_{s=1}^{K}\theta_{s,t}(\pi_t)\cdot a_{ij}(s)\cdot {\lambda_j}^{f,conditional}(s,\pi_t)$$

Compensation for residual risk

Similarly to factor risk, residual risk completely belongs to "combined risk" (not "expectation risk") and has an intra-distribution and an inter-distribution component. Its pricing is completely analogous to factor risk. The intra distribution component in regime s reads:

$$b_i(s)\cdot {\lambda_i}^{e,conditional}(s,\pi_t)$$

with

$${\lambda_i}^{e,conditional}(s,\pi_t)\equiv\frac{1}{1+r}\cdot E\left(AfR_t{}^{conditional}(s;\eta_{t+1},S_{t+1},fe_{t+1},,\pi_t)\cdot e_{i,t+1}\middle|S_t=s\right)$$

The inter-distribution component is reflected by risk-neutralized regime-probabilities. The total compensation required for residual risk, therefore, is

$$\sum\nolimits_{s=1}^{K}\theta_{s,t}(\pi_t)\cdot b_i(s)\cdot {\lambda_i}^{e,conditional}(s,\pi_t)$$

3.3.3.2.2.3 Equilibrium Risk Premia

Cash flows without lags in levels: no additional restrictions on the functional form of $D(.)$

If cash flow models belong to the subclass of this section, CARA complete information risk premia (3-73) specialize to

3-108

$$RP_t{}^{iI}(\pi_t)=-cov\left(\begin{matrix}AfR^{iI}(S_{t+1},fe_{t+1},\eta_{t+1},S_t,\pi_t)\\ ,P_{t+1}{}^{iI}(\pi_{t+1})+D_{t+1}\end{matrix}\middle|\pi_t\right)$$

with

$$AfR_{t,t+1}{}^{iI}(\eta_{t+1},S_{t+1},fe_{t+1},\pi_t)$$

$$\equiv\frac{\left\{\begin{matrix}exp\left(-\alpha_{t+1}\cdot\frac{1}{n_I}\cdot\left\{\begin{matrix}D_{t+1}{}^{market}(D_{t+1})\\+V_{t+1}{}^{market}(\pi_{t+1})\end{matrix}\right\}\right)\\ \cdot m_{t+1}{}^{iI}(\pi_{t+1})\end{matrix}\right\}}{\sum_{s=1}^{K}\pi_{st}E\left(\left\{\begin{matrix}exp\left(-\alpha_{t+1}\cdot\frac{1}{n_I}\cdot\left\{\begin{matrix}D_{t+1}{}^{market}(D_{t+1})\\+V_{t+1}{}^{market}(\pi_{t+1})\end{matrix}\right\}\right)\\ \cdot m_{t+1}{}^{iI}(\pi_{t+1})\end{matrix}\right\}\middle|S_t=s,\pi_t\right)}$$

$$D_{t+1}=D(D_t,S_t,fe_{t+1})$$

$$\pi_{t+1}=\Pi(\pi_t,D_{t+1},Sig_{t+1})$$

$$Sig_{t+1}=Sig_{t+1}(S_t,S_{t+1},fe_{t+1},\eta_{t+1})$$

Risk premia consists of a covariance between adjustment for risk and $P_{t+1}{}^{iI}(\pi_{t+1})+D_{t+1}$. Similarly to the case of prices (3-105) and opposed to the complete information case, the adjustment for risk cannot be separated multiplicatively into an adjustment for risk stemming from the source S_{t+1} an adjustment for risk stemming from the source factors and residuals fe_{t+1}.

<u>Cash flows without lags in levels: affine linear factor model</u>

If it is further assumed that cash flows follow the affine linear factor,

$$D_{t+1} = \mu(S_t) + A(S_t) \cdot fe_{t+1}$$

CARA equilibrium risk premia under complete information and general cash flow model (3-108) take the specific form

3-109

$$RP_t{}^{iI}(\pi_t) = -cov\big(AfR_{t,t+1}{}^{iI}(\eta_{t+1}, S_{t+1}, fe_{t+1}, \pi_t), P_{t+1}{}^{iI}(\pi_{t+1})\big|\pi_t\big)$$
$$- cov\big(AfR_{t,t+1}{}^{iI}(\eta_{t+1}, S_{t+1}, fe_{t+1}, \pi_t), D_{t+1}\big|\pi_t\big)$$

with

$$D_{t+1} = D(S_t, fe_{t+1})$$
$$\pi_{t+1} = \Pi(\pi_t, D_{t+1}, Sig_{t+1})$$
$$Sig_{t+1} = Sig_{t+1}(S_t, S_{t+1}, fe_{t+1}, \eta_{t+1})$$

where

$$-cov\left(\begin{matrix} AfR_{t,t+1}{}^{iI}(\eta_{t+1}, S_{t+1}, fe_{t+1}, \pi_t) \\ , D_{i,t+1} \end{matrix}\middle|\pi_t\right)$$
$$= \sum_{s=1}^{K} \left[\pi_{s,t} - \theta_{s,t}(\pi_t)\right] \cdot \mu_i(s) - (1+r)$$
$$\cdot \sum_{j=1}^{m} \sum_{s=1}^{K} \theta_{s,t}(\pi_t) \cdot a_{ij}(s) \cdot \lambda_j{}^{f,conditional}(s, \pi_t)$$
$$-(1+r) \cdot \sum_{s=1}^{K} \theta_{s,t}(\pi_t) \cdot b_i(s) \cdot \lambda_i{}^{e,conditional}(s, \pi_t)$$

The first term of (3-109) is the risk premium resulting from $P_{t+1}{}^{cI}(S_{t+1})$ and cannot be significantly simplified compared to (3-108). The second term of (3-109) is the risk premium resulting from cash flows D_{t+1} and consists of compensations for “expectation risk” (second row), factor risk component of “combined risk” (third row) and residual risk component of “combined risk” (fourth row).

4 General Equilibrium Asset Pricing

Chapter 4 comprises definitions and general results (Section 4.1), derives and discusses equilibrium asset prices for general cash flow models for a broad class of utility functions (Section 4.2), and asset prices for special cash flow models (Section 4.3).

Within this general structure, Section 4.1 serves purposes similar to the corresponding partial equilibrium Section 3.1, namely the definition of the equilibrium concept, as well as the intuitive derivation and interpretation of asset prices within a framework which does not make particular assumptions on the information scenario, the cash flow model or the utility function. In contrast to the detailed argumentation in Section 3.1, I treat briefly or omit details whenever they would merely repeat the reasoning from the partial equilibrium framework; this procedure has the double advantage of avoiding repetitions and shifting the focus to the differences between partial and general equilibrium framework.

4.1 Definitions and General Results

4.1.1 General Equilibrium and investors' decision problem

In a dynamic consumption and portfolio selection problem the market for risky assets and a one-period riskless asset will be in general equilibrium if (i) all investors behave optimally at all points in time within the planning horizon and if (ii) the demand for risky assets and the riskless asset is equal to the exogenous supply. Cash holdings are not an investment alternative. Moreover, cash flows are specified exogenously and not derived endogenously from optimal production decisions making this economy an exchange economy in the sense of Lucas (1978). Since an equilibrium relation is applied to all (risky and riskless) assets, a general equilibrium as in, e.g., Lucas (1978), is obtained.

Time horizon, investors and information scenarios are as in the partial equilibrium case.

Formally, a general equilibrium consists of (i) a process of the riskless bond price $\{B_t\}_{0\le t\le T-1}$, (ii) a process of risky asset prices $\{P_t\}_{0\le t\le T-1}$, and (iii) an optimal strategy for each of the identical investors that determines the portfolio of risky assets, the riskless investment and consumption, denoted by $\{N_t(I_t)\}_{0\le t\le T-1}$, $\{H_t(I_t)\}_{0\le t\le T-1}$, $\{C_t(I_t)\}_{0\le t\le T}$, respectively, such that the demand for risky assets and the riskless asset is equal to the exogenous market supply of these assets.

Optimality means that the strategy of each of the identical investors must solve the problem

4-1

$$max_{\{N_\tau,C_\tau\},0\le\tau\le T-1,C_T} E\left(\sum_{t=0}^{T} \frac{1}{(1+\rho)^t} \cdot U(C_t) \middle| I_0\right)$$

with wealth dynamics

4-2

$$W_{\tau+1} = [W_\tau - C_\tau] \cdot (1+r) + {N_\tau}^T \{P_{\tau+1} + D_{\tau+1} - (1+r) \cdot P_\tau\}$$
$$0 \le \tau \le T-1$$

with

$$P_T \equiv \underline{0}_n$$

Initial wealth is denoted by ${W_0}^{initial}$; all remaining wealth must be consumed at time T

4-3

$$C_T = W_T$$

The riskless investment is eliminated as a decision variable and implicitly given by (see the discussion in Section 3.1, pp. 31)

4-4

$${N_\tau}^T P_\tau + H_\tau + C_\tau = W_\tau, 0 \le \tau \le T-1$$

Market clearing means that the demand for risky assets and the riskless asset are always equal to the exogenous supply. The exogenous market portfolio of risky assets is denoted by $\overline{N}$.

4-5

$$\sum_{\nu=1}^{n_I} {N_t}^{(\nu)}(I_t) = \overline{N}\ \forall I_t$$

The riskless asset is in zero net supply:

4-6

$$\sum_{\nu=1}^{n_I} {H_t}^{(\nu)}(I_t) = 0\ \forall I_t$$

Since investors are identical, the market clearing conditions (4-5) and (4-6) imply that each investor must hold $\frac{1}{n_I}$-th of the market portfolio of risky assets and does not invest or borrow via the riskless asset:

4-7

$${N_t}^{(\nu)}(I_t) = N_t(I_t) = \frac{1}{n_I} \cdot \overline{N}\ \forall I_t$$
$$\nu = 1, \dots, n_I$$

4-8

$${H_t}^{(\nu)}(I_t) = H_t(I_t) = 0 \forall I_t$$
$$\nu = 1, \dots, n_I$$

4.1.2 Outline of the Derivation of General Equilibrium Asset Prices

Prices are characterized in the same four steps as in the partial equilibrium context: (i) an abstract model for the joint behavior of asset prices and cash flows from the point of view of investors is specified, (ii) the decision problem of investors based on the abstract model is solved, thereby deriving the demand function, (iii) equilibrium asset prices are obtained as the aggregate result of investor decisions and (iv) conditions are analyzed under which equilibrium asset prices do indeed belong to the abstract class of models specified in step (i).

4.1.2.1 Step 1: Class of Models for Equilibrium Asset Prices and Cash Flows

As in the partial equilibrium case, a class of models for the dynamics of asset prices and cash flows is exogenously specified and investors are assumed to base their decisions on this model.

Class of Models for Equilibrium Asset Prices

Asset prices are a function of information relevant to pricing at time t, ${z_t}^p$:

4-9

$$P_t\left({z_t}^p(I_t)\right) = {P_t}^I(I_t)$$
$$\forall I_t \forall t = 0, \dots, T-1$$

In contrast to the complete information case, the price of the riskless bond now also is a function of ${z_t}^p$:

4-10

$$B_t\left({z_t}^p(I_t)\right) = {B_t}^I(I_t)$$
$$\forall I_t \forall t = 0, \dots, T-1$$

Class of Cash Flow Models

The state of the cash flow process at time t is captured by ${z_t}^d$ and is defined exactly as in the partial equilibrium framework: it consists of (i) current cash flows D_t, modeled by a component ${z_t}^{d,0}$, and (ii) a sufficient statistic for the distribution of future cash flows $(D_{t+1}, .., D_T)$ conditional on information at time t, modeled by ${z_t}^{d,+}$:

4-11

$${z_t}^d = \left({z_t}^{d,0}, {z_t}^{d,+}\right)$$

The state of the cash flow process ${z_t}^d$ does not differ between partial and general equilibrium: since cash flow process and signal model are exogenous and the same under both equilibrium frameworks, only asset prices can be different in partial and general equilibrium.

Joint Dynamics of $z_t{}^p$ and $z_t{}^d$

Again as in the partial equilibrium case, the joint dynamics of the information relevant to pricing, $z_t{}^p$, and the state of the cash flow process, $z_t{}^d$ are governed by a Markov process

4-12

$$z_t \equiv (z_t{}^p, z_t{}^d)$$

with dynamics

4-13

$$z_{t+1} = f_{z,t}(z_t, \xi_{t+1})$$

where ξ_{t+1} are vector-valued and independent random variables.

Consistency Conditions

Both information relevant to pricing and the state of the cash flow process must satisfy the same conditions as under partial equilibrium, in particular they must be sufficient, irreducible, and possess a time-independent composition as well as the Markov property (for the details, see pp. 34 and pp. 35).

4.1.2.2 Step 2: The Optimization Problem of an Individual Investor

The optimal decision of each of the individual investors at time t is characterized by the Bellman equation. Based on the insights from the partial equilibrium case, it can be conjectured that the value function at time $t+1$ is a function of individual wealth, the information relevant to pricing $z_{t+1}{}^p$ and time $t+1$,

4-14

$$J(I_{t+1}, t+1) = J(W_{t+1}, z_{t+1}{}^p, t+1)$$

Note that (4-14) makes use of the insight from the partial equilibrium case that the second argument of the value function is only $z_{t+1}{}^p$ rather than the entire state variable z_{t+1}; the corresponding argument carries over to the general equilibrium case.

The Bellman equation is identical to the partial equilibrium case although the expectation is to be taken with respect to an underlying distribution of general equilibrium (rather than partial equilibrium) asset prices; intuitively, it does not matter to individual investors how the riskless interest rate observed in the market has been determined.

4-15

$$J(W_t, z_t, t) = \frac{1}{(1+\rho)^t} \cdot sup_{N,C}\left\{U(C) + \frac{1}{1+\rho} \cdot E\left(\bar{J}\left(W_{t+1}\left(z_{t+1}{}^p, z_t{}^p, z_{t+1}{}^d, W_t; C, N\right), z_{t+1}{}^p, t+1\right) \middle| z_t, W_t\right)\right\}$$

with wealth dynamics (based on (4-2))

$$W_{t+1}\left(z_{t+1}{}^p, z_t{}^p, z_{t+1}{}^d\right) = [W_t - C_t] \cdot \left(1 + r_t(z_t{}^p)\right) + N_t{}^T\left\{P_{t+1}(z_{t+1}{}^p) + D_{t+1}\left(z_{t+1}{}^d\right) - \left(1 + r_t(z_t{}^p)\right) \cdot P_t(z_t{}^p)\right\}$$

Note that the value function for time $t+1$ can without loss of generality be written in the form

4-16

$$J(W_{t+1}, z_{t+1}, t+1) \equiv \frac{1}{(1+\rho)^{t+1}} \cdot \bar{J}(W_{t+1}, z_{t+1}, t+1)$$

It is assumed that the function $\bar{J}$ is twice partially differentiable with respect to W_{t+1}, increasing in W_{t+1} and concave in W_{t+1}.

4.1.2.3 Step 3: Equilibrium Asset Prices

To obtain equilibrium prices of risky assets and the riskless asset, first-order conditions for the problem of the identical investors are derived and aggregated into equilibrium. The prices of risky assets and the riskless asset can (in this order) be obtained from the first-order conditions for the optimal portfolio and optimal consumption. While it is intuitive that equilibrium prices of risky assets can be obtained from the first-order conditions for optimal portfolio holdings of these assets, it is perhaps less clear why the price of the riskless asset is obtained from the first-order condition of consumption. The intuitive explanation is that, given individual wealth and portfolio holdings, choosing consumption is tantamount to choosing the riskless investment because both decision variables are linked through the budget equation (4-4).

Individual pricing equation

4-17

$$P_t = \frac{1}{1+r_t(z_t{}^p)} \cdot E\left(\frac{\frac{\partial}{\partial W_{t+1}} \bar{J}^{(\nu)}\left(W_{t+1}\left(z_{t+1}{}^p, z_t{}^p, z_{t+1}{}^d, W_t{}^{(\nu)}; C_t{}^{*(\nu)}, N_t{}^{*(\nu)}\right), z_{t+1}{}^p, t+1\right)}{E\left(\frac{\partial}{\partial W_{t+1}} \bar{J}^{(\nu)}\left(W_{t+1}\left(z_{t+1}{}^p, z_t{}^p, z_{t+1}{}^d, W_t{}^{(\nu)}; C_t{}^{*(\nu)}, N_t{}^{*(\nu)}\right), z_{t+1}{}^p, t+1\right) \middle| z_t, W_t{}^{(\nu)}\right)} \cdot \left\{P_{t+1}(z_{t+1}{}^p) + D_{t+1}\left(z_{t+1}{}^d\right)\right\} \middle| z_t, W_t{}^{(\nu)} \right)$$

4-18

$$B_t \equiv \frac{1}{1+r_t(z_t{}^p)} = \frac{1}{1+\rho} \cdot \frac{E\left(\frac{\partial}{\partial W_{t+1}} \bar{J}^{(\nu)}\left(W_{t+1}\left(z_{t+1}{}^p, z_t{}^p, z_{t+1}{}^d, W_t{}^{(\nu)}; C_t{}^{*(\nu)}, N_t{}^{*(\nu)}\right), z_{t+1}{}^p, t+1\right) \middle| z_t, W_t{}^{(\nu)}\right)}{U'\left(C_t{}^{*(\nu)}\right)}$$

Prices in market equilibrium

In equilibrium, wealth of each of the identical investors at time $t+1$ must be equal to $\frac{1}{n_I}$-th of equilibrium aggregate wealth $\overline{W}_{t+1}{}^{eq}$, yielding a relation between equilibrium asset prices at time t and equilibrium aggregate wealth at time $t+1$:

4-19

$$P_t(z_t) = \frac{1}{1+r_t(z_t{}^p)} \cdot E\left(\frac{\frac{\partial}{\partial W_{t+1}} \bar{J}\left(\frac{1}{n_I} \cdot \overline{W}_{t+1}{}^{eq}, z_{t+1}{}^p, t+1\right)}{E\left(\frac{\partial}{\partial W_{t+1}} \bar{J}\left(\frac{1}{n_I} \cdot \overline{W}_{t+1}{}^{eq}, z_{t+1}{}^p, t+1\right) \middle| z_t, \overline{W}_t{}^{eq}\right)} \cdot \left\{P_{t+1}(z_{t+1}{}^p) + D_{t+1}\left(z_{t+1}{}^d\right)\right\} \middle| z_t, \overline{W}_t{}^{eq} \right)$$

Moreover, if $C_t{}^*$ denotes equilibrium consumption of each of the identical investors, it is obtained for the price of the riskless bond:

4-20

$$B_t(z_t) = \frac{1}{1+r_t(z_t{}^p)} = \frac{1}{1+\rho} \cdot \frac{E\left(\frac{\partial}{\partial W_{t+1}} \bar{J}\left(\frac{1}{n_I} \cdot \overline{W}_{t+1}{}^{eq}, z_{t+1}{}^p, t+1\right) \middle| z_t, \overline{W}_t{}^{eq}\right)}{U'(C_t{}^*)}$$

Final pricing formula – existence of a closed-form solution

To derive a closed-form solution, $\overline{W}_{t+1}{}^{eq}$, $\overline{W}_t{}^{eq}$, and $C_t{}^*$ must be analyzed. Equilibrium aggregate wealth is, in principle, as in the partial equilibrium case,

$$\overline{W}_{t+1}{}^{eq} = \overline{N}^T\left\{P_{t+1}(z_{t+1}{}^p) + D_{t+1}\left(z_{t+1}{}^d\right)\right\} + \overline{H}_t{}^{eq} \cdot \left(1 + r_t(z_t{}^p)\right)$$

However, there is now a decisive difference to the partial equilibrium case: the market clearing condition for the riskless asset (4-6) implies $\overline{H}_t^{\,eq} = 0$; hence equilibrium aggregate wealth simplifies to

4-21

$$\overline{W}_{t+1}^{\,eq}(z_{t+1}) = \overline{N}^T\left\{P_{t+1}(z_{t+1}{}^{p}) + D_{t+1}\left(z_{t+1}{}^{d}\right)\right\}$$

Equilibrium consumption of each the identical investors can be found by observing that aggregating individual budget equations (4-4) yields

4-22

$$\overline{W}_t^{\,eq}(z_t) = \overline{N}^T P_t(z_t{}^{p}) + n_I \cdot C_t{}^{*}$$

and at the same time $\overline{W}_t^{\,eq}(z_t)$ possesses the same structure as $\overline{W}_{t+1}^{\,eq}(z_{t+1})$ (4-21), i.e.,

$$\overline{W}_t^{\,eq}(z_t) = \overline{N}^T\left\{P_t(z_t{}^{p}) + D_t\left(z_t{}^{d}\right)\right\}$$

The characteristic result for economies in the style of Lucas (1978) is obtained: in equilibrium, each investor consumes $\frac{1}{n_I}$-th of equilibrium aggregate cash flows,

4-23

$$C_t{}^{*} = \frac{1}{n_I} \cdot D_t^{\,market}\left(z_t{}^{d}\right)$$

with

$$D_t^{\,market} \equiv \overline{N}^T D_t\left(z_t{}^{d}\right)$$

For that reason, it is obtained

4-24

$$P_t(z_t) = \frac{1}{1 + r_t(z_t{}^{p})} \cdot E\left(\frac{\frac{\partial}{\partial W_{t+1}}\bar{J}\left(\frac{1}{n_I} \cdot \overline{W}_{t+1}^{\,eq}(z_{t+1}), z_{t+1}{}^{p}, t+1\right)}{E\left(\frac{\partial}{\partial W_{t+1}}\bar{J}\left(\frac{1}{n_I} \cdot \overline{W}_{t+1}^{\,eq}(z_{t+1}), z_{t+1}{}^{p}, t+1\right)\middle| z_t, W_t^{(\nu)}\right)} \cdot \left\{P_{t+1}(z_{t+1}{}^{p}) + D_{t+1}\left(z_{t+1}{}^{d}\right)\right\} \middle| z_t\right)$$

4-25

$$B_t(z_t) = \frac{1}{1 + r_t(z_t{}^{p})} = \frac{1}{1+\rho} \cdot \frac{E\left(\frac{\partial}{\partial W_{t+1}}\bar{J}\left(\frac{1}{n_I} \cdot \overline{W}_{t+1}^{\,eq}(z_{t+1}), z_{t+1}{}^{p}, t+1\right)\middle| z_t\right)}{U'\left(\frac{1}{n_I} \cdot D_t^{\,market}(z_t{}^{d})\right)}$$

and a closed-form solution becomes possible because equilibrium aggregate wealth $\overline{W}_{t+1}^{\,eq}$ no longer depends on quantities that are endogenously determined at time t (i.e., P_t, $\overline{H}_t^{\,eq}$ or $C_t{}^{*}$).

4.1.3 Step 4: Consistency Conditions

The conditions under which the class of asset prices specified in step (i) is consistent with actual equilibrium asset prices are derived by reasoning in the same way as in the partial equilibrium case; the details of the derivation can, therefore, be omitted and it suffices to state results: the overall structure of information relevant to pricing is the same as in the general equilibrium case (3-35)., i.e., the exogenous parts of equilibrium aggregate wealth $\overline{W}_t^{eq}$, and a sufficient statistic of the conditional cash flow distribution must be information relevant to pricing, $z_t^{d,+}$.However, there is an important difference in the composition of the exogenous part of equilibrium aggregate wealth: in the general equilibrium framework, it merely consists of aggregate cash flows paid by the market portfolio of risky assets, $D_t^{market}(z_t^d) \equiv \overline{N}^T D_t(z_t^d)$. By contrast, the contribution of equilibrium aggregate riskless investment $\overline{H}_{t-1}^{eq} \cdot (1 + r_t(z_t^p))$, the second component of exogenous aggregate wealth under partial equilibrium, is identically equal to zero in the general equilibrium framework and, thus, drops out.

In conclusion, information relevant to pricing in the general equilibrium framework takes the form

4-26
$$z_t^p = \left(D_t^{market}, z_t^{d,+}\right)$$

and

4-27
$$z_t = \left(z_t^p, z_t^d\right) = \left(D_t^{market}, z_t^{d,+}, z_t^{d,0}\right)$$

To check the Markov property of z_t, D_t^{market}, $z_t^{d,+}$ and $z_t^{d,0}$ must be analyzed in more detail. Since D_t^{market}, $z_t^{d,+}$ and $z_t^{d,0}$ are derived from or are parts of the state of the cash flow process z_t^d, is is clear that z_t will be a Markov process if z_t^d is a Markov process. This is the case for all cash flow models considered in this thesis.

4.1.4 Dependency of Equilibrium Wealth, of Equilibrium Consumption, of the Value Function, and of Reinvestment Opportunities on z_t^p

In preparation of a detailed analysis of asset prices and stochastic discount factor, it is shown that the following quantities that will be frequently needed depend on information relevant to pricing z_t^p:

Equilibrium Aggregate Wealth

Since consistency requires that aggregate cash flows paid by the market portfolio of risky assets are an element of information relevant to pricing, i.e., $D_t^{market} = D_t^{market}(z_t^p)$, it follows that equilibrium aggregate wealth is a function of z_t^p only and reads

4-28
$$\overline{W}_t^{eq}(z_t^p) = V_t^{market}(z_t^p) + D_t^{market}(z_t^p)$$

where $V_t^{market}(z_t^p) \equiv \overline{N}^T P_t(z_t^p)$ is the market value of the portfolio of risky assets.

Equilibrium Consumption

Equilibrium consumption of each of the identical investors equals $\frac{1}{n_I}$-th of aggregate cash flows paid by the market portfolio of risky assets $D_t^{market}(z_t{}^p)$ and, therefore, is a function of $z_t{}^p$ only:

4-29

$$C_t{}^*(z_t{}^p) = \frac{1}{n_I} \cdot D_t{}^{market}(z_t{}^p)$$

Value Function

It has been conjectured in step ii (4-14) (p. 98) that the second argument of the value function of each of the identical investors at time $t+1$ is $z_{t+1}{}^p$ (rather than the less restrictive z_{t+1}). Thus this conjecture would be inconsistent (and therefore invalid) if it turned out that the second argument of the value function at time t was z_t (or in fact any variable other than $z_t{}^p$); the proof is analogous to the partial equilibrium case (see p. 46).

Hence, the Bellman equation reads:

4-30

$$\begin{aligned} &J(W_t, z_t{}^p, t) \\ &= \frac{1}{(1+\rho)^t} \\ &\cdot \left\{ U\big(C_t{}^*(z_t{}^p, W_t)\big) + \frac{1}{1+\rho} \right. \\ &\left. \cdot E\left(\bar{J}\left(W_{t+1}\left(z_{t+1}{}^p, z_t{}^p, z_{t+1}{}^d, W_t; C_t{}^*(z_t{}^p, W_t), N_t{}^*(z_t{}^p, W_t) \right), z_{t+1}{}^p, t+1 \right) \middle| z_t{}^p, W_t \right) \right\} \end{aligned}$$

where $C_t{}^*(z_t{}^p, W_t)$ and $N_t{}^*(z_t{}^p, W_t)$ denote optimal consumption and the optimal portfolio of risky assets.

Reinvestment Opportunities

Recall from the partial equilibrium setting that reinvestment opportunities at time t consist of prices of risky assets at time t, the riskless interest rate at time t, and the conditional distributions of future asset prices and cash flows from the perspective of time t. As in the partial equilibrium case, prices of risky assets and the conditional distribution of future cash flows are functions $z_t{}^p$. In contrast to the partial equilibrium case, the one-period riskless interest rate is no longer exogenous; however, the one-period riskless interest rate is a function of $z_t{}^p$: the one-period riskless interest rate is related to the one-period bond price via $r_t(z_t{}^p) = B_t(z_t{}^p)^{-1} - 1$ (see (4-20)).

4.1.5 Economic Interpretation of Asset Prices

The pricing of both the riskless asset and of risky assets in general equilibrium needs to be interpreted. Since the endogenous riskless interest rate is new compared to the partial equilibrium case, it requires a detailed analysis. The structure of risky asset prices is parallel to partial equilibrium, but there are important differences in details, and it is these differences that are of interest in this section.

4.1.5.1 The One-Period Riskless Interest Rate

The one-period riskless interest rate is obtained from the price for the one-period riskless bond (4-25) and reads

4-31

$$r_t(z_t{}^p) = (1+\rho) \cdot \frac{U'\left(\frac{1}{n_I} \cdot D_t{}^{market}(z_t{}^p)\right)}{E\left(U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}(z_{t+1}{}^p)\right) \middle| z_t{}^p\right)} - 1$$

The one-period riskless interest rate consists of a compensation for time preference (represented by the parameter ρ) and a component relating to the inter-temporal consumption decision under risk (see, e.g., Cochrane (2005), p. 11-12). While the compensation for time preference is straightforward, the second part deserves a closer look. It is an increasing function of the ratio of current marginal utility of consumption to expected marginal utility of consumption at time $t+1$. Hence, the second part is larger than, identical to or smaller than one depending on whether current marginal utility of consumption is (in this order) greater to, equal to, or lower than expected marginal utility of consumption at time $t+1$. Consequently, the one-period riskless rate will be higher than, identical to or lower than the time preference rate. Observe that the interest rate may become negative in this model if $U'\left(\frac{1}{n_I} \cdot D_t{}^{market}\right)$ is sufficiently low relative to $E\left(U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}\right) \middle| z_t\right)$, i.e., if investors expect time $t+1$ to be much worse than time t. This is a consequence of the assumption that investors cannot hold cash: negative interest rates are not possible if investors can costlessly hold cash since this introduces a riskless investment opportunity with a zero interest rate, ruling out negative interest rates by arbitrage.

To see the economic mechanism behind the composition of the one-period interest rate, consider the adjustment process that would occur in the case the one-period riskless interest rate deviated from the level indicated by (4-31). If the interest rate was below the level indicated by (4-31), investors would try to choose a level of consumption at time t which exceeds $\frac{1}{n_I} \cdot D_t{}^{market}(z_t{}^p)$ by borrowing at the riskless rate. However, the riskless asset is in zero net supply and, in addition, investors cannot consume more than $\frac{1}{n_I} \cdot D_t{}^{market}(z_t{}^p)$. Hence there would be an excess supply of the riskless asset and the riskless interest rate would have to rise until investors choose to lower desired consumption to $\frac{1}{n_I} \cdot D_t{}^{market}(z_t{}^p)$, thus eliminating the desire to borrow. If the riskless interest instead exceeded (4-31), the high interest rate would induce investors to reduce consumption below $\frac{1}{n_I} \cdot D_t{}^{market}(z_t{}^p)$ and to save instead. This would create an excess demand for the riskless asset and the riskless interest rate would have to fall until investors choose to increase desired consumption to $\frac{1}{n_I} \cdot D_t{}^{market}(z_t{}^p)$, thus eliminating the desire to save.

4.1.5.2 The Stochastic Discount Factor

4.1.5.2.1 The Stochastic Discount Factor Expressed Through an Adjustment for Risk

Similar to the partial equilibrium case, the core of the stochastic discount factor in general equilibrium is the adjustment for risk

4-32

$$q_{t,t+1}(z_{t+1}{}^p, z_t{}^p) \equiv \frac{1}{1 + r_t(z_t{}^p)} \cdot AfR_{t,t+1}(z_{t+1}{}^p, z_t{}^p)$$

with

$$AfR_{t,t+1}(z_{t+1}{}^p, z_t{}^p) \equiv \frac{\frac{\partial}{\partial W_{t+1}} \bar{J}\left(\overline{W}_{t+1}{}^{eq}(z_{t+1}{}^p), z_{t+1}{}^p, t+1\right)}{E\left(\frac{\partial}{\partial W_{t+1}} \bar{J}\left(\overline{W}_{t+1}{}^{eq}(z_{t+1}{}^p), z_{t+1}{}^p, t+1\right) \middle| z_t{}^p\right)}$$

Using the envelope condition, the adjustment for risk in (4-32) can be re-expressed through marginal consumption utility

4-33

$$AfR_{t,t+1}(z_{t+1}{}^p, z_t{}^p) \equiv \frac{U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}(z_{t+1}{}^p)\right)}{E\left(U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}(z_{t+1}{}^p)\right) \middle| z_t{}^p\right)}$$

(4-33) is a major difference to the consumption-based adjustment for risk in partial equilibrium (3-48) (p. 50). Equilibrium consumption is identical to aggregate cash flows per investor, an exogenous quantity which is known. This simple result does not have a partial equilibrium analogue: equilibrium consumption under partial equilibrium does not have to coincide with aggregate cash flows per investor.

4.1.5.2.2 The Stochastic Discount Factor Expressed Through Time Preference and Relative Marginal Utilities of Consumption

Substituting the price of the riskless bond (4-25) and the adjustment for risk based on marginal utility of consumption (4-33) into the stochastic discount factor (4-32) yields:

4-34

$$q_{t,t+1}({z_{t+1}}^p, {z_t}^p) \equiv \frac{1}{1+\rho} \cdot \frac{U'\left(\frac{1}{n_I} \cdot {D_{t+1}}^{market}({z_{t+1}}^p)\right)}{U'\left(\frac{1}{n_I} \cdot {D_t}^{market}({z_t}^p)\right)}$$

Moreover, (4-34) can be generalized to multi-period stochastic discount factors:

4-35

$$q_{t,t+\tau}({z_{t+\tau}}^p, {z_t}^p) \equiv \frac{1}{(1+\rho)^\tau} \cdot \frac{U'\left(\frac{1}{n_I} \cdot {D_{t+\tau}}^{market}({z_{t+\tau}}^p)\right)}{U'\left(\frac{1}{n_I} \cdot {D_t}^{market}({z_t}^p)\right)}$$

The stochastic discount factors (4-34) and (4-35) have three advantages over (4-32): first, they consist of exogenous consumption, second, they yield the price of the riskless bond (while (4-32) is based on the one-period riskless rate) and, third, they easily lend themselves to a multi-period expression of risky asset prices.

4.1.5.3 Equilibrium Risk Premia

The general structure of risk premia is identical to partial equilibrium. However, there are two differences. First, there are two channels through which risk premia of assets are formed, (i) through the riskless interest rate and (ii) through prices of risky assets. Second, the adjustment for risk can be expressed through exogenous components only.

4.1.6 Conclusion and Consequences to the Further Analysis

The preceding discussion has shown that asset pricing in the general equilibrium framework is in many respects simpler than under the partial equilibrium framework. Asset prices can be derived in closed-form for arbitrary utility functions, provided the value functions of the identical investors is twice continuously differentiable with respect to wealth, concave in wealth, and the (aggregate version) of the envelope condition holds. These results have three implications for the further analysis: (i) the specification of a particular utility function is of less importance in the general equilibrium setting than in the partial equilibrium case.[23] For that reason, the remainder of the general equilibrium analysis is structured based on cash flow models rather than utility functions as in the partial equilibrium case. (ii) All analyses are based on stochastic discount factors expressed through time preference and relative marginal utilities of consumption only, i.e.,

4-36

$$P_t(z_t{}^p) = E\left(q_{t,t+1}(z_{t+1}{}^p, z_t{}^p) \cdot \{P_{t+1}(z_{t+1}{}^p) + D_{t+1}(z_{t+1}{}^d)\} \middle| z_t{}^p\right)$$

4-37

$$B_t(z_t{}^p) = E\left(q_{t,t+1}(z_{t+1}{}^p, z_t{}^p) \middle| z_t{}^p\right)$$

with

$$q_{t,t+1}(z_{t+1}{}^p, z_t{}^p) \equiv \frac{1}{1+\rho} \cdot \frac{U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}(z_{t+1}{}^p)\right)}{U'\left(\frac{1}{n_I} \cdot D_t{}^{market}(z_t{}^p)\right)}$$

4-38

$$RP_t(z_t{}^p) = -cov\left(AfR_{t,t+1}(z_{t+1}{}^p, z_t{}^p), P_{t+1}(z_{t+1}{}^p) + D_{t+1}(z_{t+1}{}^d) \middle| z_t{}^p\right)$$

with

$$AfR_{t,t+1}(z_{t+1}{}^p, z_t{}^p) \equiv \frac{U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}(z_{t+1}{}^p)\right)}{E\left(U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}(z_{t+1}{}^p)\right) \middle| z_t{}^p\right)}$$

(iii) Additional insights can be obtained by expressing prices of risky assets as (multi-period) discounted future cash flows:

4-39

$$P_t(z_t{}^p) = \sum_{\tau=1}^{T-t} E\left(q_{t,t+\tau}(z_{t+\tau}{}^p, z_t{}^p) \cdot D_{t+\tau}(z_{t+\tau}{}^d) \middle| z_t{}^p\right)$$

with

$$q_{t,t+\tau}(z_{t+\tau}{}^p, z_t{}^p) \equiv \frac{1}{(1+\rho)^\tau} \cdot \frac{U'\left(\frac{1}{n_I} \cdot D_{t+\tau}{}^{market}(z_{t+\tau}{}^p)\right)}{U'\left(\frac{1}{n_I} \cdot D_t{}^{market}(z_t{}^p)\right)}$$

To distinguish verbally between risky asset prices in the forms (4-36) and (4-39), I refer to (4-36) as “quasi-static asset prices”[24] and to (4-39) as “asset prices as discounted future cash flows”. Although asset prices can, in principle, also be expressed as discounted future cash flows under partial equilibrium, it only permits non-trivial results under general equilibrium because of the simple structure of stochastic discount factors.

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[23] Recall that CARA preferences had to be imposed in the partial equilibrium case to obtain tractable results.

[24] The terminology “quasi-static“ is due to Wilhelm (1983) p. 53, and is motivated by the idea that multi-period asset prices can be expressed as interdependent one-period asset pricing problems.

4.2 Equilibrium Asset Prices and Risk Premia for the General Cash Flow Model under Complete and Incomplete Information

4.2.1 All Assets Pay Cash Flows in Every Period: Information Frequency = Cash Flow Frequency

4.2.1.1 Complete Information

4.2.1.1.1 Information Relevant to Pricing $z_t{}^p$

For general cash flow models under complete information, information relevant to pricing consists of the pair of the current regime and current cash flows:

4-40

$$z_t{}^p = (S_t, D_t)$$

To see that (S_t, D_t) contains all information relevant to pricing, observe that it describes (i) aggregate cash flows paid by the market portfolio of risky assets, $D_t{}^{market}$, and (ii) also is a sufficient statistic for the conditional distribution of future cash flows: (i) is evident because $D_t{}^{market}$ is the aggregate of D_t; the argument to justify (ii) is the same as in the partial equilibrium case (3-66) (p. 60).

4.2.1.1.2 Equilibrium Asset Prices

4.2.1.1.2.1 Quasi Static Asset Prices

In the special case of complete information, the (quasi static) pricing results for general cash flow models and information scenarios (4-36) and (4-37) specialize to

4-41

$$P_t{}^{cI}(S_t, D_t) = E\left(q_{t,t+1}{}^{cI}(fe_{t+1}, S_t, D_t) \cdot \left\{P_{t+1}{}^{cI}(S_{t+1}, D_{t+1}) + D_{t+1}\right\} \middle| S_t, D_t\right)$$

4-42

$$B_t{}^{cI}(S_t, D_t) = E\left(q_{t,t+1}{}^{cI}(fe_{t+1}, S_t, D_t) \middle| S_t, D_t\right)$$

with stochastic discount factor

4-43

$$q_{t,t+1}{}^{cI}(fe_{t+1}, S_t, D_t) = \frac{1}{1+\rho} \cdot \frac{U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}(D_{t+1})\right)}{U'\left(\frac{1}{n_I} \cdot D_t{}^{market}(D_t)\right)}$$

with

$$D_{t+1} = D(D_t, S_t, fe_{t+1})$$

The pricing equations (4-41) and (4-42) demonstrate how the two sources of risk under complete information, namely (i) regimes S_{t+1} and (ii) factors and residuals fe_{t+1}, are priced. Since the pricing of risk in general depends on the relationship of asset prices and cash flows to the stochastic discount factor - and in particular on the covariance with the stochastic discount factor -, it appears promising to examine if and how the effects of sources of risk S_{t+1} and fe_{t+1} on asset prices and cash flows tend to introduce covariances with the stochastic discount factor. Such an analysis is facilitated in the general equilibrium framework of this chapter by the fact that the stochastic discount factor is known to be determined by aggregate cash flows $D_{t+1}{}^{market}$. The question of how the regime S_{t+1} and factors and residuals fe_{t+1} tend to induce covariances between asset prices and cash flows with the stochastic discount factor then amounts to analyzing how these sources of risk jointly determine asset prices and cash flows as well as aggregate cash flows.

Formally, asset prices and cash flows can be thought of as consisting of two components, (i) a part that is "explained" by aggregate cash flows $D_{t+1}{}^{market}$ and (ii) a remaining part that captures all other stochastic influences that are not due to $D_{t+1}{}^{market}$. More precisely, part (i) is the expectation of asset prices and cash flows conditional on $D_{t+1}{}^{market}(D_{t+1})$, S_t and D_t and part (ii) equals the fluctuation around this expectation:

4-44

$$P_{t+1}{}^{cI}(S_{t+1}, D_{t+1}) + D_{t+1} = \underbrace{E\left(P_{t+1}{}^{cI}(S_{t+1}, D_{t+1}) + D_{t+1}\middle| S_t, D_t, D_{t+1}{}^{market}(D_{t+1})\right)}_{part\ (i)} + \underbrace{\Delta_{t+1}}_{part\ (ii)}$$

with

$$\Delta_{t+1} \equiv P_{t+1}{}^{cI} + D_{t+1} - E\left(P_{t+1}{}^{cI} + D_{t+1}\middle| S_t, D_t, D_{t+1}{}^{market}(D_{t+1})\right)$$

and

$$D_{t+1} = D(D_t, S_t, fe_{t+1})$$

There are two important aspects about Δ_{t+1}, both of which can be established by virtue of the tower property of conditional expectations. First, Δ_{t+1} has a zero expectation conditional on information relevant to pricing S_t, D_t:

$$E(\Delta_{t+1}|S_t, D_t) = 0$$[25]

Second, Δ_{t+1} is uncorrelated[26] with the stochastic discount factor conditional on information relevant to pricing S_t, D_t. Both properties of Δ_{t+1} imply that it is not priced: it is uncorrelated with the

[25] This follows from the following equation:

$$\begin{aligned} E(\Delta_{t+1}|S_t, D_t) &= E\left(E\left(\Delta_{t+1}\middle| S_t, D_t, D_{t+1}{}^{market}(D_{t+1})\right)\middle| S_t, D_t\right) \\ &= E\left(E\left(P_{t+1}{}^{cI} + D_{t+1}\middle| S_t, D_t, D_{t+1}{}^{market}(D_{t+1})\right) - E\left(P_{t+1}{}^{cI} + D_{t+1}\middle| S_t, D_t, D_{t+1}{}^{market}(D_{t+1})\right)\middle| S_t, D_t\right) \\ &= 0 \end{aligned}$$

[26] To see this, first use the fact that $E(\Delta_{t+1}|S_t, D_t) = 0$ and apply the tower property of expectations:

stochastic discount factor and, therefore, priced risk neutrally as its expectation discounted at the riskless interest rate, but this (discounted) expectation is zero. This result can also be shown directly by first applying the tower property of conditional expectations to pricing equation (4-41), yielding

$$P_t^{cI}(S_t, D_t) = E\left(E\left(q_{t,t+1}^{cI}(fe_{t+1}, S_t, D_t) \cdot \{P_{t+1}^{cI}(S_{t+1}, D_{t+1}) + D_{t+1}\} \middle| S_t, D_t, D_{t+1}^{market}(D_{t+1})\right) \middle| S_t, D_t\right)$$

with

$$D_{t+1} = D(D_t, S_t, fe_{t+1})$$

and by then noting that the stochastic discount factor is certain conditional on the information $S_t, D_t, D_{t+1}^{market}$ and, therefore, can be factored out from the inner expectation:

4-45

$$P_t^{cI}(S_t, D_t) = E\left(\begin{matrix} q_{t,t+1}^{cI}(fe_{t+1}, S_t, D_t) \\ \cdot E\left(P_{t+1}^{cI}(S_{t+1}, D_{t+1}) + D_{t+1} \middle| S_t, D_t, D_{t+1}^{market}(D_{t+1})\right) \end{matrix} \middle| S_t, D_t \right)$$

with

$$D_{t+1} = D(D_t, S_t, fe_{t+1})$$

Solely $E\left(P_{t+1}^{cI}(S_{t+1}, D_{t+1}) + D_{t+1} \middle| S_t, D_t, D_{t+1}^{market}(D_{t+1})\right)$ is priced because Δ_{t+1} has a zero price. This conditional expectation is, conditional on S_t, D_t, non-zero (except for degenerate constellations). It will also be correlated with the stochastic discount factor: as D_{t+1}^{market} is a random variable, the conditional expectation

$$E\left(P_{t+1}^{cI}(S_{t+1}, D_{t+1}) + D_{t+1} \middle| S_t, D_t, D_{t+1}^{market}(D_{t+1})\right)$$

will be random conditional on information relevant to pricing S_t, D_t, too. Thus the conditional expectation and the stochastic discount factor are both driven by the same underlying random variable D_{t+1}^{market}. Therefore, sources of risk that affect D_{t+1}^{market} will be priced in a risk-adjusted way.

These considerations on the relevance and irrelevance of price and cash flow components allow answering how the sources of risk S_{t+1} and fe_{t+1} are priced: in a first step, the influences that cause a covariance with the stochastic discount factor can be identified. From the form of the cash flow model, it follows that D_{t+1}^{market} is a function of D_t, S_t and fe_{t+1} only, and of these variables only fe_{t+1} is stochastic under complete information at time t. It can be concluded that fe_{t+1} will, through its effect on D_{t+1}^{market}, give rise to correlations between (the relevant component of) asset prices and cash flows and the stochastic discount factor. In a second step, sources of risk that do not cause covariances with the stochastic discount factor can be identified. Perhaps surprisingly, the new

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$$cov\left(q_{t,t+1}^{cI}(fe_{t+1}, S_t, D_t), \Delta_{t+1} \middle| S_t, D_t\right) = E\left(q_{t,t+1}^{cI}(fe_{t+1}, S_t, D_t) \cdot E\left(\Delta_{t+1} \middle| S_t, D_t, D_{t+1}^{market}(D_{t+1})\right) \middle| S_t, D_t\right)$$

and further observe that $E\left(\Delta_{t+1} \middle| S_t, D_t, D_{t+1}^{market}\right) = 0$.

regime is uncorrelated with ${D_{t+1}}^{market}$ by the assumptions on the processes of regimes and factors. It, therefore, only determines Δ_{t+1}. In other words, although S_{t+1} may influence ${P_{t+1}}^{cI}$, this influence is not priced since it is not related to the stochastic discount factor. In addition to the irrelevance of S_{t+1}, it can also be argued that not all aspects of factors and residuals fe_{t+1} cause correlations with the stochastic discount factor. Loosely speaking, fe_{t+1} gives rise to a covariance with the stochastic discount factor insofar as it determines aggregate cash flow ${D_{t+1}}^{market}$. However, fe_{t+1} not only determines this aggregate of cash flows but also individual cash flows D_{t+1}. Insofar as fe_{t+1} only determines individual (but not aggregate) cash flows, it is uncorrelated with the stochastic discount factor and its influence is priced risk-neutrally. Formally, the distribution of fe_{t+1} conditional on ${D_{t+1}}^{market}$ will contribute to Δ_{t+1}.

Although it is now clear that non-zero covariances with the stochastic discount factor and, thus, risk-adjusted pricing, can only be introduced by factors and residuals fe_{t+1}, it is still unclear through which channels such correlations are introduced by fe_{t+1}. This question can be answered by analyzing the covariances of the stochastic discount factor with ${P_{t+1}}^{cI}$ and D_{t+1}:

4-46

$$cov\left({q_{t,t+1}}^{cI}(fe_{t+1}, S_t, D_t), {P_{t+1}}^{cI}(S_{t+1}, D_{t+1})\middle|S_t, D_t\right)$$

and

4-47

$$cov\left({q_{t,t+1}}^{cI}(fe_{t+1}, S_t, D_t), D_{t+1}\middle|S_t, D_t\right)$$

To obtain the desired insights into the covariance between stochastic discount factor ${q_{t,t+1}}^{cI}$ and $P_{t+1}(S_{t+1}, D_{t+1})$, it is necessary to learn more about the dependency of $P_{t+1}(S_{t+1}, D_{t+1})$ on cash flows D_{t+1}. Using the fact that asset prices are present values of future cash flows (4-39) (p. 107) reveals more about the structure the of covariance:

4-48

$$cov\left({q_{t,t+1}}^{cI}(fe_{t+1}, S_t, D_t), {P_{t+1}}^{cI}(S_{t+1}, D_{t+1})\middle|S_t, D_t\right)$$
$$= \sum_{\tau=1}^{T-t} \frac{1}{(1+\rho)^{\tau}} \cdot \frac{1}{U'\left(\frac{1}{n_I} \cdot {D_t}^{market}\right)}$$
$$\cdot\, cov\left(U'\left(\frac{1}{n_I} \cdot {D_{t+1}}^{market}\right), \frac{E\left(U'\left(\frac{1}{n_I} \cdot {D_{t+1+\tau}}^{market}\right) \cdot D_{t+1+\tau}\middle|S_{t+1}, D_{t+1}\right)}{U'\left(\frac{1}{n_I} \cdot {D_{t+1}}^{market}\right)}\middle|S_t, D_t\right)$$

with

$$D_{t+1} = D(D_t, S_t, fe_{t+1})$$

Formula (4-48) reveals three possible channels through which fe_{t+1} can cause correlations between the stochastic discount factor ${q_{t,t+1}}^{cI}$ and asset prices ${P_{t+1}}^{cI}$: channel (i) marginal utility $U'\left(\frac{1}{n_I} \cdot {D_{t+1}}^{market}\right)$. Marginal utility is the first argument of the covariance and also appears in the denominator of the second argument. Therefore, this first channel works in the direction of a negative covariance and a lower price at time t. Channels (ii) and (iii) stem from the covariance between $U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}\right)$ and $E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1+\tau}{}^{market}\right)\cdot D_{t+1+\tau}\middle|S_{t+1},D_{t+1}\right)$. The conditional expectation comprises two components: future marginal utility $U'\left(\frac{1}{n_I}\cdot D_{t+1+\tau}{}^{market}\right)$ and future cash flows $D_{t+1+\tau}$. Although both components cannot be separated mathematically, they are different from an economic perspective in that marginal utility is asset independent whereas cash flows, by definition, are asset-specific. Following the economic argument, an interrelation between $U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}\right)$ and $U'\left(\frac{1}{n_I}\cdot D_{t+1+\tau}{}^{market}\right)$ (channel (ii)) and an interrelation between $U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}\right)$ and $D_{t+1+\tau}$ (channel (iii)) can be identified. Regarding the sign of the covariance between $U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}\right)$ and $E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1+\tau}{}^{market}\right)\cdot D_{t+1+\tau}\middle|S_{t+1},D_{t+1}\right)$ nothing can be said without further restrictions on the cash flow model. Hence, channels (ii) and (iii) can increase or decrease prices at time t.

To obtain the desired insights into the covariance between stochastic discount factor $q_{t,t+1}{}^{cI}$ and D_{t+1}, plug in for $q_{t,t+1}{}^{cI}$ to obtain

4-49

$$\frac{1}{1+\rho}\cdot\frac{1}{U'\left(\frac{1}{n_I}\cdot D_t{}^{market}(D_t)\right)}\cdot cov\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right),D_{t+1}\middle|S_t,D_t\right)$$

with

$$D_{t+1}=D(D_t,S_t,fe_{t+1})$$

fe_{t+1} introduces correlations between the stochastic discount factor $q_{t,t+1}{}^{cI}$ and asset prices D_{t+1} since it contributes to marginal utility $U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right)$ via its effect on aggregate cash flows and, at the same time, influences individual cash flows. Insofar this effect is similar to channel (iii) in (4-48).

Concerning the price of the one-period riskless bond (4-42), by definition, cash flows of the one-period riskless bond are discounted at the riskless-rate. Hence, the question of the relevancy or irrelevancy of sources of risk regimes S_{t+1} as well as factors and residuals fe_{t+1} is meaningless.

4.2.1.1.2.2 Asset Prices as Discounted Future Cash Flows

In the special case of complete information, asset prices as discounted future cash flows for general cash flow models and information scenarios (4-39) specialize to

4-50

$$P_t^{cI}(S_t, D_t) = \sum_{\tau=1}^{T-t} E\left(q_{t,t+\tau}^{cI}\left(fe_{t+1,t+\tau}, S_{t+1,t+\tau-1}; S_t, D_t\right) \cdot D_{t+\tau} \middle| S_t, D_t\right)$$

with multi-period stochastic discount factors

4-51

$$q_{t,t+\tau}^{cI}\left(fe_{t+1,t+\tau}, S_{t+1,t+\tau-1}; S_t, D_t\right) = \frac{1}{(1+\rho)^\tau} \cdot \frac{U'\left(\frac{1}{n_I} \cdot D_{t+\tau}^{market}(D_{t+\tau})\right)}{U'\left(\frac{1}{n_I} \cdot D_t^{market}(D_t)\right)}$$

where cash flows at time $t + \tau$ are obtained recursively from current cash flows D_t as a function of the path of regimes $S_{t,t+\tau-1}$ and the path of factors and residuals $fe_{t+1,t+\tau}$, i.e.,

$$D_{t+\tau} = D_{t+\tau}\left(D_t, S_{t,t+\tau-1}, fe_{t+1,t+\tau}\right)$$

The pricing equation (4-50) demonstrates how the two sources of risk (i) path of future regimes $S_{t+1,t+\tau}$ - note the current regime S_t is non-stochastic - and (ii) path of factors and residuals $fe_{t+1,t+\tau}$ are priced. Since asset prices are the sum of discounted future cash flows, it suffices to consider the present value of one point in time $t + \tau$. The pricing of these sources of risk can be analyzed in the same way as in the quasi-static case through their influence on the stochastic discount factor; the only two differences compared to the quasi-static case are that only cash flows (but not asset prices) at time $t + \tau$ are discounted and that more sources of risk influence $D_{t+\tau}$ than in the quasi-static case.

Similar to the quasi-static pricing equation in the form (4-45), the tower property of expectations reveals that solely those components of cash flows $D_{t+\tau}$ are priced that can be explained through $D_{t+\tau}^{market}$, i.e., $E\left(D_{t+\tau} \middle| S_t, D_t, D_{t+\tau}^{market}(D_{t+\tau})\right)$; these components of cash flows $D_{t+\tau}$ alone are possibly correlated with the stochastic discount factor:

4-52

$$\begin{aligned} &E\left(q_{t,t+\tau}^{cI}\left(fe_{t+1,t+\tau}, S_{t+1,t+\tau-1}; S_t, D_t\right) \cdot D_{t+\tau} \middle| S_t, D_t\right) \\ &= E\left(q_{t,t+\tau}^{cI}\left(fe_{t+1,t+\tau}, S_{t+1,t+\tau-1}; S_t, D_t\right) \cdot E\left(D_{t+\tau} \middle| S_t, D_t, D_{t+\tau}^{market}(D_{t+\tau})\right) \middle| S_t, D_t\right) \end{aligned}$$

with

$$D_{t+\tau} = D_{t+\tau}\left(D_t, S_{t,t+\tau-1}, fe_{t+1,t+\tau}\right)$$

It follows that the influence of the sources of risk can be discussed in the same way as in the quasi-static case. In a first step, those sources of risk that determine $D_{t+\tau}^{market}(D_{t+\tau})$ can be identified as priced in a risk-adjusted way. Since $D_{t+\tau}^{market}$ is a function of the paths $S_{t,t+\tau-1}$ and $fe_{t+1,t+\tau}$, it can be concluded that all elements of these paths are potentially priced in a risk-adjusted way. In a second step, the sources of risk that do not cause covariances with the stochastic discount factor can be identified. Clearly, $S_{t+\tau}$ is not an element in the determination of $D_{t+\tau}^{market}$ and, therefore, is not priced. In addition to the irrelevance of $S_{t+\tau}$, it can also be argued that not all aspects of factors and residuals $fe_{t+\tau}$ cause correlations with the stochastic discount factor. First, observe that not the paths of regimes $S_{t,t+\tau-1}$ as well as factors and residuals $fe_{t+1,t+\tau}$ as such exert influence on prices but only their joint effects on $D_{t+\tau}{}^{market}$ via $D_{t+\tau-1}$ and $S_{t+\tau-1}$ are of interest. Formulated differently, the differences between two paths that lead to the same $D_{t+\tau-1}$ and $S_{t+\tau-1}$ are of no importance from a pricing perspective. Second, $fe_{t+\tau}$ is only priced insofar as it determines aggregate cash flows whereas its influence on individual cash flows is not priced.

Although it is now clear that non-zero covariances with the stochastic discount factor and, thus, risk-adjusted pricing, can only be introduced by $D_{t+\tau-1}$, $S_{t+\tau-1}$, and $fe_{t+\tau}$, it is still unclear through which channels such correlations are introduced by $D_{t+\tau-1}$, $S_{t+\tau-1}$, and $fe_{t+\tau}$: only a generalized version of the cash flow channel (4-49) exists. However, this channel is by far more complex and, hence, needs a more thorough analysis. This can be done by considering the covariance of the stochastic discount factor $q_{t,t+\tau}{}^{cI}$ with cash flows $D_{t+\tau}$ in more detail:

4-53

$$cov\left(q_{t,t+\tau}{}^{cI}\left(fe_{t+1,t+\tau}, S_{t+1,t+\tau-1}; S_t, D_t\right), D_{t+\tau} \middle| S_t, D_t\right) = cov\left(\frac{1}{(1+\rho)^\tau} \cdot \frac{U'\left(\frac{1}{n_I} \cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right)}{U'\left(\frac{1}{n_I} \cdot D_t{}^{market}(D_t)\right)}, D_{t+\tau} \middle| S_t, D_t\right)$$

Since $D_{t+\tau-1}$, $S_{t+\tau-1}$, and $fe_{t+\tau}$ possess a different reference to time, it is reasonable to break up the cash flow channel (4-53) into two sub-channels: $D_{t+\tau-1}$ and $S_{t+\tau-1}$ on the one hand and $fe_{t+\tau}$ on the other hand. This is achieved by splitting up the covariance (4-53) into a covariance due to $fe_{t+\tau}$ (with $D_{t+\tau-1}$, $S_{t+\tau-1}$ averaged out) and a second covariance due to $D_{t+\tau-1}$, $S_{t+\tau-1}$ (with $fe_{t+\tau}$ averaged out). As Appendix A3.2 shows, this leads to

4-54

$$cov\left(\frac{1}{(1+\rho)^\tau} \cdot \frac{U'\left(\frac{1}{n_I} \cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right)}{U'\left(\frac{1}{n_I} \cdot D_t{}^{market}(D_t)\right)}, D_{t+\tau} \middle| S_t, D_t\right) = \frac{1}{(1+\rho)^\tau} \cdot \frac{1}{U'\left(\frac{1}{n_I} \cdot D_t{}^{market}(D_t)\right)} \cdot$$
$$\left\{E\left(cov\left(U'\left(\frac{1}{n_I} \cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right), D_{t+\tau} \middle| S_{t+\tau-1}, D_{t+\tau-1}\right) \middle| S_t, D_t\right)\right.$$
$$\left. + cov\left(E\left(U'\left(\frac{1}{n_I} \cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right) \middle| S_{t+\tau-1}, D_{t+\tau-1}\right), E(D_{t+\tau} | S_{t+\tau-1}, D_{t+\tau-1}) \middle| S_t, D_t\right)\right\}$$

The covariance (4-54) reveals the details of risk-adjusted pricing. Consider the first term in brackets on the right-hand side of (4-54), the covariance of the stochastic discount factor and cash flows due to $fe_{t+\tau}$ conditional on $S_{t+\tau-1}, D_{t+\tau-1}$,

$$cov\left(U'\left(\frac{1}{n_I} \cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right), D_{t+\tau} \middle| S_{t+\tau-1}, D_{t+\tau-1}\right)$$

Since $S_{t+\tau-1}, D_{t+\tau-1}$ are stochastic, the expectation of these conditional covariances is taken:

$$E\left(cov\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right), D_{t+\tau}\middle|S_{t+\tau-1}, D_{t+\tau-1}\right)\middle|S_t, D_t\right)$$

From an economic point of view, taking the expectation means that $S_{t+\tau-1}$ and $D_{t+\tau-1}$ are averaged out.

The second term in brackets on the right-hand side of (4-54) can be interpreted as a covariance due to $S_{t+\tau-1}, D_{t+\tau-1}$ alone: the effect of factors and residuals $fe_{t+\tau}$ is not contained in this term because both marginal utility of consumption and cash flows are replaced by their expectations conditional on $S_{t+\tau-1}, D_{t+\tau-1}$:

$$cov\left(E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right)\middle|S_{t+\tau-1}, D_{t+\tau-1}\right), E(D_{t+\tau}|S_{t+\tau-1}, D_{t+\tau-1})\middle|S_t, D_t\right)$$

In this sense, the second term complements the first term on the right-hand side of (4-54). The order of expectation and covariance are reversed because the influence of factors and residuals is averaged out.

In conclusion, the channels through which correlations are introduced by $D_{t+\tau-1}$, $S_{t+\tau-1}$, and $fe_{t+\tau}$ are now clarified.

4.2.1.1.3 Equilibrium Risk Premia

In the special case of complete information, the result on risk premia for general cash flow models and information scenarios (4-38) specializes to

4-55

$$RP_t{}^{cI}(S_t, D_t) = -cov\left(\begin{array}{c} AfR_{t,t+1}{}^{cI}(fe_{t+1}, S_t, D_t) \\ , E\left(P_{t+1}{}^{cI}(S_{t+1}, D_{t+1}) + D_{t+1}\middle|S_t, D_t, D_{t+1}{}^{market}(D_{t+1})\right)\end{array}\middle|S_t, D_t\right)$$

with

$$AfR_{t,t+1}{}^{cI}(fe_{t+1}, S_t, D_t) \equiv \frac{U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right)\middle|S_t, D_t\right)}$$

$$D_{t+1} = D(D_t, S_t, fe_{t+1})$$

Paralleling the results for asset prices, (4-55) shows that only factors and residuals fe_{t+1} insofar as they influence $D_{t+1}{}^{market}$, give rise to a risk premium. The new regime S_{t+1} and the influence of factors and residual fe_{t+1} on structure of D_{t+1} do not contribute to the risk premium.

4.2.1.2 Incomplete Information

4.2.1.2.1 Information Relevant to Pricing ${z_t}^p$

For general cash flow models under complete information, information relevant to pricing consists of the pair of conditional regime probabilities and current cash flows:

4-56

$$ {z_\tau}^{iI,p} = (\pi_\tau, D_\tau), 0 \leq \ \tau \leq T-1 $$

The argument is similar to the complete information case, but conditional regime probabilities replace the unobservable regime in the description of the conditional distribution of future cash flows from the point of view of investors.

4.2.1.2.2 Equilibrium Asset Prices

4.2.1.2.2.1 Quasi Static Asset Prices

In the special case of incomplete information, the (quasi static) pricing results for general cash flow models and information scenarios (4-36) and (4-37) specialize to

4-57

$$ {P_t}^{iI}(\pi_t, D_t) = \sum_{s=1}^{K} \pi_{s,t} \cdot E\left(\begin{matrix} {q_{t,t+1}}^{iI}(fe_{t+1}, S_t, D_t) \\ \cdot \left\{ {P_{t+1}}^{iI}(\pi_{t+1}, D_{t+1}) + D_{t+1} \right\} \end{matrix} \middle| S_t = s, D_t \right) $$

and

4-58

$$ {B_t}^{iI}(\pi_t, D_t) = \sum_{s=1}^{K} \pi_{s,t} \cdot E\left({q_{t,t+1}}^{iI}(fe_{t+1}, S_t, D_t) \middle| S_t = s, D_t \right) $$

with stochastic discount factor

4-59

$$ {q_{t,t+1}}^{iI}(fe_{t+1}, S_t, D_t) = \frac{1}{1+\rho} \cdot \frac{U'\left(\frac{1}{n_I} \cdot {D_{t+1}}^{market}(D_{t+1}) \right)}{U'\left(\frac{1}{n_I} \cdot {D_t}^{market}(D_t) \right)} $$

with

$$ D_{t+1} = D(D_t, S_t, fe_{t+1}) $$
$$ \pi_{t+1} = \Pi(\pi_t, D_t, D_{t+1}, Sig_{t+1}) $$
$$ Sig_{t+1} = Sig_{t+1}(S_t, S_{t+1}, fe_{t+1}, \eta_{t+1}) $$

Pricing equations (4-57) and (4-58) demonstrate how the four sources of risk under incomplete information, namely (i) the unknown current regime S_t, (ii) factors and residuals fe_{t+1}, (iii) the new regime at time S_{t+1} and (iv) signal noise η_{t+1}, are priced.

The findings from the complete information case suggest that the pricing of sources of risk depends on their relation to aggregate cash flows ${D_{t+1}}^{market}$ and, in particular, the covariances of asset prices and cash flows with the stochastic discount factor. Asset prices and cash flows can be thought of as consisting of a first part that can be explained by ${D_{t+1}}^{market}$, i.e., the expectation of asset prices and cash flows conditional on ${D_{t+1}}^{market}$, and a second part that captures the fluctuation around this expectation:

4-60

$$ {P_{t+1}}^{iI}(\pi_{t+1}, D_{t+1}) + D_{t+1} = \underbrace{E\left({P_{t+1}}^{iI}(\pi_{t+1}, D_{t+1}) + D_{t+1} \middle| \pi_t, D_t, {D_{t+1}}^{market}(D_{t+1})\right)}_{part\ (i)} + \underbrace{\Delta_{t+1}}_{part\ (ii)} $$

with

$$ \Delta_{t+1} \equiv {P_{t+1}}^{iI}(\pi_{t+1}, D_{t+1}) + D_{t+1} - E\left({P_{t+1}}^{iI}(\pi_{t+1}, D_{t+1}) + D_{t+1} \middle| \pi_t, D_t, {D_{t+1}}^{market}(D_{t+1})\right) $$

and

$$ D_{t+1} = D(D_t, S_t, fe_{t+1}) $$
$$ \pi_{t+1} = \Pi(\pi_t, D_t, D_{t+1}, Sig_{t+1}) $$
$$ Sig_{t+1} = Sig_{t+1}(S_t, S_{t+1}, fe_{t+1}, \eta_{t+1}) $$

Clearly, Δ_{t+1} is uncorrelated with the stochastic discount factor and possesses a zero expectation conditional on information relevant to pricing, hence its price is zero. As a consequence, solely $E\left(\left\{{P_{t+1}}^{iI}(\pi_{t+1}, D_{t+1}) + D_{t+1}\right\} \middle| \pi_t, D_t, {D_{t+1}}^{market}(D_{t+1})\right)$ is priced:

4-61

$$ {P_t}^{iI}(\pi_t, D_t) = E\left(\begin{array}{c} {q_{t,t+1}}^{iI}(fe_{t+1}, S_t, D_t) \\ \cdot E\left(\left\{{P_{t+1}}^{iI}(\pi_{t+1}, D_{t+1}) + D_{t+1}\right\} \middle| \pi_t, D_t, {D_{t+1}}^{market}(D_{t+1})\right) \end{array} \middle| \pi_t, D_t \right) $$

with

$$ D_{t+1} = D(D_t, S_t, fe_{t+1}) $$
$$ \pi_{t+1} = \Pi(\pi_t, D_t, D_{t+1}, Sig_{t+1}) $$
$$ Sig_{t+1} = Sig_{t+1}(S_t, S_{t+1}, fe_{t+1}, \eta_{t+1}) $$

These considerations on the relevance and irrelevance of price and cash flow components allow answering how the sources of risk (i) the unknown current regime S_t, (ii) factors and residuals fe_{t+1}, (iii) the new regime at time S_{t+1} and (iv) signal noise $\eta_{t+1}$are priced. There are two sources of risk that cause covariances with the stochastic discount factor, namely (i) the unknown current regime and (ii) factors and residuals: this can be deduced from the fact that ${D_{t+1}}^{market}$ is a function of D_t, S_t and fe_{t+1}, where only D_t is non-stochastic conditional on information relevant to pricing at time t. The remaining two sources of risk do not cause covariances with the stochastic discount factor;

these sources are (iii) the new regime at time S_{t+1} and (iv) signal noise η_{t+1}. They only influence Δ_{t+1}, the part of price and cash flow fluctuations that is not priced. In addition, sources of risk (i) S_t and (ii) fe_{t+1} are not priced insofar as they only determine individual (but not aggregate) cash flows D_{t+1}. Formally, the distribution of the unobservable current regime and of factors and residuals conditional on $D_{t+1}{}^{market}$ are a source of fluctuations in Δ_{t+1} and, therefore, not priced.

As in the complete information case, one may not only want to identify the sources of risk that are priced but also learn more about how they give rise to correlations with the stochastic discount factor. Paralleling the considerations of the complete information case, there are several channels through which correlations between the stochastic discount factor $q_{t,t+1}{}^{iI}$ and asset prices $P_{t+1}{}^{iI}$ as well as cash flows D_{t+1} can occur. In analogy to (4-48) the covariance between $q_{t,t+1}{}^{iI}$ and $P_{t+1}{}^{iI}$ reads:

4-62

$$\begin{aligned} &cov\left(q_{t,t+1}{}^{iI}(fe_{t+1}, S_t, D_t), P_{t+1}{}^{iI}(\pi_{t+1}, D_{t+1})\middle|\pi_t, D_t\right) \\ &= \sum_{\tau=1}^{T-t} \frac{1}{(1+\rho)^\tau} \cdot \frac{1}{U'\left(\frac{1}{n_I} \cdot D_t{}^{market}\right)} \\ &\cdot cov\left(U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}\right), \frac{E\left(U'\left(\frac{1}{n_I} \cdot D_{t+1+\tau}{}^{market}\right) \cdot D_{t+1+\tau} \middle| \pi_{t+1}, D_{t+1}\right)}{U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}\right)} \middle| \pi_t, D_t \right) \end{aligned}$$

(4-62) reveals that the three channels identified in the complete information case still exist in the incomplete information case. However, there is one difference: these three channels are now introduced by S_t and fe_{t+1} (instead of only fe_{t+1}). Moreover, under incomplete information a fourth channel exists. The conditional regime probabilities π_{t+1} are stochastic from the perspective of time t because they depend on D_{t+1}. This dependence on D_{t+1} causes an interrelation with marginal utility.

The covariance between stochastic discount factor $q_{t,t+1}{}^{iI}$ and D_{t+1} is similar to the complete information cash flow channel (4-49) with the difference that under incomplete information both S_t and fe_{t+1} introduce the interrelation.

Concerning the price of the one-period riskless bond (4-58), by definition, cash flows of the one-period riskless bond are discounted at the riskless-rate. Hence, the question of the relevancy or irrelevancy of sources of risk regime S_{t+1}, current regime S_t, signal noise η_{t+1} as well as factors and residuals fe_{t+1} is meaningless.

4.2.1.2.2.2 Asset Prices as Discounted Future Cash Flows

In the special case of incomplete information, asset prices as discounted future cash flows for general cash flow models and information scenarios (4-39) specialize to

4-63

$$P_t{}^{iI}(\pi_t, D_t) = \sum_{s=1}^{K} \pi_{s,t} \cdot \sum_{\tau=1}^{T-t} E\big(q_{t,t+\tau}{}^{iI}\big(fe_{t+1,t+\tau}, S_{t+1,t+\tau-1}; S_t, D_t\big) \cdot D_{t+\tau} \big| S_t = s, D_t\big)$$

with stochastic discount factor

4-64

$$q_{t,t+\tau}{}^{iI}\big(fe_{t+1,t+\tau}, S_{t+1,t+\tau-1}; S_t, D_t\big) = \frac{1}{(1+\rho)^{\tau}} \cdot \frac{U'\left(\frac{1}{n_I} \cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right)}{U'\left(\frac{1}{n_I} \cdot D_t{}^{market}(D_t)\right)}$$

where cash flows at time $t+\tau$ are obtained recursively from current cash flows D_t as a function of the path of regimes $S_{t,t+\tau-1}$ and the path of factors and residuals $fe_{t+1,t+\tau}$.

Note that (4-63) can be considered a discrete-time generalization[27] of the pricing formula of Veronesi (2000), Proposition 1a, p. 813 in that it considers several risky assets, more general cash flow models, signal structures, and regime transition probabilities.

To examine how the sources of risk are priced, observe that signal noise does not enter the present value equation (4-63). The reason is that signal noise affects prices, but not cash flows. For that reason, only the pricing influence of the three other sources of risk needs to be discussed in more detail. Similar to the quasi-static pricing equation in the form (4-61), the tower property of expectations reveals that solely those components of cash flows $D_{t+\tau}$ are priced that can be explained through $D_{t+\tau}{}^{market}$, i.e., $E\left(D_{t+\tau}\middle|S_t, D_t, D_{t+\tau}{}^{market}(D_{t+\tau})\right)$; these components of cash flows $D_{t+\tau}$ alone are possibly correlated with the stochastic discount factor:

4-65

$$\begin{aligned} &E\big(q_{t,t+\tau}{}^{iI}\big(fe_{t+1,t+\tau}, S_{t+1,t+\tau-1}; S_t, D_t\big) \cdot D_{t+\tau} \big| \pi_t, D_t\big) \\ &= E\left(q_{t,t+\tau}{}^{iI}\big(fe_{t+1,t+\tau}, S_{t+1,t+\tau-1}; S_t, D_t\big) \cdot E\left(D_{t+\tau}\middle|\pi_t, D_t, D_{t+\tau}{}^{market}(D_{t+\tau})\right)\middle|\pi_t, D_t\right) \end{aligned}$$

It follows that the influence of the sources of risk can be discussed in the same way as in the complete information discounted cash flow case (4-52). The sources of risk that determine $D_{t+\tau}{}^{market}(D_{t+\tau})$ are the path of regimes $S_{t,t+\tau-1}$ and the path of factors and residuals $fe_{t+1,t+\tau}$; these sources of risk are potentially priced in a risk-adjusted way. The sources of risk that do not determine $D_{t+\tau}{}^{market}$ certainly include the regime $S_{t+\tau}$ and, therefore, $S_{t+\tau}$ is not priced. In addition to the irrelevance of $S_{t+\tau}$, it can also be argued that not all aspects of factors and residuals fe_{t+1}

[27] Veronesi (2000) uses a continuous-time framework whereas (4-56) is derived in discrete time. Strictly speaking, (4-56) is a generalization of a discrete-time analogue of Veronesi (2000).

cause correlations with the stochastic discount factor. Recall from the complete information case that, first, not the paths of regimes $S_{t,t+\tau-1}$ as well as factors and residuals $fe_{t+1,t+\tau}$ as such exert influence on prices but only their joint effects on ${D_{t+\tau}}^{market}$ via $D_{t+\tau-1}$ and $S_{t+\tau-1}$ are of interest. In particular, the unobservable regime S_t per se does not matter; however, the fact that S_t is unobservable means that it is integrated into the distribution of $S_{t+\tau-1}$ and $D_{t+\tau-1}$ conditional on information relevant to pricing. Second, $fe_{t+\tau}$ is only priced insofar as it determines aggregate cash flows whereas its influence on individual cash flows is not priced.

Although it is now clear that non-zero covariances with the stochastic discount factor and, thus, risk-adjusted pricing, can only be introduced by $D_{t+\tau-1}$, $S_{t+\tau-1}$, and $fe_{t+\tau}$, it is still unclear through which channels such correlations are introduced by $D_{t+\tau-1}$, $S_{t+\tau-1}$, and $fe_{t+\tau}$: only a generalized version of the cash flow channel of the quasi-static model exists. However, this channel is by far more complex and, hence, needs a more thorough analysis. This can be done by considering the covariance of the stochastic discount factor ${q_{t,t+\tau}}^{iI}$ with cash flows $D_{t+\tau}$ in more detail (for a derivation, see Appendix A3.2):

4-66

$$cov\left(\frac{1}{(1+\rho)^{\tau}}\cdot\frac{U'\left(\frac{1}{n_I}\cdot {D_{t+\tau}}^{market}(D_{t+\tau})\right)}{U'\left(\frac{1}{n_I}\cdot {D_t}^{market}(D_t)\right)},D_{t+\tau}\middle|\pi_t,D_t\right)=\frac{1}{(1+\rho)^{\tau}}\cdot\frac{1}{U'\left(\frac{1}{n_I}\cdot {D_t}^{market}(D_t)\right)}\cdot$$

$$\left\{E\left(cov\left(U'\left(\frac{1}{n_I}\cdot {D_{t+\tau}}^{market}(D_{t+\tau})\right)\cdot D_{t+\tau}\middle|S_{t+\tau-1},D_{t+\tau-1}\right)\middle|\pi_t,D_t\right)\right.$$

$$\left.+\,cov\left(E\left(U'\left(\frac{1}{n_I}\cdot {D_{t+\tau}}^{market}(D_{t+\tau})\right)\middle|S_{t+\tau-1},D_{t+\tau-1}\right),E(D_{t+\tau}|S_{t+\tau-1},D_{t+\tau-1})\middle|\pi_t,D_t\right)\right\}$$

The interpretation is largely analogous to the complete information case. The first channel is the covariance of the stochastic discount factor and cash flows due to $fe_{t+\tau}$ conditional on $S_{t+\tau-1},D_{t+\tau-1}$ where $S_{t+\tau-1}$ and $D_{t+\tau-1}$ are averaged out. The second channel consists of the covariance due to $S_{t+\tau-1},D_{t+\tau-1}$ alone where $fe_{t+\tau}$ is averaged out. Note only one difference to the complete information case: the distribution of $D_{t+\tau-1}$, $S_{t+\tau-1}$ is now conditional on π_t,D_t because the current regime S_t is unobservable.

4.2.1.2.3 Equilibrium Risk Premia

In the special case of incomplete information, the result on risk premia for general cash flow models and information scenarios (4-38) (p. 107) specializes to

4-67

$$RP_t{}^{iI}(\pi_t, D_t) = -cov\left(\begin{matrix} AfR_{t,t+1}{}^{iI}(fe_{t+1}, S_t, \pi_t, D_t) \\ , E\left(P_{t+1}{}^{iI}(\pi_{t+1}, D_{t+1}) + D_{t+1} \middle| \pi_t, D_t, D_{t+1}{}^{market}\right) \end{matrix} \middle| \pi_t, D_t \right)$$

with

$$AfR_{t,t+1}{}^{iI}(fe_{t+1}, S_t, \pi_t, D_t) \equiv \frac{U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}(D_{t+1})\right)}{\sum_{s=1}^{K} \pi_{s,t} \cdot E\left(U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}(D_{t+1})\right) \middle| S_t = s, D_t\right)}$$

$$D_{t+1} = D(D_t, S_t, fe_{t+1})$$

$$\pi_{t+1} = \Pi(\pi_t, D_t, D_{t+1}, Sig_{t+1})$$

$$Sig_{t+1} = Sig_{t+1}(S_t, S_{t+1}, fe_{t+1}, \eta_{t+1})$$

Risk premia arise from the unknown regime S_t as well as from factors and residuals fe_{t+1}, insofar as they influence aggregate cash flows $D_{t+1}{}^{market}$; insofar as the unknown regime S_t as well as from factors and residuals fe_{t+1} only influence the structure of cash flows D_{t+1} they do not give rise to a risk premium. Moreover, for a similar reason there is no risk premium on signal noise $\eta_{Sig,t+1}$ and the new regime S_{t+1}

4.2.1.2.4 Risk Decomposition and Consequences to Prices and Risk Premia

Parallel to partial equilibrium (Section 3.3.2.1.3.4), risk is decomposed into "inter- and intra-distribution risk". In this connection, both the quasi static and the discounted cash flow case are considered separately.

4.2.1.2.4.1 Quasi Static Case

4.2.1.2.4.1.1 Decomposition of Risky Asset Prices and Cash Flows

"Inter-distribution risk" covers the fact that under incomplete information the true probability law at time t governing future cash flows and prices is not known; it is caused by the current regime S_t (see p. 64). "Intra-distribution risk" refers to time $t+1$ in that it determines the realizations of cash flows and prices at time $t+1$ given the regime at time t. It results from factors and residuals fe_{t+1}, the regime at time $t+1$, S_{t+1}, and signal noise η_{t+1} (see p. 65). As opposed to the partial equilibrium case, in general equilibrium the risk due to S_{t+1} and η_{t+1} is not priced. Hence, "intra-distribution risk" coincides with risk from factors and residuals fe_{t+1}.

Clearly, it will not make sense from an economic perspective to decompose those parts of asset prices that have a price of zero. These parts have already been identified in (4-60) (p. 117)

4-60

$$P_{t+1}{}^{iI}(\pi_{t+1},D_{t+1})+D_{t+1}=\underbrace{E\left(P_{t+1}{}^{iI}(\pi_{t+1},D_{t+1})+D_{t+1}\middle|\pi_t,D_t,D_{t+1}{}^{market}(D_{t+1})\right)}_{part\ (i)}+\underbrace{\Delta_{t+1}}_{part\ (ii)}$$

with

$$\Delta_{t+1}\equiv P_{t+1}{}^{iI}(\pi_{t+1},D_{t+1})+D_{t+1}-E\left(P_{t+1}{}^{iI}(\pi_{t+1},D_{t+1})+D_{t+1}\middle|\pi_t,D_t,D_{t+1}{}^{market}(D_{t+1})\right)$$

and

$$D_{t+1}=D(D_t,S_t,fe_{t+1})$$
$$\pi_{t+1}=\Pi(\pi_t,D_t,D_{t+1},Sig_{t+1})$$
$$Sig_{t+1}=Sig_{t+1}(S_t,S_{t+1},fe_{t+1},\eta_{t+1})$$

For that reason, only part (i) is relevant to the risk decomposition where the decomposition itself parallels the partial equilibrium case

4-68

$$E\left(P_{t+1}{}^{iI}(\pi_{t+1},D_{t+1})+D_{t+1}\middle|\pi_t,D_t,D_{t+1}{}^{market}(D_{t+1})\right)-E\left(P_{t+1}{}^{iI}+D_{t+1}\middle|\pi_t,D_t\right)=$$
$$\underbrace{\begin{Bmatrix}E\left(P_{t+1}{}^{iI}(\pi_{t+1},D_{t+1})+D_{t+1}\middle|\pi_t,D_t,D_{t+1}{}^{market}(D_{t+1})\right)\\-E\left(\left(EP_{t+1}{}^{iI}+ED_{t+1}\right)(S_t,fe_{t+1};\pi_t,D_t)\middle|\pi_t,D_t,S_t\right)\end{Bmatrix}}_{\Delta^{comb.risk}}$$
$$+\underbrace{\left\{E\left(\left(EP_{t+1}{}^{iI}+ED_{t+1}\right)(S_t,fe_{t+1};\pi_t,D_t)\middle|\pi_t,D_t,S_t\right)-E\left(P_{t+1}{}^{iI}+D_{t+1}\middle|\pi_t,D_t\right)\right\}}_{\Delta^{exp.risk}}$$

with

$$\left(EP_{t+1}{}^{iI}+ED_{t+1}\right)(S_t,fe_{t+1};\pi_t,D_t)\equiv E\left(P_{t+1}{}^{iI}(\pi_{t+1},D_{t+1})+D_{t+1}\middle|\pi_t,D_t,D_{t+1}{}^{market}(D_{t+1})\right)$$

4.2.1.2.4.1.2 Pricing of the Parts of the Decomposition

The pricing of $\Delta^{comb.risk}$ and $\Delta^{exp..risk}$ is so similar to the partial equilibrium case that it suffices to merely state the results:

<u>Price of “expectation risk”</u>

4-69

$$E\left(q_{t,t+1}{}^{iI}(fe_{t+1},S_t,D_t)\cdot\Delta^{exp.risk}\middle|\pi_t,D_t\right)$$
$$=-\frac{1}{1+r_t{}^{iI}(\pi_t,D_t)}\cdot\sum_{s=1}^{K}\{\pi_{s,t}-\theta_{s,t}(\pi_t,D_t)\}\cdot E\left(\left(EP_{t+1}{}^{iI}+ED_{t+1}\right)(S_t,fe_{t+1};\pi_t,D_t)\middle|\pi_t,D_t,S_t=s\right)$$

with risk-neutralized regime probabilities

4-70

$$\theta_{s,t}(\pi_t,D_t)\equiv\frac{\pi_{s,t}\cdot E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}\left(D(D_t,S_t,fe_{t+1})\right)\right)\middle|S_t=s,D_t\right)}{\sum_{s'=1}^{K}\pi_{s',t}\cdot E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}\left(D(D_t,S_t,fe_{t+1})\right)\right)\middle|S_t=s',D_t\right)}$$

<u>Price of “combined risk”</u>

4-71

$$E\left(q_{t,t+1}{}^{iI}(fe_{t+1},S_t,D_t)\cdot\Delta^{comb.risk}\middle|\pi_t,D_t\right)$$
$$=\frac{1}{1+r_t{}^{iI}(\pi_t,D_t)}\cdot\sum_{s=1}^{K}\theta_{s,t}(\pi_t,D_t)\cdot E\left(AfR_t{}^{conditional}(S_t;fe_{t+1},D_t)\cdot\Delta^{comb.risk}\middle|\pi_t,D_t,S_t=s\right)$$

with

$$AfR_t{}^{conditional}(S_t;fe_{t+1},D_t)\equiv AfR_{t,t+1}{}^{cI}(fe_{t+1},S_t,D_t)=\frac{U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right)\middle|S_t,D_t\right)}$$

Based on (4-70) and (4-71), it becomes clear how “inter-distribution risk” and “intra- distribution risk” are priced: “inter-distribution risk” is priced via risk-neutralized regime probabilities $\theta_t(\pi_t,D_t)$ that are elements in both the prices of “expectation risk” and “combined risk”. “Intra-distribution risk” is priced via a conditional adjustment for risk and only appears in the price of “combined risk”. In this connection, the most notable difference to the corresponding partial equilibrium occurs. The conditional adjustment for risk now exactly coincides with the complete information conditional adjustment for risk.

4.2.1.2.4.2 Discounted Cash Flow Case

4.2.1.2.4.2.1 Decomposition of Risky Cash Flows

Since the discounted cash flow case contains different sources of risk than the quasi-static case, the sources of risk of the discounted cash flow case must be categorized into "inter- and intra-distribution risk". According to (4-64) not all sources of risk are relevant to pricing, but only the following elements: $D_{t+\tau-1}$, $S_{t+\tau-1}$, and $fe_{t+\tau}$. These three elements must be categorized.
"Intra-distribution risk" refers to future points in time which is why $D_{t+\tau-1}$, $S_{t+\tau-1}$, and $fe_{t+\tau}$ belong to "intra-distribution risk". "Inter-distribution risk" refers to time t and, hence, consists of the unobservable current regime S_t only. Compared to the quasi static case, however, S_t only plays a minor role because it is only indirectly relevant through the distribution of $S_{t+\tau-1}, D_{t+\tau-1}$ conditional on information relevant to pricing, i.e.,

4-72

$$\varphi^{iI}(S_{t+\tau-1}, D_{t+\tau-1}|\pi_t, D_t) = \sum_{s=1}^{K} \varphi^{cI}(S_{t+\tau-1}, D_{t+\tau-1}|S_t = s, D_t) \cdot \pi_{s,t}$$

where $\pi_{s,t}$ denotes the conditional regime probability for regime s, φ^{iI} the probability (or density) of $S_{t+\tau-1}, D_{t+\tau-1}$ conditional on π_t, D_t, and φ^{cI} the probability (or density) of $S_{t+\tau-1}, D_{t+\tau-1}$ conditional on S_t, D_t (i.e., information relevant to pricing under complete information).

It is known from (4-65) that only $E\left(D_{t+\tau}\middle|\pi_t, D_t, D_{t+\tau}{}^{market}\right)$ is priced and, hence, relevant to the risk decomposition. What is still unclear is the exact criterion according to which risk is decomposed. Since S_t is only of indirect relevance, it is more promising to decompose cash flows into parts relating to the sources of risk $S_{t+\tau-1}, D_{t+\tau-1}$ and $fe_{t+\tau}$. These sources of risk can be grouped together into (i) $S_{t+\tau-1}$ and $D_{t+\tau-1}$ and (ii) $fe_{t+\tau}$ based on their reference to time and, thus, mimic the idea of the decomposition in the quasi-static case:

4-73

$$\begin{aligned} E\left(D_{t+\tau}\middle|\pi_t, D_t, D_{t+\tau}{}^{market}\right) - E(D_{t+\tau}|\pi_t, D_t) = \\ \left\{\underbrace{E\left(D_{t+\tau}\middle|\pi_t, D_t, D_{t+\tau}{}^{market}\right) - E\left(E\left(D_{t+\tau}\middle|\pi_t, D_t, D_{t+\tau}{}^{market}\right)\middle|\pi_t, D_t, S_{t+\tau-1}, D_{t+\tau-1}\right)}_{\Delta^{comb.risk}}\right\} \\ + \left\{\underbrace{E\left(E\left(D_{t+\tau}\middle|\pi_t, D_t, D_{t+\tau}{}^{market}\right)\middle|\pi_t, D_t, S_{t+\tau-1}, D_{t+\tau-1}\right) - E(D_{t+\tau}|\pi_t, D_t)}_{\Delta^{exp.risk}}\right\} \end{aligned}$$

4.2.1.2.4.2.2 Pricing of the Parts of the Decomposition

Applying the stochastic discount factor to each of the three summands on the right-hand side of

$$E\left(D_{t+\tau}\middle|\pi_t, D_t, D_{t+\tau}{}^{market}\right) = E(D_{t+\tau}|\pi_t, D_t) + \Delta^{exp.risk} + \Delta^{comb.risk}$$

leads to a decomposition of the present value of $D_{t+\tau}$ into the prices of three components.

The expectations of cash flows $D_{t+\tau}$ conditional on the information relevant to pricing π_t, D_t are simply priced through riskless discounting. Although there is no explicit multi-period riskless asset, a multi-period riskless discount factor can still be computed and is given by

4-74

$$q_{t,t+\tau}{}^{iI,riskless}(\pi_t, D_t) \equiv \frac{1}{(1+\rho)^\tau} \cdot \frac{E\left(U'\left(\frac{1}{n_I} \cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right)\middle|\pi_t, D_t\right)}{U'\left(\frac{1}{n_I} \cdot D_t{}^{market}(D_t)\right)}$$

Therefore, the price of expected cash flows reads

4-75

$$E\left(q_{t,t+\tau}{}^{iI} \cdot E(D_{t+\tau}|\pi_t, D_t)\middle|\pi_t, D_t\right) = q_{t,t+\tau}{}^{iI,riskless}(\pi_t, D_t) \cdot E(D_{t+\tau}|\pi_t, D_t)$$

<u>Price of “expectation risk”</u>

Pricing “expectation risk“ yields (see Appendix A3.5.2.1.2.1)

4-76

$$\begin{aligned} &E\left(q_{t,t+\tau}{}^{iI} \cdot \Delta^{exp.risk}\middle|\pi_t, D_t\right) \\ &\qquad = E\left(q_{t,t+\tau}{}^{iI}\middle|\pi_t, D_t\right) \\ &\qquad \cdot \left\{E^{rn}\left(E\left(\widehat{D}_{t+\tau}\middle|\pi_t, D_t, S_{t+\tau-1}, D_{t+\tau-1}\right)\middle|\pi_t, D_t\right) - E(E(D_{t+\tau}|S_{t+\tau-1}, D_{t+\tau-1})|\pi_t, D_t)\right\} \end{aligned}$$

with cash flows explained by aggregate cash flows

$$\widehat{D}_{t+\tau} \equiv E\left(D_{t+\tau}\middle|\pi_t, D_t, D_{t+\tau}{}^{market}\right)$$

where E^{rn} denotes the expectation taken with respect to the risk-neutralized probability (or density) of $S_{t+\tau-1}, D_{t+\tau-1}$ conditional on π_t, D_t, i.e., based on (4-72):

4-77

$$\varphi^{rn,iI}(S_{t+\tau-1}, D_{t+\tau-1}|\pi_t, D_t) \equiv \varphi^{iI}(S_{t+\tau-1}, D_{t+\tau-1}|\pi_t, D_t) \cdot \frac{E\left(U'\left(\frac{1}{n_I} \cdot D_{t+\tau}\right)\middle|S_{t+\tau-1}, D_{t+\tau-1}\right)}{E\left(U'\left(\frac{1}{n_I} \cdot D_{t+\tau}\right)\middle|\pi_t, D_t\right)}$$

With $S_{t+\tau-1}, D_{t+\tau-1}$ in place of the unobservable regime S_t from the quasi-static case, the economic interpretation of the pricing of “expectation risk” in the discounted cash flow case (4-76) is analogous to the quasi-static case (4-69).

Price of "combined risk"

Pricing "combined risk", yields (see Appendix A3.5.2.1.2.2)

4-78

$$E\left(q_{t,t+\tau}{}^{iI}\cdot\Delta^{comb.risk}\middle|\pi_t,D_t\right)$$
$$=E\left(q_{t,t+\tau}{}^{iI}\middle|\pi_t,D_t\right)$$
$$\cdot E^{rn}\left(E\left(\frac{U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|S_{t+\tau-1},D_{t+\tau-1}\right)}\cdot\Delta^{comb.risk}\middle|S_{t+\tau-1},D_{t+\tau-1},\pi_t,D_t\right)\middle|\pi_t,D_t\right)$$

To answer the initial question of this section, namely how "inter-distribution risk" and "intra- distribution risk" are priced and whether their price effects can be separated, it can be stated: "Intra-distribution risk" equals the price of "expectation risk" and "combined risk", i.e., is priced via a conditional adjustment for risk and risk-neutralized probability (or density) of $S_{t+\tau-1},D_{t+\tau-1}$.

To make a pricing statement on "inter-distribution risk", the influence of the unobservable regime S_t must be made explicit. For that reason the risk-neutralized density is written as

4-79

$$\varphi^{rn,iI}(S_{t+\tau-1},D_{t+\tau-1}|\pi_t,D_t)=\sum_{s=1}^{K}\varphi^{rn,cI}(S_{t+\tau-1},D_{t+\tau-1}|S_t=s,D_t)\cdot\theta_s{}^{+\tau}(\pi_t,D_t)$$

where $\varphi^{rn,cI}(S_{t+\tau-1},D_{t+\tau-1}|S_t=s,D_t)$ denotes the risk-neutralization of $\varphi^{cI}(S_{t+\tau-1},D_{t+\tau-1}|S_t=s,D_t)$ and $\theta_s{}^{+\tau}(\pi_t,D_t)$ the risk-neutralization of $\pi_{s,t}$, the conditional probability of $S_t=s$.

4-80

$$\theta_s{}^{+\tau}(\pi_t,D_t)\equiv\pi_{s,t}\cdot\frac{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|S_t=s,D_t\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|\pi_t,D_t\right)}$$

The unobservable regime S_t as the source of "inter-distribution risk" is averaged out to obtain the (multi-dimensional) marginal distribution of $S_{t+\tau-1},D_{t+\tau-1}$ (conditional on information relevant to pricing), but with respect to risk-neutralized (rather than empirical) probabilities. This means that the unobservable regime S_t is at least indirectly priced as an element of the risk-neutralized distribution of $S_{t+\tau-1},D_{t+\tau-1}$.

4.2.1.2.4.3 Decomposition of Risk Premia

Incomplete information risk premia expressed as the sum of an "expectation risk" and a "combined risk" component read

4-81

$$RP_t{}^{iI}(\pi_t, D_t) = \sum_{s=1}^{K} \{\pi_{s,t} - \theta_{s,t}(\pi_t, D_t)\} \cdot E\left((EP_{t+1}{}^{iI} + ED_{t+1})(S_t, fe_{t+1}; \pi_t, D_t) \middle| \pi_t, D_t, S_t = s \right) + \sum_{s=1}^{K} \theta_{s,t}(\pi_t, D_t) \cdot RP_t{}^{conditional}(s; \pi_t, D_t)$$

with

$$RP_t{}^{conditional}(s; \pi_t, D_t) \equiv -cov\left(AfR_t{}^{conditional}(S_t; fe_{t+1}, D_t), \Delta^{comb.risk} \middle| \pi_t, D_t, S_t = s \right)$$

Analogous to the price formulation, "inter-distribution risk" is priced via risk-neutralized regime probabilities $\theta_t(\pi_t, D_t)$ that are elements in both the prices of "expectation risk" and "combined risk". "Intra-distribution risk" is priced via a conditional risk premium and only appears in the part of the risk premium attributable to "combined risk" where the conditional adjustment for risk now exactly coincides with the complete information conditional adjustment for risk.

4.2.2 Not All Assets Pay Cash Flows in Every Period: Information Frequency ≥ Cash Flow Frequency

4.2.2.1 Cash Flow Model

The cash flow model is the same as in the corresponding partial equilibrium case (Section 3.3.2.2):

$$D_{t+1}{}^{(1)} = D^{(1)}\left(D_t{}^{(1)}, S_t, fe_{t+1}\right)$$

In addition, there are n_{Δ_C} (Δ_C)-periodic assets which pay cash flows every Δ_C periods at the points of time $t_{(k)} \equiv k \cdot \Delta_C, k \epsilon \mathbb{N}_0$ and are described by the cash flow model

$$D_{t_{(k+1)}}{}^{(\Delta_C)} = D^{(\Delta_C)}\left(D_{t_{(k)}}{}^{(\Delta_C)}, S_{t_{(k)}, t_{(k+1)}-1}, fe_{t_{(k+1)}}\right)$$

In contrast to the partial equilibrium case it must be assumed that there is at least one asset paying cash flows at every point of time, i.e., $n_1 > 0$ because otherwise equilibrium consumption will be zero.

4.2.2.2 Complete Information

4.2.2.2.1 Information Relevant to Pricing $z_t{}^p$

For general cash flow models under complete information where information frequency is higher than or equal to cash flow frequency, information relevant to pricing reads:

4-82

$$z_t{}^{cI,(\Delta_C),p} = \left(S_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)$$

To see that $\left(S_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)$ contains all information relevant to pricing, observe that it describes (i) cash flows paid by the market portfolio of risky assets, $D_t{}^{market}$, and (ii) also is a sufficient statistic for the conditional distribution of future cash flows. (i) follows from the composition of aggregate cash flows: aggregate cash flows $D_t{}^{market}$ are a function of $D_t{}^{(1)}$ and, if $t = t_{(k_t)}$, of $D_{t_{(k_t)}}{}^{(\Delta_C)}$. The argument to justify (ii) is the same as in the partial equilibrium case (3-80) (p. 69).

4.2.2.2.2 Equilibrium Asset Prices

4.2.2.2.2.1 Quasi Static Asset Prices

In the special case of complete information where information frequency is higher than or equal to cash flow frequency, the (quasi static) pricing results for general cash flow models and information scenarios (4-36) and (4-37) specialize to

Prices of (Δ_C)-periodic risky assets

4-83

$$P_t{}^{cI,(\Delta_C)}\left(S_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) = E\left(\begin{array}{c} q_{t,t+1}{}^{cI,(\Delta_C)}\left(fe_{t+1}, S_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) \\ \cdot\left\{P_{t+1}{}^{cI,(\Delta_C)} + D_{t+1}{}^{(\Delta_C)}\right\}\end{array}\middle| S_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)$$

with

$$k = k_t$$

$$D_{t+1}{}^{(\Delta_C)} = \begin{cases} D^{(\Delta_C)}\left(D_{t_{(k)}}{}^{(\Delta_C)}, S_{t_{(k)},t}, fe_{t+1}\right) & t = t_{(k+1)} - 1 \\ 0 & t < t_{(k+1)} - 1 \end{cases}$$

$$P_{t+1}{}^{cI,(\Delta_C)} = \begin{cases} P_{t+1}{}^{cI,(\Delta_C)}\left(S_{t+1,t+1}, D_{t+1}{}^{(\Delta_C)}, D_{t+1}{}^{(1)}\right) & t = t_{(k+1)} - 1 \\ P_{t+1}{}^{cI,(\Delta_C)}\left(S_{t_{(k)},t+1}, D_{t_{(k)}}{}^{(\Delta_C)}, D_{t+1}{}^{(1)}\right) & t < t_{(k+1)} - 1 \end{cases}$$

<u>Prices of (1)-periodic risky assets</u>

4-84

$$P_t^{cI,(1)}\left(S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta c)}, D_t^{(1)}\right) = E\left(\begin{matrix} q_{t,t+1}^{cI,(\Delta c)}\left(fe_{t+1}, S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta c)}, D_t^{(1)}\right) \\ \cdot \left\{P_{t+1}^{cI,(1)} + D_{t+1}^{(1)}\right\}\end{matrix}\middle| S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta c)}, D_t^{(1)}\right)$$

with

$$P_{t+1}^{cI,(1)} = \begin{cases} P_{t+1}^{cI,(1)}\left(S_{t_{(k+1)},t_{(k+1)}}, D_{t_{(k+1)}}^{(\Delta c)}, D_{t_{(k+1)}}^{(1)}\right) & t+1 = t_{(k+1)} \\ P_{t+1}^{cI,(1)}\left(S_{t_{(k)},t+1}, D_{t_{(k)}}^{(\Delta c)}, D_{t+1}^{(1)}\right) & t+1 \neq t_{(k+1)} \end{cases}$$

<u>Price of the one-period riskless bond</u>

4-85

$$B_t^{cI,(\Delta c)}\left(S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta c)}, D_t^{(1)}\right) = E\left(q_{t,t+1}^{cI,(\Delta c)}\left(fe_{t+1}, S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta c)}, D_t^{(1)}\right)\middle| S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta c)}, D_t^{(1)}\right)$$

where the precise form of the stochastic discount factor depends on the position of t within the cash flow period $t_{(k)}, t_{(k+1)} - 1$:

4-86

$$q_{t,t+1}^{cI,(\Delta c)}\left(fe_{t+1}, S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta c)}, D_t^{(1)}\right) = \frac{1}{1+\rho}\cdot\frac{U'\left(\frac{1}{n_I}\cdot D_{t+1}^{market}\right)}{U'\left(\frac{1}{n_I}\cdot D_t^{market}\right)}$$

$$= \begin{cases} q_{t,t+1}^{cI,(\Delta c)}\left(fe_{t+1}, S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta c)}, D_t^{(1)}\right) & t = t_{(k+1)} - 1 \\ q_{t,t+1}^{cI,(\Delta c)}\left(fe_{t+1}, S_t, D_t^{(1)}\right) & t_{(k)} < t+1 < t_{(k+1)} \\ q_{t,t+1}^{cI,(\Delta c)}\left(fe_{t+1}, S_t, D_t^{(\Delta c)}, D_t^{(1)}\right) & t_{(k)} = t \end{cases}$$

with

$$D_{t+1}^{market} = \begin{cases} D_{t+1}^{market}\left(D_{t+1}^{(\Delta c)}, D_{t+1}^{(1)}\right) & t = t_{(k+1)} - 1 \\ D_{t+1}^{market}\left(D_{t+1}^{(1)}\right) & t_{(k)} \le t < t_{(k+1)} - 1 \end{cases}$$

$$D_t^{market} = \begin{cases} D_t^{market}\left(D_t^{(\Delta c)}, D_t^{(1)}\right) & t_{(k)} < t < t_{(k+1)} \\ D_t^{market}\left(D_t^{(\Delta c)}, D_t^{(1)}\right) & t_{(k)} = t \end{cases}$$

Similar to the partial equilibrium case there are two types of asset prices for risky assets because there are two different cash flow structures. Both prices are interdependent through the common stochastic discount factor.

In addition, the pricing of the two sources of risk can be analyzed with the help of (4-83) and (4-84). From the perspective of time t the sources of risk are (i) regimes S_{t+1}, the only stochastic element of the regime path $S_{t_{(k)},t+1}$ or, if $t+1 = t_{(k+1)}$, $S_{t_{(k+1)},t_{(k+1)}}$, and (ii) factors and residuals fe_{t+1}.

Given these identified sources of risk in the case where information frequency is higher than or equal to cash flow frequency, analogous pricing results hold to the case where information frequency equals cash flow frequency: S_{t+1} is not priced. Insofar as fe_{t+1} contributes to aggregate cash flows, it is priced in a risk-adjusted way, but if insofar as it determines individual (but not aggregate) cash flows, it is priced risk-neutrally. – The transmission channels are de facto the same as in the case where information frequency equals cash flow frequency.

Concerning the price of the one-period riskless bond (4-85), the irrelevance of regimes S_{t+1} for pricing is clear because cash flows are riskless and S_{t+1} does not contribute to the stochastic discount factor. Factors and residuals fe_{t+1}, on the other hand, enter the price of the one-period riskless bond price via the stochastic discount factor. Depending on the position of time t within the cash flow period, the price of the bond further simplifies: if, there are no (Δ_C)-periodic cash flows at time t or $t+1$, the price of the riskless bond is a function of the single regime S_t,and (1)-periodic cash flows ${D_t}^{(1)}$ only. This is a direct consequence of the form of the stochastic discount factor. If t is a payment date of (Δ_C)-periodic cash flows ($t = t_{(k_t)}$), the price of the riskless bond depends on the single regime S_t as well as the cash flows of both types of assets, ${D_t}^{(\Delta_C)}$ and ${D_t}^{(1)}$. Finally, if $t+1$ is a payment date of (Δ_C)-periodic cash flows ($t+1 = t_{(k_t+1)}$), the price of the riskless bond depends on the same information as risky assets, namely $S_{t_{(k)},t}, {D_{t_{(k)}}}^{(\Delta_C)}$ and ${D_t}^{(1)}$.

4.2.2.2.2.2 Asset Prices as Discounted Future Cash Flows

In the special case of complete information where information frequency is higher than or equal to cash flow frequency prices of risky assets expressed as discounted future cash flows (4-39) specialize to two pricing equations, one relating to each group of risky assets:

Prices of (Δ_C)-periodic risky assets

4-87

$$
\begin{aligned}
&{P_t}^{cI,(\Delta_C)}\left({D_{t_{(k)}}}^{(\Delta_C)}, {D_t}^{(1)}, S_{t_{(k)},t+\tau-1}\right) \\
&= E\left(\sum_{\tau=1}^{T-t}\left\{\begin{matrix} {q_{t,t+\tau}}^{cI,(\Delta_C)}\left(fe_{t+1,t+\tau}, S_{t_{(k)},t+\tau-1}, {D_{t_{(k)}}}^{(\Delta_C)}, {D_t}^{(1)}\right) \\ \cdot 1_{t+\tau=t_{(k_{t+\tau})}} \cdot {D_{t+\tau}}^{(\Delta_C)}\left({D_{t_{(k)}}}^{(\Delta_C)}, fe_{t+1,t+\tau}, S_{t_{(k_t)},t+\tau-1}\right)\end{matrix}\right\}\middle| {D_{t_{(k)}}}^{(\Delta_C)}, {D_t}^{(1)}, S_{t_{(k)},t}\right)
\end{aligned}
$$

Prices of (1)-periodic risky assets

4-88

$$
\begin{aligned}
&{P_t}^{cI,(1)}\left({D_{t_{(k)}}}^{(\Delta_C)}, {D_t}^{(1)}, S_{t_{(k)},t+\tau-1}\right) \\
&= E\left(\sum_{\tau=1}^{T-t}\left\{\begin{matrix} {q_{t,t+\tau}}^{cI,(\Delta_C)}\left(fe_{t+1,t+\tau}, S_{t_{(k)},t+\tau-1}, {D_{t_{(k)}}}^{(\Delta_C)}, {D_t}^{(1)}\right) \\ \cdot {D_{t+\tau}}^{(1)}\left({D_t}^{(1)}, fe_{t+1,t+\tau}, S_{t,t+\tau-1}\right)\end{matrix}\right\}\middle| {D_{t_{(k)}}}^{(\Delta_C)}, {D_t}^{(1)}, S_{t_{(k)},t}\right)
\end{aligned}
$$

with

$$k = k_t$$

with multi-period stochastic discount factor

4-89

$$q_{t,t+\tau}{}^{cI,(\Delta_C)}\left(fe_{t+1,t+\tau}, S_{t_{(k)},t+\tau-1}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) = \frac{1}{(1+\rho)^\tau}\cdot\frac{U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\left(D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}, fe_{t+1,t+\tau}, S_{t_{(k)},t+\tau-1}\right)\right)}{U'\left(\frac{1}{n_I}\cdot D_t{}^{market}\right)}$$

where $D_t{}^{market}$ and $D_{t+\tau}{}^{market}$ are given by

$$D_{t'}{}^{market} = \begin{cases} D_{t'}{}^{market}\left(D_{t'}{}^{(\Delta_C)}, D_{t'}{}^{(1)}\right) & t_{(k_{t'})} < t' < t_{(k_{t'}+1)} \\ D_{t'}{}^{market}\left(D_{t'}{}^{(\Delta_C)}, D_{t'}{}^{(1)}\right) & t_{(k_{t'})} = t' \end{cases}$$

$$t' \epsilon \{t, t+\tau\}$$

with cash flows $D_{t+\tau}{}^{(\Delta_C)}$ and $D_{t+\tau}{}^{(1)}$ recursively derived from $D_{t_{(k)}}{}^{(\Delta_C)}$ and $D_t{}^{(1)}$ as functions of the paths of factors and residuals $fe_{t+1,t+\tau}$ as well as the path of regimes $S_{t_{(k)},t+\tau-1}$.

The pricing equations (4-87) and (4-88) demonstrate how the two sources of risk (i) path of future regimes $S_{t+1,t+\tau}$ - note the current regime S_t is non-stochastic - and (ii) path of factors and residuals $fe_{t+1,t+\tau}$ are priced: the pricing is virtually identical to the case where information frequency is equal to cash flow frequency although the timing of cash flows is more complicated because cash flows are no longer paid in each point in time. Recall that in the special case where information frequency equals cash flow frequency, not the complete paths of factors and residuals as well as regimes were directly relevant but only $D_{t+\tau-1}$ and $S_{t+\tau-1}$ as well as $fe_{t+\tau}$. In the current case where information frequency is higher than or equal to cash flow frequency, the relevant aspects of the path generalize to $D_{t_{(k_{t+\tau-1})}}{}^{(\Delta_C)}$, $D_{t+\tau-1}{}^{(1)}$ and $S_{t_{(k_{t+\tau-1})},t+\tau-1}$ as well as $fe_{t+\tau}$.

4.2.2.2.3 Equilibrium Risk Premia

In the special case of complete information where information frequency is higher than or equal to cash flow frequency the result on risk premia for general cash flow models and information scenarios (4-38) takes the following form:

<u>Risk premia of (Δ_C)-periodic risky assets</u>

4-90

$$RP_t^{cI,(\Delta_C)}\left(S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}\right) = -cov\left(\begin{matrix} AfR_t^{cI,(\Delta_C)}\left(fe_{t+1}, S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}\right) \\ \cdot\left\{P_{t+1}^{cI,(\Delta_C)} + D_{t+1}^{(\Delta_C)}\right\} \end{matrix}\middle| S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}\right)$$

with

$$k = k_t$$

$$D_{t+1}^{(\Delta_C)} = \begin{cases} D^{(\Delta_C)}\left(D_{t_{(k)}}^{(\Delta_C)}, S_{t_{(k)},t}, fe_{t+1}\right) & t = t_{(k+1)} - 1 \\ 0 & t < t_{(k+1)} - 1 \end{cases}$$

$$P_{t+1}^{cI,(\Delta_C)} = \begin{cases} P_{t+1}^{cI,(\Delta_C)}\left(S_{t_{(k+1)},t_{(k+1)}}, D_{t_{(k+1)}}^{(\Delta_C)}, D_{t_{(k+1)}}^{(1)}\right) & t = t_{(k+1)} - 1 \\ P_{t+1}^{cI,(\Delta_C)}\left(S_{t_{(k)},t+1}, D_{t_{(k)}}^{(\Delta_C)}, D_{t+1}^{(1)}\right) & t < t_{(k+1)} - 1 \end{cases}$$

<u>Risk premia of (1)-periodic risky assets</u>

4-91

$$RP_t^{cI,(1)}\left(S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}\right) = -cov\left(\begin{matrix} AfR_t^{cI,(\Delta_C)}\left(fe_{t+1}, S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}\right) \\ \cdot\left\{P_{t+1}^{cI,(1)} + D_{t+1}^{(1)}\right\} \end{matrix}\middle| S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}\right)$$

with

$$P_{t+1}^{cI,(1)} = \begin{cases} P_{t+1}^{cI,(1)}\left(S_{t_{(k+1)},t_{(k+1)}}, D_{t_{(k+1)}}^{(\Delta_C)}, D_{t_{(k+1)}}^{(1)}\right) & t+1 = t_{(k+1)} \\ P_{t+1}^{cI,(1)}\left(S_{t_{(k)},t+1}, D_{t_{(k)}}^{(\Delta_C)}, D_{t+1}^{(1)}\right) & t+1 \neq t_{(k+1)} \end{cases}$$

with adjustment for risk

$$AfR_t^{cI,(\Delta_C)}\left(fe_{t+1}, S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}\right) = \frac{U'\left(\frac{1}{n_I}\cdot D_{t+1}^{market}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}^{market}\right)\middle| S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}\right)}$$

$$= \begin{cases} AfR_t^{cI,(\Delta_C)}\left(fe_{t+1}, S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}\right) & t = t_{(k+1)} - 1 \\ AfR_t^{cI,(\Delta_C)}\left(fe_{t+1}, S_t, D_t^{(1)}\right) & t_{(k)} \leq t < t_{(k+1)} - 1 \end{cases}$$

with

$$D_{t+1}^{market} = \begin{cases} D_{t+1}^{market}\left(D_{t+1}^{(\Delta_C)}, D_{t+1}^{(1)}\right) & t = t_{(k+1)} - 1 \\ D_{t+1}^{market}\left(D_{t+1}^{(1)}\right) & t_{(k)} \leq t < t_{(k+1)} - 1 \end{cases}$$

Paralleling the results for asset prices (4-83) and (4-84), (4-90), and (4-91) show that only factors and residuals fe_{t+1} insofar as they influence D_{t+1}^{market}, give rise to a risk premium. The new regime S_{t+1} and the influence of factors and residual fe_{t+1} on structure of D_{t+1} do not contribute to the risk premium.

4.2.2.3 Incomplete Information

4.2.2.3.1 Information Relevant to Pricing ${z_t}^p$

For general cash flow models under incomplete information where information frequency is higher than or equal to cash flow frequency, information relevant to pricing reads:

4-92

$$ {z_t}^{cI,(\Delta_C),p} = \left(\pi_{t_{(k_t)},t}, {D_{t_{(k_t)}}}^{(\Delta_C)}, {D_t}^{(1)}\right) $$

where $\pi_{t_{(k_t)},t}$ are conditional probabilities for the path of regimes since the last payment date for (Δ_C)-periodic cash flows and $t_{(k_t)}$ denotes the most recent cash flow payment date of (Δ_C)-periodic cash flows from the perspective of time t.

The argument is analogous to the complete information case, with conditional regime path probabilities now replacing regime paths as an element of the conditional distribution of future cash flows.

4.2.2.3.2 Equilibrium Asset Prices

4.2.2.3.2.1 Quasi Static Asset Prices

In the special case of incomplete information where information frequency is higher than or equal to cash flow frequency, the (quasi static) pricing results for general cash flow models and information scenarios (4-36) and (4-37) specialize to

Prices of (Δ_C)-periodic risky assets

4-93

$$ {P_t}^{iI,(\Delta_C)}\left(\pi_{t_{(k)},t}, {D_{t_{(k)}}}^{(\Delta_C)}, {D_t}^{(1)}\right) = E\left(\begin{matrix} {q_{t,t+1}}^{iI,(\Delta_C)}\left(fe_{t+1}, S_{t_{(k)},t}, {D_{t_{(k)}}}^{(\Delta_C)}, {D_t}^{(1)}\right) \\ \cdot\left\{{P_{t+1}}^{iI,(\Delta_C)} + {D_{t+1}}^{(\Delta_C)}\right\} \end{matrix}\middle| \pi_{t_{(k)},t}, {D_{t_{(k)}}}^{(\Delta_C)}, {D_t}^{(1)}\right) $$

with

$$ k = k_t $$

$$ {D_{t+1}}^{(\Delta_C)} = \begin{cases} D^{(\Delta_C)}\left({D_{t_{(k)}}}^{(\Delta_C)}, S_{t_{(k)},t}, fe_{t+1}\right) & t = t_{(k+1)} - 1 \\ 0 & t < t_{(k+1)} - 1 \end{cases} $$

$$ {P_{t+1}}^{iI,(\Delta_C)} = \begin{cases} {P_{t+1}}^{iI,(\Delta_C)}\left(\pi_{t+1,t+1}, {D_{t+1}}^{(\Delta_C)}, {D_{t+1}}^{(1)}\right) & t = t_{(k+1)} - 1 \\ {P_{t+1}}^{iI,(\Delta_C)}\left(\pi_{t_{(k)},t+1}, {D_{t_{(k)}}}^{(\Delta_C)}, {D_{t+1}}^{(1)}\right) & t < t_{(k+1)} - 1 \end{cases} $$

$$ \pi_{t_{(k)},t+1} = \Pi_0\left(\pi_{t_{(k)},t+1}, {D_t}^{(1)}, {D_{t+1}}^{(1)}, Sig_{t+1}\right), t+1 \neq t_{(k+1)} $$

$$\pi_{t_{(k+1)},t_{(k+1)}} = \Pi_1\begin{pmatrix}\pi_{t_{(k)},t_{(k+1)}-1}, D_{t_{(k)}}{}^{(\Delta c)}, D_{,t_{(k+1)}-1}{}^{(1)} \\ , D_{t_{(k+1)}}{}^{(\Delta c)}, D_{t_{(k+1)}}{}^{(1)}, Sig_{t_{(k+1)}}\end{pmatrix}$$

$$Sig_{t+1} = Sig_{t+1}\left(S_{t_{(k)},t}, S_{t+1}, fe_{t+1}, \eta_{t+1}\right)$$

Prices of (1)-periodic risky assets

4-94

$$P_t{}^{iI,(1)}\left(\pi_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta c)}, D_t{}^{(1)}\right) = E\left(\begin{matrix} q_{t,t+1}{}^{iI,(\Delta_C)}\left(fe_{t+1}, S_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta c)}, D_t{}^{(1)}\right) \\ \cdot\left\{P_{t+1}{}^{iI,(1)} + D_{t+1}{}^{(1)}\right\}\end{matrix}\middle| \pi_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta c)}, D_t{}^{(1)}\right)$$

with

$$P_{t+1}{}^{iI,(1)} = \begin{cases} P_{t+1}{}^{iI,(1)}\left(\pi_{t+1,t+1}, D_{t+1}{}^{(\Delta c)}, D_{t+1}{}^{(1)}\right) & t = t_{(k+1)} - 1 \\ P_{t+1}{}^{iI,(1)}\left(\pi_{t_{(k)},t+1}, D_{t_{(k)}}{}^{(\Delta c)}, D_{t+1}{}^{(1)}\right) & t+1 \neq t_{(k+1)}\end{cases}$$

Price of the one-period riskless bond

4-95

$$B_t{}^{iI,(\Delta c)}\left(\pi_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta c)}, D_t{}^{(1)}\right) = E\left(q_{t,t+1}{}^{iI,(\Delta c)}\left(fe_{t+1}, S_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta c)}, D_t{}^{(1)}\right)\middle|\pi_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta c)}, D_t{}^{(1)}\right)$$

where the precise form of the stochastic discount factor depends on the position of t within the cash flow period $t_{(k)}, t_{(k+1)} - 1$:

4-96

$$q_{t,t+1}{}^{cI,(\Delta c)}\left(fe_{t+1}, S_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta c)}, D_t{}^{(1)}\right) = \frac{1}{1+\rho}\cdot\frac{U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}\right)}{U'\left(\frac{1}{n_I}\cdot D_t{}^{market}\right)}$$

$$= \begin{cases} q_{t,t+1}{}^{cI,(\Delta c)}\left(fe_{t+1}, S_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta c)}, D_t{}^{(1)}\right) & t = t_{(k+1)} - 1 \\ q_{t,t+1}{}^{cI,(\Delta c)}\left(fe_{t+1}, S_t, D_t{}^{(1)}\right) & t_{(k)} < t+1 < t_{(k+1)} \\ q_{t,t+1}{}^{cI,(\Delta c)}\left(fe_{t+1}, S_t, D_t{}^{(\Delta c)}, D_t{}^{(1)}\right) & t_{(k)} = t\end{cases}$$

with

$$D_{t+1}{}^{market} = \begin{cases} D_{t+1}{}^{market}\left(D_{t+1}{}^{(\Delta c)}, D_{t+1}{}^{(1)}\right) & t = t_{(k+1)} - 1 \\ D_{t+1}{}^{market}\left(D_{t+1}{}^{(1)}\right) & t_{(k)} \le t < t_{(k+1)} - 1\end{cases}$$

$$D_t{}^{market} = \begin{cases} D_t{}^{market}\left(D_t{}^{(\Delta c)}, D_t{}^{(1)}\right) & t_{(k)} < t < t_{(k+1)} \\ D_t{}^{market}\left(D_t{}^{(\Delta c)}, D_t{}^{(1)}\right) & t_{(k)} = t\end{cases}$$

As in the complete information case with information frequency higher than or equal to cash flow frequency, there are two groups of prices that are interdependent. There are two underlying reasons for this interdependence: first, and similar to the complete information case, the stochastic discount factor; second, in addition to the complete information case, an information argument: (1)-periodic cash flows, together with signals, provide information on the true path of regimes that is relevant to pricing (Δ_C)- periodic assets.

In addition, the pricing of the four sources of risk can be analyzed with the help of (4-93) and (4-94). From the perspective of time t the sources of risk are (i) the current regime path $S_{t_{(k)},t}$, (ii) factors and residuals fe_{t+1}, (iii) the regime at time $t+1$, S_{t+1}, as the next addition to the regime path, and (iv) signal noise η_{t+1}.

Given these identified sources of risk in the case where information frequency is higher than or equal to cash flow frequency, analogous pricing results hold to the case where information frequency equals cash flow frequency: S_{t+1} and η_{t+1} are not priced. Insofar as $S_{t_{(k)},t}$ and fe_{t+1} contribute to aggregate cash flows, they are correlated with marginal utility and are priced in a risk-adjusted way. However, insofar as $S_{t_{(k)},t}$ and fe_{t+1} only determine individual (but not aggregate) cash flows, they are one component of the conditional expectation and, thus, priced risk-neutrally. – The transmission channels are de facto the same as in the case where information frequency equals cash flow frequency.

Concerning the price of the one-period riskless bond (4-95), the irrelevance of S_{t+1} and η_{t+1} for pricing is clear because cash flows are riskless and S_{t+1} and η_{t+1} do not contribute to the stochastic discount factor. The current regime path $S_{t_{(k)},t}$ as well as factors and residuals fe_{t+1} enter the price of the one-period riskless bond price via the stochastic discount factor. Depending on the position of time t within the cash flow period, the price of the bond further simplifies: if, there are no (Δ_C)-periodic cash flows at time t or $t+1$, the price of the riskless bond is a function of the probabilities π_t for the single regime S_t, and (1)-periodic cash flows ${D_t}^{(1)}$ only. This is a direct consequence of the form of the stochastic discount factor and the fact that S_t is unobservable. If t is a payment date of (Δ_C)-periodic cash flows ($t = t_{(k_t)}$), the price of the riskless bond probabilities π_t for the single regime S_t as well as the cash flows of both types of assets, ${D_t}^{(\Delta_C)}$ and ${D_t}^{(1)}$. Finally, if $t+1$ is a payment date of (Δ_C)-periodic cash flows ($t+1 = t_{(k_t+1)}$), the price of the riskless bond depends on the same information as risky assets, namely $\pi_{t_{(k)},t}$, ${D_{t_{(k)}}}^{(\Delta_C)}$ and ${D_t}^{(1)}$.

4.2.2.3.2.2 Asset Prices as Discounted Future Cash Flows

In the special case of incomplete information where information frequency is higher than or equal to cash flow frequency, prices of risky assets expressed as discounted future cash flows (4-39) specialize to two pricing equations, one relating to each group of risky assets:

Prices of (Δ_C)-periodic risky assets

4-97

$$P_t^{iI,(\Delta_C)}\left(D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}, \pi_{t_{(k)},t}\right)$$

$$= \sum_{s_{t_{(k)},t}} \pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right)$$

$$\cdot E\left(\sum_{\tau=1}^{T-t}\left\{\begin{matrix} q_{t,t+\tau}^{iI,(\Delta_C)}\left(fe_{t+1,t+\tau}, S_{t_{(k)},t+\tau-1}, D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}\right) \\ \cdot 1_{t+\tau=t_{(k_{t+\tau})}} \cdot D_{t+\tau}^{(\Delta_C)}\left(D_{t_{(k)}}^{(\Delta_C)}, fe_{t+1,t+\tau}, S_{t_{(k_t)},t+\tau-1}\right)\end{matrix}\right\}\middle| D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}, S_{t_{(k)},t} = s_{t_{(k)},t}\right)$$

Prices of (1)-periodic risky assets

4-98

$$P_t^{iI,(1)}\left(D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}, \pi_{t_{(k)},t}\right)$$

$$= \sum_{s_{t_{(k)},t}} \pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right)$$

$$\cdot E\left(\sum_{\tau=1}^{T-t}\left\{\begin{matrix} q_{t,t+\tau}^{iI,(\Delta_C)}\left(fe_{t+1,t+\tau}, S_{t_{(k)},t+\tau-1}, D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}\right) \\ \cdot D_{t+\tau}^{(1)}\left(D_t^{(1)}, fe_{t+1,t+\tau}, S_{t,t+\tau-1}\right)\end{matrix}\right\}\middle| D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}, S_{t_{(k)},t} = s_{t_{(k)},t}\right)$$

with

$$k = k_t$$

with multi-period stochastic discount factor

4-99

$$q_{t,t+\tau}^{iI,(\Delta_C)}\left(fe_{t+1,t+\tau}, S_{t_{(k)},t+\tau-1}, D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}\right)$$

$$= \frac{1}{(1+\rho)^{\tau}} \cdot \frac{U'\left(\frac{1}{n_I} \cdot D_{t+\tau}^{market}\left(D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}, fe_{t+1,t+\tau}, S_{t_{(k)},t+\tau-1}\right)\right)}{U'\left(\frac{1}{n_I} \cdot D_t^{market}\right)}$$

where D_t^{market} and $D_{t+\tau}^{market}$ are given by

$$D_{t'}^{market} = \begin{cases} D_{t'}^{market}\left(D_{t'}^{(\Delta_C)}, D_{t'}^{(1)}\right) & t_{(k_{t'})} < t' < t_{(k_{t'}+1)} \\ D_{t'}^{market}\left(D_{t'}^{(\Delta_C)}, D_{t'}^{(1)}\right) & t_{(k_{t'})} = t' \end{cases}$$

$$t' \epsilon \{t, t+\tau\}$$

with cash flows $D_{t+\tau}^{(\Delta_C)}$ and $D_{t+\tau}^{(1)}$ recursively derived from $D_{t_{(k)}}^{(\Delta_C)}$ and $D_t^{(1)}$ as functions of the paths of factors and residuals $fe_{t+1,t+\tau}$ as well as the path of regimes $S_{t_{(k)},t+\tau-1}$.

Observe that signal noise η_{t+1} does not enter the present value equations (4-97) and (4-98). The reason is that signal noise affects prices, but not cash flows. For that reason, only the pricing influence of the three other sources of risk, (i) the current regime path $S_{t_{(k)},t}$,(ii) path of factors and residuals $fe_{t+1,t+\tau}$, (iii) the regime at time $t+1$ S_{t+1} as the next addition to the regime path must be analyzed. The results parallel those obtained in the cases that have been analyzed so far: the relevant aspects of the paths of factors and residuals as well as regimes are $D_{t_{(k_{t+\tau-1})}}{}^{(\Delta_C)}$, $D_{t+\tau-1}{}^{(1)}$ and $S_{t_{(k_{t+\tau-1})},t+\tau-1}$ as well as $fe_{t+\tau}$. In particular, the unobservable regime S_t per se does not matter; however, the fact that S_t is unobservable means that it is integrated into the distribution of $D_{t_{(k_{t+\tau-1})}}{}^{(\Delta_C)}$, $D_{t+\tau-1}{}^{(1)}$ and $S_{t_{(k_{t+\tau-1})},t+\tau-1}$ conditional on information relevant to pricing. Second, $fe_{t+\tau}$ is only priced insofar as it determines aggregate cash flows whereas its influence on individual cash flows is not priced.

4.2.2.3.3 Equilibrium Risk Premia

In the special case of incomplete information, where information frequency is higher than or equal to cash flow frequency, the result on risk premia for general cash flow models and information scenarios (4-38) takes the following form:

Risk premia of (Δ_C)-periodic risky assets

4-100

$$RP_t{}^{iI,(\Delta_C)}\left(\pi_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) = \left\{-cov\left(\begin{matrix} AfR_t{}^{iI,(\Delta_C)}\left(fe_{t+1}, S_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}, \pi_{t_{(k)},t}\right) \\ \cdot\left\{P_{t+1}{}^{iI,(\Delta_C)} + D_{t+1}{}^{(\Delta_C)}\right\}\end{matrix}\middle| \begin{matrix}\pi_{t_{(k)},t} \\ , D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}\end{matrix}\right)\right.$$

with

$$k = k_t$$

$$D_{t+1}{}^{(\Delta_C)} = \begin{cases} D^{(\Delta_C)}\left(D_{t_{(k)}}{}^{(\Delta_C)}, S_{t_{(k)},t}, fe_{t+1}\right) & t = t_{(k+1)} - 1 \\ 0 & t < t_{(k+1)} - 1 \end{cases}$$

$$P_{t+1}{}^{iI,(\Delta_C)} = \begin{cases} P_{t+1}{}^{iI,(\Delta_C)}\left(\pi_{t_{(k+1)},t_{(k+1)}}, D_{t_{(k+1)}}{}^{(\Delta_C)}, D_{t_{(k+1)}}{}^{(1)}\right) & t = t_{(k+1)} - 1 \\ P_{t+1}{}^{iI,(\Delta_C)}\left(\pi_{t_{(k)},t+1}, D_{t_{(k)}}{}^{(\Delta_C)}, D_{t+1}{}^{(1)}\right) & t < t_{(k+1)} - 1 \end{cases}$$

Risk premia of (1)-periodic risky assets

4-101

$$RP_t{}^{iI,(1)}\left(\pi_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) = -cov\left(\begin{matrix} AfR_t{}^{iI,(\Delta_C)}\left(fe_{t+1}, S_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}, \pi_{t_{(k)},t}\right) \\ \cdot\left\{P_{t+1}{}^{iI,(1)} + D_{t+1}{}^{(1)}\right\}\end{matrix}\middle| \begin{matrix}\pi_{t_{(k)},t} \\ , D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}\end{matrix}\right)$$

with

$$P_{t+1}{}^{iI,(1)} = \begin{cases} P_{t+1}{}^{iI,(1)}\left(\pi_{t_{(k+1)},t_{(k+1)}}, D_{t_{(k+1)}}{}^{(\Delta_C)}, D_{t_{(k+1)}}{}^{(1)}\right) & t+1 = t_{(k+1)} \\ P_{t+1}{}^{iI,(1)}\left(\pi_{t_{(k)},t+1}, D_{t_{(k)}}{}^{(\Delta_C)}, D_{t+1}{}^{(1)}\right) & t+1 \neq t_{(k+1)} \end{cases}$$

with adjustment for risk

$$AfR_t^{iI,(\Delta_C)}\left(fe_{t+1}, S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}, \pi_{t_{(k)},t}\right)$$

$$= \frac{U'\left(\frac{1}{n_I}\cdot D_{t+1}^{market}\right)}{\sum_{s_{t_{(k)},t}} \pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right)\cdot E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}^{market}\right)\middle| S_{t_{(k)},t} = s_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}\right)}$$

$$= \begin{cases} AfR_t^{cI,(\Delta_C)}\left(fe_{t+1}, S_{t_{(k)},t}, D_{t_{(k)}}^{(\Delta_C)}, D_t^{(1)}, \pi_{t_{(k)},t}\right) & t = t_{(k+1)} - 1 \\ AfR_t^{cI,(\Delta_C)}\left(fe_{t+1}, S_t, D_t^{(1)}, \pi_t\right) & t_{(k)} \le t < t_{(k+1)} - 1 \end{cases}$$

with

$$D_{t+1}^{market} = \begin{cases} D_{t+1}^{market}\left(D_{t+1}^{(\Delta_C)}, D_{t+1}^{(1)}\right) & t = t_{(k+1)} - 1 \\ D_{t+1}^{market}\left(D_{t+1}^{(1)}\right) & t_{(k)} \le t < t_{(k+1)} - 1 \end{cases}$$

Paralleling the results for asset prices (4-93) and (4-94), (4-100), and (4-101) show that only the current regime path $S_{t_{(k)},t}$ as well as factors and residuals fe_{t+1} give rise to a risk premium insofar as they influence D_{t+1}^{market}. The new regime S_{t+1}, signal noise η_{t+1}, and the influence on structure of cash flows through the current regime path $S_{t_{(k)},t}$ as well as factors and residual fe_{t+1} do not contribute to the risk premium.

4.2.2.3.4 Risk Decomposition and Consequences to Prices and Risk Premia

The results on the pricing of "inter-distribution risk" and "intra-distribution risk" generalize without major change to the case where information frequency is higher than or equal to cash flow frequency: regimes merely have to be replaced by paths of regimes. Since the results otherwise parallel those obtained for the special case of identical information and cash flow frequencies, the details are only stated in the Appendices A3.5.2.2.2.1.1 and A3.5.2.2.2.1.2.

4.2.3 Comparison of Asset Prices and Risk Premia across Information Structures

Two questions relating to asset prices and risk premia cross differing information structures have been addressed in the partial equilibrium framework (Section 3.3.2.3) and are of equal interest in the general equilibrium framework as well:

- How do complete information asset prices and risk premia relate to their incomplete information counterparts?
  The focus here is slightly different from the partial equilibrium framework: there, the main question was whether incomplete information asset prices are lower than the corresponding complete information prices. The answer has turned out to be negative and, clearly, it would be negative in the general equilibrium framework for similar reasons. Therefore, the question is reformulated more generally to characterize the relationship between complete and incomplete information asset prices.
- What is the effect of signal quality on asset prices and risk premia? How do asset prices change across information structures with high and low signal qualities?

4.2.3.1.1 Comparison of Asset Prices and Risk Premia under Complete and Incomplete Information

4.2.3.1.1.1 Asset Prices under Complete and Incomplete Information

There is a very close relationship between complete and incomplete information asset prices: incomplete information asset prices are simply expectations of complete information asset prices with respect to empirical probabilities (as opposed to risk-neutralized probabilities) for regimes.

To see this relation, observe that the stochastic discount factors under incomplete information (4-64) coincide with the complete information discount factor (4-51)

4-102

$$q_{t,t+\tau}{}^{iI}\left(fe_{t+1,t+\tau}, S_{t+1,t+\tau-1}; S_t, D_t\right) = q_{t,t+\tau}{}^{cI}\left(fe_{t+1,t+\tau}, S_{t+1,t+\tau-1}; S_t, D_t\right)$$
$$= \frac{1}{(1+\rho)^\tau} \cdot \frac{U'\left(\frac{1}{n_I} \cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right)}{U'\left(\frac{1}{n_I} \cdot D_t{}^{market}(D_t)\right)}$$

Moreover, cash flow models under complete and incomplete information are the same, i.e.,

$$D_{t+\tau} = D(D_{t+\tau-1}, S_{t+\tau-1}, fe_{t+\tau})$$

The reason is that cash flow models merely describe the relation between cash flows $D_{t+\tau-1}$ and $D_{t+\tau}$ but do not make any assumptions on whether their inputs are observable.

Identical discount factors together with identical cash flow models imply that the present value of cash flows at time $t+\tau$ under incomplete information (4-63) simply is an expectation of complete information present values of cash flows at time $t+\tau$ (4-50):

$$\sum_{s=1}^{K} \pi_{s,t} \cdot E\left(q_{t,t+\tau}{}^{iI}\left(fe_{t+1,t+\tau}, S_{t+1,t+\tau-1}; S_t, D_t\right) \cdot D_{t+\tau} \middle| S_t = s, D_t\right)$$
$$= \sum_{s=1}^{K} \pi_{s,t} \cdot E\left(q_{t,t+\tau}{}^{cI}\left(fe_{t+1,t+\tau}, S_{t+1,t+\tau-1}; S_t, D_t\right) \cdot D_{t+\tau} \middle| S_t = s, D_t\right)$$

Finally, asset prices are the sum of discounted future cash flows and, therefore, incomplete information asset prices must be expectations of complete information asset prices.

4-103

$$P_t{}^{iI}(\pi_t, D_t) = \sum_{s=1}^{K} \pi_{s,t} \cdot P_t{}^{cI}(S_t = s, D_t)$$

In the case of the one-period riskless bond, similar reasoning yields an analogous result:

4-104

$$B_t{}^{iI}(\pi_t, D_t) = \sum_{s=1}^{K} \pi_{s,t} \cdot B_t{}^{cI}(S_t = s, D_t)$$

Note that this relation between complete and incomplete information asset prices generalizes similar findings of Veronesi (2000), pp. 813-814.

Two aspects of (4-103) and (4-104) are worthwhile pointing out: (i) the relation between complete and incomplete information prices is completely different from partial equilibrium. (ii) (4-103) and (4-104) seem to imply a risk-neutral pricing of the unobservable regime's risk because prices under incomplete information are expectations (under the empirical measure) of complete information prices.

The key to answering (i) is that the stochastic discount factors under complete and incomplete information are identical. This result can be attributed to two core characteristics of a Lucas (1978) economy. First, cash flows are exogenous and, hence, do not depend on available information. If they were endogenously determined (production economy), production would depend on available information and, therefore, differ under complete and incomplete information. Second, in general equilibrium, aggregate consumption must be equal to aggregate cash flow because the riskless asset is in zero net supply and holding cash is impossible.

To see that (ii) is actually false, it is necessary to compare the pricing of risk under complete and incomplete information in more detail:

Complete Information Risk Correction

The present value of cash flows (under either complete or incomplete information) is rewritten as the sum of risk-neutrally discounted cash flows and a price of risk,

$$E\left(q_{t,t+\tau} \cdot D_{t+\tau} \middle| S_t, D_t\right) = E\left(q_{t,t+\tau} \middle| S_t, D_t\right) \cdot E(D_{t+\tau}|S_t, D_t) + cov\left(q_{t,t+\tau}, D_{t+\tau} \middle| S_t, D_t\right)$$

Since $E\left(q_{t,t+\tau} \middle| S_t, D_t\right)$ is a multi-period riskless discount factor, $E\left(q_{t,t+\tau} \middle| S_t, D_t\right) \cdot E(D_{t+\tau}|S_t, D_t)$ is the risklessly discounted expected cash flow. The risk correction $cov\left(q_{t,t+\tau}, D_{t+\tau} \middle| S_t, D_t\right)$ is responsible for the pricing of risk and consists of the price of risk discounted risklessly to t:

4-105

$$cov\left(q_{t,t+\tau}, D_{t+\tau}\middle|S_t, D_t\right) = E\left(q_{t,t+\tau}\middle|S_t, D_t\right) \cdot cov\left(AfR_{t,t+\tau}{}^{cI}, D_{t+\tau}\middle|S_t, D_t\right)$$

with

$$AfR_{t,t+\tau}{}^{cI} \equiv \frac{q_{t,t+\tau}}{E\left(q_{t,t+\tau}\middle|S_t, D_t\right)} \equiv \frac{U'\left(\frac{1}{n_I} \cdot D_{t+\tau}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I} \cdot D_{t+\tau}{}^{market}\right)\middle|S_t, D_t\right)}$$

Incomplete Information Risk Correction

The incomplete analogue of (4-105) is

4-106

$$cov\left(q_{t,t+\tau}, D_{t+\tau}\middle|\pi_t, D_t\right) = E\left(q_{t,t+\tau}\middle|\pi_t, D_t\right) \cdot cov\left(AfR_{t,t+\tau}{}^{iI}, D_{t+\tau}\middle|\pi_t, D_t\right)$$

with

$$AfR_{t,t+\tau}{}^{iI} \equiv \frac{U'\left(\frac{1}{n_I} \cdot D_{t+\tau}{}^{market}\right)}{\sum_{s=1}^{K} \pi_{s,t} \cdot E\left(U'\left(\frac{1}{n_I} \cdot D_{t+\tau}{}^{market}\right)\middle|S_t = s, D_t\right)}$$

Relation between Complete and Incomplete Risk Corrections

Based on (4-105) and (4-106), it is now possible to answer the question of how the pricing of risk under complete and incomplete information and, in particular, whether the risk of unobservable regimes is priced risk-neutrally: The risk correction at time $t+\tau$ under incomplete information, i.e., $cov\left(AfR_{t,t+\tau}{}^{iI}, D_{t+\tau}\middle|\pi_t, D_t\right)$, is closely related to its complete information counterpart, $cov\left(AfR_{t,t+\tau}{}^{cI}, D_{t+\tau}\middle|S_t, D_t\right)$:

4-107

$$\begin{aligned} &cov\left(AfR_{t,t+\tau}{}^{iI}, D_{t+\tau}\middle|\pi_t, D_t\right) \\ &\qquad = \sum_{s=1}^{K} \theta_{s,t}(\pi_t, D_t) \cdot cov\left(AfR_{t,t+\tau}{}^{cI}, D_{t+\tau}\middle|S_t, D_t\right) \\ &\qquad + \sum_{s=1}^{K} \left\{\theta_{s,t}(\pi_t, D_t) - \pi_{s,t}\right\} \cdot E(D_{t+\tau}|S_t = s, D_t) \end{aligned}$$

where $\theta_{s,t}(\pi_t, D_t)$ is the risk-neutralized probability of regime s defined in (4-70), i.e.,

$$\theta_{s,t}(\pi_t, D_t) \equiv \frac{\pi_{s,t} \cdot E\left(U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}\left(D(D_t, S_t, fe_{t+1})\right)\right)\middle|S_t = s, D_t\right)}{\sum_{s'=1}^{K} \pi_{s',t} \cdot E\left(U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}\left(D(D_t, S_t, fe_{t+1})\right)\right)\middle|S_t = s', D_t\right)}$$

(4-107) demonstrates that the unobservable regime's risk is not priced risk-neutrally. Instead, the risk corrections under complete information $cov\left(AfR_{t,t+\tau}{}^{cI}, D_{t+\tau}\middle|S_t, D_t\right)$ are weighted using risk-neutralized regime probabilities $\theta_{s,t}(\pi_t, D_t)$. Moreover, the conditional expectation $E(D_{t+\tau}|S_t, D_t)$ is random and, therefore, an additional correction for risk is needed that is absent in the complete information case.

Information frequency equal to or higher than cash flow frequency

Both results derived from the case where case where information frequency is equal to cash flow frequency transfer to this case. First, incomplete information asset prices are simply expectations with respect to empirical probabilities (as opposed to risk-neutralized probabilities). The crucial point that stochastic discount factors are the same under complete (4-89) and incomplete information (4-99) still holds. Second, unobservable regime's risk, the path of regimes $S_{t(k_t),t}$ is not priced risk-neutrally. The risk corrections under complete information are weighted using risk-neutralized regime path probabilities. Moreover, the conditional expectation of cash flows conditional on the regime path is random and, therefore, an additional correction for risk is needed that is absent in the complete information case.

4.2.3.1.1.2 Risk Premia under Complete and Incomplete Information

If asset prices under incomplete information are expectations of complete information asset prices, one may suspect that this might also hold for risk premia. However, this is not the case. Instead, conditions resemble the relation between complete and incomplete information risk corrections. To see this, it is best to use the decomposition (4-81) (p. 127):

4-81

$$RP_t^{iI}(\pi_t, D_t) = \sum_{s=1}^{K} \{\pi_{s,t} - \theta_{s,t}(\pi_t, D_t)\} \cdot E\left(\left(EP_{t+1}^{iI} + ED_{t+1}\right)(S_t, fe_{t+1}; \pi_t, D_t) \middle| \pi_t, D_t, S_t = s\right)$$
$$+ \sum_{s=1}^{K} \theta_{s,t}(\pi_t, D_t) \cdot RP_t^{conditional}(s; \pi_t, D_t)$$

with

$$RP_t^{conditional}(s; \pi_t, D_t) \equiv -cov\left(AfR_t^{conditional}(S_t; fe_{t+1}, D_t), \Delta^{comb.risk} \middle| \pi_t, D_t, S_t = s\right)$$

and

4-68

$$\underbrace{\begin{Bmatrix} E\left(P_{t+1}^{iI}(\pi_{t+1}, D_{t+1}) + D_{t+1} \middle| \pi_t, D_t, D_{t+1}^{market}(D_{t+1})\right) \\ -E\left(\left(EP_{t+1}^{iI} + ED_{t+1}\right)(S_t, fe_{t+1}; \pi_t, D_t) \middle| \pi_t, D_t, S_t\right) \end{Bmatrix}}_{\Delta^{comb.risk}}$$

4-71

$$AfR_t^{conditional}(S_t; fe_{t+1}, D_t) \equiv AfR_{t,t+1}^{cI}(fe_{t+1}, S_t, D_t) = \frac{U'\left(\frac{1}{n_I} \cdot D_{t+1}^{market}(D_{t+1})\right)}{E\left(U'\left(\frac{1}{n_I} \cdot D_{t+1}^{market}(D_{t+1})\right) \middle| S_t, D_t\right)}$$

To give an economic intuition behind this result, recall that the first term in (4-81) equals the risk premium due to "expectation risk" and the second "combined risk". Since "expectation risk" has no complete information counterpart, incomplete information risk premia cannot be expressed meaningfully through complete information risk premia.

It could still be hoped that clear a relation between the "combined risk" part of the risk premium and complete information risk premia holds in that conditional risk premia coincide with complete information risk premia. However, this is not the case either: Even though complete and incomplete information adjustments for risk are identical, this is not the case for the second argument of the covariances. $E\left(P_{t+1}{}^{cI}(S_{t+1},D_{t+1})+D_{t+1}\middle|S_t=s,D_t,D_{t+1}{}^{market}(D_{t+1})\right)$ (second argument of the covariance under complete information (4-55)) and $\Delta^{comb.risk}$ (second argument of the covariance under complete information) are different because they are based on complete and incomplete information prices respectively.

Finally, there cannot be a clear-cut relation between complete and incomplete information risk premia in the case where information frequency is higher than or equal to cash flow frequency if no such relationship holds in the simpler case where information frequency equals cash flow frequency.

4.2.3.1.2 Asset Prices and Risk Premia under Differing Signal Qualities

Recall from the corresponding partial equilibrium discussion for CARA utility (Section 3.3.2.3) that signal quality can influence asset prices and risk premia via two channels, (i) dynamics of information relevant to pricing and (ii) the functional relation which transforms this information into prices, the price function $P_t{}^{iI}(.)$ or $P_t{}^{iI,(\nu)}(.), \nu\epsilon\{1,\Delta_C\}$.

Dynamics of the information relevant to pricing under different signal qualities

The central insight concerning information relevant to pricing is that it is identical with information relevant to pricing under partial equilibrium and CARA utility: information relevant to pricing is (π_t, D_t) or, more generally, $\left(\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)$. Since cash flows and signals are exogenous and do not depend on the equilibrium concept, it is clear that the dynamics of information relevant to pricing are exactly as in the partial equilibrium framework with CARA utility.

Price and risk premium functions under different signal qualities

As a direct implication of the relation between complete and incomplete information asset prices, it follows immediately that signal quality does not influence price functions: (4-103) (p. 140) implies

$$P_t(\pi_t, D_t; \mathfrak{s}) = P_t(\pi_t, D_t) = \sum_{s=1}^{K} \pi_{s,t} \cdot P_t{}^{cI}(s, D_t)$$

where $\mathfrak{s}$ denotes any signal quality

since the right-hand side is independent of the signal quality $\mathfrak{s}$.

By the same reasoning, it is concluded that the price function for the one-period riskless bond does not depend on signal quality (4-104) (p. 140):

$$B_t(\pi_t, D_t; \mathfrak{s}) = B_t(\pi_t, D_t) = \sum_{s=1}^{K} \pi_{s,t} \cdot B_t{}^{cI}(s, D_t)$$

Risk premium functions are likewise independent of signal quality: one way to see this is through the definition of risk premia:

$$RP_t(\pi_t, D_t; \mathfrak{s}) = E(P_{t+1}(\pi_{t+1}, D_{t+1}; \mathfrak{s}) + D_{t+1}|\pi_t, D_t) - \left(1 + r_t(\pi_t, D_t; \mathfrak{s})\right) \cdot P_t(\pi_t, D_t; \mathfrak{s})$$

with

$$D_{t+1} = D(D_t, S_t, fe_{t+1})$$
$$\pi_{t+1} = \Pi(\pi_t, D_t, D_{t+1}, Sig_{t+1})$$
$$Sig_{t+1} = Sig_{t+1}(S_t, S_{t+1}, fe_{t+1}, \eta_{t+1})$$

Since prices of both risky and the riskless asset and, therefore the one-period riskless interest rate, are independent of the signal structure, it is clear that the second summand of the risk premium is independent of signal quality:

$$\left(1 + r_t(\pi_t, D_t; \mathfrak{s})\right) \cdot P_t(\pi_t, D_t; \mathfrak{s}) = \left(1 + r_t(\pi_t, D_t)\right) \cdot P_t(\pi_t, D_t)$$

It is less evident that the first summand, $E(P_{t+1}(\pi_{t+1}, D_{t+1}; \mathfrak{s}) + D_{t+1}|\pi_t, D_t)$, does not depend on signal quality: prices at time $t+1$ depend on conditional regime probabilities, and these conditional regime probabilities depend on signals. To see this, substitute the relation between incomplete and complete information asset prices (4-103) applied to time $t+1$, $\sum_{s'=1}^{K} \pi_{t+1,s'} \cdot P_{t+1}{}^{cI}(s', D_{t+1})$, into $E(P_{t+1}(\pi_{t+1}, D_{t+1}; \mathfrak{s}) + D_{t+1}|\pi_t, D_t)$ to obtain

$$E\left(\sum_{s'=1}^{K} \pi_{t+1,s'} \cdot P_{t+1}{}^{cI}(s', D_{t+1}) + D_{t+1}\middle|\pi_t, D_t\right)$$

with

$$\pi_{t+1} = \Pi(\pi_t, D_t, D_{t+1}, Sig_{t+1})$$

Clearly $P_{t+1}{}^{cI}(s', D_{t+1})$ depends on cash flows only (but not signals) whereas π_{t+1} is a function of both cash flows and signal. Using this fact in combination with the tower property of conditional expectations, it is obtained

$$E\left(\sum_{s'=1}^{K} E\left(\pi_{t+1,s'}\middle|\pi_t, D_t, D_{t+1}\right) \cdot P_{t+1}{}^{cI}(s, D_{t+1}) + D_{t+1}\middle|\pi_t, D_t\right)$$

Intuitively, the conditional expectation $E\left(\pi_{t+1,s'}\middle|\pi_t, D_t, D_{t+1}\right)$ averages out the effect of signals Sig_{t+1}. Moreover, it is shown in Appendix A3.6 that $E\left(\pi_{t+1,s'}\middle|\pi_t, D_t, D_{t+1}\right)$ is independent not only of a particular signal realization but even of the signal quality $\mathfrak{s}$ itself.[28] For $E\left(\pi_{t+1,s'}\middle|\pi_t, D_t, D_{t+1}\right)$

[28] Note that the fact that the expectation $E\left(\pi_{t+1,s'}\middle|\pi_t, D_t, D_{t+1}\right)$ does not depend on a particular signal realization does not in itself imply functional independence of the signal quality $\mathfrak{s}$. The expectation could still be a function of the distribution of signal quality $\mathfrak{s}$. For example, an expectation over normally distributed signals usually differs from an expectation over t-distributed signals.

can be interpreted as a particular form of conditional probability for regime $S_{t+1} = s'$ that is obtained by updating regime probabilities π_t for regime S_t by including new cash flows D_{t+1} but without recourse to signals.

In sum,

$$E\left(\sum_{s'=1}^{K} \pi_{t+1,s'} \cdot P_{t+1}{}^{cI}(s', D_{t+1}) + D_{t+1} \middle| \pi_t, D_t\right)$$

and, consequently, risk premium functions do not depend on the signal quality $\mathfrak{s}$.

Finally, in the generalized case where information frequency can be higher than cash flow frequency, price and risk premium functions are still independent of signal quality. The crucial point is that incomplete information prices are still expectations of complete information prices.

Signal quality and its effect on asset prices and risk premia

In conclusion, it is possible to give a general answer to the question of this section, namely the effect of signal quality on prices and risk premia. The dynamics of information relevant to pricing depends on signal quality as in the partial equilibrium framework. In contrast to the partial equilibrium framework both price and risk premium functions are independent of signal quality.

4.3 Equilibrium Asset Prices and Risk Premia for Special Cash Flow Models under Complete and Incomplete Information

4.3.1 Motivation for the Cases to Analyze

The purpose of this section is to analyze the asset pricing implications of the special cash flow models introduced in Chapter 2: cash flows without lags in levels and cash flows without lags in growth rates. A detailed discussion of these cash flow models is only justified if significant insights are obtained in addition to the results for general cash flow models (Section 4.2).

As in the partial equilibrium framework, the selection criterion is whether information relevant to pricing is simplified relative to the general case. First, in the case of cash flows without lags in levels information relevant to pricing for all assets is simplified; second, in the case of cash flows without lags in growth rates under constant relative risk aversion (CRRA) information relevant to pricing one asset i is simplified.

4.3.2 Cash Flow Models without Lags in Levels

4.3.2.1 All Assets Pay Cash Flows in Every Period: Information Frequency = Cash Flow Frequency

Cash flow models without lags in levels read by definition

2-7

$$D_{t+1} = D(S_t, fe_{t+1})$$

In addition, a special functional form of $D(.)$, an affine linear factor model, is specified

2-8

$$D_{i,t+1} = \mu_{i,t} + \sum_{j=1}^{m} a_{ij,t} \cdot f_{j,t+1} + b_{i,t} \cdot e_{i,t+1}, i = 1, \ldots, n$$

The discussion is organized in the same way as in the partial equilibrium framework (Section 3.3.3.2): the implications that follow from the more general form $D_{t+1} = D(S_t, fe_{t+1})$ alone are discussed before the more special affine linear factor model.

4.3.2.1.1 Complete Information

4.3.2.1.1.1 Information Relevant to Pricing $z_t{}^p$

For cash flow models without lags in levels under complete information, information relevant to pricing consists of the pair of the current regime and aggregate cash flows paid by the market portfolio of risky assets and is, hence, simpler than (S_t, D_t), information relevant to pricing for general cash flows models:

4-108

$$z_t{}^p = \left(S_t, D_t{}^{market}\right)$$

To see that $\left(S_t, D_t{}^{market}\right)$ contains all information relevant to pricing, observe that it contains (i), by definition, aggregate cash flows paid by the market portfolio of risky assets and (ii) also contains a sufficient statistic for the conditional distribution of future cash flows: recall from the corresponding partial equilibrium case (Section 3.3.3.2, p. 84) that future cash flows are no longer functionally related to current cash flows, implying that the conditional distribution of future cash flows is completely described by the current regime.

4.3.2.1.1.2 Equilibrium Asset Prices

4.3.2.1.1.2.1 Quasi-Static Asset Prices

Cash flows without lags in levels: no additional restrictions on the functional form of $D(.)$

If cash flow models belong to the subclass of this section, quasi-static asset prices under complete information and general cash flow models (4-41)and (4-42) (see p. 108) simplify to

4-109

$$P_t^{cI}(S_t, D_t^{market}) = E\left(q_{t,t+1}^{cI}(fe_{t+1}, S_t, D_t^{market}) \cdot \left\{P_{t+1}^{cI}\left(S_{t+1}, D_{t+1}^{market}(D_{t+1})\right) + D_{t+1}\right\} \middle| S_t, D_t^{market}\right)$$

4-110

$$B_t^{cI}(S_t, D_t^{market}) = E\left(q_{t,t+1}^{cI}(fe_{t+1}, S_t, D_t^{market}) \middle| S_t, D_t^{market}\right)$$

with stochastic discount factor

4-111

$$q_{t,t+1}^{cI}(fe_{t+1}, S_t, D_t^{market}) = \frac{1}{1+\rho} \cdot \frac{U'\left(\frac{1}{n_I} \cdot D_{t+1}^{market}(D_{t+1})\right)}{U'\left(\frac{1}{n_I} \cdot D_t^{market}\right)}$$

with

$$D_{t+1} = D(S_t, fe_{t+1})$$

This means – contrary to the case for general cash flow models – both D_t^{market} and D_{t+1}^{market} are no longer interrelated via D_t. As a consequence, the influence of D_t^{market} on the stochastic discount factor can be multiplicatively separated from the influence of D_{t+1}^{market} on the stochastic discount factor.

This separation of D_t^{market} and D_{t+1}^{market} translates from stochastic discount factors to asset prices and the pricing equations (4-109), (4-110), and (4-111) can be written as:

4-112

$$P_t^{cI}(S_t, D_t^{market}) = \frac{P_t^{reg}(S_t)}{U'\left(\frac{1}{n_I} \cdot D_t^{market}\right)}$$

with

$$P_t^{reg}(S_t) = E\left(\frac{1}{1+\rho} \cdot U'\left(\frac{1}{n_I} \cdot D_{t+1}^{market}(D_{t+1})\right) \cdot \left\{P_{t+1}^{cI}\left(S_{t+1}, D_{t+1}^{market}(D_{t+1})\right) + D_{t+1}\right\} \middle| S_t\right)$$

4-113

$$B_t{}^{cI}\left(S_t, D_t{}^{market}\right) = \frac{B_t{}^{reg}(S_t)}{U'\left(\frac{1}{n_I}\cdot D_t{}^{market}\right)}$$

with

$$B_t{}^{reg}(S_t) = \frac{1}{1+\rho}\cdot E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right)\middle| S_t\right)$$

$$D_{t+1} = D(S_t, fe_{t+1})$$

With the help of Equations (4-109) and (4-110) the pricing statement of the general cash flows model (4-41) (p. 108) regarding the pricing of the two sources of risk S_{t+1} and fe_{t+1} can be further specified. To that end, recall that prices and cash flows can be decomposed into (i) a part that is "explained" by aggregate cash flows $D_{t+1}{}^{market}$ and (ii) a remaining part that captures all other stochastic influences that are not due to $D_{t+1}{}^{market}$, see (4-44) (p. 109):

4-114

$$\frac{P_{t+1}{}^{reg}(S_{t+1})}{U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}\right)} + D_{t+1} = \underbrace{E\left(\frac{P_{t+1}{}^{reg}(S_{t+1})}{U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}\right)} + D_{t+1}\middle| S_t, D_{t+1}{}^{market}(D_{t+1})\right)}_{part\ (i)} + \underbrace{\Delta_{t+1}}_{part\ (ii)}$$

with

$$\Delta_{t+1} \equiv P_{t+1}{}^{cI} + D_{t+1} - E\left(P_{t+1}{}^{cI} + D_{t+1}\middle| S_t, D_{t+1}{}^{market}(D_{t+1})\right)$$

$$D_{t+1} = D(S_t, fe_{t+1})$$

I discuss parts (i) and (ii) separately for asset prices and cash flows.

In the case of $P_{t+1}{}^{cI}$, part (ii) is entirely due to the regime S_{t+1} and reads

4-115

$$\frac{P_{t+1}{}^{reg}(S_{t+1}) - E(P_{t+1}{}^{reg}(S_{t+1})|S_t)}{U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}\right)}$$

It is known from the general cash flow case that S_{t+1} is not priced. However, in contrast to the general cash flow case, the effect of fe_{t+1} on individual (as opposed to aggregate) cash flows does not appear in part (ii). The reason is that asset prices are only a function of aggregate cash flow, see (4-109).

The priced part of $P_{t+1}{}^{cI}$, part (i), reads:

4-116

$$E\left(P_{t+1}{}^{cI}\middle| S_t, D_{t+1}{}^{market}\right) = \frac{E(P_{t+1}{}^{reg}(S_{t+1})|S_t)}{U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}\right)}$$

The main difference to the general case is that aggregate cash flows $D_{t+1}{}^{market}$ only affect this part via marginal utility of consumption at time $t+1$ and not additionally via the conditional expectation in the numerator. The expectation $E(P_{t+1}{}^{reg}(S_{t+1})|S_t)$ does not depend on $D_{t+1}{}^{market}$ because conditional distribution of future cash flows at $t+1$ is completely described by the regime S_{t+1}.

On the pricing of cash flows D_{t+1} (as opposed to asset prices $P_{t+1}{}^{cI}$) not much more than for general cash flow models can be said. In particular, part (ii) still depends on factors and residuals via their effect on individual (as opposed to aggregate) cash flows:

4-117

$$D_{t+1} - E\left(D_{t+1}\middle|S_t, D_{t+1}{}^{market}\right)$$

Part (i) consists of the expectation of cash flows conditional on $D_{t+1}{}^{market}$ and S_t:

4-118

$$E\left(D_{t+1}\middle|S_t, D_{t+1}{}^{market}\right)$$

The difference to the general case is that this expectation is no longer conditional on D_t.

The channels through which factors and residuals fe_{t+1} exert price influence in this special cash flow model without lags in cash flows are of a particularly simple form. Out of the three channels for prices – covariances of the stochastic discount factor with $P_{t+1}{}^{cI}$ – only the first channel exists, i.e., the fact that marginal utility of consumption at time $U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}\right)$ appears in both the numerator of the stochastic discount factor and the denominator of prices $P_{t+1}{}^{cI}$. However, neither marginal utility of consumption nor cash flows at some future point of time $t+\tau, \tau > 1$ are correlated with the stochastic discount factor if there are no lags in cash flows (channels two and three). In other words, asset prices are always negatively correlated with the stochastic discount factor[29] and, therefore, discounted at a rate that is higher than the riskless rate. Thus the price of asset prices $P_{t+1}{}^{cI}$ must always be less (or, in degenerate cases, equal to) the "risk-neutral" discounting of $E\left(P_{t+1}{}^{cI}\left(S_{t+1}, D_{t+1}{}^{market}\right)\middle|S_t\right)$ at the riskless rate:

4-119

$$E\left(q_{t,t+1}{}^{cI}\left(fe_{t+1}, S_t, D_t{}^{market}\right) \cdot P_{t+1}{}^{cI}\left(S_{t+1}, D_{t+1}{}^{market}(D_{t+1})\right)\middle|S_t, D_t{}^{market}\right)$$
$$\leq \frac{1}{1 + r_t{}^{cI}\left(S_t, D_t{}^{market}\right)} \cdot E\left(P_{t+1}{}^{cI}\left(S_{t+1}, D_{t+1}{}^{market}\right)\middle|S_t\right)$$

The cash flow channel - covariances of the stochastic discount factor with D_{t+1} -, however, is unchanged.

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[29] Formally, a convexity argument via Jensen's inequality leads to this result: if f is any convex function, then $f\left(E(X)\right) \leq E\left(f(X)\right)$ (see, e.g., Chow/Teicher (1997), p. 104). Applying this to the convex function $f(x) = \frac{1}{x}$, defined on the set of positive real numbers, with $x = U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}(D_{t+1})\right)$, shows $\frac{1}{E\left(U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}(D_{t+1})\right)\middle|S_t\right)} \leq E\left(\frac{1}{U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}(D_{t+1})\right)}\middle|S_t\right)$

Concerning the price of the one-period riskless bond (4-112), the special cash flow model without lags in levels cannot contribute over what is already knows from general cash flow models.

Cash flows without lags in cash flows: affine linear factor model

If it is further assumed that cash flows follow the affine linear factor,

$$D_{i,t+1} = \mu_{i,t} + \sum_{j=1}^{m} a_{ij,t} \cdot f_{j,t+1} + b_{i,t} \cdot e_{i,t+1}, i = 1, \dots, n$$

equilibrium asset prices of risky assets read

4-120

$$P_t^{cI}\left(S_t, D_t^{market}\right) = \frac{1}{1 + r_t^{cI}\left(S_t, D_t^{market}\right)} \cdot \frac{E\left(P_{t+1}^{reg}(S_{t+1})|S_t\right)}{E\left(U'\left(\frac{1}{n_I} \cdot D_{t+1}^{market}(D_{t+1})\right)\middle|S_t\right)} + \frac{1}{1 + r_t^{cI}\left(S_t, D_t^{market}\right)} \cdot E\left(AfR_{t,t+1}^{cI}(fe_{t+1}, S_t) \cdot D_{t+1}\middle|S_t\right)$$

with

$$AfR_{t,t+1}^{cI}(fe_{t+1}, S_t) = \frac{U'\left(\frac{1}{n_I} \cdot D_{t+1}^{market}(D_{t+1})\right)}{E\left(U'\left(\frac{1}{n_I} \cdot D_{t+1}^{market}(D_{t+1})\right)\middle|S_t\right)}$$

$$D_{t+1} = \mu(S_t) + A(S_t) \cdot fe_{t+1}$$

where

4-121

$$\frac{1}{1 + r_t^{cI}\left(S_t, D_t^{market}\right)} \cdot E\left(AfR_{t,t+1}^{cI}(fe_{t+1}, S_t) \cdot D_{i,t+1}\middle|S_t\right) = \frac{1}{1 + r_t^{cI}\left(S_t, D_t^{market}\right)} \cdot \mu_i(S_t) + \sum_{j=1}^{m} a_{ij}(S_t) \cdot \frac{1}{1 + r_t^{cI}\left(S_t, D_t^{market}\right)} \cdot E\left(AfR_{t,t+1}^{cI}(fe_{t+1}, S_t) \cdot f_{j,t+1}\middle|S_t\right) + b_i(S_t) \cdot \frac{1}{1 + r_t^{cI}\left(S_t, D_t^{market}\right)} \cdot E\left(AfR_{t,t+1}^{cI}(fe_{t+1}, S_t) \cdot e_{i,t+1}\middle|S_t\right)$$

$$i = 1, \dots, n$$

No significant additional insights on the price of the one-period riskless bond or on the part of risky assets that prices $P_{t+1}^{cI}\left(S_{t+1}, D_{t+1}^{market}\right)$ are obtainable. However the price of new cash flows D_{t+1} can be analyzed in more detail. The structure of the price of D_{t+1} is very similar to partial equilibrium case with CARA utility: expected cash flows are discounted at the riskless rate (i.e., $\frac{1}{1+r_t^{cI}\left(S_t, D_t^{market}\right)} \cdot \mu_i(S_t)$). Moreover, there is a compensation for factor risk and one for residual risk. – A compensation for risk always consists of the quantity of risk multiplied by the price of risk.

Compensation for factor risk

The quantity of factor j risk of asset i equals $a_{ij}(s)$, the price of factor j risk is $\lambda_j^f\left(s, D_t^{market}\right)$

4-122

$$\lambda_j^f\left(s, D_t^{market}\right) \equiv \frac{1}{1 + r_t^{cI}\left(S_t, D_t^{market}\right)} \cdot E\left(AfR_{t,t+1}^{cI}(fe_{t+1}, S_t) \cdot f_{j,t+1} \middle| S_t = s\right)$$

$$j = 1, \dots, m$$

Compensation for residual risk

The quantity of residual risk of asset i equals $b_i(s)$, the price of residual risk is $\lambda_i^e\left(s, D_t^{market}\right)$

4-123

$$\lambda_i^e\left(s, D_t^{market}\right) \equiv \frac{1}{1 + r_t^{cI}\left(S_t, D_t^{market}\right)} \cdot E\left(AfR_{t,t+1}^{cI}(fe_{t+1}, S_t) \cdot e_{i,t+1} \middle| S_t = s\right)$$

The economic interpretation of compensations for factor and residual risk is parallel to the partial equilibrium case with CARA preferences; however, here results are not based on any particular utility function.

In addition to these parallel results to the partial equilibrium case, there is one result specific to the general equilibrium case: factors and residuals fe_{t+1} will only be priced insofar as they contribute to aggregate (as opposed to individual) cash flows can be expressed in a particularly simple form: the relevant part of $f_{j,t+1}$ (or $e_{i,t+1}$) can be identified as the conditional expectation $E\left(f_{j,t+1} \middle| S_t, D_{t+1}^{market}(D_{t+1})\right)$ (or $E\left(e_{i,t+1} \middle| S_t, D_{t+1}^{market}(D_{t+1})\right)$). Formally, this can be derived using the tower property of conditional expectations: the price the price of factor j risk /residual risk reads:

4-124

$$\begin{aligned} &E\left(AfR_{t,t+1}^{cI}(fe_{t+1}, S_t) \cdot f_{j,t+1} \middle| S_t\right) \\ &\qquad = E\left(AfR_{t,t+1}^{cI}(fe_{t+1}, S_t) \cdot E\left(f_{j,t+1} \middle| S_t, D_{t+1}^{market}(D_{t+1})\right) \middle| S_t\right) \end{aligned}$$

4-125

$$\begin{aligned} &E\left(AfR_{t,t+1}^{cI}(fe_{t+1}, S_t) \cdot e_{i,t+1} \middle| S_t\right) \\ &\qquad = E\left(AfR_{t,t+1}^{cI}(fe_{t+1}, S_t) \cdot E\left(e_{i,t+1} \middle| S_t, D_{t+1}^{market}(D_{t+1})\right) \middle| S_t\right) \end{aligned}$$

with

$$D_{t+1} = \mu(S_t) + A(S_t) \cdot fe_{t+1}$$

In economic terms, this means that the truly relevant risk factor is not $f_{j,t+1}$ but only a certain part, namely the part that can be explained by D_{t+1}^{market}. These truly relevant risk factors are the expectations of factors conditional on S_t and $D_{t+1}^{market}(D_{t+1})$, $E\left(f_{j,t+1} \middle| S_t, D_{t+1}^{market}(D_{t+1})\right)$. The general result that only fluctuations explained by aggregate cash flows are priced, therefore, takes a very intuitive form.

4.3.2.1.1.2.2 Asset Prices as Discounted Future Cash Flows

Cash flows without lags in cash flows: no additional restrictions on the functional form of $D(.)$

If cash flow models belong to the subclass of this section, asset prices as discounted future cash flows under complete information (4-50) simplify to

4-126

$$P_t^{cI}\left(S_t, D_t^{market}\right) = \sum_{\tau=1}^{T-t} E\left(q_{t,t+\tau}{}^{cI}\left(fe_{t+\tau}, S_{t+\tau-1}; D_t^{market}\right) \cdot D_{t+\tau} \middle| S_t, D_t^{market}\right)$$

with multi-period stochastic discount factors

4-127

$$q_{t,t+\tau}{}^{cI}\left(fe_{t+\tau}, S_{t+\tau-1}; D_t^{market}\right) = \frac{1}{(1+\rho)^\tau} \cdot \frac{U'\left(\frac{1}{n_I} \cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right)}{U'\left(\frac{1}{n_I} \cdot D_t^{market}\right)}$$

with

$$D_{t+\tau} = D(S_{t+\tau-1}, fe_{t+\tau})$$

The pricing equation (4-126) and the multi-period stochastic discount factor (4-127) demonstrate how the two sources of risk (i) path of future regimes $S_{t+1,t+\tau}$ - note the current regime S_t is non-stochastic - and (ii) path of factors and residuals $fe_{t+1,t+\tau}$ are priced. First, recall from the general cash flow model that not the paths of regimes $S_{t,t+\tau-1}$ as well as factors and residuals $fe_{t+1,t+\tau}$ as such exert influence on prices but only their joint effects on $D_{t+\tau-1}$, $S_{t+\tau-1}$, and $fe_{t+\tau}$. Second, the multi-period stochastic discount factor in the special cash flow model without lags in levels clarifies that only $S_{t+\tau-1}$, and $fe_{t+\tau}$ remain. The underlying reason is that cash flows $D_{t+\tau-1}$ do - by definition - not affect cash flows at time $t+\tau$ if there are no lags in cash flow levels and, therefore, differences between paths of factors and residuals as well as regimes are irrelevant as long as they lead to the same regime $S_{t+\tau-1}$ and $fe_{t+\tau}$.

$S_{t+\tau-1}$ and $fe_{t+\tau}$ affect risk-adjusted pricing via a generalized version of the cash flow channel (4-49). This generalized cash flow channel can be described by means of the covariance between the stochastic discount factor and cash flows, which leads in the special case of cash flows without lags in levels to the following simplified version of (4-54):

4-128

$$cov\left(\frac{1}{(1+\rho)^\tau} \cdot \frac{U'\left(\frac{1}{n_I} \cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right)}{U'\left(\frac{1}{n_I} \cdot D_t^{market}\right)}, D_{t+\tau} \middle| S_t, D_t^{market}\right) = \frac{1}{(1+\rho)^\tau} \cdot \frac{1}{U'\left(\frac{1}{n_I} \cdot D_t^{market}\right)} \cdot$$

$$\left\{\sum_{s'=1}^{K} P(S_{t+\tau-1}=s'|S_t)\cdot cov\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right),D_{t+\tau}\middle|S_{t+\tau-1}\right)\right.$$
$$\left.+cov\left(E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right)\middle|S_{t+\tau-1}\right),E(D_{t+\tau}|S_{t+\tau-1})\middle|S_t\right)\right\}$$

The covariance (4-128) reveals the details of risk-adjusted pricing. The first term in brackets on the right-hand side of (4-128) is the covariance of the stochastic discount factor and cash flows due to $fe_{t+\tau}$. The conditioning information simplifies to $S_{t+\tau-1}$ (instead of $S_{t+\tau-1},D_{t+\tau-1}$ as for general cash flow models)) and is averaged out by the transition probabilities $P(S_{t+\tau-1}=s'|S_t),s'=1,\dots,K$. The second term in brackets captures the covariances of the stochastic discount factor and cash flows due to $S_{t+\tau-1}$ (instead of $S_{t+\tau-1},D_{t+\tau-1}$), with factors and residuals $fe_{t+\tau}$ averaged out.

<u>Cash flows without lags in cash flows: affine linear factor model</u>

Cash flows that follow the affine linear factor read

$$D_{i,t+1}=\mu_{i,t}+\sum_{j=1}^{m} a_{ij,t}\cdot f_{j,t+1}+b_{i,t}\cdot e_{i,t+1}, i=1,\dots,n$$

To prepare for the analysis of the affine linear factor model, the present value of cash flows (4-126) is re-formulated as the sum of expected cash flows discounted at a riskless rate and a risk correction,

4-129

$$E\left(q_{t,t+\tau}{}^{cI}\cdot D_{t+\tau}\middle|S_t,D_t{}^{market}\right)$$
$$=E\left(q_{t,t+\tau}{}^{cI}\middle|S_t,D_t{}^{market}\right)$$
$$\cdot\left\{E(D_{t+\tau}|S_t)+cov\left(\frac{q_{t,t+\tau}{}^{cI}}{E\left(q_{t,t+\tau}{}^{cI}\middle|S_t,D_t{}^{market}\right)},D_{t+\tau}\middle|S_t,D_t{}^{market}\right)\right\}$$

The first part, expected cash flows discounted at a riskless rate, reads in the affine linear factor model

4-130

$$E\left(q_{t,t+\tau}{}^{cI}\middle|S_t,D_t{}^{market}\right)\cdot E(D_{t+\tau}|S_t)=E\left(q_{t,t+\tau}{}^{cI}\middle|S_t,D_t{}^{market}\right)\cdot\sum_{s'=1}^{K} P(S_{t+\tau-1}=s'|S_t)\cdot\mu(s')$$

The second part, the risk correction, specifies in the affine linear factor model to

4-131

$$E\left(q_{t,t+\tau}{}^{cI}\middle|S_t,D_t{}^{market}\right)\cdot cov\left(\frac{q_{t,t+\tau}{}^{cI}}{E\left(q_{t,t+\tau}{}^{cI}\middle|S_t,D_t{}^{market}\right)},D_{t+\tau}\middle|S_t,D_t{}^{market}\right)$$
$$=E\left(q_{t,t+\tau}{}^{cI}\middle|S_t,D_t{}^{market}\right)$$
$$\cdot\sum_{s'=1}^{K}\theta_{s'}{}^{+\tau}(S_t)\cdot cov\left(AfR_{t,t+\tau}{}^{cI}(fe_{t+\tau},S_{t+\tau-1}),D_{t+\tau}\middle|S_{t+\tau-1}=s'\right)$$
$$+\sum_{s'=1}^{K}\left\{\theta_{s'}{}^{+\tau}(S_t)-P(S_{t+\tau-1}=s'|S_t)\right\}\cdot E(D_{t+\tau}|S_{t+\tau-1}=s')$$

with

$$AfR_{t,t+\tau}{}^{cI}(fe_{t+\tau},S_{t+\tau-1}) \equiv \frac{U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|S_{t+\tau-1}\right)}$$

with risk-neutralized probabilities for a transition from regime S_t to $S_{t+\tau-1}$

$$\theta_{s'}{}^{+\tau}(S_t) \equiv P(S_{t+\tau-1}=s'|S_t)\cdot\frac{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|S_{t+\tau-1}=s'\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|S_t\right)}$$

$$s'=1,\dots,K$$

The risk correction (4-131) shows how the two sources of risk $S_{t+\tau-1}$ and $fe_{t+\tau}$ are priced. The risk of $S_{t+\tau-1}$ is priced by risk-neutralized probabilities for a transition from regime S_t to $S_{t+\tau-1}$. The source of risk $fe_{t+\tau}$ is priced by covariances conditional on a given regime $S_{t+\tau-1}$,

$$cov\left(AfR_{t,t+\tau}{}^{cI}(fe_{t+\tau},S_{t+\tau-1}),D_{t+\tau}\middle|S_{t+\tau-1}=s'\right)$$

The assumption of an affine linear factor model leads to a specific form of these covariances:

4-132

$$\begin{aligned} &cov\left(AfR_{t,t+\tau}{}^{cI}(fe_{t+\tau},S_{t+\tau-1}),D_{i,t+\tau}\middle|S_{t+\tau-1}=s'\right) \\ &\quad=\sum_{j=1}^{K} a_{ij}(s')\cdot E\left(AfR_{t,t+\tau}{}^{cI}(fe_{t+\tau},S_{t+\tau-1})\cdot f_{j,t+\tau}\middle|S_{t+\tau-1}=s'\right)+b_i(s') \\ &\quad\cdot E\left(AfR_{t,t+\tau}{}^{cI}(fe_{t+\tau},S_{t+\tau-1})\cdot e_{i,t+\tau}\middle|S_{t+\tau-1}=s'\right)\end{aligned}$$

(4-132) shows that the pricing of the source of risk $fe_{t+\tau}$ in the present value of cash flows is the same as in the quasi-static case: the risk correction for $fe_{t+\tau}$ consists of quantities of risk multiplied by the prices of risk; the quantities of factor j risk and residual risk in the cash flow of asset i are conditional on regime $S_{t+\tau-1}=s'$ and read $a_{ij}(s')$ and $b_i(s')$, respectively. The prices of factor j risk and residual risk are also conditional on regime $S_{t+\tau-1}=s'$and read

$$E\left(AfR_{t,t+\tau}{}^{cI}(fe_{t+\tau},S_{t+\tau-1})\cdot f_{j,t+\tau}\middle|S_{t+\tau-1}=s'\right)$$

and

$$E\left(AfR_{t,t+\tau}{}^{cI}(fe_{t+\tau},S_{t+\tau-1})\cdot e_{i,t+\tau}\middle|S_{t+\tau-1}=s'\right)$$

Observe that both quantities and prices of factors and residuals are exactly the same as in the quasi-static framework because the regime process, the process of factors and residuals and, finally, the cash flow function are time-homogeneous.

4.3.2.1.1.3 Equilibrium Risk Premia

<u>Cash flows without lags in cash flows: no additional restrictions on the functional form of $D(.)$</u>

If cash flow models belong to the subclass of this section, risk premia under complete information and general cash flow models (4-55) simplify to

4-133

$$RP_t^{cI}(S_t) = -cov\left(\frac{AfR_{t,t+1}^{cI}(fe_{t+1}, S_t)}{,\frac{E(P_{t+1}^{reg}(S_{t+1})|S_t)}{U'\left(\frac{1}{n_I}\cdot D_{t+1}^{market}(D_{t+1})\right)}} + E\left(D_{t+1}\middle|S_t, D_{t+1}^{market}(D_{t+1})\right)\middle| S_t\right)$$

with

$$AfR_{t,t+1}^{cI}(fe_{t+1}, S_t) \equiv \frac{U'\left(\frac{1}{n_I}\cdot D_{t+1}^{market}(D_{t+1})\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}^{market}(D_{t+1})\right)\middle|S_t\right)}$$

$$D_{t+1} = D(S_t, fe_{t+1})$$

Observe that risk premia depend on the regime S_t only and, therefore, on less information than asset prices, which depend, by definition, on information relevant to pricing $z_t^p = (S_t, D_t^{market})$. The underlying reason is that risk premia only depend on the conditional distribution of D_{t+1}, but this conditional distribution is summarized by S_t alone for cash flow models without lags in levels.

The risk premium consists of two parts stemming from asset prices P_{t+1}^{cI} (first term) and cash flows D_{t+1} (second term). While not much can be said on the risk premium part stemming from cash flows in addition to what is already known for general cash flow models, the risk premium part that stems from asset prices is unequivocally non-negative:

4-134

$$-cov\left(\frac{AfR_{t,t+1}^{cI}(fe_{t+1}, S_t)}{,\frac{E(P_{t+1}^{reg}(S_{t+1})|S_t)}{U'\left(\frac{1}{n_I}\cdot D_{t+1}^{market}(D_{t+1})\right)}}\middle| S_t\right)$$

$$= E(P_{t+1}^{reg}(S_{t+1})|S_t)$$

$$\cdot \left\{ E\left(\frac{1}{U'\left(\frac{1}{n_I}\cdot D_{t+1}^{market}(D_{t+1})\right)}\middle| S_t\right) - \frac{1}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}^{market}(D_{t+1})\right)\middle|S_t\right)}\right\}$$

$$\geq 0$$ [30]

The reason is that the channel of marginal utility of consumption at time t (channel (i)) introduces a negative correlation between the stochastic discount factor and asset prices. This is formally shown by the same convexity argument as in Footnote 29, p. 149.

[30] It is implicitly assumed that $E(P_{t+1}^{reg}(S_{t+1})|S_t) \geq 0$. Note that this assumption is necessary because this analysis is not restricted to assets with limited liability.

Cash flows without lags in cash flows: affine linear factor model

If it is further assumed that cash flows follow the affine linear factor,

$$D_{t+1} = \mu(S_t) + A(S_t) \cdot fe_{t+1}$$

the risk premium part resulting from D_{t+1} is linear in prices for factor and residual risk

4-135

$$-cov\left(\begin{matrix} AfR_{t,t+1}{}^{cI}(fe_{t+1}, S_t) \\ , E\left(D_{i,t+1}\middle|S_t, D_{t+1}{}^{market}(D_{t+1})\right) \end{matrix}\middle|S_t\right)$$
$$= -exp\left(r_t{}^{cI}\left(S_t, D_t{}^{market}\right)\right) \cdot \sum_{j=1}^{m} a_{ij}(S_t) \cdot \lambda_j{}^f\left(S_t, D_t{}^{market}\right)$$
$$- exp\left(r_t{}^{cI}\left(S_t, D_t{}^{market}\right)\right) \cdot \sum_{j=1}^{m} b_i(S_t) \cdot \lambda_i{}^e\left(S_t, D_t{}^{market}\right)$$
$$i = 1, \dots, n$$

where $\lambda_j{}^f\left(s, D_t{}^{market}\right)$ and $\lambda_i{}^e\left(s, D_t{}^{market}\right)$ are the prices of factor j risk (4-122) and the price of residual risk (4-123).

Observe that aggregate cash flows $D_t{}^{market}$ are contained in both $exp\left(r_t{}^{cI}\left(S_t, D_t{}^{market}\right)\right)$ and $\lambda_j{}^f\left(s, D_t{}^{market}\right)$ or $\lambda_i{}^e\left(s, D_t{}^{market}\right)$. However, $D_t{}^{market}$ cancels out, i.e., there is no contradiction to the assertion that risk premia are not a function of $D_t{}^{market}$. With the now clarified structure of the risk premium part stemming from cash flows, it is evident that the interpretation is essentially the same as under partial equilibrium with CARA preferences (3-103) (p. 89) and can, therefore, be omitted.

4.3.2.1.2 Incomplete Information

4.3.2.1.2.1 Information Relevant to Pricing $z_t{}^p$

For cash flow models without lags in levels under incomplete information, information relevant to pricing consists of conditional regime probabilities and aggregate cash flows paid by the market portfolio of risky assets and is, hence, simpler than (π_t, D_t), information relevant to pricing for general cash flows models:

4-136

$$z_t{}^p = \left(\pi_t, D_t{}^{market}\right)$$

To see that $\left(\pi_t, D_t{}^{market}\right)$ contains all information relevant to pricing, observe that it contains (i), by definition, aggregate cash flows paid by the market portfolio of risky assets and (ii) also contains a sufficient statistic for the conditional distribution of future cash flows: recall from the corresponding partial equilibrium case (Section 3.3.3.2, p. 84) that future cash flows are no longer functionally related to current cash flows, implying that the conditional distribution of future cash flows is completely described by current regime probabilities.

4.3.2.1.2.2 Equilibrium Asset Prices

4.3.2.1.2.2.1 Quasi-Static Asset Prices

Cash flows without lags in levels: no additional restrictions on the functional form of $D(.)$

If cash flow models belong to the subclass of this section, quasi-static asset prices under incomplete information and general cash flow models (4-57) and (4-58) (see p. 116) simplify to

4-137

$$P_t^{iI}(\pi_t, D_t^{market}) = \sum_{s=1}^{K} \pi_{s,t} \cdot E\left(q_{t,t+1}^{iI}(fe_{t+1}, S_t, D_t^{market}) \cdot \left\{ \frac{\sum_{s'=1}^{K} \pi_{s',t+1} \cdot P_{t+1}^{reg}(s')}{U'\left(\frac{1}{n_I} \cdot D_{t+1}^{market}\right)} + D_{t+1} \right\} \middle| S_t = s, D_t^{market} \right)$$

and

4-138

$$B_t^{iI}(\pi_t, D_t) = \sum_{s=1}^{K} \pi_{s,t} \cdot E\left(q_{t,t+1}^{iI}(fe_{t+1}, S_t, D_t^{market}) \middle| S_t = s, D_t\right)$$

with stochastic discount factor

4-139

$$q_{t,t+1}^{iI}(fe_{t+1}, S_t, D_t^{market}) = \frac{1}{1+\rho} \cdot \frac{U'\left(\frac{1}{n_I} \cdot D_{t+1}^{market}(D_{t+1})\right)}{U'\left(\frac{1}{n_I} \cdot D_t^{market}\right)}$$

with

$$D_{t+1} = D(S_t, fe_{t+1})$$
$$\pi_{t+1} = \Pi(\pi_t, D_t, D_{t+1}, Sig_{t+1})$$
$$Sig_{t+1} = Sig_{t+1}(S_t, S_{t+1}, fe_{t+1}, \eta_{t+1})$$

All results concerning the pricing of the various sources of risk derived for general cash flow models must evidently hold for the special case; in particular, only the risk sources S_t and fe_{t+1} are priced according to their relation to aggregate cash flows.

I discuss the various channels through which this relation between the stochastic discount factor and aggregate cash flows is introduced separately for asset prices P_{t+1}^{iI} and cash flows D_{t+1}.

Recall the four channels between stochastic discount factor $q_{t,t+1}^{iI}$ and asset prices P_{t+1}^{iI} from the general cash flow model. In this special case only channels (i) marginal utility and (iv) stochastic conditional probabilities remain relevant. Channels (ii) marginal utility at future points in time and (iii) cash flow do no longer exist.

To understand this result, plug in for the ${P_{t+1}}^{iI}$, i.e., combine the results on the relation between complete and incomplete information asset prices (4-103) (p. 140) with the properties of complete information asset prices for cash flow models without lags in levels (4-112):

4-140

$$ {P_{t+1}}^{iI} = \frac{\sum_{s'=1}^{K} \pi_{s',t+1} \cdot {P_{t+1}}^{reg}(s')}{U'\left(\frac{1}{n_I} \cdot {D_{t+1}}^{market}\right)} $$

From the decomposition (4-60) (p. 117), it is know that not all components of (4-140) are priced but only

4-141

$$ E\left({P_{t+1}}^{iI} \middle| \pi_t, {D_{t+1}}^{market}\right) = \frac{\sum_{s'=1}^{K} E\left(\pi_{s',t+1} \middle| \pi_t, {D_{t+1}}^{market}\right) \cdot {P_{t+1}}^{reg}(s')}{U'\left(\frac{1}{n_I} \cdot {D_{t+1}}^{market}\right)} $$

(4-141) demonstrates that in addition to the channel of marginal utility of consumption that is already present in the complete information case (channel (i) of the general case), correlations between asset prices and the stochastic discount factor are now also introduced through stochastic probabilities π_{t+1} (channel (iv) of the general case). The priced part of these stochastic probabilities is $E\left(\pi_{s',t+1} \middle| \pi_t, {D_{t+1}}^{market}\right)$ and has a simple interpretation: $E\left(\pi_{s',t+1} \middle| \pi_t, {D_{t+1}}^{market}\right)$ is a particular conditional probability for $S_{t+1} = s'$ that is formed by updating regime probabilities π_t by information on aggregate (but not individual) cash flows:

4-142

$$ E\left(\pi_{s',t+1} \middle| \pi_t, {D_{t+1}}^{market}\right) = P\left(S_{t+1} = s' \middle| \pi_t, {D_{t+1}}^{market}\right) $$ [31]

The channel of stochastic conditional probabilities (channel (iv)) can either reinforce or dampen the effect of the marginal utility channel. Consider a regime $S_{t+1} = s'$ that is likely to be the true regime if ${D_{t+1}}^{market}$ is high (and unlikely if ${D_{t+1}}^{market}$ is low). Then both the inverse of marginal utility of consumption, $\left[U'\left(\frac{1}{n_I} \cdot {D_{t+1}}^{market}\right)\right]^{-1}$(channel (i)), and the regime probability (4-142), $E\left(\pi_{s',t+1} \middle| \pi_t, {D_{t+1}}^{market}\right)$ (channel (iv)), will move in the same direction as aggregate cash flows and, therefore, introduce a negative covariance with the stochastic discount factor. On the other hand, if $S_{t+1} = s'$ is a regime that is unlikely to be the true regime if ${D_{t+1}}^{market}$ is high, $\left[U'\left(\frac{1}{n_I} \cdot\right.\right.$

[31] To see this, first observe that the probability of some event A is the expectation of the indicator function $\underline{1}_A$, i.e., $P(A) = E(\underline{1}_A)$. Applying this to the left-hand side of (4-142) yields $E\left(\pi_{s',t+1} \middle| \pi_t, {D_{t+1}}^{market}\right) = E\left(E\left(1_{S_{t+1}=s'} \middle| \pi_t, D_{t+1}, Sig_{t+1}\right) \middle| \pi_t, {D_{t+1}}^{market}\right)$. Further observe that knowledge of D_{t+1} implies knowledge of ${D_{t+1}}^{market}$, i.e., $\{\pi_t, D_{t+1}, Sig_{t+1}\} = \{\pi_t, D_{t+1}, {D_{t+1}}^{market}, Sig_{t+1}\}$, and apply the tower property of expectations.

${D_{t+1}}^{market}\big)\big]^{-1}$ (channel (i)) will move in the same direction as ${D_{t+1}}^{market}$ while $E\left(\pi_{s',t+1}\middle|\pi_t, {D_{t+1}}^{market}\right)$ (channel (iv)) will move in the opposite direction, with an unclear overall effect. Observe that the total effect of the marginal utility channel (channel (i)) and the channel of stochastic conditional regime probabilities (channel (iv)) on asset prices is captured by K asset-independent pricing components,

4-143

$$E\left({q_{t,t+1}}^{iI}\left(fe_{t+1}, S_t, {D_t}^{market}\right) \cdot \frac{E\left(\pi_{s',t+1}\middle|\pi_t, {D_{t+1}}^{market}\right)}{U'\left(\frac{1}{n_I} \cdot {D_{t+1}}^{market}\right)} \middle| \pi_t, {D_t}^{market} \right)$$

The price of asset prices ${P_{t+1}}^{iI}$ is a linear combination of the complete information price components ${P_{t+1}}^{reg}(s'), s' = 1, \dots, K,$ with the asset-independent components that price the sources of risk, (4-143):

4-144

$$E\left({q_{t,t+1}}^{iI}\left(fe_{t+1}, S_t, {D_t}^{market}\right) \cdot {P_{t+1}}^{iI}\left(\pi_{t+1}, {D_{t+1}}^{market}\right)\middle|\pi_t, {D_t}^{market}\right)$$
$$= \sum_{s'=1}^{K} E\left({q_{t,t+1}}^{iI}\left(fe_{t+1}, S_t, {D_t}^{market}\right) \cdot \frac{E\left(\pi_{s',t+1}\middle|\pi_t, {D_{t+1}}^{market}\right)}{U'\left(\frac{1}{n_I} \cdot {D_{t+1}}^{market}\right)} \middle| \pi_t, {D_t}^{market} \right) \cdot {P_{t+1}}^{reg}(s')$$

Since there are two channels in the incomplete information case, the price of ${P_{t+1}}^{iI}$ at time t therefore is no longer necessarily less than the "risk-neutral" discounting of $E\left({P_{t+1}}^{iI}\left(\pi_{t+1}, {D_{t+1}}^{market}\right)\middle|\pi_t\right)$ at the riskless rate as it is the case under complete information (4-119).

Concerning the pricing of cash flows D_{t+1} and the price of the one-period bond (4-58) the assumption of cash flow models without lags in levels does not lead to insights that go beyond general results and is, therefore, omitted.

Cash flows without lags in cash flows: affine linear factor model

Since the assumption of an affine linear factor model is merely more specific (compared to the more abstract form $D(S_t, fe_{t+1})$) with regard to the effect of factors and residuals on cash flows given a particular true regime S_t, not much would be gained by detailed discussion of the effect of an affine linear factor model on asset prices under incomplete information: incomplete information is concerned with the fact that the regime S_t is unobservable, whereas the affine linear factor model takes S_t as given.

4.3.2.1.2.2.2 Asset Prices as Discounted Future Cash Flows

Cash flows without lags in levels: no additional restrictions on the functional form of $D(.)$

Asset prices as discounted future cash flows are fully described by the relation between complete and incomplete information asset prices (4-103) (p. 140) in combination with complete information asset prices expressed as discounted future cash flows (4-126) (p. 152):

$$P_t^{iI}\left(\pi_t, D_t^{market}\right) = \sum_{s=1}^{K} \pi_{s,t} \cdot P_t^{cI}\left(s, D_t^{market}\right)$$

$$P_t^{cI}\left(S_t, D_t^{market}\right) = \sum_{\tau=1}^{T-t} E\left(q_{t,t+\tau}^{cI}\left(fe_{t+\tau}, S_{t+\tau-1}; D_t^{market}\right) \cdot D_{t+\tau} \middle| S_t, D_t^{market}\right)$$

The only remaining question of interest therefore is the pricing of the sources of risk compared to the complete information case. $S_{t+\tau-1}$, and $fe_{t+\tau}$ can be identified as the only relevant sources of risk(as opposed to path of future regimes and path of factors and residuals) for the same reason as under complete information. The difference to the complete information case lies in the distribution of $S_{t+\tau-1}$ since the current regime S_t is unobservable.

$S_{t+\tau-1}$ and $fe_{t+\tau}$ affect risk-adjusted pricing via a generalized version of the cash flow channel (4-49). This generalized cash flow channel can be described by means of the covariance between the stochastic discount factor and cash flows, which leads in the special case of cash flows without lags in levels to the following simplified version of (4-66):

4-145

$$cov\left(\frac{1}{(1+\rho)^\tau} \cdot \frac{U'\left(\frac{1}{n_I} \cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right)}{U'\left(\frac{1}{n_I} \cdot D_t{}^{market}\right)}, D_{t+\tau} \middle| \pi_t, D_t^{market}\right) = \frac{1}{(1+\rho)^\tau} \cdot \frac{1}{U'\left(\frac{1}{n_I} \cdot D_t{}^{market}\right)} \cdot$$

$$\left\{\sum_{s'=1}^{K} P(S_{t+\tau-1} = s'|\pi_t) \cdot cov\left(U'\left(\frac{1}{n_I} \cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right), D_{t+\tau} \middle| S_{t+\tau-1} = s'\right)\right.$$

$$\left. + cov\left(E\left(U'\left(\frac{1}{n_I} \cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right) \middle| S_{t+\tau-1}\right), E(D_{t+\tau}|S_{t+\tau-1}) \middle| \pi_t\right)\right\}$$

The covariance (4-145) reveals the details of risk-adjusted pricing. The first term in brackets on the right-hand side of (4-145) is the covariance of the stochastic discount factor and cash flows due to $fe_{t+\tau}$. The conditioning information simplifies to $S_{t+\tau-1}$ (instead of $S_{t+\tau-1}, D_{t+\tau-1}$ as for general cash flow models)) and is averaged out by transition probabilities $P(S_{t+\tau-1} = s'|\pi_t), s' = 1, \dots, K$. The second term in brackets captures the covariances of the stochastic discount factor and cash flows due to $S_{t+\tau-1}$ (instead of $S_{t+\tau-1}, D_{t+\tau-1}$), with factors and residuals $fe_{t+\tau}$ averaged out.

Cash flows without lags in cash flows: affine linear factor model

If it is further assumed that cash flows exhibit an affine linear factor model, the covariance conditional on $S_{t+\tau-1} = s'$ in (4-145),

$$cov\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right), D_{t+\tau}\middle|S_{t+\tau-1} = s'\right)$$

takes on a factor structure. However, since this covariance is conditional on $S_{t+\tau-1} = s'$, its specific form is not related to incomplete information. Similar reasoning holds for the expectations of cash flows conditional on $S_{t+\tau-1} = s'$ in (4-145),

$$E(D_{t+\tau}|S_{t+\tau-1})$$

Therefore, the implications of the assumptions of an affine linear factor model need not be further elaborated and can be omitted.

4.3.2.1.2.3 Equilibrium Risk Premia

Cash flows without lags in cash flows: no additional restrictions on the functional form of $D(.)$

If cash flow models belong to the subclass of this section, risk premia under incomplete information and general cash flow models (4-67) simplify to

$$RP_t{}^{iI}(\pi_t) = -cov\left(, \frac{AfR_{t,t+1}{}^{iI}(fe_{t+1}, S_t, \pi_t)}{\frac{\sum_{s'=1}^{K} E\left(\pi_{s',t+1}|\pi_t, D_{t+1}{}^{market}\right)\cdot P_{t+1}{}^{reg}(s')}{U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}\right)}} + E\left(D_{t+1}|\pi_t, D_{t+1}{}^{market}\right)\middle|\pi_t\right)$$

with

$$AfR_{t,t+1}{}^{iI}(fe_{t+1}, S_t, \pi_t) \equiv \frac{U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right)}{\sum_{s=1}^{K}\pi_{s,t}\cdot E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right)\middle|S_t = s\right)}$$

$$D_{t+1} = D(S_t, fe_{t+1})$$

$$\pi_{t+1} = \Pi(\pi_t, D_t, D_{t+1}, Sig_{t+1})$$

$$Sig_{t+1} = Sig_{t+1}(S_t, S_{t+1}, fe_{t+1}, \eta_{t+1})$$

Risk premia depend on conditional regime probabilities π_t and, therefore, on less information than asset prices, which depend, by definition, on information relevant to pricing $z_t{}^p = (\pi_t, D_t{}^{market})$. The underlying reason is similar to the complete information case: risk premia only depend on the conditional distribution of D_{t+1}, but this conditional distribution is summarized by π_t alone for cash flow models without lags in levels.

The risk premium consists of two parts stemming from asset prices $P_{t+1}{}^{cI}$ (first term) and cash flows D_{t+1} (second term). The risk premium part stemming from asset prices is governed by the channels of marginal utility of consumption at time t (channel (i)) and the channel of stochastic conditional regime probabilities (channel (iv)). Due to the interrelation between two channels, the risk premium is no longer necessarily non-negative as is the case under complete information (4-134). The risk premium part that stems from cash flows can be expressed through (i) risk-neutralized regime probabilities, capturing the risk of the unobservable regime, and (ii) complete information covariances and expectations, capturing the risk of factors and residuals (see the decomposition of risk premia (4-81) (p. 127)):

4-146

$$-cov\left(\begin{matrix} AfR_{t,t+1}{}^{iI}(fe_{t+1}, S_t, \pi_t) \\ , E\left(D_{t+1} \middle| \pi_t, D_{t+1}{}^{market}\right) \end{matrix} \middle| \pi_t \right)$$
$$= -\sum_{s=1}^{K} \{\theta_{s,t}(\pi_t) - \pi_{s,t}\} \cdot E\left(E\left(D_{t+1} \middle| S_t, D_{t+1}{}^{market}(D_{t+1})\right) \middle| S_t = s\right)$$
$$-\sum_{s=1}^{K} \theta_{s,t}(\pi_t)$$
$$\cdot cov\left(AfR_{t,t+1}{}^{cI}(fe_{t+1}, S_t), E\left(D_{t+1} \middle| S_t, D_{t+1}{}^{market}(D_{t+1})\right) \middle| S_t = s\right)$$

Cash flows without lags in cash flows: affine linear factor model

Since the assumption of an affine linear factor model is merely more specific (compared to the more abstract form $D(S_t, fe_{t+1})$) with regard to the effect of factors and residuals on cash flows given a particular true regime S_t, not much would be gained by detailed discussion of the effect of an affine linear factor model on asset prices under incomplete information: incomplete information is concerned with the fact that the regime S_t is unobservable, whereas the affine linear factor model takes S_t as given.

4.3.3 Cash Flow Models without Lags in Growth Rates under Constant Relative Risk Aversion

Cash flow models without lags in growth rates read by definition

2-9

$$D_{i,t+1} = D_{i,t} \cdot [1 + d_i(S_t, fe_{t+1})]$$
$$i = 1, \dots, n$$

It is further assumed that cash flow growth rates cannot be less than -1, i.e., special case of limited liability is solely considered, and that initial cash flows are positive for all assets:[32]

[32] The restriction to nonnegative cash flows is motivated by economic intuition and not necessary to derive the pricing results for cash flow models without lags in growth rates. To understand the argument of the economic intuition, consider the case of a negative cash flow. A positive 1 + growth rate – something that has a positive connotation - combined with a negative cash flows means that the cash flows becomes even more

$$d_i(S_t, fe_{t+1}) \geq -1$$
$$D_{i,0} > 0$$
$$i = 1, \dots, n$$

These assumptions jointly imply that cash flows are always non-negative and, in particular, that investors cannot lose more than their initial investments. In view of these properties, cash flows in this section can be interpreted and will be referred to as dividends.

4.3.3.1 Complete Information

4.3.3.1.1 Information Relevant to Pricing $z_t{}^p$

For cash flow models without lags in growth rates under constant relative risk aversion (CRRA) and complete information, information relevant to pricing for all assets remains unchanged relative to the general case, i.e.,

4-147
$$z_t{}^{cI,p} = (S_t, D_t)$$

However, information relevant to pricing for an individual asset i is simplified considerably relative to the general case and reads

4-148
$$z_{i,t}{}^{cI,p} = \left(S_t, D_{i,t}, \delta_t\right)$$

where $\delta_t \equiv \left(\delta_{1,t}, \dots, \delta_{n,t}\right)$ and $\delta_{j,t}$ denotes dividends paid by company j relative to dividends paid by the market portfolio of risky assets,

4-149
$$\delta_{j,t} \equiv \frac{\bar{N}_j \cdot D_{j,t}}{\sum_{v=1}^{n} \bar{N}_v \cdot D_{v,t}} = \frac{\bar{N}_j \cdot D_{j,t}}{D_t{}^{market}}$$
$$j = 1, \dots, n$$

which, for brevity, will be referred to as relative dividend contributions.

In words, information relevant to pricing for an individual asset i includes its own dividend payment $D_{i,t}$ and the relative dividend contributions of all risky assets, but neither dividend levels of other individual assets nor the level of aggregate dividends.

It is more difficult to explain the composition of $z_{i,t}{}^p$ intuitively than in the previous cases. This difficulty is due to the fact that the simplification to relative dividend contributions results as much from the form of the utility function (CRRA) as from the special properties of the cash flow function. Constant relative risk aversion leads to stochastic discount factors that only depend on dividend

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negative; a negative1 + growth rate (negative connotation) combined with a negative cash flow results in a positive cash flow.

growth rates and relative dividend contributions. The special dividend model of this section means that dividend growth rates are independent of past dividend levels. These properties of utility and special cash flow models jointly imply that stochastic discount factors only depend on relative dividend contributions but not in any way on dividend levels.

4.3.3.1.2 Equilibrium Asset Prices

4.3.3.1.2.1 Quasi-Static Asset Prices

If cash flow models belong to the subclass of this section and investors exhibit CRRA preferences, quasi-static asset prices under complete information and general cash flow models (4-41) and (4-42) simplify to (for the derivation, see Appendix A3.7)

4-150

$$P_{i,t}{}^{cI}\left(S_t, D_{i,t}, \delta_t\right)$$
$$= D_{i,t} \cdot E\left(q_{t,t+1}{}^{cI}(fe_{t+1}, S_t, \delta_t) \cdot [1 + d_i(S_t, fe_{t+1})] \cdot \left\{ \left(\frac{P}{D}\right)_{i,t+1}^{cI} (S_{t+1}, \delta_{t+1}) + 1 \right\} \middle| S_t, \delta_t \right)$$
$$i = 1, \dots, n$$

4-151

$$B_t{}^{cI}(S_t, \delta_t) = E\left(q_{t,t+1}{}^{cI}(fe_{t+1}, S_t, \delta_t) \middle| S_t, \delta_t\right)$$

with stochastic discount factor

4-152

$$q_{t,t+1}{}^{cI}(fe_{t+1}, S_t, \delta_t) = \frac{1}{1+\rho} \cdot \left\{\left[1 + d_{t+1}{}^{market}(\delta_t, S_t, fe_{t+1})\right]\right\}^{-\gamma}$$

with aggregate dividend growth

4-153

$$d_{t+1}{}^{market}(\delta_t, S_t, fe_{t+1}) \equiv \sum_{\nu=1}^{n} \delta_{\nu,t} \cdot d_\nu(S_t, fe_{t+1})$$

with dynamics of relative dividend contributions

4-154

$$\delta_{j,t+1}(\delta_t, S_t, fe_{t+1}) = \frac{\delta_{j,t} \cdot \left[1 + d_j(S_t, fe_{t+1})\right]}{\sum_{\nu=1}^{n} \delta_{\nu,t} \cdot [1 + d_\nu(S_t, fe_{t+1})]}$$
$$j = 1, \dots, n$$

and with price dividend ratio, i.e., the inverse of the dividend yield, at time $t+1$

4-155

$$\left(\frac{P}{D}\right)_{i,t+1}^{cI} (S_{t+1}, \delta_{t+1}) \equiv \begin{cases} \dfrac{P_{i,t+1}\left(S_{t+1}, D_{i,t+1}, \delta_{t+1}\right)}{D_{i,t+1}} & D_{i,t+1} > 0 \\ 0 & D_{i,t+1} = 0 \end{cases}$$
$$i = 1, \dots, n$$

The stochastic discount factor (4-152) depends on the growth rate of aggregate dividends that, in turn, is a weighted sum of the individual dividend growth rates where the weights are relative dividend contributions. Given individual dividend growth rates $d_1, \dots, d_n$, a change in relative dividend contributions such that contributions of assets with high dividend growth increase relative to assets with low dividend growth will result in higher aggregate dividend growth ${d_{t+1}}^{market}$. Hence, the stochastic discount factor decreases because marginal utility at time $t + 1$ decreases. In contrast to general cash flow models the stochastic discount factor (4-152) does not depend on dividend levels. To understand why dividend levels do not matter, consider a proportional increase of dividends of all assets in time t. Then the structure of the cash flow model without lags in growth rates implies that dividends at time $t + 1$ change by the same proportion, and the properties of constant relative risk aversion further imply that the relation of marginal utility at time $t + 1$ to marginal utility at time t remains constant.

The fact that aggregate dividend levels do not play a role for the stochastic discount factor translates to the risky asset prices (4-150) and the price of the one-period riskless bond (4-151). To illustrate this fact, tautologically re-express the price for general cash flow models (4-41) in the form of a price dividend ratio

$$ {P_{i,t+1}}^{cI} = \underbrace{D_{i,t}}_{\substack{current\ dividend \\ level}} \cdot \underbrace{\left\{\frac{D_{i,t+1}}{D_{i,t}}\right\}}_{dividend\ growth} \cdot \underbrace{\left\{\frac{{P_{i,t+1}}^{cI}}{D_{i,t+1}}\right\}}_{price\ dividend\ ratio} $$

For dividends without lags in growth rates, both dividend growth rates and price dividend ratios take special forms

4-156

$$ {P_{i,t+1}}^{cI} = \underbrace{D_{i,t}}_{\substack{current\ dividend \\ level}} \cdot \underbrace{[1 + d_i(S_t, fe_{t+1})]}_{dividend\ growth} \cdot \underbrace{\left(\frac{P}{D}\right)_{i,t+1}^{cI} (S_{t+1}, \delta_{t+1})}_{price\ dividend\ ratio} $$

In this special case neither dividend growth rates nor the price dividend ratio depend on dividend levels. While this is clear in the case of dividend growth rates by definition, some justification is needed for the price dividend ratios. Intuitively, if the dividend of asset i at time t is multiplied by a constant factor, dividends at future dates change by the same factor. Therefore, the price of asset i at time t – the present value of future dividends - will change by the same factor as well. This in turn means that the price dividend ratio is constant.

Concerning the price of the one-period riskless bond (4-151), the special cash flow model without lags in growth rates shows that the one-period riskless bond price depends on relative dividend contributions via the stochastic discount factor. Furthermore, this special cash flow model cannot contribute over what is already knows from general cash flow models.

With the help of Equations (4-150) and (4-151) the pricing statement of the general cash flows model (4-41) (p. 108) regarding the pricing of the two sources of risk S_{t+1} and fe_{t+1} can be further specified. To that end, recall that the pricing of the sources of risk in general depends on their relation to aggregate cash flows. For the subclass of cash flows without lags in growth rates, these general results can be reformulated in terms of growth rates of individual and aggregate dividends (4-153). Only that part of capital gains and dividend growth that can be explained through the growth rate of aggregate cash flows is priced:

4-157

$$P_{i,t}{}^{cI}(S_t, D_{i,t}, \delta_t) = D_{i,t} \cdot E\left(\begin{array}{c} q_{t,t+1}{}^{cI}(fe_{t+1}, S_t, \delta_t) \\ \cdot E\left([1 + d_i(S_t, fe_{t+1})] \cdot \left\{ \left(\frac{P}{D}\right)_{i,t+1}^{cI} (S_{t+1}, \delta_{t+1}) + 1 \right\} \middle| S_t, \delta_t, d_{t+1}{}^{market} \right) \end{array} \middle| S_t, \delta_t \right)$$

$$i = 1, \ldots, n$$

All channels through which factors and residuals fe_{t+1} exert influence on $P_{i,t}{}^{cI}$ that have been identified in the general cash flow model exist in this special cash flow model without lags in growth rates (in contrast to special cash flow models without lags in levels). Since covariances of the stochastic discount factor with $P_{t+1}{}^{cI}$ and covariances of the stochastic discount factor with D_{t+1} are not immediately visible in (4-157), they must be located first.

The price channel - covariances of the stochastic discount factor with $P_{t+1}{}^{cI}$ - reads in this special case:

covariance of the stochastic discount factor with $D_{i,t} \cdot [1 + d_i(S_t, fe_{t+1})] \cdot \left(\frac{P}{D}\right)_{i,t+1}^{cI} (S_{t+1}, \delta_{t+1})$;

the cash flow channel - covariances of the stochastic discount factor with D_{t+1} - is: covariance of the stochastic discount factor with $D_{i,t} \cdot \left(1 + d_i(S_t, fe_{t+1})\right)$.

Within the three channels of the price channel, the channel of marginal utility of consumption at time $t + 1$ (channel (i)) reads

$$U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}\right) = \left\{\frac{1}{n_I} \cdot D_{t+1}{}^{market}\right\}^{-\gamma}$$

and the channel of future marginal utility of consumption in points of time $t + 1 + \tau, \tau \geq 1$ (channel (ii)) reads

$$U'\left(\frac{1}{n_I} \cdot D_{t+1+\tau}{}^{market}\right) = \left\{\frac{1}{n_I} \cdot D_{t+1}{}^{market} \cdot \sum_{\nu=1}^{n} \delta_{\nu,t+1} \cdot \left[1 + d_\nu{}^{+(\tau)}\left(S_{t+1,t+\tau-1}, fe_{t+1,t+1+\tau}\right)\right]\right\}^{-\gamma}$$

with

$$d_\nu{}^{+(\tau)}\left(S_{t+1,t+\tau-1}, fe_{t+1,t+1+\tau}\right) = \prod_{\zeta=1}^{\tau} \left[1 + d_\nu\left(S_{t+1+\zeta-1}, fe_{t+1+\zeta}\right)\right] - 1$$

Since the stochastic discount factor is a ratio of channel (ii) divided by channel (i), the effects of dividend levels ($\left\{\frac{1}{n_I} \cdot D_{t+1}{}^{market}\right\}^{-\gamma}$) cancel out. Instead, only relative dividend contributions matter. The channel of future dividends $D_{t+1+\tau}$ (channel (iii)) remains unchanged compared to the general case flow case in that the level of individual dividends still is decisive because $D_{i,t+1+\tau} = D_{i,t+1} \cdot \prod_{\zeta=1}^{\tau}\left[1 + d_i\left(S_{t+1+\zeta-1}, fe_{t+1+\zeta}\right)\right]$ cannot be simplified.

For a similar reason the cash flow channel - covariances of the stochastic discount factor with $1 + d_i(S_t, fe_{t+1})$ -, is unchanged.

4.3.3.1.2.2 Asset Prices as Discounted Future Cash Flows

If cash flow models belong to the subclass of this section, asset prices as discounted future cash flows under complete information (4-50) simplify to

4-158

$$P_{i,t}{}^{cI}\left(S_t, D_{i,t}, \delta_t\right)$$
$$= D_{i,t} \cdot \sum_{\tau=1}^{T-t} E\left(q_{t,t+\tau}{}^{cI}\left(fe_{t+1,t+\tau}, S_{t+1,t+\tau-1}; S_t, \delta_t\right) \cdot \left[1 + d_i{}^{+\tau}\left(S_{t,t+\tau-1}, fe_{t+1,t+\tau}\right)\right] \middle| S_t, \delta_t\right)$$
$$i = 1, \dots, n$$

with multi-period stochastic discount factors

4-159

$$q_{t,t+\tau}{}^{cI}\left(fe_{t+1,t+\tau}, S_{t+1,t+\tau-1}; S_t, \delta_t\right) = \frac{1}{(1+\rho)^{\tau}} \cdot \left\{\sum_{\nu=1}^{n} \delta_{\nu,t} \cdot \left[1 + d_\nu{}^{+(\tau)}\left(S_{t,t+\tau-1}, fe_{t+1,t+\tau}\right)\right]\right\}^{-\gamma}$$

with

$$d_\nu{}^{+(\tau)}\left(S_{t+1,t+\tau-1}, fe_{t+1,t+1+\tau}\right) = \prod_{\zeta=1}^{\tau}\left[1 + d_\nu\left(S_{t+\zeta-1}, fe_{t+1+\zeta}\right)\right] - 1$$

The pricing equation (4-158) and the multi-period stochastic discount factor (4-159) demonstrate how the two sources of risk (i) path of future regimes $S_{t+1,t+\tau}$ - note the current regime S_t is non-stochastic - and (ii) path of factors and residuals $fe_{t+1,t+\tau}$ are priced. Recall from the general cash flow model that not the paths of regimes $S_{t,t+\tau-1}$ as well as factors and residuals $fe_{t+1,t+\tau}$ as such exert influence on prices but only their joint effects on $D_{t+\tau-1}$, $S_{t+\tau-1}$, and $fe_{t+\tau}$. For cash flow model without lags in growth rates, cash flows $D_{t+\tau-1}$ are merely replaced by the more specific $\tau - 1$-period growth rates $d_\nu{}^{+(\tau-1)}, \nu = 1, \dots, n$. In other words, assuming a cash flow model without lags in growth rates does not simplify the situation as much as in the case of cash flows without lags in levels: for cash flow model without lags in levels, cash flows are only inter-temporally related via the regime; by contrast, for cash flow model without lags in growth rates, future cash flows are still directly related to current cash flows (in addition to the indirect relation via the regime).

The channel through which ${d_v}^{+(\tau-1)}$, $S_{t+\tau-1}$ and $fe_{t+\tau}$ exert price influence is the generalized cash flow channel (4-49) and is described by means of the covariance between the stochastic discount factor and cash flows, which leads in the special case of cash flows without lags in growth rates to the following simplified version of (4-54):

4-160

$$cov\left(\begin{matrix}\frac{1}{(1+\rho)^\tau}\cdot\left\{\sum_{v=1}^{n}\delta_{v,t}\cdot\left[1+{d_v}^{+(\tau)}\left(S_{t,t+\tau-1},fe_{t+1,t+\tau}\right)\right]\right\}^{-\gamma} \\ ,\left[1+{d_i}^{+(\tau)}\left(S_{t,t+\tau-1},fe_{t+1,t+1+\tau}\right)\right]\end{matrix}\middle|S_t,\delta_t\right)$$

$$=\frac{1}{(1+\rho)^\tau}\cdot$$

$$\left\{E\left(cov\left(\begin{matrix}\left\{\sum_{v=1}^{n}\delta_{v,t}\cdot\left[1+{d_v}^{+(\tau)}\left(S_{t+1,t+\tau-1},fe_{t+1,t+\tau}\right)\right]\right\}^{-\gamma} \\ ,\left[1+{d_i}^{+(\tau)}\left(S_{t+1,t+\tau-1},fe_{t+1,t+1+\tau}\right)\right]\end{matrix}\middle|S_{t+\tau-1},d^{+(\tau-1)},\delta_t\right)\middle|S_t,\delta_t\right)\right.$$

$$\left.+\,cov\left(\begin{matrix}E\left(\left\{\sum_{v=1}^{n}\delta_{v,t}\cdot\left[1+{d_v}^{+(\tau)}\left(S_{t+1,t+\tau-1},fe_{t+1,t+\tau}\right)\right]\right\}^{-\gamma}\middle|S_{t+\tau-1},d^{+(\tau-1)},\delta_t\right) \\ ,E\left({d_i}^{+(\tau-1)}\middle|S_{t+\tau-1},d^{+(\tau-1)},\delta_t\right)\end{matrix}\middle|S_t,\delta_t\right)\right\}$$

The covariance (4-160) reveals the details of risk-adjusted pricing. The first term in brackets on the right-hand side of (4-160) is the covariance of the stochastic discount factor and cash flows due to $fe_{t+\tau}$. The conditioning information simplifies to $S_{t+\tau-1},d^{+(\tau-1)},\delta_t$ (instead of $S_{t+\tau-1},D_{t+\tau-1}$ as for general cash flow models)). The second term in brackets captures the covariances of the stochastic discount factor and cash flows due to $S_{t+\tau-1}$ and $d^{+(\tau-1)}$ (instead of $S_{t+\tau-1},D_{t+\tau-1}$), with factors and residuals $fe_{t+\tau}$ averaged out.

4.3.3.1.3 Equilibrium Risk Premia

If cash flow models belong to the subclass of this section, risk premia under complete information and general cash flow models (4-55) simplify to

4-161

$$RP_t^{cI}\left(S_t,D_{i,t},\delta_t\right)=-D_{i,t}\cdot cov\left(\begin{matrix}AfR_{t,t+1}{}^{cI}\left(fe_{t+1},S_t,\delta_t\right) \\ ,\left[1+d_i\left(S_t,fe_{t+1}\right)\right]\cdot\left\{\left(\frac{P}{D}\right)_{i,t+1}^{cI}\left(S_{t+1},\delta_{t+1}\right)+1\right\}\end{matrix}\middle|S_t,\delta_t\right)$$

with

$$AfR_{t,t+1}{}^{cI}\left(fe_{t+1},S_t,\delta_t\right)\equiv\frac{\left\{\sum_{v=1}^{n}\delta_{v,t}\cdot d_v\left(S_t,fe_{t+1}\right)\right\}^{-\gamma}}{E\left(\left\{\sum_{v=1}^{n}\delta_{v,t}\cdot d_v\left(S_t,fe_{t+1}\right)\right\}^{-\gamma}\middle|S_t,\delta_t\right)}$$

$$d_{t+1}=d_v\left(S_t,fe_{t+1}\right)$$

Like quasi-static asset prices, the risk premium is the product of the current dividend level $D_{i,t}$ and a structure term that depends on relative dividend contributions only. The risk premium is governed by the three price channels and the one cash flow channel in a way that is parallel to quasi-static asset prices.

4.3.3.2 Incomplete Information

4.3.3.2.1 Information Relevant to Pricing ${z_t}^p$

For cash flow models without lags in growth rates under constant relative risk aversion (CRRA) and incomplete information, information relevant to pricing for all assets remains unchanged relative to the general case, i.e.,

4-162
$$ {z_t}^{iI,p} = (\pi_t, D_t) $$

However, information relevant to pricing for an individual asset i is simplified considerably relative to the general case and reads

4-163
$$ {z_{i,t}}^{iI,p} = \left(\pi_t, D_{i,t}, \delta_t\right) $$

The argument is parallel to the complete information case, with conditional regime probabilities π_t replacing their complete information counterparts, observable current regimes S_t.

4.3.3.2.2 Equilibrium Asset Prices

4.3.3.2.2.1 Quasi-Static Asset Prices

If cash flow models belong to the subclass of this section and investors exhibit CRRA preferences, quasi-static asset prices under incomplete information and general cash flow models (4-57) and (4-58) (see p. 116) simplify to

4-164
$$ {P_{i,t}}^{iI}\left(\pi_t, D_{i,t}, \delta_t\right) = D_{i,t} \cdot \sum_{s=1}^{K} \pi_{s,t} \cdot E\left(\begin{matrix} {q_{t,t+1}}^{iI}(fe_{t+1}, S_t, \delta_t) \\ \cdot [1 + d_i(S_t, fe_{t+1})] \cdot \left\{ \sum_{s'=1}^{K} \pi_{s',t+1} \cdot \left(\frac{P}{D}\right)_{i,t+1}^{cI} (s', \delta_{t+1}) + 1 \right\} \end{matrix} \middle| S_t = s, \delta_t \right) $$

4-165

$$B_t{}^{iI}(\pi_t, \delta_t) = \sum_{s=1}^{K} \pi_{s,t} \cdot E\left(q_{t,t+1}{}^{iI}(fe_{t+1}, S_t, \delta_t) \middle| S_t, \delta_t\right)$$

with stochastic discount factor

4-166

$$q_{t,t+1}{}^{iI}(fe_{t+1}, S_t, \delta_t) = \frac{1}{1+\rho} \cdot \left\{\left[1 + d_{t+1}{}^{market}(\delta_t, S_t, fe_{t+1})\right]\right\}^{-\gamma}$$

with

$$\pi_{t+1} = \Pi(\pi_t, d_{t+1}, Sig_{t+1})$$

$$Sig_{t+1} = Sig_{t+1}(S_t, S_{t+1}, fe_{t+1}, \eta_{t+1})$$

with aggregate dividend growth (4-153)

$$d_{t+1}{}^{market}(\delta_t, S_t, fe_{t+1}) \equiv \sum_{\nu=1}^{n} \delta_{\nu,t} \cdot d_\nu(S_t, fe_{t+1})$$

with dynamics of relative dividend contributions and complete information price dividend ratios given in (4-154) and (4-155).

All results concerning the pricing of the various sources of risk derived for general cash flow models must evidently hold for the special case; in particular, only the risk sources S_t and fe_{t+1} are priced according to their relation to aggregate cash flows.

The four channels that introduce correlations between the stochastic discount factor and asset prices identified for general cash flow models under incomplete information still exist. In addition to the three channels already present in the complete information case, stochastic conditional probabilities introduce another source of correlations between the stochastic discount factor and asset prices. Jointly with relative dividend contributions δ_{t+1} (and, therefore, channels (ii) and (iii)), conditional regime probabilities affect the price dividend ratio at time $t+1$:

4-167

$$P_{i,t+1}{}^{iI} = \underbrace{D_{i,t}}_{\substack{current\ dividend \\ level}} \cdot \underbrace{[1 + d_i(S_t, fe_{t+1})]}_{dividend\ growth} \cdot \underbrace{\sum_{s'=1}^{K} \pi_{s',t+1} \cdot \left(\frac{P}{D}\right)_{i,t+1}^{cI} (s', \delta_{t+1})}_{price\ dividend\ ratio}$$

Concerning the pricing of dividends D_{t+1} and the one-period riskless bond (4-165), the special cash flow model without lags in growth rates shows that the one-period riskless bond price depends on relative dividend contributions via the stochastic discount factor. Furthermore, this special cash flow model cannot contribute over what is already knows from general cash flow models.

4.3.3.2.2.2 Asset Prices as Discounted Future Cash Flows

If cash flow models belong to the subclass of this section, asset prices as discounted future cash flows under incomplete information (4-63) simplify to

4-168

$$P_{i,t}{}^{iI}(\pi_t, D_{i,t}, \delta_t)$$
$$= D_{i,t}$$
$$\cdot \sum_{s=1}^{K} \pi_{s,t}$$
$$\cdot \sum_{\tau=1}^{T-t} E\left(q_{t,t+\tau}{}^{iI}(fe_{t+1,t+\tau}, S_{t+1,t+\tau-1}; S_t, \delta_t) \cdot \left[1 + d_i{}^{+\tau}(S_{t,t+\tau-1}, fe_{t+1,t+\tau})\right] \middle| S_t = s, \delta_t\right)$$
$$i = 1, \dots, n$$

with multi-period stochastic discount factors

4-169

$$q_{t,t+\tau}{}^{iI}(fe_{t+1,t+\tau}, S_{t+1,t+\tau-1}; S_t, \delta_t) = \frac{1}{(1+\rho)^\tau} \cdot \left\{\sum_{v=1}^{n} \delta_{v,t} \cdot \left[1 + d_v{}^{+(\tau)}(S_{t,t+\tau-1}, fe_{t+1,t+\tau})\right]\right\}^{-\gamma}$$

with

$$d_v{}^{+(\tau)}(S_{t+1,t+\tau-1}, fe_{t+1,t+1+\tau}) = \prod_{\zeta=1}^{\tau}\left[1 + d_v(S_{t+\zeta-1}, fe_{t+1+\zeta})\right] - 1$$

The channel through which $d_v{}^{+(\tau-1)}$, $S_{t+\tau-1}$ and $fe_{t+\tau}$ exert price influence is the generalized cash flow channel (4-49) and is described by means of the covariance between the stochastic discount factor and cash flows, which leads in the special case of cash flows without lags in growth rates to the following simplified version of (4-54):

4-170

$$cov\left(\begin{array}{c}\frac{1}{(1+\rho)^\tau} \cdot \left\{\sum_{v=1}^{n} \delta_{v,t} \cdot \left[1 + d_v{}^{+(\tau)}(S_{t,t+\tau-1}, fe_{t+1,t+\tau})\right]\right\}^{-\gamma} \\ ,\left[1 + d_i{}^{+(\tau)}(S_{t,t+\tau-1}, fe_{t+1,t+1+\tau})\right]\end{array}\middle| \pi_t, \delta_t\right)$$
$$= \frac{1}{(1+\rho)^\tau} \cdot$$
$$E\left(cov\left(\begin{array}{c}\left\{\sum_{v=1}^{n} \delta_{v,t} \cdot \left[1 + d_v{}^{+(\tau)}(S_{t+1,t+\tau-1}, fe_{t+1,t+\tau})\right]\right\}^{-\gamma} \\ ,\left[1 + d_i{}^{+(\tau)}(S_{t+1,t+\tau-1}, fe_{t+1,t+1+\tau})\right]\end{array}\middle| S_{t+\tau-1}, d^{+(\tau-1)}, \delta_t\right)\middle| \pi_t, \delta_t\right)$$
$$+ cov\left(\begin{array}{c}E\left(\left\{\sum_{v=1}^{n} \delta_{v,t} \cdot \left[1 + d_v{}^{+(\tau)}(S_{t+1,t+\tau-1}, fe_{t+1,t+\tau})\right]\right\}^{-\gamma} \middle| S_{t+\tau-1}, d^{+(\tau-1)}, \delta_t\right) \\ ,E\left(d_i{}^{+(\tau-1)} \middle| S_{t+\tau-1}, d^{+(\tau-1)}, \delta_t\right)\end{array}\middle| \pi_t, \delta_t\right)$$

The pricing of these sources of risk is not much simplified compared to general cash flow models and, hence, the same economic interpretations hold. The covariance of the stochastic discount factor with dividends can be attributed to a covariance due to $fe_{t+\tau}$ on the one hand and a covariance due to $d_v{}^{+(\tau-1)}, v = 1, \dots, n$, and $S_{t+\tau-1}$ on the other hand.

4.3.3.2.3 Equilibrium Risk Premia

If cash flow models belong to the subclass of this section, risk premia under complete information and general cash flow models (4-67) simplify to

4-171

$$RP_t{}^{iI}(\pi_t, D_{i,t}, \delta_t) = -D_{i,t} \cdot cov\left(\begin{array}{c} AfR_{t,t+1}{}^{iI}(fe_{t+1}, S_t, \delta_t, \pi_t) \\ , [1 + d_i(S_t, fe_{t+1})] \cdot \left\{ \sum_{s'=1}^{K} \pi_{s',t+1} \cdot \left(\frac{P}{D}\right)_{i,t+1}^{cI} (s', \delta_{t+1}) + 1 \right\} \end{array} \middle| \pi_t, \delta_t \right)$$

with

$$AfR_{t,t+1}{}^{iI}(fe_{t+1}, S_t, \delta_t, \pi_t) \equiv \frac{\left\{\sum_{v=1}^{n} \delta_{v,t} \cdot d_v(S_t, fe_{t+1})\right\}^{-\gamma}}{\sum_{s=1}^{K} \pi_{s,t} \cdot E\left(\left\{\sum_{v=1}^{n} \delta_{v,t} \cdot d_v(S_t, fe_{t+1})\right\}^{-\gamma} \middle| S_t = s, \delta_t\right)}$$

$$d_{t+1} = d_v(S_t, fe_{t+1})$$

$$\delta_{j,t+1}(\delta_t, S_t, fe_{t+1}) = \frac{\delta_{j,t} \cdot [1 + d_j(S_t, fe_{t+1})]}{\sum_{v=1}^{n} \delta_{v,t} \cdot [1 + d_v(S_t, fe_{t+1})]}$$

$$j = 1, \dots, n$$

$$\pi_{t+1} = \Pi(\pi_t, D_t, D_{t+1}, Sig_{t+1})$$

$$Sig_{t+1} = Sig_{t+1}(S_t, S_{t+1}, fe_{t+1}, \eta_{t+1})$$

As under complete information, the risk premium is the product of the current dividend level $D_{i,t}$ and a structural term that depends on relative dividend contributions only. In addition to the three price channels and the one cash flow channel that cause covariances of the stochastic discount factor with asset prices under complete information, the fourth price channel of stochastic conditional regime probabilities is now also present under incomplete information.

4.4 Asset Prices for Large Time Horizons

4.4.1 Motivation and Methodology

Asset prices at some point of time t have so far been derived given a finite remaining time horizon $T - t$. However, it would be desirable to extend the analysis to cases where the time horizon is large, if possible, infinite to cope with, e.g., dividends paid by stocks.

Infinite time horizons are tackled in the dynamic programming literature in two ways. First, when a direct utility function is given, a value function for the infinite time horizon is guessed and verified by means of the Bellman equation and a transversality condition (see, e.g., Ingersoll (1987) , p. 274).

Second, the value function for infinite time horizons is obtained, if feasible, from iterating the optimal value function of the corresponding finite case, i.e., using so called "value iteration", (see Bertsekas (2007), p. 8, 9). However, the "guess and verify approach" is only applicable for specific utility functions and, hence, is not a valid alternative for my analysis because a concrete utility function is not specified. "Value iteration" is confronted with the problem that it is only valid under fairly general, but not all conditions (see Bertsekas (2007), p. 9). These conditions are cumbersome and also depend on the concrete specification of the utility function (see Bertsekas (2007), p. 124 and p. 135). The "guess and verify approach" is completely infeasible without specifying a concrete utility function, whereas "value iteration" can at least be implemented but not verified whether the conditions for convergence to a proper infinite horizon problem are met. Therefore, I use "value iteration" and deliberately leave open the problem of convergence conditions.

More formally, "value iteration" is used to analyze asset prices for large time horizons. The idea of a large time horizon is formalized by the limit $lim_{T-t\to\infty} P_t({z_t}^p; T-t)$ where $T-t$ denotes the remaining time horizon. Of course this limit does not necessarily need to exist, precisely because the cash flows model is fairly general. For that reason, the special cash flow models without lags in levels (Section 4.3.2) and without lags in growth rates (Section 4.3.3) are considered for the case where information frequency is equal to cash flow frequency.

One final remark is in order. It suffices to analyze complete information asset prices because incomplete information asset prices are expectations of complete information asset prices and, therefore, finite if all complete information asset prices are finite.

4.4.2 The Case of Cash Flows without Lags in Levels

If cash flow models do not exhibit lags in levels, asset prices as discounted future cash flows read ((4-126) with (4-127)):

$$ {P_t}^{cI}\left(S_t, {D_t}^{market}; T-t\right) = \sum_{\tau=1}^{T-t} E\left({q_{t,t+\tau}}^{cI}\left(fe_{t+\tau}, S_{t+\tau-1}; {D_t}^{market}\right) \cdot D_{t+\tau} \middle| S_t, {D_t}^{market}\right) $$

with multi-period stochastic discount factors

$$ {q_{t,t+\tau}}^{cI}\left(fe_{t+\tau}, S_{t+\tau-1}; {D_t}^{market}\right) = \frac{1}{(1+\rho)^\tau} \cdot \frac{U'\left(\frac{1}{n_I} \cdot {D_{t+\tau}}^{market}(D_{t+\tau})\right)}{U'\left(\frac{1}{n_I} \cdot {D_t}^{market}\right)} $$

with

$$ D_{t+\tau} = D(S_{t+\tau-1}, fe_{t+\tau}) $$

To show that prices converge as the time horizon goes to infinity, it suffices to show that $E\left(U'\left(\frac{1}{n_I} \cdot {D_{t+\tau}}^{market}(D_{t+\tau})\right) \cdot D_{t+\tau} \middle| S_t\right)$ is bounded for all possible values of τ. Discounting with the subjective preference term $\frac{1}{(1+\rho)^{\tau}}\cdot$ then leads to a convergent series. To that end, rewrite these conditional expectations into the form

4-172

$$E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right)\cdot D_{t+\tau}\middle|S_t\right)$$
$$=\sum_{s'=1}^{K} P(S_{t+\tau-1}=s'|S_t)\left\{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right)\cdot D_{t+\tau}\middle|S_{t+\tau-1}=s'\right)\right\}$$

with

$$D_{t+\tau}= D(S_{t+\tau-1},fe_{t+\tau})$$

Because both the cash flow function and the regime process have been assumed to be time-homogenous, it can be concluded that the conditional expectations

4-173

$$(EU'D)(s')\equiv E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right)\cdot D_{t+\tau}\middle|S_{t+\tau-1}=s'\right)$$

are independent of the time index, i.e.,

$$(EU'D)(s')=E\left(U'\left(\frac{1}{n_I}\cdot D_{t_1+1}{}^{market}(D_{t_1+1})\right)\cdot D_{t_1+1}\middle|S_{t_1}=s'\right)$$
$$=E\left(U'\left(\frac{1}{n_I}\cdot D_{t_2+1}{}^{market}(D_{t_2+1})\right)\cdot D_{t_2+1}\middle|S_{t_2}=s'\right)$$

for any two time indices t_1 and t_2. It follows that the expectations (4-172) are for all values of τ linear combinations of the K conditional expectations $(EU'D)(s'), s'=1,\dots,K$. Moreover, because probabilities cannot exceed one, it follows that the conditional expectations (4-172) are bounded provided the K conditional expectations (4-173) exist

4-174

$$(EU'D)(s')<\infty$$ [33]
$$s'=1,\dots,K$$

Returning to the problem of price convergence, it can first be concluded that the present value of cash flows at time $t+\tau$ reads

$$E\left(q_{t,t+\tau}{}^{cI}\left(fe_{t+\tau},S_{t+\tau-1};D_t{}^{market}\right)\cdot D_{t+\tau}\middle|S_t,D_t{}^{market}\right)$$
$$=\sum_{s'=1}^{K} P(S_{t+\tau-1}=s'|S_t)\cdot exp(-\rho\cdot\tau)\cdot\frac{(EU'D)(s')}{U'\left(\frac{1}{n_I}\cdot D_t{}^{market}\right)}$$

and this in turn implies that asset prices are linear combinations of the conditional tions$(EU'D)(s'), s'=1,\dots,K$:

---

[33] Note that no particular distribution of factors and residuals has been imposed. This implies that the expectations may not exist. In this case it is clear that neither finite nor infinite time horizon asset prices exist.

4-175

$$P_t^{cI}\left(S_t, D_t^{market}; T-t\right) = \frac{\sum_{s'=1}^{K} \omega(s', S_t; T-t) \cdot (EU'D)(s')}{U'\left(\frac{1}{n_I} \cdot D_t^{market}\right)}$$

with

$$\omega(s', S_t; T-t) \equiv \sum_{\tau=1}^{T-t} \frac{1}{(1+\rho)^{\tau}} \cdot P(S_{t+\tau-1} = s' | S_t)$$

(4-175) provides the answer to the problem of convergence: large time horizon asset price are finite (provided condition (4-174) is met) and read

4-176

$$lim_{T-t\to\infty} P^{cI}\left(S_t, D_t^{market}; T-t\right) = \frac{\sum_{s'=1}^{K} \omega(s', S_t) \cdot (EU'D)(s')}{U'\left(\frac{1}{n_I} \cdot D_t^{market}\right)}$$

with

$$\omega(s', S_t) \equiv \sum_{\tau=1}^{\infty} \frac{1}{(1+\rho)^{\tau}} \cdot P(S_{t+\tau-1} = s' | S_t)$$

because the limits of $\omega(s', S_t; T-t)$ (as $T-t$ goes to infinity) are well-defined and given by $\omega(s', S_t)$.[34]

From an economic point of view, the result that asset prices always converge to finite values is due to the fact that expected discounted utility-weighted cash flows do not exceed a certain maximum ($\frac{max_{s' \epsilon \{1,\ldots,K\}}(EU'D)(s')}{U'\left(\frac{1}{n_I} \cdot D_t^{market}\right)}$), hence the time preference rate dominates.[35]

4.4.3 The Case of Non-Negative Cash Flows without Lags in Growth Rates under CRRA Utility

If cash flows do not exhibit lags in growth rates and are non-negative ($D_{i,0} > 0, d_i(S_t, fe_{t+1}) > -1, i = 1, .., n$)[36], asset prices as discounted future cash flows read for CRRA utility functions ((4-158) with (4-159)):

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[34] This follows because $\frac{1}{(1+\rho)^{\tau-1}} \cdot P\left(S_{t+\tau-1} = s' \middle| S_t\right) \geq 0$, hence the series that defines $\omega(s', S_t)$ must either converge to a finite limit or diverge to $+\infty$, but divergence to $+\infty$ can be excluded because the positive time preference parameter ($\rho > 0$) and the fact that probabilities are less than one ($P(S_{t+\tau-1} = s' | S_t) \leq 1$) show that $\omega(s', S_t)$ must be less than a convergent geometric series:

$$\omega(s', S_t) \leq \sum_{\tau=1}^{\infty} \frac{1}{(1+\rho)^{\tau-1}} < \infty$$

[35] This result in the context of on cash flow levels in a regime-switching model parallels the findings of Ingersoll (1987), p. 275 for growth rates.

[36] Note that the cash flow model is slightly more restrictive than in Section 4.3.3 because the case $d_i(S_t, fe_{t+1}) = -1$ is excluded because some of the quantities introduced below could otherwise not be meaningfully defined.

$$P_{i,t}{}^{cI}(S_t, D_{i,t}, \delta_t; T-t)$$
$$= D_{i,t} \cdot \sum_{\tau=1}^{T-t} E\left(q_{t,t+\tau}{}^{cI}(fe_{t+1,t+\tau}, S_{t+1,t+\tau-1}; S_t, \delta_t) \cdot [1 + d_i{}^{+\tau}(S_{t,t+\tau-1}, fe_{t+1,t+\tau})] \middle| S_t, \delta_t\right)$$

with

$$q_{t,t+\tau}{}^{cI}(fe_{t+1,t+\tau}, S_{t+1,t+\tau-1}; S_t, \delta_t) = \frac{1}{(1+\rho)^\tau} \cdot \left\{\sum_{\nu=1}^{n} \delta_{\nu,t} \cdot [1 + d_\nu{}^{+(\tau)}(S_{t,t+\tau-1}, fe_{t+1,t+\tau})]\right\}^{-\gamma}$$

Asset prices (4-158) can either diverge to +∞ or converge to finite values as the time horizon goes to infinity because growth rates $1 + d_i{}^{+\tau}$ (due to limited liability) and stochastic discounted factors are non-negative (due to no-arbitrage). Intuitively, convergence and divergence depend on whether the time preference rate dominates the growth of expected discounted utility-weighted dividends. Checking this condition now is more complicated than in the previous case because, first, convergence must be checked for all possible combinations of the current regime S_t and relative dividend contributions δ_t and, second, expected discounted utility-weighted dividends are no longer necessarily bounded[37].

The first problem turns out to be less problematic than it may first appear because the price dividend function can be shown to be convex in relative dividend contributions for each current regime (see Appendix A3.7). This implies that it suffices to consider the n "corner points" where all dividends come from one single asset, i.e., $\left(\frac{P}{D}\right)_{i,t}^{cI}(S_t, e_j), j = 1, \dots, n$, where e_j is the degenerate situation[38] where all dividends are paid by asset j. It can be deduced that, given a current regime S_t, price dividend ratios take finite values for all possible constellations of relative dividend contributions if and only if this is the case for the K degenerate cases $\left(\frac{P}{D}\right)_{i,t}^{cI}(S_t, e_j; T-t), j = 1, \dots, n$.

The second problem (unbounded cash flows) means that asset prices can diverge to $+\infty$, hence conditions must be found that decide whether asset prices under a given cash flow model and regime process converge or diverge. To that end note that "corner" price dividend ratios can be expressed through a generalized geometric series (with matrices instead of real numbers) (see Appendix A3.8):

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[37] For example, consider the following non-stochastic special case of the dividend model:$D_{i,t+1} = D_{i,t} \cdot (1 + \mu_i)$ with $D_{i,0} > 0$, i.e., $[1 + d_i(S_t, fe_{t+1})] = [1 + d_i] = (1 + \mu_i)$, i.e., dividends grow exponentially and will go to infinity as the time horizon goes to infinity.

[38] Note that the assumptions of positive initial dividends $D_{i,0} > 0, i = 1, \dots, n$ and dividend growth rates and dividend growth rates $d_i(S_t, fe_{t+1}) > -1, i = 1, \dots, n$, imply that these degenerate cases can never be reached. They must be interpreted as limiting cases as the relative dividend contribution of one asset goes to one.

4-177

$$\left(\frac{P}{D}\right)_{i,t}^{cI}(e_j;T-t) \equiv \begin{pmatrix} \left(\frac{P}{D}\right)_{i,t}^{cI}(1,e_j) \\ \dots \\ \left(\frac{P}{D}\right)_{i,t}^{cI}(K,e_j) \end{pmatrix} = \left\{\sum_{\tau=1}^{T-t} \frac{1}{(1+\rho)^{\tau-1}} \cdot M_{i,j}{}^{[\tau-1]}\right\} \cdot (Eqd)$$

where $M_{i,j}$ is the KxK-matrix

$$M_{i,j} = \left(m_{i,j}(i,j)_{s,s'}\right)$$

with

$$m_{i,j}(i,j)_{s,s'} \equiv p_{s,s'} \cdot E\left(\{1 + d_v(S_t, fe_{t+1})\}^{-\gamma} \cdot \{1 + d_i(S_{t,}, fe_{t+1})\} \middle| S_t = s\right)$$

where $M_{i,j}{}^{\tau}$ is the τ-th power of $M_{i,j}$ and (Eqd) is the K-dimensional column vector

$$(Eqd) \equiv \frac{1}{1+\rho} \cdot \begin{pmatrix} E\left(\{1 + d_v(S_t, fe_{t+1})\}^{-\gamma} \cdot \{1 + d_i(S_{t,}, fe_{t+1})\} \middle| S_t = 1\right) \\ \dots \\ E\left(\{1 + d_v(S_t, fe_{t+1})\}^{-\gamma} \cdot \{1 + d_i(S_{t,}, fe_{t+1})\} \middle| S_t = K\right) \end{pmatrix}$$

A common approach to decide whether expressions involving powers of matrices converge is to use Eigenvalue decompositions. Only the case where $M_{i,j}$ admits such a decomposition is considered, i.e.

$$M_{i,j} = V \cdot D \cdot V^{-1}$$

where D is the diagonal KxK-matrix with the Eigenvalues of $M_{i,j}$ on the main diagonal and where V is an invertible KxK-matrix. Some manipulations (see Appendix A3.8.3) show that the prices $lim_{T-t} \to \infty \left(\frac{P}{D}\right)_{i,t}^{cI}(e_j;T-t)$ are linear combinations of geometric series of the factors $\lambda(i,j)_s \cdot \frac{1}{1+\rho}, s = 1, \dots, K$, where $\lambda(i,j)_s, s = 1, \dots, K$, are the Eigenvalues of matrix $M_{i,j}$. It can be concluded that these prices exist if

4-178

$$|\lambda(i,j)_s| \cdot \frac{1}{1+\rho} < 1$$
$$s = 1, \dots K$$
$$j = 1, \dots, n$$

From an economic point of view, the result (4-178) is the condition that the time preference rate dominates the growth of expected discounted utility-weighted dividends. [39]

[39] This result in the context of dividend growth in a regime-switching model parallels the findings of Ingersoll (1987), p. 275 for growth rates.

5 Numerical Computations of General Equilibrium Asset Prices and Risk Premia

The analysis in the Chapters 3 and 4 is theoretical and, as such, can only establish the existence of certain effects. The magnitude and, therefore, the economic importance of these effects for empirically plausible parameter constellations remain unclear, however. Numerical analysis can fill this gap (see, e.g., Judd (1998), p. 6).

Against this background, the following three questions regarding prices and risk premia under complete and incomplete information arise:

- Question 1: How relevant is the effect of incomplete information on risk premia compared to complete information?
- Question 2: Do the extensions of Veronesi (2000) introduced in this thesis - namely more than one risky asset, a richer class of regime processes, and incomplete information on both first and second order moments of dividends – translate into substantially different risk premia than in Veronesi (2000) or do these extensions not fundamentally alter incomplete information risk premia?
- Question 3: For what parameter constellations are the differences identified in Questions 1 and 2 particularly pronounced?

Not all models analyzed in Chapters 3 and 4 are suitable to address these three question. The model must be sufficiently comparable to the model in Veronesi (2000) but, at the same time, allow for extensions mentioned in Question 2. Only general equilibrium models with "large" time horizon, constant relative risk aversion, dividends given by a model without lags in growth rates, and normally distributed factors and residuals are suitable, i.e., one particular version of the model of Section 4.4.3 is used. If the cash flow function differed too much (such as cash flow without lags in levels), or if an altogether different type of utility function (such as constant absolute risk aversion) were considered, results would be hard to compare. At the same time, this model framework is sufficiently flexible to address the extensions outlined in Question 2.

Since Veronesi (2000) focuses on return-based risk premia (the cash flow-based risk premium divided by the price of the risky asset) a return-based risk premium is used in this chapter as well. To answer Questions 1,2, and 3, four steps are necessary: (i) the risk premium defined in (4-55) and (4-67) must be translated into a return-based formulation (Section 5.1) (ii) the continuous-time dividend model in Veronesi (2000) must be translated into a discrete-time framework, and dividend models for the extensions must be formulated (Section 5.2); (iii) empirically plausible parameter constellations must be identified (Section 5.3); (iv) numerical methods must be chosen since analytical results cannot be obtained (Section 5.4). Section 5.5 finally describes numerical results and answers the questions.

5.1 Return-Based Risk Premia

5.1.1 Definition of Return-Based Risk Premia

Dividing asset i's risk premia (4-171) by its price yields its return-based risk premium

5-1

$$RP_{i,t}{}^{iI,ret}(\pi_t,\delta_t) \equiv \frac{RP_{i,t}{}^{iI}(\pi_t, D_{i,t}, \delta_t)}{P_{i,t}{}^{iI}(\pi_t, D_{i,t}, \delta_t)}$$

$$= -cov\left(\begin{array}{c} AfR_{t,t+1}{}^{iI}(fe_{t+1}, S_t, \delta_t, \pi_t), \\ [1+d_i(S_t, fe_{t+1})] \cdot \dfrac{\left\{\sum_{s'=1}^{K} \pi_{s',t+1} \cdot \left(\frac{P}{D}\right)_{i,t+1}^{cI}(s', \delta_{t+1}) + 1\right\}}{\left(\frac{P}{D}\right)_{i,t}^{iI}(\pi_t, \delta_t)} \end{array} \middle| \pi_t, \delta_t \right)$$

with

$$AfR_{t,t+1}{}^{iI}(fe_{t+1}, S_t, \delta_t, \pi_t) \equiv \frac{\left\{\sum_{v=1}^{n} \delta_{v,t} \cdot d_v(S_t, fe_{t+1})\right\}^{-\gamma}}{\sum_{s=1}^{K} \pi_{s,t} \cdot E\left(\left\{\sum_{v=1}^{n} \delta_{v,t} \cdot d_v(S_t, fe_{t+1})\right\}^{-\gamma} \middle| S_t = s, \delta_t\right)}$$

In a similar way, a return-based risk premium $RP_{i,t}{}^{cI,ret}(S_t,\delta_t)$ is obtained from complete information (4-161):

5-2

$$RP_{i,t}{}^{cI,ret}(S_t,\delta_t) = -cov\left(\begin{array}{c} AfR_{t,t+1}{}^{cI}(fe_{t+1}, S_t, \delta_t) \\ ,[1+d_i(S_t, fe_{t+1})] \cdot \left\{ \dfrac{\sum_{s'=1}^{K} p_{S_t,s'} \cdot \left(\frac{P}{D}\right)_{i,t+1}^{cI}(s', \delta_{t+1}) + 1}{\left(\frac{P}{D}\right)_{i,t}^{cI}(s', \delta_t)} \right\} \end{array} \middle| S_t, \delta_t \right)$$

with

$$AfR_{t,t+1}{}^{cI}(fe_{t+1}, S_t, \delta_t) \equiv \frac{\left\{\sum_{v=1}^{n} \delta_{v,t} \cdot d_v(S_t, fe_{t+1})\right\}^{-\gamma}}{E\left(\left\{\sum_{v=1}^{n} \delta_{v,t} \cdot d_v(S_t, fe_{t+1})\right\}^{-\gamma} \middle| S_t, \delta_t\right)}$$

5.1.2 The Components of Asset Returns

The interaction of various components of asset returns is crucial to the interpretation of numerical results and, therefore, to answering the Questions 1, 2, and 3. It is helpful to identify these components of the return on risky assets for the model of cash flows without lags in growth rates under CRRA utility. The total return on asset i reads

5-3

$$Return_{i,t,t+1} = [1 + d_i(S_t, fe_{t+1})] \cdot \frac{\left\{\sum_{s'=1}^{K} \pi_{s',t+1} \cdot \left(\frac{P}{D}\right)_{i,t+1}^{cI} (s', \delta_{t+1}) + 1\right\}}{\left(\frac{P}{D}\right)_{i,t}^{iI} (\pi_t, \delta_t)}$$

This return can be thought of as consisting of capital gains and dividend yield.

The first return component, (gross) capital gains, is given by

5-4

$$Cap.Gains_{i,t,t+1} \equiv \frac{P_{i,t+1}{}^{iI}(\pi_{t+1}, D_{i,t+1}, \delta_{t+1})}{P_{i,t}{}^{iI}(\pi_t, D_{i,t}, \delta_t)}$$

$$= [1 + d_i(S_t, fe_{t+1})] \cdot \frac{\left\{\sum_{s'=1}^{K} \pi_{s',t+1} \cdot \left(\frac{P}{D}\right)_{i,t+1}^{cI} (s', \delta_{t+1}) + 1\right\}}{\left(\frac{P}{D}\right)_{i,t+1}^{iI} (\pi_t, \delta_t)}$$

This return component can be interpreted as the product of (gross) dividend growth $[1 + d_i(S_t, fe_{t+1})]$ and incomplete information price dividend ratio growth,

5-5

$$\frac{\left(\frac{P}{D}\right)_{i,t+1}^{iI} (\pi_{t+1}, \delta_{t+1})}{\left(\frac{P}{D}\right)_{i,t}^{iI} (\pi_t, \delta_t)} = \frac{\left\{\sum_{s'=1}^{K} \pi_{s',t+1} \cdot \left(\frac{P}{D}\right)_{i,t+1}^{cI} (s', \delta_{t+1})\right\}}{\left(\frac{P}{D}\right)_{i,t+1}^{iI} (\pi_t, \delta_t)}$$

For example, if dividends grow by 5 percent and the price dividend ratio increases from 12 to 15, incomplete information price dividend ratio growth is 1.25 (where a value of 1 means no change), and capital gains are $1{,}05 \cdot 1.25 = 1.3125$.

The second return component, dividend yield, reads

$$Div.Yield_{i,t,t+1} \equiv \frac{D_{i,t+1}}{P_{i,t}{}^{iI}(\pi_t, D_{i,t}, \delta_t)} = \frac{[1 + d_i(S_t, fe_{t+1})]}{\left(\frac{P}{D}\right)_{i,t}^{iI} (\pi_t, \delta_t)}$$

Continuing the above example, dividend yield would be 0.0875 if dividend growth is 5 percent and the price dividend ratio at time t is 12.

5.2 Dividend Models for the Numerical Analysis

5.2.1 A Discretized Version of the Veronesi Model

Veronesi (2000), p. 810, models continuous-time dividends as a generalized geometric Brownian motion: the drift parameter of dividend growth is, in contrast to a geometric Brownian motion, regime-dependent; the volatility of dividend growth is constant as in a geometric Brownian motion. Veronesi's model must be discretized in order to answer Questions 1, 2, and 3:

5-6

$$D_{t+1} = D_t \cdot exp(\mu_t + b \cdot e_{t+1})$$

where $\{e_\tau\}$ is a process of independent standard normal random variables, b is a constant, and μ_t can take K values as a function of the regime S_t

5-7

$$\mu_t = \mu(S_t)$$

The regime S_t takes K possible values and is governed by the following two-stage process: in every time interval, there is a probability p_{Draw} that a regime is drawn. If drawing occurs, the probabilities of the K regimes are $p_{\mu,s}, s = 1, \dots, K$ (referred to as the conditional transition probabilities). The transition probabilities $p_{s,s'}$ from the regime process in my model then read:

5-8

$$p_{s,s'} = \begin{cases} p_{Draw} \cdot p_{\mu,s'} & s \neq s' \\ (1 - p_{Draw}) + p_{Draw} \cdot p_{\mu,s} & s = s' \end{cases}$$

Note that the term $p_{Draw} \cdot p_{\mu,s}$ in the case $s = s'$ captures the possibility that a drawing occurs but no change in regimes occurs, i.e., the regime s is drawn again.

This discretized model has the disadvantage that the parameter μ_t cannot be interpreted as expected dividend growth $E_t\left(\frac{D_{t+1}}{D_t}\right)$:

$$E_t\left(\frac{D_{t+1}}{D_t}\right) = E\big(exp(\mu_t + b \cdot e_{t+1})\big) = exp(\mu_t + 0.5 \cdot b^2)$$

i.e., expected dividend growth is a function of both μ_t and b.

This problem can be solved by using a slightly different model that has regimes in expected dividend growth (as opposed to regimes in log dividend growth) and a constant standard deviation of dividend growth conditional on the regime (as opposed to constant standard deviation of log dividend growth conditional on the regime). Both models are comparable in the sense that the modified model still reproduces the essential results (see Section 5.5.1.1, p. 199) found in Veronesi (2000) (Lemma 3, p. 816, incomplete information lowers risk premia if the risk aversion parameter γ exceeds one). Formally, the modified model can be described as follows:

There are K different regimes in expected dividend growth $E_t\left(\frac{D_{t+1}}{D_t}\right)$:

5-9

$$E_t\left(\frac{D_{t+1}}{D_t}\right) = E\left(\frac{D_{t+1}}{D_t}\middle| S_t = s\right)$$

$$s = 1, \dots., K$$

The standard deviation σ_D of dividend growth conditional on the regime is assumed to be constant across regimes:

5-10

$$\sigma_D = Stddev_t\left(\frac{D_{t+1}}{D_t}\right) = Stddev\left(\frac{D_{t+1}}{D_t}\middle|S_t = s\right)$$
$$s = 1, \dots., K$$

Transition probabilities are defined as in the discretized Veronesi model: a draw of regimes in expected dividend growth rates occurs with probability $p_{Mean,Draw}$, and the conditional transition probabilities (conditional on the occurrence of a draw) are $p_{Mean,s}, s = 1, \dots, K$. The transition probabilities of the regime process are, therefore, given by

5-11

$$p_{s,s'} = \begin{cases} p_{Mean,Draw} \cdot p_{Mean,s'} & s \neq s' \\ (1 - p_{Mean,Draw}) + p_{Mean,Draw} \cdot p_{Mean,s} & s = s' \end{cases}$$

Initial probabilities are set equal to $p_{Mean,s}, s = 1, \dots, K$. The idea here is that the first drawing occurs at time zero.

Since expectation and standard deviation of dividend growth conditional on the regime are model inputs, the parameters $\mu(S_t)$ and $b(S_t)$ must be chosen to match these inputs. Given expected dividend growth $E\left(\frac{D_{t+1}}{D_t}\middle|S_t = s\right)$ (5-9) and the conditional standard deviation σ_D (5-10), it is obtained[40]

5-12

$$\mu(S_t) = ln\left(E\left(\frac{D_{t+1}}{D_t}\middle|S_t = s\right)\right) - 0.5 \cdot b(S_t)^2$$
$$s = 1, \dots., K$$

with

5-13

$$b(S_t) = \sqrt{ln\left(\frac{{\sigma_D}^2}{E^2\left(\frac{D_{t+1}}{D_t}\middle|S_t = s\right)} + 1\right)}$$

Here the difference to the discretized Veronesi model becomes visible: the assumption of regime-dependent expectations and a regime-independent conditional standard deviation imply that both μ and b are regime-dependent (as opposed to μ only). Conversely, the assumption in the discretized Veronesi model that only μ but not b is regime-dependent yields both regime-dependent conditional expectations and standard deviations.

[40] These parameters can be found and verified by virtue of $E(exp(X)) = exp(E(X) + 0.5 \cdot Var(X))$ where X is a normally distributed random variable.

5.2.2 Extension of the Discretized Veronesi Model

Extension of the Veronesi model include incomplete information on both first and second order moments of dividends as well as more than one risky asset.

5.2.2.1 Single Risky Asset: Incomplete Information on Both First and Second Order Moments of Dividends

5.2.2.1.1 Motivation

Veronesi (2000) only considers incomplete information about the drift parameter μ of the diffusion process that governs dividend growth; since dividends are paid continuously in his model and, as a consequence, there always exists an infinite number of dividend observations over any finite time interval, this assumption is reasonable within a continuous time framework: investors can always precisely infer the standard deviation parameter of the diffusion process. However, the assumption of a continuous stream of dividends is clearly unrealistic. If there is only a finite number of dividends, as is the case in my model, uncertainty about second-order moments of dividend growth becomes a potentially important problem to investors.

5.2.2.1.2 Formulation of the Model

It is assumed that there are regimes in means and standard deviations conditional on regimes where standard deviation regimes are independent of mean regimes.

To formalize this idea, I first define two independent regime processes for means and standard deviations and then combine both processes into a single regime process.

<u>Regimes in means</u>

5-14

$$E_t\left(\frac{D_{t+1}}{D_t}\right) = E\left(\frac{D_{t+1}}{D_t}\middle|{S_t}^{Means} = s_m\right)$$

$$s_m = 1, \ldots., K_{means}$$

<u>Regimes in standard deviations conditional on regimes</u>

5-15

$$Stddev_t\left(\frac{D_{t+1}}{D_t}\right) = Stddev\left(\frac{D_{t+1}}{D_t}\middle|{S_t}^{Stddevs} = s_s\right)$$

$$s_s = 1, \ldots., K_{Stddevs}$$

Combination of both processes into a single regime process

The total number of regimes K then is the product of the number of mean and standard deviation regimes, i.e., $K = K_{Means} \cdot K_{Stddevs}$. The first $K_{Stddevs}$ regimes ($s = 1, \dots, K_{Stddevs}$) correspond to mean regime 1 and the standard deviation regimes 1 to $K_{Stddevs}$, the next $K_{Stddevs}$ regimes ($s = 1 + K_{Stddevs}, \dots, 2 \cdot K_{Stddevs}$) correspond to mean regime 2 and 1 to $K_{Stddevs}$, and so on. More formally, the combination of mean regime s_m and standard deviation regime s_s are assigned a regime number $s(s_m, s_s)$ via

5-16

$$s = s(s_m, s_s) = K_{Stddevs} \cdot (s_m - 1) + s_s$$
$$s_m = 1, \dots, K_{Means}$$
$$s_s = 1, \dots, K_{Stddevs}$$

Since mean regimes and standard deviation regimes are assumed to be stochastically independent, the probability for a transition from $s = s(s_m, s_s)$ to $s' = s(s_m', s_s')$ must be

5-17

$$p_{s(s_m,s_s),s(s_m',s_s')} = p_{s_m,s_m'} \cdot p_{s_s,s_s'}$$

with

$$p_{s_m,s_m'} = \begin{cases} p_{Mean,Draw} \cdot p_{Mean,s_m'} & s_m \neq s_m' \\ (1 - p_{Mean,Draw}) + p_{Mean,Draw} \cdot p_{Mean,s_m} & s_m = s_m' \end{cases}$$
$$p_{s_s,s_s'} = \begin{cases} p_{Stddev,Draw} \cdot p_{s_s,s_s'} & s_s \neq s_s' \\ (1 - p_{Stddev,Draw}) + p_{Stddev,Draw} \cdot p_{Stddev,s_s} & s_s = s_s' \end{cases}$$

Likewise, the initial probability of regime $s(s_m, s_s)$ is $p_{mean,s_m} \cdot p_{stddev,s_s}$, again because the first drawing occurs at time zero.

Determination of the parameters $\mu(S_t)$ and $b(S_t)$

Since expectation and standard deviation of dividend growth conditional on the regime are model inputs, the parameters $\mu(S_t)$ and $b(S_t)$ must be chosen to match these inputs. Given expected dividend growth $E\left(\frac{D_{t+1}}{D_t}\middle|S_t = s\right)$ (5-14) and the conditional standard deviation $Stddev_t\left(\frac{D_{t+1}}{D_t}\right)$ (5-15), it is obtained

5-18

$$\mu(S_t) = ln\left(E\left(\frac{D_{t+1}}{D_t}\middle|S_t = s\right)\right) - 0.5 \cdot b(S_t)^2$$
$$s = 1, \dots., K$$

5-19

$$b(S_t) = \sqrt{ln\left(\frac{Var\left(\frac{D_{t+1}}{D_t}\middle|S_t = s\right)}{E^2\left(\frac{D_{t+1}}{D_t}\middle|S_t = s\right)} + 1\right)}$$

$$s = 1, \ldots., K$$

5.2.2.2 Two Risky Asset: Regimes in Asset 2 Only

5.2.2.2.1 Motivation

In models with a single risky asset, the dividends of the risky asset also completely determine the stochastic discount factor. This assumption is unrealistic since investors tend to have multiple sources of income. At the same time, the fact that dividends provide the stochastic discount factor drives many of the central results for models with a single risky asset, in particular the result that the incomplete information risk premium is less than the complete information risk premium. Thus, if the price of a risky asset that is not the sole source of income is to be studied, it is desirable to have a model with multiple risky assets. I use two risky assets, where one risky asset (asset 2) will be the main object of interest and the other risky asset (asset 1) should be interpreted as lumping together a range of other assets.

If there are two risky assets, dividends of all assets could potentially exhibit regimes. For ease of interpretation, however, I define a model where only dividend growth of asset 2 exhibits regimes while dividend growth for asset 1 is regime-independent. To facilitate comparisons with Veronesi's model (Section 5.2.1) and the first extension of Veronesi's model (Section 5.2.2.1), regimes in asset 2 are as in the first extension of Veronesi's model.

5.2.2.2.2 Formulation of the Model

I assume that asset 2 exhibits regimes in means and standard deviation in dividend growth of the type described in the previous model. Expectation and standard deviation of dividends of asset 1 as well as the correlation in dividend growth of both assets conditional on the regime are assumed to be regime-independent.

Dividend model

Dividends of assets 1 and 2 are given by

5-20
$$D_{1,t+1} = D_{1,t} \cdot exp(\mu_1 + a_1 \cdot f_{t+1} + b_1 \cdot e_{1,t+1})$$

5-21
$$D_{2,t+1} = D_{2,t} \cdot exp(\mu_{2,t} + a_{2,t} \cdot f_{t+1} + b_{2,t} \cdot e_{2,t+1})$$

with regime-independent parameters μ_1, a_1 and b_1 for asset 1 and with regime-dependent parameters $\mu_{2,t}$, $a_{2,t}$ and $b_{2,t}$ for asset 2:

$$\mu_{2,t} = \mu_2(S_t)$$
$$a_{2,t} = a_2(S_t)$$
$$b_{2,t} = b_2(S_t)$$

The correlation in dividend growth of both assets conditional on the regime is regime-independent and denoted by $corr_{d_1,d_2}$:

5-22

$$corr\left(\frac{D_{1,t+1}}{D_{1,t}}, \frac{D_{2,t+1}}{D_{2,t}}\middle| S_t = s\right) = corr_{d_1,d_2}$$
$$s = 1, \dots, K$$

Regime process

Since the regime S_t is supposed to consist of independent mean and standard deviation components, it is described by (5-14), (5-15), (5-16), and (5-17).

Determination of the parameters μ_1, a_1, b_1 and $\mu_2(S_t), a_2(S_t), b_2(S_t)$

Since expectations, standard deviations and correlations of dividend growth conditional on the regime are model inputs, the parameters μ_1, a_1, b_1 and $\mu_2(S_t)$, $a_2(S_t)$, $b_2(S_t)$ must be chosen to match these inputs. Given (5-14), (5-15), (5-16), (5-17), and (5-22), unique solutions are found for μ_1, $\mu_2(s), s = 1, \dots, K$, and for the sums ${a_1}^2 + {b_1}^2$ and $a_2(s)^2 + b_1(s)^2, s = 1, \dots, K$:

5-23

$$\mu_1 = ln\left(E\left(\frac{D_{1,t+1}}{D_{1,t}}\right)\right) - 0.5 \cdot \{{a_1}^2 + {b_1}^2\}$$
$$s = 1, \dots., K$$

5-24

$$\sqrt{{a_1}^2 + {b_1}^2} = \sqrt{ln\left(\frac{Var\left(\frac{D_{1,t+1}}{D_{1,t}}\right)}{E^2\left(\frac{D_{1,t+1}}{D_{1,t}}\right)} + 1\right)}$$

5-25

$$\mu_2(s) = ln\left(E\left(\frac{D_{1,t+1}}{D_{1,t}}\middle| S_t = s\right)\right) - 0.5 \cdot \{a_2(s)^2 + b_2(s)^2\}$$

5-26

$$\sqrt{a_2(s)^2 + b_2(s)^2} = \sqrt{ln\left(\frac{Var\left(\frac{D_{2,t+1}}{D_{2,t}}\middle| S_t = s\right)}{E^2\left(\frac{D_{2,t+1}}{D_{2,t}}\middle| S_t = s\right)} + 1\right)}$$

However, there are various possible solutions for a_1, b_1 and $a_2(s), b_2(s), s = 1, \dots, K$. These solutions must satisfy the restrictions imposed by the assumption of regime-independent dividend correlations (5-22):

5-27

$$corr_{d_1,d_2} = Corr_t\left(\frac{D_{1,t+1}}{D_{1,t}}, \frac{D_{2,t+1}}{D_{2,t}}\right) = \frac{exp\big(a_1 \cdot a_2(s)\big) - 1}{\sqrt{\{\{exp(a_1{}^2) - 1\}\} \cdot \{exp(a_2(s)^2 + b_2(s)^2) - 1\}}}$$

Based on (5-24), a_1 can be chosen to take any value that satisfies

$$0 \leq a_1 \leq \sqrt{ln\left(\frac{Var\left(\frac{D_{1,t+1}}{D_{1,t}}\right)}{E^2\left(\frac{D_{1,t+1}}{D_{1,t}}\right)} + 1\right)}$$

$|b_1|$ is then found via (5-24), and all further parameters $(a_2(s), b_2(s), s = 1, \dots, K)$ are found from (5-27).

Observe that correlations in dividends translate intuitively into the elements of the affine linear factor model for log dividend growth: Assuming (without loss of generality) that a_1 is positive, correlations of dividends are positive if and only if the coefficients $a_2(s), s = 1, \dots, K$ are also positive. This means that the coefficients for the common factor f_{t+1} must possess the same sign and can be deduced from the right-hand side of (5-27). Similarly, it can be concluded that dividend growth of both assets is uncorrelated if and only if the coefficients $a_2(s), s = 1, \dots, K$ are all zero.

5.3 Arguments of the Risk Premium Function and Parameter Values

In order to obtain numerical results, two things must be specified: (i) values for the arguments of the risk premium function, i.e., regime probabilities π_t and, if there are several risky assets, relative dividend contributions δ_t; (ii) specific values for the various model parameters. While model parameters are exogenous, the arguments of the risk premium function are, in principle, endogenous (and thus are no model parameters).

5.3.1 Choice of the Arguments of the Risk Premium Function

Incomplete Information risk premia are functions of regime probabilities π_t and, if there are several risky assets, of relative dividend contributions δ_t. This leads to the question of which values are to be chosen for the numerical analysis.

In the case of relative dividend contributions, this is comparatively straightforward because only the case with two risky assets is considered. The interval [0;1] in which $\delta_{1,t}$ takes values can, for example, simply be divided into equidistant points. If the second asset is to be interpreted as "small" relative to the market, the limit case where $\delta_{1,t} \approx 1$ can be considered.

In the case of regime probabilities, the space of possible values is much larger. There are two types of conditional regime probabilities that are interesting choices as arguments for risk premia:

(i) Invariant (steady state) regime probabilities

Under certain conditions, the regime process possesses unique invariant regime probabilities which possess several interesting properties. One of these properties is that these probabilities can be thought of as acting like a center of gravity toward which conditional regime probabilities can be expected to return. More formally, the expectation of regime probabilities at time $t + \tau$ conditional on information available at time t, $E_t(\pi_{t+\tau})$, converges to a limit if the regime chain is aperiodic and irreducible as τ goes to infinity, and this limit coincides with (unique) invariant probabilities (see Appendix A4.1 for details). Therefore, if the current regime probability $\pi_{s,t}$ is higher (lower) than the invariant probability of regime s, the regime probability of regime s can be expected to decline (increase) in future time periods.

(ii) Initial regime probabilities

Alternatively, initial probabilities can be considered. Initial probabilities then represent investors' initial information, whereas complete information (for which risk premia are also computed) represents the limit case as information becomes very precise over time. This approach ensures that risk premia are also reported for models that do not possess a unique invariant distribution. This is, for example, the case in the discretized Veronesi-model without switches in regimes ($p_{Draw} = 0$) where investors essentially learn about an unobservable (but constant) parameter.

In the following only initial probabilities are considered: for the models specified above, it turns out that whenever invariant regime probabilities exist, they coincide with initial probabilities. If $p_{Draw} = 0$, no unique invariant regime probabilities exist and only initial probabilities can be considered.

5.3.2 Choice of Parameter Values

A range of values for parameters is considered: in this way, risk premia are not only obtained for given parameter values, but additional insights regarding robustness are gained on how risk premia change in response to changes in parameters. The range of values should be empirically plausible, although no attempt is made to calibrate the model to empirical data. Finally, I have conducted a preliminary analysis in which a wider range of parameters were considered than those discussed below. Mostly divergent parameter constellations as well as parameter constellations that did not provide substantial additional insights could be eliminated with the help of the preliminary analysis.

5.3.2.1 Preference Parameters

Preference parameters comprise the time preference parameter ρ and the risk aversion parameter γ.

Time Preference Parameter ρ

The values $\rho = 0.025$ and $\rho = 0.05$ are considered for the time preference parameter ρ. I exclude lower values of ρ (such as $\rho = 0.01$) from the analysis because price dividend ratios frequently diverge for such values. On the other hand, high values of ρ (say, $\rho = 0.1$) are omitted because they lead to very high interest rates and to very low price dividend ratios.

Risk Aversion Parameter γ

The risk aversion parameter γ takes a value from the set $\{0.5; 1; 2; 3; 5\}$. The seminal paper Friend/Blume (1975) finds relative risk aversion parameters γ in excess of one, and thus several values greater than one are considered. Friend/Blume (1975), p. 920, emphasize the special role of logarithmic utility ($\gamma = 1$) which also plays a crucial role in Veronesi (2000). Therefore, the case $\gamma = 1$ is also considered and one parameter value with less risk aversion than logarithmic utility is included for robustness reasons.

5.3.2.2 Parameters of Regime Process and Dividend Function

5.3.2.2.1 General Remarks on the Parameter Choice

The structure of the regime processes are similar in all of the three models that are considered. For example, switches in regimes are always modeled as a two-stage random experiment where it is first decided if a drawing occurs (first stage, captured by the parameter p_{Draw}) and, if a drawing occurs, which regime becomes the new regime (second stage, modeled by conditional transition probabilities $p_{Mean,s}$ and/or $p_{Stddev,s}$). This suggests that parameters are kept identical across the three models if possible in order to ensure that results can easily be compared.

5.3.2.2.2 Parameters for the Discretized Veronesi Model (Regimes in Expectation of Dividend Growth)

5.3.2.2.2.1 Expected Dividend Growth Regimes

There are assumed to be eleven expected dividend growth regimes: These regimes form an equidistant partition of the interval from 0 to 10 percent expected dividend growth. In other words, the expected dividend growth regimes are

5-28

$$E\left(\frac{D_{t+1}-D_t}{D_t}\middle|S_t=s\right)=\frac{[s-1]}{10}\cdot 0.1$$

$$s=1,\dots,11$$

The choice of 0 and 10 percent are to be interpreted as lower and upper bounds, respectively. Several symmetric probability distributions centered on a mean of 5 percent will be defined on this range.

5.3.2.2.2.2 Standard Deviation of Dividend Growth Conditional on the Regime

The (regime-independent) standard deviation of dividend growth conditional on the regime, $\sigma_D \equiv Stddev\left(\frac{D_{t+1}}{D_t}\middle|S_t=s\right)$, takes values in the set $\{0.01,0.025,0.05,0.075,0.1\}$. At the lower end of this parameter range, the standard deviation of dividend growth takes a value that is empirically plausible for annual aggregate consumption growth per year. Higher values of up to ten percent could represent the standard deviation of annual aggregate dividend growth (see, e.g., Campbell (2006), Table 2, p. 814 for volatilities of aggregate consumption and dividend growth).

5.3.2.2.2.3 Probability of a Drawing of Regimes

The parameter p_{Draw} (the probability that a regime in expected dividend growth is drawn) will take one of the following values:

$$p_{Draw}\epsilon\{0,0.04,0.1,0.25\}$$

If $p_{Mean,Draw}$ equals 0, the probability of a regime switch is zero. This case is of interest as it represents the case with one single regime under complete information and the case with uncertainty about a constant mean parameter under incomplete information. Under incomplete information, this model can thus be seen as a form of parameter uncertainty.

The parameter values for $p_{Mean,Draw}$ $(0.04, 0.1, 0.25)$ are intended to represent various degrees of regime stability: since the expected time until a (possibly new) regime is drawn is ${p_{Mean,Draw}}^{-1}$, the parameters imply that a regime is drawn on average every 25 periods ($p_{Mean,Draw} = 0.04$), every ten periods ($p_{Mean,Draw} = 0.1$), and every four periods ($p_{Mean,Draw} = 0.25$).

5.3.2.2.2.4 Conditional Transition Probabilities

Conditional transition probabilities of the 11 regimes, $p_{Mean,s}, s = 1, \dots, 11$ are modeled in two ways: (i) lower probability for more extreme regimes (based on a normal distribution); (ii) uniform distribution.

Model Based on the Normal Distribution

I define classes (similar to a histogram) with each of the expectation regimes as middle-points of one class. The probability of the regime then is the probability of the class under a normal distribution with expectation $\mu_{Means} \equiv 0.05$ and standard deviation parameter σ_{Means}; this standard deviation is a model parameter and can take various values. The probability of regime s is defined by

$$p_{Mean,s} = \begin{cases} \Phi_{\mu_{Means},\sigma_{Means}}\left(Ed_s + \frac{h}{2}\right) & s = 1 \\ \Phi_{\mu_{Means},\sigma_{means}}\left(Ed_s + \frac{h}{2}\right) - \Phi_{\mu_{Means},\sigma_{Means}}\left(Ed_s - \frac{h}{2}\right) & 1 < s < K \\ 1 - \Phi_{\mu_{Means},\sigma_{Means}}\left(Ed_s - \frac{h}{2}\right) & s = K \end{cases}$$

with

$$Ed_s \equiv E(d_{t+1} | S_t = s)$$

where $\Phi_{\mu_{Means},\sigma_{Means}}$ denotes the distribution function of a normal distribution with mean μ_{Means} and standard deviation σ_{Means}.

The advantage of this models is that it can be summarized by one single parameter (σ_{Means}) that is easy to interpret: A high values of σ_{Means} implies that extreme expected dividend growth regimes are comparatively likely (relative to more moderate regimes close to the middle point of the interval for expected dividend growth).

Uniform Distribution

If σ_{Means} exceeds a certain threshold, the discretized normal probability ceases to be a reasonable model because the most extreme regimes (0 and 10 percent expected dividend growth) would ultimately be assigned the highest probabilities, resulting in a U-shaped probability distribution (as the preliminary analysis has shown). Such a model would be hard to interpret in economic terms. As a better alternative, I use a discrete uniform distribution as the model with the highest level of dispersion of conditional transition probabilities/initial probabilities.

Figure 5-1 illustrates the model for $p_{Mean,s}$ for values of $\sigma_{Means} \epsilon \{0.005; 0.01; 0.025\}$ and for the interval ranging from 0 percent to 10 percent.

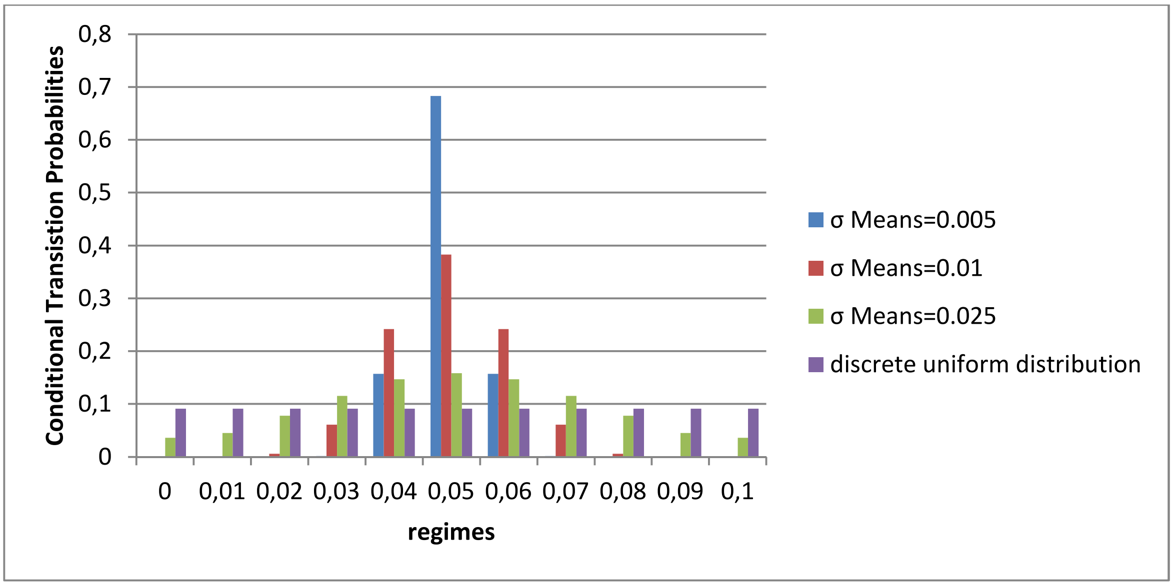

Figure 5-1: Various models for conditional transition probabilities

5.3.2.2.3 Parameters for the Extended Veronesi Model (Regimes in Expectation and Standard Deviation of Dividend Growth)

5.3.2.2.3.1 Expected Dividend Growth Regimes

The same range of expectation regimes as in the discretized Veronesi model is considered, i.e., eleven expected dividend growth regimes range from 0 percent to 10 percent and are chosen to yield an equidistant partition of this interval.

5.3.2.2.3.2 Standard Deviation of Dividend Growth Regimes

Three standard deviation regimes are considered. I primarily consider standard deviations ranging from 1 percent to 10 percent and obtain the regimes for the standard deviation of dividend growth of 1 percent, 5.5 percent, and 10 percent (equidistant regimes). In addition, narrower bands of standard deviation regimes (1 percent to 2 percent, 1 percent to 5 percent, and 1 percent to 7 percent, each case with three equidistant regimes) are also analyzed in order to obtain insights on the effect of the range of standard deviation regimes on risk premia. This range of regimes replaces the (regime-independent) parameter σ_D from the discretized Veronesi model.

5.3.2.2.3.3 Probability of a Drawing of Regimes

The probability of a drawing in expectation regimes and the probability of a drawing in the standard deviation regime are both parameterized by $p_{Draw}\epsilon\{0, 0.04, 0.1, 0.25\}$. To interpret this parameter correctly, recall from Section 5.2.2.1.2 that expectation and standard deviation regimes are assumed to be stochastically independent. For example, the probability that both expectation regimes and standard deviation regimes are drawn is ${p_{Draw}}^2$ (rather than p_{Draw}). Similarly, the probability that a drawing either only in regimes or only in standard deviations occurs is $2 \cdot p_{Draw}(1 - p_{Draw})$.

5.3.2.2.3.4 Conditional Transition Probabilities

The conditional transition probabilities of expected dividend growth regimes are given either by one of the three models based on the normal distribution (with parameter $\sigma_{Means}\epsilon\{0.005, 0.01, 0.025\}$) or by the uniform distribution.

The conditional transition probabilities for the standard deviation of dividend growth regimes are also parameterized either based on a normal distribution with a parameter $\sigma_{Stddevs}$ (which takes one of three possible values) or a uniform distribution. However, $\sigma_{Stddevs}$ should not be set equal to σ_{Means} because both parameters refer to different intervals (0 to 10 percent for σ_{Means}, but intervals with various ranges for $\sigma_{Stddevs}$). For the case of standard deviation regimes ranging from 1 percent to 10 percent, σ_{Means} takes values $\{0.025, 0.03, 0.04\}$. The resulting conditional transition probabilities are depicted in Figure 5-2. For the case of narrower bands of standard deviation regimes, $\sigma_{Stddevs}$ is chosen to match the conditional transition probabilities from the case with standard deviations ranging from 1 percent to 10 percent. For example, the regime with the standard deviation of 3 percent in the case of regimes ranging from 1 percent to 5 percent is assigned the same conditional transition probability as the 5.5 percent standard deviation regime in the case with standard deviations ranging from 1 percent to 5 percent. This approach facilitates comparing results because only the range of standard deviation regimes changes while conditional transition probabilities remain unchanged. Formally, the values for $\sigma_{Stddevs}$ from the case with standard deviations ranging from 1 percent to 10 percent are scaled down in the case of standard deviation regimes ranging from 1 percent to 5 percent by the term $\frac{0.02}{0.045}$ (the relative length of intervals between regimes in both cases) and $\sigma_{Stddevs}$ takes one of the values $\left\{0.025 \cdot \frac{0.02}{0.045}, 0.03 \cdot \frac{0.02}{0.045}, 0.04 \cdot \frac{0.02}{0.045}\right\}$. The remaining bands of standard deviation regimes are treated in an analogous way.

Finally, I pair the various levels of dispersion of conditional transition probabilities for means (σ_{Means}) and standard deviations ($\sigma_{Stddevs}$) such that the lowest/middle/highest value of σ_{Means} is always associated with the lowest/middle/highest value of $\sigma_{Stddevs}$, respectively. Similarly, the cases of uniform transition probabilities are always paired. Although it would be possible to consider "unpaired" variables, for example, a model with a low value for σ_{Means} and a high value of $\sigma_{Stddevs}$, the additional insights are limited as the preliminary analysis has shown.

In order to simplify and unify notation, the resulting four levels of dispersion of conditional transition probabilities (for expectation and standard deviation regimes, and for all bands of standard deviation regimes) are denoted by σ_{low}, σ_{middle}, σ_{high}, and "uniform".

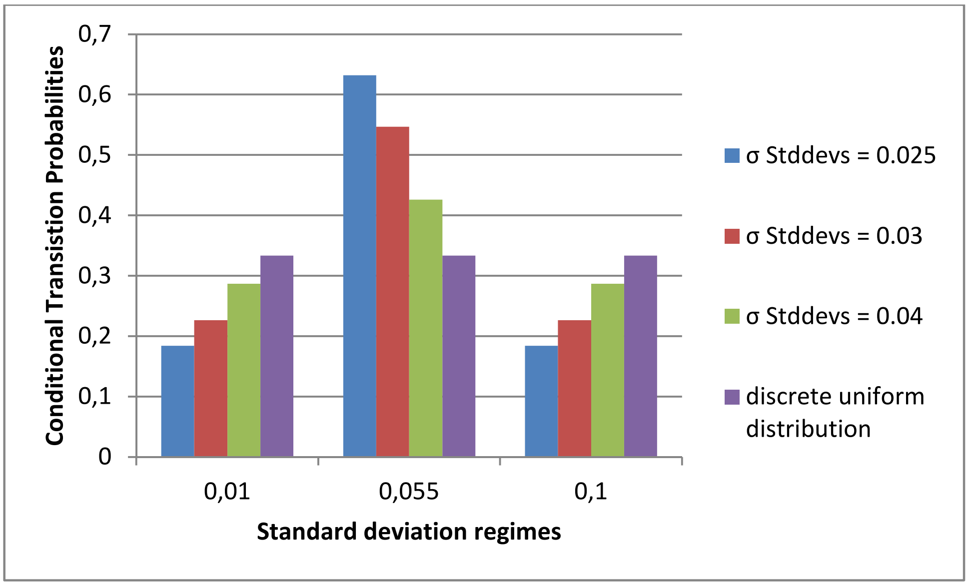

Figure 5-2: Various models for conditional transition probabilities for standard deviation regimes

5.3.2.2.4 Parameters for the Model with Two Risky Assets

5.3.2.2.4.1 Parameters for Asset 2

Recall that asset 2 is the risky asset with regimes in dividend growth (while dividends of asset 1 are regime-independent). Since these regimes are exactly as in the extended Veronesi model, the parameters (expectation and standard deviation regimes, probability of a draw in regimes, conditional transition probabilities) are chosen as in the extended Veronesi model. However, only the case with standard deviation regimes ranging from 1 percent to 10 percent is included into the analysis: the main object of interest is the existence of a second risky asset, and the additional insights obtained by considering the cases with narrower bands of standard deviation regimes would be limited.

5.3.2.2.4.2 Parameters for Asset 1

There are only two parameters that must be specified for dividend growth of asset 1: the regime-independent expectation and the regime-independent standard deviation of dividend growth of asset 1. Both parameters are chosen to match the expectation and standard deviation of dividend growth of asset 2 conditional on incomplete information (rather than conditional on a particular regime). The idea is to avoid any substantial decline or increase in the level of aggregate dividend growth as the relative dividend contribution of asset 1 goes from 0 (i.e., all dividends paid by asset 2) to 1 (all dividends paid by asset 1). If, for example, the standard deviation of dividend growth of asset 1 (conditional on incomplete information) was substantially less or more than that of asset 2, a change in the level of the risk premium would result from this difference in standard deviation levels alone as the relative dividend contribution of asset 1 ($\delta_{1,t}$) increases or decreases. By equating standard deviation levels for the extreme cases where one asset provides all dividends ($\delta_{1,t} = 0$ or $\delta_{1,t} = 1$), this effect is mostly eliminated, and it becomes possible to focus on changes in correlations of asset returns and the adjustment for risk.

5.3.2.2.4.3 Correlation of Dividend Growth Conditional on the Regime

Since expectation and standard deviation of asset 1 have been set to match asset 2, the only new parameter in this model with two risky assets is the correlation of dividend growth rates of both assets conditional on the true regime (5-22),

$$corr\left(\frac{D_{1,t+1}}{D_{1,t}}, \frac{D_{2,t+1}}{D_{2,t}} \middle| S_t = s\right) = corr_{d_1,d_2}$$
$$s = 1, \dots, K$$

which has been assumed to be regime-independent.[41] I consider uncorrelated dividend growth rates ($corr_{d_1,d_2} = 0$) as well as positive and negative correlations. Positive values are 0.6, 0.8 and 0.99, ranging from intermediate to very high correlations. Similarly, negative values are -0.6, -0.8 and -0.99.

5.4 Numerical Aspects

There are three main computational problems that must be solved to answer the questions outlined in (Section 5.1): (i) check for convergence of price dividend ratios in the case of “large time

[41] Similar to the standard deviation of dividend growth, it is necessary to distinguish between the correlation conditional on the true regime and the correlation conditional on incomplete information.

horizon"; (ii) computation of complete information price dividend ratios; (iii) computation of complete and incomplete information risk premia. It turns out (see Sections 5.4.2, 5.4.3, and 5.4.4) that these problems cannot be solved analytically, thus numerical methods must be employed.

5.4.1 Software Implementation

There is no (commercial or free) software available that contains inbuilt routines to solve problems (i) to (iii): although algorithms and routines for Eigenvalue decompositions, Gaussian quadrature, spline interpolation and Monte Carlo simulation exist in software packages, it is still necessary to develop an overall algorithm that combines these techniques. I have developed and implemented a suitable overall algorithm in the programming language C++.

5.4.2 Problem 1: Check for Convergence of Price Dividend Ratios

Convergence of the models can easily be examined by employing the Eigenvalue-based approached outlined in Section 4.4.3. This approach has the advantage that it can be decided whether price dividend ratios converge or diverge by analyzing "degenerate" cases where all dividends are paid by a single asset, i.e., it is not necessary to compute the entire price dividend ratio function for all relative dividend contributions. Formally, this approach checks the condition (4-178) and involves the computation of the matrix $M_{1,1}$ in the case of a single asset and the matrices $M_{1,1}, M_{1,2}, M_{2,1}, M_{2,2}$ in the case with two single assets) described in (4-177). The elements of these matrices can be found analytically by using the normality assumption on factors and residuals and the form of the dividend growth function,

$(1 + d_i) = exp(\mu_i(S_t) + A_i(S_t) \cdot fe_{t+1})$

where $A_i(S_t)$ is the row vector of regime-dependent coefficients of factors and residuals, and by using the identity

$$E\big(exp(X)\big) = exp\big(E(X) + 0.5 \cdot Var(X)\big)$$

that holds for a normally distributed random variable X.

Algorithms from the Eigen C++ library (Version 3.2.0) have been used for the Eigenvalue decompositions of the matrices $M_{i,j}$.

5.4.3 Problem 2: Computation of Complete Information Price Dividend Ratios

5.4.3.1 Case with a Single Risky Asset

In the case of a single risky asset, dividends are, by definition, always paid by one single asset. In other words, there only is the “degenerate case”, and all price dividend ratios are obtained as a by-product of the check for convergence of price dividend ratios. Price dividend ratios can be obtained as a by-product of the Eigenvalue-based check for model convergence (see Appendix A3.8.3.2.2.2 for the details).[42]

5.4.3.2 Case with Two Risky Assets

In the case with two risky assets, complete information price dividend ratios are also needed for non-degenerate constellations of relative dividend ratios. As a consequence, the Eigenvalue-based approach only checks convergence and yields price dividend ratios for special constellations. The computation of complete information price dividend ratios for “large time horizons” can be thought of as consisting of two sub-problems: (i) compute complete information price dividend ratios for a finite remaining time horizon $T-t$; (ii), approximately take the limit as $T-t$ goes to infinity.

<u>Solution to sub-problem 1</u>

Denote by $\left(\frac{P}{D}\right)_i^{(n)}(s,\delta_1)$ the complete information price dividend ratio for a remaining time horizon of n periods. I show in Appendix A4.2.1.1 that $\left(\frac{P}{D}\right)_i^{(n)}(s,\delta_1), s=1,\dots,K, \delta_1\epsilon[0;1]$ can be recursively obtained from $\left(\frac{P}{D}\right)_i^{(n-1)}(s,\delta_1), s=1,\dots,K, \delta_1\epsilon[0;1]$, where $\left(\frac{P}{D}\right)_i^{(0)}\equiv 0$ starts the recursion[43]:

5-29

$$\left(\frac{P}{D}\right)_i^{(n)}(s,\delta_1)=E\left(\begin{array}{c} q_{t,t+1}{}^{cI}\left(fe_{t+1},S_t,\delta_{1,t}\right)\cdot[1+d_i(S_t,fe_{t+1})] \\ \cdot\left\{\sum_{s'=1}^{K}p_{S_ts'}\cdot\left(\frac{P}{D}\right)_i^{(n-1)}\left(s',\delta\left(\delta_{1,t},S_t,fe_{t+1}\right)\right)+1\right\}\end{array}\middle|S_t=s,\delta_{1,t}=\delta_1\right)$$

$$s=1,\dots,K$$
$$i=1,2$$

[42] Alternatively, complete information price dividend ratios could be found from a linear equation system with the K complete information price dividend ratios as unknowns: quasi-static asset prices (4-150) simplify to a linear structure because the relative dividend contribution of the single asset is always equal to one.

[43] Intuitively, the asset price in a model with a time horizon of zero periods must be zero.

Solution to sub-problem 2

The price dividend ratio is iterated until a distance measure for the functions $\left(\frac{P}{D}\right)_i^{(n+1)}$ and $\left(\frac{P}{D}\right)_i^{(n)}$ falls below a certain error tolerance level (for details see the Appendix A4.2.1.2). Since the iteration (5-29) is based on the equation for quasi-static asset prices, this procedure leads to an approximate solution of the integral equation defined by quasi-static asset prices.

Numerical details on the solutions to sub-problems 1 and 2

The computation of the iterations and the check for approximate convergence entail various numerical operations: (i) the expectation (i.e., an integral) on the right-hand side of (5-29) has to be evaluated for selected values of δ_1 and (ii) the resulting function values have to be interpolated to the interval $[0,1]$. I choose equidistant partitions of $[0,1]$ to define the values of δ_1 at which the expectations are taken.

Concerning (i), the integrals are evaluated by Gauss-Hermite quadrature with the "probabilist" weight function $\omega(x) = exp(-0.5 \cdot x^2)$ and with 32 weights/abscissas (for details, see the Appendix A4.2.2. Abscissas and weights for the quadrature rules can be found at Burkardt). Stoer/Bulirsch (2000) , p. 181, state that Gaussian integration methods give the most accurate results compared to simple rules such as the Newton-Cotes formulas and also compared to extrapolation methods for given computational effort. Moreover, Gaussian quadrature of the Hermite type is particularly suitable to a setting with integrals over the entire real line $(-\infty, +\infty)$ that is relevant in the present context, whereas most other methods are for finite intervals, thus leading to additional problems. To check the accuracy of integration by Gaussian quadrature using a number of 32 weights/abscissas, I use integrals that have an analytical solution and, in addition, use Monte Carlo simulation where no analytical solution is available.

Concerning (ii), cubic splines are used (see Stoer/Bulirsch (2000), Section 2.4.2). These splines yield smooth functions and are computationally simple. They also have the advantage that the approximation error can be made arbitrarily small if a sufficient number of interpolation points is chosen (see Stoer/Bulirsch (2000), p. 101 for smoothness, and Section 2.4.3 for the discussion of the interpolation error).

5.4.4 Problem 3: Computation of Risk Premia

Although risk premia can be found analytically if there is a single risky asset and information is complete, risk premia under incomplete must always be computed numerically. This again involves the computation of integrals. The same methods as for price dividend ratios (Gaussian quadrature of the Hermite type) can be used for risk premia. In addition to Gaussian quadrature, I also use Monte Carlo simulation in order check the accuracy of results and, more importantly, gain insights into the various channels that introduce correlations between asset returns and the adjustment for risk (see the discussion in connection with (4-62)).

5.5 Results

5.5.1 Discretized Veronesi Model: Single Risky Asset Model with Incomplete Information about Expected Dividend Growth

5.5.1.1 Description of Results and Answers to Questions

- Answer to Question 1 (relevance of incomplete information to risk premia compared to complete information)

The answer to Question 1 is due to Veronesi (2000), Lemma 3, p. 816: Incomplete information risk premia differ from complete information risk premia: while complete information risk premia are always positive, as implied by theory, incomplete information risk premia can become negative for certain parameter constellations if the risk aversion parameter γ exceeds one (see Table 5-1 and Table 5-2). Moreover, even if incomplete information risk premia are positive, they are almost always below complete information risk premia and are of a significantly different size (see Figure 5-3):

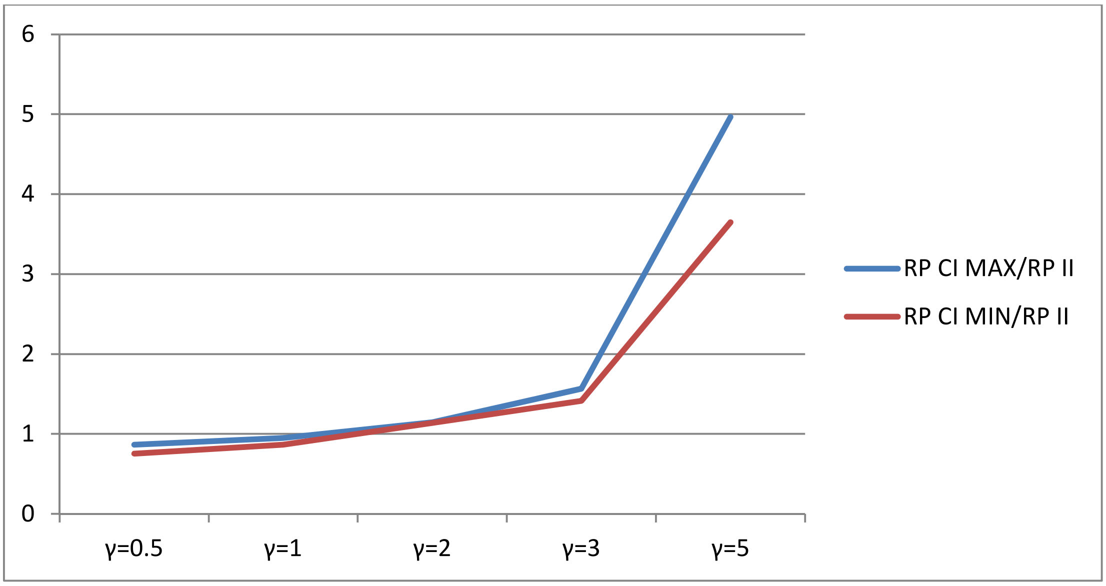

Figure 5-3: Ratio of complete to incomplete information risk premium as a function of the risk aversion parameter γ for uniform conditional transition probabilities, ($\sigma_D = 0.1, p_{Draw} = 0.25$, $\rho = 0.05$). The ratio is computed for the maximum and minimum complete information risk premium across regimes.
Note that risk premiums ratios are functions of a finite set of risk aversion parameters but have been connected by a polygonal path for readability.

- Answer to Question 2 (relevance of extensions to Veronesi (2000))

Since the model of this section is a discretized version of Veronesi (2000), results cannot differ from Veronesi (2000).

- Answer to Question 3 (relevance of model parameters)

This question is not fully addressed in Veronesi (2000). Based on my numerical results (see Table 5-1 and Table 5-2), the parameters γ (risk aversion), $p_{Mean,Draw}$ (probability that a drawing of regimes occurs), σ_D (standard deviation of dividend growth conditional on the true regime), and σ_{Means} (Dispersion of conditional transition probabilities/initial probabilities) exert a strong influence on incomplete information risk premia. Of lesser importance is the time preference parameter ρ.

Incomplete information risk premia are often negative if γexceeds one, particularly if $p_{Mean,Draw}$ is low and/or the dispersion of conditional transition probabilities (σ_{Means}) is very high. Whenever incomplete information risk premia are positive, their magnitude is low, but negative incomplete information risk premia can be substantial, in particular if σ_D, the standard deviation of dividend growth conditional on the true regime, is low (1 percent). For a high value of $\sigma_D = 0.1$, negative risk premia are observed less often and are less pronounced. Incomplete information risk premia tend to be higher than complete information risk premia if γ is below one. If $\gamma > 1$, incomplete information risk premia are almost always below complete information risk premia in all regimes.

Incomplete information risk premia ($\sigma_D = 0.01$)				
$p_{Mean.Draw} = 0$				
	$\sigma_{Means} = 0.005$	$\sigma_{Means} = 0.0$	$\sigma_{Means} = 0.025$	uniform
$\rho = 0.025$				
$\gamma = 0.5$	DIV	DIV	DIV	DIV
$\gamma = 1$	0.01%	0.02%	0.07%	0.11%
$\gamma = 2$	-0.06%	-0.26%	-1.81%	-3.03%
$\gamma = 3$	-0.12%	-0.51%	-4.01%	-6.46%
$\gamma = 5$	-0.25%	-1.02%	-9.75%	-14.66%
$\rho = 0.05$				
$\gamma = 0.5$	0.04%	0.13%	1.34%	1.87%
$\gamma = 1$	0.01%	0.02%	0.07%	0.11%
$\gamma = 2$	-0.04%	-0.18%	-1.20%	-2.03%
$\gamma = 3$	-0.10%	-0.41%	-2.84%	-4.72%
$\gamma = 5$	-0.21%	-0.90%	-7.17%	-11.40%
$p_{Mean.Draw} = 0.25$				
	$\sigma_{Means} = 0.005$	$\sigma_{Means} = 0.0$	$\sigma_{Means} = 0.025$	uniform
$\rho = 0.025$				
$\gamma = 0.5$	0.01%	0.02%	DIV	DIV
$\gamma = 1$	0.01%	0.02%	0.07%	0.11%
$\gamma = 2$	0.01%	-0.01%	-0.14%	-0.25%
$\gamma = 3$	0.00%	-0.07%	-0.55%	-0.95%
$\gamma = 5$	-0.04%	-0.27%	-1.85%	-3.15%
$\rho = 0.05$				
$\gamma = 0.5$	0.01%	0.02%	0.07%	0.12%
$\gamma = 1$	0.01%	0.02%	0.07%	0.11%
$\gamma = 2$	0.01%	0.00%	-0.12%	-0.22%
$\gamma = 3$	0.00%	-0.06%	-0.50%	-0.88%
$\gamma = 5$	-0.03%	-0.26%	-1.75%	-2.98%

Table 5-1: Incomplete information risk premia for the discretized Veronesi model (case $\sigma_D = 0.01$). 'DIV' refers to a parameter constellation where the complete information price dividend ratio diverges to infinity in at least one regime. Negative risk premia are highlighted in red print.

Incomplete information risk premia ($\sigma_D = 0.1$)				
$p_{Mean.Draw} = 0$				
	$\sigma_{Means} = 0.005$	$\sigma_{Means} = 0.0$	$\sigma_{Means} = 0.025$	uniform
$\rho = 0.025$				
$\gamma = 0.5$	DIV	DIV	DIV	DIV
$\gamma = 1$	0.97%	0.98%	1.03%	1.07%
$\gamma = 2$	1.91%	1.67%	-0.42%	-1.94%
$\gamma = 3$	DIV	DIV	DIV	DIV
$\gamma = 5$	DIV	DIV	DIV	DIV
$\rho = 0.05$				
$\gamma = 0.5$	0.52%	0.60%	1.54%	2.07%
$\gamma = 1$	0.99%	1.00%	1.05%	1.09%
$\gamma = 2$	1.98%	1.82%	0.62%	-0.35%
$\gamma = 3$	3.00%	2.58%	-1.51%	-4.21%
$\gamma = 5$	DIV	DIV	DIV	DIV
$p_{Mean.Draw} = 0.25$				
	$\sigma_{Means} = 0.005$	$\sigma_{Means} = 0.0$	$\sigma_{Means} = 0.025$	uniform
$\rho = 0.025$				
$\gamma = 0.5$	0.48%	0.49%	0.54%	0.59%
$\gamma = 1$	0.97%	0.98%	1.03%	1.07%
$\gamma = 2$	2.00%	1.97%	1.84%	1.72%
$\gamma = 3$	3.05%	2.97%	2.43%	1.96%
$\gamma = 5$	5.15%	4.83%	2.63%	0.86%
$\rho = 0.05$				
$\gamma = 0.5$	0.49%	0.50%	0.55%	0.60%
$\gamma = 1$	0.99%	1.00%	1.05%	1.09%
$\gamma = 2$	2.04%	2.03%	1.90%	1.80%
$\gamma = 3$	3.13%	3.05%	2.55%	2.12%
$\gamma = 5$	5.28%	4.98%	2.93%	1.26%

Table 5-2: Incomplete information risk premia for the discretized Veronesi model (case $\sigma_D = 0.1$). 'DIV' refers to a parameter constellation where the complete information price dividend ratio diverges to infinity in at least one regime. Negative risk premia are highlighted in red print.

Complete information risk premia vary little across regimes and depend positively on σ_D, the standard deviation of dividend growth conditional on the true regime, and the risk aversion parameter γ (see Figure 5-4 and Figure 5-5). The other model parameters exert a negligible influence.

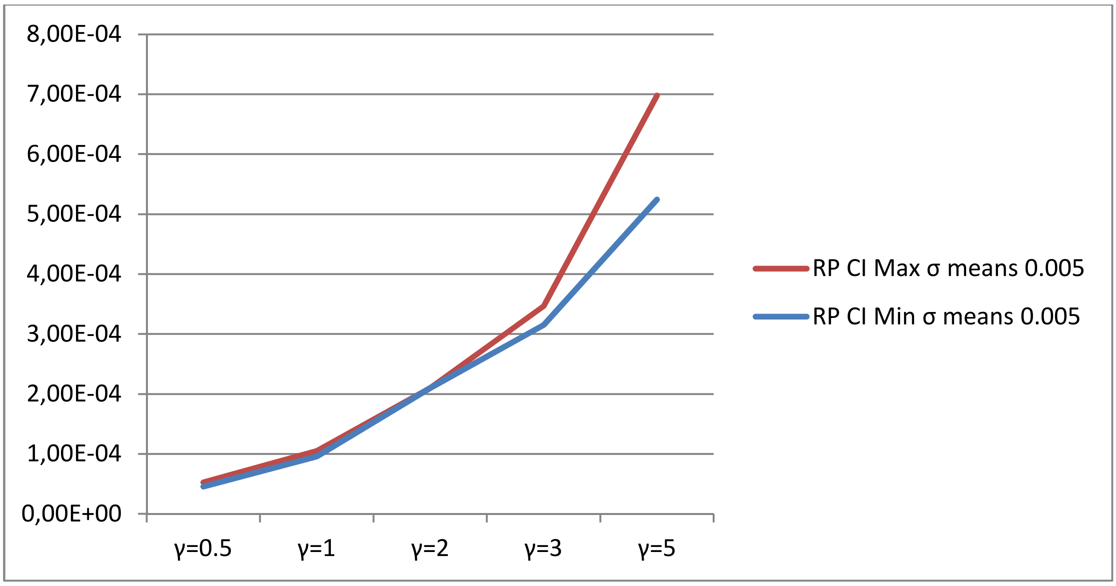

Figure 5-4: Maximum and minimum complete information risk premia across regimes as a function of the risk aversion parameter γ ($\sigma_D = 0.01, p_{Draw} = 0.25, \rho = 0.05$)
Note that risk premium ratios are functions of a finite set of risk aversion parameters but have been connected by a polygonal path for readability.

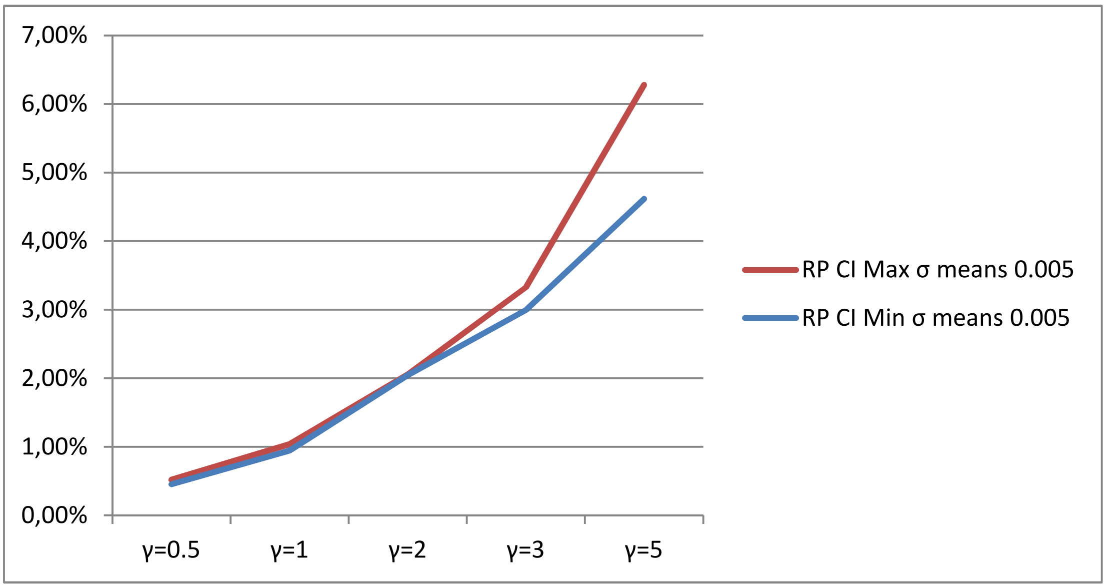

Figure 5-5: Maximum and minimum complete information risk premia across regimes as a function of the risk aversion parameter γ ($\sigma_D = 0.1, p_{Draw} = 0.25, \rho = 0.05$)
Note that risk premium ratios are functions of a finite set of risk aversion parameters but have been connected by a polygonal path for readability.

5.5.1.2 Interpretation of Results

5.5.1.2.1 Explanation of the Answer to Question 1

Return-based incomplete information risk premia read for the special case of the discretized Veronesi model

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$$RP_t^{iI,ret}(\pi_t) = -cov\left(\begin{array}{c} AfR_{t,t+1}{}^{iI}(fe_{t+1}, S_t, \pi_t), \\ [1 + d(S_t, fe_{t+1})] \cdot \dfrac{\left\{ \sum_{s'=1}^{K} \pi_{s',t+1} \cdot \left(\frac{P}{D}\right)_{t+1}^{cI}(s') + 1 \right\}}{\left(\frac{P}{D}\right)_t^{iI}(\pi_t)} \end{array} \middle| \pi_t \right)$$

with

$$AfR_{t,t+1}{}^{iI}(fe_{t+1}, S_t, \pi_t) = \frac{\left(1 + d(S_t, fe_{t+1})\right)^{-\gamma}}{\sum_{s=1}^{K} \pi_{s,t} \cdot E\left(\left(1 + d(S_t, fe_{t+1})\right)^{-\gamma} \middle| S_t = s\right)}$$

$$\pi_{t+1} = \Pi(\pi_t, d_{t+1})$$

Negative incomplete information risk premia occur whenever the correlation of the adjustment for risk with the return on the risky asset is positive; positive incomplete information risk premia arise when the covariance is negative. The sign of this correlation depends crucially on stochastic regime probabilities π_{t+1}. Since these regime probabilities are functions of dividend growth $1 + d$, a correlation with the adjustment for risk – which is proportional to $(1 + d)^{-\gamma}$ - is introduced (channel of stochastic regime probabilities, see the discussion of (4-62), p. 118).

Observe that all components of the correlation between return on the risky asset and adjustment for risk are driven by dividend growth $1 + d$, a fact that can be summarized in the following Table 5-3:

	Reaction to an increase in dividend growth $1 + d$:			Sign of the risk premium:
	Adjustment for risk $(1 + d)^{-\gamma}$	Dividend yield $\frac{(1 + d)}{\left(\frac{P}{D}\right)_t^{iI}(\pi_t)}$	Incomplete information Price Dividend Ratio growth $\frac{\left\{\sum_{s'=1}^{K} \pi_{s',t+1} \cdot \left(\frac{P}{D}\right)_{t+1}^{cI}(s')\right\}}{\left(\frac{P}{D}\right)_t^{iI}(\pi_t)}$	$RP_t^{iI,ret}(\pi_t)$
$0 < \gamma < 1$	-	+	+	> 0
$\gamma = 1$	-	+	0	> 0
$\gamma > 1$	-	+	-	>=< 0

Table 5-3: Responses of risk premium components to positive dividend growth and implication for the sign of the incomplete information risk premium.

While the responses of the adjustment for risk and dividend yield to high dividend growth are obvious, the response of incomplete information price dividend ratio growth is not immediately clear and needs a more detailed explanation. This ratio consists of conditional regime probabilities π_{t+1} and complete information price dividend ratios $\left(\frac{P}{D}\right)_{t+1}^{cI}(s), s = 1, \dots, K$. Both components interact in a way that depends on the risk aversion parameter γ. To see how this interaction works, consider how the risk aversion parameter γ and the expected dividend growth regime jointly form complete information price dividend ratios. The complete information price dividend ratio increases or decreases in the expected dividend growth regime, depending on whether γ is less than or greater than one. In the remaining case of logarithmic utility ($\gamma = 1$), complete information price dividend ratios do not depend on regimes. The underlying economic reason for this behavior for different γ is standard: high future dividend growth has two conflicting effects on the price dividend ratio; (i) the direct effect of high dividend growth is that future dividends are high, but the price dividend ratio equals discounted future dividend growth. (ii) However, high future dividend growth also has an indirect effect on price dividend ratios: high dividend growth is tantamount to high future consumption, hence low marginal utility of consumption, and, finally, low stochastic discount factors. The relative strength of the direct and the indirect effect depends on γ: If γ is less than one, the direct effect prevails, whereas the indirect effect via stochastic discount factors dominates for $\gamma > 1$. In the remaining case, log utility, both effects always exactly offset.

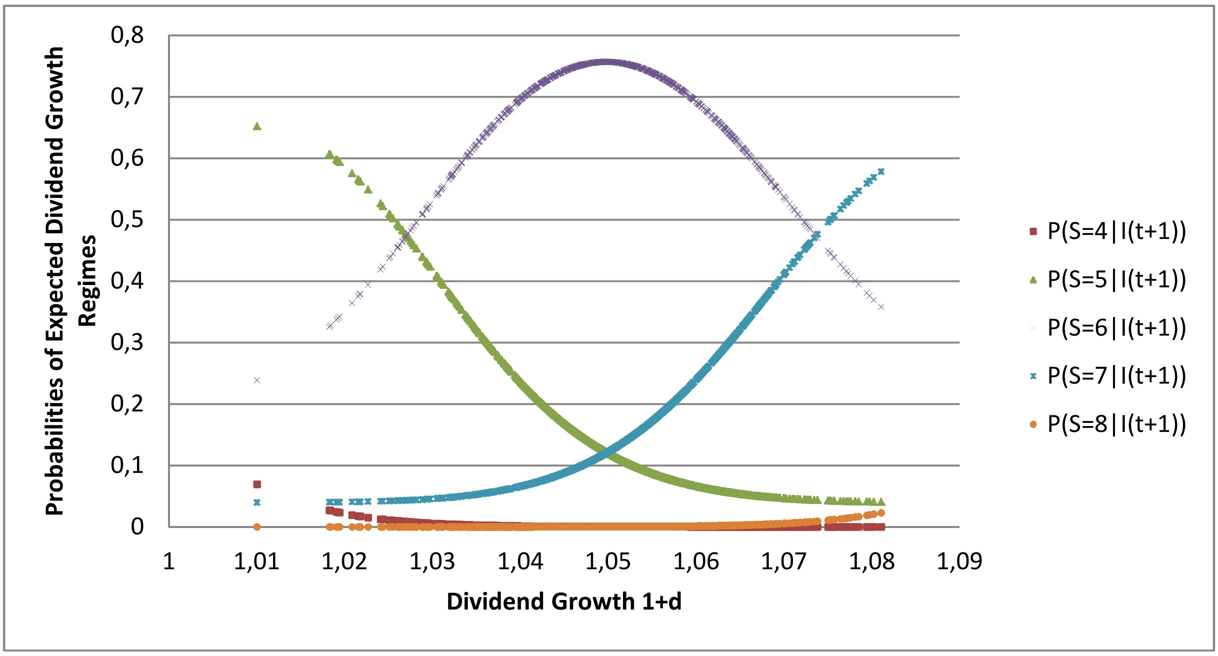

Figure 5-6: Probabilities of various expected dividend growth regimes (E(d)) as functions of dividend growth ($p_{Draw} = 0.25$, $\sigma_{Means} = 0.005$, $\sigma_D = 0.01$): simulation with 10,000 dividend growth realizations

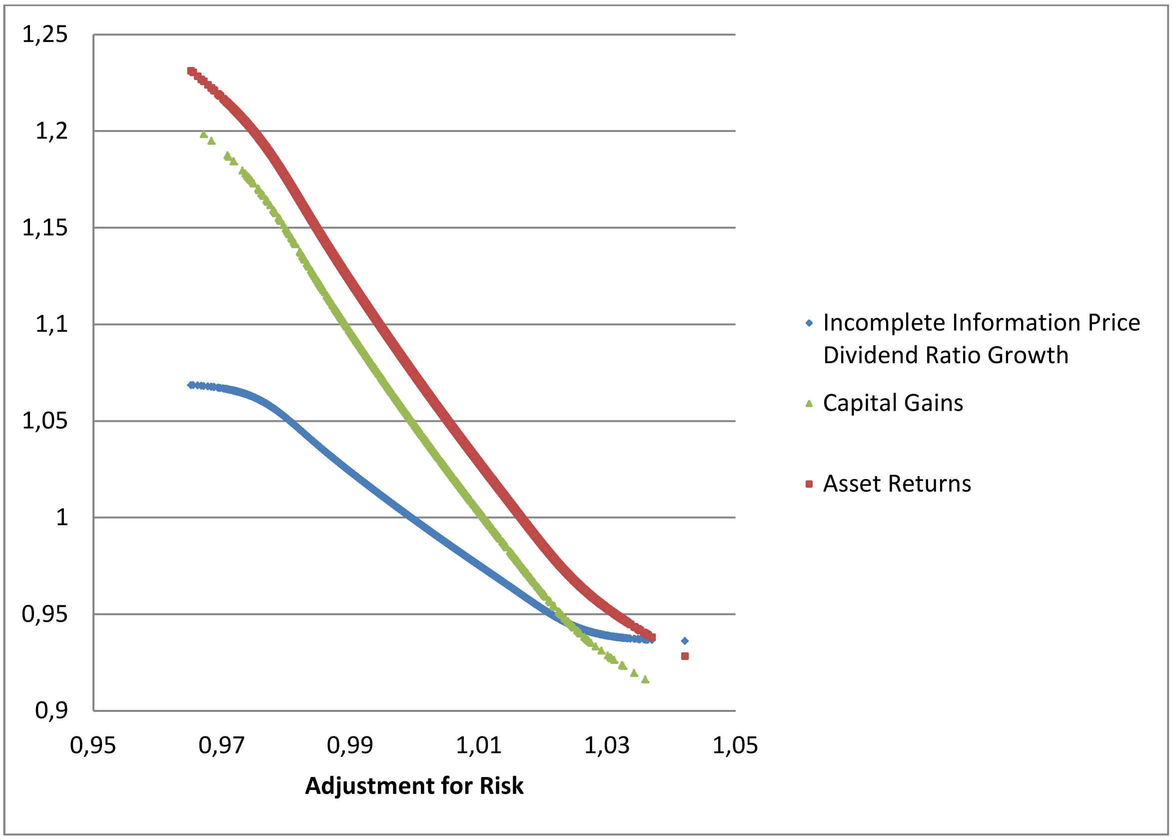

Figure 5-7: Case $\gamma < 1$, incomplete information price dividend ratio growth, assets returns, and capital gains as functions of adjustment or risk ($p_{Draw} = 0.25$, $\sigma_{Means} = 0.005$, $\sigma_D = 0.01$, $\rho = 0.05$, $\gamma = 0.5$): simulation with 10,000 adjustment for risk realizations.

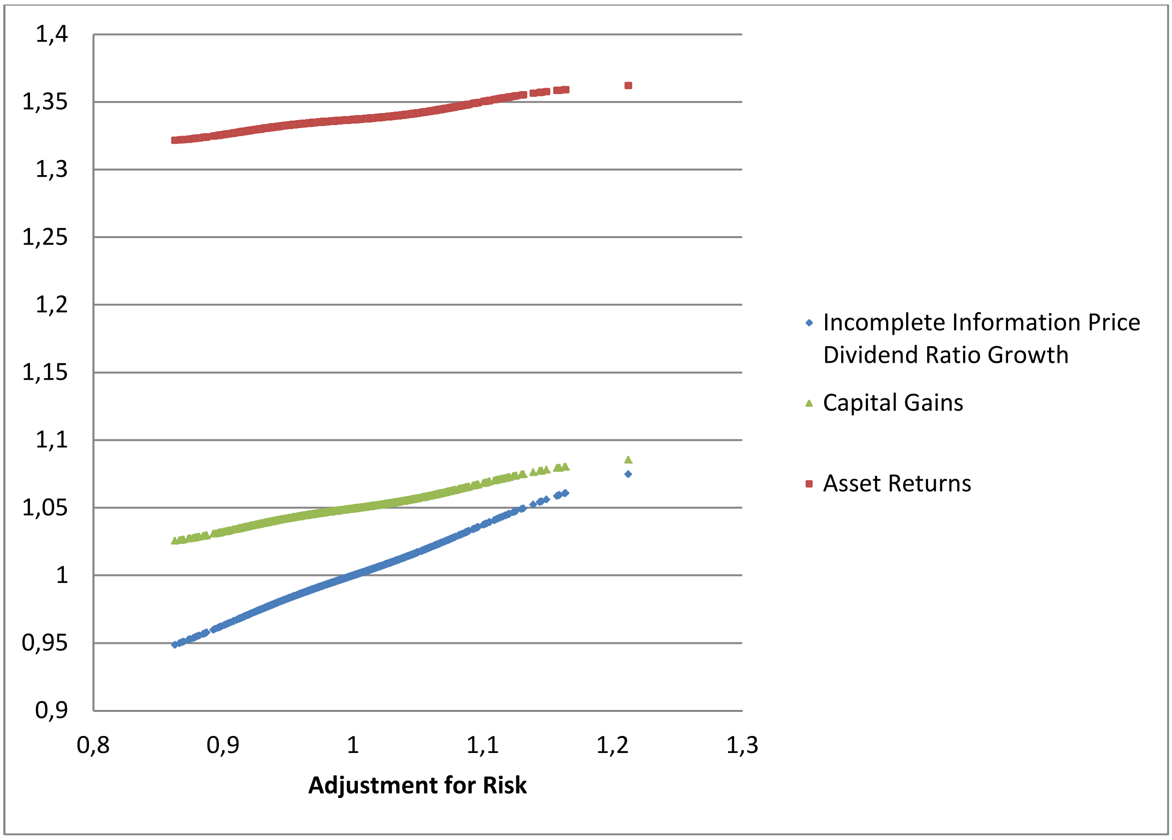

Figure 5-8: Case $\gamma > 1$, incomplete information price dividend ratio growth, assets returns, and capital gains as functions of adjustment or risk ($p_{Draw} = 0.25$, $\sigma_{Means} = 0.005$, $\sigma_D = 0.01$, $\rho = 0.05$, $\gamma = 5$): simulation with 10,000 adjustment for risk realizations.

Now observe how complete information price dividend ratios and conditional regime probabilities interact depending on γ. The numerator of incomplete information price dividend ratio growth, i.e., the incomplete information price dividend ratio at time $t + 1$ ($\sum_{s'=1}^{K} \pi_{s',t+1} \cdot \left(\frac{P}{D}\right)_{t+1}^{cI} (s')$), is a weighted average of complete information price dividend ratios, with stochastic regime probabilities serving as weights. Consider the effect of high dividend growth on asset returns: investors will update conditional regime probabilities: the probabilities of regimes with high expected dividend growth will increase, whereas the probabilities of low growth regimes will decrease. This implies that the incomplete information price dividend ratio will be similar to complete information price dividend ratios in regimes with high expected dividend growth. Now combine this with the interaction of the complete information price dividend ratio with γ: if $\gamma < 1$, complete information price dividend ratios in high expected dividend growth regimes are highest. This implies that incomplete information price dividend ratio growth in response to high dividend growth will be a high value. If $\gamma > 1$, one gets the opposite case and incomplete information price dividend ratio growth will take a low value: incomplete information price dividend ratio growth will then be roughly equal to the lowest complete information price dividend ratios (which are associated with the highest expected dividend growth regimes for $\gamma > 1$).

5.5.1.2.2 Explanation of the Answer to Question 3

Since the risk premium is proportional to the covariance of the adjustment for risk with asset returns, the effect of the various parameters on risk premia can, in principle, be discussed by examining how the covariance depends on these parameters. Several parameters, however, affect the covariance in multiple ways, and it is thus helpful to break down the covariance into the following three components: the covariance is the product of (i) the standard deviation of the adjustment for risk, (ii) the standard deviation of asset returns and (iii) the correlation of the adjustment for risk with asset returns. Any parameter that strongly affects at least one of these components of risk premia can, therefore, have a strong effect on risk premia.

5.5.1.2.2.1 Incomplete Information

The effects of model parameters on the standard deviation of the adjustment for risk

Numerical computations (see Appendix A4.2.3) show that the standard deviation can roughly be thought of as the product of the risk aversion parameter γ and the standard deviation of dividend growth conditional on incomplete information:

$$Stddev(AfR|\pi_t) \approx \gamma \cdot Stddev(1+d|\pi_t)$$

The standard deviation of dividend growth $Stddev(1+d|\pi_t) = \sqrt{\sigma_D{}^2 + Var\left(E\left(\frac{D_{t+1}}{D_t}\middle|S_t\right)\middle|\pi_t\right)}$ consists of σ_D, the standard deviation of dividend growth conditional on the regime, and a second component which captures the uncertainty about the true regime. This uncertainty about the regime is captured by conditional regime probabilities at time t, π_t, which are initial probabilities.[44] The level of dispersion of these initial probabilities is quantified by σ_{Means}.

Note that $p_{Mean,Draw}$ does not have any effect on the standard deviation because it neither describes current regime probabilities nor the distribution of (one-period) dividend growth $1+d$ between times t and $t+1$.

In conclusion, the following parameter influences can be identified: the standard deviation increases in γ, in σ_D, and in σ_{Means}.

The effects of model parameters on the standard deviation of asset returns

The standard deviation of asset returns based on (5-3) reads

[44] More precisely, the probabilities π_t are either initial probabilities or, if existing, unique invariant regime probabilities. However, it turns out that unique invariant regime probabilities only exist if $p_{Mean,Draw}$ is positive, and that they coincide with initial probabilities in this case. If $p_{Mean,Draw}$ is zero, only initial probabilities can be considered. In short, it suffices to consider initial probabilities in all cases.

$$Stddev\left(Return_{t,t+1}\right) = Stddev\left([1 + d(S_t, fe_{t+1})] \cdot \frac{\left\{\sum_{s'=1}^{K} \pi_{s',t+1} \cdot \left(\frac{P}{D}\right)_{t+1}^{cI} (s') + 1\right\}}{\left(\frac{P}{D}\right)_{t}^{iI} (\pi_t)} \right)$$

Since dividend growth and incomplete information price dividend ratio growth are typically highly correlated, this standard deviation is not easily expressed as a sum or product of standard deviations of dividend growth and incomplete information price dividend ratio growth. For that reason, I heuristically discuss in a first step the fluctuations of both components individually and then consider their interactions in a second step, but do not formally compute the standard deviation of returns.

Fluctuation of incomplete information price dividend ratio growth

The fluctuation of incomplete information price dividend ratio growth $\sum_{s'=1}^{K} \pi_{s',t+1} \cdot \left(\frac{P}{D}\right)_{i,t+1}^{cI} (s')$ increases as (i) the range between the minimum and maximum complete information price dividend ratios increases,

$$\frac{max_{s\epsilon\{1,\dots,K\}} \left(\frac{P}{D}\right)_{t+1}^{cI} (s)}{min_{s\epsilon\{1,\dots,K\}} \left(\frac{P}{D}\right)_{t+1}^{cI} (s)}$$

and (ii) as the fluctuation of stochastic regime probabilities π_{t+1} increases.

Complete information price dividend ratios differ across regimes if $p_{Mean,Draw}$ is low, if γ is high, and if σ_D is high. The intuition for these effects is as follows:

- $p_{Mean,Draw}$

If the regime is an extreme one (say, 0 expected dividend growth or, on the other end, 10 percent expected dividend), then this regime is likely, or even certain (if $p_{Mean,Draw} = 0$), to last for many periods, and thus highly influences future dividend growth. By contrast, if $p_{Mean,Draw}$ is relatively high, an extreme regime will probably be replaced by a more moderate regime within the next few periods, and its influence on future dividend growth is more limited.

- γ

Minimum and maximum complete information price dividend ratios will be found in the most extreme regimes. The subjective impact of the most extreme regimes increases with risk aversion γ. Consequently, the range of complete information price dividend ratios increases in the risk aversion parameter γ.

- σ_D

If σ_D is high, the probability of obtaining extreme dividend growth outcomes increases in all regimes. Since investors are risk averse, this effect is more felt in regimes with low expected dividend growth than in regimes with high expected dividend growth. Therefore, the range of complete information price dividend ratios increases.

- σ_{Means}

  σ_{Means} modestly affects the absolute levels of complete information price dividend ratios. However, if one divides the complete information price dividend ratio in regime 11 (10 percent expected dividend growth) by the complete information price dividend ratio in regime 1 (0 percent expected dividend growth), σ_{Means} has a negligible effect in this relative sense.

The fluctuation of stochastic regime probabilities decreases in $p_{Mean,Draw}$ as well as σ_D, and increases in σ_{Means}, and decreases in $p_{Mean,Draw}$ and σ_D. The effect of σ_D and σ_{Means} are particularly pronounced. Note that stochastic regime probabilities do not depend on the risk aversion parameter γ.

The intuition for these effects is as follows:

- $p_{Mean,Draw}$

  Recall that π_{t+1} are probabilities of the regime S_{t+1}, while dividend growth only depends on the current regime S_t.[45] Nevertheless, learning about S_t also conveys indirect information on S_{t+1}. However, the link between S_t and S_{t+1} becomes weak if $p_{Mean,Draw}$ is high and, consequently, the probability of a regime switch is high. By contrast, if $p_{Mean,Draw}$ is zero, S_t and S_{t+1} are identical, and dividend growth provides direct information on S_{t+1}. Therefore, the probabilities π_{t+1} will react strongly to dividend growth if $p_{Mean,Draw}$ draw is low and dividend growth contains much information about S_{t+1}. The opposite holds if $p_{Mean,Draw}$ is high.

- σ_D and σ_{Means}

  The effect of σ_D and σ_{Means} is best understood in combination. σ_{Means} can be interpreted as the level of uncertainty on the regime before new information in the form of dividend growth arises (with a high value of σ_{Means} corresponding to high uncertainty), while σ_D is inversely related to the quality of this new information (i.e., the information provided by dividend growth). For example, if σ_{Means} is low and σ_D is very high, investors are relatively certain about the current regime, and will learn little about the current regime because dividends are a noisy source of information (σ_D is high). If, by contrast, σ_{Means} is high (or, in the most extreme case, regime probabilities are even uniform), and if σ_D is low, little is known about the regime S_t before dividend growth is observed, and dividend growth provides much information about S_t. Therefore, stochastic regime probabilities react strongly

[45] Dividend growth over the time interval from t to $t+1$ is a function of the current regime S_t as well as factors and residuals fe_{t+1}, but not S_{t+1}.

to dividend growth if σ_{Means} is high relative to σ_D, and the opposite is true if σ_{Means} is low relative to σ_D.[46]

Fluctuation of dividend growth

The fluctuation of dividend growth increases in (i) the standard deviation of dividend growth conditional on the true regime, σ_D, and (ii) the uncertainty about the true regime σ_{Means}. The parameter $p_{Mean,Draw}$ and γ have no effect.

Fluctuation of asset returns (total effect)

The standard deviation of asset returns depends on the interaction of dividend growth and incomplete information price dividend ratio growth.

If γ is less than one, both of these influencing variables tend to move into the same direction. Any parameter that increases fluctuations in one or both influencing variables should then increase the standard deviation of asset returns.

If γ exceeds one, dividend growth and incomplete information price dividend ratio growth tend to move in opposite directions. This typically leads to a U-shaped relationship between model parameters and the standard deviation of returns. To understand this U-shaped relation, consider σ_D, the other parameters behave in a similar way. If σ_D is very low, dividend growth exhibits little variation, and the standard deviation of asset returns is almost the same as the standard deviation of incomplete information price dividend ratio growth (which can be substantial even for low σ_D). As σ_D increases, incomplete information price dividend ratio growth is to some degree compensated by dividend growth[47] and, hence, the standard deviation of asset returns is less than the standard deviations of its two components. From a certain level of σ_D on, the standard deviation of asset returns mostly consists of the standard deviation of dividend growth and any further increases in σ_D will now lead to an increase in the standard deviation of asset returns.

Effect of model parameters on the correlation of asset returns with the adjustment for risk

The correlation of asset returns with the adjustment for risk

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[46] The intuition is similar to Bayesian statistics: if inference about an unknown parameter is to be drawn, the effect of an observation on posterior probabilities depends on the relative precision of the prior distribution (i.e., the information prior to the observation, here inversely related to σ_{Means}) and that of the observation (here the inverse of σ_D), see, e.g., Box/Tiao (1973) p. 17.

[47] Note that an increase in σ_D not only leads to higher fluctuations of dividend growth but also to lower fluctuations of incomplete information price dividend ratio growth: a higher level of σ_D means that dividend growth contains less information on the regime and thus the reaction of stochastic regime probabilities to dividend growth is less pronounced.

$$corr\left(\left[\underbrace{1+d(S_t,fe_{t+1})}_{term\ i}\right]\cdot\underbrace{\frac{\left\{\sum_{s'=1}^{K}\pi_{s',t+1}\cdot\left(\frac{P}{D}\right)_{t+1}^{cI}(s')+1\right\}}{\left(\frac{P}{D}\right)_t^{iI}(\pi_t)}}_{term\ ii}\ \middle|\ \pi_t\right)^{AfR_{t,t+1}{}^{iI}(fe_{t+1},S_t,\pi_t),}$$

depends on whether γ is greater or less than 1. If $\gamma \leq 1$, the correlation is always negative. If $\gamma > 1$, the correlation depends on the interplay between dividend growth (term i, tends to reduce correlation) and incomplete information price dividend ratio growth (term ii, tends to increase correlation); these two terms exert conflicting influences. The term that fluctuates more coins the correlation. If p_{Draw} increases, term ii fluctuates less (see p. 209) and term i is unaffected resulting in a decreasing correlation (less positive or more negative). If σ_D and σ_{Means}, increase, term i fluctuates more; σ_D reduces the fluctuation of term ii while σ_{Means} has an increasing effect (see p. 210). In other words, an increasing σ_D makes term i more important than term ii and, thus, tends to reduce reduced correlation of asset returns with the adjustment for risk. The effect of increases in σ_{Means} is unclear because both terms fluctuate more.

<u>Effect of model parameters on the risk premium</u>

Combining the three statements derived, model parameters influence the risk premium as in Table 5-1 and Table 5-2.

5.5.1.2.2.2 Complete Information

The return-based complete information risk premium for the single asset case reads (special case of (5-2))

5-31

$$RP_t^{cI,ret}(S_t) = -cov\left(AfR_{t,t+1}{}^{cI}(fe_{t+1},S_t),\ [1+d(S_t,fe_{t+1})]\cdot\frac{\left\{\sum_{s'=1}^{K}p_{S_ts'}\cdot\left(\frac{P}{D}\right)_{t+1}^{cI}(s')+1\right\}}{\left(\frac{P}{D}\right)_t^{cI}(S_t)}\ \middle|\ S_t\right)$$

with

$$AfR_{t,t+1}{}^{cI}(fe_{t+1},S_t) = \frac{\left(1+d(S_t,fe_{t+1})\right)^{-\gamma}}{E\left(\left(1+d(S_t,fe_{t+1})\right)^{-\gamma}\middle|S_t\right)}$$

Under complete information, the term

$$\frac{\left\{\sum_{s'=1}^{K}p_{S_ts'}\cdot\left(\frac{P}{D}\right)_{t+1}^{cI}(s')+1\right\}}{\left(\frac{P}{D}\right)_t^{cI}(S_t)}$$

is expected price dividend ratio growth, not random, and, hence, can be factored out of the covariance.

$$RP_t^{cI,ret}(S_t) = -cov\left(\begin{matrix} AfR_{t,t+1}{}^{cI}(fe_{t+1}, S_t), \\ [1 + d(S_t, fe_{t+1})] \end{matrix} \middle| S_t\right) \cdot \frac{\left\{\sum_{s'=1}^{K} p_{S_t s'} \cdot \left(\frac{P}{D}\right)_{t+1}^{cI}(s') + 1\right\}}{\left(\frac{P}{D}\right)_t^{cI}(S_t)}$$

The covariance is determined by the risk aversion parameter γ (via the adjustment for risk) and the standard deviation of dividend growth σ_D which drives both the adjustment for risk and dividend growth $[1 + d(S_t, fe_{t+1})]$. All other parameters ($p_{Mean,Draw}$, σ_{Means}) only influence the fraction; moreover, since these parameters change both numerator and denominator, the overall effect is comparatively small.

To see why the covariance increases in both the risk aversion parameter γ and the standard deviation of dividend growth σ_D, note that the covariance in (5-31) can be expressed as

$$cov\left(\begin{matrix} AfR_{t,t+1}{}^{cI}(fe_{t+1}, S_t), \\ [1 + d(S_t, fe_{t+1})] \end{matrix} \middle| S_t\right)$$
$$= stdev\left(AfR_{t,t+1}{}^{cI}(fe_{t+1}, S_t) \middle| S_t\right) \cdot corr\left(\begin{matrix} AfR_{t,t+1}{}^{cI}(fe_{t+1}, S_t), \\ [1 + d(S_t, fe_{t+1})] \end{matrix} \middle| S_t\right)$$
$$\cdot\, stdev([1 + d(S_t, fe_{t+1})]|S_t)$$

where

$$stdev([1 + d(S_t, fe_{t+1})]|S_t) = \sigma_D$$

The standard deviation of the adjustment for risk is well approximated (analogue to Appendix A4.2.3) by the product of the risk aversion parameter γ and the standard deviation of dividend growth:

$$stdev\left(AfR_{t,t+1}{}^{cI}(fe_{t+1}, S_t) \middle| S_t\right) \approx \gamma \cdot \sigma_D$$

The risk premium, thus, approximately reads

$$RP_t^{cI,ret}(S_t) \approx \gamma \cdot \sigma_D{}^2 \cdot \left\{-corr\left(\begin{matrix} AfR_{t,t+1}{}^{cI}(fe_{t+1}, S_t), \\ [1 + d(S_t, fe_{t+1})] \end{matrix} \middle| S_t\right)\right\} \cdot \frac{\left\{\sum_{s'=1}^{K} p_{S_t s'} \cdot \left(\frac{P}{D}\right)_{t+1}^{cI}(s') + 1\right\}}{\left(\frac{P}{D}\right)_t^{cI}(S_t)}$$

For that reason, an increase in γ and in σ_D increases the risk premium. Note that $-corr\left(\begin{matrix} AfR_{t,t+1}{}^{cI}(fe_{t+1}, S_t), \\ [1 + d(S_t, fe_{t+1})] \end{matrix} \middle| S_t\right)$ is positive.

5.5.2 Extended Veronesi Model: Single Risky Asset Model with Incomplete Information about Expectation and Standard Deviation of Dividend Growth

5.5.2.1 Description of Results and Answers to Questions

- Answer to Question 1 (relevance of incomplete information to risk premia compared to complete information):

Incomplete information risk premia differ significantly from complete information risk premia for this extension of Veronesi's model. Incomplete information risk premia are not necessarily lower than complete information risk premia (see Table 5-4 and Table 5-5). However, incomplete information risk premia can be negative if the risk aversion parameter γ exceeds one whereas complete information risk premia are positive.

- Answer to Question 2 (relevance of extensions to Veronesi (2000))

This extension of Veronesi's model leads to incomplete information risk premia that can be greater or less than complete information risk premia (see Figure 5-5). Since incomplete information risk premia are always less than complete information risk premia in the discretized Veronesi model, this extension of Veronesi (2000) is non-negligible. However, the force underlying Veronesi's result, the positive correlation of incomplete information price dividend ratio with the adjustment for risk, remains the same.

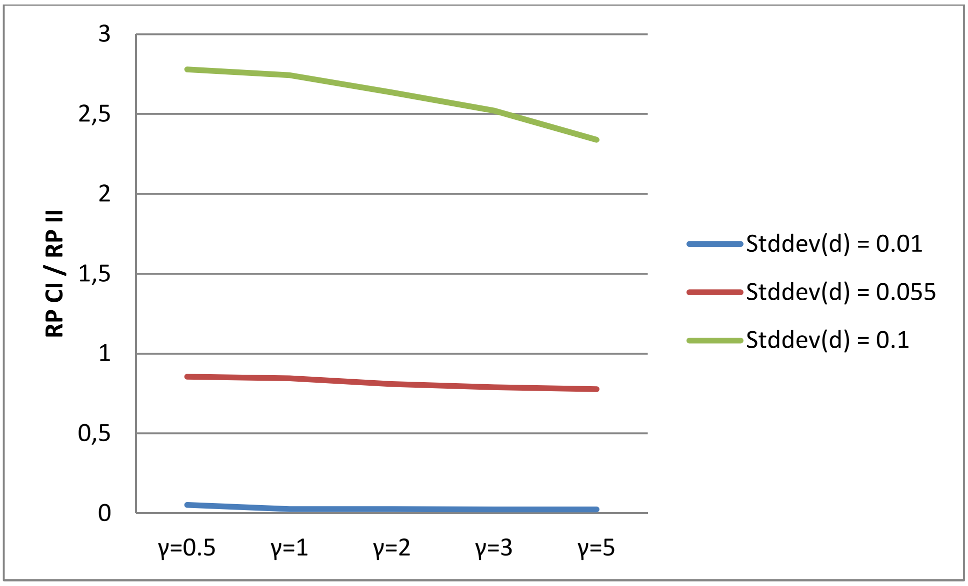

Figure 5-9: Complete information risk premia divided by incomplete information risk premium in various standard deviation regimes and for various values of the risk aversion parameter γ (expected dividend growth regime = 0.01), where $p_{Draw} = 0.25$, conditional transition probabilities σ_{low}, and $\rho = 0.05$.
Note that complete information price dividend ratios are functions of a finite set of regimes but have been connected by a polygonal path for readability.

- Answer to Question 3 (relevance of model parameters)

Based on my numerical results (see Table 5-4), the parameters γ (risk aversion), $p_{Mean,Draw}$ (probability that a drawing of regimes occurs), and σ_{Means} (dispersion of conditional transition probabilities/initial probabilities). Finally, the dispersion of dividend growth rates has an important effect. This parameter is defined as the maximum difference in standard deviations of dividend growth across regimes (e.g., $10\% - 1\% \ = \ 9\%$ in the model with standard deviation regimes 1%, 5.5%, and 10%) and is, for brevity, denoted by $\Delta{\sigma_D}^{max}$; it replaces the parameter σ_D (standard deviation of dividend growth conditional on the regime) from the discretized Veronesi model. It turns out that this parameter has an effect on risk premia similar to the effect exerted on risk premia by the parameter σ_D in the discretized Veronesi model. In particular, negative risk premia (in the case $\gamma > 1$) are observed less often as the difference of standard deviations of dividend growth increases. Of lesser importance is the time preference parameter ρ.

Incomplete information risk premia				
$p_{Draw} = 0$				
	σ_{low}	σ_{middle}	σ_{high}	uniform
$\rho = 0.025$				
$\gamma = 0.5$	DIV	DIV	DIV	DIV
$\gamma = 1$	0.37%	0.39%	0.47%	0.52%
$\gamma = 2$	0.67%	0.48%	-1.26%	-2.63%
$\gamma = 3$	DIV	DIV	DIV	DIV
$\gamma = 5$	DIV	DIV	DIV	DIV
	σ_{low}	σ_{middle}	σ_{high}	uniform
$\rho = 0.05$				
$\gamma = 0.5$	0.22%	0.31%	1.43%	2.01%
$\gamma = 1$	0.38%	0.40%	0.48%	0.54%
$\gamma = 2$	0.71%	0.60%	-0.45%	-1.32%
$\gamma = 3$	1.03%	0.71%	-2.46%	-4.91%
$\gamma = 5$	DIV	DIV	DIV	DIV
$p_{Draw} = 0.25$				
	σ_{low}	σ_{middle}	σ_{high}	uniform
$\rho = 0.025$				
$\gamma = 0.5$	0.18%	0.20%	0.27%	0.33%
$\gamma = 1$	0.37%	0.39%	0.47%	0.52%
$\gamma = 2$	0.76%	0.77%	0.69%	0.61%
$\gamma = 3$	1.16%	1.13%	0.70%	0.33%
$\gamma = 5$	1.90%	1.71%	-0.05%	-1.52%
$\rho = 0.05$				
$\gamma = 0.5$	0.19%	0.20%	0.27%	0.33%
$\gamma = 1$	0.38%	0.40%	0.48%	0.54%
$\gamma = 2$	0.78%	0.80%	0.73%	0.66%
$\gamma = 3$	1.19%	1.17%	0.78%	0.44%
$\gamma = 5$	1.97%	1.79%	0.15%	-1.23%

Table 5-4: Incomplete information risk premia for the extended Veronesi model (standard deviation regimes ranging from 1 percent to 10 percent). 'DIV' refers to a parameter constellation where the complete information price dividend ratio diverges to infinity in at least one regime. Negative risk premia are highlighted in red print.

Complete information risk premia increase substantially in standard deviation regimes; expectation regimes have a much lesser influence. In addition, complete information risk premia increase in the risk aversion parameter γ:

Complete information risk premia in minimum and maximum expected dividend growth regimes, each expectation regime combined with all three standard deviation regimes						
$p_{Draw} = 0$ (the dispersion of conditional transition probabilities is irrelevant for $p_{Draw} = 0$)						
$E(1 + d\|S)$	0			0.1		
$Stddev(1 + d\|S)$	0.01	0.055	0.1	0.01	0.055	0.1
$\rho = 0.025$						
$\gamma = 0.5$	DIV	DIV	DIV	DIV	DIV	DIV
$\gamma = 1$	0.01%	0.31%	1.01%	0.01%	0.28%	0.92%
$\gamma = 2$	0.02%	0.62%	2.00%	0.02%	0.62%	2.01%
$\gamma = 3$	DIV	DIV	DIV	DIV	DIV	DIV
$\gamma = 5$	DIV	DIV	DIV	DIV	DIV	DIV
$\rho = 0.05$						
$\gamma = 0.5$	0.01%	0.16%	0.52%	0.00%	0.14%	0.45%
$\gamma = 1$	0.01%	0.32%	1.04%	0.01%	0.29%	0.95%
$\gamma = 2$	0.02%	0.63%	2.05%	0.02%	0.63%	2.06%
$\gamma = 3$	0.03%	0.94%	3.00%	0.03%	1.04%	3.33%
$\gamma = 5$	DIV	DIV	DIV	DIV	DIV	DIV
$p_{Draw} = 0.25$ (dispersion of conditional transition probabilities = σ_{low})						
$E(1 + d\|S)$	0			0.1		
$Stddev(1 + d\|S)$	0.01	0.055	0.1	0.01	0.055	0.1
$\rho = 0.025$						
$\gamma = 0.5$	0.01%	0.15%	0.51%	0.00%	0.13%	0.44%
$\gamma = 1$	0.01%	0.31%	1.01%	0.01%	0.28%	0.92%
$\gamma = 2$	0.02%	0.62%	2.00%	0.02%	0.62%	2.01%
$\gamma = 3$	0.03%	0.92%	2.93%	0.03%	1.01%	3.25%
$\gamma = 5$	0.05%	1.49%	4.50%	0.07%	2.00%	6.13%
$\rho = 0.05$						
$\gamma = 0.5$	0.01%	0.16%	0.52%	0.00%	0.14%	0.45%
$\gamma = 1$	0.01%	0.32%	1.04%	0.01%	0.29%	0.95%
$\gamma = 2$	0.02%	0.63%	2.05%	0.02%	0.63%	2.06%
$\gamma = 3$	0.03%	0.94%	3.00%	0.03%	1.04%	3.33%
$\gamma = 5$	0.05%	1.53%	4.61%	0.07%	2.05%	6.28%

Table 5-5: Complete information risk premia for the extended Veronesi model (standard deviation regimes ranging from 1 percent to 10 percent) for the minimum and maximum expected dividend growth regimes, each in combination with all three standard deviation regimes. 'DIV' refers to a parameter constellation where the complete information price dividend ratio diverges to infinity in at least one regime.

5.5.2.2 Interpretation of Results

5.5.2.2.1 Explanation of the Answer to Question 1

Why are incomplete information risk premia are not necessarily less than complete information risk premia?

This result is due to the standard deviation of dividend growth that is now subject to regimes. For example, if the lowest standard deviation regime is 1 percent and the highest standard deviation regime is 10 percent, and if it is further assumed that the probability of the 10 percent standard deviation regime is, say, 99 percent under incomplete information, then the incomplete information risk premium should still be higher than the complete information risk premium in the complete information regime with 1 percent standard deviation.

Why can incomplete information risk premia be negative?

The channel of stochastic regime probabilities is responsible for this result: if the risk aversion parameter γ exceeds one, the correlation of the adjustment for risk with asset returns can become positive, and the risk premium will be negative.

To see how this channel works, first note that the general structure of the risk premium is the same as in the discretized Veronesi model:

5-32

$$RP_t^{iI,ret}(\pi_t) = -cov\left(\begin{array}{c} AfR_{t,t+1}{}^{iI}(fe_{t+1}, S_t, \pi_t), \\ [1 + d(S_t, fe_{t+1})] \cdot \dfrac{\left\{\sum_{s'=1}^{K} \pi_{s',t+1} \cdot \left(\frac{P}{D}\right)_{t+1}^{cI}(s') + 1\right\}}{\left(\frac{P}{D}\right)_t^{iI}(\pi_t)} \end{array} \middle| \pi_t \right)$$

with

$$AfR_{t,t+1}{}^{iI}(fe_{t+1}, S_t, \pi_t) = \frac{\left(1 + d(S_t, fe_{t+1})\right)^{-\gamma}}{\sum_{s=1}^{K} \pi_{s,t} \cdot E\left(\left(1 + d(S_t, fe_{t+1})\right)^{-\gamma} \middle| S_t = s\right)}$$

$$\pi_{t+1} = \Pi(\pi_t, d_{t+1})$$

However, incomplete information price dividend ratio growth

$$\frac{\left\{\sum_{s'=1}^{K} \pi_{s',t+1} \cdot \left(\frac{P}{D}\right)_{t+1}^{cI}(s')\right\}}{\left(\frac{P}{D}\right)_t^{iI}(\pi_t)}$$

– the crucial part in the discretized Veronesi model - differs.

We know, however, from the discretized Veronesi model that the reaction of incomplete information price dividend ratio growth to an increase in dividend growth $1 + d$ is exactly the part that is responsible for the potentially negative sign of the incomplete information risk premium. For that reason, the reaction of incomplete information price dividend ratio growth to an increase in dividend growth $1 + d$ should be analyzed more thoroughly.

Recall this term consists of (i) complete information price dividend ratios $\left(\frac{P}{D}\right)_{t+1}^{cI}(s), s = 1, \ldots, K$ and (ii) stochastic regime probabilities π_{t+1}. Both components interact in a way that depends on the risk aversion parameter γ.

Complete information price dividend ratio $\left(\frac{P}{D}\right)_{t+1}^{cI}(s)$

The complete information price dividend ratio depends on both expectation regimes and standard deviation regimes of dividend growth. The form of this relation depends on the risk aversion parameter γ, is rather complex, and, hence, is best understood by Figure 5-10:

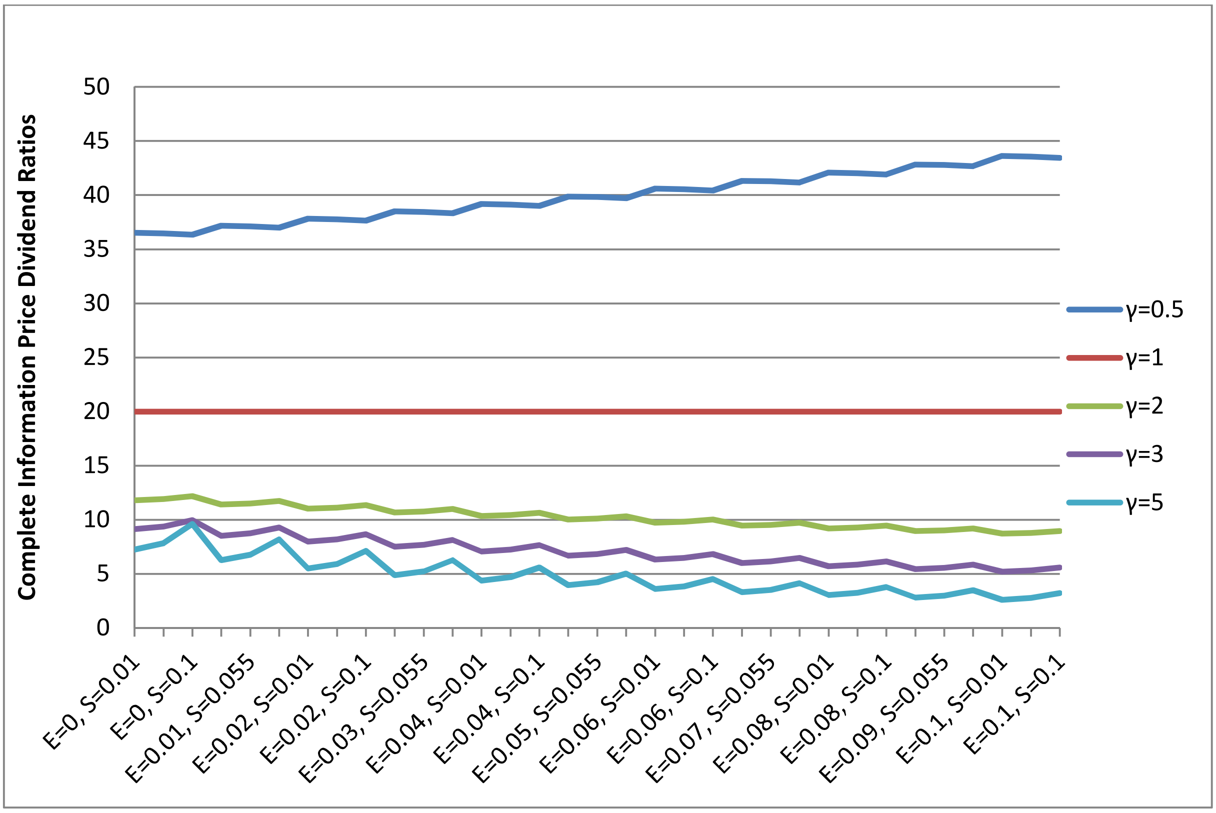

Figure 5-10: Complete information price dividend ratios as function of expectation (E) and standard deviation regimes (S) and for various risk aversion parameters γ where $p_{Draw} = 0.25$, $\sigma_{low} = 0.005$, and $\rho = 0.05$
Note that complete information price dividend ratios are functions of a finite set of regimes but have been connected by a polygonal path for readability.

Figure 5-10 illustrates:

(i) If $\gamma < 1$, the complete information price dividend ratio increases in the expectation regime and decreases in the standard deviation regime of dividend growth.

(ii) If $\gamma = 1$, the complete information price dividend ratio is independent of expectation regimes and standard deviation regimes of dividend growth.

(iii) If $\gamma > 1$, the complete information price dividend ratio decreases in the expectation regime and increases in the standard deviation regime of dividend growth.

(iv) The effect of the expectation regime is stronger than the effect of the standard deviation regime, leading to an overall positive effect for $\gamma < 1$ and an overall negative effect for $\gamma > 1$.

Stochastic regime probabilities π_{t+1}

The computation of regime probabilities π_{t+1} is more complicated than in the discretized Veronesi model where only expectations are unobservable; for that reason, partially a graphical illustration is used: high dividend growth can either be due to high expected dividend growth or a high standard deviation of dividend growth while high dividend growth always suggests a high expected dividend growth regime in the discretized Veronesi model. Consider the probabilities of expected dividend growth regimes (Figure 5-11): as in the discretized Veronesi model, the probability of a particular expected dividend growth regime reaches a local maximum value at dividend growth rates that equal this expectation. However, if dividend growth rates are sufficiently above or below the expectation in question, the probability rises again as it becomes clear that the standard deviation regime is one with a high conditional standard deviation (see Figure 5-12). This effect does not exist in the discretized Veronesi model.

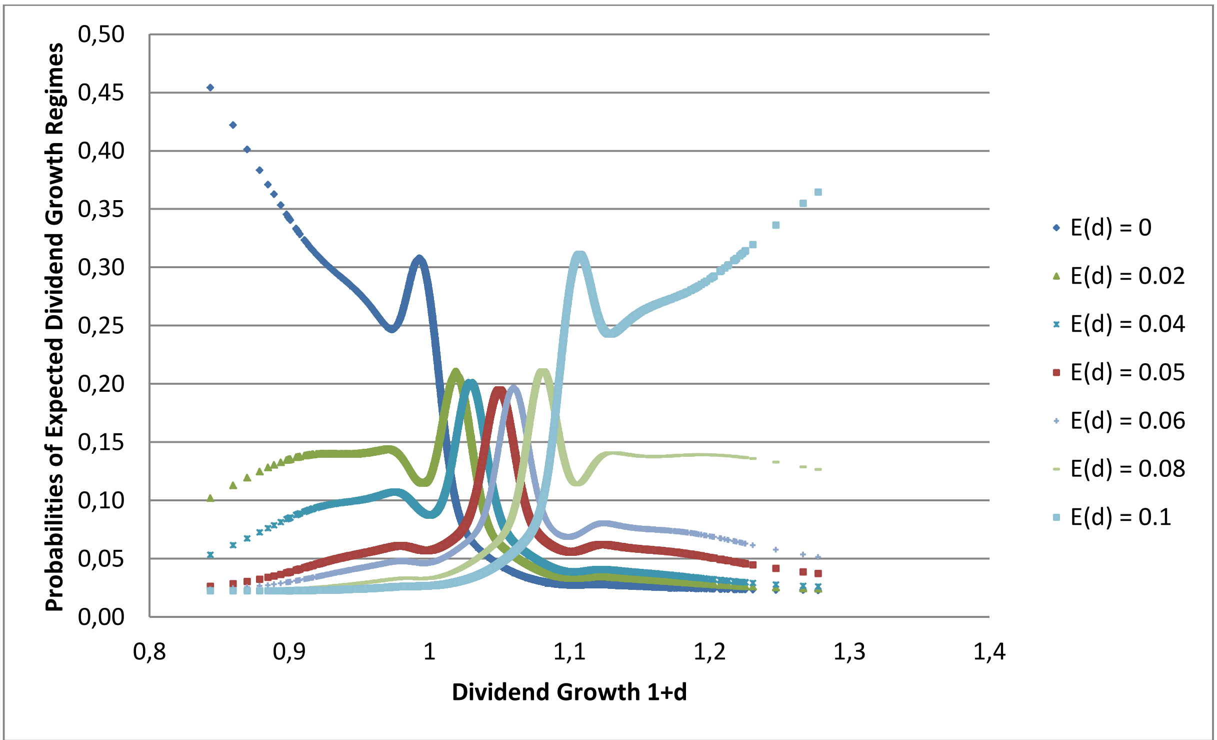

Figure 5-11: Probabilities of various expected dividend growth regimes (E(d)) as functions of dividend growth ($p_{Draw} = 0.25$, uniform conditional transition probabilities): simulation with 10,000 dividend growth realizations.

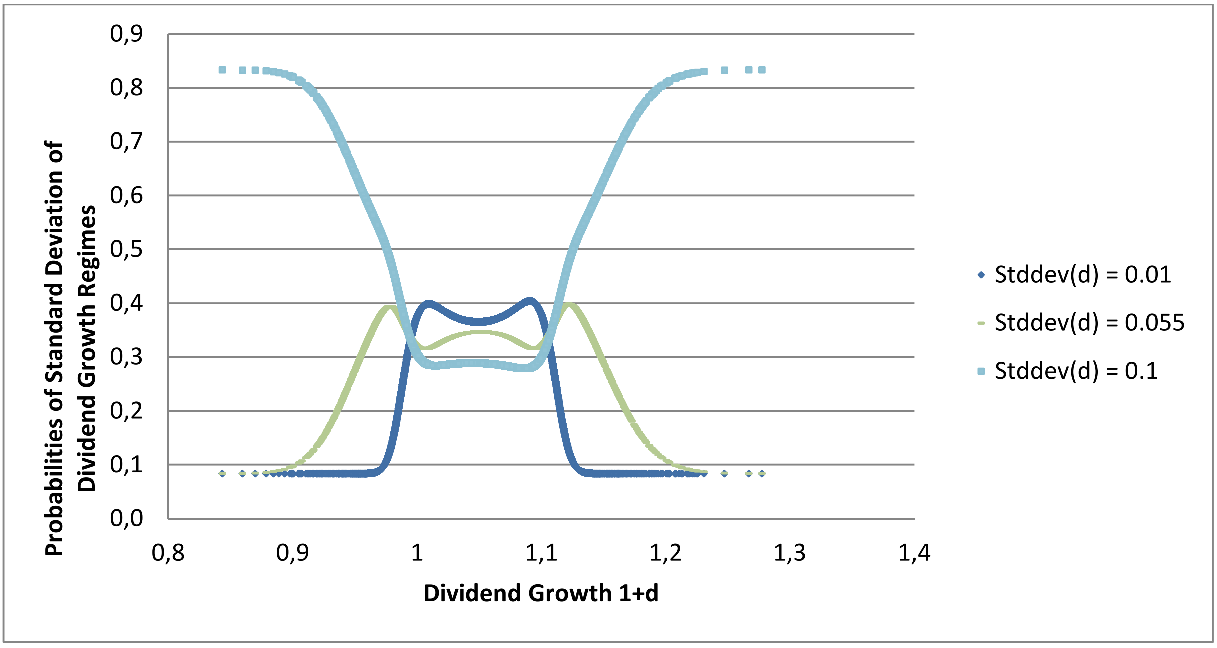

Figure 5-12: Probabilities of various standard deviation (stddev(d)) of dividend growth regimes as functions of dividend growth ($p_{Draw} = 0.25$, uniform conditional transition probabilities): simulation with 10,000 dividend growth realizations.

Reaction of incomplete information price dividend ratio growth to an increase in dividend growth $1 + d$ depending on γ

Figure 5-13and Figure 5-14 plot simulated dividend growth rates against price dividend ratio growth rates for the cases $\gamma > 1$ (Figure 5-13) and $\gamma < 1$ (Figure 5-14). The cases $\gamma > 1$ and $\gamma < 1$ differ in that price dividend ratio growth reacts negatively to dividend growth for $\gamma > 1$ and positively for $\gamma < 1$. However, in both cases the relation consists of three different regions: a relatively steep middle region is situated between two flatter outer regions. If one looks at the probabilities of the standard deviation regimes (Figure 5-14), it becomes clear that the transition from the steep middle regions to the flat outer regions in Figure 5-13 and Figure 5-14 coincide with the sharp rise of the probability of the highest standard deviation regime. In the steep middle region, it is mostly the probabilities of expectation regimes that react strongly to dividend growth (see Figure 5-13) and the probabilities of the three standard deviation regimes are comparatively stable (again see Figure 5-14). Thus there is a middle region that mostly behaves in a way similar to the discretized Veronesi model, and there are two outer regions that are characterized by a sharp increase in the probability of the highest standard deviation regime.

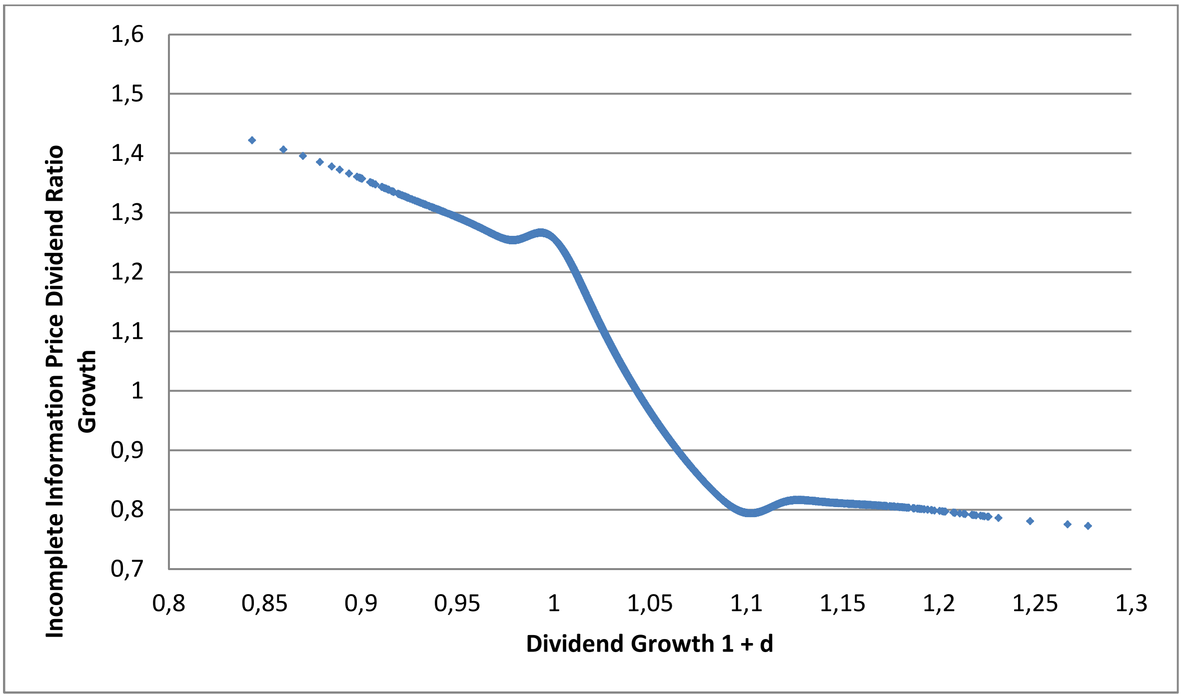

Figure 5-13: Case $\gamma > 1$: incomplete information price dividend ratio growth as a function of dividend growth ($p_{Draw} = 0.25$, uniform conditional transition probabilities, $\rho = 0.05$, $\gamma = 5$): simulation with 10,000 dividend growth realizations.

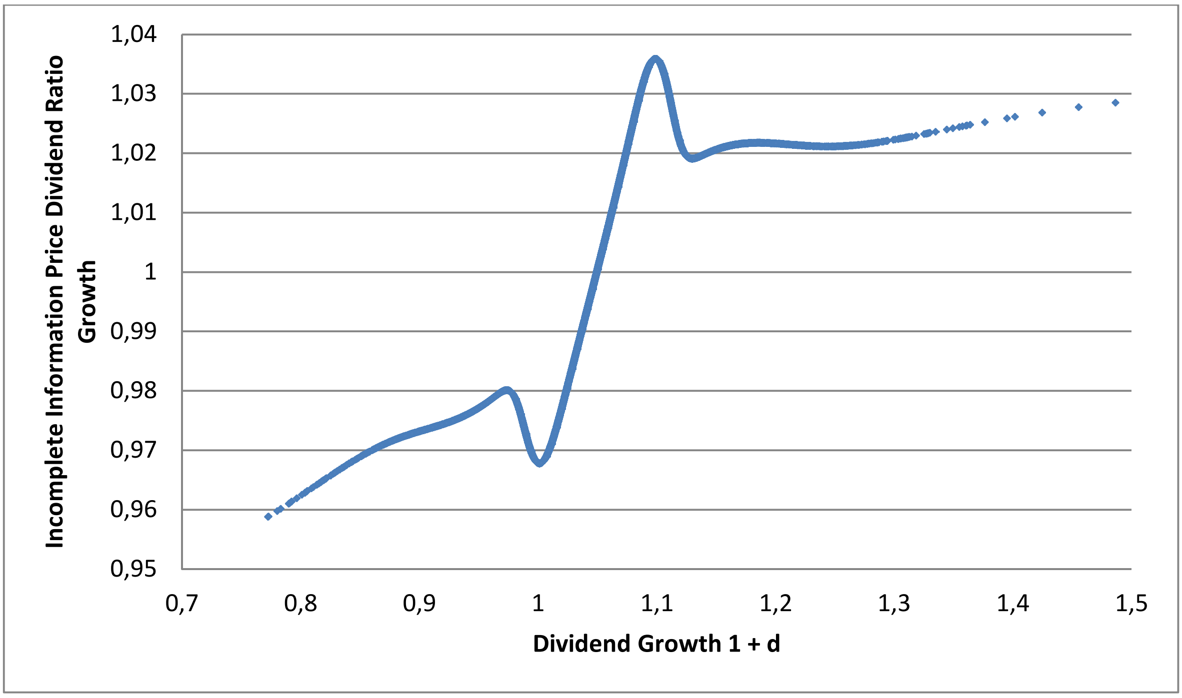

Figure 5-14: Case $\gamma < 1$: incomplete information price dividend ratio growth as a function of dividend growth ($p_{Draw} = 0.25$, uniform conditional transition probabilities, $\rho = 0.05$, $\gamma = 0.5$): simulation with 10,000 dividend growth realizations.

Connection between adjustment for risk and return on the risky asset $[1 + d(S_t, fe_{t+1})] \cdot \frac{\left\{\sum_{s'=1}^{K} \pi_{s',t+1} \cdot \left(\frac{P}{D}\right)_{t+1}^{cI}(s')+1\right\}}{\left(\frac{P}{D}\right)_{t}^{iI}(\pi_t)}$ and sign of the risk premium

It is now possible to use the insights on the reaction of price dividend ratio growth to dividend growth to discuss the sign of the risk premium. Recall that the return on the risky asset consists of dividend growth and price dividend ratio growth. The various effects of an increase in dividend growth are parallel to the discretized Veronesi model and can be summarized by the following table:

	Reaction to an increase in dividend growth $1+d$:			Sign of the risk premium:
	Adjustment for risk $(1+d)^{-\gamma}$	Dividend yield $\frac{(1+d)}{\left(\frac{P}{D}\right)_t^{iI}(\pi_t)}$	Incomplete information the Price Dividend Ratio growth $\frac{\left\{\sum_{s'=1}^{K}\pi_{s',t+1}\cdot\left(\frac{P}{D}\right)_{t+1}^{cI}(s')\right\}}{\left(\frac{P}{D}\right)_t^{iI}(\pi_t)}$	$RP_t^{iI,ret}(\pi_t)$
$0<\gamma<1$	-	+	+	> 0
$\gamma=1$	-	+	0	> 0
$\gamma>1$	-	+	-	>=< 0

Table 5-6: Responses of risk premium components to positive dividend growth and implication for the sign of the incomplete information risk premium.

The details of the interplay between the components of Table 5-6 can best be analyzed graphically: Figure 5-15 illustrates this relationship for the case $\gamma > 1$ where a clear positive relationship between the adjustment for risk and the return on the risky asset is evident in the middle region. In the outer region, this relation reverses from positive to negative: dividend growth over-compensates price dividend ratio growth in these regions where dividend growth is so extreme that it most likely comes from the high standard deviation regime. Note that the risk premium is still negative in the case depicted in Figure 5-15 despite the negative relation between adjustment for risk and asset returns in the outer regions: the covariance is a special form of an expectation, and as such is weighted by a density function, implying that the middle region has a much higher impact on the risk premium than the outer regions. Figure 5-16 depicts an alternative parameter constellation with $\gamma > 1$ where the risk premium is positive: the reaction of price dividend ratio growth in response to dividend growth is very weak in this instance and dividend growth dominates, resulting in a positive risk premium.

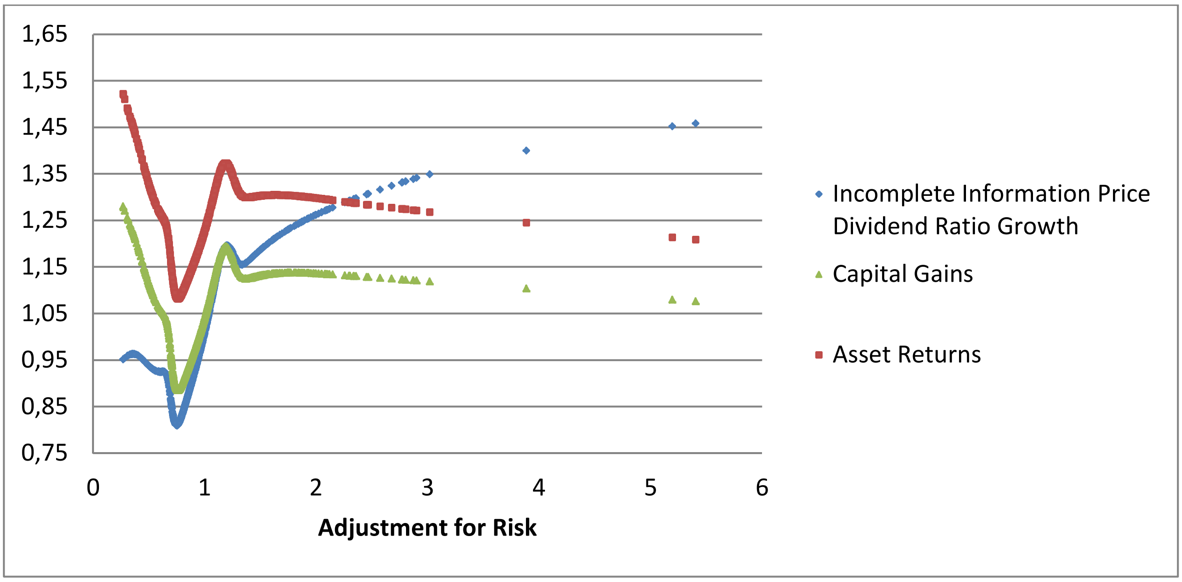

Figure 5-15: Case $\gamma > 1$, parameter constellation with a negative risk premium: incomplete information price dividend ratio growth, assets returns, and capital gains ass functions of adjustment or risk ($p_{Draw} = 0.25$, uniform conditional transition probabilities, $\rho = 0.05$, $\gamma = 5$): simulation with 10,000 adjustment for risk realizations.

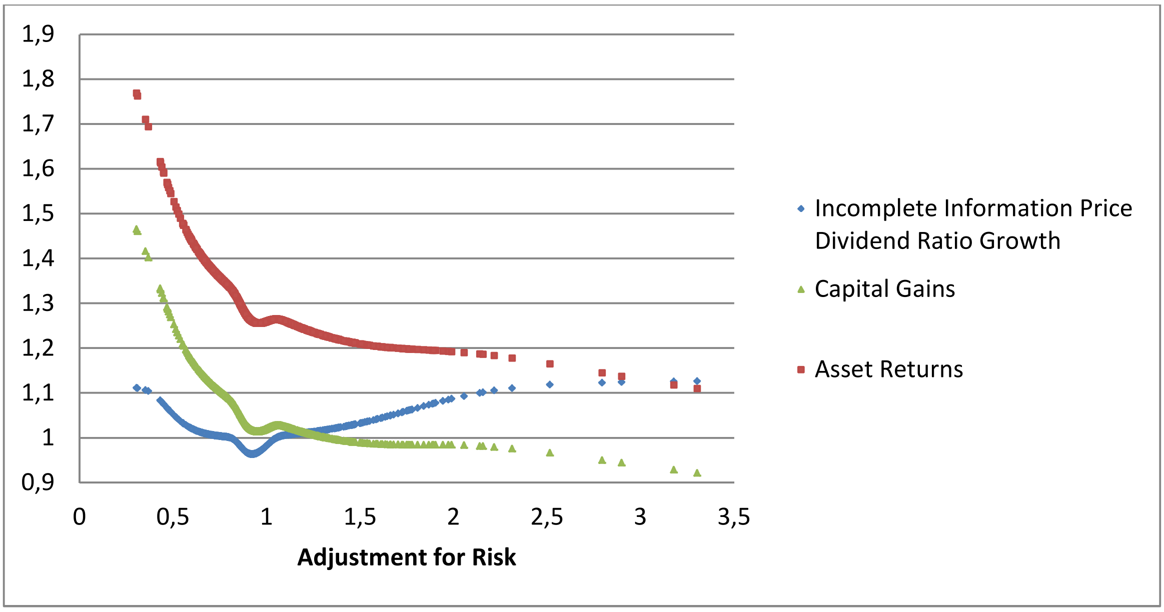

Figure 5-16: Case $\gamma < 1$, parameter constellation with a positive risk premium: incomplete information price dividend ratio growth, assets returns, and capital gains as functions of adjustment or risk ($p_{Draw} = 0.25$, σ_{middle}, $\rho = 0.05$, $\gamma = 5$): simulation with 10,000 adjustment for risk realizations.

5.5.2.2.2 Explanation of the Answer to Question 2

Why can incomplete information risk premia be greater or less than complete information risk premia?

See Explanation of the Answer to Question 1.

What mechanism is responsible for that fact that incomplete information risk premia are less than complete information risk premia still exists in the extended Veronesi model?

The channel of stochastic regime probabilities still exists in the extended Veronesi model, see Explanation of the Answer to Question 1.

5.5.2.2.3 Explanation of the Answer to Question 3

The economic mechanisms behind the effects of the risk aversion parameter γ, the drawing probability p_{Draw},, and the dispersion of the conditional transition probabilities are very similar to the discretized Veronesi model and, hence, are omitted.

The difference between the highest and lowest standard deviations of dividend growth (${\Delta\sigma_D}^{max}$) is new and should be explained. To that end, the risk premium as a covariance is broken up into the following three components: the covariance is the product of (i) the standard deviation of the adjustment for risk, (ii) the standard deviation of asset returns and (iii) the correlation of the adjustment for risk with asset returns. Any parameter that strongly affects at least one of these components of risk premia can, therefore, have a strong effect on risk premia.

5.5.2.2.3.1 Incomplete Information

Effect of ${\Delta\sigma_D}^{max}$ on the standard deviation of the adjustment for risk

If ${\Delta\sigma_D}^{max}$ is high, dividends conditional on incomplete information will exhibit high standard deviation which, in turn, leads to a high standard deviation of the adjustment for risk (first element of the risk premium).

Effect of ${\Delta\sigma_D}^{max}$ on the standard deviation of asset returns

${\Delta\sigma_D}^{max}$ has various effects on the standard deviation of asset returns (second element of the risk premium): recall that returns possess the structure

$$[1 + d(S_t, fe_{t+1})] \cdot \frac{\left\{\sum_{s'=1}^{K} \pi_{s',t+1} \cdot \left(\frac{P}{D}\right)_{t+1}^{cI} (s') + 1\right\}}{\left(\frac{P}{D}\right)_{t}^{iI} (\pi_t)}$$

If ${\Delta\sigma_D}^{max}$ is high, dividend growth $[1 + d(S_t, fe_{t+1})]$ will exhibit substantial fluctuation.

Incomplete information price dividend ratio growth is influenced by ${\Delta\sigma_D}^{max}$ as follows: first, the higher ${\Delta\sigma_D}^{max}$, the less regime probabilities $\pi_{s',t+1}$ react to dividend growth (see p. 210), thereby decreasing the fluctuation of incomplete information price dividend ratio growth. Second, via the ratio of maximum to minimum complete information price dividend ratios across regimes,

$$\frac{max_{s\epsilon\{1,\ldots,K\}} \left(\frac{P}{D}\right)_{t+1}^{cI} (s)}{min_{s\epsilon\{1,\ldots,K\}} \left(\frac{P}{D}\right)_{t+1}^{cI} (s)}$$

The spikes that standard deviation regimes introduce into complete information price dividend ratios (see Figure 5-10) become more pronounced as ${\Delta\sigma_D}^{max}$ increases; this second effect tends to increase the fluctuation of incomplete information price dividend ratio growth. Concerning the total effect of ${\Delta\sigma_D}^{max}$ on standard deviation of incomplete information price dividend ratio growth, Table 5-7 suggest a U-shaped influence:

Standard deviation of incomplete information price dividend ratio growth				
	Dispersion of conditional transition probabilities			
${\Delta\sigma_D}^{max}$	σ_{low}	σ_{middle}	σ_{high}	uniform
2%-1%	1.40%	4.17%	15.26%	20.92%
5%-1%	0.97%	2.63%	11.12%	16.83%
7%-1%	0.62%	1.95%	9.48%	15.07%
10%-1%	2.95%	3.87%	9.51%	14.17%

Table 5-7: Standard deviation of incomplete information price dividend ratio growth as a function of the difference between the highest and lowest standard deviations of dividend growth across regimes (${\Delta\sigma_D}^{max}$) and for various levels of dispersion of the conditional transition probabilities ($\rho = 0.05, \gamma = 5, p_{Draw} = 0.25$).

The joint effect of ${\Delta\sigma_D}^{max}$ on the standard deviation of asset returns depends on the risk aversion parameter γ in a way that is analogous to the discretized Veronesi model. For $\gamma \leq 1$ an increasing ${\Delta\sigma_D}^{max}$ typically leads to an increase in standard deviation of asset return. For $\gamma > 1$ there is a U-shaped relation between ${\Delta\sigma_D}^{max}$ and the standard deviation of asset returns (illustrated in Table 5-8Table 5-8): for parameter constellations where incomplete information price dividend ratio growth outweighs the effect of dividend growth, an increase in ${\Delta\sigma_D}^{max}$ reduces the standard deviation of asset returns (e.g., see the case of uniform conditional transition probabilities in Table 5-8)). From a certain level of ${\Delta\sigma_D}^{max}$ on, the effect of ${\Delta\sigma_D}^{max}$ on dividend growth overcompensates the effect on incomplete information price dividend ratio growth leading to an increase in the standard deviation of asset returns (see the case σ_{high} in Table 5-8).

Standard deviation of incomplete information asset returns				
	Dispersion of conditional transition probabilities			
${\Delta\sigma_D}^{max}$	σ_{low}	σ_{middle}	σ_{high}	uniform
2%-1%	0.70%	2.00%	12.20%	17.58%
5%-1%	3.51%	2.45%	7.00%	12.39%
7%-1%	4.56%	3.44%	4.33%	10.14%
10%-1%	7.79%	7.79%	7.01%	9.76%

Table 5-8: Standard deviation of incomplete information asset returns as a function of the difference between the highest and lowest standard deviations of dividend growth across regimes (${\Delta\sigma_D}^{max}$) and for various levels of dispersion of the conditional transition probabilities ($\rho = 0.05, \gamma = 5, p_{Draw} = 0.25$).

<u>Effect of ${\Delta\sigma_D}^{max}$ on the correlation of asset returns with the adjustment for risk</u>

The correlation of asset returns with the adjustment for risk

$$corr\left(\begin{array}{c} {AfR_{t,t+1}}^{iI}(fe_{t+1}, S_t, \pi_t), \\ \left[\underbrace{1 + d(S_t, fe_{t+1})}_{term\ i}\right] \cdot \underbrace{\frac{\left\{\sum_{s'=1}^{K} \pi_{s',t+1} \cdot \left(\frac{P}{D}\right)_{t+1}^{cI}(s') + 1\right\}}{\left(\frac{P}{D}\right)_t^{iI}(\pi_t)}}_{term\ ii} \end{array}\middle| \pi_t\right)$$

depends on whether γ is greater or less than 1. If $\gamma \leq 1$, the correlation is always negative. If $\gamma > 1$, the correlation depends on the interplay between dividend growth (term i, tends to reduce correlation) and incomplete information price dividend ratio growth (term ii, tends to increase correlation). The term that fluctuates more coins the correlation. An increase in $\Delta\sigma_D{}^{max}$ always leads to an increase in the fluctuation of term i and has a mixed effect on term ii. Typically, this leads to a reduced (less positive or more negative) correlation of asset returns with the adjustment for risk.

Effect of $\Delta\sigma_D{}^{max}$ on the risk premium

Combining the three statements derived on the compenents of the risk premium as a covariance, an increase in $\Delta\sigma_D{}^{max}$ leads to an increase in the incomplete information risk premium (more positive or less negative) as Table 5-9 shows:

Incomplete information risk premia					
		Dispersion of conditional transition probabilities			
$\gamma = 0.5$	$\Delta\sigma_D{}^{max}$	σ_{low}	σ_{middle}	σ_{high}	uniform
	2%-1%	0.02%	0.02%	0.08%	0.13%
	5%-1%	0.05%	0.07%	0.12%	0.17%
	7%-1%	0.08%	0.09%	0.15%	0.22%
	10%-1%	0.20%	0.20%	0.27%	0.33%
$\gamma = 5$	$\Delta\sigma_D{}^{max}$	σ_{low}	σ_{middle}	σ_{high}	uniform
	2%-1%	0.05%	-0.17%	-1.67%	-2.91%
	5%-1%	0.53%	0.32%	-1.20%	-2.46%
	7%-1%	0.88%	0.67%	-0.86%	-2.00%
	10%-1%	1.79%	1.79%	0.15%	-1.23%

Table 5-9: Incomplete information risk premia as a function of the difference between the highest and lowest standard deviations of dividend growth across regimes ($\Delta\sigma_D{}^{max}$) for $\gamma = 0.5$ and $\gamma = 5$ as well as for various levels of dispersion of the conditional transition probabilities ($\rho = 0.05$, $p_{Draw} = 0.25$).

5.5.2.2.3.2 Complete Information

The return-based complete information risk premium for the single asset case is formally identical to 5-31. For that reason,

$$\frac{\left\{\sum_{S'=1}^{K} p_{S_t S'} \cdot \left(\frac{P}{D}\right)_{t+1}^{cI} (s') + 1\right\}}{\left(\frac{P}{D}\right)_t^{cI} (S_t)}$$

can be factored out simplifying the risk premium to:

5-33

$$RP_t^{cI,ret}(S_t) = -cov\left(\begin{matrix} AfR_{t,t+1}{}^{cI}(fe_{t+1}, S_t), \\ [1 + d(S_t, fe_{t+1})] \end{matrix}\middle| S_t\right) \cdot \frac{\left\{\sum_{s'=1}^{K} p_{S_t s'} \cdot \left(\frac{P}{D}\right)_{t+1}^{cI}(s') + 1\right\}}{\left(\frac{P}{D}\right)_t^{cI}(S_t)}$$

The covariance is determined by the risk aversion parameter γ (via the adjustment for risk) and the regime-dependent standard deviation of dividend growth which drives both the adjustment for risk and dividend growth $[1 + d(S_t, fe_{t+1})]$, but expected dividend growth regimes do not play an important role. To see this, note that the covariance in (5-33) can be expressed as

$$\begin{aligned} &cov\left(\begin{matrix} AfR_{t,t+1}{}^{cI}(fe_{t+1}, S_t), \\ [1 + d(S_t, fe_{t+1})] \end{matrix}\middle| S_t\right) \\ &\qquad = stdev\left(AfR_{t,t+1}{}^{cI}(fe_{t+1}, S_t)\middle|S_t\right) \cdot corr\left(\begin{matrix} AfR_{t,t+1}{}^{cI}(fe_{t+1}, S_t), \\ [1 + d(S_t, fe_{t+1})] \end{matrix}\middle| S_t\right) \\ &\qquad \cdot stdev([1 + d(S_t, fe_{t+1})]|S_t) \end{aligned}$$

The standard deviation of the adjustment for risk is well approximated (analogous to Appendix A4.2.3) by the product of the risk aversion parameter γ and the standard deviation of dividend growth,

$$stdev\left(AfR_{t,t+1}{}^{cI}(fe_{t+1}, S_t)\middle|S_t\right) \approx \gamma \cdot stdev([1 + d(S_t, fe_{t+1})]|S_t)$$

The risk premium, thus, approximately reads

$$\begin{aligned} RP_t^{cI,ret}(S_t) &\approx \gamma \cdot \{stdev([1 + d(S_t, fe_{t+1})]|S_t)\}^2 \cdot \left\{-corr\left(\begin{matrix} AfR_{t,t+1}{}^{cI}(fe_{t+1}, S_t), \\ [1 + d(S_t, fe_{t+1})] \end{matrix}\middle| S_t\right)\right\} \\ &\cdot \frac{\left\{\sum_{s'=1}^{K} p_{S_t s'} \cdot \left(\frac{P}{D}\right)_{t+1}^{cI}(s') + 1\right\}}{\left(\frac{P}{D}\right)_t^{cI}(S_t)} \end{aligned}$$

For that reason, an increase in γ and in $stdev([1 + d(S_t, fe_{t+1})]|S_t)$ increases the risk premium. Note that $-corr\left(\begin{matrix} AfR_{t,t+1}{}^{cI}(fe_{t+1}, S_t), \\ [1 + d(S_t, fe_{t+1})] \end{matrix}\middle| S_t\right)$ is positive.

5.5.3 Regimes in Asset 2 Only

5.5.3.1 Description of Results and Answers to Questions

- Answer to Question 1 (relevance of incomplete information to risk premia compared to complete information)

Incomplete information risk premia differ significantly from complete information risk premia for this extension of Veronesi's model. Incomplete information risk premia are not necessarily lower than complete information risk premia (see, e.g., Figure 5-17 and Figure 5-18). However, incomplete information risk premia can be negative if the risk aversion parameter γ exceeds one whereas complete information risk premia are positive.

Since only the dividends of asset 2, but not those of asset 1, are subject to regimes, the relevance of incomplete information risk premia should be analyzed separately for both assets. In this connection note that dividends of asset 2 follow the same model as dividends in the models with a single risky asset (extended Veronesi model); it is instructive to consider asset 2 first:

Asset 2

First consider the case of incomplete information (see Figure 5-17).

If $\delta_{1,t} \to 0$ (all dividends are paid by asset 2), prices and risk premia of asset 2 correspond to a model with a single risky asset with regimes in expectations and standard deviations of dividend growth; hence all results that have been described for the case with a single risky asset must hold for this special constellation: (i) incomplete information risk premia are significantly different from complete information risk premia; (ii) incomplete information risk premia can be negative for $\gamma > 1$. Note that the correlation of dividend growth rates of assets 1 and 2 is irrelevant to risk premia of asset 2 for $\delta_{1,t} = 0$ because both the stochastic discount factor and dividends are completely provided by asset 2 and do not depend in any way on dividends of asset 1.

As the dividend contribution of asset 1, $\delta_{1,t}$, increases relative to asset 2, the risk premium of asset 2 increases for positively correlated dividend growth of both assets and the incomplete information risk premium will be positive for all $\delta_{1,t}$ that exceed a certain threshold $\delta_{1,t}^*$[48]. If dividend growth of both assets is negatively correlated, the incomplete information risk premium first increases in $\delta_{1,t}$ but then decreases from a certain value of $\delta_{1,t}$ on, with a negative risk premium for $\delta_{1,t} \to 1$[49].

Second consider the case of complete information (see Figure 5-17).

Complete information risk premia are positive for all analyzed parameter constellations; note that negative complete information risk premia are (in contrast to the single asset case) theoretically possible.

[48] $\delta_{1,t}^*$ can be zero because the incomplete information risk premium can be positive even if all dividends are provided by asset 2 (see the results for the single asset models).

[49] Of course, asset 2 would not have a risk premium if it did not pay any dividends (i.e., $\delta_{1,t} = 1$) because the price of such an asset has to be zero at the present and all future points of time. However, $\delta_{1,t} = 1$ should be interpreted as a limit case, i.e., the contribution of asset 2 is "very small" relative to the market to the point that it is negligible.

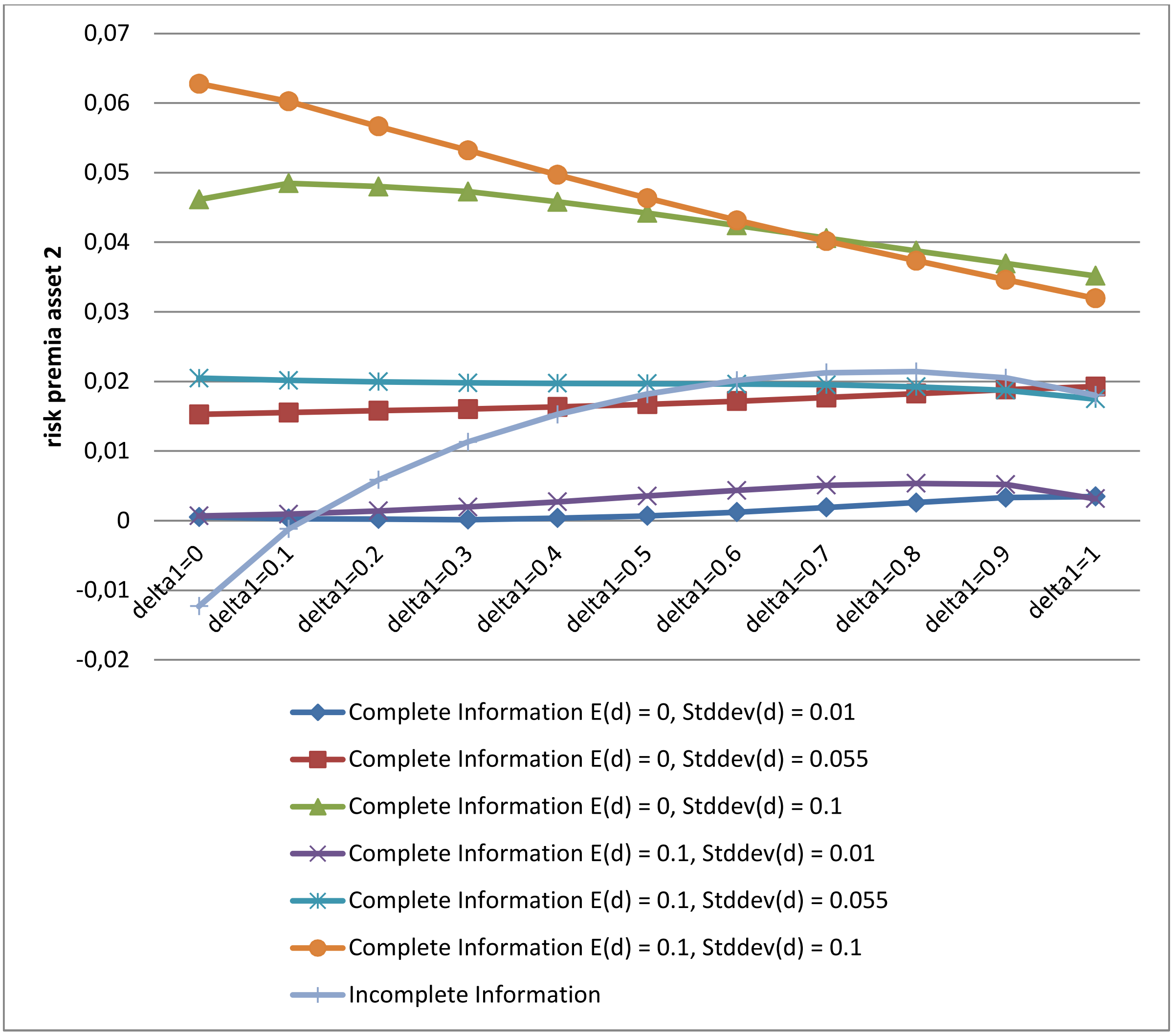

Figure 5-17: Risk premia of asset 2 for selected complete information regimes and for incomplete information for the case of positive correlation (Parameter values: $\gamma = 5$, $\rho = 0.05$ $p_{Draw} = 0.25$, correlation of dividend growth conditional on the regime = 80 percent).
Risk premia have been computed at equidistant intervals of 10 percent for $\delta_{1,t}$ and interpolated to the interval [0;1].

<u>Asset 1</u>

First consider the case of incomplete information (see Figure 5-18).

If $\delta_{1,t} \to 0$, incomplete information leads to a low and possible negative risk premium of asset 1. This effect is quite large for $\gamma > 1$.

As $\delta_{1,t}$ increases to 1, the incomplete information risk premium increases.

Second consider the case of complete information (see Figure 5-18).

Complete information risk premia are positive for all analyzed parameter constellations; note that negative complete information risk premia are (in contrast to the single asset case) theoretically possible. As $\delta_{1,t}$ increases to 1, the various complete information risk premia converge to the same value (which also is the incomplete information risk premium). Note that this behavior is not observed in asset 2.

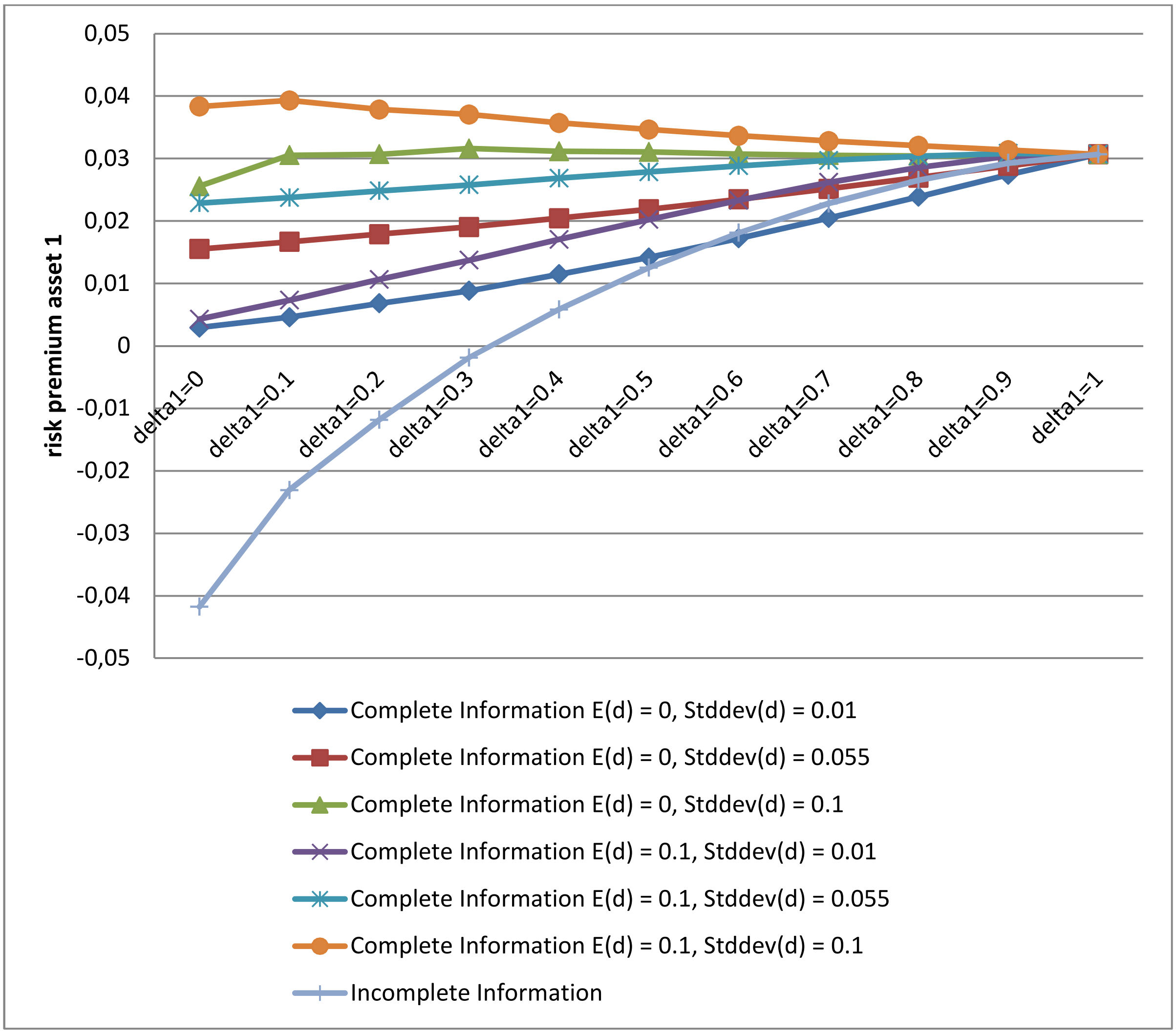

Figure 5-18: Risk premia of asset 1 for selected complete information regimes and for incomplete information for the case of positive correlation (Parameter values: $\gamma = 5$, $\rho = 0.05$ $p_{Draw} = 0.25$, correlation of dividend growth conditional on the regime = 80 percent).
Risk premia have been computed at equidistant intervals of 10 percent for $\delta_{1,t}$ and interpolated to the interval [0;1].

- Answer to Question 2 (relevance of extensions to Veronesi (2000))

Extending Veronesi's model to two assets leads to incomplete information risk premia that can be greater or less than complete information risk premia. Since incomplete information risk premia are always less than complete information risk premia in the discretized Veronesi model, this extension of Veronesi (2000) is non-negligible. In addition, the force underlying Veronesi's result, the correlation of incomplete information price dividend ratio with the adjustment for risk, reverses from positive to negative as $\delta_{1,t}$ increases from 0 to 1. In other words, incomplete information has the opposite effect as in Veronesi's model if $\delta_{1,t}$ is close to 1.

- Answer to Question 3 (relevance of model parameters)

Asset 2

If $\delta_{1,t} \to 0$ (asset 2 pays all dividends), the model essentially corresponds to the extended Veronesi model. From that perspective, the correlation of dividend growth rates of both assets conditional on the regime is the only parameter of interest. It turns out that this correlation neither affects the complete nor incomplete information risk premium on asset 2.

If $\delta_{1,t} \to 1$ (asset 1 pays all dividends), the regime parameters p_{Draw} and dispersion of conditional transition probabilities have a small effect on complete and incomplete information risk premia of asset 2. The effect of the risk aversion parameter γ depends on the correlation of dividend growth rates of both assets conditional on the regime. If this correlation is positive, both complete and incomplete information risk premia increase in γ; for negative correlation both complete and incomplete information risk premia decrease in γ. The standard deviation of dividend growth has conflicting effects on incomplete information risk premia; under complete information, the standard deviation of dividend growth has an effect that is similar to the risk aversion parameter γ.

For intermediate cases where both assets pay at least some dividends ($0 < \delta_{1,t} < 1$), the most interesting parameter is the correlation of dividend growth rates conditional on the regime. Figure 5-19 shows that the incomplete information risk premium of asset 2 increases in this correlation for $0 < \delta_{1,t} < 1$. The incomplete information risk premium of asset 2 first increases and then decreases in $\delta_{1,t}$. The relative dividend contribution at which the incomplete information risk premium takes a maximum value increases in the correlation of dividend growth rates conditional on the regime. The complete information risk premium of asset 2 increases in the correlation of dividend growth rates of both assets.

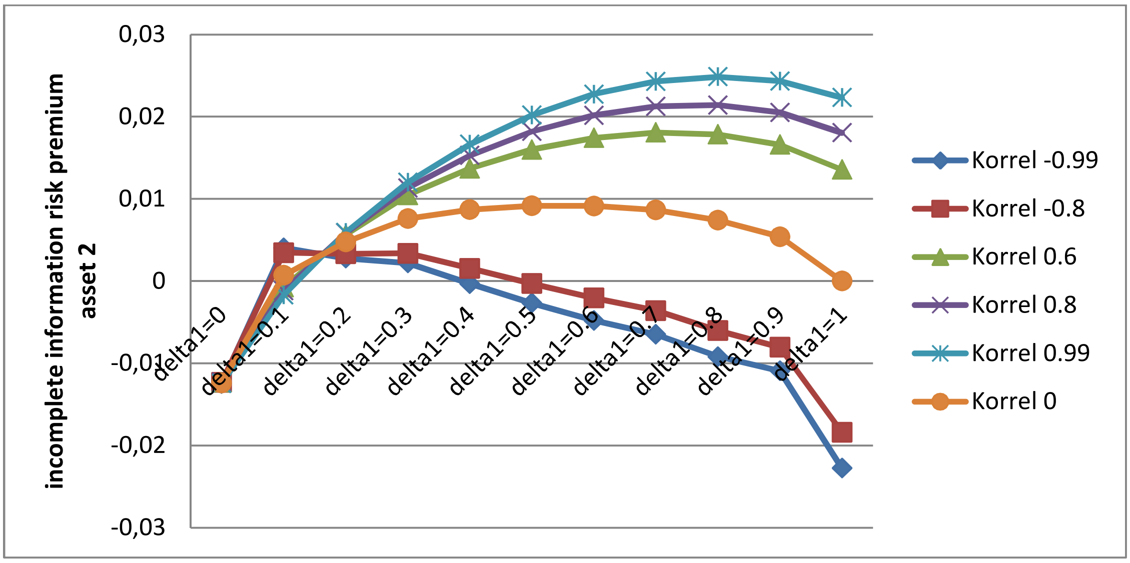

Figure 5-19: Incomplete information risk premia on asset 2 as a function of $\delta_{1,t}$ and various correlations of dividend growth rates conditional on the true regime ($p_{Draw} = 0.25$, uniform conditional transition probabilities, $\rho = 0.05$, $\gamma = 5$)

<u>Asset 1</u>

If $\delta_{1,t} \to 1$ (asset 1 pays all dividends), complete and incomplete information risk premia coincide. The risk premium on asset 1 does not depend on the correlation of dividend growth rates conditional on the regime.

If $\delta_{1,t} \to 0$ (asset 2 pays all dividends), the incomplete information risk premium on asset 1 increases in p_{Draw} and decreases in the dispersion of conditional transition probabilities. However, p_{Draw} and the dispersion of conditional transition probabilities have no effect on complete information risk premia. The risk aversion parameter γ has conflicting effects on the incomplete information risk premium on asset 1. The complete information risk premium of asset 1 becomes more extreme with an increase in γ: negative risk premia get even more negative, positive risk premia more positive. The correlation of dividend growth rates conditional on the regime increases both complete and incomplete information risk premia on asset 1.

For intermediate values of relative dividend contributions ($0 < \delta_{1,t} < 1$), the incomplete information risk premium on asset 1 increases in the correlation of dividend growth rates conditional on the regime (see Figure 5-20). Similarly, the complete information risk premium of asset 1 increases in the correlation of dividend growth rates of both assets.

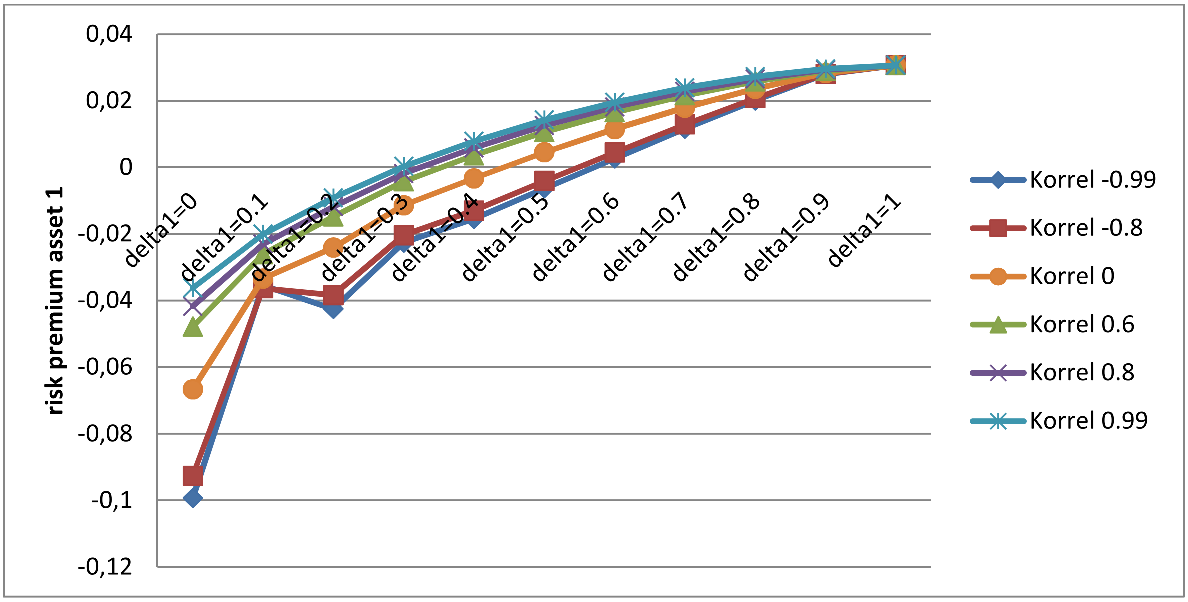

Figure 5-20: Incomplete information risk premia on asset 1 as a function of $\delta_{1,t}$ and various correlations of dividend growth rates conditional on the true regime ($p_{Draw} = 0.25$, uniform conditional transition probabilities, $\rho = 0.05$, $\gamma = 5$).

5.5.3.2 Interpretation of Results

5.5.3.2.1 Explanation of the Answer to Question 1

Why are incomplete information risk premia not necessarily less than complete information risk premia?

The reason why incomplete information risk premia can be greater or less than complete information risk premia is the same as in the extended Veronesi model with regimes in standard deviations and expectations of dividend growth: the standard deviation of dividend growth is now subject to regimes.

Why can incomplete information risk premia be negative?

The channel of stochastic regime probabilities is responsible for potentially negative incomplete information risk premia. It should therefore be analyzed how this channel depends on a change in the relative dividend contribution of asset 1, $\delta_{1,t}$. First note that the general structure of the risk premium is given by specializing (5-1) to the case with two risky assets,

$$RP_{i,t}{}^{iI,ret}(\pi_t,\delta_{1,t}) \equiv \frac{RP_{i,t}{}^{iI}(\pi_t,D_{i,t},\delta_{1,t})}{P_{i,t}{}^{iI}(\pi_t,D_{i,t},\delta_{1,t})}$$

$$= -cov\left(\begin{matrix} AfR_{t,t+1}{}^{iI}(fe_{t+1},S_t,\delta_{1,t},\pi_t), \\ [1+d_i(S_t,fe_{t+1})]\cdot\dfrac{\left\{\sum_{s'=1}^{K}\pi_{s',t+1}\cdot\left(\frac{P}{D}\right)_{i,t+1}^{cI}(s',\delta_{1,t+1})+1\right\}}{\left(\frac{P}{D}\right)_{i,t}^{iI}(\pi_t,\delta_{1,t})} \end{matrix} \middle| \pi_t,\delta_{1,t}\right)$$

$$i = 1,2$$

with

$$AfR_{t,t+1}{}^{iI}(fe_{t+1},S_t,\delta_{1,t},\pi_t)$$
$$= \frac{\{\delta_{1,t}\cdot(1+d_1(S_t,fe_{t+1}))+(1-\delta_{1,t})\cdot(1+d_2(S_t,fe_{t+1}))\}^{-\gamma}}{\sum_{s=1}^{K}\pi_{s,t}\cdot E\left(\{\delta_{1,t}\cdot(1+d_1(S_t,fe_{t+1}))+(1-\delta_{1,t})\cdot(1+d_2(S_t,fe_{t+1}))\}^{-\gamma}\middle|S_t=s,\delta_{1,t}\right)}$$

As in the models with a single risky asset, (i) stochastic regime probabilities and (ii) complete information price dividend ratios jointly form incomplete information price dividend ratio growth,

$$\frac{\sum_{s'=1}^{K}\pi_{s',t+1}\cdot\left(\frac{P}{D}\right)_{i,t+1}^{cI}(s',\delta_{1,t+1})}{\left(\frac{P}{D}\right)_{i,t}^{iI}(\pi_t,\delta_{1,t})}$$

$$i = 1,2$$

hence (i) and (ii) should first be analyzed individually. In a second step, the reaction of incomplete information price dividend ratio growth in response to dividend growth and the consequences for the correlation of the adjustment for risk with asset returns should be discussed.

Complete information price dividend ratio $\left(\frac{P}{D}\right)_{i,t+1}^{cI}(s,\delta_1), i = 1,2$

Asset 1

Complete information price dividend ratios of asset 1 decrease in the expectation and increase in the standard deviation of dividend growth of asset 2 (see Figure 5-21). Note that this pattern does not reverse for $\gamma < 1$ (see Figure 5-22) because regimes are now limited to the stochastic discount factor due to asset 2, whereas dividend growth of asset 1 does not exhibit regimes; hence there is no conflicting effect of discounting and dividends as in the case of asset 2.

An increase in the relative dividend contribution of asset 1, δ_1, lowers the differences in price dividend ratios across regimes to the point where the price dividend ratio does not depend on the regime if all dividends are paid by asset 1 ($\delta_1 = 1$): The stochastic discount factor becomes less regime-dependent because more aggregate dividends are paid by the regime-independent dividends of asset 1.

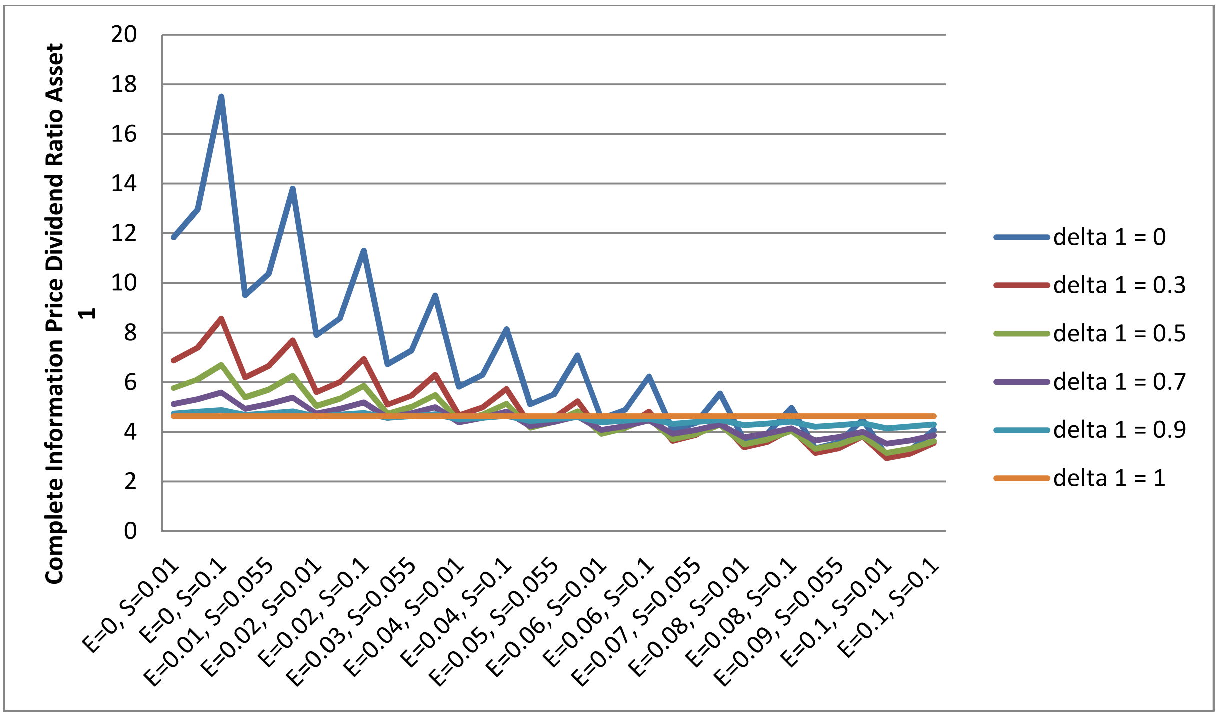

Figure 5-21: Complete information price dividend ratios of asset 1 as a function of expectation (E) and standard deviation regimes (S) and for various relative dividend contributions δ_1 and for risk aversion parameter $\gamma = 5$, with $p_{Draw} = 0.25$, uniform conditional transition probabilities, and $\rho = 0.05$ as well as correlation of dividend growth conditional on the regime = 80 percent.
Note that complete information price dividend ratios are functions of a finite set of regimes but have been connected by a polygonal path for readability.

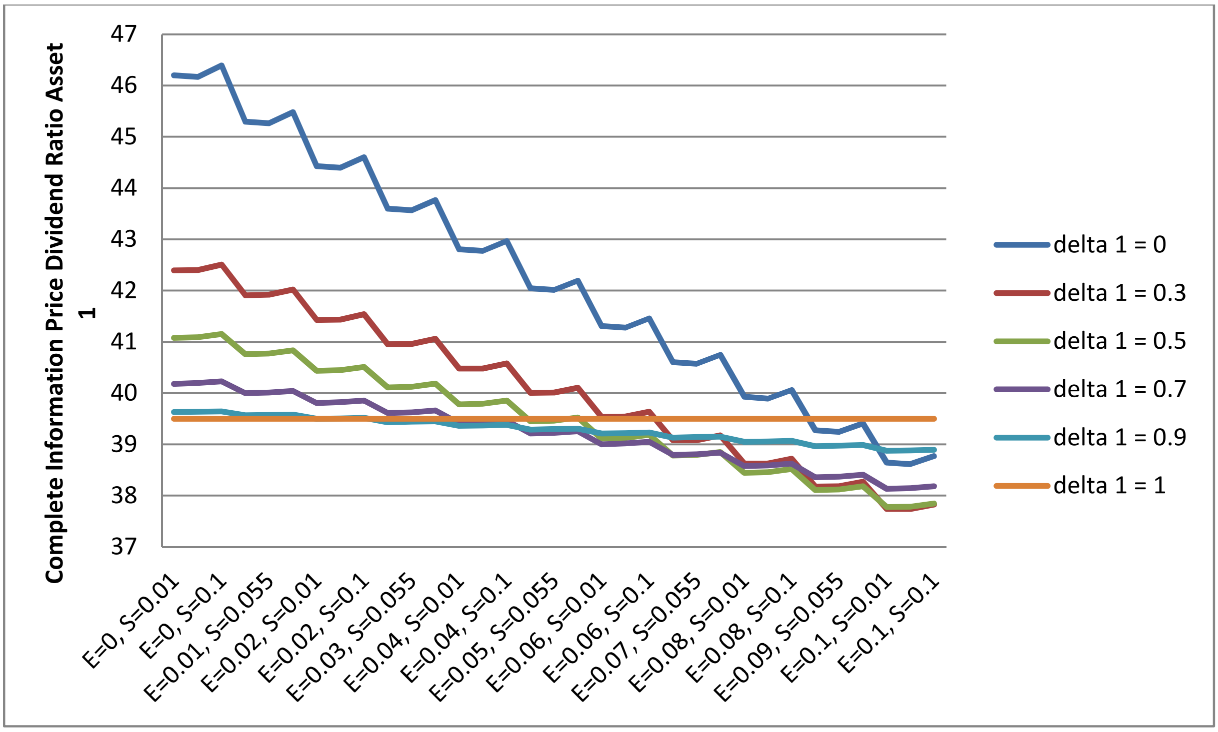

Figure 5-22: Complete information price dividend ratios of asset 1 as a function of expectation (E) and standard deviation regimes (S) and for various relative dividend contributions δ_1 and for risk aversion parameter $\gamma = 0.05$, with $p_{Draw} = 0.25$, uniform conditional transition probabilities, and $\rho = 0.05$ as well as correlation of dividend growth conditional on the regime = 80 percent.
Note that complete information price dividend ratios are functions of a finite set of regimes but have been connected by a polygonal path for readability.

Asset 2

If $\gamma > 1$ and if δ_1 is low (i.e., most dividends are paid by asset 2, the complete information price dividend ratio decreases in the expectation and increases in the standard deviation of dividend growth regimes of asset 2 (see Figure 5-23), i.e., the complete information price dividend ratio behaves as in the extended Veronesi model. However, as δ_1 increases (still with $\gamma > 1$), this pattern first weakens and finally reverses into a positive relation between expected dividend growth regime and the complete information price dividend ratio; the effect of the standard deviation of dividend growth regime also switches from positive to negative for high δ_1. As in the case of asset 1, the effect of a high value δ_1 on complete information price dividend ratios is due to the mostly regime-independent stochastic discount factor. Therefore, high expected dividends of asset 2 are no longer associated with low future marginal utility of consumption as would be the case for low values of δ_1, and the complete information price dividend ratio as discounted future dividend growth increases in expected dividend growth.

If $\gamma < 1$, the price dividend ratio of asset 2 always increases in the expectation regime, and decreases in the standard deviation regime of dividend growth of asset 2 and this increase is reinforced as δ_1 increases. The reason for this pattern is that the effect of regimes on dividends always dominates the effect of regimes on the stochastic discount factor for $\gamma < 1$, and this dominance becomes more pronounced as the stochastic discount factor is increasingly determined by regime independent dividend growth of asset 1.

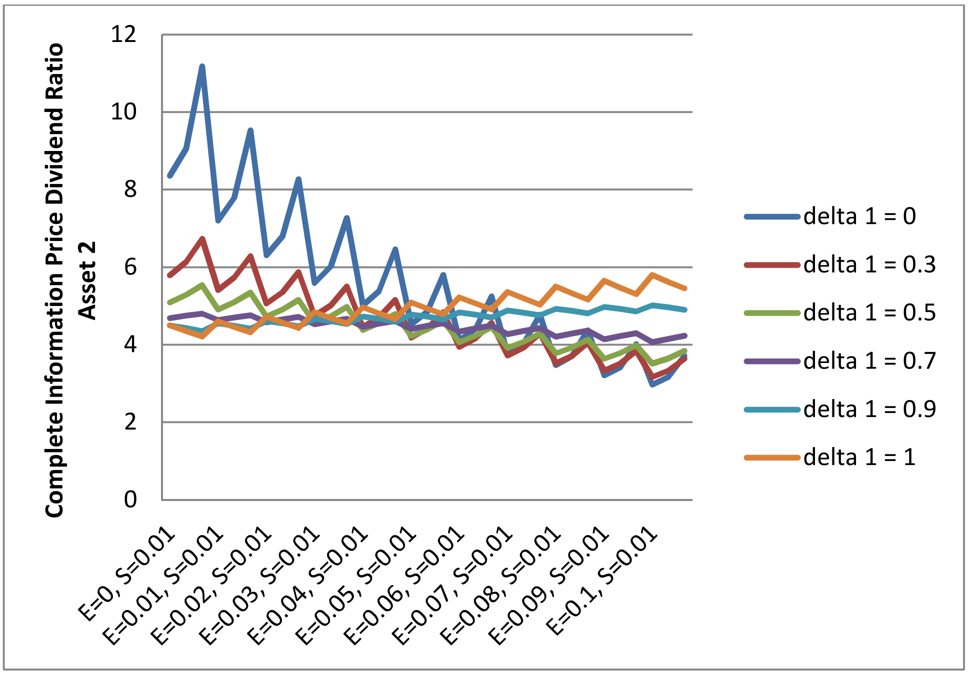

Figure 5-23: Complete information price dividend ratios of asset 2 as a function of expectation (E) and standard deviation regimes (S) and for various relative dividend contributions δ_1 and for risk aversion parameter $\gamma = 5$, with $p_{Draw} = 0.25$, uniform conditional transition probabilities, and $\rho = 0.05$ as well as correlation of dividend growth conditional on the regime = 80 percent.
Note that complete information price dividend ratios are functions of a finite set of regimes but have been connected by a polygonal path for readability.

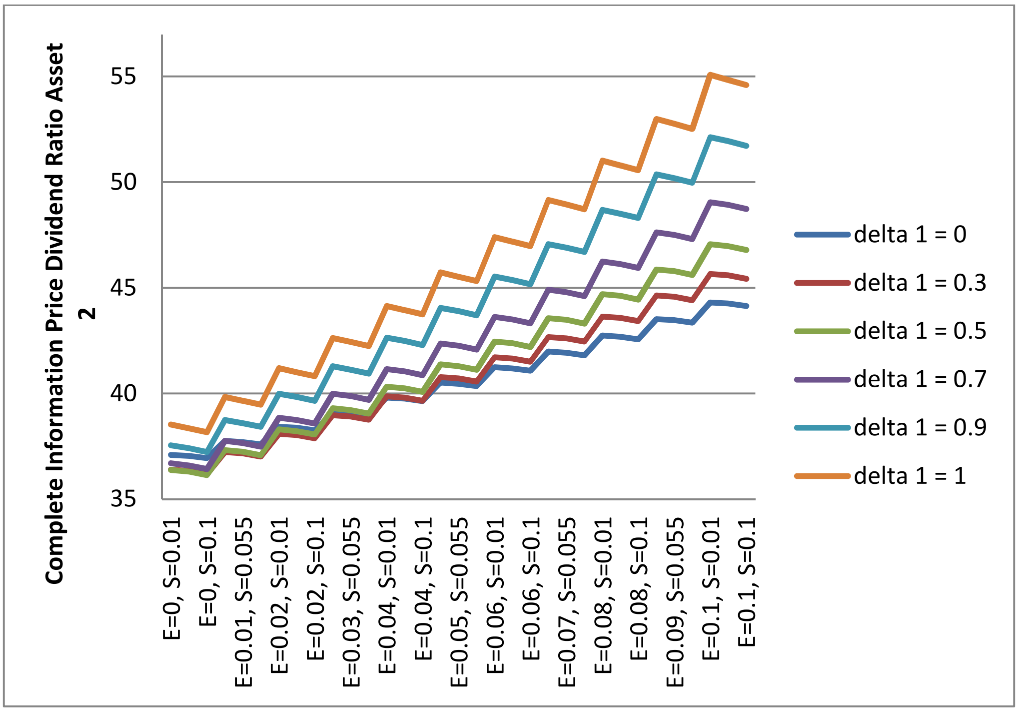

Figure 5-24: Complete information price dividend ratios of asset 2 as a function of expectation (E) and standard deviation regimes (S) and for various relative dividend contributions δ_1 and for risk aversion parameter $\gamma = 0.05$, with $p_{Draw} = 0.25$, uniform conditional transition probabilities, and $\rho = 0.05$ as well as correlation of dividend growth conditional on the regime = 80 percent.

Note that complete information price dividend ratios are functions of a finite set of regimes but have been connected by a polygonal path for readability.

Stochastic regime probabilities π_{t+1}

Stochastic regime probabilities π_{t+1} depend on dividend growth rates of both assets: (i) dividend growth of asset 2 is regime-dependent and, thus, provides direct information; (ii) dividend growth of asset 1 is regime-independent, but still provides indirect information via the common factor f_{t+1} (see the dividend models (5-20) and (5-21)):

$$D_{1,t+1} = D_{1,t} \cdot exp(\mu_1 + a_1 \cdot f_{t+1} + b_1 \cdot e_{1,t+1})$$

$$D_{2,t+1} = D_{2,t} \cdot exp(\mu_2(S_t) + a_2(S_t) \cdot f_{t+1} + b_2(S_t) \cdot e_{2,t+1})$$

The direct information effect is illustrated by Figure 5-25 which depicts the probability of one expected dividend growth regime (expected dividend growth = 5 percent): this plot is roughly similar to Figure 5-11 (expected dividend growth in the case of one asset). Similarly, Figure 5-26 illustrates the relation between dividend growth of asset 2 and the probability of the high standard deviation regime, which resembles Figure 5-12(probabilities of all standard deviation regimes in the case with a single risky asset).

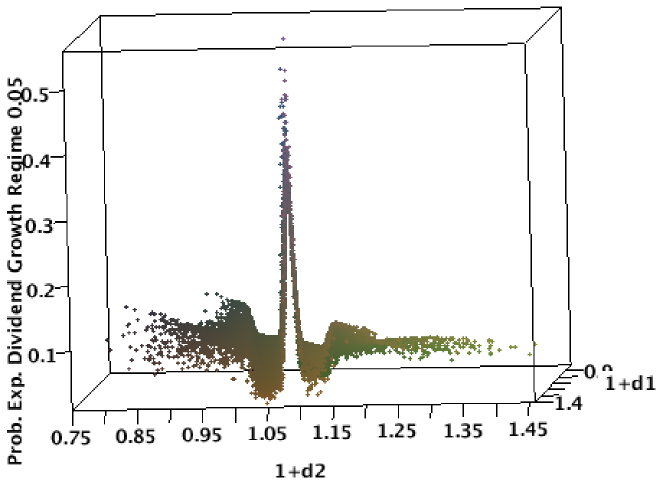

Figure 5-25: Probability of the expectation regime with 5 percent expected dividend growth and dividend growth of asset 2 (correlation of dividend growth conditional on the regime = 80 percent, uniform conditional transition probabilities, $p_{Draw} = 0.25$): simulation with 10,000 dividend growth realizations.

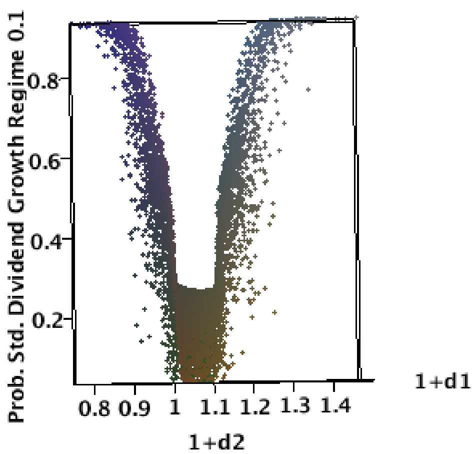

Figure 5-26: Probability of the high standard deviation regime and dividend growth of asset 2 (correlation of dividend growth conditional on the regime = 80 percent, uniform conditional transition probabilities, $p_{Draw} = 0.25$): simulation with 10,000 dividend growth realizations.

If the indirect information effect is added to the direct information effect, the total effect arises. Since the total effect is less intuitive than the direct information effect, it is illustrated by two examples for expectation regimes and standard deviation regimes:

Example 1 (effect of dividend growth of asset 1 on the probability of a 5 percent expected dividend growth regime): Figure 5-27 depicts the total effect of dividend growth rates of both assets on the probability of the 5 percent expected dividend growth regime. If dividend growth of asset 2 is close to its expectation of five percent, dividend growth of asset 1 has a U-shaped effect on the probability of the expected dividend growth regime. Intuitively, very high or very low dividend growth of asset 1 is most likely due to an extreme realization of the common factor f_{t+1}. However, since dividend growth of asset 2 is close to the expectation regime of 5 percent, it can be inferred that the impact of the common factor on dividend growth of asset 2, $a_2(S_t) \cdot f_{t+1}$ cannot be large, and thus $a_2(S_t)$ must be small. This, in turn, is the case in the regime with low standard deviation of dividend growth.

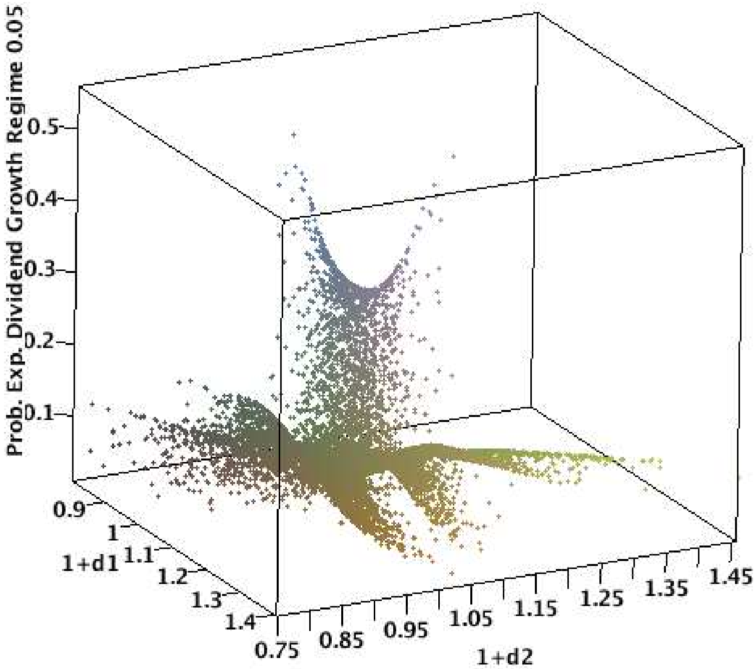

Figure 5-27: Probability of the expectation regime with 5 percent expected dividend growth and dividend growth of asset 2 (correlation of dividend growth conditional on the regime = 80 percent, uniform conditional transition probabilities, $p_{Draw} = 0.25$): simulation with 10,000 dividend growth realizations.

Example 2 (effect of dividend growth of asset 1 on the probability of the high standard deviation regime): consider the effect that dividend growth of asset 1 has in combination with dividend growth of asset 2 on the probability of the high standard deviation regime (Figure 5-28): if dividend growth of asset 2 is moderate, the probability of the high standard deviation regime is low; if dividend growth of asset 1 is very high or very low, it can be inferred (as in the example with expected dividend growth) that $a_2(S_t) \cdot f_{t+1}$ is most likely low, suggesting a low standard deviation regime. Hence high dividend growth of asset 1 further reduces the probability of the high standard deviation regime. By contrast, if dividend growth of asset 1 is moderate, it is likely that the moderate value of dividend growth of asset 2 is at least partially the result of a factor realization f_{t+1} close to zero, and hence the probability of the high standard deviation regime is somewhat higher.

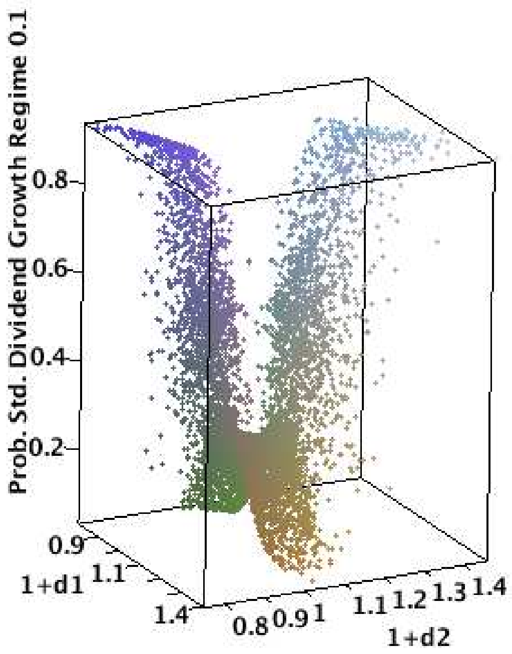

Figure 5-28: Probability of the regime with high standard deviation of dividend growth of asset 2 (correlation of dividend growth conditional on the regime = 80 percent, uniform conditional transition probabilities, $p_{Draw} = 0.25$): simulation with 10,000 dividend growth realizations.

Reaction of incomplete information price dividend ratio growth to an increase in dividend growth $1 + d$ depending on γ

Asset 1

If all dividends are paid by asset 1 ($\delta_{1,t}$=1) and dividend growth rates of both assets are either positively or negatively correlated, the price dividend ratio growth of asset 1 is regime-independent ($\frac{(PD)_{1,t+1}}{(PD)_{1,t}} = 1$) and thus does not react to dividend growth of asset 1.

If all dividends are paid by asset 2 ($\delta_{1,t}$=0) incomplete information price dividend ratio growth reacts only weakly to an increase in dividend growth of asset 1 for both $\gamma > 1$ and $\gamma < 1$. For positively correlated dividend growth of both assets, this reaction is weakly positive, for negative correlation it is weakly negative (see Figure 5-29 and Figure 5-30)

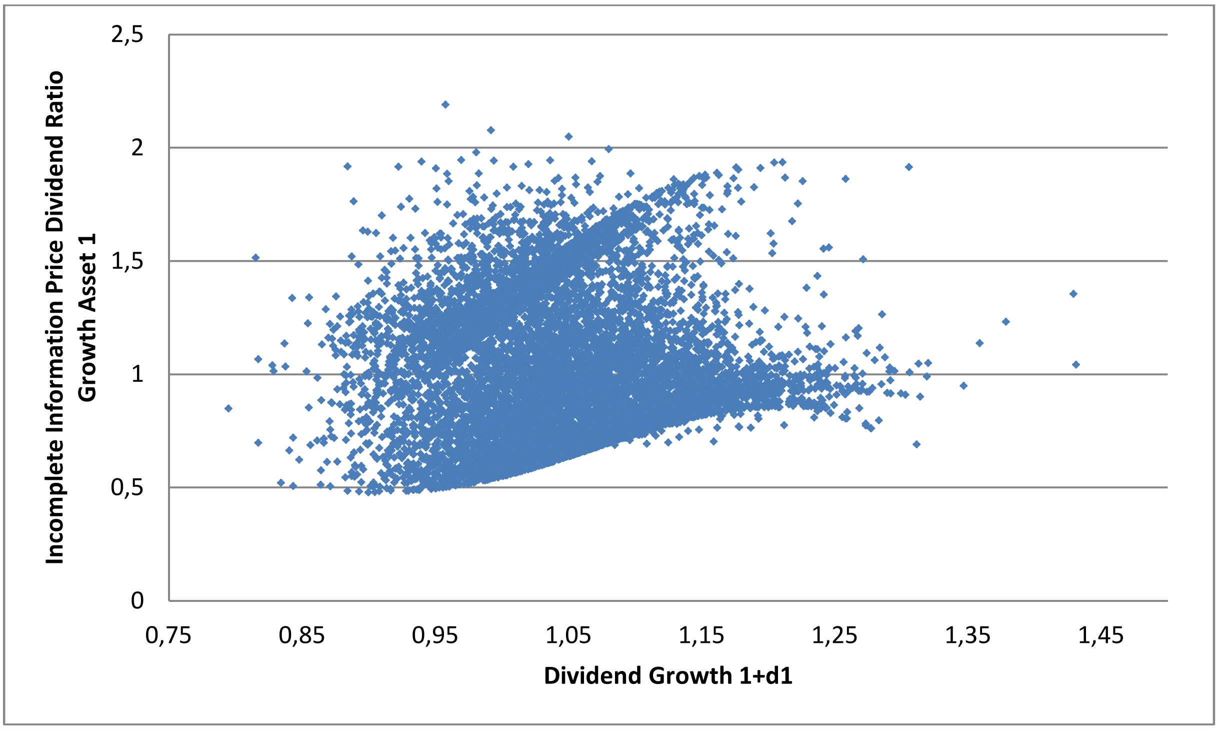

Figure 5-29: Incomplete information price dividend ratio growth of asset 1 and dividend growth of asset 1: case $\delta_{1,t}$=0 and positive correlation of dividend growth rates ($p_{Draw} = 0.1$, uniform conditional transition probabilities, $\rho = 0.05$, $\gamma = 3$, correlation of dividend growth conditional on the regime = 80%.): simulation with 10,000 dividend growth realizations.

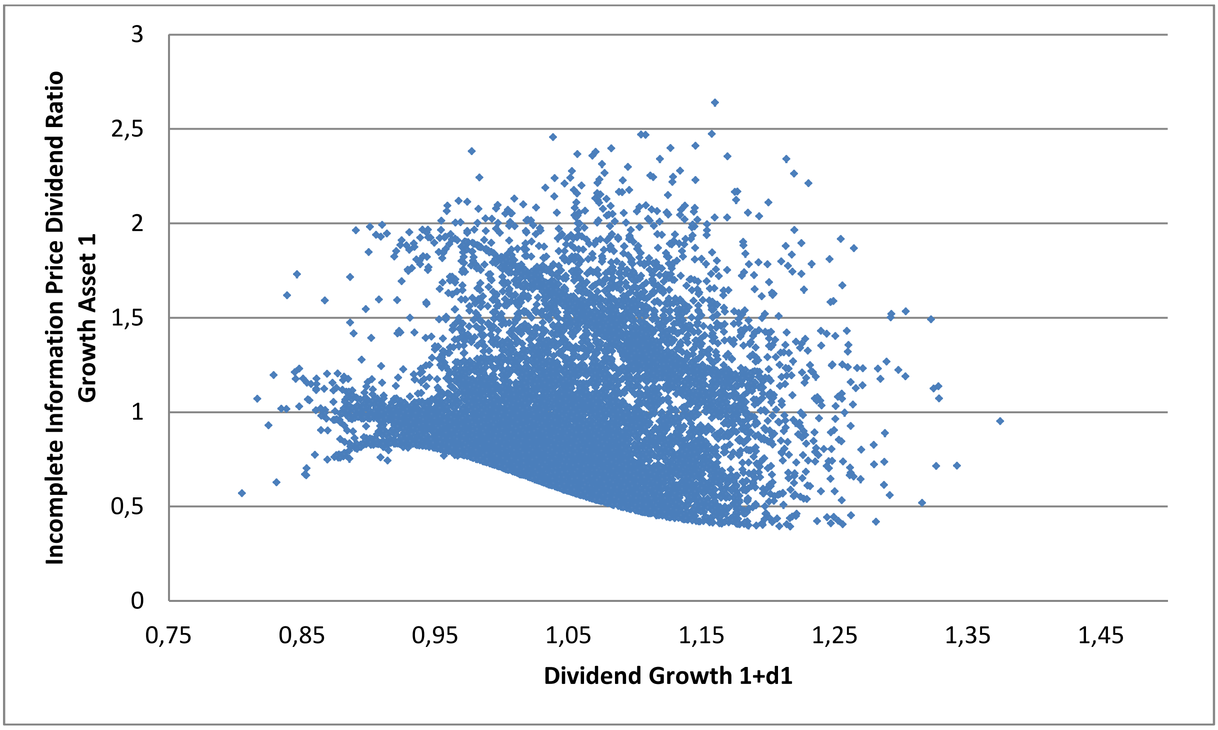

Figure 5-30: Incomplete information price dividend growth rate of asset 1: case $\delta_{1,t}$=0 and negative correlation of dividend growth rates ($p_{Draw} = 0.1$, uniform conditional transition probabilities, $\rho = 0.05$, $\gamma = 3$, correlation of dividend growth conditional on the regime = -80%.) : simulation with 10,000 dividend growth realizations.

For intermediate constellations with $0 < \delta_{1,t} < 1$, the reaction of incomplete information price dividend ratio growth to dividend growth of asset 1 is intermediate between the situations for $\delta_{1,t} \to 0$ and $\delta_{1,t} \to 1$ (for all values of γ and for positive and negative correlations of dividend growth rates). The lower $\delta_{1,t}$, the more dispersed the scatter plots become.

Asset 2

If all dividends are paid by asset 2 ($\delta_{1,t} \to 0$) and dividend growth rates of both assets are positively correlated, there is no strictly decreasing reaction of incomplete information price dividend ratio growth to dividend growth of asset 2. Instead the three regions known from the extended Veronesi case are observable (see Figure 5-31 for $\gamma > 1$ and Figure 5-32 for $\gamma < 1$)

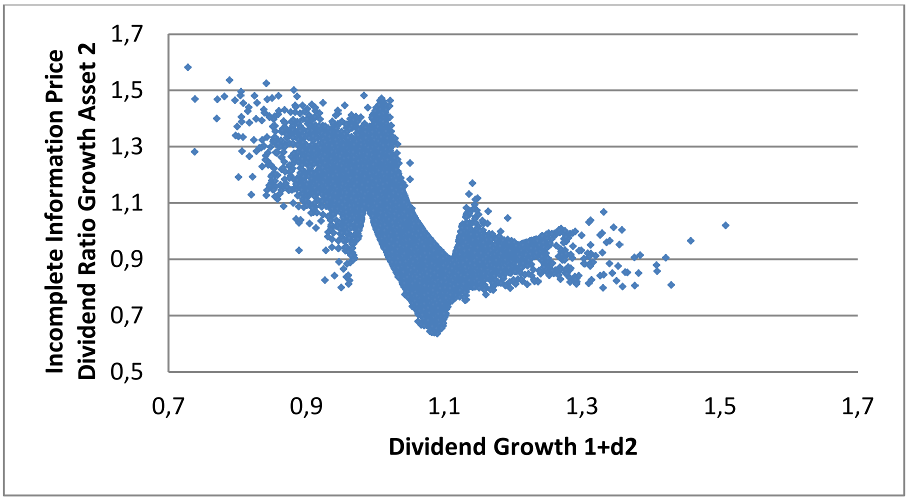

Figure 5-31: Incomplete information price dividend ratio growth of asset 2 and dividend growth of asset 2: case $\delta_{1,t}$=0, $\gamma > 1$, and positive correlation of dividend growth rates ($p_{Draw} = 0.1$, uniform conditional transition probabilities, $\rho = 0.05$, $\gamma = 3$, correlation of dividend growth conditional on the regime = +80 percent.): simulation with 10,000 dividend growth realizations.

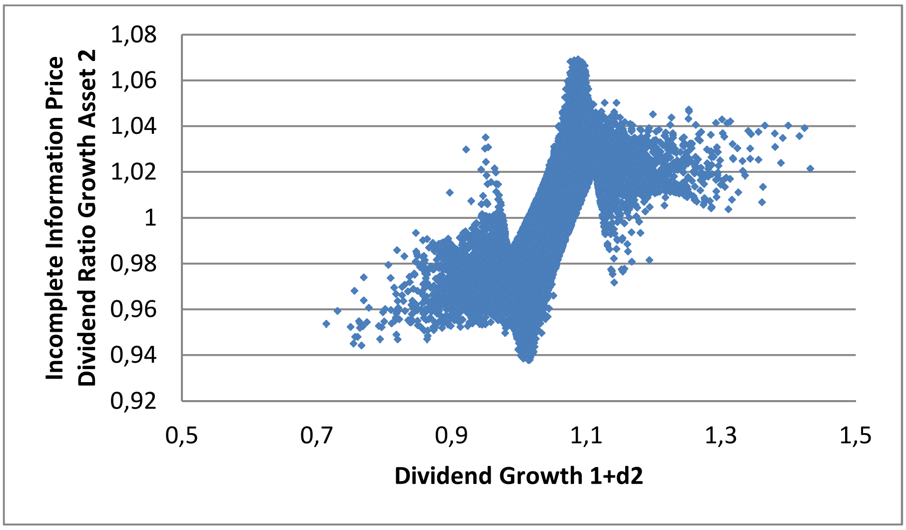

Figure 5-32: Incomplete information price dividend ratio growth of asset 2 and dividend growth of asset 2: case $\delta_{1,t}$=0, $\gamma < 1$, and positive correlation of dividend growth rates ($p_{Draw} = 0.1$, uniform conditional transition probabilities, $\rho = 0.05$, $\gamma = 0.5$, correlation of dividend growth conditional on the regime = 80 percent.): simulation with 10,000 dividend growth realizations.

If all dividends are paid by asset 2 ($\delta_{1,t} \to 0$) and dividend growth rates of both assets are negatively correlated, the relation between price dividend ratio growth and dividend growth of asset 2 does not change markedly (not depicted) because the stochastic discount factor is a function of dividend growth of asset 2; dividend growth of asset 1 only affects stochastic regime probabilities but not the stochastic discount factor.

If all dividends are paid by asset 1 ($\delta_{1,t} \to 1$) and dividend growth rates of both assets are positively correlated, the reaction of incomplete information price dividend ratio growth to dividend growth of asset 2 is reversed compared to the case $\delta_{1,t} \to 0$ if $\gamma > 1$; it looks roughly similar to Figure 5-32.

If γ is less than one, no such reversal occurs in complete information price dividend ratios (see the complete information price dividend ratios in Figure 5-22), and consequently the slope in the relation between incomplete information price dividend ratio growth and dividend growth remains positive as for $\delta_{1,t} = 0$ (not depicted).

If all dividends are paid by asset 1 ($\delta_{1,t} \to 1$) and dividend growth rates of both assets are negatively correlated, the relation between incomplete information price dividend ratio growth and dividend growth of asset 2 is very similar to the case with positively correlated dividend growth rates for both $\gamma < 1$ and $\gamma > 1$ (not depicted).

For intermediate constellations with $0 < \delta_{1,t} < 1$, the reaction of incomplete information price dividend ratio growth to dividend growth of asset 2 for $\gamma > 1$ switches from the behavior for $\delta_{1,t} \to 0$ to the one for $\delta_{1,t} \to 1$. The switching occurs when $\delta_{1,t}$ is high enough (around $\delta_{1,t} = 0.7$ in the example depicted in Figure 5-23).

Connection between adjustment for risk and return on the risky assets $[1 + d_i(S_t, fe_{t+1})] \cdot \frac{\left\{\sum_{s'=1}^{K} \pi_{s',t+1} \cdot \left(\frac{P}{D}\right)_{i,t+1}^{cI}(s', \delta_{t+1}) + 1\right\}}{\left(\frac{P}{D}\right)_{i,t}^{iI}(\pi_t, \delta_t)}$ and the sign of risk premia

Asset 1

If all dividends are paid by asset 1 ($\delta_{1,t}$=1) and dividend growth rates of both assets are either positively or negatively correlated, the sign of the risk premium is positive: incomplete information price dividend ratio growth does not react to dividend growth of asset 1; the sign of the risk premium is solely determined by the reactions of the adjustment for risk and of dividend yield to dividend growth of asset 1.

$\delta_1 \to 1$ $corr(d_1, d_2)$ positive or negative	Reaction to an increase in dividend growth $1 + d_1$ (driver of the adjustment for risk):			Sign of the risk premium:
	Adjustment for risk $(1 + d_1)^{-\gamma}$	Dividend yield $\frac{(1 + d_1)}{\left(\frac{P}{D}\right)_{1,t}^{iI} (\pi_t, 1)}$	Incomplete Information Price Dividend Ratio Growth $\frac{\left\{\sum_{s'=1}^{K} \pi_{s',t+1} \cdot \left(\frac{P}{D}\right)_{1,t+1}^{cI} (s', 1)\right\}}{\left(\frac{P}{D}\right)_{1,t}^{iI} (\pi_t, 1)}$	${RP_{1,t}}^{iI,ret}(\pi_t, 1)$
$0 < \gamma < 1$	-	+	0	> 0
$\gamma = 1$	-	+	0	> 0
$\gamma > 1$	-	+	0	> 0

Table 5-10: Responses of risk premium components of asset 1 to positive dividend growth of asset 1 and implication for the sign of the incomplete information risk premium of asset 1.

If all dividends are paid by asset 2 ($\delta_{1,t} \to 0$) and dividend growth rates of both assets are positively correlated, the sign of the risk premium is ambiguous: incomplete information price dividend ratio growth reacts negatively, but dividend yield reacts positively to dividend growth of asset 1; hence the sign of the risk premium depends on the relative strength of both components.

$\delta_1 \to 0$ $corr(d_1, d_2)$ > 0	Reaction to an increase in dividend growth $1 + d_2$ (driver of the adjustment for risk)			Sign of the risk premium:
	Adjustment for risk $(1 + d_2)^{-\gamma}$	Dividend yield $\frac{(1 + d_1)}{\left(\frac{P}{D}\right)_{1,t}^{iI} (\pi_t, 0)}$	Incomplete Information Price Dividend Ratio Growth $\frac{\left\{\sum_{s'=1}^{K} \pi_{s',t+1} \cdot \left(\frac{P}{D}\right)_{1,t+1}^{cI} (s', 0)\right\}}{\left(\frac{P}{D}\right)_{1,t}^{iI} (\pi_t, 0)}$	${RP_{1,t}}^{iI,ret}(\pi_t, 0)$
$0 < \gamma < 1$	-	+	-	>=< 0
$\gamma = 1$	-	+	-	>=< 0
$\gamma > 1$	-	+	-	>=< 0

Table 5-11: Case of positive correlation of dividend growth rates; responses of risk premium components of asset 1 to positive dividend growth of asset 2 and implication for the sign of the incomplete information risk premium of asset 1.

In the case of negative correlation (still for $\delta_{1,t} \to 0$), a negative sign of the risk premium results: both dividend yield and incomplete information price dividend ratio growth react negatively to dividend growth of asset 2, as does the adjustment for risk.

$\delta_1 \to 0$ $corr(d_1, d_2) < 0$	Reaction to an increase in dividend growth $1 + d_2$ (driver of the adjustment for risk)			Sign of the risk premium:
	Adjustment for risk $(1 + d_2)^{-\gamma}$	Dividend yield $\frac{(1 + d_1)}{\left(\frac{P}{D}\right)_{1,t}^{iI}(\pi_t, 0)}$	Incomplete Information Price Dividend Ratio Growth $\frac{\left\{\sum_{s'=1}^{K} \pi_{s',t+1} \cdot \left(\frac{P}{D}\right)_{1,t+1}^{cI}(s', 0)\right\}}{\left(\frac{P}{D}\right)_{1,t}^{iI}(\pi_t, 0)}$	$RP_{1,t}^{iI,ret}(\pi_t, 0)$
$0 < \gamma < 1$	-	-	-	< 0
$\gamma = 1$	-	-	-	< 0
$\gamma > 1$	-	-	-	< 0

Table 5-12: Case of negative correlation of dividend growth rates: responses of risk premium components of asset 1 to positive dividend growth of asset 2 and implication for the sign of the incomplete information risk premium of asset 1.

Asset 2

If all dividends are paid by asset 2 ($\delta_{1,t} \to 0$) and dividend growth rates of both assets are positively or negatively correlated, the sign of the risk premium depends on γ: for $\gamma \leq 1$, both dividend yield and incomplete information price dividend ratio growth react positively to dividend growth of asset 2 and, thus, are inversely related to the reaction of the adjustment for risk (positive risk premium). For $\gamma > 1$, incomplete information price dividend ratio growth reacts negatively, but dividend yield reacts positively to dividend growth of asset 2; hence the sign of the risk premium depends on the relative strength of both components (positive or negative risk premium). Note that this case ($\delta_{1,t} \to 0$, asset 2) is very close to the extended Veronesi single asset case, with the only difference that dividend growth of asset 1 provides additional information. This information depends on the strength of the correlation of dividend growth rates, but not on the sign of this correlation.

$\delta_1 \to 0$ $corr(d_1, d_2)$ positive or negative	Reaction to an increase in dividend growth $1 + d_2$ (driver of the adjustment for risk)			Sign of the risk premium:
	Adjustment for risk $(1 + d_2)^{-\gamma}$	Dividend yield $\frac{(1 + d_2)}{\left(\frac{P}{D}\right)_{2,t}^{iI}(\pi_t, 0)}$	Incomplete Information Price Dividend Ratio Growth $\frac{\left\{\sum_{s'=1}^{K} \pi_{s',t+1} \cdot \left(\frac{P}{D}\right)_{2,t+1}^{cI}(s', 0)\right\}}{\left(\frac{P}{D}\right)_{2,t}^{iI}(\pi_t, 0)}$	$RP_{2,t}{}^{iI,ret}(\pi_t, 0)$
$0 < \gamma < 1$	-	+	+	> 0
$\gamma = 1$	-	+	0	> 0
$\gamma > 1$	-	+	-	>=< 0

Table 5-13: Responses of risk premium components of asset 2 to positive dividend growth of asset 2 and implication for the sign of the incomplete information risk premium of asset 1.

If all dividends are paid by asset 1 ($\delta_{1,t}$=1) and dividend growth rates of both assets are positively correlated, the sign of the risk premium is positive: incomplete information price dividend ratio growth does not react to dividend growth of asset 1; the sign of the risk premium is solely determined by the reactions of the adjustment for risk and of dividend yield to dividend growth of asset 1.

$\delta_1 \to 1$ $corr(d_1, d_2) > 0$	Reaction to an increase in dividend growth $1 + d_1$ (driver of the adjustment for risk)			Sign of the risk premium:
	Adjustment for risk $(1 + d_1)^{-\gamma}$	Dividend yield $\frac{(1 + d_2)}{\left(\frac{P}{D}\right)_{2,t}^{iI}(\pi_t, 1)}$	Incomplete Information Price Dividend Ratio Growth $\frac{\left\{\sum_{s'=1}^{K} \pi_{s',t+1} \cdot \left(\frac{P}{D}\right)_{2,t+1}^{cI}(s', 1)\right\}}{\left(\frac{P}{D}\right)_{2,t}^{iI}(\pi_t, 1)}$	$RP_{2,t}{}^{iI,ret}(\pi_t, 1)$
$0 < \gamma < 1$	-	+	0	> 0
$\gamma = 1$	-	+	0	> 0
$\gamma > 1$	-	+	0	>0

Table 5-14: Case of positive correlation of dividend growth rates: responses of risk premium components of asset 2 to positive dividend growth of asset 1 and implication for the sign of the incomplete information risk premium of asset 1.

If all dividends are paid by asset 1 ($\delta_{1,t} \to 1$) and dividend growth rates of both assets are negatively correlated, the sign of the risk premium is negative: incomplete information price dividend ratio growth does not react to dividend growth of asset 1; the sign of the risk premium is solely determined by the reactions of the adjustment for risk and of dividend yield to dividend growth of asset 1.

Negative correlation between dividend growth of both assets means high dividend growth of asset 2 is mostly associated with low dividend growth of asset 1 and vice versa.

$\delta_1 \to 1$ $corr(d_1, d_2) < 0$	Reaction to an increase in dividend growth $1 + d_1$ (driver of the adjustment for risk)			Sign of the risk premium:
	Adjustment for risk $(1 + d_1)^{-\gamma}$	Dividend yield $\frac{(1 + d_2)}{\left(\frac{P}{D}\right)_{2,t}^{iI}(\pi_t, 1)}$	Incomplete Information Price Dividend Ratio Growth $\frac{\left\{\sum_{s'=1}^{K} \pi_{s',t+1} \cdot \left(\frac{P}{D}\right)_{2,t+1}^{cI}(s', 1)\right\}}{\left(\frac{P}{D}\right)_{2,t}^{iI}(\pi_t, 1)}$	$RP_{2,t}{}^{iI,ret}(\pi_t, 1)$
$0 < \gamma < 1$	-	-	0	< 0
$\gamma = 1$	-	-	0	< 0
$\gamma > 1$	-	-	0	<0

Table 5-15: Case of negative correlation of dividend growth rates: responses of risk premium components of asset 2 to positive dividend growth of asset 1 and implication for the sign of the incomplete information risk premium of asset 1.

Now consider combinations of $\delta_{1,t}$ that are intermediate between the extreme cases considered so far ($\delta_{1,t} \to 0$ and $\delta_{1,t} \to 1$). The sign of risk premia depends on the interplay between adjustment for risk, dividend yield, and incomplete information price dividend ratio growth. Since there is an infinite number of intermediate cases ($0 < \delta_{1,t} < 1$), a graphical analysis instead of a tabular approach is suited. To be more precise, correlations between adjustment of risk and dividend yield as well as between adjustment for risk and incomplete information price dividend ratio growth is analyzed graphically.

Asset 1

If dividend growth rates of both assets are positively correlated, the sign of the risk premium is ambiguous with the exception of $\delta_{1,t}$ very close to one: the correlation between dividend yield and adjustment for risk is negative, the one between adjustment for risk and incomplete information price dividend ratio growth is positive (except for $\delta_{1,t}$ very close to one) as Figure 5-33 and Figure 5-34 illustrate.

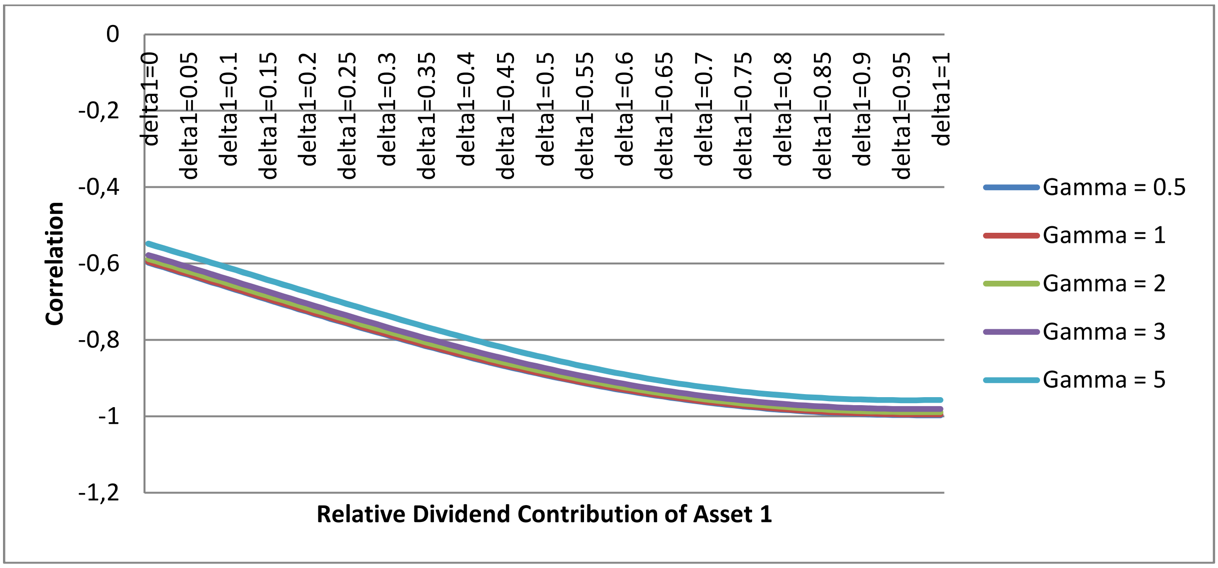

Figure 5-33: Correlation between dividend yield of asset 1 with adjustment for risk as a function of $\delta_{1,t}$, various values of γ, and positive correlation of dividend growth rates ($p_{Draw} = 0.1$, uniform conditional transition probabilities, $\rho = 0.05$, correlation of dividend growth conditional on the regime = 80 percent)

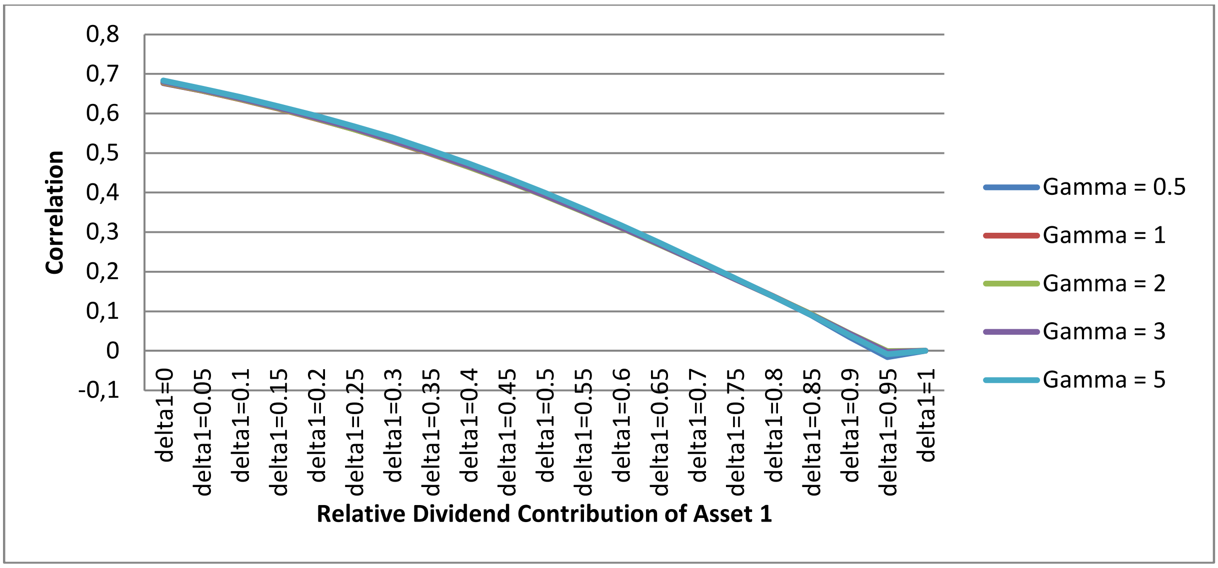

Figure 5-34: Correlation between incomplete information price dividend ratio growth of asset 1 with adjustment for risk as a function of $\delta_{1,t}$, various values of γ, and positive correlation of dividend growth rates ($p_{Draw} = 0.1$, uniform conditional transition probabilities, $\rho = 0.05$, correlation of dividend growth conditional on the regime = 80 percent)

If dividend growth rates of both assets are negatively correlated, the sign of the risk premium switches from negative for low values of $\delta_{1,t}$ to positive for high values of $\delta_{1,t}$: both the correlations between dividend yield and adjustment for risk as well as the correlation between adjustment for risk and incomplete information price dividend ratio growth change sign from positive to negative as $\delta_{1,t}$ increases as Figure 5-35 and Figure 5-36 illustrate.

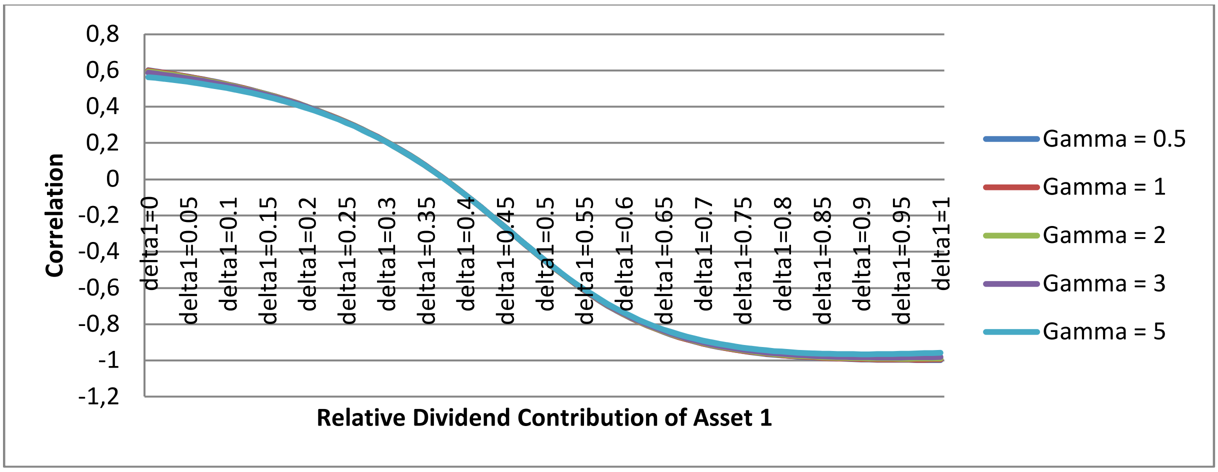

Figure 5-35: Correlation between dividend yield of asset 1 with adjustment for risk as a function of $\delta_{1,t}$, various values of γ, and negative correlation of dividend growth rates ($p_{Draw} = 0.1$, uniform conditional transition probabilities, $\rho = 0.05$, correlation of dividend growth conditional on the regime = -80 percent)

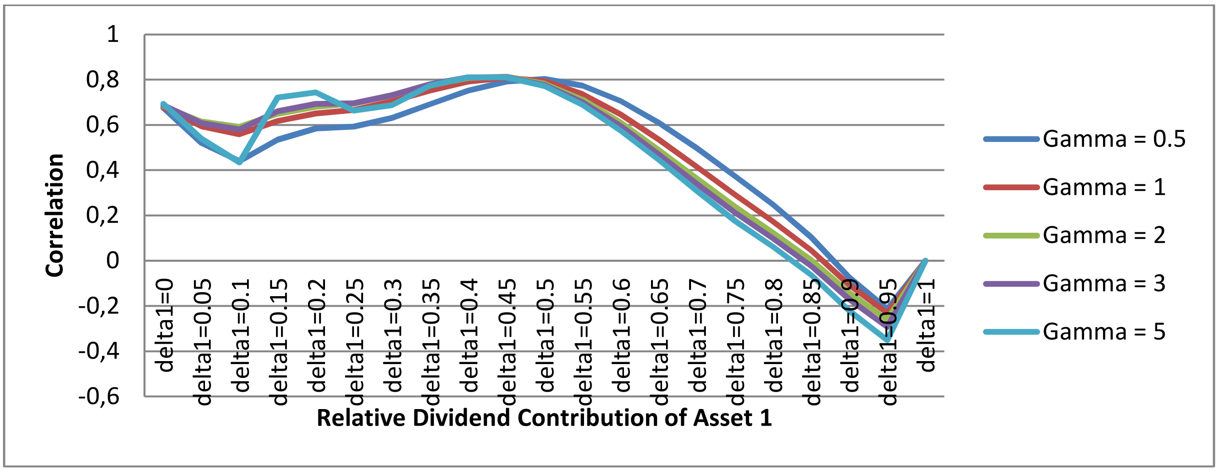

Figure 5-36: Correlation between incomplete information price dividend ratio growth of asset 1 with adjustment for risk as a function of $\delta_{1,t}$, various values of γ, and negative correlation of dividend growth rates ($p_{Draw} = 0.1$, uniform conditional transition probabilities, $\rho = 0.05$, correlation of dividend growth conditional on the regime = -80 percent)

Asset 2

If dividend growth rates of both assets are positively correlated, the sign of the risk premium depends on γ: for $\gamma \leq 1$, the risk premium is positive for all values of $\delta_{1,t}$, but for $\gamma > 1$, the risk premium changes from ambiguous for low values of $\delta_{1,t}$ to positive for high values of $\delta_{1,t}$: if $\gamma \leq 1$, both dividend yield and incomplete information price dividend ratio growth are negatively correlated with the adjustment for risk, resulting in a positive risk premium. If $\gamma > 1$, dividend yield is correlated negatively with the adjustment for risk, whereas incomplete information price dividend ratio growth is correlated positively with the adjustment for risk. This implies that the sign of the risk premium is ambiguous for low values of $\delta_{1,t}$. However, if $\delta_{1,t}$ exceeds a certain threshold, the sign of the correlation of incomplete information price dividend ratio growth with the adjustment for risk switches from positive to negative, and the risk premium becomes positive as Figure 5-37 and Figure 5-38 illustrate.

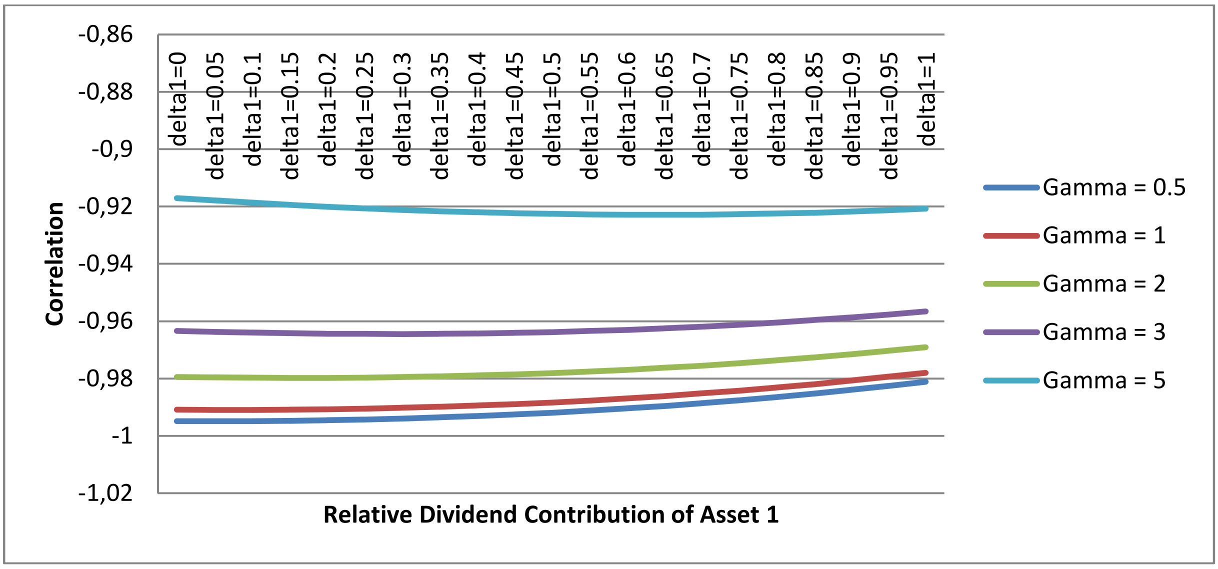

Figure 5-37: Correlation between dividend yield of asset 2 with adjustment for risk as a function of $\delta_{1,t}$, various values of γ, and positive correlation of dividend growth rates ($p_{Draw} = 0.1$, uniform conditional transition probabilities, $\rho = 0.05$, correlation of dividend growth conditional on the regime = 80 percent)

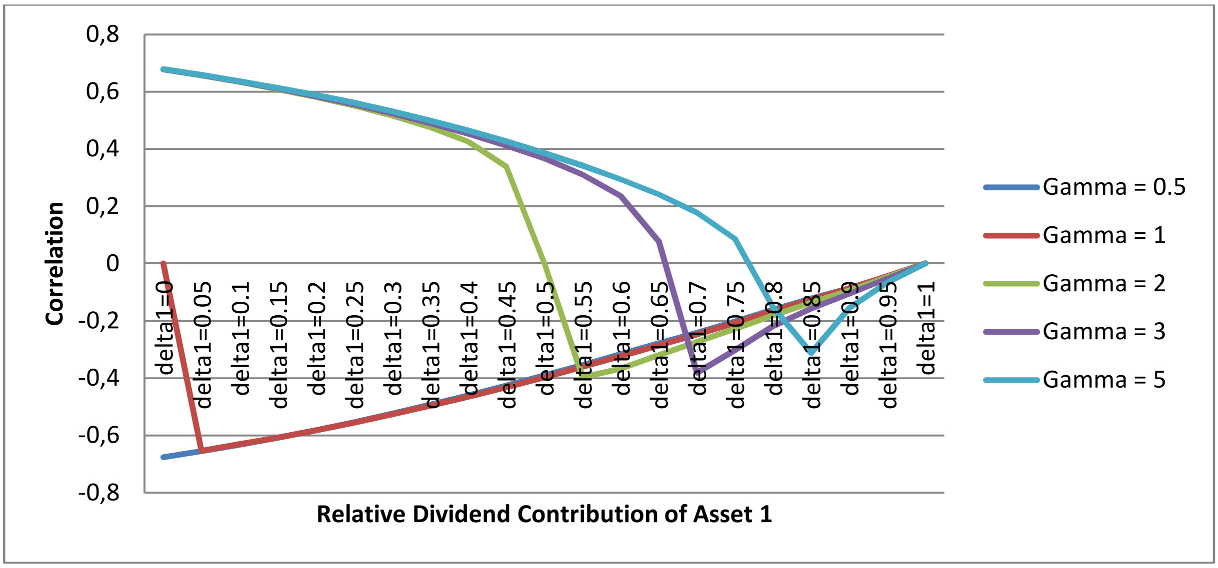

Figure 5-38: Correlation between incomplete information price dividend ratio growth of asset 2 with adjustment for risk as a function of $\delta_{1,t}$, various values of γ, and positive correlation of dividend growth rates ($p_{Draw} = 0.1$, uniform conditional transition probabilities, $\rho = 0.05$, correlation of dividend growth conditional on the regime = 80 percent)

If dividend growth rates of both assets are negatively correlated, the sign of the risk premium depends on γ: for $\gamma \leq 1$, the risk premium is positive for low values of $\delta_{1,t}$ and ambiguous for values of $\delta_{1,t}$ that are high but less than 1. If $\delta_{1,t} \rightarrow 1$, the sign of the risk premium is negative. For $\gamma > 1$, the sign of the risk premium is always ambiguous with the exception of $\delta_{1,t} = 1$ where the sign of the risk premium is negative. If $\gamma \leq 1$, the sign of the correlation of dividend yield with adjustment for risk changes from negative to positive; the sign of the correlation of incomplete information price dividend ratio growth with adjustment for risk is always negative (with the exception of $\delta_{1,t} \to 1$ and, if $\gamma = 1$, $\delta_{1,t} = 0$). This implies that the risk premium is positive as long as both dividend yield and incomplete information price dividend ratio growth are negatively correlated with adjustment for risk. As the sign of the correlation of dividend yield with adjustment for risk reverses, the total effect on the risk premium becomes unclear, with the exception of $\delta_{1,t} \to 1$ where incomplete information price dividend ratio growth is uncorrelated with adjustment for risk; the risk premium is determined by dividend yield alone for $\delta_{1,t} \to 1$ and, thus, is negative. If $\gamma > 1$, the sign of the correlation of dividend yield with the adjustment for risk is always opposite to the sign of the correlation of incomplete information price dividend ratio growth with adjustment for risk leading to an ambiguous sign of the risk premium (with the exception of $\delta_{1,t} \to 1$ where only dividend yield matters and the risk premium is negative) as Figure 5-39 and Figure 5-40 illustrate.

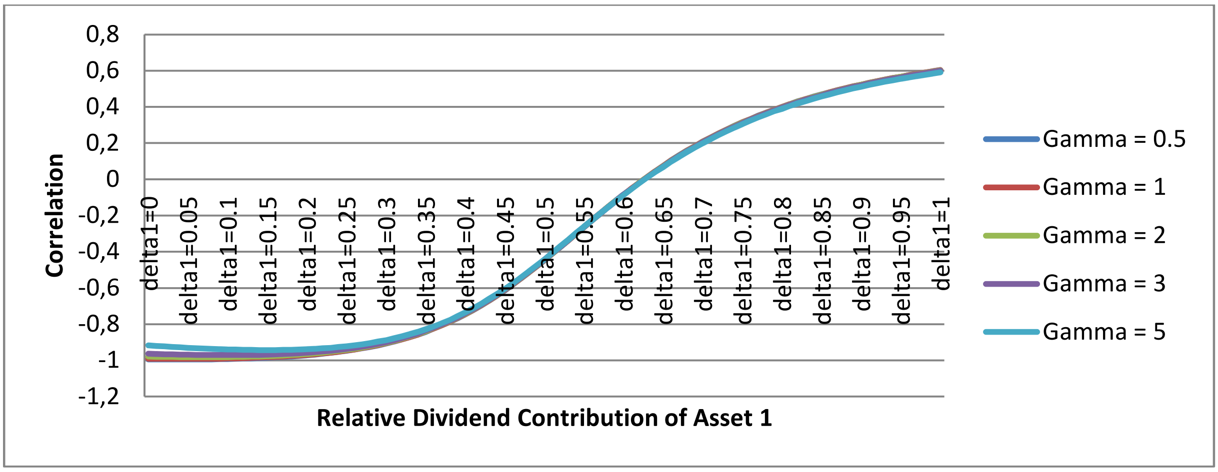

Figure 5-39: Correlation between dividend yield of asset 2 with adjustment for risk as a function of $\delta_{1,t}$, various values of γ, and negative correlation of dividend growth rates ($p_{Draw} = 0.1$, uniform conditional transition probabilities, $\rho = 0.05$, ,correlation of dividend growth conditional on the regime = -80 percent

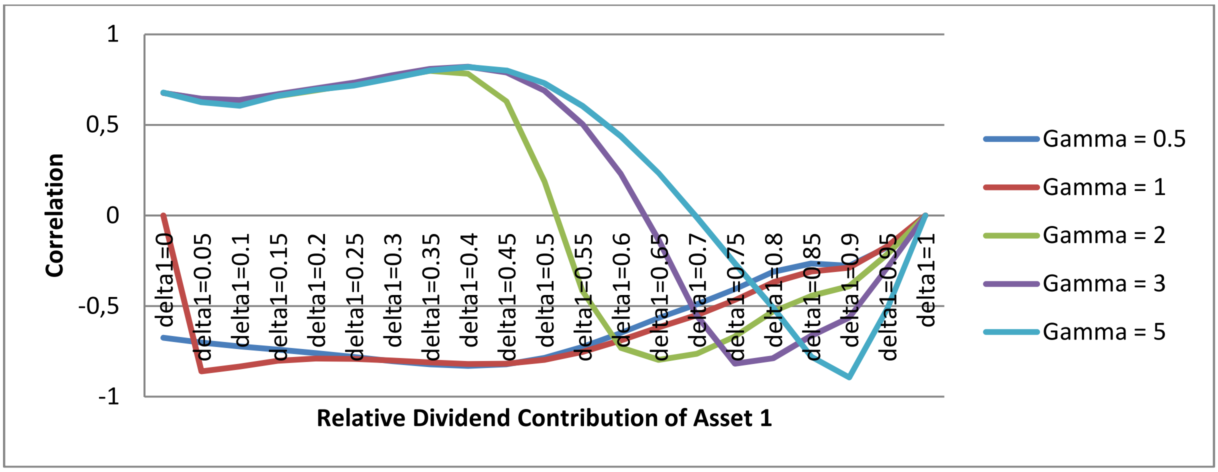

Figure 5-40: Correlation between incomplete information price dividend ratio growth of asset 2 with adjustment for risk as a function of $\delta_{1,t}$, various values of γ, and negative correlation of dividend growth rates ($p_{Draw} = 0.1$, uniform conditional transition probabilities, $\rho = 0.05$, correlation of dividend growth conditional on the regime = -80 percent)

5.5.3.2.2 Explanation of the Answer to Question 2

Why can incomplete information risk premia be greater or less than complete information risk premia?

See Explanation of the Answer to Question 1.

Why does the force underlying Veronesi's results, the correlation of incomplete information price dividend ratio with the adjustment for risk, reverse from positive to negative as $\delta_{1,t}$ increases from 0 to 1?

The answer can be found in Figure 5-38 and Figure 5-40: a negative correlation of incomplete information price dividend ratio growth with adjustment for risk is visible for values of $\delta_{1,t}$ that are high but less than 1. For these values, first, dividend growth of asset 2 contributes at least a part of the stochastic discount factor (as opposed to the extreme case $\delta_{1,t} \to 1$). Second, the relation between complete information price dividend ratios of asset 2 and expected dividend growth of asset 2 becomes a positive (rather than negative) one if $\delta_{1,t}$ is high enough (see Figure 5-23 where this reversal from negative to positive occurs for $\delta_{1,t} > 0.7$). This implies that high dividend growth of asset 2 is associated with high incomplete information price dividend ratio growth even if $\gamma > 1$.

5.5.3.2.3 Explanation of the Answer to Question 3

5.5.3.2.3.1 Incomplete Information

Asset 2

$\delta_{1,t} \to 0$

If $\delta_{1,t} \to 0$, the independence of the incomplete information risk premium of the correlation of dividend growth rates conditional on the regime must be explained. This result is somewhat surprising because dividend growth of asset 1 is a source of information on the true regime due to the common factor f_{t+1}.

However, it turns out that dividend growth of asset 1 is not relevant because the adjustment for risk consists of dividend growth of asset 2 only. Therefore, the effect of dividend growth of asset 1 on asset returns of asset 2 (as a source of information) is uncorrelated with the adjustment for risk, and not priced.

$\delta_{1,t} \to 1$

The regime parameters p_{Draw} and dispersion of conditional transition probabilities do not substantially influence incomplete information risk premia of asset 2 although the distribution of dividend growth of asset 2 depends on both of these parameters. The reason is that the return on asset 2 is priced only insofar as it is correlated with the adjustment for risk which, for $\delta_{1,t} \to 1$, is completely determined by the regime-independent dividend growth of asset 1.

It should, however, be mentioned that there is a pseudo-dependence of (complete and incomplete information) risk premia of asset 2 on the dispersion of conditional transition probabilities: prima facie, risk premia of asset 2 increase in this dispersion at $\delta_{1,t} \to 1$ (see the last columns in Table 5-16 and Table 5-17); however, recall that the standard deviation of dividend growth of asset 1 at $\delta_{1,t} \to 1$ has been chosen to match the standard deviation of dividend growth of asset 2 at $\delta_{1,t} \to 0$, and this latter standard deviation increases in the dispersion of conditional transition probabilities. Hence risk premia on asset 2 increase in the dispersion of conditional transition probabilities because the adjustment for risk (which is determined by dividend growth of asset 1) exhibits a higher standard deviation.

The risk aversion parameter γ has a strong positive effect on the standard deviation of the adjustment for risk (recall that this standard deviation roughly corresponds to the product of γ with the standard deviation of dividend growth). There also exists a small effect of γ on incomplete information price dividend ratio growth of asset 2 which, however, is mostly uncorrelated with the adjustment for risk and, thus, not priced.

If dividend growth rates of both assets are positively correlated, dividend growth of asset 2 is negatively correlated with the adjustment for risk. An increase in the correlation of dividend growth rates then results in an increased positive risk premium. If dividend growth rates of both assets are negatively correlated, dividend growth of asset 2 is positively correlated with the adjustment for risk; an increase in the correlation of dividend growth rates (i.e., a less negative correlation) increases the risk premium by making it less negative. If dividend growth rates of both assets are uncorrelated, the risk premium of asset 2 is (very close to) zero.

$0 \leq \delta_{1,t} \leq 1$

Adjustment for risk consists of a part contributed by asset 1 and another part contributed by asset 2 where the part of asset 2 becomes less important if $\delta_{1,t}$ increases. The part of asset 2 always induces a negative relation with dividend growth of asset 2. If dividend growth rates of both assets are either uncorrelated or negatively correlated, asset 1 compensates or at least dilutes this negative relation between the adjustment for risk and dividend growth of asset 2, a decreasing risk premium of asset 2 results. For positive correlations, the part of the adjustment for risk provided by asset 1 also stands in a negative relation to dividend growth of asset 2 working in the direction of an increasing risk premium.

Incomplete information risk premia of asset 2 ($\gamma = 3$)											
$p_{Draw} = 0.1$											
	$\delta_{1,t} \to 0$	$\delta_{1,t} = 0.1$	$\delta_{1,t} = 0.2$	$\delta_{1,t} = 0.3$	$\delta_{1,t} = 0.4$	$\delta_{1,t} = 0.5$	$\delta_{1,t} = 0.6$	$\delta_{1,t} = 0.7$	$\delta_{1,t} = 0.8$	$\delta_{1,t} = 0.9$	$\delta_{1,t} \to 1$
σ_{low}	1.13%	1.13%	1.12%	1.10%	1.08%	1.06%	1.03%	1.00%	0.96%	0.92%	0.88%
σ_{middle}	1.02%	1.07%	1.09%	1.10%	1.10%	1.09%	1.07%	1.04%	1.01%	0.96%	0.90%
σ_{high}	-0.04%	0.40%	0.67%	0.87%	1.01%	1.11%	1.17%	1.18%	1.16%	1.10%	0.97%
uniform	-0.93%	-0.16%	0.30%	0.67%	0.93%	1.12%	1.24%	1.30%	1.29%	1.21%	1.02%
$p_{Draw} = 0.25$											
	$\delta_{1.t} \to 0$	$\delta_{1.t} = 0.1$	$\delta_{1.t} = 0.2$	$\delta_{1.t} = 0.3$	$\delta_{1.t} = 0.4$	$\delta_{1.t} = 0.5$	$\delta_{1.t} = 0.6$	$\delta_{1.t} = 0.7$	$\delta_{1.t} = 0.8$	$\delta_{1.t} = 0.9$	$\delta_{1.t} \to 1$
σ_{low}	1.19%	1.17%	1.15%	1.12%	1.09%	1.06%	1.03%	0.99%	0.96%	0.92%	0.88%
σ_{middle}	1.17%	1.17%	1.17%	1.15%	1.13%	1.10%	1.07%	1.04%	1.00%	0.96%	0.90%
σ_{high}	0.78%	0.94%	1.04%	1.12%	1.16%	1.19%	1.19%	1.17%	1.14%	1.07%	0.97%
uniform	0.44%	0.74%	0.94%	1.08%	1.19%	1.25%	1.28%	1.28%	1.25%	1.17%	1.03%

Table 5-16: Incomplete information risk premia on asset 2 for as a function of the relative dividend contribution of asset 1 ($\delta_{1,t}$) and the dispersion of conditional transition probabilities ($\gamma = 3$, $\rho = 0.05$, correlation of dividend growth rates conditional on the regime = +80 percent). Negative risk premia are highlighted in red print.

Incomplete information risk premia of asset 2 ($\gamma = 0.5$)											
$p_{Draw} = 0.1$											
	$\delta_{1.t} \to 0$	$\delta_{1.t} = 0.1$	$\delta_{1.t} = 0.2$	$\delta_{1.t} = 0.3$	$\delta_{1.t} = 0.4$	$\delta_{1.t} = 0.5$	$\delta_{1.t} = 0.6$	$\delta_{1.t} = 0.7$	$\delta_{1.t} = 0.8$	$\delta_{1.t} = 0.9$	$\delta_{1.t} \to 1$
σ_{low}	0.19%	0.19%	0.18%	0.18%	0.17%	0.16%	0.16%	0.15%	0.15%	0.14%	0.13%
σ_{middle}	0.22%	0.21%	0.20%	0.20%	0.19%	0.18%	0.17%	0.16%	0.16%	0.15%	0.14%
σ_{high}	0.34%	0.33%	0.31%	0.30%	0.28%	0.26%	0.24%	0.22%	0.20%	0.18%	0.15%
uniform	0.44%	0.42%	0.41%	0.38%	0.36%	0.33%	0.30%	0.27%	0.24%	0.20%	0.15%
$p_{Draw} = 0.25$											
	$\delta_{1.t} \to 0$	$\delta_{1.t} = 0.1$	$\delta_{1.t} = 0.2$	$\delta_{1.t} = 0.3$	$\delta_{1.t} = 0.4$	$\delta_{1.t} = 0.5$	$\delta_{1.t} = 0.6$	$\delta_{1.t} = 0.7$	$\delta_{1.t} = 0.8$	$\delta_{1.t} = 0.9$	$\delta_{1.t} \to 1$
σ_{low}	0.19%	0.18%	0.18%	0.17%	0.17%	0.16%	0.16%	0.15%	0.14%	0.14%	0.13%
σ_{middle}	0.20%	0.20%	0.19%	0.19%	0.18%	0.17%	0.17%	0.16%	0.15%	0.14%	0.14%
σ_{high}	0.27%	0.26%	0.25%	0.24%	0.23%	0.22%	0.21%	0.19%	0.18%	0.16%	0.15%
uniform	0.33%	0.32%	0.30%	0.29%	0.27%	0.26%	0.24%	0.22%	0.20%	0.18%	0.16%

Table 5-17: Incomplete information risk premia on asset 2 for as a function of the relative dividend contribution of asset 1 ($\delta_{1,t}$) and the dispersion of conditional transition probabilities ($\gamma = 0.5$, $\rho = 0.05$, correlation of dividend growth rates conditional on the regime = +80 percent).

Asset 1

$\delta_{1,t} \to 1$

If $\delta_{1,t} \to 1$, there is essentially complete information on asset 1. It is clear that neither the correlation of dividend growth rates nor the regime parameters have any effect on the risk premium on asset 1 in this case.[50]

$\delta_{1,t} \to 0$

Under incomplete information, the regime parameters p_{Draw} and dispersion of conditional transition probabilities are highly relevant to the risk premia of asset 1 via the channel of stochastic regime probabilities. If $\delta_{1,t} \to 0$, complete information price dividend ratios of asset 1 are negatively related to expected dividend growth of asset 2 (recall that the stochastic discount factor is completely determined by asset 2). This yields a strong positive correlation of incomplete information price dividend ratio growth of asset 1 and the adjustment for risk (see Table 5-11 and Table 5-12). A low value of p_{Draw}, or a high dispersion of conditional regime probabilities both reinforce this positive correlation of the adjustment for risk with incomplete information price dividend ratio growth of asset 2 for reasons that are analogous to the case with a single risky asset (recall, however, that positive correlation exists for asset 1 even if $\gamma \leq 1$ in contrast to the single risky asset case).

The risk aversion parameter γ has conflicting effects on the risk premium of asset 1 because both the standard deviation of the adjustment for risk and incomplete information price dividend ratio growth of asset 1 are affected by γ and, typically, in different direction.

Incomplete information risk premia of asset 1 increase in the correlation of dividend growth rates conditional on the regime for a reason that is analogous to the case with asset 2 and $\delta_{1,t} \to 1$.

$0 \leq \delta_{1,t} \leq 1$

The reasoning for the effect of the correlation is identical to the case of asset 2.

[50] As in the case of asset 2, complete and incomplete information risk premia increase in the dispersion of conditional transition probabilities; this increase, however, is a mere artifact (see the discussion of asset 2).

Incomplete information risk premia of asset 1 ($\gamma = 3$)											
$p_{Draw} = 0.1$											
	$\delta_{1.t} = 0$	$\delta_{1.t} = 0.1$	$\delta_{1.t} = 0.2$	$\delta_{1.t} = 0.3$	$\delta_{1.t} = 0.4$	$\delta_{1.t} = 0.5$	$\delta_{1.t} = 0.6$	$\delta_{1.t} = 0.7$	$\delta_{1.t} = 0.8$	$\delta_{1.t} = 0.9$	$\delta_{1.t} = 1$
σ_{low}	0.67%	0.76%	0.84%	0.91%	0.97%	1.02%	1.07%	1.12%	1.16%	1.20%	1.24%
σ_{middle}	0.38%	0.55%	0.69%	0.81%	0.92%	1.01%	1.09%	1.16%	1.22%	1.28%	1.32%
σ_{high}	-1.87%	-0.89%	-0.38%	0.09%	0.44%	0.74%	0.98%	1.19%	1.35%	1.47%	1.54%
uniform	-3.65%	-1.98%	-1.27%	-0.48%	0.03%	0.51%	0.89%	1.20%	1.46%	1.64%	1.74%
$p_{Draw} = 0.25$											
	$\delta_{1,t} = 0$	$\delta_{1,t} = 0.1$	$\delta_{1,t} = 0.2$	$\delta_{1,t} = 0.3$	$\delta_{1,t} = 0.4$	$\delta_{1,t} = 0.5$	$\delta_{1,t} = 0.6$	$\delta_{1,t} = 0.7$	$\delta_{1,t} = 0.8$	$\delta_{1,t} = 0.9$	$\delta_{1,t} = 1$
σ_{low}	0.78%	0.84%	0.90%	0.95%	1.00%	1.04%	1.08%	1.13%	1.17%	1.20%	1.24%
σ_{middle}	0.65%	0.75%	0.84%	0.92%	0.99%	1.06%	1.12%	1.18%	1.23%	1.28%	1.32%
σ_{high}	-0.27%	0.09%	0.37%	0.61%	0.81%	0.99%	1.15%	1.28%	1.39%	1.48%	1.54%
uniform	-1.05%	-0.46%	-0.03%	0.34%	0.65%	0.93%	1.16%	1.36%	1.53%	1.66%	1.74%

Table 5-18: Incomplete information risk premia on asset 1 for as a function of the relative dividend contribution of asset 1 ($\delta_{1,t}$) and the dispersion of conditional transition probabilities ($\gamma = 3$, $\rho = 0.05$, correlation of dividend growth rates conditional on the regime = +80 percent). Negative risk premia are highlighted in red print.

Incomplete information risk premia of asset 1 ($\gamma = 0.5$)											
$p_{Draw} = 0.1$											
	$\delta_{1,t} = 0$	$\delta_{1,t} = 0.1$	$\delta_{1,t} = 0.2$	$\delta_{1,t} = 0.3$	$\delta_{1,t} = 0.4$	$\delta_{1,t} = 0.5$	$\delta_{1,t} = 0.6$	$\delta_{1,t} = 0.7$	$\delta_{1,t} = 0.8$	$\delta_{1,t} = 0.9$	$\delta_{1,t} = 1$
σ_{low}	0.13%	0.13%	0.14%	0.14%	0.15%	0.16%	0.16%	0.17%	0.17%	0.18%	0.19%
σ_{middle}	0.12%	0.13%	0.14%	0.15%	0.15%	0.16%	0.17%	0.18%	0.18%	0.19%	0.20%
σ_{high}	0.04%	0.08%	0.10%	0.13%	0.15%	0.17%	0.18%	0.20%	0.21%	0.22%	0.23%
uniform	-0.02%	0.04%	0.08%	0.11%	0.14%	0.17%	0.19%	0.21%	0.23%	0.25%	0.26%
$p_{Draw} = 0.25$											
	$\delta_{1.t} = 0$	$\delta_{1.t} = 0.1$	$\delta_{1.t} = 0.2$	$\delta_{1.t} = 0.3$	$\delta_{1.t} = 0.4$	$\delta_{1.t} = 0.5$	$\delta_{1.t} = 0.6$	$\delta_{1.t} = 0.7$	$\delta_{1.t} = 0.8$	$\delta_{1.t} = 0.9$	$\delta_{1.t} = 1$
σ_{low}	0.13%	0.14%	0.14%	0.15%	0.15%	0.16%	0.16%	0.17%	0.17%	0.18%	0.19%
σ_{middle}	0.13%	0.14%	0.14%	0.15%	0.16%	0.17%	0.17%	0.18%	0.18%	0.19%	0.20%
σ_{high}	0.11%	0.13%	0.14%	0.16%	0.17%	0.18%	0.19%	0.20%	0.21%	0.22%	0.23%
uniform	0.09%	0.12%	0.14%	0.16%	0.18%	0.19%	0.21%	0.22%	0.24%	0.25%	0.26%

Table 5-19: Incomplete information risk premia on asset 1 for as a function of the relative dividend contribution of asset 1 ($\delta_{1,t}$) and the dispersion of conditional transition probabilities ($\gamma = 0.5$, $\rho = 0.05$, correlation of dividend growth rates conditional on the regime = +80 percent). Negative risk premia are highlighted in red print.

5.5.3.2.3.2 Complete Information

The complete information risk premium in the case with two risky assets reads

$$RP_{i,t}^{cI,ret}(S_t, \delta_{1,t}) = -cov\left(AfR_{t,t+1}^{cI}(fe_{t+1}, S_t, \delta_t), [1 + d_i(S_t, fe_{t+1})] \cdot \left\{ \frac{\sum_{s'=1}^{K} p_{S_t,s'} \cdot \left(\frac{P}{D}\right)_{i,t+1}^{cI}(s', \delta_{1,t+1}) + 1}{\left(\frac{P}{D}\right)_{i,t}^{cI}(s', \delta_{1,t})} \right\} \middle| S_t, \delta_{1,t} \right)$$

In contrast to the case with a single risky asset, the term

$$\frac{\sum_{s'=1}^{K} p_{S_t,s'} \cdot \left(\frac{P}{D}\right)_{i,t+1}^{cI}(s', \delta_{1,t+1}) + 1}{\left(\frac{P}{D}\right)_{i,t}^{cI}(s', \delta_{1,t})}$$

can no longer be factored out of the covariance because the relative dividend contribution δ_{t+1} is stochastic and correlated with the adjustment for risk. As a consequence, complete information risk premia become dependent to some extent on both p_{Draw} and the dispersion of conditional regime probabilities. These regime parameters influence complete information price dividend ratios and, therefore, the correlation of the term

$$\frac{\sum_{s'=1}^{K} p_{S_t,s'} \cdot \left(\frac{P}{D}\right)_{i,t+1}^{cI} (s', \delta_{1,t+1}) + 1}{\left(\frac{P}{D}\right)_{i,t}^{cI} (s', \delta_{1,t})}$$

with the adjustment for risk.

However, the correlation induced by δ_{t+1} is small.

Similar to the single risky asset cases, the covariance can be approximated by

$$\begin{aligned} RP_{i,t}^{cI,ret}(S_t, \delta_{1,t}) &\approx \gamma \cdot stddev\left(\left[\delta_{1,t} \cdot \{1 + d_1(S_t, fe_{t+1})\} + \left(1 - \delta_{1,t}\right) \cdot \{1 + d_2(S_t, fe_{t+1})\}\right] \middle| S_t, \delta_{1,t}\right) \\ &\cdot stddev\left([1 + d_i(S_t, fe_{t+1})] \cdot \frac{\sum_{s'=1}^{K} p_{S_t,s'} \cdot \left(\frac{P}{D}\right)_{i,t+1}^{cI} (s', \delta_{1,t+1}) + 1}{\left(\frac{P}{D}\right)_{i,t}^{cI} (s', \delta_{1,t})} \middle| S_t, \delta_{1,t} \right) \\ &\cdot \left\{ -corr\left(\begin{array}{c} AfR_{t,t+1}^{cI}(fe_{t+1}, S_t, \delta_{1,t}) \\ , [1 + d_i(S_t, fe_{t+1})] \cdot \frac{\sum_{s'=1}^{K} p_{S_t,s'} \cdot \left(\frac{P}{D}\right)_{i,t+1}^{cI} (s', \delta_{1,t+1}) + 1}{\left(\frac{P}{D}\right)_{i,t}^{cI} (s', \delta_{1,t})} \end{array} \middle| S_t, \delta_{1,t} \right) \right\} \end{aligned}$$

Risk premia can be expresses as the product of three terms.

The standard deviation of the adjustment for risk (first term) is approximated by (see the cases with a single risky asset)

$$\gamma \cdot stddev\left(\left[\delta_{1,t} \cdot \{1 + d_1(S_t, fe_{t+1})\} + \left(1 - \delta_{1,t}\right) \cdot \{1 + d_2(S_t, fe_{t+1})\}\right] \middle| S_t, \delta_{1,t}\right)$$

and, hence, increases in γ. Moreover, the standard deviation in this product reads

$$stddev\left(\left[\delta_{1,t} \cdot \{1 + d_1(S_t, fe_{t+1})\} + \left(1 - \delta_{1,t}\right) \cdot \{1 + d_2(S_t, fe_{t+1})\}\right] \middle| S_t, \delta_{1,t}\right) =$$

$$\sqrt{\begin{array}{c} \delta_{1,t}{}^2 \cdot var(d_1(S_t, fe_{t+1})|S_t) + \left(1 - \delta_{1,t}\right)^2 \cdot var(d_2(S_t, fe_{t+1})|S_t) \\ +2 \cdot \delta_{1,t} \cdot \left(1 - \delta_{1,t}\right) \cdot stddev(d_1(S_t, fe_{t+1})|S_t) \cdot stddev(d_2(S_t, fe_{t+1})|S_t) \cdot corr_{d_1,d_2} \end{array}}.$$

where $corr(d_1(S_t, fe_{t+1}), d_2(S_t, fe_{t+1})|S_t = s) = corr_{d_1,d_2}$ is the correlation of dividend growth rates

This standard deviation and, hence, the standard deviation of the adjustment for risk increases in $corr_{d_1,d_2}$.

The standard deviation of asset returns (second term)

$$stddev\left([1 + d_i(S_t, fe_{t+1})] \cdot \frac{\sum_{s'=1}^{K} p_{S_t,s'} \cdot \left(\frac{P}{D}\right)_{i,t+1}^{cI} (s', \delta_{1,t+1}) + 1}{\left(\frac{P}{D}\right)_{i,t}^{cI} (s', \delta_{1,t})} \middle| S_t, \delta_{1,t} \right)$$

is mostly determined by the regime-dependent standard deviation of dividend growth and increases in $stdev\left([1 + d_i(S_t, fe_{t+1})] \middle| S_t, \delta_{1,t}\right)$. p_{Draw} and the dispersion of conditional transition probabilities only affect the fraction consisting of complete information price dividend ratios. Since they have influences on numerator and denominator that mainly offset each other, their overall effect is typically small.

In addition, the sign of (third term)

$$-corr\left(\begin{array}{c} AfR_{t,t+1}{}^{cI}\left(fe_{t+1},S_t,\delta_{1,t}\right) \\ ,[1+d_i(S_t,fe_{t+1})]\cdot\dfrac{\sum_{s'=1}^{K}p_{S_t,s'}\cdot\left(\frac{P}{D}\right)_{i,t+1}^{cI}\left(s',\delta_{1,t+1}\right)+1}{\left(\frac{P}{D}\right)_{i,t}^{cI}\left(s',\delta_{1,t}\right)}\end{array}\middle| S_t,\delta_{1,t}\right)$$

mostly depends on the correlation of $AfR_{t,t+1}{}^{cI}\left(fe_{t+1},S_t,\delta_{1,t}\right)$ and $1+d_i(S_t,fe_{t+1})$ which, in turn, depends on (i) the relative dividend contribution of asset 1 $\delta_{1,t}$ and (ii) the correlation of dividend growth rates of both assets. In contrast to the case with a single risky asset, the sign of the third term can be negative if dividend growth rates of both assets are negatively correlated and, in addition, asset i does not contribute to the adjustment for risk (for example, asset $i=2$ is considered at $\delta_{1,t}\to 1$).

Putting the insights on the three terms together, an increase in γ or the regime-dependent standard deviation of dividend growth will increase a positive risk premium, and decrease a negative risk premium. Whether the risk premium is positive or negative, in turn, depends on the combination of the correlation of dividend growth rates and the relative dividend contribution $\delta_{1,t}$.

6 Conclusion

Starting point of our analysis was the observation that decision makers do not perfectly know the stochastic process of financial figures, in particular corporate cash flows and that this incomplete information is modeled with the help of an unobservable underlying regime model.

Veronesi (2000) has shown that incomplete information leads to an unintuitive asset pricing outcome: the incomplete information risk premium (two sources of risk) is below its complete information counterpart (one source of risk) for typical values of risk aversion parameters. This result of Veronesi (2000) has been derived in a narrow model framework: CRRA utility, mere expectation regimes, and only one asset.

This thesis found:

(i) Incomplete information exerts a massive influence on risk premia for all models considered in this thesis - CARA and CRRA utility functions, richer class of regime processes, various forms of cash flow model, and more than one risky asset - as the analytical analyses demonstrate. Core of all pricing approaches is the covariance between stochastic discount factor and asset return. Incomplete information fundamentally alters this covariance.

(ii) The numerical analyses illustrate that the theoretical pricing results are also relevant from an economic point of view: incomplete information risk premia are significantly different from complete information risk premia and the different model versions also translate into significantly different risk premia.

In other words, this thesis demonstrates that Veronesi (2000) results are rather robust, i.e., hold in a much broader model framework.

A1 Appendix to Section 2.3.4.1 and 2.3.4.2: Recursions for Conditional Regime Probabilities

A1.1 Information Frequency = Cash Flow Frequency

A1.1.1 Formulation of the Problem

The problem to be solved in this section of the appendix is the determination of conditional regime probabilities for all points of time conditional on the information I_t available to each of the individual investors:

$$\pi_{\nu,t} \equiv P(S_t = \nu | I_t)$$
$$\nu = 1, \dots, K$$

where S_t denotes the regime at time t and K is the number of possible regimes. Information includes the history of past cash flows and, optionally, a history of past signals.

(a) <u>Case with signals</u>

The most general signal model used in this thesis reads

A 1-1

$$Sig_{t+1} = Sig(S_t, S_{t+1}, fe_{t+1}, \eta_{t+1})$$

where η_{t+1} is a vector of i.i.d. "white noise". Information available to investors comes from observing

$$\overrightarrow{\widehat{D}_t}, \overrightarrow{\widehat{Sig}_t}$$

where $\overrightarrow{D_t}$ and $\overrightarrow{Sig_t}$ are the histories of dividends and signals up to time t

$$\overrightarrow{\widehat{D}_t} \equiv \left(\widehat{D}_0, \dots, \widehat{D}_t\right)$$
$$\overrightarrow{\widehat{Sig}_t} \equiv \left(\widehat{Sig}_1, \dots, \widehat{Sig}_t\right)$$

(throughout, I denote a realization of some random variable X_t by $\widehat{X}_t$).

(b) <u>Case without signals (Section 2.3.4.1)</u>

The case without signals can be interpreted as a special case of signal model (A 1-1). For example, signals Sig_t can be set to a constant. Evidently, such signals reveal no information and are therefore equivalent to a situation without any signals. Information available to investors comes from observing

$$\overrightarrow{\widehat{D}_t}$$

A1.1.2 Results

A1.1.2.1 Case with Signals

Define conditional regime probabilities at times t and $t+1$ by π_t and π_{t+1}, respectively:

$$\pi_t \equiv \begin{pmatrix} P\left(S_t = 1 \middle| \overrightarrow{\widehat{D}_t}, \overrightarrow{\widehat{Sig}_t}\right) \\ \dots \\ P\left(S_t = K \middle| \overrightarrow{\widehat{D}_t}, \overrightarrow{\widehat{Sig}_t}\right) \end{pmatrix}$$

Similar to Hamilton (1994), p. 693, conditional regime probabilities at time $t+1$ are recursively obtained from conditional regime probabilities at time t via

A 1-2

$$P\left(S_{t+1} = \hat{S}_{t+1} \middle| \overrightarrow{\widehat{D}_{t+1}}, \overrightarrow{\widehat{Sig}_{t+1}}\right)$$

$$= \frac{\sum_{\hat{S}_t=1}^{K} \left\{ \begin{array}{c} P\left(D_{t+1} = \widehat{D}_{t+1}, Sig_{t+1} = \widehat{Sig}_{t+1} \middle| S_{t+1} = \hat{S}_{t+1}, S_t = \hat{S}_t, D_t = \widehat{D}_t\right) \\ \cdot p_{\hat{S}_t, \hat{S}_{t+1}} \cdot P\left(S_t = \hat{S}_t \middle| \overrightarrow{\widehat{D}_t}, \overrightarrow{\widehat{Sig}_t}\right) \end{array} \right\}}{\sum_{s_{+1}=1}^{K} \sum_{s=1}^{K} \left\{ \begin{array}{c} P\left(D_{t+1} = \widehat{D}_{t+1}, Sig_{t+1} = \widehat{Sig}_{t+1} \middle| S_{t+1} = s_{+1}, S_t = s, D_t = \widehat{D}_t\right) \\ \cdot p_{s, s_{+1}} \cdot P\left(S_t = s \middle| \overrightarrow{\widehat{D}_t}, \overrightarrow{\widehat{Sig}_t}\right) \end{array} \right\}}$$

$$\hat{S}_{t+1} = 1, \dots, K$$

This recursion is of the abstract form

A 1-3

$$\pi_{t+1} = \Pi(\pi_t, D_t, D_{t+1}, Sig_{t+1})$$

where $\Pi(.)$ is the functional from defined on the right-hand side of (A 1-2).

Observe that the term (first term in the sum (A 1-2))

A 1-4

$$p\left(\widehat{D}_{t+1}, \widehat{Sig}_{t+1}; \hat{S}_{t+1}, \hat{S}_t, \widehat{D}_t\right) \equiv P\left(D_{t+1} = \widehat{D}_{t+1}, Sig_{t+1} = \widehat{Sig}_{t+1} \middle| S_{t+1} = \hat{S}_{t+1}, S_t = \hat{S}_t, D_t = \widehat{D}_t\right)$$

possesses the function of adding new information $\widehat{D}_{t+1}$ and $\widehat{Sig}_{t+1}$ to conditional regime probabilities.

A1.1.2.2 Case without Signals

If there are no signals (and general cash flow models), (A 1-4) simplifies as follows:

A 1-5

$$p\left(\widehat{D}_{t+1}, \widehat{Sig}_{t+1}; \hat{S}_{t+1}, \hat{S}_t, \widehat{D}_t\right) = p\left(\widehat{D}_{t+1}; \hat{S}_t, \widehat{D}_t\right) = P\left(D_{t+1} = \widehat{D}_{t+1} \middle| S_t = \hat{S}_t, D_t = \widehat{D}_t\right)$$

A1.1.3 Proof

A1.1.3.1 Case with Signals

Step 1: Use the definition of conditional probability to separate new information $\widehat{D}_{t+1}$ and $\widehat{Sig}_{t+1}$ from information $\overrightarrow{\widehat{D}_t}, \overrightarrow{\widehat{Sig}_t}$ that is already included in regime probabilities for time t

$$P\left(S_{t+1} = \hat{S}_{t+1} \middle| \overrightarrow{\widehat{D}_{t+1}}, \overrightarrow{\widehat{Sig}_{t+1}}\right) = \frac{P\left(S_{t+1} = \hat{S}_{t+1}, D_{t+1} = \widehat{D}_{t+1}, Sig_{t+1} = \widehat{Sig}_{t+1} \middle| \overrightarrow{\widehat{D}_t}, \overrightarrow{\widehat{Sig}_t}\right)}{P\left(D_{t+1} = \widehat{D}_{t+1}, Sig_{t+1} = \widehat{Sig}_{t+1} \middle| \overrightarrow{\widehat{D}_t}, \overrightarrow{\widehat{Sig}_t}\right)}$$

Step 2: Since all information comes from $\widehat{D}_{t+1}$ and $\widehat{Sig}_{t+1}$ which both depend on the regime S_t, this dependence is made explicit by rewriting the numerator as a sum over all possible values of S_t

$$P\left(S_{t+1} = \hat{S}_{t+1}, D_{t+1} = \widehat{D}_{t+1}, Sig_{t+1} = \widehat{Sig}_{t+1} \middle| \overrightarrow{\widehat{D}_t}, \overrightarrow{\widehat{Sig}_t}\right)$$
$$= \sum_{\hat{S}_t=1}^{K} P\left(S_{t+1} = \hat{S}_{t+1}, S_t = \hat{S}_t, D_{t+1} = \widehat{D}_{t+1}, Sig_{t+1} = \widehat{Sig}_{t+1} \middle| \overrightarrow{\widehat{D}_t}, \overrightarrow{\widehat{Sig}_t}\right)$$

Step 3: Rewrite the numerator in a Bayesian fashion to separate regime probabilities at time t ("prior probability") from the probability of new information given regimes ("likelihood")

$$P\left(S_{t+1} = \hat{S}_{t+1}, D_{t+1} = \widehat{D}_{t+1}, Sig_{t+1} = \widehat{Sig}_{t+1} \middle| \overrightarrow{\widehat{D}_t}, \overrightarrow{\widehat{Sig}_t}\right)$$
$$= \sum_{\hat{S}_t=1}^{K} P\left(S_{t+1} = \hat{S}_{t+1}, S_t = \hat{S}_t, D_{t+1} = \widehat{D}_{t+1}, Sig_{t+1} = \widehat{Sig}_{t+1} \middle| \overrightarrow{\widehat{D}_t}, \overrightarrow{\widehat{Sig}_t}\right)$$
$$= \sum_{\hat{S}_t=1}^{K} P\left(D_{t+1} = \widehat{D}_{t+1}, Sig_{t+1} = \widehat{Sig}_{t+1} \middle| S_{t+1} = \hat{S}_{t+1}, S_t = \hat{S}_t, \overrightarrow{\widehat{D}_t}, \overrightarrow{\widehat{Sig}_t}\right)$$
$$\cdot P\left(S_{t+1} = \hat{S}_{t+1}, S_t = \hat{S}_t \middle| \overrightarrow{\widehat{D}_t}, \overrightarrow{\widehat{Sig}_t}\right)$$

The cash flow model together with the assumptions on the processes of regimes, factors and residuals, and signal noise imply that the conditional distribution of D_{t+1} and Sig_{t+1} only depend on the regimes S_t and S_{t+1} and cash flows D_t so that the first term in the sum can be considerably simplified to

$$P\left(D_{t+1} = \widehat{D}_{t+1}, Sig_{t+1} = \widehat{Sig}_{t+1} \middle| S_{t+1} = \hat{S}_{t+1}, S_t = \hat{S}_t, \overrightarrow{\widehat{D}_t}, \overrightarrow{\widehat{Sig}_t}\right)$$
$$= P\left(D_{t+1} = \widehat{D}_{t+1}, Sig_{t+1} = \widehat{Sig}_{t+1} \middle| S_{t+1} = \hat{S}_{t+1}, S_t = \hat{S}_t, D_t = \widehat{D}_t\right)$$

In addition, the Markov property and the assumption that the process of regime is independent of the processes of factors and residuals as well as signal noise together imply for the second term in the sum:

$$P\left(S_{t+1}=\hat{S}_{t+1},S_t=\hat{S}_t\middle|\overrightarrow{\widehat{D}_t},\overrightarrow{\widehat{Sig}_t}\right)=p_{\hat{S}_t,\hat{S}_{t+1}}\cdot P\left(S_t=\hat{S}_t\middle|\overrightarrow{\widehat{D}_t},\overrightarrow{\widehat{Sig}_t}\right)$$

where $p_{\hat{S}_t,\hat{S}_{t+1}}$ is the probability of a transition from regime $\hat{S}_t$ to regime $\hat{S}_{t+1}$.

Combining the Steps 1 to 3 yields the intermediate result:

A 1-6

$$P\left(S_{t+1}=\hat{S}_{t+1}\middle|\overrightarrow{\widehat{D}_{t+1}},\overrightarrow{\widehat{Sig}_{t+1}}\right)=\frac{\sum_{\hat{S}_t=1}^{K}\left\{\begin{matrix}P\left(D_{t+1}=\widehat{D}_{t+1},Sig_{t+1}=\widehat{Sig}_{t+1}\middle|S_{t+1}=\hat{S}_{t+1},S_t=\hat{S}_t,D_t=\widehat{D}_t\right)\\ \cdot p_{\hat{S}_t,\hat{S}_{t+1}}\cdot P\left(S_t=\hat{S}_t\middle|\overrightarrow{\widehat{D}_t},\overrightarrow{\widehat{Sig}_t}\right)\end{matrix}\right\}}{P\left(D_{t+1}=\widehat{D}_{t+1},Sig_{t+1}=\widehat{Sig}_{t+1}\middle|\overrightarrow{\widehat{D}_t},\overrightarrow{\widehat{Sig}_t}\right)}$$

Step 4: Determine the denominator in (A 1-6) from the condition that conditional regime probabilities at time $t+1$ must add to one

$$P\left(D_{t+1}=\widehat{D}_{t+1},Sig_{t+1}=\widehat{Sig}_{t+1}\middle|\overrightarrow{\widehat{D}_t},\overrightarrow{\widehat{Sig}_t}\right)=\sum_{s_{+1}=1}^{K}\sum_{s=1}^{K}P\left(D_{t+1}=\widehat{D}_{t+1},Sig_{t+1}=\widehat{Sig}_{t+1}\middle|S_{t+1}=s_{+1},S_t=s,D_t=\widehat{D}_t\right)\cdot p_{s,s_{+1}}\cdot P\left(S_t=s\middle|\overrightarrow{\widehat{D}_t},\overrightarrow{\widehat{Sig}_t}\right)$$

Step 5: Final result

Plugging Step 4 into (A 1-6) yields the desired recursion between conditional regime probabilities (A 1-2):

$$P\left(S_{t+1}=\hat{S}_{t+1}\middle|\overrightarrow{\widehat{D}_{t+1}},\overrightarrow{\widehat{Sig}_{t+1}}\right)=\frac{\sum_{\hat{S}_t=1}^{K}\left\{\begin{matrix}P\left(D_{t+1}=\widehat{D}_{t+1},Sig_{t+1}=\widehat{Sig}_{t+1}\middle|S_{t+1}=\hat{S}_{t+1},S_t=\hat{S}_t,D_t=\widehat{D}_t\right)\\ \cdot p_{\hat{S}_t,\hat{S}_{t+1}}\cdot P\left(S_t=\hat{S}_t\middle|\overrightarrow{\widehat{D}_t},\overrightarrow{\widehat{Sig}_t}\right)\end{matrix}\right\}}{\sum_{s_{+1}=1}^{K}\sum_{s=1}^{K}\left\{\begin{matrix}P\left(D_{t+1}=\widehat{D}_{t+1},Sig_{t+1}=\widehat{Sig}_{t+1}\middle|S_{t+1}=s_{+1},S_t=s,D_t=\widehat{D}_t\right)\\ \cdot p_{s,s_{+1}}\cdot P\left(S_t=s\middle|\overrightarrow{\widehat{D}_t},\overrightarrow{\widehat{Sig}_t}\right)\end{matrix}\right\}}$$

which can be abbreviated as in (A 1-3):

$$\pi_{t+1}=\Pi(\pi_t,D_t,D_{t+1},Sig_{t+1})$$

A1.1.3.2 Case without Signals

If there are no signals, (A 1-4)

$$p\left(\widehat{D}_{t+1},\widehat{Sig}_{t+1};\hat{S}_{t+1},\hat{S}_t,\widehat{D}_t\right)\equiv P\left(D_{t+1}=\widehat{D}_{t+1},Sig_{t+1}=\widehat{Sig}_{t+1}\middle|S_{t+1}=\hat{S}_{t+1},S_t=\hat{S}_t,D_t=\widehat{D}_t\right)$$

simplifies to

$$p\left(\widehat{D}_{t+1},\widehat{Sig}_{t+1};\hat{S}_{t+1},\hat{S}_t,\widehat{D}_t\right)=P\left(D_{t+1}=\widehat{D}_{t+1}\middle|S_t=\hat{S}_t,D_t=\widehat{D}_t\right)\equiv p\left(\widehat{D}_{t+1};\hat{S}_t,\widehat{D}_t\right)$$

Observe that the regime $\hat{S}_{t+1}$ drops out of the conditioning information on the right-hand side of this previous equation because cash flows D_{t+1} do only functionally depend on the regime S_t: $D_{t+1} = D(D_t, S_t, fe_{t+1})$.

A1.2 Appendix to Section 2.3.4.2: Information Frequency ≥ Cash Flow Frequency

A1.2.1 Formulation of the Problem

The problem to be solved in this section of the appendix is the determination of conditional regime path probabilities for all points of time conditional on the information I_t available to each of the individual investors:

$$\pi_{t_{(k_t)},t}\left(\hat{S}_{t_{(k_t)},t}\right) \equiv P\left(S_{t_{(k_t)},t} = \hat{S}_{t_{(k_t)},t} \middle| I_t\right)$$

where $S_{t_{(k_t)},t}$ is the path of regimes from the most recent payment date of (Δ_C)-periodic cash flows $t_{(k_t)}$ to the current point of time t. Information includes the history of past cash flows and a history of past signals.

Signals are of the form

$$Sig_{t+1} = Sig_{t+1}\left(S_{t_{(k)},t}, S_{t+1}, fe_{t+1}, \eta_{t+1}\right)$$

where η_{t+1} is a vector of i.i.d. “white noise”. Information available to investors comes from observing

$$\overrightarrow{\hat{D}_{t_{(k_t)}}^{(\Delta_C)}}, \overrightarrow{\hat{D}_t^{(1)}}, \overrightarrow{\widehat{Sig}_t}$$

where $\overrightarrow{\hat{D}_{t_{(k_t)}}^{(\Delta_C)}}$, $\overrightarrow{\hat{D}_t^{(1)}}$ and $\overrightarrow{\widehat{Sig}_t}$ are the histories of (Δ_C)-periodic and (1)-periodic cash flows and of signals up to time t:

$$\overrightarrow{\hat{D}_{t_{(k_t)}}^{(\Delta_C)}} \equiv \left(\hat{D}_{t_{(0)}}^{(\Delta_C)}, \hat{D}_{t_{(1)}}^{(\Delta_C)}, \dots, \hat{D}_{t_{(k_t)}}^{(\Delta_C)}\right)$$

$$\overrightarrow{\hat{D}_t^{(1)}} \equiv \left(\hat{D}_0^{(1)}, \dots, \hat{D}_t^{(1)}\right)$$

$$\overrightarrow{\widehat{Sig}_t} \equiv \left(\widehat{Sig}_1, \dots, \widehat{Sig}_t\right)$$

A1.2.2 Results

Conditional regime path probabilities at time $t+1$, $t_{(k+1)} \geq t+1 > t_{(k)}$ are obtained by the following system of equations:

<u>Case 1</u>: No (Δ_C)-periodic cash flows at the next point of time $t+1$: $t_{(k+1)} > t+1 > t_{(k)}$

A 1-7

$$P\left(\hat{S}_{t_{(k_t)},t+1}\middle|\overrightarrow{\widehat{D}_{t_{(k_t)}}^{(\Delta_C)}},\overrightarrow{\widehat{D}_t^{(1)}},\overrightarrow{\widehat{Sig}_t}\right)$$
$$=\frac{P\left(\widehat{D}_{t+1}^{(1)},\widehat{Sig}_{t+1}\middle|\hat{S}_{t_{(k_t)},t+1},\widehat{D}_t^{(1)}\right)\cdot p_{\hat{S}_t,\hat{S}_{t+1}}\cdot P\left(\hat{S}_{t_{(k_t)},t}\middle|\overrightarrow{\widehat{D}_{t_{(k_t)}}^{(\Delta_C)}},\overrightarrow{\widehat{D}_t^{(1)}},\overrightarrow{\widehat{Sig}_t}\right)}{\sum_{\bar{\bar{S}}_{t_{(k)},t+1}}P\left(\widehat{D}_{t+1}^{(1)},\widehat{Sig}_{t+1}\middle|\bar{\bar{S}}_{t_{(k)},t+1},\widehat{D}_t^{(1)}\right)\cdot p_{\bar{\bar{S}}_t,\bar{\bar{S}}_{t+1}}\cdot P\left(\bar{\bar{S}}_{t_{(k)},t}\middle|\overrightarrow{\widehat{D}_{t_{(k_t)}}^{(\Delta_C)}},\overrightarrow{\widehat{D}_t^{(1)}},\overrightarrow{\widehat{Sig}_t}\right)}$$

<u>Case 2</u>: No (Δ_C)-periodic cash flows at the next point of time $t+1=t_{(k+1)}$

A 1-8

$$P\left(\hat{S}_{t_{(k+1)}}\middle|\overrightarrow{\widehat{D}_{t_{(k+1)}}^{(\Delta_C)}},\overrightarrow{\widehat{D}_{t_{(k+1)}}^{(1)}},\overrightarrow{\widehat{Sig}_{t_{(k+1)}}}\right)$$
$$=\frac{\sum_{s_{t_{(k)},t}}\left\{\begin{array}{c}P\left(\widehat{D}_{t_{(k+1)}}^{(\Delta_C)},\widehat{D}_{t_{(k+1)}}^{(1)},\widehat{Sig}_{t_{(k+1)}}\middle|\hat{S}_{t_{(k+1)}},S_{t_{(k)},t}=s_{t_{(k)},t},\widehat{D}_{t_{(k)}}^{(\Delta_C)},\widehat{D}_t^{(1)}\right)\\ \cdot p_{\hat{S}_{t_{(k+1)}-1},\hat{S}_{t_{(k+1)}}}\cdot P\left(S_{t_{(k)},t}=s_{t_{(k)},t}\middle|\overrightarrow{\widehat{D}_{t_{(k)}}^{(\Delta_C)}},\overrightarrow{\widehat{D}_t^{(1)}},\overrightarrow{\widehat{Sig}_t}\right)\end{array}\right\}}{\sum_{s_{t_{(k+1)}}=1}^{K}\sum_{s_{t_{(k)},t}}\left\{\begin{array}{c}P\left(\widehat{D}_{t_{(k+1)}}^{(\Delta_C)},\widehat{D}_{t_{(k+1)}}^{(1)},\widehat{Sig}_{t_{(k+1)}}\middle|\begin{array}{c}S_{t_{(k+1)}}=s_{t_{(k+1)}},S_{t_{(k)},t}=s_{t_{(k)},t}\\ ,\widehat{D}_{t_{(k)}}^{(\Delta_C)},\widehat{D}_t^{(1)}\end{array}\right)\\ \cdot p_{\hat{S}_{t_{(k+1)}-1},\hat{S}_{t_{(k+1)}}}\cdot P\left(S_{t_{(k)},t}=s_{t_{(k)},t}\middle|\overrightarrow{\widehat{D}_{t_{(k)}}^{(\Delta_C)}},\overrightarrow{\widehat{D}_t^{(1)}},\overrightarrow{\widehat{Sig}_t}\right)\end{array}\right\}}$$

Finally, an abbreviated form that comprises both cases, the recursion of conditional regime path probabilities reads:

A 1-9

$$\pi_{t_{(k)},t+1}=\begin{cases}\Pi_0\left(\pi_{t_{(k)},t},D_t^{(1)},D_{t+1}^{(1)},Sig_{t+1}\right) & t+1<t_{(k+1)}\\ \Pi_1\left(\begin{array}{c}\pi_{t_{(k)},t},D_{t_{(k)}}^{(\Delta_C)},D_{,t}^{(1)}\\ ,D_{t_{(k+1)}}^{(\Delta_C)},D_{t_{(k+1)}}^{(1)},Sig_{t_{(k+1)}}\end{array}\right) & t+1=t_{(k+1)}\end{cases}$$

A1.2.3 Proof

A1.2.3.1 Case 1: No (Δ_C)-periodic cash flows at the next point of time $t+1$: $t_{(k+1)} > t+1$

Step 1: Use the definition of conditional probability to separate new information $\widehat{D}_{t+1}{}^{(1)}$ and $\widehat{Sig}_{t+1}$ from information $\overrightarrow{\widehat{D}_{t_{(k_t)}}{}^{(\Delta_C)}}, \overrightarrow{\widehat{D}_t{}^{(1)}}, \overrightarrow{\widehat{Sig}_t}$ that is already included in regime path probabilities for time t

$$P\left(\hat{S}_{t_{(k_t)},t+1} \middle| \overrightarrow{\widehat{D}_{t_{(k_t)}}{}^{(\Delta_C)}}, \overrightarrow{\widehat{D}_{t+1}{}^{(1)}}, \overrightarrow{\widehat{Sig}_{t+1}}\right) = \frac{P\left(\hat{S}_{t_{(k_t)},t+1}, \widehat{D}_{t+1}{}^{(1)}, \widehat{Sig}_{t+1} \middle| \overrightarrow{\widehat{D}_{t_{(k_t)}}{}^{(\Delta_C)}}, \overrightarrow{\widehat{D}_t{}^{(1)}}, \overrightarrow{\widehat{Sig}_t}\right)}{P\left(\widehat{D}_{t+1}{}^{(1)}, \widehat{Sig}_{t+1} \middle| \overrightarrow{\widehat{D}_{t_{(k_t)}}{}^{(\Delta_C)}}, \overrightarrow{\widehat{D}_t{}^{(1)}}, \overrightarrow{\widehat{Sig}_t}\right)}$$

Step 2: Rewrite the numerator in a Bayesian fashion to separate regime path probabilities at time t ("prior probability") from the probability of new information given regime paths ("likelihood")

$$P\left(\hat{S}_{t_{(k_t)},t+1}, \widehat{D}_{t+1}{}^{(1)}, \widehat{Sig}_{t+1} \middle| \overrightarrow{\widehat{D}_{t_{(k_t)}}{}^{(\Delta_C)}}, \overrightarrow{\widehat{D}_t{}^{(1)}}, \overrightarrow{\widehat{Sig}_t}\right) =$$

$$= P\left(\widehat{D}_{t+1}{}^{(1)}, \widehat{Sig}_{t+1} \middle| \hat{S}_{t_{(k_t)},t+1}, \overrightarrow{\widehat{D}_{t_{(k_t)}}{}^{(\Delta_C)}}, \overrightarrow{\widehat{D}_t{}^{(1)}}, \overrightarrow{\widehat{Sig}_t}\right)$$

$$\cdot P\left(\hat{S}_{t_{(k_t)},t+1} \middle| \overrightarrow{\widehat{D}_{t_{(k_t)}}{}^{(\Delta_C)}}, \overrightarrow{\widehat{D}_t{}^{(1)}}, \overrightarrow{\widehat{Sig}_t}\right)$$

The cash flow model together with the assumptions on the processes of regimes, factors and residuals, and signal noise imply that the conditional distribution of D_{t+1} and Sig_{t+1} only depend on the regimes S_t and S_{t+1} and cash flows D_t. Then the first term of the above product simplifies to

$$P\left(\widehat{D}_{t+1}{}^{(1)}, \widehat{Sig}_{t+1} \middle| \hat{S}_{t_{(k_t)},t+1}, \overrightarrow{\widehat{D}_{t_{(k_t)}}{}^{(\Delta_C)}}, \overrightarrow{\widehat{D}_t{}^{(1)}}, \overrightarrow{\widehat{Sig}_t}\right) = P\left(\widehat{D}_{t+1}{}^{(1)}, \widehat{Sig}_{t+1} \middle| \hat{S}_{t_{(k_t)},t+1}, \widehat{D}_t{}^{(1)}\right)$$

In addition, the Markov property and the assumption that the process of regime is independent of the processes of factors and residuals as well as signal noise together imply for the second term in the product:

$$P\left(\hat{S}_{t_{(k_t)},t+1}\middle|\overrightarrow{\widehat{D}_{t_{(k_t)}}^{(\Delta_C)}},\overrightarrow{\widehat{D}_t^{(1)}},\overrightarrow{\widehat{Sig}_t}\right)$$
$$=P\left(\hat{S}_{t+1}\middle|\hat{S}_{t_{(k_t)},t},\overrightarrow{\widehat{D}_{t_{(k_t)}}^{(\Delta_C)}},\overrightarrow{\widehat{D}_t^{(1)}},\overrightarrow{\widehat{Sig}_t}\right)\cdot P\left(\hat{S}_{t_{(k_t)},t}\middle|\overrightarrow{\widehat{D}_{t_{(k_t)}}^{(\Delta_C)}},\overrightarrow{\widehat{D}_t^{(1)}},\overrightarrow{\widehat{Sig}_t}\right)$$
$$=P\left(\hat{S}_{t+1}\middle|\hat{S}_t\right)\cdot P\left(\hat{S}_{t_{(k_t)},t}\middle|\overrightarrow{\widehat{D}_{t_{(k_t)}}^{(\Delta_C)}},\overrightarrow{\widehat{D}_t^{(1)}},\overrightarrow{\widehat{Sig}_t}\right)$$
$$=p_{\hat{S}_t,\hat{S}_{t+1}}\cdot P\left(\hat{S}_{t_{(k_t)},t}\middle|\overrightarrow{\widehat{D}_{t_{(k_t)}}^{(\Delta_C)}},\overrightarrow{\widehat{D}_t^{(1)}},\overrightarrow{\widehat{Sig}_t}\right)$$

where $p_{\hat{S}_t,\hat{S}_{t+1}}$ is the probability of a transition from regime $\hat{S}_t$ to regime $\hat{S}_{t+1}$.

Combining the Steps 1 to 2 yields the intermediate result:

A 1-10

$$P\left(\hat{S}_{t_{(k_t)},t+1}\middle|\overrightarrow{\widehat{D}_{t_{(k_t)}}^{(\Delta_C)}},\overrightarrow{\widehat{D}_{t+1}^{(1)}},\overrightarrow{\widehat{Sig}_{t+1}}\right)$$
$$=\frac{P\left(\widehat{D}_{t+1}^{(1)},\widehat{Sig}_{t+1}\middle|\hat{S}_{t_{(k_t)},t+1},\widehat{D}_t^{(1)}\right)\cdot p_{\hat{S}_t,\hat{S}_{t+1}}\cdot P\left(\hat{S}_{t_{(k_t)},t}\middle|\overrightarrow{\widehat{D}_{t_{(k_t)}}^{(\Delta_C)}},\overrightarrow{\widehat{D}_t^{(1)}},\overrightarrow{\widehat{Sig}_t}\right)}{P\left(\widehat{D}_{t+1}^{(1)},\widehat{Sig}_{t+1}\middle|\overrightarrow{\widehat{D}_{t_{(k_t)}}^{(\Delta_C)}},\overrightarrow{\widehat{D}_t^{(1)}},\overrightarrow{\widehat{Sig}_t}\right)}$$

<u>Step 3</u>: Determine the denominator in (A 1-10) from the condition that conditional regime path probabilities at time $t+1$ must add to one

$$P\left(\widehat{D}_{t+1}^{(1)},\widehat{Sig}_{t+1}\middle|\overrightarrow{\widehat{D}_{t_{(k_t)}}^{(\Delta_C)}},\overrightarrow{\widehat{D}_t^{(1)}},\overrightarrow{\widehat{Sig}_t}\right)$$
$$=\sum_{\hat{S}_{t_{(k)},t+1}}P\left(\widehat{D}_{t+1}^{(1)},\widehat{Sig}_{t+1}\middle|\hat{S}_{t_{(k_t)},t+1},\widehat{D}_t^{(1)}\right)\cdot p_{\hat{S}_t,\hat{S}_{t+1}}$$
$$\cdot P\left(\hat{S}_{t_{(k_t)},t}\middle|\overrightarrow{\widehat{D}_{t_{(k_t)}}^{(\Delta_C)}},\overrightarrow{\widehat{D}_t^{(1)}},\overrightarrow{\widehat{Sig}_t}\right)$$

<u>Step 4</u>: Final result

Plugging Step 3 into (A 1-10) yields the desired recursion between conditional regime path probabilities (A 1-7):

A 1-11

$$P\left(\hat{S}_{t_{(k_t)},t+1}\middle|\overrightarrow{\widehat{D}_{t_{(k_t)}}^{(\Delta_C)}},\overrightarrow{\widehat{D}_t^{(1)}},\overrightarrow{\widehat{Sig}_t}\right)$$
$$=\frac{P\left(\widehat{D}_{t+1}^{(1)},\widehat{Sig}_{t+1}\middle|\hat{S}_{t_{(k_t)},t+1},\widehat{D}_t^{(1)}\right)\cdot p_{\hat{S}_t,\hat{S}_{t+1}}\cdot P\left(\hat{S}_{t_{(k_t)},t}\middle|\overrightarrow{\widehat{D}_{t_{(k_t)}}^{(\Delta_C)}},\overrightarrow{\widehat{D}_t^{(1)}},\overrightarrow{\widehat{Sig}_t}\right)}{\sum_{\bar{\bar{S}}_{t_{(k)},t+1}}P\left(\widehat{D}_{t+1}^{(1)},\widehat{Sig}_{t+1}\middle|\bar{\bar{S}}_{t_{(k)},t+1},\widehat{D}_t^{(1)}\right)\cdot p_{\bar{\bar{S}}_t,\bar{\bar{S}}_{t+1}}\cdot P\left(\bar{\bar{S}}_{t_{(k)},t}\middle|\overrightarrow{\widehat{D}_{t_{(k_t)}}^{(\Delta_C)}},\overrightarrow{\widehat{D}_t^{(1)}},\overrightarrow{\widehat{Sig}_t}\right)}$$

A1.2.3.2 Case 2: (Δ_C)-periodic cash flows at the next point of time $t+1=t_{(k+1)}$

If $t+1=t_{(k+1)}$, then the path of regimes at time $t+1$ consists of the single regime $S_{t_{(k+1)}}$. The conditional probability for this degenerate regime path can be obtained recursively from the conditional probabilities for the regime path $\mathrm{S}_{t_{(\mathrm{k})},t_{(\mathrm{k+1})}-1}$ as follows:

Step 1: Use the definition of conditional probability to separate new information $\widehat{D}_{t_{(\mathrm{k+1})}}{}^{(\Delta_C)}, \widehat{D}_{t_{(\mathrm{k+1})}}{}^{(1)}$ and $\widehat{Sig}_{t_{(\mathrm{k+1})}}$ from information $\overrightarrow{\widehat{D}_{t_{(\mathrm{k})}}{}^{(\Delta_C)}}, \overrightarrow{\widehat{D}_{\mathrm{t}}{}^{(1)}}, \overrightarrow{\widehat{Sig}_t}$ that is already included in regime path probabilities for time t

$$P\left(\hat{S}_{t_{(\mathrm{k+1})}} \middle| \overrightarrow{\widehat{D}_{t_{(\mathrm{k+1})}}{}^{(\Delta_C)}}, \overrightarrow{\widehat{D}_{t_{(\mathrm{k+1})}}{}^{(1)}}, \overrightarrow{\widehat{Sig}_{t_{(\mathrm{k+1})}}}\right)$$
$$= \frac{P\left(\hat{S}_{t_{(\mathrm{k+1})}}, \widehat{D}_{t_{(\mathrm{k+1})}}{}^{(\Delta_C)}, \widehat{D}_{t_{(\mathrm{k+1})}}{}^{(1)}, \widehat{Sig}_{t_{(\mathrm{k+1})}} \middle| \overrightarrow{\widehat{D}_{t_{(\mathrm{k})}}{}^{(\Delta_C)}}, \overrightarrow{\widehat{D}_{\mathrm{t}}{}^{(1)}}, \overrightarrow{\widehat{Sig}_t}\right)}{P\left(\widehat{D}_{t_{(\mathrm{k+1})}}{}^{(\Delta_C)}, \widehat{D}_{t_{(\mathrm{k+1})}}{}^{(1)}, \widehat{Sig}_{t_{(\mathrm{k+1})}} \middle| \overrightarrow{\widehat{D}_{t_{(k_t)}}{}^{(\Delta_C)}}, \overrightarrow{\widehat{D}_t{}^{(1)}}, \overrightarrow{\widehat{Sig}_t}\right)}$$

Step 2: Since all information comes from $\widehat{D}_{t_{(\mathrm{k+1})}}{}^{(\Delta_C)}, \widehat{D}_{t_{(\mathrm{k+1})}}{}^{(1)}$, and $\widehat{Sig}_{t+1}$ which all depend on the regime path $\mathrm{S}_{t_{(\mathrm{k})},\mathrm{t}}$, this dependence is made explicit by rewriting the numerator as a sum over all possible values of $\mathrm{S}_{t_{(\mathrm{k})},\mathrm{t}}$

$$P\left(\hat{S}_{t_{(\mathrm{k+1})}}, \widehat{D}_{t_{(\mathrm{k+1})}}{}^{(\Delta_C)}, \widehat{D}_{t_{(\mathrm{k+1})}}{}^{(1)}, \widehat{Sig}_{t_{(\mathrm{k+1})}} \middle| \overrightarrow{\widehat{D}_{t_{(k)}}{}^{(\Delta_C)}}, \overrightarrow{\widehat{D}_t{}^{(1)}}, \overrightarrow{\widehat{Sig}_t}\right)$$
$$= \sum_{s_{t_{(k)},t}} P\left(\hat{S}_{t_{(\mathrm{k+1})}}, \mathrm{S}_{t_{(\mathrm{k})},\mathrm{t}} = s_{t_{(k)},t}, \widehat{D}_{t_{(\mathrm{k+1})}}{}^{(\Delta_C)}, \widehat{D}_{t_{(\mathrm{k+1})}}{}^{(1)}, \widehat{Sig}_{t_{(\mathrm{k+1})}} \middle| \overrightarrow{\widehat{D}_{t_{(k)}}{}^{(\Delta_C)}}, \overrightarrow{\widehat{D}_t{}^{(1)}}, \overrightarrow{\widehat{Sig}_t}\right)$$

Step 3: Rewrite the numerator in a Bayesian fashion to separate regime path probabilities at time t ("prior probability") from the probability of new information given regime paths ("likelihood")

$$P\left(\hat{S}_{t_{(\mathrm{k+1})}}, \mathrm{S}_{t_{(\mathrm{k})},\mathrm{t}} = s_{t_{(k)},t}, \widehat{D}_{t_{(\mathrm{k+1})}}{}^{(\Delta_C)}, \widehat{D}_{t_{(\mathrm{k+1})}}{}^{(1)}, \widehat{Sig}_{t_{(\mathrm{k+1})}} \middle| \overrightarrow{\widehat{D}_{t_{(k)}}{}^{(\Delta_C)}}, \overrightarrow{\widehat{D}_t{}^{(1)}}, \overrightarrow{\widehat{Sig}_t}\right)$$
$$= P\left(\widehat{D}_{t_{(\mathrm{k+1})}}{}^{(\Delta_C)}, \widehat{D}_{t_{(\mathrm{k+1})}}{}^{(1)}, \widehat{Sig}_{t_{(\mathrm{k+1})}} \middle| \hat{S}_{t_{(\mathrm{k+1})}}, \mathrm{S}_{t_{(\mathrm{k})},\mathrm{t}} = s_{t_{(k)},t}, \overrightarrow{\widehat{D}_{t_{(k)}}{}^{(\Delta_C)}}, \overrightarrow{\widehat{D}_t{}^{(1)}}, \overrightarrow{\widehat{Sig}_t}\right)$$
$$\cdot P\left(\hat{S}_{t_{(\mathrm{k+1})}}, \mathrm{S}_{t_{(\mathrm{k})},\mathrm{t}} = s_{t_{(k)},t} \middle| \overrightarrow{\widehat{D}_{t_{(k)}}{}^{(\Delta_C)}}, \overrightarrow{\widehat{D}_t{}^{(1)}}, \overrightarrow{\widehat{Sig}_t}\right)$$

The cash flow model together with the assumptions on the processes of regimes, factors and residuals, and signal noise imply that the conditional distribution of $\widehat{D}_{t_{(\mathrm{k+1})}}{}^{(\Delta_C)}, \widehat{D}_{t_{(\mathrm{k+1})}}{}^{(1)}$ and $\widehat{Sig}_{t_{(\mathrm{k+1})}}$

only depends on the regime path $S_{t_{(k)},t_{(k+1)}}$ and cash flows $\widehat{D}_{t_{(k+1)}-1}^{(1)}$ and $\widehat{D}_{t_{(k)}}^{(\Delta_C)}$. Hence, the first term of the above product simplifies to:

$$P\left(\widehat{D}_{t_{(k+1)}}^{(\Delta_C)}, \widehat{D}_{t_{(k+1)}}^{(1)}, \widehat{Sig}_{t_{(k+1)}} \middle| \hat{S}_{t_{(k+1)}}, S_{t_{(k)},t} = s_{t_{(k)},t}, \overrightarrow{\widehat{D}_{t_{(k)}}^{(\Delta_C)}}, \overrightarrow{\widehat{D}_t^{(1)}}, \overrightarrow{\widehat{Sig}_t}\right)$$
$$= P\left(\widehat{D}_{t_{(k+1)}}^{(\Delta_C)}, \widehat{D}_{t_{(k+1)}}^{(1)}, \widehat{Sig}_{t_{(k+1)}} \middle| \hat{S}_{t_{(k+1)}}, S_{t_{(k)},t} = s_{t_{(k)},t}, \widehat{D}_{t_{(k)}}^{(\Delta_C)}, \widehat{D}_{t_{(k+1)}-1}^{(1)}\right)$$

In addition, the Markov property and the assumption that the process of regime is independent of the processes of factors and residuals as well as signal noise together imply for the second term in the product:

$$P\left(\hat{S}_{t_{(k+1)}}, S_{t_{(k)},t} = s_{t_{(k)},t} \middle| \overrightarrow{\widehat{D}_{t_{(k)}}^{(\Delta_C)}}, \overrightarrow{\widehat{D}_t^{(1)}}, \overrightarrow{\widehat{Sig}_t}\right)$$
$$= P\left(\hat{S}_{t_{(k+1)}} \middle| S_{t_{(k)},t} = s_{t_{(k)},t}, \overrightarrow{\widehat{D}_{t_{(k)}}^{(\Delta_C)}}, \overrightarrow{\widehat{D}_t^{(1)}}, \overrightarrow{\widehat{Sig}_t}\right)$$
$$\cdot P\left(S_{t_{(k)},t} = s_{t_{(k)},t} \middle| \overrightarrow{\widehat{D}_{t_{(k)}}^{(\Delta_C)}}, \overrightarrow{\widehat{D}_t^{(1)}}, \overrightarrow{\widehat{Sig}_t}\right)$$
$$= p_{\hat{S}_t, \hat{S}_{t_{(k+1)}}} \cdot P\left(S_{t_{(k)},t} = s_{t_{(k)},t} \middle| \overrightarrow{\widehat{D}_{t_{(k)}}^{(\Delta_C)}}, \overrightarrow{\widehat{D}_t^{(1)}}, \overrightarrow{\widehat{Sig}_t}\right)$$

where $p_{\hat{S}_t, \hat{S}_{t_{(k+1)}}}$ is the probability of a transition from regime $\hat{S}_t$ to regime $\hat{S}_{t_{(k+1)}}$.

Combining the Steps 1 to 3 yields the intermediate result:

A 1-12

$$P\left(\hat{S}_{t_{(k+1)}} \middle| \overrightarrow{\widehat{D}_{t_{(k+1)}}^{(\Delta_C)}}, \overrightarrow{\widehat{D}_{t_{(k+1)}}^{(1)}}, \overrightarrow{\widehat{Sig}_{t_{(k+1)}}}\right)$$
$$= \frac{\sum_{s_{t_{(k)},t}} \left\{ \begin{array}{c} P\left(\widehat{D}_{t_{(k+1)}}^{(\Delta_C)}, \widehat{D}_{t_{(k+1)}}^{(1)}, \widehat{Sig}_{t_{(k+1)}} \middle| \hat{S}_{t_{(k+1)}}, S_{t_{(k)},t} = s_{t_{(k)},t}, \widehat{D}_{t_{(k)}}^{(\Delta_C)}, \widehat{D}_t^{(1)}\right) \\ \cdot p_{\hat{S}_{t_{(k+1)}-1}, \hat{S}_{t_{(k+1)}}} \cdot P\left(S_{t_{(k)},t} = s_{t_{(k)},t} \middle| \overrightarrow{\widehat{D}_{t_{(k)}}^{(\Delta_C)}}, \overrightarrow{\widehat{D}_t^{(1)}}, \overrightarrow{\widehat{Sig}_t}\right) \end{array} \right\}}{P\left(\widehat{D}_{t_{(k+1)}}^{(\Delta_C)}, \widehat{D}_{t_{(k+1)}}^{(1)}, \widehat{Sig}_{t_{(k+1)}} \middle| \overrightarrow{\widehat{D}_{t_{(k_t)}}^{(\Delta_C)}}, \overrightarrow{\widehat{D}_t^{(1)}}, \overrightarrow{\widehat{Sig}_t}\right)}$$

<u>Step 4</u>: Determine the denominator in (A 1-12) from the condition that conditional regime path probabilities at time $t+1$ must add to one

$$P\left(\widehat{D}_{t_{(k+1)}}^{(\Delta_C)}, \widehat{D}_{t_{(k+1)}}^{(1)}, \widehat{Sig}_{t_{(k+1)}} \middle| \overrightarrow{\widehat{D}_{t_{(k_t)}}^{(\Delta_C)}}, \overrightarrow{\widehat{D}_t^{(1)}}, \overrightarrow{\widehat{Sig}_t}\right)$$
$$= \sum_{s_{t_{(k+1)}}=1}^{K} \sum_{s_{t_{(k)},t}} \left\{ \begin{array}{c} P\left(\widehat{D}_{t_{(k+1)}}^{(\Delta_C)}, \widehat{D}_{t_{(k+1)}}^{(1)}, \widehat{Sig}_{t_{(k+1)}} \middle| \hat{S}_{t_{(k+1)}} = s_{t_{(k+1)}}, S_{t_{(k)},t} = s_{t_{(k)},t}, \widehat{D}_{t_{(k)}}^{(\Delta_C)}, \widehat{D}_t^{(1)}\right) \\ \cdot p_{\hat{S}_{t_{(k+1)}-1}, \hat{S}_{t_{(k+1)}}} \cdot P\left(S_{t_{(k)},t} = s_{t_{(k)},t} \middle| \overrightarrow{\widehat{D}_{t_{(k)}}^{(\Delta_C)}}, \overrightarrow{\widehat{D}_t^{(1)}}, \overrightarrow{\widehat{Sig}_t}\right) \end{array} \right\}$$

Step 5: Final result

Plugging Step 4 into (A 1-12) yields the desired recursion between conditional regime probabilities (A 1-8):

A 1-13

$$P\left(\hat{S}_{t_{(k+1)}}\middle|\overrightarrow{\widehat{D}_{t_{(k+1)}}{}^{(\Delta_C)}},\overrightarrow{\widehat{D}_{t_{(k+1)}}{}^{(1)}},\overrightarrow{\widehat{Sig}_{t_{(k+1)}}}\right)$$

$$= \frac{\sum_{s_{t_{(k)},t}}\left\{\begin{array}{c} P\left(\widehat{D}_{t_{(k+1)}}{}^{(\Delta_C)},\widehat{D}_{t_{(k+1)}}{}^{(1)},\widehat{Sig}_{t_{(k+1)}}\middle|\hat{S}_{t_{(k+1)}},\mathrm{S}_{t_{(k)},\mathrm{t}}=s_{t_{(k)},t},\widehat{D}_{t_{(k)}}{}^{(\Delta_C)},\widehat{D}_{t}{}^{(1)}\right) \\ \cdot\, p_{\hat{S}_{t_{(k+1)}-1},\hat{S}_{t_{(k+1)}}}\cdot P\left(\mathrm{S}_{t_{(k)},\mathrm{t}}=s_{t_{(k)},t}\middle|\overrightarrow{\widehat{D}_{t_{(k)}}{}^{(\Delta_C)}},\overrightarrow{\widehat{D}_{t}{}^{(1)}},\overrightarrow{\widehat{Sig}_{t}}\right)\end{array}\right\}}{\sum_{s_{t_{(k+1)}}=1}^{K}\sum_{s_{t_{(k)},t}}\left\{\begin{array}{c} P\left(\widehat{D}_{t_{(k+1)}}{}^{(\Delta_C)},\widehat{D}_{t_{(k+1)}}{}^{(1)},\widehat{Sig}_{t_{(k+1)}}\middle|\begin{array}{c}\hat{S}_{t_{(k+1)}}=s_{t_{(k+1)}},\mathrm{S}_{t_{(k)},\mathrm{t}}=s_{t_{(k)},t}\\ ,\widehat{D}_{t_{(k)}}{}^{(\Delta_C)},\widehat{D}_{t}{}^{(1)}\end{array}\right) \\ \cdot\, p_{\hat{S}_{t_{(k+1)}-1},\hat{S}_{t_{(k+1)}}}\cdot P\left(\mathrm{S}_{t_{(k)},\mathrm{t}}=s_{t_{(k)},t}\middle|\overrightarrow{\widehat{D}_{t_{(k)}}{}^{(\Delta_C)}},\overrightarrow{\widehat{D}_{t}{}^{(1)}},\overrightarrow{\widehat{Sig}_{t}}\right)\end{array}\right\}}$$

(A 1-11) and (A 1-13) can be combined and abbreviated as in (A 1-9):

$$\pi_{t_{(k)},t+1} = \begin{cases} \Pi_0\left(\pi_{t_{(k)},t},D_t{}^{(1)},D_{t+1}{}^{(1)},Sig_{t+1}\right) & t+1 < t_{(k+1)} \\ \Pi_1\left(\begin{array}{c}\pi_{t_{(k)},t},D_{t_{(k)}}{}^{(\Delta_C)},D_{,t}{}^{(1)}\\ ,D_{t_{(k+1)}}{}^{(\Delta_C)},D_{t_{(k+1)}}{}^{(1)},Sig_{t_{(k+1)}}\end{array}\right) & t+1 = t_{(k+1)} \end{cases}$$

A2 Appendix to Section 3.3: Partial Equilibrium Asset Pricing with CARA Preferences

A2.1 Appendix to Section 3.1.2.2.3: Concavity of the Maximand in the Bellman Equation

A2.1.1 Formulation of the Problem

The Bellman equation of each of the identical investors leads to a problem of the following kind: a function of the form

A 2-1

$$h(C,N;W_t,z_t) \equiv U(C) + \frac{1}{1+\rho} \cdot E\left(J\left(W_{t+1}\left(N,C;W_t,{z_t}^p,{z_{t+1}}^p,{z_{t+1}}^d\right),{z_{t+1}}^p,t+1\right)\middle|{z_t}^p\right)$$

with

$$W_{t+1}\left(N,C;W_t,{z_t}^p,{z_{t+1}}^p,{z_{t+1}}^d\right) = [W_t - C]\cdot(1+r) + N^T\left\{P_{t+1}({z_{t+1}}^p) + D_{t+1}\left({z_{t+1}}^d\right) - (1+r)\cdot P_t({z_t}^p)\right\}$$

must be maximized by choosing N (portfolio holdings) and C (consumption). The function J is twice partially differentiable with respect to wealth W_{t+1} with

$$\frac{\partial}{\partial W_{t+1}} J > 0$$

$$\frac{\partial^2}{\partial {W_{t+1}}^2} J < 0$$

i.e., J is increasing and concave in wealth given ${z_{t+1}}^p$ and ${z_t}^p$. Similarly, the utility function $U(C)$ is increasing and concave in C

$$U' > 0$$

$$U'' < 0$$

The problem is to show that these assumptions (concavity of J and U in W_{t+1} and C, respectively) imply concavity of $h(C,N;W_t,z_t)$ in $C, N \epsilon \mathbb{R}^{n+1}$. This is important because any combination of C and N that solves the first-order conditions $\frac{\partial h}{\partial C} = 0$ and $\frac{\partial h}{\partial N} = \underline{0}_n$ then maximizes (A 2-1).

A2.1.2 Proof

A2.1.2.1 Idea of the Proof

The following characterization of concave functions is used (see Geiger/Kanzow (1999), Proposition 3.7a, p. 15):

If $X \subseteq \mathbb{R}^n$ is an open and convex set and $f: X \to \mathbb{R}$ is continuously differentiable, then f is concave if and only if

A 2-2
$$0 \geq [x - x']^{\mathrm{T}} \cdot \{\nabla f(x) - \nabla f(x')\} \; \forall x, x' \epsilon X$$

where $\nabla f(x)$ is the gradient of the function f at x,

$$\nabla f(x) \equiv \begin{pmatrix} \frac{\partial f}{\partial x_1}(x) \\ \cdots \\ \frac{\partial f}{\partial x_n}(x) \end{pmatrix}$$

A2.1.2.2 Details of the Proof

Step 1: Find suitable choices for the function f and the set X in the consumption/portfolio context

Choose $h(.\,; W_t, z_t)$ as the function f, with arguments $\binom{C}{N}$ from $X \equiv \mathbb{R}^{n+1}$. Let $\binom{C}{N}$ and $\binom{C'}{N'}$ be any two combinations of consumption/portfolio holdings (corresponding to arbitrary vectors x and x' in (A 2-2).

Step 2: Specify the right hand side of the inequality (A 2-2)

A 2-3
$$\left[\binom{C}{N} - \binom{C'}{N'}\right]^{\mathrm{T}} \cdot \left\{\nabla h\left(\binom{C}{N}\right) - \nabla h\left(\binom{C'}{N'}\right)\right\}$$
$$= \{C - C'\} \cdot \left\{\frac{\partial h}{\partial C}(C, N) - \frac{\partial h}{\partial C}(C', N')\right\} + \{N - N'\}^T \left\{\frac{\partial h}{\partial N}(C, N) - \frac{\partial h}{\partial N}(C', N')\right\}$$

where

$$\nabla h(C, N) \equiv \begin{pmatrix} \frac{\partial h}{\partial C}(C, N) \\ \cdots \\ \frac{\partial h}{\partial N}(C, N) \end{pmatrix}$$

For brevity, I write $W_{t+1}(N, C; W_t)$ instead of $W_{t+1}(N, C; W_t, {z_t}^p, {z_{t+1}}^p, {z_{t+1}}^d)$ and $P_{t+1} + D_{t+1} - (1 + r) \cdot P_t$ instead of $P_{t+1}({z_{t+1}}^p) + D_{t+1}({z_{t+1}}^d) - (1 + r) \cdot P_t({z_t}^p)$

In this notation, we have

A 2-4

$$\frac{\partial h}{\partial C}(C,N) = U'(C) - \frac{1+r}{1+\rho} \cdot E\left(\frac{\partial}{\partial W_{t+1}} J(W_{t+1}(N,C;W_t), z_{t+1}{}^p, t+1) \middle| z_t{}^p\right)$$

A 2-5

$$\frac{\partial h}{\partial N}(C,N) = \frac{1}{1+\rho} \cdot E\left(\frac{\partial}{\partial W_{t+1}} J(W_{t+1}(N,C;W_t), z_{t+1}{}^p, t+1) \cdot \{P_{t+1} + D_{t+1} - (1+r)\cdot P_t\} \middle| z_t{}^p\right)$$

Moreover, we have

A 2-6

$$W_{t+1}(N,C;W_t) - W_{t+1}(N',C';W_t) = -(1+r)\cdot\{C - C'\} + \{N - N'\}^T\{P_{t+1} + D_{t+1} - (1+r)\cdot P_t\}$$

Plugging (A 2-4) and (A 2-5) into (A 2-3) yields

A 2-7

$$\left[\binom{C}{N} - \binom{C'}{N'}\right]^{\mathrm{T}} \cdot \left\{\nabla h\left(\binom{C}{N}\right) - \nabla h\left(\binom{C'}{N'}\right)\right\}$$
$$= \{C - C'\}$$
$$\cdot\left\{U'(C) - \frac{1+r}{1+\rho}\cdot E\left(\frac{\partial}{\partial W_{t+1}} J(W_{t+1}(N,C;W_t), z_{t+1}{}^p, t+1)\middle| z_t{}^p\right) - U'(C') + \frac{1+r}{1+\rho}\right.$$
$$\left.\cdot E\left(\frac{\partial}{\partial W_{t+1}} J(W_{t+1}(N',C';W_t), z_{t+1}{}^p, t+1)\middle| z_t{}^p\right)\right\}$$
$$+ \{N - N'\}^T\left\{\frac{1}{1+\rho}\right.$$
$$\cdot E\left(\frac{\partial}{\partial W_{t+1}} J(W_{t+1}(N,C;W_t), z_{t+1}{}^p, t+1)\cdot\{P_{t+1} + D_{t+1} - (1+r)\cdot P_t\}\middle| z_t{}^p\right)$$
$$- \frac{1}{1+\rho}$$
$$\left.\cdot E\left(\frac{\partial}{\partial W_{t+1}} J(W_{t+1}(N',C';W_t), z_{t+1}{}^p, t+1)\cdot\{P_{t+1} + D_{t+1} - (1+r)\cdot P_t\}\middle| z_t{}^p\right)\right\}$$

Step 3: Show the non-positivity of (A 2-7)

Step 3a: Show that term involving the utility function U are non-positive

Collecting all terms that include the function U yields:

$$\{C - C'\}\cdot\{U'(C) - U'(C')\} \leq 0$$

To see why this term is non-positive, first assume that $C \geq C'$ (i.e., $\{C - C'\} \geq 0$). Then the fact that $U'' < 0$ means that U' decreases in its argument, hence $\{U'(C) - U'(C')\} \leq 0$ and $\{C - C'\}\cdot\{U'(C) - U'(C')\}$. In the second case $C < C'$, similar reasoning yields $\{C - C'\}\cdot\{U'(C) - U'(C')\} < 0$.

Step 3b: Show that terms involving the function J are non-positive

Collecting all terms that include the function J yields

$$-(1+r)\cdot\{C-C'\}$$
$$\cdot\left\{\frac{1}{1+\rho}\cdot E\left(\frac{\partial}{\partial W_{t+1}}J(W_{t+1}(N,C;W_t),z_{t+1}{}^p,t+1)\middle|z_t{}^p\right)-\frac{1}{1+\rho}\right.$$
$$\left.\cdot E\left(\frac{\partial}{\partial W_{t+1}}J(W_{t+1}(N',C';W_t),z_{t+1}{}^p,t+1)\middle|z_t{}^p\right)\right\}$$
$$+\{N-N'\}^T\left\{\frac{1}{1+\rho}\right.$$
$$\cdot E\left(\frac{\partial}{\partial W_{t+1}}J(W_{t+1}(N,C;W_t),z_{t+1}{}^p,t+1)\cdot\{P_{t+1}+D_{t+1}-(1+r)\cdot P_t\}\middle|z_t{}^p\right)$$
$$-\frac{1}{1+\rho}$$
$$\left.\cdot E\left(\frac{\partial}{\partial W_{t+1}}J(W_{t+1}(N',C';W_t),z_{t+1}{}^p,t+1)\cdot\{P_{t+1}+D_{t+1}-(1+r)\cdot P_t\}\middle|z_t{}^p\right)\right\}$$

and, by (A 2-6),

$$=\frac{1}{1+\rho}$$
$$\cdot E\left(\begin{matrix}\left\{\frac{\partial}{\partial W_{t+1}}J(W_{t+1}(N,C;W_t),z_{t+1}{}^p,t+1)-\frac{\partial}{\partial W_{t+1}}J(W_{t+1}(N',C';W_t),z_{t+1}{}^p,t+1)\right\}\\ \cdot\{W_{t+1}(N,C;W_t)-W_{t+1}(N',C';W_t)\}\end{matrix}\middle|z_t{}^p\right)\leq 0$$

To see non-positivity, apply an argument similar to the terms involving the function U to the terms in the expectation: if $W_{t+1}(N,C;W_t)\geq W_{t+1}(N',C';W_t)$, concavity of J in W_{t+1} means

$$\frac{\partial}{\partial W_{t+1}}J(W_{t+1}(N,C;W_t),z_{t+1}{}^p,t+1)-\frac{\partial}{\partial W_{t+1}}J(W_{t+1}(N',C';W_t),z_{t+1}{}^p,t+1)\leq 0$$

and if $W_{t+1}(N,C;W_t)<W_{t+1}(N',C';W_t)$, then

$$\frac{\partial}{\partial W_{t+1}}J(W_{t+1}(N,C;W_t),z_{t+1}{}^p,t+1)-\frac{\partial}{\partial W_{t+1}}J(W_{t+1}(N',C';W_t),z_{t+1}{}^p,t+1)>0$$

In conclusion, both summands that make up the right-hand side of (A 2-7), and therefore the sum itself, are non-positive, as was to be shown:

$$\left[\binom{C}{N}-\binom{C'}{N'}\right]^{\mathrm{T}}\cdot\left\{\nabla h\left(\binom{C}{N}\right)-\nabla h\left(\binom{C'}{N'}\right)\right\}\leq 0$$

A2.2 Appendix to Section 3.3.1.1: Value Function of Each of the Identical Investors

A2.2.1 Formulation of the Problem

The problem is to find the value function for the following consumption and portfolio selection problem of a CARA investor:

$$E\left(\sum\nolimits_{t=0}^{T} \frac{1}{(1+\rho)^t} \cdot U(C_t) \,\middle|\, z_0, W_0{}^{initial}\right)$$

with

$$U(C) = -exp(-\alpha \cdot C)$$
$$\alpha > 0$$

by choosing a portfolio holding of risky assets N_τ, $0 \leq \tau \leq T-1$, and consumption C_τ, $0 \leq \tau \leq T$ with wealth dynamics

$$W_{\tau+1} = [W_\tau - C_\tau] \cdot exp(r) + N_\tau{}^T\{P_{\tau+1} + D_{\tau+1} - (1+r) \cdot P_\tau\}$$
$$0 \leq \tau \leq T-1$$

where all remaining wealth is consumed at time T

$$W_T = C_T$$

and with asset prices and cash flows as functions of a Markov process $z_\tau \equiv \left(z_\tau{}^p, z_\tau{}^d\right)$:

A 2-8

$$P_\tau = \begin{cases} \underline{0} & \tau = T \\ P_\tau(z_\tau{}^p) & 0 \leq \tau \leq T-1 \end{cases}$$
$$D_\tau = D_\tau\left(z_\tau{}^d\right)$$

Note that it is not yet assumed here that the process $\{z_\tau\}$ is the partial equilibrium price process described in Chapter 3. Instead, $\{z_\tau\}$ is any Markov process of the form

$$z_{\tau+1} = f_{z,\tau}(z_\tau, \xi_{\tau+1})$$
$$\tau = 0, \dots, T-1$$

where $\{\xi_\tau\}$ is a vector-valued i.i.d. process and where $z_t{}^p$ must include a sufficient statistic for $z_{t+1}{}^d$ which is denoted by $z_t{}^{d+}$.

However, it is implicitly assumed that the price process is sufficiently “well-behaved” in the sense that it does not allow arbitrage opportunities and there always exists at least one interior solution to the problems posed by the Bellman equation.

A2.2.2 Results

The value function takes the form

A 2-9

$$J(W_t, z_t{}^p, t) = -\frac{1}{(1+\rho)^t} \cdot exp(-\alpha_t \cdot W_t) \cdot m_t(z_t{}^p)$$

where α_t and $m_t(z_t{}^p)$ are recursively defined by

A 2-10

$$\alpha_t \equiv \begin{cases} \alpha & t = T \\ \dfrac{\alpha \cdot \alpha_{t+1} \cdot (1+r)}{[\alpha + \alpha_{t+1} \cdot (1+r)]} & 0 \leq t \leq T-1 \end{cases}$$

A 2-11

$$m_t(z_t{}^p) \equiv \begin{cases} 1 & t = T \\ \left\{ exp\left(-\dfrac{\alpha_{t+1} \cdot (1+r)}{[\alpha + \alpha_{t+1} \cdot (1+r)]} \cdot ln\left(\dfrac{\alpha_{t+1}}{\alpha} \cdot \dfrac{1+r}{1+\rho}\right) - \dfrac{\alpha_{t+1} \cdot (1+r)}{[\alpha + \alpha_{t+1} \cdot (1+r)]} \cdot ln\big(G_t(N_t{}^*(z_t{}^p); z_t{}^p)\big) \right) \cdot G_t(N_t{}^*(z_t{}^p); z_t{}^p) \cdot \left\{\dfrac{\alpha_{t+1}}{\alpha} \cdot \dfrac{1+r}{1+\rho} + 1\right\} \right\} & 0 \leq t \leq T-1 \end{cases}$$

with

$$G_t(N_t{}^*(z_t{}^p); z_t{}^p) \equiv E\Big(exp\big(-\alpha_{t+1} \cdot N_t{}^*(z_t{}^p)^T \{P_{t+1}(z_{t+1}{}^p) + D_{t+1}(z_{t+1}{}^d) - (1+r) \cdot P_t(z_t{}^p)\}\big) \cdot m_{t+1}(z_{t+1}{}^p) \Big| z_t{}^p\Big)$$

Here α_t is a constant that is strictly positive and $m_t(.)$ is positive for all possible values of $z_t{}^p$. $N_t{}^*(z_t{}^p)$ is a portfolio characterized by the following optimality conditions:

Any combination of portfolio holdings $N_t{}^*(z_t{}^p)$ and consumption $C_t{}^*(z_t{}^p, W_t)$ that solves the following first-order conditions will be optimal due to the concavity of the value function in wealth:

A 2-12

$$E\left(\begin{matrix} exp\big(-\alpha_{t+1} \cdot N_t{}^*(z_t{}^p)^T \{P_{t+1}(z_{t+1}{}^p) + D_{t+1}(z_{t+1}{}^d)\}\big) \cdot m_{t+1}(z_{t+1}{}^p) \\ \cdot \{P_{t+1}(z_{t+1}{}^p) + D_{t+1}(z_{t+1}{}^d) - (1+r) \cdot P_t(z_t{}^p)\} \end{matrix} \middle| z_t{}^p \right) = \underline{0}$$

A 2-13

$$C_t{}^*(z_t{}^p, W_t) = -\frac{ln\left(\frac{\alpha_{t+1}}{\alpha} \cdot \frac{1+r}{1+\rho}\right)}{[\alpha + \alpha_{t+1} \cdot (1+r)]} + \frac{\alpha_{t+1} \cdot (1+r)}{[\alpha + \alpha_{t+1} \cdot (1+r)]} \cdot W_t - \frac{ln\big(G_t(N_t{}^*(z_t{}^p); z_t{}^p)\big)}{[\alpha + \alpha_{t+1} \cdot (1+r)]}$$

A2.2.3 Proof

A2.2.3.1 Idea of the Proof

The proof is by induction over the remaining time horizon up to the final point of time T, i.e., $T - t$. I prove the form of the value function ((A 2-9) with (A 2-10) and (A 2-11), including the positivity of α_t and $m_t(z_t{}^p)$).

A2.2.3.2 Details of the Proof

Base case: $T - t = 0$

At time T, the value function must coincide with the direct utility function by the nature of dynamic programming:

$$J(W_T, z_T{}^p, T) = J(W_T, T) = -\frac{1}{(1+\rho)^T} \cdot exp(-\alpha \cdot W_T)$$

By setting

$$m_T(z_T{}^p) \equiv 1$$

and

$$\alpha_T \equiv \alpha$$

it is evident that (A 2-9) is correct for time $t = T$.

Inductive step

Inductively assume that (A 2-9) is true for time $t + 1$. Then the Bellman equation for time t reads

A 2-14

$$J(W_t, z_t, t) = sup_{C,N} \left\{ -\frac{1}{(1+\rho)^t} \cdot exp(-\alpha \cdot C) \right.$$
$$\left. + E\left(-\frac{1}{(1+\rho)^{t+1}} \cdot exp(-\alpha_{t+1} \cdot W_{t+1}) \cdot m_{t+1}(z_{t+1}{}^p) \middle| z_t, W_t \right) \right\}$$

with

$$W_{t+1} = [W_t - C] \cdot (1+r) + N^T \{ P_{t+1}(z_{t+1}{}^p) + D_{t+1}(z_{t+1}{}^d) - (1+r) \cdot P_t(z_t{}^p) \}$$

In order to determine the precise form of $J(W_t, z_t{}^p, t)$ in (A 2-14), the optimization problem on the right-hand side of (A 2-14) must be solved. To that end, first substitute the wealth dynamics into the Bellman equation. In a next step, first, factor out $-\frac{1}{(1+\rho)^t}$ and, second, rewrite the expectation $E\left(-\frac{1}{(1+\rho)^{t+1}} \cdot exp(-\alpha_{t+1} \cdot W_{t+1}) \cdot m_{t+1}(z_{t+1}{}^p) \middle| z_t, W_t \right)$:

A 2-15

$$J(W_t, z_t{}^p, t) = -\frac{1}{(1+\rho)^t}$$
$$\cdot inf_{C,N} \left\{ exp(-\alpha \cdot C) + \frac{1}{1+\rho} \cdot exp\big(-\alpha_{t+1} \cdot [W_t - C] \cdot (1+r)\big) \cdot G_t(N; z_t{}^p) \right\}$$

with

$$G_t(N; z_t{}^p)$$
$$\equiv E\Big(exp\big(-\alpha_{t+1} \cdot N^T \{ P_{t+1}(z_{t+1}{}^p) + D_{t+1}(z_{t+1}{}^d) - (1+r) \cdot P_t(z_t{}^p) \}\big) \cdot m_{t+1}(z_{t+1}{}^p) \Big| z_t{}^p\Big)$$

Observe that $G_t(N; z_t{}^p)$ is strictly positive because both terms in the expectation are strictly positive (m_{t+1} by inductive assumption), hence the expectation must also be strictly positive. Also note that G_t is a function of $z_t{}^p$ (rather than all of z_t) because $z_t{}^p$ must include a sufficient statistic for $z_{t+1}{}^d$ which is denoted by $z_t{}^{d+}$.

In order to determine the precise form of $J(W_t, z_t{}^p, t)$ in (A 2-15), the optimization problem

$$inf_{C,N}\left\{exp(-\alpha\cdot C)+\frac{1}{1+\rho}\cdot exp\big(-\alpha_{t+1}\cdot[W_t-C]\cdot(1+r)\big)\cdot G_t(N;z_t{}^p)\right\}$$

must be solved. This in turn can be accomplished in two steps: (i) minimize $G_t(N; z_t{}^p)$ with respect to N and (ii) minimize $exp(-\alpha\cdot C)+\frac{1}{1+\rho}\cdot exp\big(-\alpha_{t+1}\cdot[W_t-C]\cdot(1+r)\big)\cdot G_t(N_t{}^*(z_t{}^p);z_t{}^p)$, where$N_t{}^*(z_t{}^p)$ is a portfolio that attains a minimal value for $G_t(N; z_t{}^p)$.

Concerning the first step, the first order condition reads

$$E\left(\begin{matrix}exp\big(-\alpha_{t+1}\cdot N^T\{P_{t+1}(z_{t+1}{}^p)+D_{t+1}(z_{t+1}{}^d)\}\big)\cdot m_{t+1}(z_{t+1}{}^p)\\ \cdot\{P_{t+1}(z_{t+1}{}^p)+D_{t+1}(z_{t+1}{}^d)-(1+r)\cdot P_t(z_t{}^p)\}\end{matrix}\middle| z_t{}^p\right)=\underline{0}$$

Any pair of consumption and portfolio holdings that solve the first-order conditions attain optimal values because the value function is concave in wealth (see Appendix A2.1). Note, however, that a solution does not necessarily exist for all possible price and cash flow processes described by (A 2-8). However, here it can be assumed that there exists some portfolio $N_t{}^*(z_t{}^p)$ that minimizes $G_t(N; z_t{}^p)$. because later on the object of interest will only be a particular price process, namely the process of partial equilibrium CARA asset prices that will, by construction, solve the first-order conditions.

Given a solution $N_t{}^*(z_t{}^p)$ to the first sub-problem, the first-order condition for consumption reads:

$$-\alpha\cdot exp(-\alpha\cdot C)+\alpha_{t+1}\cdot\frac{1+r}{1+\rho}\cdot exp\big(-\alpha_{t+1}\cdot[W_t-C]\cdot(1+r)\big)\cdot G_t(N_t{}^*(z_t{}^p);z_t{}^p)=0$$

Some simple manipulations yield a solution $C_t{}^*(z_t{}^p, W_t)$ (as a function of $N_t{}^*(z_t{}^p)$):

$$\Leftrightarrow$$

A 2-16

$$exp(-\alpha\cdot C)=\frac{\alpha_{t+1}}{\alpha}\cdot\frac{1+r}{1+\rho}\cdot exp\big(-\alpha_{t+1}\cdot[W_t-C]\cdot(1+r)\big)\cdot G_t(N_t{}^*(z_t{}^p);z_t{}^p)$$

$$\Leftrightarrow$$

$$-\alpha\cdot C=ln\left(\frac{\alpha_{t+1}}{\alpha}\cdot\frac{1+r}{1+\rho}\right)-\alpha_{t+1}\cdot[W_t-C]\cdot(1+r)+ln\big(G_t(N_t{}^*(z_t{}^p);z_t{}^p)\big)$$

$$\Leftrightarrow$$

$$-[\alpha+\alpha_{t+1}\cdot(1+r)]\cdot C=ln\left(\frac{\alpha_{t+1}}{\alpha}\cdot\frac{1+r}{1+\rho}\right)-\alpha_{t+1}\cdot(1+r)\cdot W_t+ln\big(G_t(N_t{}^*(z_t{}^p);z_t{}^p)\big)$$

$$\Leftrightarrow$$

$$C=-\frac{ln\left(\frac{\alpha_{t+1}}{\alpha}\cdot\frac{1+r}{1+\rho}\right)}{[\alpha+\alpha_{t+1}\cdot(1+r)]}+\frac{\alpha_{t+1}\cdot(1+r)}{[\alpha+\alpha_{t+1}\cdot(1+r)]}\cdot W_t-\frac{ln\big(G_t(N_t{}^*(z_t{}^p);z_t{}^p)\big)}{[\alpha+\alpha_{t+1}\cdot(1+r)]}$$

To finalize the proof of the form of the value function, the optimal controls ${C_t}^*({z_t}^p, W_t)$ and ${N_t}^*({z_t}^p)$ are plugged into the Bellman equation:

$$J(W_t, {z_t}^p, t) = -\frac{1}{(1+\rho)^t} \cdot \left\{exp\left(-\alpha \cdot {C_t}^*({z_t}^p, W_t)\right) + exp\left(-\alpha_{t+1} \cdot [W_t - {C_t}^*({z_t}^p, W_t)] \cdot (1+r)\right) \cdot G_t({N_t}^*({z_t}^p); {z_t}^p)\right\}$$

First use the first-order consumption condition (A 2-16) to simplify the Bellman equation (A 2-14):

A 2-17

$$J(W_t, {z_t}^p, t) = -\frac{1}{(1+\rho)^t} \cdot \left\{exp\left(-\alpha_{t+1} \cdot [W_t - {C_t}^*({z_t}^p, W_t)] \cdot (1+r)\right) \cdot G_t({N_t}^*({z_t}^p); {z_t}^p)\right\} \cdot \left\{\frac{\alpha_{t+1}}{\alpha} \cdot \frac{1+r}{1+\rho} + 1\right\}$$

The remaining problem is to substitute ${C_t}^*({z_t}^p, W_t)$ into $-\alpha_{t+1} \cdot [W_t - {C_t}^*({z_t}^p, W_t)] \cdot (1+r)$ and then plug the latter term into the value function:

$$W_t - {C_t}^*({z_t}^p, W_t) = \frac{ln\left(\frac{\alpha_{t+1}}{\alpha} \cdot \frac{1+r}{1+\rho}\right)}{[\alpha + \alpha_{t+1} \cdot (1+r)]} + W_t \cdot \frac{\alpha}{[\alpha + \alpha_{t+1} \cdot (1+r)]} + \frac{ln\left(G_t({N_t}^*({z_t}^p); {z_t}^p)\right)}{[\alpha + \alpha_{t+1} \cdot (1+r)]}$$

(observe that $\frac{\alpha_{t+1}}{\alpha} \cdot G_t({N_t}^*({z_t}^p); {z_t}^p)$ is positive since $G_t({N_t}^*({z_t}^p); {z_t}^p)$ is positive, hence, the logarithm is well-defined)

$$\Leftrightarrow$$

$$-\alpha_{t+1} \cdot [W_t - {C_t}^*({z_t}^p, W_t)] \cdot (1+r) = -\frac{\alpha_{t+1} \cdot (1+r)}{[\alpha + \alpha_{t+1} \cdot (1+r)]} \cdot ln\left(\frac{\alpha_{t+1}}{\alpha} \cdot \frac{1+r}{1+\rho}\right) - \frac{\alpha \cdot \alpha_{t+1} \cdot (1+r)}{[\alpha + \alpha_{t+1} \cdot (1+r)]} \cdot W_t - \frac{\alpha_{t+1} \cdot (1+r)}{[\alpha + \alpha_{t+1} \cdot (1+r)]} \cdot ln\left(G_t({N_t}^*({z_t}^p); {z_t}^p)\right)$$

Define α_t as the coefficient of W_t on the right-hand side of the preceding equation:

$$\alpha_t \equiv \frac{\alpha \cdot \alpha_{t+1} \cdot (1+r)}{[\alpha + \alpha_{t+1} \cdot (1+r)]}$$

Observe that α_t is positive because $(1+r)$, α_{t+1} (by inductive assumption) and α (risk averse investor) are all positive.

Plugging $-\alpha_{t+1} \cdot [W_t - C_t^*(z_t{}^p, W_t)] \cdot (1+r)$ into the Bellman equation in the intermediate form (A 2-17) then shows that the value function is of the following form:

$$J(W_t, z_t{}^p, t) = -\frac{1}{(1+\rho)^t} \cdot exp(-\alpha_t \cdot W_t) \cdot m_t(z_t{}^p)$$

with

$$m_t(z_t{}^p) \equiv exp\left(-\frac{\alpha_{t+1} \cdot (1+r)}{[\alpha + \alpha_{t+1} \cdot (1+r)]} \cdot ln\left(\frac{\alpha_{t+1}}{\alpha} \cdot \frac{1+r}{1+\rho}\right) - \frac{\alpha_{t+1} \cdot (1+r)}{[\alpha + \alpha_{t+1} \cdot (1+r)]} \cdot ln\left(G_t(N_t{}^*(z_t{}^p); z_t{}^p)\right)\right) \cdot G_t(N_t{}^*(z_t{}^p); z_t{}^p) \cdot \left\{\frac{\alpha_{t+1}}{\alpha} \cdot \frac{1+r}{1+\rho} + 1\right\} > 0$$

This concludes the proof by induction of the form of the value function.

A2.3 Appendix to Section 3.3.1.2: Partial Equilibrium Asset Prices

A2.3.1 Formulation of the Problem

Consider a cash flow process of the form

$$D_t = D_t(z_t{}^d)$$
$$z_t{}^d = (z_t{}^{d,0}, z_t{}^{d,+})$$

where $z_t{}^{d,+}$ is a sufficient statistic for the conditional distribution of future cash flows (and, in particular, $z_{t+1}{}^d$) and $z_t{}^{d,0}$ is a component that describes current cash flows. The dynamics of $z_\tau{}^d$ are given by

$$z_{\tau+1}{}^d = f_{z,\tau}{}^d(z_\tau{}^d, \xi_{\tau+1}{}^d)$$
$$\tau = 0, \dots, T-1$$

where $\xi_{\tau+1}{}^d$ are vector-valued i.i.d. random variables.

Partial equilibrium asset prices possess two characteristics: first, information relevant to pricing consists of a sufficient statistic for the conditional distribution of future cash flows, i.e., $z_t{}^p \equiv z_t{}^{d,+}$; second, the partial equilibrium price process under CARA utility is recursively defined starting from time T where the price process is zero:

A 2-18

$$P_t(z_t{}^p) = \begin{cases} \underline{0} & t = T \\ E\left(q_{t,t+1}(z_{t+1}{}^p, z_{t+1}{}^d, z_t{}^p) \cdot \{P_{t+1}(z_{t+1}{}^p) + D_{t+1}(z_{t+1}{}^d)\} \middle| z_t{}^p\right) & 0 \le t \le T-1 \end{cases}$$

with

$$q_{t,t+1}\left(z_{t+1}{}^{p}, z_{t+1}{}^{d}, z_t{}^{p}\right)$$

$$= exp(-r) \cdot \frac{exp\left(-\alpha_{t+1} \cdot \frac{1}{n_I} \cdot \left\{D_{t+1}{}^{market}\left(z_{t+1}{}^{d}\right) + V_{t+1}{}^{market}\left(z_{t+1}{}^{p}\right)\right\}\right) \cdot \overline{m}_{t+1}\left(z_{t+1}{}^{p}\right)}{E\left(exp\left(-\alpha_{t+1} \cdot \frac{1}{n_I} \cdot \left\{D_{t+1}{}^{market}\left(z_{t+1}{}^{d}\right) + V_{t+1}{}^{market}\left(z_{t+1}{}^{p}\right)\right\}\right) \cdot \overline{m}_{t+1}\left(z_{t+1}{}^{p}\right) \middle| z_t{}^{p}\right)}$$

where $D_{t+1}{}^{market}$ is the aggregate cash flows paid by all risky assets in the market portfolio

$$D_{t+1}{}^{market}\left(z_{t+1}{}^{p}\right) \equiv \overline{N}^{T} D_{t+1}\left(z_{t+1}{}^{d}\right)$$

and where $V_{t+1}{}^{market}$ is the aggregate value of the market portfolio of risky assets

$$V_{t+1}{}^{market}\left(z_{t+1}{}^{p}\right) \equiv \overline{N}^{T} P_{t+1}\left(z_{t+1}{}^{p}\right)$$

with α_{t+1} given by the recursion (A 2-10),

$$\alpha_t \equiv \begin{cases} \alpha & t = T \\ \dfrac{\alpha \cdot \alpha_{t+1} \cdot (1+r)}{[\alpha + \alpha_{t+1} \cdot (1+r)]} & 0 \le t \le T-1 \end{cases}$$

and with $\overline{m}_{t+1}\left(z_{t+1}{}^{p}\right)$ given by the following partial equilibrium analogue to (A 2-11)

A 2-19

$$\overline{m}_t\left(z_t{}^{p}\right) \equiv \begin{cases} 1 & t = T \\ \left\{ exp\left(-\dfrac{\alpha_{t+1} \cdot (1+r)}{[\alpha + \alpha_{t+1} \cdot (1+r)]} \cdot ln\left(\dfrac{\alpha_{t+1}}{\alpha} \cdot \dfrac{1+r}{1+\rho}\right) - \dfrac{\alpha_{t+1} \cdot (1+r)}{[\alpha + \alpha_{t+1} \cdot (1+r)]} \cdot ln\left(\overline{G}_t\left(z_t{}^{p}\right)\right) \right) \cdot \overline{G}_t\left(z_t{}^{p}\right) \cdot \left\{\dfrac{\alpha_{t+1}}{\alpha} \cdot \dfrac{1+r}{1+\rho} + 1\right\} \right\} & 0 \le t \le T-1 \end{cases}$$

with

$$\overline{G}_t\left(z_t{}^{p}\right) \equiv G_t\left(\frac{1}{n_I} \cdot \overline{N}; z_t{}^{p}\right)$$

$$\equiv E\left(exp\left(-\alpha_{t+1} \cdot \overline{N}^{T}\left\{P_{t+1}\left(z_{t+1}{}^{p}\right) + D_{t+1}\left(z_{t+1}{}^{d}\right) - (1+r) \cdot P_t\left(z_t{}^{p}\right)\right\}\right) \cdot \overline{m}_{t+1}\left(z_{t+1}{}^{p}\right) \middle| z_t{}^{p}\right)$$

A2.3.2 Proof

A2.3.2.1 Idea of the Proof

In order to demonstrate that this price process is indeed a partial equilibrium, it must be shown that the n_I identical investors always behave optimally by holding $\frac{1}{n_I}$-th of the market portfolio of risky assets $\overline{N}$. This in turn is established by deriving the value function of one of the identical investors acting under the price process (A 2-18) with (A 2-10) and (A 2-19); this value function will be referred to as the equilibrium value function. The proof is by induction over the remaining time horizon up to the final point of time T, i.e., $T - t$.

Evidently, if each of the n_I identical investors holds $\frac{1}{n_I}$-th of the market portfolio of risky assets $\overline{N}$, the market for risky asset clears.

A2.3.2.2 Details of the Proof

I inductively prove that the equilibrium value function of one of the identical investors acting under the partial equilibrium price process (A 2-18) with (A 2-19) is given by

A 2-20

$$J(W_t, {z_t}^p, t) = -\frac{1}{(1+\rho)^t} \cdot \cdot exp(-\alpha_t \cdot W_t) \cdot \overline{m}_t({z_t}^p)$$

with $\overline{m}_t$ and α_t given by (A 2-19) and (A 2-10), respectively:

$$\overline{m}_t({z_t}^p) \equiv \begin{cases} 1 & t = T \\ \left\{ exp\left(-\frac{\alpha_{t+1} \cdot (1+r)}{[\alpha + \alpha_{t+1} \cdot (1+r)]} \cdot ln\left(\frac{\alpha_{t+1}}{\alpha} \cdot \frac{1+r}{1+\rho}\right) - \frac{\alpha_{t+1} \cdot (1+r)}{[\alpha + \alpha_{t+1} \cdot (1+r)]} \cdot ln\left(\overline{G_t}({z_t}^p)\right) \right) \cdot \overline{G_t}({z_t}^p) \cdot \left\{\frac{\alpha_{t+1}}{\alpha} \cdot \frac{1+r}{1+\rho} + 1\right\} \right\} & 0 \le t \le T-1 \end{cases}$$

with

$$\overline{G}_t({z_t}^p) \equiv G_t\left(\frac{1}{n_I} \cdot \overline{N}; {z_t}^p\right)$$
$$\equiv E\left(exp\left(-\alpha_{t+1} \cdot \overline{N}^T \left\{ P_{t+1}({z_{t+1}}^p) + D_{t+1}\left({z_{t+1}}^d\right) - (1+r) \cdot P_t({z_t}^p) \right\} \right) \cdot \overline{m}_{t+1}({z_{t+1}}^p) \middle| {z_t}^p \right)$$

and

$$\alpha_t \equiv \begin{cases} \alpha & t = T \\ \frac{\alpha \cdot \alpha_{t+1} \cdot (1+r)}{[\alpha + \alpha_{t+1} \cdot (1+r)]} & 0 \le t \le T-1 \end{cases}$$

where α_t is a positive constant and $\overline{m}_{t+1}(.)$ is a function that only takes positive values.

If this equilibrium value function characterizes optimal behavior of investors, then holding $\frac{1}{n_I}$-th of the market portfolio of risky assets $\overline{N}$ at all points of time solves the first-order condition of portfolio holdings and, hence, is optimal.

Inductive proof of the form of the equilibrium value function

Base case: $T - t = 0$

If $t = T$, it is clear that the equilibrium value function simply reads

$$J(W_T, z_T{}^p, T) = J(W_T, T) = -\frac{1}{(1+\rho)^T} \cdot exp(-\alpha \cdot W_T)$$

with

$$m_T(z_T{}^p) \equiv 1$$

and

$$\alpha_T \equiv \alpha$$

(A 2-20) is true.

Inductive step

Inductively assume that (A 2-20) is true for time $t+1$. Then the necessary conditions for portfolio holdings (A 2-12) read in partial equilibrium

$$E\left(\begin{matrix} exp\left(-\alpha_{t+1} \cdot N_t{}^*(z_t{}^p)^T\{P_{t+1}(z_{t+1}{}^p) + D_{t+1}\left(z_{t+1}{}^d\right)\}\right) \cdot \overline{m}_{t+1}(z_{t+1}{}^p) \\ \cdot \left\{P_{t+1}(z_{t+1}{}^p) + D_{t+1}\left(z_{t+1}{}^d\right) - (1+r) \cdot P_t(z_t{}^p)\right\} \end{matrix} \middle| z_t{}^p \right) = \underline{0}$$

By definition of the price process, $N_t{}^*(z_t{}^p) = \frac{1}{n_I} \cdot \overline{N}$ is a solution of the first-order conditions for optimal portfolio holdings (and the corresponding consumption is given by (A 2-13)) in partial equilibrium. Second-order conditions need not be checked because of the concavity of the equilibrium value function. From the discussion of the portfolio selection problem under CARA preferences, it follows that the equilibrium value function at time t is indeed given by (A 2-9). Since the optimal portfolio is $N_t{}^*(z_t{}^p) = \frac{1}{n_I} \cdot \overline{N}$ and, by inductive assumption, $m_{t+1}(z_{t+1}{}^p) = \overline{m}_{t+1}(z_{t+1}{}^p)$, the recursion for $m_t(.)$ (A 2-11) implies that we also have $m_t(z_t{}^p) = \overline{m}_t(z_t{}^p)$ at time t This completes the proof by induction of (A 2-20).

A2.4 Appendix to Prices of "Expectation Risk" and "Combined Risk"

A2.4.1 Appendix to Section 3.3.2.1.3.4.2: Information Frequency = Cash Flow Frequency

A2.4.1.1 Formulation of the Problem

The problem is to show that the prices of "expectation risk" and "combined risk" read as follows:

Price of "Expectation Risk"

3-76

$$E\left(q_{t,t+1}{}^{iI}\cdot\Delta^{exp.risk}\middle|\pi_t,D_t\right)=-\frac{1}{1+r}\cdot\sum_{s=1}^{K}\{\pi_{s,t}-\theta_{s,t}(\pi_t,D_t)\}\cdot E\left(P_{t+1}{}^{iI}+D_{t+1}\middle|S_t=s,\pi_t,D_t\right)$$

with

$$\Delta^{exp.risk}\equiv E\left(P_{t+1}{}^{iI}+D_{t+1}\middle|S_t,\pi_t,D_t\right)-E\left(P_{t+1}{}^{iI}+D_{t+1}\middle|\pi_t,D_t\right)$$

$$\theta_{s,t}(\pi_t,D_t)$$

$$\equiv\frac{\pi_{s,t}\cdot E\left(\begin{array}{c}exp\left(-\alpha_{t+1}\cdot\frac{1}{n_I}\cdot\{D_{t+1}{}^{market}(D_{t+1})+V_{t+1}{}^{market}(\pi_{t+1},D_{t+1})\}\right)\\ \cdot m_{t+1}{}^{iI}(\pi_{t+1},D_{t+1})\end{array}\middle|S_t=s,D_t\right)}{\sum_{s'=1}^{K}\pi_{s',t}\cdot E\left(\begin{array}{c}exp\left(-\alpha_{t+1}\cdot\frac{1}{n_I}\cdot\{D_{t+1}{}^{market}(D_{t+1})+V_{t+1}{}^{market}(\pi_{t+1},D_{t+1})\}\right)\\ \cdot m_{t+1}{}^{iI}(\pi_{t+1},D_{t+1})\end{array}\middle|S_t=s',D_t\right)}$$

$$s=1,\dots,K$$

Price of "Combined Risk"

3-78

$$E\left(q_{t,t+1}{}^{iI}\cdot\Delta^{comb.risk}\middle|\pi_t,D_t\right)$$

$$=\frac{1}{1+r}$$

$$\cdot\sum_{s=1}^{K}\theta_{s,t}(\pi_t,D_t)$$

$$\cdot E\left(AfR_t{}^{conditional}(s;\eta_{t+1},S_{t+1},fe_{t+1},S_t,\pi_t,D_t)\cdot\Delta^{comb.risk}\middle|S_t=s,D_t,\pi_t\right)$$

with $\theta_{s,t}(\pi_t,D_t)$ as in (3-77)

$$AfR_t{}^{conditional}(s;\eta_{t+1},S_{t+1},fe_{t+1},S_t,\pi_t,D_t)$$

$$\equiv\frac{exp\left(-\alpha_{t+1}\cdot\frac{1}{n_I}\cdot\left\{\begin{array}{c}D_{t+1}{}^{market}(D_{t+1})\\+V_{t+1}{}^{market}(\pi_{t+1},D_{t+1})\end{array}\right\}\right)\cdot m_{t+1}{}^{iI}(\pi_{t+1},D_{t+1})}{E\left(exp\left(-\alpha_{t+1}\cdot\frac{1}{n_I}\cdot\left\{\begin{array}{c}D_{t+1}{}^{market}(D_{t+1})\\+V_{t+1}{}^{market}(\pi_{t+1},D_{t+1})\end{array}\right\}\right)\cdot m_{t+1}{}^{iI}(\pi_{t+1},D_{t+1})\middle|S_t=s,D_t\right)}$$

A2.4.1.2 Solution

A2.4.1.2.1 Price of "Expectation Risk"

Using the definition of $\Delta^{exp.risk}$ in (3-74), the price of "expectation risk" reads

$$E\left(q_{t,t+1}{}^{iI} \cdot \Delta^{exp.risk} \middle| \pi_t, D_t\right)$$
$$= E\left(q_{t,t+1}{}^{iI} \cdot \left\{E\left(P_{t+1}{}^{iI} + D_{t+1} \middle| S_t, \pi_t, D_t\right) - E\left(P_{t+1}{}^{iI} + D_{t+1} \middle| \pi_t, D_t\right)\right\} \middle| \pi_t, D_t\right)$$

The computation of the price of "expectation risk" is developed as follows.

<u>Step 1</u>: Rewrite the expectation conditional on π_t, D_t as an expectation over all possible values of the unobservable current regime S_t

A 2-21

$$E\left(q_{t,t+1}{}^{iI} \cdot \left\{E\left(P_{t+1}{}^{iI} + D_{t+1} \middle| S_t, \pi_t, D_t\right) - E\left(P_{t+1}{}^{iI} + D_{t+1} \middle| \pi_t, D_t\right)\right\} \middle| \pi_t, D_t\right)$$
$$= \sum_{s=1}^{K} \pi_{s,t}$$
$$\cdot E\left(q_{t,t+1}{}^{iI} \cdot \left\{E\left(P_{t+1}{}^{iI} + D_{t+1} \middle| S_t, \pi_t, D_t\right) - E\left(P_{t+1}{}^{iI} + D_{t+1} \middle| \pi_t, D_t\right)\right\} \middle| S_t = s, \pi_t, D_t\right)$$

<u>Step 2</u>: Re-express the first term on the right-hand side of the previous equation,

$$\sum_{s=1}^{K} \pi_{s,t} \cdot E\left(q_{t,t+1}{}^{iI} \cdot \left\{E\left(P_{t+1}{}^{iI} + D_{t+1} \middle| S_t, \pi_t, D_t\right)\right\} \middle| S_t = s, \pi_t, D_t\right)$$

with the help of risk-neutralized regime probabilities:

For brevity, first define

A 2-22

$$\frac{\partial J^{iI}}{\partial W_{t+1}} \equiv exp\left(-\alpha_{t+1} \cdot \frac{1}{n_I} \cdot \left\{D_{t+1}{}^{market}(D_{t+1}) + V_{t+1}{}^{market}(\pi_{t+1}, D_{t+1})\right\}\right)$$
$$\cdot\, m_{t+1}{}^{iI}(\pi_{t+1}, D_{t+1})$$

In this notation, the stochastic discount factor reads

$$q_{t,t+1}{}^{iI} = \frac{1}{1+r} \cdot \frac{\frac{\partial J^{iI}}{\partial W_{t+1}}}{E\left(\frac{\partial J^{iI}}{\partial W_{t+1}} \middle| \pi_t, D_t\right)}$$

Then the first term on the right-hand side of (A 2-21) can be rewritten as

$$\sum_{s=1}^{K} \pi_{s,t} \cdot E\left(q_{t,t+1}{}^{iI} \cdot E\left(P_{t+1}{}^{iI} + D_{t+1} \middle| S_t, \pi_t, D_t\right) \middle| S_t = s, \pi_t, D_t\right)$$

$$= \frac{1}{1+r}$$

$$\cdot \sum_{s=1}^{K} \frac{\pi_{s,t} \cdot E\left(\frac{\partial J^{iI}}{\partial W_{t+1}} \middle| S_t = s, \pi_t, D_t\right)}{E\left(\frac{\partial J^{iI}}{\partial W_{t+1}} \middle| \pi_t, D_t\right)}$$

$$\cdot E\left(\frac{\frac{\partial J^{iI}}{\partial W_{t+1}}}{E\left(\frac{\partial J^{iI}}{\partial W_{t+1}} \middle| S_t = s, \pi_t, D_t\right)} \cdot E\left(P_{t+1}{}^{iI} + D_{t+1} \middle| S_t, \pi_t, D_t\right) \middle| S_t = s, \pi_t, D_t\right)$$

Risk-neutralized regime probabilities read in this context of partial equilibrium under incomplete information

$$\theta_{s,t}(\pi_t, D_t) = \frac{\pi_{s,t} \cdot E\left(\frac{\partial J^{iI}}{\partial W_{t+1}} \middle| S_t = s, \pi_t, D_t\right)}{E\left(\frac{\partial J^{iI}}{\partial W_{t+1}} \middle| \pi_t, D_t\right)}$$

i.e.,

A 2-23

$$\sum_{s=1}^{K} \pi_{s,t} \cdot E\left(q_{t,t+1}{}^{iI} \cdot E\left(P_{t+1}{}^{iI} + D_{t+1} \middle| S_t, \pi_t, D_t\right) \middle| S_t = s, \pi_t, D_t\right)$$

$$= \frac{1}{1+r}$$

$$\cdot \sum_{s=1}^{K} \theta_{s,t}(\pi_t, D_t)$$

$$\cdot E\left(\frac{\frac{\partial J^{iI}}{\partial W_{t+1}}}{E\left(\frac{\partial J^{iI}}{\partial W_{t+1}} \middle| S_t = s, \pi_t, D_t\right)} \cdot E\left(P_{t+1}{}^{iI} + D_{t+1} \middle| S_t, \pi_t, D_t\right) \middle| S_t = s, \pi_t, D_t\right)$$

<u>Step 3</u>: Simplify the conditional expectations on the right hand side of (A 2-23)

The conditional expectation $E\left(P_{t+1}{}^{iI} + D_{t+1} \middle| S_t, \pi_t, D_t\right)$ is non-stochastic conditional on $S_t = s, \pi_t, D_t$. Hence, it can be factored out:

$$E\left(\frac{\frac{\partial J^{iI}}{\partial W_{t+1}}}{E\left(\frac{\partial J^{iI}}{\partial W_{t+1}} \middle| S_t = s, \pi_t, D_t\right)} \cdot E\left(P_{t+1}{}^{iI} + D_{t+1} \middle| S_t, \pi_t, D_t\right) \middle| S_t = s, \pi_t, D_t\right)$$

$$= E\left(\frac{\frac{\partial J^{iI}}{\partial W_{t+1}}}{E\left(\frac{\partial J^{iI}}{\partial W_{t+1}} \middle| S_t = s, \pi_t, D_t\right)} \middle| S_t = s, \pi_t, D_t\right) \cdot E\left(P_{t+1}{}^{iI} + D_{t+1} \middle| S_t = s, \pi_t, D_t\right)$$

$$= E\left(P_{t+1}{}^{iI} + D_{t+1} \middle| S_t = s, \pi_t, D_t\right)$$

Note that $E\left(\frac{\frac{\partial J^{iI}}{\partial W_{t+1}}}{E\left(\frac{\partial J^{iI}}{\partial W_{t+1}}\middle|S_t=s,\pi_t,D_t\right)}\middle|S_t = s, \pi_t, D_t\right)$ equals one.

Putting Steps 2 and 3 together, the first term on the right-hand side of (A 2-21) reads:

$$\sum_{s=1}^{K} \pi_{s,t} \cdot E\left(q_{t,t+1}{}^{iI} \cdot E\left(P_{t+1}{}^{iI} + D_{t+1}\middle|S_t, \pi_t, D_t\right)\middle|S_t = s, \pi_t, D_t\right)$$
$$= \frac{1}{1+r} \cdot \sum_{s=1}^{K} \theta_{s,t}(\pi_t, D_t) \cdot E\left(P_{t+1}{}^{iI} + D_{t+1}\middle|S_t, \pi_t, D_t\right)$$

<u>Step 4</u>: Evaluate the second term

$$-\sum_{s=1}^{K} \pi_{s,t} \cdot E\left(q_{t,t+1}{}^{iI} \cdot E\left(P_{t+1}{}^{iI} + D_{t+1}\middle|\pi_t, D_t\right)\middle|S_t = s, \pi_t, D_t\right)$$

on the right-hand side of (A 2-21):

First, observe that the conditional expectation $E\left(P_{t+1}{}^{iI} + D_{t+1}\middle|\pi_t, D_t\right)$ is non-stochastic conditional on $S_t = s, \pi_t, D_t$. Hence, it can be factored out:

$$-\sum_{s=1}^{K} \pi_{s,t} \cdot E\left(q_{t,t+1}{}^{iI} \cdot E\left(P_{t+1}{}^{iI} + D_{t+1}\middle|\pi_t, D_t\right)\middle|S_t = s, \pi_t, D_t\right)$$
$$= -\sum_{s=1}^{K} \pi_{s,t} \cdot E\left(q_{t,t+1}{}^{iI}\middle|S_t = s, \pi_t, D_t\right) \cdot E\left(P_{t+1}{}^{iI} + D_{t+1}\middle|\pi_t, D_t\right)$$

Second, the expectation of the stochastic discount factor evaluates to $\frac{1}{1+r}$, yielding

$$\sum_{s=1}^{K} \pi_{s,t} \cdot E\left(q_{t,t+1}{}^{iI} \cdot \left\{-E\left(P_{t+1}{}^{iI} + D_{t+1}\middle|\pi_t, D_t\right)\right\}\middle|S_t = s, \pi_t, D_t\right)$$
$$= -\frac{1}{1+r} \cdot E\left(P_{t+1}{}^{iI} + D_{t+1}\middle|\pi_t, D_t\right)$$

Third, rewrite the expectation conditional on π_t, D_t as an expectation over all possible values of the unobservable current regime S_t

$$-\sum_{s=1}^{K} \pi_{s,t} \cdot E\left(q_{t,t+1}{}^{iI} \cdot E\left(P_{t+1}{}^{iI} + D_{t+1}\middle|\pi_t, D_t\right)\middle|S_t = s, \pi_t, D_t\right)$$
$$= -\frac{1}{1+r} \cdot \sum_{s=1}^{K} \pi_{s,t} \cdot E\left(P_{t+1}{}^{iI} + D_{t+1}\middle|S_t = s, \pi_t, D_t\right)$$

Step 5: Final result

Putting Steps 2 and 4 together, the right-hand side of (A 2-21) reads:

$$E\left(q_{t,t+1}{}^{iI} \cdot \Delta^{exp.risk} \middle| \pi_t, D_t\right)$$
$$= E\left(q_{t,t+1}{}^{iI} \cdot \left\{E\left(P_{t+1}{}^{iI} + D_{t+1} \middle| S_t, \pi_t, D_t\right) - E\left(P_{t+1}{}^{iI} + D_{t+1} \middle| \pi_t, D_t\right)\right\} \middle| \pi_t, D_t\right)$$
$$= \frac{1}{1+r}$$
$$\cdot \left\{\sum_{s=1}^{K} \theta_{s,t}(\pi_t, D_t) \cdot E\left(P_{t+1}{}^{iI} + D_{t+1} \middle| S_t = s, \pi_t, D_t\right)\right.$$
$$\left. - \sum_{s=1}^{K} \pi_{s,t} \cdot E\left(P_{t+1}{}^{iI} + D_{t+1} \middle| S_t = s, \pi_t, D_t\right)\right\}$$
$$= -\frac{1}{1+r} \cdot \sum_{s=1}^{K} \left\{\pi_{s,t} - \theta_{s,t}(\pi_t, D_t)\right\} \cdot E\left(P_{t+1}{}^{iI} + D_{t+1} \middle| S_t = s, \pi_t, D_t\right)$$

as was to be shown (3-76).

A2.4.1.2.2 Price of "Combined Risk"

Step 1: Rewrite the expectation conditional on π_t, D_t as an expectation over all possible values of the unobservable current regime S_t

$$E\left(q_{t,t+1}{}^{iI} \cdot \Delta^{comb.risk} \middle| \pi_t, D_t\right) = \sum_{s=1}^{K} \pi_{s,t} \cdot E\left(q_{t,t+1}{}^{iI} \cdot \Delta^{comb.risk} \middle| S_t = s, \pi_t, D_t\right)$$

Step 2: Re-express the right-hand side of the previous equation with the help of risk-neutralized regime probabilities

Using

$$q_{t,t+1}{}^{iI} = \frac{1}{1+r} \cdot \frac{\frac{\partial J^{iI}}{\partial W_{t+1}}}{E\left(\frac{\partial J^{iI}}{\partial W_{t+1}} \middle| \pi_t, D_t\right)}$$

yields

$$E\left(q_{t,t+1}{}^{iI} \cdot \Delta^{comb.risk} \middle| \pi_t, D_t\right)$$
$$= \frac{1}{1+r} \cdot \sum_{s=1}^{K} \pi_{s,t} \cdot E\left(\frac{\frac{\partial J^{iI}}{\partial W_{t+1}}}{E\left(\frac{\partial J^{iI}}{\partial W_{t+1}} \middle| \pi_t, D_t\right)} \cdot \Delta^{comb.risk} \middle| S_t = s, \pi_t, D_t\right)$$

Factoring out $E\left(\frac{\partial J^{iI}}{\partial W_{t+1}} \middle| \pi_t, D_t\right)$ and expanding by $E\left(\frac{\partial J^{iI}}{\partial W_{t+1}} \middle| S_t = s, \pi_t, D_t\right)$ yields

$$E\left(q_{t,t+1}{}^{iI}\cdot\Delta^{comb.risk}\middle|\pi_t,D_t\right)$$
$$=\frac{1}{1+r}$$
$$\cdot\sum_{s=1}^{K}\theta_{s,t}(\pi_t,D_t)\cdot E\left(\frac{\frac{\partial J^{iI}}{\partial W_{t+1}}}{E\left(\frac{\partial J^{iI}}{\partial W_{t+1}}\middle|S_t=s,\pi_t,D_t\right)}\cdot\Delta^{comb.risk}\middle|S_t=s,\pi_t,D_t\right)$$

with risk-neutralized regime probabilities

$$\theta_{s,t}(\pi_t,D_t)=\frac{\pi_{s,t}\cdot E\left(\frac{\partial J^{iI}}{\partial W_{t+1}}\middle|S_t=s,\pi_t,D_t\right)}{E\left(\frac{\partial J^{iI}}{\partial W_{t+1}}\middle|\pi_t,D_t\right)}$$

Step 3: Integrate the adjustment for risk

Using

$$AfR_t{}^{conditional}(s;\eta_{t+1},S_{t+1},fe_{t+1},S_t,\pi_t,D_t)=\frac{\frac{\partial J^{iI}}{\partial W_{t+1}}}{E\left(\frac{\partial J^{iI}}{\partial W_{t+1}}\middle|S_t=s,\pi_t,D_t\right)}$$

the price of “combined risk” (3-78) is obtained as follows:

$$=\frac{1}{1+r}\cdot\sum_{s=1}^{K}\theta_{s,t}(\pi_t,D_t)$$
$$\cdot E\left(AfR_t{}^{conditional}(s;\eta_{t+1},S_{t+1},fe_{t+1},S_t,\pi_t,D_t)\cdot\Delta^{comb.risk}\middle|S_t=s,\pi_t,D_t\right)$$

A2.4.2 Appendix to Section 3.3.2.2.3.4: Information Frequency $\geq$ Cash Flow Frequency

A2.4.2.1 Formulation of the Problem

Definition and price of “expectation risk” and “combined risk” for the case where information frequency is higher than or equal to cash flow frequency have been omitted in the main text because they are almost entirely analogous to the special case where information frequency equals cash flow frequency. For the sake of completeness, “expectation risk” and “combined risk” are defined and priced in this section of the appendix.

A2.4.2.1.1 Definition of “expectation risk” and “combined risk” in the case where information frequency is higher than or equal to cash flow frequency

Total risks

Total risk is defined as the deviation of risky asset prices and cash flows from their expectations conditional on information relevant to pricing, $z_t^{iI,(\Delta_C),p} = \left(\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)$:

For (Δ_C)-periodic risky assets

Case 1: (Δ_C)-periodic cash flows are paid in $t+1$ $(t+1 = t_{(k_t+1)})$

$$P_{t+1}{}^{iI,(\Delta_C)}\left(\pi_{t+1,t+1}, D_{t+1}{}^{(\Delta_C)}, D_{t+1}{}^{(1)}\right) + D_{t_{(k+1)}}{}^{(\Delta_C)} - E_t\left(P_{t+1}{}^{iI,(\Delta_C)}\left(\pi_{t+1,t+1}, D_{t+1}{}^{(\Delta_C)}, D_{t+1}{}^{(1)}\right) + D_{t_{(k+1)}}{}^{(\Delta_C)} \middle| \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)$$

Case 2: no (Δ_C)-periodic cash flows in $t+1$ $(t_{(k_t)} \leq t < t_{(k_t+1)})$

$$P_{t+1}{}^{iI,(\Delta_C)}\left(\pi_{t_{(k_t)},t+1}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_{t+1}{}^{(1)}\right) - E_t\left(P_{t+1}{}^{iI,(\Delta_C)}\left(\pi_{t_{(k_t)},t+1}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_{t+1}{}^{(1)}\right) \middle| \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)$$

where cash flows and the recursion for conditional regime path probabilities are given by

$$D_{t+1}{}^{(\Delta_C)} = \begin{cases} D^{(\Delta_C)}\left(D_{t_{(k)}}{}^{(\Delta_C)}, \pi_{t_{(k)},t}, fe_{t+1}\right) & t+1 = t_{(k+1)} \\ 0 & t+1 \neq t_{(k+1)} \end{cases}$$

$$D_{t+1}{}^{(1)} = D^{(1)}\left(D_t{}^{(1)}, S_t, fe_{t+1}\right)$$

$$\pi_{t_{(k+1)},t_{(k+1)}} = \Pi_1\begin{pmatrix} \pi_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta_C)}, D_{,t}{}^{(1)} \\ , D_{t+1}{}^{(\Delta_C)}, D_{t+1}{}^{(1)}, Sig_{t+1} \end{pmatrix}$$

$$Sig_{t+1} = Sig_{t+1}\left(S_{t_{(k)},t}, S_{t+1}, fe_{t+1}, \eta_{t+1}\right)$$

For brevity, I adopt the following notation that omits the details of the dynamics of cash flows and regime path probabilities to cover both cases:

A 2-24
$$P_{t+1}{}^{iI,(\Delta_C)} + 1_{t+1=t_{(k+1)}} \cdot D_{t+1}{}^{(\Delta_C)} - E_t\left(P_{t+1}{}^{iI,(\Delta_C)} + 1_{t+1=t_{(k+1)}} \cdot D_{t+1}{}^{(\Delta_C)} \middle| \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)$$

with

$$1_{t+1=t_{(k+1)}} \cdot D_{t+1}{}^{(\Delta_C)} = \begin{cases} 0 & t+1 \neq t_{(k+1)} \\ D_{t_{(k+1)}}{}^{(\Delta_C)} & t+1 = t_{(k+1)} \end{cases}$$

In similar notation, the deviation of asset prices and cash flows from their information conditional on information relevant to pricing reads:

For (1)-periodic risky assets

A 2-25

$$P_{t+1}{}^{iI,(1)} + D_{t+1}{}^{(1)} - E_t\left(P_{t+1}{}^{iI,(1)} + D_{t+1}{}^{(1)} \middle| \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)$$

Definition of "expectation risk" and "combined risk"

As in the special case with information frequency equal to cash flow frequency, I decompose (A 2-24) and (A 2-25) into an "expectation risk" component and a "combined risk" component:

A 2-26

$$\begin{aligned}
&P_{t+1}{}^{iI,(\Delta_C)} + 1_{t+1=t_{(k+1)}} \cdot D_{t+1}{}^{(\Delta_C)} - E_t\left(P_{t+1}{}^{iI,(\Delta_C)} + 1_{t+1=t_{(k+1)}} \cdot D_{t+1}{}^{(\Delta_C)} \middle| \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) \\
&= \underbrace{\left\{\begin{matrix} P_{t+1}{}^{iI,(\Delta_C)} + 1_{t+1=t_{(k+1)}} \cdot D_{t+1}{}^{(\Delta_C)} \\ -E\left(P_{t+1}{}^{iI,(\Delta_C)} + 1_{t+1=t_{(k+1)}} \cdot D_{t+1}{}^{(\Delta_C)} \middle| S_{t_{(k_t)},t}, \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) \end{matrix}\right\}}_{\Delta^{comb.risk}} \\
&+ \underbrace{\left\{\begin{matrix} E\left(P_{t+1}{}^{iI,(\Delta_C)} + 1_{t+1=t_{(k+1)}} \cdot D_{t+1}{}^{(\Delta_C)} \middle| S_{t_{(k_t)},t}, \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) \\ -E\left(P_{t+1}{}^{iI,(\Delta_C)} + 1_{t+1=t_{(k+1)}} \cdot D_{t+1}{}^{(\Delta_C)} \middle| \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) \end{matrix}\right\}}_{\Delta^{exp.risk}}
\end{aligned}$$

A 2-27

$$\begin{aligned}
&P_{t+1}{}^{iI,(1)} + D_{t+1}{}^{(1)} - E_t\left(P_{t+1}{}^{iI,(1)} + D_{t+1}{}^{(1)} \middle| \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) \\
&= \underbrace{\left\{P_{t+1}{}^{iI,(1)} + D_{t+1}{}^{(1)} - E\left(P_{t+1}{}^{iI,(1)} + D_{t+1}{}^{(1)} \middle| S_{t_{(k_t)},t}, \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)\right\}}_{\Delta^{comb.risk}} \\
&+ \underbrace{\left\{\begin{matrix} E\left(P_{t+1}{}^{iI,(1)} + D_{t+1}{}^{(1)} \middle| S_{t_{(k_t)},t}, \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) \\ -E\left(P_{t+1}{}^{iI,(1)} + D_{t+1}{}^{(1)} \middle| \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) \end{matrix}\right\}}_{\Delta^{exp.risk}}
\end{aligned}$$

A2.4.2.1.2 Price of "expectation risk" and "combined risk"

Price of "expectation risk"

The price of "expectation risk" (defined in (A 2-26) and (A 2-27)) for both (Δ_C)-periodic and (1)-periodic assets reads

A 2-28

$$\begin{aligned}
&E\left(q_{t,t+1}{}^{iI,(\Delta_C)}\left(\eta_{t+1}, S_{t+1}, fe_{t+1}, S_{t_{(k)},t}, \pi_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) \cdot \Delta^{exp.risk} \middle| \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) \\
&= -\frac{1}{1+r} \\
&\cdot \sum_{s_{t_{(k)},t}} \left\{\pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right) - \theta_t\left(s_{t_{(k)},t}; \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)\right\} \\
&\cdot E\left(P_{t+1}{}^{iI,(\nu)} + D_{t+1}{}^{(\nu)} \middle| S_{t_{(k_t)},t} = s_{t_{(k)},t}, \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)
\end{aligned}$$

with

$$E\left(P_{t+1}{}^{iI,(\nu)}+D_{t+1}{}^{(\nu)}\middle|S_{t_{(k_t)},t}=s_{t_{(k)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)$$
$$=\begin{cases}E\left(P_{t+1}{}^{iI,(\Delta_C)}+1_{t+1=t_{(k+1)}}\cdot D_{t+1}{}^{(\Delta_C)}\middle|S_{t_{(k_t)},t}=s_{t_{(k)},t},\pi_{t_{(k_t)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\right) & \nu=\Delta_C\\ E\left(P_{t+1}{}^{iI,(1)}+D_{t+1}{}^{(1)}\middle|S_{t_{(k_t)},t}=s_{t_{(k)},t},\pi_{t_{(k_t)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\right) & \nu=1\end{cases}$$

and with risk-neutralized regime-path probabilities

$$\theta_t\left(s_{t_{(k)},t};\pi_{t_{(k_t)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)$$
$$\equiv\frac{\pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right)\cdot E\left(\begin{Bmatrix}exp\left(-\alpha_{t+1}\cdot\left\{D_{t+1}{}^{market}+V_{t+1}{}^{market}\right\}\right)\\ \cdot m_{t+1}{}^{iI,(\Delta_C)}\end{Bmatrix}\middle|\begin{matrix}S_{t_{(k)},t}=s_{t_{(k)},t}\\ ,\pi_{t_{(k_t)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\end{matrix}\right)}{\sum_{s_{t_{(k)},t}{}'}\pi_{t_{(k)},t}\left(s_{t_{(k)},t}{}'\right)\cdot E\left(\begin{Bmatrix}exp\left(-\alpha_{t+1}\cdot\left\{D_{t+1}{}^{market}+V_{t+1}{}^{market}\right\}\right)\\ \cdot m_{t+1}{}^{iI,(\Delta_C)}\end{Bmatrix}\middle|\begin{matrix}S_{t_{(k)},t}=s_{t_{(k)},t}{}'\\ ,\pi_{t_{(k_t)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\end{matrix}\right)}$$

with

$$D_{t+1}{}^{market}=\begin{cases}D_{t+1}{}^{market}\left(D_{t+1}{}^{(\Delta_C)},D_{t+1}{}^{(1)}\right) & t+1=t_{(k+1)}\\ D_{t+1}{}^{market}\left(D_{t+1}{}^{(1)}\right) & t+1\neq t_{(k+1)}\end{cases}$$

$$V_{t+1}{}^{market}=\begin{cases}V_{t+1}{}^{market}\left(\pi_{t+1t+1},D_{t+1}{}^{(\Delta_C)},D_{t+1}{}^{(1)}\right) & t+1=t_{(k+1)}\\ V_{t+1}{}^{market}\left(\pi_{t_{(k)},t+1},D_{t_{(k)}}{}^{(\Delta_C)},D_{t+1}{}^{(1)}\right) & t+1\neq t_{(k+1)}\end{cases}$$

$$m_{t+1}{}^{iI,(\Delta_C)}=\begin{cases}m_{t+1}{}^{iI,(\Delta_C)}\left(\pi_{t+1,t+1},D_{t+1}{}^{(\Delta_C)},D_{t+1}{}^{(1)}\right) & t+1=t_{(k+1)}\\ m_{t+1}{}^{iI,(\Delta_C)}\left(\pi_{t_{(k)},t+1},D_{t_{(k)}}{}^{(\Delta_C)},D_{t+1}{}^{(1)}\right) & t+1\neq t_{(k+1)}\end{cases}$$

<u>Price of “combined risk”</u>

The price of “combined risk” (defined in (A 2-26) and (A 2-27)) for both (Δ_C)-periodic and (1)-periodic assets reads

A 2-29

$$E\left(q_{t,t+1}{}^{iI,(\Delta_C)}\left(\eta_{t+1},S_{t+1},fe_{t+1},S_{t_{(k)},t},\pi_{t_{(k)},t},D_{t_{(k)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)\cdot\Delta^{comb.risk}\middle|\pi_{t_{(k_t)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)$$
$$=\frac{1}{1+r}$$
$$\cdot\sum_{s_{t_{(k)},t}}\theta_t\left(s_{t_{(k)},t};\pi_{t_{(k_t)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)$$
$$\cdot E\left(AfR_t{}^{conditional}\left(\eta_{t+1},S_{t+1},fe_{t+1},S_{t_{(k)},t},\pi_{t_{(k)},t},D_{t_{(k)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)\cdot\Delta^{comb.risk}\middle|\begin{matrix}S_{t_{(k_t)},t}=s_{t_{(k)},t}\\ ,\pi_{t_{(k_t)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\end{matrix}\right)$$

with

$$AfR_t{}^{conditional}\left(\eta_{t+1},S_{t+1},fe_{t+1},S_{t_{(k)},t},\pi_{t_{(k)},t},D_{t_{(k)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)$$
$$\equiv\frac{\begin{matrix}exp\left(-\alpha_{t+1}\cdot\left\{D_{t+1}{}^{market}+V_{t+1}{}^{market}\right\}\right)\\ \cdot m_{t+1}{}^{iI,(\Delta_C)}\end{matrix}}{E\left(\begin{matrix}exp\left(-\alpha_{t+1}\cdot\left\{D_{t+1}{}^{market}+V_{t+1}{}^{market}\right\}\right)\\ \cdot m_{t+1}{}^{iI,(\Delta_C)}\end{matrix}\middle|S_{t_{(k_t)},t},\pi_{t_{(k_t)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)}$$

with

$$D_{t+1}{}^{market} = \begin{cases} D_{t+1}{}^{market}\left(D_{t+1}{}^{(\Delta_C)}, D_{t+1}{}^{(1)}\right) & t+1 = t_{(k+1)} \\ D_{t+1}{}^{market}\left(D_{t+1}{}^{(1)}\right) & t+1 \neq t_{(k+1)} \end{cases}$$

$$V_{t+1}{}^{market} = \begin{cases} V_{t+1}{}^{market}\left(\pi_{t+1t+1}, D_{t+1}{}^{(\Delta_C)}, D_{t+1}{}^{(1)}\right) & t+1 = t_{(k+1)} \\ V_{t+1}{}^{market}\left(\pi_{t_{(k)},t+1}, D_{t_{(k)}}{}^{(\Delta_C)}, D_{t+1}{}^{(1)}\right) & t+1 \neq t_{(k+1)} \end{cases}$$

$$m_{t+1}{}^{iI,(\Delta_C)} = \begin{cases} m_{t+1}{}^{iI,(\Delta_C)}\left(\pi_{t+1,t+1}, D_{t+1}{}^{(\Delta_C)}, D_{t+1}{}^{(1)}\right) & t+1 = t_{(k+1)} \\ m_{t+1}{}^{iI,(\Delta_C)}\left(\pi_{t_{(k)},t+1}, D_{t_{(k)}}{}^{(\Delta_C)}, D_{t+1}{}^{(1)}\right) & t+1 \neq t_{(k+1)} \end{cases}$$

A2.4.2.2 Proof

A2.4.2.2.1 Idea of the proof

The proof follows the steps of the case where information frequency is equal to cash flow frequency. The single regime S_t must merely be replaced by the path of regimes since the last payment date of (Δ_C)-periodic assets, $S_{t_{(k_t)},t}$.

A2.4.2.2.2 Details of the proof

A2.4.2.2.2.1 Price of "Expectation Risk"

Step 1: Rewrite the expectation conditional on $\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}$ as an expectation over all possible values of the unobservable current regime path $S_{t_{(k_t)},t}$

A 2-30

$$E\left(q_{t,t+1}{}^{iI,(\Delta_C)} \cdot \left\{ \begin{array}{c} E\left(P_{t+1}{}^{iI} + D_{t+1} \middle| S_{t_{(k_t)},t}, \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) \\ -E\left(P_{t+1}{}^{iI} + D_{t+1} \middle| \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) \end{array} \right\} \middle| \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)} \right)$$

$$= \sum_{s_{t_{(k)},t}} \pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right)$$

$$\cdot E\left(q_{t,t+1}{}^{iI,(\Delta_C)} \cdot \left\{ \begin{array}{c} E\left(P_{t+1}{}^{iI,(\nu)} + D_{t+1}{}^{(\nu)} \middle| S_{t_{(k_t)},t}, \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) \\ -E\left(P_{t+1}{}^{iI,(\nu)} + D_{t+1}{}^{(\nu)} \middle| \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) \end{array} \right\} \middle| \begin{array}{c} S_{t_{(k_t)},t} = s_{t_{(k)},t} \\ , \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)} \end{array} \right)$$

with

$$P_{t+1}{}^{iI,(\nu)} + D_{t+1}{}^{(\nu)}$$
$$\equiv \begin{cases} E\left(P_{t+1}{}^{iI,(\Delta_C)} + 1_{t+1=t_{(k+1)}} \cdot D_{t+1}{}^{(\Delta_C)} \middle| S_{t_{(k_t)},t} = s_{t_{(k)},t}, \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) & \nu = \Delta_C \\ E\left(P_{t+1}{}^{iI,(1)} + D_{t+1}{}^{(1)} \middle| S_{t_{(k_t)},t} = s_{t_{(k)},t}, \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) & \nu = 1 \end{cases}$$

Step 2: Re-express the first term on the right-hand side of the previous equation,

$$\sum_{s_{t_{(k)},t}} \pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right) \cdot E\left(q_{t,t+1}{}^{iI,(\Delta_C)} \cdot E\left(P_{t+1}{}^{iI,(\nu)} + D_{t+1}{}^{(\nu)} \middle| S_{t_{(k_t)},t}, \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) \middle| \begin{matrix} S_{t_{(k_t)},t} = s_{t_{(k)},t} \\ , \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)} \end{matrix} \right)$$

with the help of risk-neutralized regime path probabilities:

For brevity, first define

A 2-31

$$\frac{\partial J^{iI}}{\partial W_{t+1}} \equiv exp\left(-\alpha_{t+1} \cdot \left\{D_{t+1}{}^{market} + V_{t+1}{}^{market}\right\}\right) \cdot m_{t+1}{}^{iI,(\Delta_C)}$$

In this notation, the stochastic discount factor reads

$$q_{t,t+1}{}^{iI,(\Delta_C)} = \frac{1}{1+r} \cdot \frac{\frac{\partial J^{iI}}{\partial W_{t+1}}}{E\left(\frac{\partial J^{iI}}{\partial W_{t+1}} \middle| \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)}$$

and risk-neutralized regime path probabilities read

$$\theta_t\left(s_{t_{(k)},t}; \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) \equiv \frac{\pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right) \cdot E\left(\frac{\partial J^{iI}}{\partial W_{t+1}} \middle| \begin{matrix} S_{t_{(k)},t} = s_{t_{(k)},t} \\ , \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)} \end{matrix}\right)}{\sum_{s_{t_{(k)},t}{}'} \pi_{t_{(k)},t}\left(s_{t_{(k)},t}{}'\right) \cdot E\left(\frac{\partial J^{iI}}{\partial W_{t+1}} \middle| \begin{matrix} S_{t_{(k)},t} = s_{t_{(k)},t}{}' \\ , \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)} \end{matrix}\right)}$$

Then the first term on the right-hand side of (A 2-30) can be rewritten as

$$\sum_{s_{t_{(k)},t}} \pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right) \cdot E\left(q_{t,t+1}{}^{iI,(\Delta_C)} \cdot E\left(P_{t+1}{}^{iI,(\nu)} + D_{t+1}{}^{(\nu)} \middle| S_{t_{(k_t)},t}, \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) \middle| \begin{matrix} S_{t_{(k_t)},t} = s_{t_{(k)},t} \\ , \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)} \end{matrix} \right)$$

$$= \frac{1}{1+r} \cdot \sum_{s_{t_{(k)},t}} \pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right) \cdot E\left(\begin{matrix} \frac{\frac{\partial J^{iI}}{\partial W_{t+1}}}{E\left(\frac{\partial J^{iI}}{\partial W_{t+1}} \middle| \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)} \\ \cdot E\left(P_{t+1}{}^{iI,(\nu)} + D_{t+1}{}^{(\nu)} \middle| S_{t_{(k_t)},t}, \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) \end{matrix} \middle| \begin{matrix} S_{t_{(k_t)},t} = s_{t_{(k)},t} \\ , \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)} \end{matrix} \right)$$

Rearranging terms yields:

$$\sum_{s_{t(k),t}} \pi_{t(k),t}\left(s_{t(k),t}\right)$$

$$\cdot E\left(q_{t,t+1}{}^{iI,(\Delta_C)} \cdot E\left(P_{t+1}{}^{iI,(\nu)} + D_{t+1}{}^{(\nu)} \middle| S_{t(k_t),t}, \pi_{t(k_t),t}, D_{t(k_t)}{}^{(\Delta_C)}, D_t{}^{(1)}\right) \middle| \begin{matrix} S_{t(k_t),t} = s_{t(k),t} \\ , \pi_{t(k_t),t}, D_{t(k_t)}{}^{(\Delta_C)}, D_t{}^{(1)} \end{matrix}\right)$$

$$= \frac{1}{1+r}$$

$$\cdot \sum_{s_{t(k),t}} \theta_t\left(s_{t(k),t}; \pi_{t(k_t),t}, D_{t(k_t)}{}^{(\Delta_C)}, D_t{}^{(1)}\right) \cdot E\left(P_{t+1}{}^{iI,(\nu)} + D_{t+1}{}^{(\nu)} \middle| S_{t(k_t),t}, \pi_{t(k_t),t}, D_{t(k_t)}{}^{(\Delta_C)}, D_t{}^{(1)}\right)$$

<u>Step 3</u>: Evaluate the second term

$$-\sum_{s_{t(k),t}} \pi_{t(k),t}\left(s_{t(k),t}\right)$$

$$\cdot E\left(q_{t,t+1}{}^{iI,(\Delta_C)} \cdot E\left(P_{t+1}{}^{iI,(\nu)} + D_{t+1}{}^{(\nu)} \middle| \pi_{t(k_t),t}, D_{t(k_t)}{}^{(\Delta_C)}, D_t{}^{(1)}\right) \middle| \begin{matrix} S_{t(k_t),t} = s_{t(k),t} \\ , \pi_{t(k_t),t}, D_{t(k_t)}{}^{(\Delta_C)}, D_t{}^{(1)} \end{matrix}\right)$$

on the right-hand side of (A 2-30)

Since $\left(P_{t+1}{}^{iI,(\nu)} + D_{t+1}{}^{(\nu)} \middle| \pi_{t(k_t),t}, D_{t(k_t)}{}^{(\Delta_C)}, D_t{}^{(1)}\right)$ is non-stochastic conditional on $S_{t(k_t),t} = s_{t(k),t}, \pi_{t(k_t),t}, D_{t(k_t)}{}^{(\Delta_C)}, D_t{}^{(1)}$, it can be factored out of the outer expectation:

$$-\sum_{s_{t(k),t}} \pi_{t(k),t}\left(s_{t(k),t}\right)$$

$$\cdot E\left(q_{t,t+1}{}^{iI,(\Delta_C)} \cdot E\left(P_{t+1}{}^{iI,(\nu)} + D_{t+1}{}^{(\nu)} \middle| \pi_{t(k_t),t}, D_{t(k_t)}{}^{(\Delta_C)}, D_t{}^{(1)}\right) \middle| \begin{matrix} S_{t(k_t),t} = s_{t(k),t} \\ , \pi_{t(k_t),t}, D_{t(k_t)}{}^{(\Delta_C)}, D_t{}^{(1)} \end{matrix}\right)$$

$$= -\sum_{s_{t(k),t}} \pi_{t(k),t}\left(s_{t(k),t}\right) \cdot E\left(q_{t,t+1}{}^{iI,(\Delta_C)} \middle| \begin{matrix} S_{t(k_t),t} = s_{t(k),t} \\ , \pi_{t(k_t),t}, D_{t(k_t)}{}^{(\Delta_C)}, D_t{}^{(1)} \end{matrix}\right)$$

$$\cdot E\left(P_{t+1}{}^{iI,(\nu)} + D_{t+1}{}^{(\nu)} \middle| \pi_{t(k_t),t}, D_{t(k_t)}{}^{(\Delta_C)}, D_t{}^{(1)}\right)$$

Moreover, the first term on the right-hand side of this previous equation is the price of a riskless bond,

$$\sum_{s_{t(k),t}} \pi_{t(k),t}\left(s_{t(k),t}\right) \cdot E\left(q_{t,t+1}{}^{iI,(\Delta_C)} \middle| \begin{matrix} S_{t(k_t),t} = s_{t(k),t} \\ , \pi_{t(k_t),t}, D_{t(k_t)}{}^{(\Delta_C)}, D_t{}^{(1)} \end{matrix}\right) = \frac{1}{1+r}$$

By rewriting $E\left(P_{t+1}{}^{iI,(\nu)} + D_{t+1}{}^{(\nu)} \middle| \pi_{t(k_t),t}, D_{t(k_t)}{}^{(\Delta_C)}, D_t{}^{(1)}\right)$ as an expectation over all possible unobservable regime paths $S_{t(k_t),t}$, the second term on the right-hand side of (A 2-30) reads:

$$- \sum_{s_{t_{(k)},t}} \pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right)$$

$$\cdot E\left(q_{t,t+1}{}^{iI,(\Delta_C)} \cdot E\left(P_{t+1}{}^{iI,(\nu)} + D_{t+1}{}^{(\nu)} \middle| \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)} \right) \middle| \begin{matrix} S_{t_{(k_t)},t} = s_{t_{(k)},t} \\ , \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)} \end{matrix} \right)$$

$$= -\frac{1}{1+r} \cdot \sum_{s_{t_{(k)},t}} \pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right) \cdot E\left(P_{t+1}{}^{iI,(\nu)} + D_{t+1}{}^{(\nu)} \middle| \begin{matrix} S_{t_{(k_t)},t} = s_{t_{(k)},t} \\ , \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)} \end{matrix} \right)$$

Step 4: Final result

Putting Steps 2 and 3 together, the right-hand side of (A 2-30) reads:

$$E\left(q_{t,t+1}{}^{iI,(\Delta_C)} \cdot \Delta^{exp.risk} \middle| \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)} \right)$$

$$= -\frac{1}{1+r}$$

$$\cdot \sum_{s_{t_{(k)},t}} \left\{ \pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right) - \theta_t\left(s_{t_{(k)},t}; \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) \right\}$$

$$\cdot E\left(P_{t+1}{}^{iI,(\nu)} + D_{t+1}{}^{(\nu)} \middle| \begin{matrix} S_{t_{(k_t)},t} = s_{t_{(k)},t} \\ , \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)} \end{matrix} \right)$$

which is just (A 2-28).

A2.4.2.2.2.2 Price of "Combined Risk"

Step 1: Rewrite the expectation conditional on $\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}$ as an expectation over all possible values of the unobservable current regime path $S_{t_{(k_t)},t}$

$$E\left(q_{t,t+1}{}^{iI,(\Delta_C)} \cdot \Delta^{comb.risk} \middle| \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)} \right)$$

$$= \sum_{s_{t_{(k)},t}} \pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right) \cdot E\left(q_{t,t+1}{}^{iI,(\Delta_C)} \cdot \Delta^{comb.risk} \middle| \begin{matrix} S_{t_{(k_t)},t} = s_{t_{(k)},t} \\ , \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)} \end{matrix} \right)$$

Step 2: Re-express the right-hand side of the previous equation with the help of risk-neutralized regime path probabilities

Using

$$q_{t,t+1}{}^{iI} = \frac{1}{1+r} \cdot \frac{\dfrac{\partial J^{iI}}{\partial W_{t+1}}}{E\left(\dfrac{\partial J^{iI}}{\partial W_{t+1}} \middle| \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)} \right)}$$

yields

$$E\left(q_{t,t+1}{}^{iI,(\Delta_C)}\cdot\Delta^{comb.risk}\middle|\pi_{t_{(k_t)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)$$

$$=\frac{1}{1+r}$$

$$\cdot\sum_{s_{t_{(k)},t}}\pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right)$$

$$\cdot E\left(\frac{\frac{\partial J^{iI}}{\partial W_{t+1}}}{E\left(\frac{\partial J^{iI}}{\partial W_{t+1}}\middle|\pi_{t_{(k_t)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)}\cdot\Delta^{comb.risk}\middle|\begin{matrix}S_{t_{(k_t)},t}=s_{t_{(k)},t}\\,\pi_{t_{(k_t)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\end{matrix}\right)$$

$$=\frac{1}{1+r}$$

$$\cdot\sum_{s_{t_{(k)},t}}\theta_t\left(s_{t_{(k)},t};\pi_{t_{(k_t)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)$$

$$\cdot E\left(AfR_t{}^{conditional}\begin{pmatrix}\eta_{t+1},S_{t+1},fe_{t+1},S_{t_{(k)},t}\\,\pi_{t_{(k)},t},D_{t_{(k)}}{}^{(\Delta_C)},D_t{}^{(1)}\end{pmatrix}\cdot\Delta^{comb.risk}\middle|\begin{matrix}S_{t_{(k_t)},t}=s_{t_{(k)},t}\\,\pi_{t_{(k_t)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\end{matrix}\right)$$

with

$$AfR_t{}^{conditional}\left(\eta_{t+1},S_{t+1},fe_{t+1},S_{t_{(k)},t},\pi_{t_{(k)},t},D_{t_{(k)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)$$

$$\equiv\frac{\frac{\partial J^{iI}}{\partial W_{t+1}}}{E\left(\frac{\partial J^{iI}}{\partial W_{t+1}}\middle|\begin{matrix}S_{t_{(k_t)},t}=s_{t_{(k)},t}\\,\pi_{t_{(k_t)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\end{matrix}\right)}$$

This is just the desired result (A 2-29).

A3 Appendix to Section 4.1: General Equilibrium Asset Pricing with CRRA Preferences

Chapter 4 derived equilibrium asset prices by a recursive argument. The purpose of this section of the appendix is to illustrate the derivation of asset prices by the “guess and verify” approach: the equilibrium price process is stated (“guessed”) and then verified by solving the optimization problem of the identical investors. However, this is only possible if a concrete utility function is specified because the exact form of the value function of the identical investors must be known for the optimization. To that end, I assume a utility function that exhibits constant relative risk aversion (CRRA) because the argumentation can be easily implemented for CRRA utility functions.

A3.1 Appendix to Section 4.1.2.2: Concavity of the Maximand in the Bellman Equation

A3.1.1 Formulation of the Problem

The Bellman equation of each of the identical investors leads to a problem of the following kind: a function of the form

A 3-1

$$h(C,w;W_t,{z_t}^p) \equiv U(C) + \frac{1}{1+\rho} \cdot E\Big(J\big(W_{t+1}\big(w,C;W_t,{z_t}^p,{z_{t+1}}^p,{z_{t+1}}^d\big),{z_{t+1}}^p,t+1\big)\Big|{z_t}^p\Big)$$

with

$$W_{t+1}\big(w,C;W_t,{z_t}^p,{z_{t+1}}^p,{z_{t+1}}^d\big) = [W_t - C] \cdot \Big(1 + R^{Pf}\big({z_t}^p,{z_{t+1}}^p,{z_{t+1}}^d;w\big)\Big)$$

$$R^{Pf}\big({z_t}^p,{z_{t+1}}^p,{z_{t+1}}^d;w\big) = \Big\{r_t({z_t}^p) + \sum_{i=1}^{n} w_{i,t} \cdot \big\{R_{i,t,t+1}\big({z_t}^p,{z_{t+1}}^p,{z_{t+1}}^d\big) - r_t({z_t}^p)\big\}\Big\}$$

The function J is twice partially differentiable with respect to wealth W_{t+1} with

$$\frac{\partial}{\partial W_{t+1}} J > 0$$

$$\frac{\partial^2}{\partial {W_{t+1}}^2} J < 0$$

i.e., J is increasing and concave in wealth given ${z_{t+1}}^p$ and ${z_t}^p$. Similarly, the utility function $U(C)$ is increasing and concave in C

$$U' > 0$$

$$U'' < 0$$

The problem is to show that $h(C,w;W_t,z_t{}^p)$, $(C,w)\epsilon\mathbb{R}^{n+1}$, is a concave function. This is important because any combination of C and w that solves the first-order conditions $\frac{\partial h}{\partial C}=0$ and $\frac{\partial h}{\partial w}=\underline{0}_n$ then maximizes (A 3-1).

A3.1.2 Proof

A3.1.2.1 Idea of the Proof

The following characterization of concave functions is used (see Geiger/Kanzow (1999), Proposition 3.7a, p. 15):

If $X\subseteq\mathbb{R}^n$ is an open and convex set and $f{:}\,X\to\mathbb{R}$ is continuously differentiable, then f is concave if and only if

A 3-2

$$0\geq[x-x']^{\mathrm{T}}\cdot\{\nabla f(x)-\nabla f(x')\}\ \forall x,x'\epsilon X$$

where $\nabla f(x)$ is the gradient of the function f at x,

$$\nabla f(x)\equiv\begin{pmatrix}\frac{\partial f}{\partial x_1}(x)\\ \cdots\\ \frac{\partial f}{\partial x_n}(x)\end{pmatrix}$$

A3.1.2.2 Details of the Proof

Step 1: Find suitable choices for the function f and the set X in the consumption/portfolio context

Choose $h(.\,;W_t,z_t{}^p)$ as the function f, with arguments $\binom{C}{w}$ from $X\equiv\mathbb{R}^{n+1}$. Let $\binom{C}{w}$ and $\binom{C'}{w'}$ be any two combinations of consumption/portfolio holdings (corresponding to arbitrary vectors x and x' in (A 3-2)).

Step 2: Specify the right hand side of the inequality (A 3-2)

A 3-3

$$\left[\binom{C}{w}-\binom{C'}{w'}\right]^{\mathrm{T}}\cdot\left\{\nabla h\left(\binom{C}{w}\right)-\nabla h\left(\binom{C'}{w'}\right)\right\}$$

$$=\{C-C'\}\cdot\left\{\frac{\partial h}{\partial C}(C,w)-\frac{\partial h}{\partial C}(C',w')\right\}+\{w-w'\}^T\left\{\frac{\partial h}{\partial w}(C,w)-\frac{\partial h}{\partial w}(C',w')\right\}$$

where

$$\nabla h(C,w) \equiv \begin{pmatrix} \frac{\partial h}{\partial C}(C,w) \\ \dots \\ \frac{\partial h}{\partial w}(C,w) \end{pmatrix}$$

For brevity, I write $W_{t+1}(w,C;W_t)$ instead of $W_{t+1}\left(w,C;W_t,{z_t}^p,{z_{t+1}}^p,{z_{t+1}}^d\right)$, $R^{Pf}(w)$ instead of $R^{Pf}\left({z_t}^p,{z_{t+1}}^p,{z_{t+1}}^d;w\right)$ and $R_{t,t+1}-r_t$ instead of $R_{t,t+1}\left({z_t}^p,{z_{t+1}}^p,{z_{t+1}}^d\right)-r_t({z_t}^p)$.

In this notation, we have

A 3-4

$$\frac{\partial h}{\partial C}(C,w) = U'(C) - \frac{1}{1+\rho}\cdot E\left(\frac{\partial}{\partial W_{t+1}}J(W_{t+1}(w,C;W_t),{z_{t+1}}^p,t+1)\cdot\left(1+R^{Pf}(w)\right)\middle|{z_t}^p\right)$$

A 3-5

$$\frac{\partial h}{\partial w}(C,w) = \frac{1}{1+\rho}\cdot E\left(\frac{\partial}{\partial W_{t+1}}J(W_{t+1}(w,C;W_t),{z_{t+1}}^p,t+1)\cdot[W_t-C]\cdot\{R_{t,t+1}-r_t\}\middle|{z_t}^p\right)$$

Moreover, we have

$$\begin{aligned} &W_{t+1}(w,C;W_t) - W_{t+1}(w',C';W_t) \\ &\quad = [W_t-C]\cdot\left(1+r_t+w^T\{R_{t,t+1}-r_t\}\right) - [W_t-C']\cdot\left(1+r_t+w'^T\{R_{t,t+1}-r_t\}\right) \end{aligned}$$

The previous equation can be tautologically re-written by subtracting the term corresponding to the pair C and w'

$$[W_t-C]\cdot\left(1+r_t+w'^T\{R_{t,t+1}-r_t\}\right)$$

from both elements of the difference in the previous equation:

$$\begin{aligned} &W_{t+1}(w,C;W_t) - W_{t+1}(w',C';W_t) \\ &\quad = \left\{[W_t-C]\cdot\left(1+r_t+w^T\{R_{t,t+1}-r_t\}\right) - [W_t-C]\cdot\left(1+r_t+w'^T\{R_{t,t+1}-r_t\}\right)\right\} \\ &\quad - \left\{[W_t-C']\cdot\left(1+r_t+w'^T\{R_{t,t+1}-r_t\}\right) - [W_t-C]\right. \\ &\quad \left.\cdot\left(1+r_t+w'^T\{R_{t,t+1}-r_t\}\right)\right\} \\ &\quad = [W_t-C]\cdot\{w-w'\}^T\{R_{t,t+1}-r_t\} - \{[W_t-C']-[W_t-C]\}\cdot R^{Pf}(w') \\ &\quad = [W_t-C]\cdot\{w-w'\}^T\{R_{t,t+1}-r_t\} - \{C-C'\}\cdot R^{Pf}(w') \end{aligned}$$

Repeating the same argument with the alternative pair C' and w $[W_t-C']\cdot\left(1+r_t+w^T\{R_{t,t+1}-r_t\}\right)$ yields:

$$\begin{aligned} &W_{t+1}(w,C;W_t) - W_{t+1}(w',C';W_t) \\ &\quad = [W_t-C]\cdot\left(1+r_t+w^T\{R_{t,t+1}-r_t\}\right) - [W_t-C']\cdot\left(1+r_t+w'^T\{R_{t,t+1}-r_t\}\right) \\ &\quad = \left\{[W_t-C]\cdot\left(1+r_t+w^T\{R_{t,t+1}-r_t\}\right) - [W_t-C']\cdot\left(1+r_t+w^T\{R_{t,t+1}-r_t\}\right)\right\} \\ &\quad - \left\{[W_t-C']\cdot\left(1+r_t+w'^T\{R_{t,t+1}-r_t\}\right) - [W_t-C']\right. \\ &\quad \left.\cdot\left(1+r_t+w^T\{R_{t,t+1}-r_t\}\right)\right\} \\ &\quad = -\{C-C'\}\cdot R^{Pf}(w) + [W_t-C']\cdot\{w-w'\}^T\{R_{t,t+1}-r_t\} \end{aligned}$$

In short, it is obtained:

A 3-6

$$W_{t+1}(w,C;W_t)-W_{t+1}(w',C';W_t)=[W_t-C]\cdot\{w-w'\}^T\{R_{t,t+1}-r_t\}-\{C-C'\}\cdot R^{Pf}(w')$$

A 3-7

$$W_{t+1}(w,C;W_t)-W_{t+1}(w',C';W_t)=-\{C-C'\}\cdot R^{Pf}(w)+[W_t-C']\cdot\{w-w'\}^T\{R_{t,t+1}-r_t\}$$

Plugging (A 3-4), (A 3-5) into (A 3-3) yields

A 3-8

$$\left[\begin{pmatrix}C\\w\end{pmatrix}-\begin{pmatrix}C'\\w'\end{pmatrix}\right]^{\mathrm{T}}\cdot\left\{\nabla h\left(\begin{pmatrix}C\\w\end{pmatrix}\right)-\nabla h\left(\begin{pmatrix}C'\\w'\end{pmatrix}\right)\right\}$$

$$=\{C-C'\}$$

$$\cdot\left\{U'(C)-\frac{1}{1+\rho}\cdot E\left(\frac{\partial}{\partial W_{t+1}}J(W_{t+1}(w,C;W_t),z_{t+1}{}^p,t+1)\cdot\left(1+R^{Pf}(w)\right)\middle|z_t{}^p\right)\right.$$

$$-U'(C')+\frac{1}{1+\rho}$$

$$\left.\cdot E\left(\frac{\partial}{\partial W_{t+1}}J(W_{t+1}(w',C';W_t),z_{t+1}{}^p,t+1)\cdot\left(1+R^{Pf}(w')\right)\middle|z_t{}^p\right)\right\}$$

$$+\{w-w'\}^T\left\{\frac{1}{1+\rho}\right.$$

$$\cdot E\left(\frac{\partial}{\partial W_{t+1}}J(W_{t+1}(w,C;W_t),z_{t+1}{}^p,t+1)\cdot[W_t-C]\cdot\{R_{t,t+1}-r_t\}\middle|z_t{}^p\right)-\frac{1}{1+\rho}$$

$$\left.\cdot E\left(\frac{\partial}{\partial W_{t+1}}J(W_{t+1}(w',C';W_t),z_{t+1}{}^p,t+1)\cdot[W_t-C']\cdot\{R_{t,t+1}-r_t\}\middle|z_t{}^p\right)\right\}$$

Step 3: Show the non-positivity of (A 3-8)

Step 3a: Show that terms involving the utility function U are non-positive

Collecting all terms that include the function U yields:

$$\{C-C'\}\cdot\{U'(C)-U'(C')\}\leq 0$$

To see why this term is non-positive, first assume that $C\geq C'$ (i.e., $\{C-C'\}\geq 0$). Then the fact that $U''<0$ means that U' decreases in its argument, hence $\{U'(C)-U'(C')\}\leq 0$ and $\{C-C'\}\cdot\{U'(C)-U'(C')\}$. In the second case $C<C'$, similar reasoning yields $\{C-C'\}\cdot\{U'(C)-U'(C')\}<0$.

Step 3b: Show that terms involving the function J are non-positive

Collecting all terms that include the function J yields

$$-\frac{1}{1+\rho}\cdot E\left(\left.\begin{array}{c}\frac{\partial}{\partial W_{t+1}}J(W_{t+1}(w,C;W_t),z_{t+1}{}^p,t+1)\\ \cdot\left\{\{C-C'\}\cdot\left(1+R^{Pf}(w)\right)-\{w-w'\}^T[W_t-C]\cdot\{R_{t,t+1}-r_t\}\right\}\end{array}\right|z_t{}^p\right)+\frac{1}{1+\rho}$$

$$\cdot E\left(\left.\begin{array}{c}\frac{\partial}{\partial W_{t+1}}J(W_{t+1}(w',C';W_t),z_{t+1}{}^p,t+1)\\ \cdot\left\{\{C-C'\}\cdot\left(1+R^{Pf}(w')\right)-\{w-w'\}^T\cdot[W_t-C']\cdot\{R_{t,t+1}-r_t\}\right\}\end{array}\right|z_t{}^p\right)$$

By (A 3-6) and (A 3-7), this is identical to

$$\frac{1}{1+\rho}\cdot E\left(\left.\begin{array}{c}\left\{\frac{\partial}{\partial W_{t+1}}J(W_{t+1}(w,C;W_t),z_{t+1}{}^p,t+1)-\frac{\partial}{\partial W_{t+1}}J(W_{t+1}(w',C';W_t),z_{t+1}{}^p,t+1)\right\}\\ \cdot\{W_{t+1}(w,C;W_t)-W_{t+1}(w',C';W_t)\}\end{array}\right|z_t{}^p\right)$$

To see non-positivity, apply an argument similar to the terms involving the function U to the terms in the expectation: if $W_{t+1}(w,C;W_t)\geq W_{t+1}(w',C';W_t)$, concavity of J in W_{t+1} means

$$\frac{\partial}{\partial W_{t+1}}J(W_{t+1}(w,C;W_t),z_{t+1}{}^p,t+1)-\frac{\partial}{\partial W_{t+1}}J(W_{t+1}(w',C';W_t),z_{t+1}{}^p,t+1)\leq 0$$

and if $W_{t+1}(w,C;W_t)<W_{t+1}(w',C';W_t)$, then

$$\frac{\partial}{\partial W_{t+1}}J(W_{t+1}(w,C;W_t),z_{t+1}{}^p,t+1)-\frac{\partial}{\partial W_{t+1}}J(W_{t+1}(w',C';W_t),z_{t+1}{}^p,t+1)>0$$

In conclusion, the both summands that make up the right-hand side of (A 3-8), and therefore the sum itself, are non-positive, as was to be shown:

$$\left[\binom{C}{w}-\binom{C'}{w'}\right]^{\mathrm{T}}\cdot\left\{\nabla h\left(\binom{C}{w}\right)-\nabla h\left(\binom{C'}{w'}\right)\right\}\leq 0$$

A3.2 Value Function of Each of the Identical Investors

A3.2.1 Formulation of the Problem

The problem is to find the value function for the following consumption and portfolio selection problem of a CRRA investor:

$$E\left(\sum_{t=0}^{T} \frac{1}{(1+\rho)^t} \cdot U(C_t) \,\middle|\, z_0, {W_0}^{initial}\right)$$

with[51]

$$U(C) = \frac{C^{1-\gamma}}{1-\gamma}$$

$$\gamma \neq 1$$

by choosing portfolio weights of risky assets w_τ, $0 \le \tau \le T-1$ and consumption C_τ, $0 \le \tau \le T$

Note that in contrast to the partial equilibrium case, portfolio weights w_τ and portfolio/asset returns are used here instead of portfolio holdings N_τ and portfolio/asset cash flows. The reason is merely technical: the form of the value function is comparatively easy to find if the portfolio selection and consumption problem is formulated in terms of portfolio weights, while this is not the case with portfolio holdings.

Wealth dynamics read

A 3-9

$$W_{\tau+1} = [W_\tau - C_\tau] \cdot \left(1 + {R_{\tau,\tau+1}}^{Pf}(w)\right)$$

with portfolio return

$${R_{\tau,\tau+1}}^{Pf}(w) \equiv \left\{r_\tau + \sum_{i=1}^{n} w_{i,\tau} \cdot \{R_{i,\tau,\tau+1} - r_\tau\}\right\}$$

with return of asset i

$$R_{i,\tau,\tau+1} \equiv \frac{\{P_{i,\tau+1} + D_{i,\tau+1}\}}{P_{i,\tau}} - 1$$

$$i = 1, \ldots, n$$

where all remaining wealth is consumed at time T

$$W_T = C_T$$

---

[51] For $\gamma \to 1$, the limit of the utility function $U(C) = \frac{C^{1-\gamma}}{1-\gamma}$ does not exist. Of course, this problem could be solved by using the utility function $\widetilde{U}_\gamma(C) = \frac{C^{1-\gamma}-1}{1-\gamma}$ which approaches the logarithmic utility function for $\gamma \to 1$. However, $\widetilde{U}_\gamma(C)$ and $U(C)$ are equivalent from an economic point of view because utility functions are unique only up to a positive linear transformation. I do not separately study the case of logarithmic utility ($\gamma = 1$): first, because omitting the constant $\frac{-1}{1-\gamma}$ simplifies the exposition; second, because all price processes will be well defined in the limit $\gamma \to 1$.

Prices of risky assets, the price of the riskless bond and cash flows are functions of a Markov process $z_\tau \equiv \left(z_\tau{}^p, z_\tau{}^d\right)$:

$$P_\tau = \begin{cases} \underline{0} & \tau = T \\ P_\tau(z_\tau{}^p) & 0 \le \tau \le T-1 \end{cases}$$

$$B_\tau = \begin{cases} \underline{0} & \tau = T \\ B_\tau(z_\tau{}^p) & 0 \le \tau \le T-1 \end{cases}$$

$$D_\tau = D_\tau\left(z_\tau{}^d\right)$$

and, therefore,

$$R_{i,\tau,\tau+1}\left(z_\tau{}^p, z_{\tau+1}{}^p, z_{\tau+1}{}^d\right) = \frac{\left\{P_{i,\tau+1}(z_{\tau+1}{}^p) + D_{i,\tau+1}\left(z_{\tau+1}{}^d\right)\right\}}{P_{i,\tau}(z_\tau{}^p)}$$

$$R^{Pf}\left(z_\tau{}^p, z_{\tau+1}{}^p, z_{\tau+1}{}^d; w\right) = \left\{r_\tau(z_\tau{}^p) + \sum_{i=1}^n w_{i,\tau} \cdot \left\{R_{i,\tau,\tau+1}\left(z_\tau{}^p, z_{\tau+1}{}^p, z_{\tau+1}{}^d\right) - r_\tau(z_\tau{}^p)\right\}\right\}$$

Note that it is not yet assumed here that the process $\{z_\tau\}$ is the general equilibrium price process described in Chapter 4. Instead, $\{z_\tau\}$ is any Markov process of the form

$$z_{\tau+1} = f_{z,\tau}(z_\tau, \xi_{\tau+1})$$

$$\tau = 0, \dots, T-1$$

where $\{\xi_\tau\}$ is a vector-valued i.i.d. process and where $z_t{}^p$ must include a sufficient statistic for $z_{t+1}{}^d$ which is denoted by $z_t{}^{d+}$.

However, it is implicitly assumed that the price process is sufficiently “well-behaved” in the sense that it does not allow arbitrage opportunities and there always exists at least one interior solution to the problems posed by the Bellman equation.

A3.2.2 Results

The value function takes the form

A 3-10

$$J(W_t, z_t{}^p, t) = \frac{1}{(1+\rho)^t} \cdot \frac{W_t{}^{1-\gamma}}{1-\gamma} \cdot m_t(z_t{}^p)$$

where $m_t(z_t{}^p)$ is a function that is recursively defined by

A 3-11

$$m_t(z_t{}^p) \equiv \begin{cases} 1 & t = T \\ \{c_t{}^*(z_t{}^p)\}^{1-\gamma} + \dfrac{1}{1+\rho} \cdot \{1 - c_t{}^*(z_t{}^p)\}^{1-\gamma} \cdot G_t(w_t{}^*(z_t{}^p); z_t{}^p) & 0 \le t \le T-1 \end{cases}$$

with

$$c_t{}^*(z_t{}^p) \equiv \frac{1}{1 + \left\{\dfrac{1}{1+\rho} \cdot G_t(w_t{}^*(z_t{}^p); z_t{}^p)\right\}^{\frac{1}{\gamma}}}$$

and

$$G_t(w_t^*(z_t^p); z_t^p) \equiv E\left(\left(1 + R^{Pf}\left(z_t^p, z_{t+1}^p, z_{t+1}^d; w_t^*(z_t^p)\right)\right)^{1-\gamma} \cdot m_{t+1}(z_{t+1}^p) \middle| z_t^p\right)$$

The function $m_t(.)$ is positive for all possible values of z_t^p. $w_t^*(z_t^p)$ is a vector of portfolio holdings characterized by the following optimality conditions:

Any combination of portfolio holdings $w_t^*(z_t^p)$ and consumption $C_t^*(z_t^p, W_t)$ that solves the following first-order conditions will be optimal due to the concavity of the value function in wealth:

A 3-12

$$E\left(\begin{matrix}\left(1 + R^{Pf}(z_t^p, z_{t+1}^p, z_{t+1}^d; w)\right)^{-\gamma} \cdot m_{t+1}(z_{t+1}^p) \\ \cdot \left[R_{i,t,t+1}(z_t^p, z_{t+1}^p, z_{t+1}^d) - r_t(z_t^p)\right]\end{matrix} \middle| z_t, W_t\right) = 0$$

$$i = 1, \dots, n$$

$$C_t^*(z_t^p, W) = c_t^*(z_t^p) \cdot W_\tau$$

A3.2.3 Proof

A3.2.3.1 Idea of the Proof

The proof is by induction over the remaining time horizon up to the final point of time T, i.e., $T - t$. I prove the form of the value function ((A 3-10) with (A 3-11), including the positivity of $m_t(z_t^p)$).

A3.2.3.2 Details of the Proof

Base case: $T - t = 0$

At time T, the value function must coincide with the direct utility function by the nature of dynamic programming:

$$J(W_T, z_T^p, T) = J(W_T, T) = \frac{1}{(1+\rho)^T} \cdot \frac{W_T^{1-\gamma}}{1-\gamma}$$

By setting

$$m_T(z_T^p) \equiv 1$$

it is evident that (A 3-10) is correct for time $t = T$.

Inductive step

Inductively assume that (A 3-10) is true for time $t+1$. Then the Bellman equation for time t reads

A 3-13

$$J(W_t, {z_t}^p, t) = sup_{C,w}\left\{\frac{1}{(1+\rho)^t}\cdot\frac{C^{1-\gamma}}{1-\gamma} + E\left(\frac{1}{(1+\rho)^{t+1}}\cdot\frac{{W_{t+1}}^{1-\gamma}}{1-\gamma}\cdot m_{t+1}({z_{t+1}}^p)\middle| {z_t}^p, W_t\right)\right\}$$

with wealth dynamics (A 3-9)

$$W_{t+1} = [W_t - C]\cdot\left(1 + R^{Pf}\left(R_{i,t,t+1}\left({z_t}^p, {z_{t+1}}^p, {z_{t+1}}^d\right) - r_t({z_t}^p); w\right)\right)$$

In order to determine the precise form of $J(W_t, {z_t}^p, t)$ in (A 3-13), the optimization problem on the right-hand side of (A 3-13) must be solved. To that end, first substitute the wealth dynamics into the Bellman equation, factor out $\frac{1}{(1+\rho)^t}$

$$J(W_t, z_t, t) = \frac{1}{(1+\rho)^t}$$
$$\cdot sup_{C,w}\left\{\frac{C^{1-\gamma}}{1-\gamma} + \frac{1}{1+\rho}\cdot\{W_t - C\}^{1-\gamma}\right.$$
$$\left.\cdot E\left(\frac{\left(1+R^{Pf}\left({z_t}^p, {z_{t+1}}^p, {z_{t+1}}^d; w\right)\right)^{1-\gamma}}{1-\gamma}\cdot m_{t+1}({z_{t+1}}^p)\middle| {z_t}^p\right)\right\}$$

and then derive first-order conditions for portfolio weights and consumption:

First-order conditions for portfolio weights

$$E\left(\begin{matrix}\left(1+R^{Pf}\left({z_t}^p, {z_{t+1}}^p, {z_{t+1}}^d; w\right)\right)^{-\gamma}\cdot m_{t+1}({z_{t+1}}^p) \\ \left[R_{i,t,t+1}\left({z_t}^p, {z_{t+1}}^p, {z_{t+1}}^d\right) - r_t({z_t}^p)\right]\end{matrix}\middle| z_t, W_t\right) = 0$$
$$i = 1, \dots, n$$

It is assumed that there is at least one portfolio ${w_t}^*({z_t}^p)$ that solves this first-order condition.

First-order condition for consumption

$$C^{-\gamma} - \frac{1}{1+\rho}\cdot\{W_t - C\}^{-\gamma}\cdot G_t({w_t}^*({z_t}^p); {z_t}^p) = 0$$

with

$$G_t(w; {z_t}^p) \equiv E\left(\left(1+R^{Pf}\left({z_t}^p, {z_{t+1}}^p, {z_{t+1}}^d; w\right)\right)^{1-\gamma}\cdot m_{t+1}({z_{t+1}}^p)\middle| {z_t}^p\right)$$

Observe that $G_t(w; {z_t}^p)$ is strictly positive because, by inductive assumption, $m_{t+1}({z_{t+1}}^p)$ is strictly positive.

Consumption can be obtained as a function of wealth W_τ and optimal portfolio weights ${w_t}^*({z_t}^p)$:

$$C^{-\gamma} = \frac{1}{1+\rho}\cdot\{W_t - C\}^{-\gamma}\cdot G_t({w_t}^*({z_t}^p); {z_t}^p)$$

$$\Leftrightarrow$$

$$\left\{\frac{W_t}{C}-1\right\}^{-\gamma}=\frac{1}{\frac{1}{1+\rho}\cdot G_t(w_t{}^*(z_t{}^p);z_t{}^p)}$$

$$\Rightarrow$$

$$\frac{W_t}{C}=1+\left\{\frac{1}{1+\rho}\cdot G_t(w_t{}^*(z_t{}^p);z_t{}^p)\right\}^{\frac{1}{\gamma}}$$

$$\Leftrightarrow$$

$$C=\frac{1}{1+\left\{\frac{1}{1+\rho}\cdot G_t(w_t{}^*(z_t{}^p);z_t{}^p)\right\}^{\frac{1}{\gamma}}}\cdot W_t$$

For brevity, define

$$c_t(z_t{}^p;w)\equiv\frac{1}{1+\left\{\frac{1}{1+\rho}\cdot G_t(w_t{}^*(z_t{}^p);z_t{}^p)\right\}^{\frac{1}{\gamma}}}$$

$$c_t{}^*(z_t{}^p)\equiv c_t\big(z_t{}^p;w_t{}^*(z_t{}^p)\big)$$

In this notation, consumption reads

$$C_t{}^*(z_t{}^p,W)=c_t{}^*(z_t{}^p)\cdot W_t$$

To finalize the problem of determining the precise form of the value function, substitute consumption $C_t{}^*(z_t{}^p,W)$ and portfolio weights $w_t{}^*(z_t{}^p)$ into the maximand on the right-hand side of (A 3-13):

$$J(W_t,z_t,t)=\frac{1}{(1+\rho)^t}\cdot\left\{\frac{\{c_t{}^*(z_t{}^p)\cdot W_t\}^{1-\gamma}}{1-\gamma}+\frac{1}{1+\rho}\cdot\frac{\big\{W_t\cdot\{1-c_t{}^*(z_t{}^p)\}\big\}^{1-\gamma}}{1-\gamma}\cdot G_t(w_t{}^*(z_t{}^p);z_t{}^p)\right\}$$

Slightly rearranging terms yields

$$J(W_t,z_t,t)=\frac{1}{(1+\rho)^t}\cdot\frac{W_t{}^{1-\gamma}}{1-\gamma}\cdot m_t(z_t{}^p)$$

with

$$m_t(z_t{}^p)\equiv\{c_t{}^*(z_t{}^p)\}^{1-\gamma}+\frac{1}{1+\rho}\cdot\{1-c_t{}^*(z_t{}^p)\}^{1-\gamma}\cdot G_t(w_t{}^*(z_t{}^p);z_t{}^p)$$

Evidently, $c_t{}^*(z_t{}^p)$ takes a value between zero and one, hence $\{c_t{}^*(z_t{}^p)\}^{1-\gamma}$ and $\{1-c_t{}^*(z_t{}^p)\}^{1-\gamma}$ are well-defined (and can only be positive). As has been remarked above, $G_t(w_t{}^*(z_t{}^p);z_t{}^p)$ is strictly positive, hence $m_t(z_t{}^p)$ must be strictly positive. This concludes the proof of the form of the value function.

A3.3 General Equilibrium Asset Prices

A3.3.1 Formulation of the Problem

Consider a cash flow process of the form

$$D_t = D_t\left({z_t}^d\right)$$
$${z_t}^d = \left({z_t}^{d,0}, {z_t}^{d,+}\right)$$

where ${z_t}^{d,+}$ is a sufficient statistic for the conditional distribution of future cash flows (and, in particular, ${z_{t+1}}^d$) and ${z_t}^{d,0}$ is a component that describes current cash flows.

The dynamics of ${z_\tau}^d$ are given by

$${z_{\tau+1}}^d = {f_{z,\tau}}^d\left({z_\tau}^d, {\xi_{\tau+1}}^d\right)$$
$$\tau = 0, \ldots, T-1$$

where ${\xi_{\tau+1}}^d$ are vector-valued i.i.d. random variables. Define ${z_\tau}^p \equiv \left({D_\tau}^{market}, {z_\tau}^{d,+}\right)$ with ${D_\tau}^{market} \equiv \overline{N}^T D_\tau\left({z_\tau}^d\right)$.

The general equilibrium price processes of risky assets and the riskless assets under CRRA utility are recursively defined starting from time T where the price process is zero:

A 3-14

$$P_t\left({z_t}^p\right) = \begin{cases} \underline{0} & t = T \\ E\left(q_{t,t+1}\left({z_{t+1}}^d, {z_t}^p\right) \cdot \left\{P_{t+1}\left({z_{t+1}}^p\right) + D_{t+1}\left({z_{t+1}}^d\right)\right\} \middle| {z_t}^p\right) & 0 \le t \le T-1 \end{cases}$$

with

$$q_{t,t+1}\left({z_{t+1}}^d, {z_t}^p\right) \equiv \frac{1}{1+\rho} \cdot \left\{\frac{\overline{N}^T D_{t+1}\left({z_{t+1}}^d\right)}{{D_t}^{market}}\right\}^{-\gamma}$$

and

A 3-15

$$B_t\left({z_t}^p\right) = \frac{1}{1 + r_t\left({z_t}^p\right)} = \begin{cases} \underline{0} & t = T \\ E\left(q_{t,t+1}\left({z_{t+1}}^d, {z_t}^p\right) \middle| {z_t}^p\right) & 0 \le t \le T-1 \end{cases}$$

A3.3.2 Proof

A3.3.2.1 Idea of the Proof

In order to demonstrate that this price process is indeed a general equilibrium, it must be shown that the n_I identical investors always behave optimally by holding $\frac{1}{n_I}$-th of the market portfolio of risky assets $\overline{N}$ and holding a zero position in the riskless bond. Re-formulated in terms of portfolio weights this means: individual portfolio weights equal market portfolio weights. This in turn is established by deriving the value function of one of the identical investors acting under the price processes (A 3-14) and (A 3-15); this value function will be referred to as the equilibrium value function.. The proof is by induction over the remaining time horizon up to the final point of time T, i.e., $T-t$.

Evidently, if each of the n_I identical investors holds $\frac{1}{n_I}$-th of the market portfolio of risky assets $\overline{N}$ and does not buy or sell the riskless bond, markets for risky assets and the riskless asset clear.

A3.3.2.2 Details of the Proof

I inductively prove that the equilibrium value function of one of the identical investors acting under the partial equilibrium price process (A 3-14) is given by

A 3-16

$$J(W_t, z_t{}^p, t) = \frac{1}{(1+\rho)^t} \cdot \frac{W_t{}^{1-\gamma}}{1-\gamma} \cdot \overline{m}_t(z_t{}^p)$$

with

$$\overline{m}_t(z_t{}^p) \equiv \left\{1 + \frac{V_t{}^{market}(z_t{}^p)}{D_t{}^{market}(z_t{}^p)}\right\}^\gamma$$

with

$$V_t{}^{market}(z_t{}^p) \equiv \overline{N}^T P_t(z_t{}^p)$$

If this equilibrium value function characterizes optimal behavior of investors, then choosing individual portfolio weights that are equal to the portfolio weights of the market portfolio , including a zero position in the riskless asset, at all points of time solves the first-order condition of portfolio weights and, hence, is optimal.

Inductive proof of the form of the equilibrium value function

Base case: $T-t=0$

If $t=T$, it is clear that the equilibrium value function simply reads

$$J(W_T, z_T{}^p, T) = J(W_T, T) = \frac{1}{(1+\rho)^T} \cdot \frac{W_T{}^{1-\gamma}}{1-\gamma}$$

Observe that all assets have prices of zero at time T. The value of the market portfolio of risky assets must then also be zero at time T, thus we have $\overline{m}_T(z_t{}^p) = 1$ and the value function at time T is indeed given by (A 3-16).

Inductive Step

Inductively assume that (A 3-16) is true for time $t+1$. Then the necessary conditions of portfolio weights (A 3-12) read in general equilibrium

A 3-17

$$E\left(\left.\begin{array}{c}\left(1+R^{Pf}\left(z_t{}^p,z_{t+1}{}^p,z_{t+1}{}^d;w\right)\right)^{-\gamma}\cdot\overline{m}_{t+1}(z_{t+1}{}^p)\\ \cdot\left[R_{i,t,t+1}\left(z_t{}^p,z_{t+1}{}^p,z_{t+1}{}^d\right)-r_t(z_t{}^p)\right]\end{array}\right|z_t,W_t\right)=0$$

$$i=1,\dots,n$$

Since this first-order condition is expressed in terms of portfolio weights but general equilibrium has been defined in terms of portfolio holdings, both views must be made compatible. The equilibrium condition reads: the portfolio of risky assets is $N_t{}^*(z_t{}^p)=\frac{1}{n_I}\cdot\overline{N}$ and a zero position in the riskless bond $H_t{}^*(z_t{}^p)=0$ is held. Re-expressing these portfolio holdings in terms of portfolio weights yields

$$w_{i,t}{}^*(z_t{}^p,W_t-C)\equiv\frac{\frac{1}{n_I}\cdot\overline{N}_i\cdot P_{i,t}(z_t{}^p)}{W_t-C}$$

$$i=1,\dots,n$$

Moreover, if $H_t{}^*(z_t{}^p)=0$, then the weights $w_{i,t}{}^*(z_t{}^p,W_t-C)$ must sum to one,

$$1=\underline{1}^T w_t{}^*(z_t{}^p,W_t-C)=\frac{1}{n_I}\cdot\frac{\overline{N}^T\cdot P_t(z_t{}^p)}{W_t-C}$$

i.e., $W_t-C=\overline{N}^T\cdot P_{i,t}(z_t{}^p)$, hence portfolio weights can be written in terms of asset prices and the portfolio $\frac{1}{n_I}\cdot\overline{N}$ only (without recourse to W_t and C):

A 3-18

$$w_{i,t}{}^*(z_t{}^p)\equiv\frac{\overline{N}_i\cdot P_{i,t}(z_t{}^p)}{\overline{N}^T\cdot P_t(z_t{}^p)}$$

$$i=1,\dots,n$$

Now substitute portfolio weights (A 3-18) into the left-hand side of the first-order condition of portfolio weights in general equilibrium (A 3-17) and show that it evaluates to zero. Because the part of the value function $\overline{m}_{t+1}(z_{t+1}{}^p)\equiv\left\{1+\frac{V_{t+1}{}^{market}(z_{t+1}{}^p)}{D_{t+1}{}^{market}(z_{t+1}{}^p)}\right\}^{\gamma}$ is expressed in terms of portfolio holdings, it is convenient to rewrite the portfolio return through the wealth of the market portfolio:

$$R^{Pf}\left(z_t{}^p,z_{t+1}{}^p,z_{t+1}{}^d;w_t{}^*(z_t{}^p)\right)=r_t+\sum_{i=1}^{n}w_{i,t}{}^*(z_t{}^p)\cdot\left\{R_{i,t,t+1}\left(z_t{}^p,z_{t+1}{}^p,z_{t+1}{}^d\right)-r_t(z_t{}^p)\right\}$$

$$=\sum_{i=1}^{n}w_{i,t}{}^*(z_t{}^p)\cdot R_{i,t,t+1}\left(z_t{}^p,z_{t+1}{}^p,z_{t+1}{}^d\right)=\frac{\overline{N}^T\cdot\left\{P_{t+1}(z_{t+1}{}^p)+D_{t+1}\left(z_{t+1}{}^d\right)-P_t(z_t{}^p)\right\}}{\overline{N}^T\cdot P_t(z_t{}^p)}$$

Hence the first-order condition of portfolio weights in general equilibrium (A 3-17) simplifies to:

$$E\left(\left.\begin{array}{c}\left\{\overline{N}^T\cdot D_{t+1}\left(z_{t+1}{}^d\right)\right\}^{-\gamma}\\ \cdot\left[\frac{P_{i,t+1}(z_{\tau+1}{}^p)+D_{i,t+1}\left(z_{t+1}{}^d\right)-P_{i,t}(z_t{}^p)}{P_{i,t}(z_t{}^p)}-r_t(z_t{}^p)\right]\end{array}\right|z_t{}^p\right)=0$$

$$i=1,\dots,n$$

Solving for the price $P_{i,t}(z_t{}^p)$ yields

$$\Leftrightarrow$$

A 3-19

$$P_{i,t}(z_t{}^p) = \frac{1}{1+r_t(z_t{}^p)} \cdot E\left(\frac{\left\{\overline{N}^T \cdot D_{t+1}\left(z_{t+1}{}^d\right)\right\}^{-\gamma}}{E\left(\left\{\overline{N}^T \cdot D_{t+1}\left(z_{t+1}{}^d\right)\right\}^{-\gamma} \middle| z_t{}^p\right)} \cdot \left\{P_{i,t+1}(z_{\tau+1}{}^p) + D_{i,t+1}\left(z_{t+1}{}^d\right)\right\} \middle| z_t{}^p \right)$$

Finally, the price of the riskless bond $\frac{1}{1+r_t(z_t{}^p)}$ is

$$\frac{1}{1+r_t(z_t{}^p)} = \frac{1}{1+\rho} \cdot E\left(\left\{\frac{\overline{N}^T D_t\left(z_t{}^d\right)}{D_t{}^{market}}\right\}^{-\gamma} \middle| z_t{}^p \right)$$

Plugging the bond price into (A 3-19) yields

$$P_{i,t}(z_t{}^p) = \frac{1}{1+\rho} \cdot E\left(\frac{\left\{\overline{N}^T \cdot D_{t+1}\left(z_{t+1}{}^d\right)\right\}^{-\gamma}}{\left\{D_t{}^{market}\right\}^{-\gamma}} \cdot \left\{P_{i,t+1}(z_{\tau+1}{}^p) + D_{i,t+1}\left(z_{t+1}{}^d\right)\right\} \middle| z_t{}^p \right)$$

but this is true because it is precisely the definition of general equilibrium asset prices according to (A 3-14).

To complete the proof, the value function for time t must be computed. We can use the results from the portfolio selection problem of the individual investor as point of departure:

(A 3-10)

$$J(W_t, z_t{}^p, t) = \frac{1}{(1+\rho)^t} \cdot \frac{W_t{}^{1-\gamma}}{1-\gamma} \cdot m_t(z_t{}^p)$$

with (A 3-11)

$$m_t(z_t{}^p) = \{c_t{}^*(z_t{}^p)\}^{1-\gamma} + \frac{1}{1+\rho} \cdot \{1 - c_t{}^*(z_t{}^p)\}^{1-\gamma} \cdot G_t(w_t{}^*(z_t{}^p); z_t{}^p)$$

where the fraction of wealth that is consumed is given by

$$c_t{}^*(z_t{}^p) \equiv \frac{1}{1 + \left\{\frac{1}{1+\rho} \cdot G_t(w_t{}^*(z_t{}^p); z_t{}^p)\right\}^{\frac{1}{\gamma}}}$$

and with

$$G_t(z_t{}^p) \equiv E\left(\left(1 + R^{Pf}\left(z_t{}^p, z_{t+1}{}^p, z_{t+1}{}^d; w_t{}^*(z_t{}^p)\right)\right)^{1-\gamma} \cdot \overline{m}_{t+1}(z_{t+1}{}^p) \middle| z_t{}^p \right)$$

The equilibrium value function must possess the structure (A 3-10). To see this, start with $G_t(z_t{}^p)$: plugging in $\overline{m}_{t+1}$ and the form of $1 + R^{Pf}$ yields

$$G_t(z_t{}^p) = \left\{\frac{\overline{N}^T D_t\left(z_t{}^d\right)}{\overline{N}^T P_t(z_t{}^p)}\right\}^{-\gamma} \cdot E\left(\frac{\overline{N}^T \left\{P_{t+1}(z_{t+1}{}^p) + D_{t+1}\left(z_{t+1}{}^d\right)\right\}}{\overline{N}^T P_t(z_t{}^p)} \cdot \frac{\left\{\overline{N}^T D_{t+1}\left(z_{t+1}{}^d\right)\right\}^{-\gamma}}{\left\{\overline{N}^T D_t\left(z_t{}^d\right)\right\}^{-\gamma}} \middle| z_t{}^p \right)$$

Since the price process implies that the value of the market portfolio of risky assets is

$$\overline{N}^T P_t(z_t{}^p) = \frac{1}{1+\rho} \cdot E\left(\overline{N}^T \{P_{t+1}(z_{t+1}{}^p) + D_{t+1}(z_{t+1}{}^d)\} \cdot \frac{\left\{\overline{N}^T D_{t+1}(z_{t+1}{}^d)\right\}^{-\gamma}}{\left\{\overline{N}^T D_t(z_t{}^d)\right\}^{-\gamma}} \middle| z_t{}^p \right)$$

the expression for $G_t(z_t{}^p)$ can be simplified to

A 3-20

$$G_t(z_t{}^p) = (1+\rho) \cdot \left\{ \frac{\overline{N}^T D_t(z_t{}^d)}{\overline{N}^T P_t(z_t{}^p)} \right\}^{-\gamma}$$

With $G_t(z_t{}^p)$ known, it is now possible to compute $c_t{}^*(z_t{}^p)$:

$$c_t{}^*(z_t{}^p) = \frac{1}{1 + \left\{ \frac{1}{1+\rho} \cdot G_t(w_t{}^*(z_t{}^p); z_t{}^p) \right\}^{\frac{1}{\gamma}}} = \frac{1}{1 + \left\{ \frac{\overline{N}^T D_t(z_t{}^d)}{\overline{N}^T P_t(z_t{}^p)} \right\}^{-1}} = \frac{\overline{N}^T D_t(z_t{}^d)}{\overline{N}^T D_t(z_t{}^d) + \overline{N}^T P_t(z_t{}^p)}$$

i.e.,

A 3-21

$$c_t{}^*(z_t{}^p) = \frac{\overline{N}^T D_t(z_t{}^d)}{\overline{N}^T D_t(z_t{}^d) + \overline{N}^T P_t(z_t{}^p)}$$

and

A 3-22

$$\left(1 - c_t{}^*(z_t{}^p)\right) = \frac{\overline{N}^T P_t(z_t{}^p)}{\overline{N}^T D_t(z_t{}^d) + \overline{N}^T P_t(z_t{}^p)}$$

Based on (A 3-20), (A 3-21) and (A 3-22), $m_t(z_t{}^p)$ it becomes possible to compute:

$$m_t(z_t{}^p) = \{c_t{}^*(z_t{}^p)\}^{1-\gamma} + \frac{1}{1+\rho} \cdot \{1 - c_t{}^*(z_t{}^p)\}^{1-\gamma} \cdot G_t(w_t{}^*(z_t{}^p); z_t{}^p)$$

$$= \left\{ \frac{\overline{N}^T D_t(z_t{}^d)}{\overline{N}^T D_t(z_t{}^d) + \overline{N}^T P_t(z_t{}^p)} \right\}^{1-\gamma} + \left\{ \frac{\overline{N}^T P_t(z_t{}^p)}{\overline{N}^T D_t(z_t{}^d) + \overline{N}^T P_t(z_t{}^p)} \right\}^{1-\gamma} \cdot \left\{ \frac{\overline{N}^T D_t(z_t{}^d)}{\overline{N}^T P_t(z_t{}^p)} \right\}^{-\gamma}$$

$$= \left\{\overline{N}^T D_t(z_t{}^d)\right\}^{-\gamma} \cdot \frac{\overline{N}^T D_t(z_t{}^d) + \overline{N}^T P_t(z_t{}^p)}{\left\{\overline{N}^T D_t(z_t{}^d) + \overline{N}^T P_t(z_t{}^p)\right\}^{1-\gamma}} = \left\{ 1 + \frac{\overline{N}^T P_t(z_t{}^p)}{\overline{N}^T D_t(z_t{}^d)} \right\}^{\gamma} = \overline{m}_t(z_t{}^p)$$

This completes the proof of the form of the equilibrium value function.

A3.4 Appendix to Sections 4.2.1.1.2.2 and 4.2.1.2.2.2: Decomposition of the Covariance of the Multi-Period Stochastic Discount Factor and Cash Flows

A3.4.1 Information Frequency = Cash Flow Frequency

A3.4.1.1 Formulation of the Problem

The problem is to show that the covariance of the stochastic discount factor with cash flows $D_{t+\tau}$ can be decomposed into (i) a part due to $fe_{t+\tau}$ conditional on $S_{t+\tau-1}, D_{t+\tau-1}$ where $S_{t+\tau-1}$ and $D_{t+\tau-1}$ are averaged out and (ii) a part due to $S_{t+\tau-1}, D_{t+\tau-1}$ alone where $fe_{t+\tau}$ is averaged out:

Complete Information

4-54

$$cov\left(\frac{1}{(1+\rho)^{\tau}}\cdot\frac{U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right)}{U'\left(\frac{1}{n_I}\cdot D_t{}^{market}(D_t)\right)}, D_{t+\tau}\middle| S_t, D_t\right) = \frac{1}{(1+\rho)^{\tau}}\cdot\frac{1}{U'\left(\frac{1}{n_I}\cdot D_t{}^{market}(D_t)\right)}\cdot$$
$$\left\{E\left(cov\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right), D_{t+\tau}\middle| S_{t+\tau-1}, D_{t+\tau-1}\right)\middle| S_t, D_t\right)\right.$$
$$\left.+\,cov\left(E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right)\middle| S_{t+\tau-1}, D_{t+\tau-1}\right), E(D_{t+\tau}|S_{t+\tau-1}, D_{t+\tau-1})\middle| S_t, D_t\right)\right\}$$

Incomplete Information

4-66

$$cov\left(\frac{1}{(1+\rho)^{\tau}}\cdot\frac{U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right)}{U'\left(\frac{1}{n_I}\cdot D_t{}^{market}(D_t)\right)}, D_{t+\tau}\middle| \pi_t, D_t\right) = \frac{1}{(1+\rho)^{\tau}}\cdot\frac{1}{U'\left(\frac{1}{n_I}\cdot D_t{}^{market}(D_t)\right)}\cdot$$
$$\left\{E\left(cov\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right)\cdot D_{t+\tau}\middle| S_{t+\tau-1}, D_{t+\tau-1}\right)\middle| \pi_t, D_t\right)\right.$$
$$\left.+\,cov\left(E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right)\middle| S_{t+\tau-1}, D_{t+\tau-1}\right), E(D_{t+\tau}|S_{t+\tau-1}, D_{t+\tau-1})\middle| \pi_t, D_t\right)\right\}$$

A3.4.1.2 Solution

Since the derivation is parallel under complete and incomplete information, the symbol ${z_t}^p$ is used to either denote ${z_t}^{cI,p} = (S_t, D_t)$ or ${z_t}^{iI,p} = (\pi_t, D_t)$.

Step 1: Apply the identity $cov(X,Y) = E(X \cdot Y) - E(X) \cdot E(Y)$

Omitting the deterministic terms $\frac{1}{(1+\rho)^\tau} \cdot \frac{1}{U'\left(\frac{1}{n_I} \cdot {D_t}^{market}(D_t)\right)}$, the covariance reads:

$$cov\left(U'\left(\frac{1}{n_I} \cdot {D_{t+\tau}}^{market}(D_{t+\tau})\right), D_{t+\tau} \middle| {z_t}^p\right)$$

$$= E\left(U'\left(\frac{1}{n_I} \cdot {D_{t+\tau}}^{market}(D_{t+\tau})\right) \cdot D_{t+\tau} \middle| {z_t}^p\right)$$

$$- E\left(U'\left(\frac{1}{n_I} \cdot {D_{t+\tau}}^{market}(D_{t+\tau})\right) \middle| {z_t}^p\right) \cdot E(D_{t+\tau} | {z_t}^p)$$

Step 2: Use the tower property of conditional expectations to modify the right-hand side of Step 1

$$E\left(E\left(U'\left(\frac{1}{n_I} \cdot {D_{t+\tau}}^{market}(D_{t+\tau})\right) \cdot D_{t+\tau} \middle| {z_t}^p, S_{t+\tau-1}, D_{t+\tau-1}\right) \middle| {z_t}^p\right)$$

$$- E\left(E\left(U'\left(\frac{1}{n_I} \cdot {D_{t+\tau}}^{market}(D_{t+\tau})\right) \middle| {z_t}^p, S_{t+\tau-1}, D_{t+\tau-1}\right) \middle| {z_t}^p\right)$$

$$\cdot E(E(D_{t+\tau} | {z_t}^p, S_{t+\tau-1}, D_{t+\tau-1}) | {z_t}^p)$$

Step 3: Eliminating redundant variables from the information that conditions the expectations yields

$$E\left(E\left(U'\left(\frac{1}{n_I} \cdot {D_{t+\tau}}^{market}(D_{t+\tau})\right) \cdot D_{t+\tau} \middle| S_{t+\tau-1}, D_{t+\tau-1}\right) \middle| {z_t}^p\right)$$

$$- E\left(E\left(U'\left(\frac{1}{n_I} \cdot {D_{t+\tau}}^{market}(D_{t+\tau})\right) \middle| S_{t+\tau-1}, D_{t+\tau-1}\right) \middle| {z_t}^p\right)$$

$$\cdot E(E(D_{t+\tau} | S_{t+\tau-1}, D_{t+\tau-1}) | {z_t}^p)$$

(since the regime is a Markov chain and factors and residuals are i.i.d., $S_{t+\tau-1}, D_{t+\tau-1}$ completely describes the distribution of $D_{t+\tau}$ and ${D_{t+\tau}}^{market}$, rendering ${z_t}^p$ redundant)

Step 4: Re-express $E\left(U'\left(\frac{1}{n_I} \cdot {D_{t+\tau}}^{market}(D_{t+\tau})\right) \cdot D_{t+\tau} \middle| S_{t+\tau-1}, D_{t+\tau-1}\right)$ as

$$cov\left(U'\left(\frac{1}{n_I} \cdot {D_{t+\tau}}^{market}(D_{t+\tau})\right) \cdot D_{t+\tau} \middle| S_{t+\tau-1}, D_{t+\tau-1}\right)$$

$$+ E\left(U'\left(\frac{1}{n_I} \cdot {D_{t+\tau}}^{market}(D_{t+\tau})\right) \middle| S_{t+\tau-1}, D_{t+\tau-1}\right) \cdot E(D_{t+\tau} | S_{t+\tau-1}, D_{t+\tau-1})$$

and plug it into the second term of Step 3

$$E\left(\begin{matrix} cov\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right), D_{t+\tau}\middle|S_{t+\tau-1}, D_{t+\tau-1}\right) \\ +E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right)\middle|S_{t+\tau-1}, D_{t+\tau-1}\right)\cdot E(D_{t+\tau}|S_{t+\tau-1}, D_{t+\tau-1}) \end{matrix}\middle| z_t{}^p\right)$$

$$-E\left(E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right)\middle|S_{t+\tau-1}, D_{t+\tau-1}\right)\middle|z_t{}^p\right)$$

$$\cdot E(E(D_{t+\tau}|S_{t+\tau-1}, D_{t+\tau-1})|z_t{}^p)$$

<u>Step 5</u>: Rearranging terms leads to

$$E\left(cov\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right), D_{t+\tau}\middle|S_{t+\tau-1}, D_{t+\tau-1}\right)\middle|z_t{}^p\right)$$

$$+E\left(E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right)\middle|S_{t+\tau-1}, D_{t+\tau-1}\right)\cdot E(D_{t+\tau}|S_{t+\tau-1}, D_{t+\tau-1})\middle|z_t{}^p\right)$$

$$-E\left(E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right)\middle|S_{t+\tau-1}, D_{t+\tau-1}\right)\middle|z_t{}^p\right)\cdot E(E(D_{t+\tau}|S_{t+\tau-1}, D_{t+\tau-1})|z_t{}^p)$$

(note that the second term is the covariance of $\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right)\middle|S_{t+\tau-1}, D_{t+\tau-1}\right)$ and $E(D_{t+\tau}|S_{t+\tau-1}, D_{t+\tau-1})$)

$$= E\left(cov\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right), D_{t+\tau}\middle|S_{t+\tau-1}, D_{t+\tau-1}\right)\middle|z_t{}^p\right)$$

$$+ cov\left(E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right)\middle|S_{t+\tau-1}, D_{t+\tau-1}\right), E(D_{t+\tau}|S_{t+\tau-1}, D_{t+\tau-1})\middle|z_t{}^p\right)$$

This is the desired result.

A3.4.2 Information Frequency ≥ Cash Flow Frequency

A3.4.2.1 Formulation of the Problem

For completeness, a generalization of the covariance decompositions (4-54) and (4-66) to the case where information frequency is higher than or equal to cash flow frequency is stated in this section of the appendix. The covariance of the stochastic discount factor with cash flows of assets $D_{t+\tau}{}^{(\nu)}, \nu\epsilon\{1, \Delta_C\}$ can be decomposed into (i) a part due to $fe_{t+\tau}$ conditional on $D_{t_{(k_{t+\tau-1})}}{}^{(\Delta_C)}$, $D_{t+\tau-1}{}^{(1)}$ and $S_{t_{(k_{t+\tau-1})},t+\tau-1}$ where $D_{t_{(k_{t+\tau-1})}}{}^{(\Delta_C)}$, $D_{t+\tau-1}{}^{(1)}$ and $S_{t_{(k_{t+\tau-1})},t+\tau-1}$ are averaged out and (ii) a part due to $D_{t_{(k_{t+\tau-1})}}{}^{(\Delta_C)}$, $D_{t+\tau-1}{}^{(1)}$ and $S_{t_{(k_{t+\tau-1})},t+\tau-1}$ alone where $fe_{t+\tau}$ is averaged out:

Complete Information

A 3-23

$$cov\left(\frac{1}{(1+\rho)^{\tau}}\cdot\frac{U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)}{U'\left(\frac{1}{n_I}\cdot D_t{}^{market}\right)}, D_{t+\tau}\middle| z_t{}^{cI,(\Delta_C),p}\right) = \frac{1}{(1+\rho)^{\tau}}\cdot\frac{1}{U'\left(\frac{1}{n_I}\cdot D_t{}^{market}\right)}\cdot$$

$$\left\{E\left(cov\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right), D_{t+\tau}{}^{(\nu)}\middle| z_{t+\tau-1}{}^{cI,(\Delta_C),p}\right)\middle| z_t{}^{cI,(\Delta_C),p}\right)\right.$$

$$\left.+\, cov\left(E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle| z_{t+\tau-1}{}^{cI,(\Delta_C),p}\right), E\left(D_{t+\tau}{}^{(\nu)}\middle| z_{t+\tau-1}{}^{cI,(\Delta_C),p}\right)\middle| z_t{}^{cI,(\Delta_C),p}\right)\right\}$$

$$\nu\epsilon\{1,\Delta_C\}$$

with

$$z_t{}^{cI,(\Delta_C),p} = \left(S_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)$$

$$z_{t+\tau-1}{}^{cI,(\Delta_C),p} = \left(S_{t_{(k_{t+\tau-1})},t+\tau-1}, D_{t_{(k_{t+\tau-1})}}{}^{(\Delta_C)}, D_{t+\tau-1}{}^{(1)}\right)$$

with

$$D_{t+\tau}{}^{market} = \begin{cases} D_{t+\tau}{}^{market}\left(D_{t+\tau}{}^{(\Delta_C)}, D_{t+\tau}{}^{(1)}\right) & t = t_{(k_{t+\tau}+1)} - 1 \\ D_{t+\tau}{}^{market}\left(D_{t+\tau}{}^{(1)}\right) & t_{(k_{t+\tau})} \le t < t_{(k_{t+\tau}+1)} - 1 \end{cases}$$

$$D_t{}^{market} = \begin{cases} D_t{}^{market}\left(D_t{}^{(\Delta_C)}, D_t{}^{(1)}\right) & t_{(k_t)} < t < t_{(k_t+1)} \\ D_t{}^{market}\left(D_t{}^{(\Delta_C)}, D_t{}^{(1)}\right) & t_{(k_t)} = t \end{cases}$$

Incomplete Information

A 3-24

$$cov\left(\frac{1}{(1+\rho)^{\tau}}\cdot\frac{U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)}{U'\left(\frac{1}{n_I}\cdot D_t{}^{market}\right)}, D_{t+\tau}\middle| z_t{}^{iI,(\Delta_C),p}\right) = \frac{1}{(1+\rho)^{\tau}}\cdot\frac{1}{U'\left(\frac{1}{n_I}\cdot D_t{}^{market}\right)}\cdot$$

$$\left\{E\left(cov\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right), D_{t+\tau}{}^{(\nu)}\middle| z_{t+\tau-1}{}^{cI,(\Delta_C),p}\right)\middle| z_t{}^{iI,(\Delta_C),p}\right)\right.$$

$$\left.+\, cov\left(E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle| z_{t+\tau-1}{}^{cI,(\Delta_C),p}\right), E\left(D_{t+\tau}{}^{(\nu)}\middle| z_{t+\tau-1}{}^{cI,(\Delta_C),p}\right)\middle| z_t{}^{iI,(\Delta_C),p}\right)\right\}$$

$$\nu\epsilon\{1,\Delta_C\}$$

with

$$z_t{}^{iI,(\Delta_C),p} = \left(\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)$$

$$z_{t+\tau-1}{}^{cI,(\Delta_C),p} = \left(S_{t_{(k_{t+\tau-1})},t+\tau-1}, D_{t_{(k_{t+\tau-1})}}{}^{(\Delta_C)}, D_{t+\tau-1}{}^{(1)}\right)$$

A3.4.2.2 Solution

Since the derivation is parallel under complete and incomplete information, the symbol ${z_t}^p$ is used to either denote ${z_t}^{cI,(\Delta_C),p}$ or ${z_t}^{iI,(\Delta_C),p}$.

Step 1: Apply the identity $cov(X,Y) = E(X \cdot Y) - E(X) \cdot E(Y)$

Omitting the deterministic terms $\frac{1}{(1+\rho)^\tau} \cdot \frac{1}{U'\left(\frac{1}{n_I} \cdot {D_t}^{market}\right)}$, the covariance reads:

$$cov\left(U'\left(\frac{1}{n_I} \cdot {D_{t+\tau}}^{market}\right), {D_{t+\tau}}^{(v)} \middle| {z_t}^p\right) = E\left(U'\left(\frac{1}{n_I} \cdot {D_{t+\tau}}^{market}\right) \cdot {D_{t+\tau}}^{(v)} \middle| {z_t}^p\right) - E\left(U'\left(\frac{1}{n_I} \cdot {D_{t+\tau}}^{market}\right) \middle| {z_t}^p\right) \cdot E\left({D_{t+\tau}}^{(v)} \middle| {z_t}^p\right)$$

Step 2: Use the tower property of conditional expectations to modify the right-hand side of Step 1

$$E\left(E\left(U'\left(\frac{1}{n_I} \cdot {D_{t+\tau}}^{market}\right) \cdot {D_{t+\tau}}^{(v)} \middle| {z_t}^p, {z_{t+\tau-1}}^{cI,(\Delta_C),p}\right) \middle| {z_t}^p\right) - E\left(E\left(U'\left(\frac{1}{n_I} \cdot {D_{t+\tau}}^{market}\right) \middle| {z_t}^p, {z_{t+\tau-1}}^{cI,(\Delta_C),p}\right) \middle| {z_t}^p\right) \cdot E\left(E\left({D_{t+\tau}}^{(v)} \middle| {z_t}^p, {z_{t+\tau-1}}^{cI,(\Delta_C),p}\right) \middle| {z_t}^p\right)$$

Step 3: Eliminating redundant variables from the information that conditions the expectations yields

$$E\left(E\left(U'\left(\frac{1}{n_I} \cdot {D_{t+\tau}}^{market}\right) \cdot {D_{t+\tau}}^{(v)} \middle| {z_{t+\tau-1}}^{cI,(\Delta_C),p}\right) \middle| {z_t}^p\right) - E\left(E\left(U'\left(\frac{1}{n_I} \cdot {D_{t+\tau}}^{market}\right) \middle| {z_{t+\tau-1}}^{cI,(\Delta_C),p}\right) \middle| {z_t}^p\right) \cdot E\left(E\left({D_{t+\tau}}^{(v)} \middle| {z_{t+\tau-1}}^{cI,(\Delta_C),p}\right) \middle| {z_t}^p\right)$$

(since the regime is a Markov chain and factors and residuals are i.i.d., ${z_{t+\tau-1}}^{cI,(\Delta_C),p} = \left(S_{t_{(k_{t+\tau-1})},t+\tau-1}, {D_{t_{(k_{t+\tau-1})}}}^{(\Delta_C)}, {D_{t+\tau-1}}^{(1)}\right)$ completely describes the distribution of ${D_{t+\tau}}^{(v)}$ and ${D_{t+\tau}}^{market}$, rendering ${z_t}^p$ redundant).

Step 4: Re-express $E\left(U'\left(\frac{1}{n_I} \cdot {D_{t+\tau}}^{market}\right) \cdot {D_{t+\tau}}^{(v)} \middle| {z_{t+\tau-1}}^{cI,(\Delta_C),p}\right)$ as

$$cov\left(U'\left(\frac{1}{n_I} \cdot {D_{t+\tau}}^{market}\right) \cdot {D_{t+\tau}}^{(v)} \middle| {z_{t+\tau-1}}^{cI,(\Delta_C),p}\right) + E\left(U'\left(\frac{1}{n_I} \cdot {D_{t+\tau}}^{market}\right) \middle| {z_{t+\tau-1}}^{cI,(\Delta_C),p}\right) \cdot E\left({D_{t+\tau}}^{(v)} \middle| {z_{t+\tau-1}}^{cI,(\Delta_C),p}\right)$$

and plug it into the second term of Step 3

$$E\left(\begin{array}{c} cov\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right), D_{t+\tau}{}^{(\nu)}\middle|z_{t+\tau-1}{}^{cI,(\Delta_C),p}\right) \\ +E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|z_{t+\tau-1}{}^{cI,(\Delta_C),p}\right)\cdot E\left(D_{t+\tau}{}^{(\nu)}\middle|z_{t+\tau-1}{}^{cI,(\Delta_C),p}\right)\end{array}\middle|z_t{}^p\right)$$

$$-E\left(E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|z_{t+\tau-1}{}^{cI,(\Delta_C),p}\right)\middle|z_t{}^p\right)$$

$$\cdot E\left(E\left(D_{t+\tau}{}^{(\nu)}\middle|z_{t+\tau-1}{}^{cI,(\Delta_C),p}\right)\middle|z_t{}^p\right)$$

<u>Step 5</u>: Rearranging terms leads to

$$E\left(cov\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right), D_{t+\tau}{}^{(\nu)}\middle|z_{t+\tau-1}{}^{cI,(\Delta_C),p}\right)\middle|z_t{}^p\right)$$

$$+E\left(E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|z_{t+\tau-1}{}^{cI,(\Delta_C),p}\right)\cdot E\left(D_{t+\tau}{}^{(\nu)}\middle|z_{t+\tau-1}{}^{cI,(\Delta_C),p}\right)\middle|z_t{}^p\right)$$

$$-E\left(E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|z_{t+\tau-1}{}^{cI,(\Delta_C),p}\right)\middle|X_t, D_t\right)$$

$$\cdot E\left(E\left(D_{t+\tau}{}^{(\nu)}\middle|z_{t+\tau-1}{}^{cI,(\Delta_C),p}\right)\middle|z_t{}^p\right)$$

(note that the second term is the covariance of $\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|z_{t+\tau-1}{}^{cI,(\Delta_C),p}\right)$ and $E\left(D_{t+\tau}{}^{(\nu)}\middle|z_{t+\tau-1}{}^{cI,(\Delta_C),p}\right)$)

$$=E\left(cov\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right), D_{t+\tau}{}^{(\nu)}\middle|z_{t+\tau-1}{}^{cI,(\Delta_C),p}\right)\middle|z_t{}^p\right)$$

$$+cov\left(E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|z_{t+\tau-1}{}^{cI,(\Delta_C),p}\right), E\left(D_{t+\tau}{}^{(\nu)}\middle|z_{t+\tau-1}{}^{cI,(\Delta_C),p}\right)\middle|z_t{}^p\right)$$

This is the desired result.

A3.5 Prices of "Expectation Risk" and "Combined Risk"

A3.5.1 Quasi-static Case

A3.5.1.1 Appendix to Section 4.2.1.2.4: Information Frequency = Cash Flow Frequency

A3.5.1.1.1 Formulation of the Problem

The problem is to show that the prices of "expectation risk" and "combined risk" read as follows:

<u>Price of "Expectation Risk"</u>

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$$E\left(q_{t,t+1}{}^{iI}(fe_{t+1}, S_t, D_t) \cdot \Delta^{exp.risk} \middle| \pi_t, D_t\right)$$
$$= -\frac{1}{1 + r_t{}^{iI}(\pi_t, D_t)} \cdot \sum_{s=1}^{K} \{\pi_{s,t} - \theta_{s,t}(\pi_t, D_t)\} \cdot E\left(\left(EP_{t+1}{}^{iI} + ED_{t+1}\right)(S_t, fe_{t+1}; \pi_t, D_t) \middle| \pi_t, D_t, S_t = s\right)$$

with

$$\Delta^{exp.risk} \equiv E\left(\left(EP_{t+1}{}^{iI} + ED_{t+1}\right)(S_t, fe_{t+1}; \pi_t, D_t) \middle| \pi_t, D_t, S_t\right) - E\left(P_{t+1}{}^{iI} + D_{t+1} \middle| \pi_t, D_t\right)$$

with

$$\left(EP_{t+1}{}^{iI} + ED_{t+1}\right)(S_t, fe_{t+1}; \pi_t, D_t) \equiv E\left(P_{t+1}{}^{iI}(\pi_{t+1}, D_{t+1}) + D_{t+1} \middle| \pi_t, D_t, D_{t+1}{}^{market}(D_{t+1})\right)$$

with risk-neutralized regime probabilities

$$\theta_{s,t}(\pi_t, D_t) \equiv \frac{\pi_{s,t} \cdot E\left(U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}\left(D(D_t, S_t, fe_{t+1})\right)\right) \middle| S_t = s, D_t\right)}{\sum_{s'=1}^{K} \pi_{s',t} \cdot E\left(U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}\left(D(D_t, S_t, fe_{t+1})\right)\right) \middle| S_t = s', D_t\right)}$$
$$s = 1, \ldots, K$$

Price of "Combined Risk"

4-71

$$E\left(q_{t,t+1}{}^{iI}(fe_{t+1},S_t,D_t)\cdot\Delta^{comb.risk}\middle|\pi_t,D_t\right)$$
$$=\frac{1}{1+r_t{}^{iI}(\pi_t,D_t)}$$
$$\cdot\sum_{s=1}^{K}\theta_{s,t}(\pi_t,D_t)\cdot E\left(AfR_t{}^{conditional}(S_t;fe_{t+1},D_t)\cdot\Delta^{comb.risk}\middle|\pi_t,D_t,S_t=s\right)$$

with

$$\Delta^{comb.risk}\equiv E\left(P_{t+1}{}^{iI}(\pi_{t+1},D_{t+1})+D_{t+1}\middle|\pi_t,D_t,D_{t+1}{}^{market}(D_{t+1})\right)$$
$$-E\left((EP_{t+1}{}^{iI}+ED_{t+1})(S_t,fe_{t+1};\pi_t,D_t)\middle|\pi_t,D_t,S_t\right)$$

with

$$AfR_t{}^{conditional}(S_t;fe_{t+1},D_t)\equiv AfR_{t,t+1}{}^{cI}(fe_{t+1},S_t,D_t)=\frac{U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right)\middle|S_t,D_t\right)}$$

A3.5.1.1.2 Solution

A3.5.1.1.2.1 Price of "Expectation Risk"

Using the definition of $\Delta^{exp.risk}$ in (4-68), the price of "expectation risk" reads

$$E\left(q_{t,t+1}{}^{iI}\cdot\Delta^{exp.risk}\middle|\pi_t,D_t\right)$$
$$=E\left(q_{t,t+1}{}^{iI}\cdot\begin{Bmatrix}E\left((EP_{t+1}{}^{iI}+ED_{t+1})(S_t,fe_{t+1};\pi_t,D_t)\middle|\pi_t,D_t,S_t\right)\\-E\left(P_{t+1}{}^{iI}+D_{t+1}\middle|\pi_t,D_t\right)\end{Bmatrix}\middle|\pi_t,D_t\right)$$

The computation of the price of "expectation risk" is developed as follows.

Step 1: Rewrite the expectation conditional on π_t,D_t as an expectation over all possible values of the unobservable current regime S_t

A 3-25

$$E\left(q_{t,t+1}{}^{iI}\cdot\begin{Bmatrix}E\left((EP_{t+1}{}^{iI}+ED_{t+1})(S_t,fe_{t+1};\pi_t,D_t)\middle|\pi_t,D_t,S_t\right)\\-E\left(P_{t+1}{}^{iI}+D_{t+1}\middle|\pi_t,D_t\right)\end{Bmatrix}\middle|\pi_t,D_t\right)$$
$$=\sum_{s=1}^{K}\pi_{s,t}$$
$$\cdot E\left(q_{t,t+1}{}^{iI}\cdot\begin{Bmatrix}E\left((EP_{t+1}{}^{iI}+ED_{t+1})(S_t,fe_{t+1};\pi_t,D_t)\middle|\pi_t,D_t,S_t\right)\\-E\left(P_{t+1}{}^{iI}+D_{t+1}\middle|\pi_t,D_t\right)\end{Bmatrix}\middle|S_t=s,\pi_t,D_t\right)$$

Step 2: Re-express the first term on the right-hand side of the previous equation,

$$\sum_{s=1}^{K} \pi_{s,t} \cdot E\left(q_{t,t+1}{}^{iI} \cdot \left\{E\left(\left(EP_{t+1}{}^{iI} + ED_{t+1}\right)(S_t, fe_{t+1}; \pi_t, D_t) \middle| \pi_t, D_t, S_t\right)\right\} \middle| S_t = s, \pi_t, D_t\right)$$

with the help of risk-neutralized regime probabilities

$$\sum_{s=1}^{K} \pi_{s,t} \cdot E\left(q_{t,t+1}{}^{iI} \cdot \left\{E\left(\left(EP_{t+1}{}^{iI} + ED_{t+1}\right)(S_t, fe_{t+1}; \pi_t, D_t) \middle| \pi_t, D_t, S_t\right)\right\} \middle| S_t = s, \pi_t, D_t\right)$$

$$= \frac{1}{1 + r_t{}^{iI}(\pi_t, D_t)}$$

$$\cdot \sum_{s=1}^{K} \frac{\pi_{s,t} \cdot E\left(U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}(D_{t+1})\right) \middle| S_t = s, \pi_t, D_t\right)}{E\left(U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}(D_{t+1})\right) \middle| \pi_t, D_t\right)}$$

$$\cdot E\left(\begin{array}{c} \dfrac{U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}(D_{t+1})\right)}{E\left(U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}(D_{t+1})\right) \middle| S_t = s, \pi_t, D_t\right)} \\ \cdot E\left(\left(EP_{t+1}{}^{iI} + ED_{t+1}\right)(S_t, fe_{t+1}; \pi_t, D_t) \middle| \pi_t, D_t, S_t\right) \end{array} \middle| S_t = s, \pi_t, D_t\right)$$

Note that risk-neutralized regime probabilities read (according to (4-70))

$$\theta_{s,t}(\pi_t, D_t) = \frac{\pi_{s,t} \cdot E\left(U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}(D_{t+1})\right) \middle| S_t = s, \pi_t, D_t\right)}{E\left(U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}(D_{t+1})\right) \middle| \pi_t, D_t\right)}$$

Using this definition of risk-neutralized regime probabilities, it is obtained

A 3-26

$$\sum_{s=1}^{K} \pi_{s,t} \cdot E\left(q_{t,t+1}{}^{iI} \cdot \left\{E\left(\left(EP_{t+1}{}^{iI} + ED_{t+1}\right)(S_t, fe_{t+1}; \pi_t, D_t) \middle| \pi_t, D_t, S_t\right)\right\} \middle| S_t = s, \pi_t, D_t\right)$$

$$= \frac{1}{1 + r_t{}^{iI}(\pi_t, D_t)}$$

$$\cdot \sum_{s=1}^{K} \theta_{s,t}(\pi_t, D_t)$$

$$\cdot E\left(\begin{array}{c} \dfrac{U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}(D_{t+1})\right)}{E\left(U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}(D_{t+1})\right) \middle| S_t = s, \pi_t, D_t\right)} \\ \cdot E\left(\left(EP_{t+1}{}^{iI} + ED_{t+1}\right)(S_t, fe_{t+1}; \pi_t, D_t) \middle| \pi_t, D_t, S_t\right) \end{array} \middle| S_t = s, \pi_t, D_t\right)$$

<u>Step 3</u>: Simplify the conditional expectations on the right hand side of (A 3-26)

The conditional expectation $E\left(\left(EP_{t+1}{}^{iI} + ED_{t+1}\right)(S_t, fe_{t+1}; \pi_t, D_t)\middle|\pi_t, D_t, S_t\right)$ is non-stochastic conditional on $S_t = s, \pi_t, D_t$. Hence, it can be factored out:

$$E\left(\frac{U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right)\middle|S_t = s, \pi_t, D_t\right)} \cdot E\left(\left(EP_{t+1}{}^{iI} + ED_{t+1}\right)(S_t, fe_{t+1}; \pi_t, D_t)\middle|\pi_t, D_t, S_t\right)\middle|S_t = s, \pi_t, D_t\right)$$

$$= E\left(\frac{U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right)\middle|S_t = s, \pi_t, D_t\right)}\middle|S_t = s, \pi_t, D_t\right) \cdot E\left(\left(EP_{t+1}{}^{iI} + ED_{t+1}\right)(S_t, fe_{t+1}; \pi_t, D_t)\middle|\pi_t, D_t, S_t\right)$$

$$= E\left(\left(EP_{t+1}{}^{iI} + ED_{t+1}\right)(S_t, fe_{t+1}; \pi_t, D_t)\middle|\pi_t, D_t, S_t\right)$$

Note that $E\left(\frac{U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right)\middle|S_t = s, \pi_t, D_t\right)}\middle|S_t = s, \pi_t, D_t\right)$ equals one.

Putting Steps 2 and 3 together, the first term on the right-hand side of (A 3-25) reads:

$$\sum_{s=1}^{K} \pi_{s,t} \cdot E\left(q_{t,t+1}{}^{iI} \cdot \left\{E\left(\left(EP_{t+1}{}^{iI} + ED_{t+1}\right)(S_t, fe_{t+1}; \pi_t, D_t)\middle|\pi_t, D_t, S_t\right)\right\}\middle|S_t = s, \pi_t, D_t\right)$$

$$= \frac{1}{1 + r_t{}^{iI}(\pi_t, D_t)} \cdot \sum_{s=1}^{K} \theta_{s,t}(\pi_t, D_t) \cdot E\left(\left(EP_{t+1}{}^{iI} + ED_{t+1}\right)(S_t, fe_{t+1}; \pi_t, D_t)\middle|\pi_t, D_t, S_t\right)$$

<u>Step 4</u>: Evaluate the second term

$$-\sum_{s=1}^{K} \pi_{s,t} \cdot E\left(q_{t,t+1}{}^{iI} \cdot E\left(P_{t+1}{}^{iI} + D_{t+1}\middle|\pi_t, D_t\right)\middle|S_t = s, \pi_t, D_t\right)$$

on the right-hand side of (A 3-25)

First, observe that the conditional expectation $E\left(P_{t+1}{}^{iI} + D_{t+1}\middle|\pi_t, D_t\right)$ is non-stochastic conditional on $S_t = s, \pi_t, D_t$. Hence, it can be factored out:

$$-\sum_{s=1}^{K} \pi_{s,t} \cdot E\left(q_{t,t+1}{}^{iI} \cdot E\left(P_{t+1}{}^{iI} + D_{t+1}\middle|\pi_t, D_t\right)\middle|S_t = s, \pi_t, D_t\right)$$

$$= -\sum_{s=1}^{K} \pi_{s,t} \cdot E\left(q_{t,t+1}{}^{iI}\middle|S_t = s, \pi_t, D_t\right) \cdot E\left(P_{t+1}{}^{iI} + D_{t+1}\middle|\pi_t, D_t\right)$$

Second, use the tower property of conditional expectations to write

$$E\left(P_{t+1}{}^{iI} + D_{t+1} \middle| \pi_t, D_t\right) = E\left(E\left(P_{t+1}{}^{iI}(\pi_{t+1}, D_{t+1}) + D_{t+1} \middle| \pi_t, D_t, D_{t+1}{}^{market}(D_{t+1})\right) \middle| \pi_t, D_t\right)$$
$$\equiv E\left(\left(EP_{t+1}{}^{iI} + ED_{t+1}\right)(S_t, fe_{t+1}; \pi_t, D_t) \middle| \pi_t, D_t\right)$$

Third, the expectation of the stochastic discount factor evaluates to $\frac{1}{1+r_t{}^{iI}(\pi_t, D_t)}$, yielding

$$-\sum_{s=1}^{K} \pi_{s,t} \cdot E\left(q_{t,t+1}{}^{iI} \cdot E\left(P_{t+1}{}^{iI} + D_{t+1} \middle| \pi_t, D_t\right) \middle| S_t = s, \pi_t, D_t\right)$$
$$= -\frac{1}{1 + r_t{}^{iI}(\pi_t, D_t)} \cdot E\left(P_{t+1}{}^{iI} + D_{t+1} \middle| \pi_t, D_t\right)$$

Fourth, rewrite the expectation conditional on π_t, D_t as an expectation over all possible values of the unobservable current regime S_t

$$-\sum_{s=1}^{K} \pi_{s,t} \cdot E\left(q_{t,t+1}{}^{iI} \cdot E\left(\left(EP_{t+1}{}^{iI} + ED_{t+1}\right)(S_t, fe_{t+1}; \pi_t, D_t) \middle| \pi_t, D_t\right) \middle| S_t = s, \pi_t, D_t\right)$$
$$= -\frac{1}{1 + r_t{}^{iI}(\pi_t, D_t)}$$
$$\cdot \sum_{s=1}^{K} \pi_{s,t} \cdot E\left(\left(EP_{t+1}{}^{iI} + ED_{t+1}\right)(S_t, fe_{t+1}; \pi_t, D_t) \middle| S_t = s, \pi_t, D_t\right)$$

<u>Step 5</u>: Final result

Putting Steps 2 and 4 together, the right-hand side of (A 3-25) reads:

$$E\left(q_{t,t+1}{}^{iI} \cdot \Delta^{exp.risk} \middle| \pi_t, D_t\right)$$
$$= -\frac{1}{1 + r_t{}^{iI}(\pi_t, D_t)}$$
$$\cdot \sum_{s=1}^{K} \left\{\pi_{s,t} - \theta_{s,t}(\pi_t, D_t)\right\}$$
$$\cdot E\left(\left(EP_{t+1}{}^{iI} + ED_{t+1}\right)(S_t, fe_{t+1}; \pi_t, D_t) P_{t+1}{}^{iI} + D_{t+1} \middle| S_t = s, \pi_t, D_t\right)$$

as was to be shown (4-69).

A3.5.1.1.2.2 Price of "Combined Risk"

<u>Step 1</u>: Rewrite the expectation conditional on π_t, D_t as an expectation over all possible values of the unobservable current regime S_t

$$E\left(q_{t,t+1}{}^{iI}(fe_{t+1}, S_t, D_t) \cdot \Delta^{comb.risk} \middle| \pi_t, D_t\right)$$
$$= \sum_{s=1}^{K} \pi_{s,t} \cdot E\left(q_{t,t+1}{}^{iI}(fe_{t+1}, S_t, D_t) \cdot \Delta^{comb.risk} \middle| S_t = s, \pi_t, D_t\right)$$

Step 2: Re-express the right-hand side of the previous equation with the help of risk-neutralized regime probabilities

$$E\left(q_{t,t+1}{}^{iI}(fe_{t+1}, S_t, D_t) \cdot \Delta^{comb.risk} \middle| \pi_t, D_t\right)$$
$$= \frac{1}{1 + r_t{}^{iI}(\pi_t, D_t)}$$
$$\cdot \sum_{s=1}^{K} \pi_{s,t} \cdot E\left(\frac{U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}(D_{t+1})\right)}{E\left(U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}(D_{t+1})\right) \middle| \pi_t, D_t\right)} \cdot \Delta^{comb.risk} \middle| S_t = s, \pi_t, D_t\right)$$

Factoring out $E\left(U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}(D_{t+1})\right) \middle| \pi_t, D_t\right)$ and expanding by

$$E\left(U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}(D_{t+1})\right) \middle| S_t = s, \pi_t, D_t\right)$$

yields

$$E\left(q_{t,t+1}{}^{iI}(fe_{t+1}, S_t, D_t) \cdot \Delta^{comb.risk} \middle| \pi_t, D_t\right)$$
$$= \frac{1}{1 + r_t{}^{iI}(\pi_t, D_t)}$$
$$\cdot \sum_{s=1}^{K} \theta_{s,t}(\pi_t, D_t)$$
$$\cdot E\left(\frac{U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}(D_{t+1})\right)}{E\left(U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}(D_{t+1})\right) \middle| S_t = s, \pi_t, D_t\right)} \cdot \Delta^{comb.risk} \middle| S_t = s, \pi_t, D_t\right)$$

with risk-neutralized regime probabilities

$$\theta_{s,t}(\pi_t, D_t) = \frac{\pi_{s,t} \cdot E\left(U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}(D_{t+1})\right) \middle| S_t = s, \pi_t, D_t\right)}{E\left(U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}(D_{t+1})\right) \middle| \pi_t, D_t\right)}$$

Step 3: Integrate the adjustment for risk

Using

$$AfR_t{}^{conditional}(S_t; fe_{t+1}, D_t) \equiv AfR_{t,t+1}{}^{cI}(fe_{t+1}, S_t, D_t) = \frac{U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}(D_{t+1})\right)}{E\left(U'\left(\frac{1}{n_I} \cdot D_{t+1}{}^{market}(D_{t+1})\right) \middle| S_t, D_t\right)}$$

the price of "combined risk" (4-71) is obtained as follows:

$$= \frac{1}{1 + r} \cdot \sum_{s=1}^{K} \theta_{s,t}(\pi_t, D_t) \cdot E\left(AfR_t{}^{conditional}(S_t; fe_{t+1}, D_t) \cdot \Delta^{comb.risk} \middle| S_t = s, \pi_t, D_t\right)$$

A3.5.1.2 Appendix to Section 4.2.2.3.4: Information Frequency ≥ Cash Flow Frequency

A3.5.1.2.1 Formulation of the Problem

Definition and price of “expectation risk” and “combined risk” for the case where information frequency is higher than or equal to cash flow frequency have been omitted in the main text because they are almost entirely analogous to the special case where information frequency equals cash flow frequency. For the sake of completeness, “expectation risk” and “combined risk” are defined and priced in this section of the appendix.

A3.5.1.2.1.1 Definition of “expectation risk” and “combined risk” in the case where information frequency is higher than or equal to cash flow frequency

As in the special case where information frequency is equal to cash flow frequency, cash flows and asset prices can be decomposed into a part that can be explained by aggregate cash flows and another part that is uncorrelated with aggregate cash flows. More formally, in generalization of (4-60), asset prices and cash flows of both (Δ_C)-periodic and (1)-periodic assets are the sum of a prices part (part (i)) and a non-priced part (part (ii)):

A 3-27

$$\begin{aligned}&{P_{t+1}}^{iI,(\Delta_C)} + 1_{t+1=t_{(k+1)}} \cdot {D_{t+1}}^{(\Delta_C)} \\ &= \underbrace{E\left({P_{t+1}}^{iI,(\Delta_C)} + 1_{t+1=t_{(k+1)}} \cdot {D_{t+1}}^{(\Delta_C)} \middle| \pi_{t_{(k_t)},t}, {D_{t_{(k_t)}}}^{(\Delta_C)}, {D_t}^{(1)}, {D_{t+1}}^{market}\right)}_{part\ (i)} \\ &+ \underbrace{{\Delta_{t+1}}^{iI,(\Delta_C)}}_{part\ (ii)}\end{aligned}$$

with

$$\begin{aligned}{\Delta_{t+1}}^{iI,(\Delta_C)} &\equiv {P_{t+1}}^{iI,(\Delta_C)} + 1_{t+1=t_{(k+1)}} \cdot {D_{t+1}}^{(\Delta_C)} \\ &- E\left({P_{t+1}}^{iI,(\Delta_C)} + 1_{t+1=t_{(k+1)}} \cdot {D_{t+1}}^{(\Delta_C)} \middle| \pi_{t_{(k_t)},t}, {D_{t_{(k_t)}}}^{(\Delta_C)}, {D_t}^{(1)}, {D_{t+1}}^{market}\right)\end{aligned}$$

and

A 3-28

$$ {P_{t+1}}^{iI,(1)} + {D_{t+1}}^{(1)} = \underbrace{E\left({P_{t+1}}^{iI,(1)} + {D_{t+1}}^{(1)} \middle| \pi_{t_{(k_t)},t}, {D_{t_{(k_t)}}}^{(\Delta_C)}, {D_t}^{(1)}, {D_{t+1}}^{market}\right)}_{part\ (i)} + \underbrace{{\Delta_{t+1}}^{iI,(\Delta_C)}}_{part\ (ii)} $$

with

$$ {\Delta_{t+1}}^{iI,(1)} \equiv {P_{t+1}}^{iI,(1)} + {D_{t+1}}^{(1)} - E\left({P_{t+1}}^{iI,(1)} + {D_{t+1}}^{(1)} \middle| \pi_{t_{(k_t)},t}, {D_{t_{(k_t)}}}^{(\Delta_C)}, {D_t}^{(1)}, {D_{t+1}}^{market}\right) $$

Definition of “expectation risk” and “combined risk”

Since part (ii) is not priced, only part (i) is further decomposed into “expectation risk” and “combined risk” parts. To this end, the expectations of asset prices and cash flows conditional on the information $z_t^{iI,(\Delta_C),p} = \left(\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}^{(\Delta_C)}, D_t^{(1)}\right)$ are first subtracted from part (i) in order to separate an expectation component from a risk component with zero expectation. In generalization of (4-68), “expectation risk” and “combined risk” can thus be defined as follows:

(Δ_C)-periodic assets

A 3-29

$$E\left(P_{t+1}^{iI,(\Delta_C)} + 1_{t+1=t_{(k+1)}} \cdot D_{t+1}^{(\Delta_C)} \middle| \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}^{(\Delta_C)}, D_t^{(1)}, D_{t+1}^{market}\right) - E\left(P_{t+1}^{iI,(\Delta_C)} + 1_{t+1=t_{(k+1)}} \cdot D_{t+1}^{(\Delta_C)} \middle| \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}^{(\Delta_C)}, D_t^{(1)}\right)$$

$$= \underbrace{\left\{\begin{matrix} \left(EP_{t+1}^{iI} + ED_{t+1}\right)^{(\Delta_C)}\left(S_{t_{(k_t)},t}, fe_{t+1}; \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}^{(\Delta_C)}, D_t^{(1)}\right) \\ -E\left(\left(EP_{t+1}^{iI} + ED_{t+1}\right)^{(\Delta_C)}\left(S_t, fe_{t+1}; \pi_t, D_t\right) \middle| \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}^{(\Delta_C)}, D_t^{(1)}, S_{t_{(k_t)},t}\right) \end{matrix}\right\}}_{\Delta^{comb.risk,(\Delta_C)}}$$

$$+ \underbrace{\left\{\begin{matrix} E\left(\left(EP_{t+1}^{iI} + ED_{t+1}\right)^{(\Delta_C)}\left(S_{t_{(k_t)},t}, fe_{t+1}; \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}^{(\Delta_C)}, D_t^{(1)}\right) \middle| \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}^{(\Delta_C)}, D_t^{(1)}, S_{t_{(k_t)},t}\right) \\ -E\left(P_{t+1}^{iI,(\Delta_C)} + 1_{t+1=t_{(k+1)}} \cdot D_{t+1}^{(\Delta_C)} \middle| \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}^{(\Delta_C)}, D_t^{(1)}\right) \end{matrix}\right\}}_{\Delta^{exp.risk,(\Delta_C)}}$$

with

$$\left(EP_{t+1}^{iI} + ED_{t+1}\right)^{(\Delta_C)}\left(S_{t_{(k_t)},t}, fe_{t+1}; \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}^{(\Delta_C)}, D_t^{(1)}\right) \equiv E\left(P_{t+1}^{iI,(\Delta_C)} + 1_{t+1=t_{(k+1)}} \cdot D_{t+1}^{(\Delta_C)} \middle| \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}^{(\Delta_C)}, D_t^{(1)}, D_{t+1}^{market}\right)$$

(1)-periodic assets

A 3-30

$$E\left(P_{t+1}^{iI,(1)} + D_{t+1}^{(1)} \middle| \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}^{(\Delta_C)}, D_t^{(1)}, D_{t+1}^{market}\right) - E\left(P_{t+1}^{iI,(1)} + D_{t+1}^{(1)} \middle| \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}^{(\Delta_C)}, D_t^{(1)}\right)$$

$$= \underbrace{\left\{\begin{matrix} \left(EP_{t+1}^{iI} + ED_{t+1}\right)^{(1)}\left(S_{t_{(k_t)},t}, fe_{t+1}; \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}^{(\Delta_C)}, D_t^{(1)}\right) \\ -E\left(\left(EP_{t+1}^{iI} + ED_{t+1}\right)^{(1)}\left(S_t, fe_{t+1}; \pi_t, D_t\right) \middle| \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}^{(\Delta_C)}, D_t^{(1)}, S_{t_{(k_t)},t}\right) \end{matrix}\right\}}_{\Delta^{comb.risk,(1)}}$$

$$+ \underbrace{\left\{\begin{matrix} E\left(\left(EP_{t+1}^{iI} + ED_{t+1}\right)^{(1)}\left(S_{t_{(k_t)},t}, fe_{t+1}; \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}^{(\Delta_C)}, D_t^{(1)}\right) \middle| \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}^{(\Delta_C)}, D_t^{(1)}, S_{t_{(k_t)},t}\right) \\ -E\left(P_{t+1}^{iI,(1)} + D_{t+1}^{(1)} \middle| \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}^{(\Delta_C)}, D_t^{(1)}\right) \end{matrix}\right\}}_{\Delta^{exp.risk,(1)}}$$

with

$$\left(EP_{t+1}{}^{iI}+ED_{t+1}\right)^{(1)}(S_t,fe_{t+1};\pi_t,D_t)$$
$$\equiv E\left(P_{t+1}{}^{iI,(1)}+D_{t+1}{}^{(1)}\middle|\pi_{t_{(k_t)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)},D_{t+1}{}^{market}\right)$$

A3.5.1.2.1.2 Price of "expectation risk" and "combined risk"

The problem is to show that the prices of "expectation risk" and "combined risk" read as follows:

Price of "expectation risk"

A 3-31

$$E\left(q_{t,t+1}{}^{iI,(\Delta_C)}\left(fe_{t+1},S_{t_{(k)},t},D_{t_{(k)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)\cdot\Delta^{exp.risk,(\nu)}\middle|\pi_{t_{(k)},t},D_{t_{(k)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)$$
$$=-\frac{1}{1+r_t{}^{iI,(\Delta_C)}\left(\pi_{t_{(k)},t},D_{t_{(k)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)}$$
$$\cdot\sum_{s_{t_{(k)},t}}\left\{\pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right)-\theta_t\left(s_{t_{(k)},t};\pi_{t_{(k_t)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)\right\}$$
$$\cdot E\left(\left(EP_{t+1}{}^{iI}+ED_{t+1}\right)^{(\nu)}\left(S_{t_{(k_t)},t},fe_{t+1};\pi_{t_{(k_t)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)\middle|\begin{array}{c}S_{t_{(k_t)},t}=s_{t_{(k)},t}\\,\pi_{t_{(k_t)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\end{array}\right)$$
$$\nu\epsilon\{1,\Delta_C\}$$

with risk-neutralized regime-path probabilities

$$\theta_t\left(s_{t_{(k)},t};\pi_{t_{(k_t)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)$$
$$\equiv\frac{\pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right)\cdot E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}\right)\middle|\begin{array}{c}S_{t_{(k)},t}=s_{t_{(k)},t}\\,D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\end{array}\right)}{\sum_{s_{t_{(k)},t}{}'}\pi_{t_{(k)},t}\left(s_{t_{(k)},t}{}'\right)\cdot E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}\right)\middle|\begin{array}{c}S_{t_{(k)},t}=s_{t_{(k)},t}{}'\\,D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\end{array}\right)}$$

with

$$D_{t+1}{}^{market}=\begin{cases}D_{t+1}{}^{market}\left(D_{t+1}{}^{(\Delta_C)},D_{t+1}{}^{(1)}\right) & t+1=t_{(k+1)}\\ D_{t+1}{}^{market}\left(D_{t+1}{}^{(1)}\right) & t+1\neq t_{(k+1)}\end{cases}$$

Price of "combined risk"

A 3-32

$$E\left(q_{t,t+1}{}^{iI,(\Delta_C)}\left(fe_{t+1},S_{t_{(k)},t},D_{t_{(k)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)\cdot\Delta^{comb.risk,(v)}\middle|\pi_{t_{(k)},t},D_{t_{(k)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)$$
$$=\frac{1}{1+r_t{}^{iI,(\Delta_C)}\left(\pi_{t_{(k)},t},D_{t_{(k)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)}$$
$$\cdot\sum_{s_{t_{(k)},t}}\theta_t\left(s_{t_{(k)},t};\pi_{t_{(k_t)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)$$
$$\cdot E\left(AfR_t{}^{conditional}\left(fe_{t+1},S_{t_{(k)},t},D_{t_{(k)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)\cdot\Delta^{comb.risk,(v)}\middle|\begin{matrix}S_{t_{(k_t)},t}=s_{t_{(k)},t}\\ ,\pi_{t_{(k_t)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\end{matrix}\right)$$

with

$$AfR_t{}^{conditional}\left(fe_{t+1},S_{t_{(k)},t},D_{t_{(k)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)=\frac{U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}\right)\middle|S_{t_{(k_t)},t}=s_{t_{(k)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)}$$

A3.5.1.3 Solution

A3.5.1.3.1 Idea of the proof

The proof follows the steps of the case where information frequency is equal to cash flow frequency. The single regime S_t must merely be replaced by the path of regimes since the last payment date of (Δ_C)-periodic assets, $S_{t_{(k_t)},t}$.

A3.5.1.3.2 Details of the proof

A3.5.1.3.2.1 Price of “Expectation Risk”

Step 1: Rewrite the expectation conditional on $\pi_{t_{(k_t)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}$ as an expectation over all possible values of the unobservable current regime path $S_{t_{(k_t)},t}$

A 3-33

$$E\left(\begin{array}{c} q_{t,t+1}{}^{iI,(\Delta_C)}\left(fe_{t+1},S_{t_{(k)},t},D_{t_{(k)}}{}^{(\Delta_C)},D_t{}^{(1)}\right) \\ \cdot\left\{\begin{array}{c} E\left(\left(EP_{t+1}{}^{iI}+ED_{t+1}\right)^{(\nu)}\left(S_{t_{(k_t)},t}\right)\middle|\pi_{t_{(k_t)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)},S_{t_{(k_t)},t}\right) \\ -E\left(P_{t+1}{}^{iI,(\nu)}+D_{t+1}{}^{(\nu)}\middle|\pi_{t_{(k_t)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)\end{array}\right\}\end{array}\middle|\pi_{t_{(k)},t},D_{t_{(k)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)$$

$$=\sum_{s_{t_{(k)},t}}\pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right)$$

$$\cdot E\left(\begin{array}{c} q_{t,t+1}{}^{iI,(\Delta_C)}\left(fe_{t+1},S_{t_{(k)},t},D_{t_{(k)}}{}^{(\Delta_C)},D_t{}^{(1)}\right) \\ \cdot\left\{\begin{array}{c} E\left(\left(EP_{t+1}{}^{iI}+ED_{t+1}\right)^{(\nu)}\left(S_{t_{(k_t)},t}\right)\middle|\pi_{t_{(k_t)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)},S_{t_{(k_t)},t}\right) \\ -E\left(P_{t+1}{}^{iI,(\nu)}+D_{t+1}{}^{(\nu)}\middle|\pi_{t_{(k_t)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)\end{array}\right\}\end{array}\middle|\begin{array}{c} S_{t_{(k_t)},t}=s_{t_{(k)},t} \\ ,\pi_{t_{(k_t)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\end{array}\right)$$

with

$$P_{t+1}{}^{iI,(\nu)}+D_{t+1}{}^{(\nu)}=\begin{cases}P_{t+1}{}^{iI,(\Delta_C)}+1_{t+1=t_{(k+1)}}\cdot D_{t+1}{}^{(\Delta_C)} & \nu=\Delta_C \\ P_{t+1}{}^{iI,(1)}+D_{t+1}{}^{(1)} & \nu=1\end{cases}$$

where, for brevity, most of the arguments of $\left(EP_{t+1}{}^{iI}+ED_{t+1}\right)^{(\nu)}$ have been suppressed,

$$\left(EP_{t+1}{}^{iI}+ED_{t+1}\right)^{(\nu)}\left(S_{t_{(k_t)},t}\right)$$
$$\equiv E\left(\left(EP_{t+1}{}^{iI}+ED_{t+1}\right)^{(\nu)}\left(S_{t_{(k_t)},t},fe_{t+1};\pi_{t_{(k_t)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)\middle|\pi_{t_{(k_t)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)},S_{t_{(k_t)},t}\right)$$

<u>Step 2</u>: Re-express the first term on the right-hand side of the previous equation,

$$\sum_{s_{t_{(k)},t}} \pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right) \cdot E\left(\begin{array}{c} q_{t,t+1}{}^{iI,(\Delta_C)} \\ \cdot E\left(\left(EP_{t+1}{}^{iI}+ED_{t+1}\right)^{(\nu)}\left(S_{t_{(k_t)},t}\right)\middle|\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}, S_{t_{(k_t)},t}\right) \end{array} \middle| \begin{array}{c} S_{t_{(k_t)},t}=s_{t_{(k)},} \\ ,\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)} \end{array}\right)$$

with the help of risk-neutralized regime path probabilities

$$\sum_{s_{t_{(k)},t}} \pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right) \cdot E\left(\begin{array}{c} q_{t,t+1}{}^{iI,(\Delta_C)} \\ \cdot E\left(\left(EP_{t+1}{}^{iI}+ED_{t+1}\right)^{(\nu)}\left(S_{t_{(k_t)},t}\right)\middle|\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}, S_{t_{(k_t)},t}\right) \end{array} \middle| \begin{array}{c} S_{t_{(k_t)},t}=s_{t_{(k)},t} \\ ,\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)} \end{array}\right)$$

$$= \frac{1}{1+r_t{}^{iI,(\Delta_C)}\left(\pi_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)}$$

$$\cdot \sum_{s_{t_{(k)},t}} \pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right) \cdot \frac{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right)\middle| \begin{array}{c} S_{t_{(k_t)},t}=s_{t_{(k)},t} \\ ,D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)} \end{array}\right)}{\sum_{s_{t_{(k)},t}{}'} \pi_{t_{(k)},t}\left(s_{t_{(k)},t}{}'\right)\cdot E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right)\middle| \begin{array}{c} S_{t_{(k_t)},t}=s_{t_{(k)},t}{}' \\ ,D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)} \end{array}\right)}$$

$$\cdot E\left(\begin{array}{c} \dfrac{U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right)\middle| \begin{array}{c} S_{t_{(k_t)},t}=s_{t_{(k)},t} \\ ,D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)} \end{array}\right)} \\ \cdot E\left(\left(EP_{t+1}{}^{iI}+ED_{t+1}\right)^{(\nu)}\left(S_{t_{(k_t)},t}\right)\middle|\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}, S_{t_{(k_t)},t}\right) \end{array} \middle| \begin{array}{c} S_{t_{(k_t)},t}=s_{t_{(k)},t} \\ ,\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)} \end{array}\right)$$

Note that risk-neutralized regime path probabilities read (according to (A 3-31)) read

$$\theta_t\left(s_{t_{(k)},t}; \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)$$

$$\equiv \frac{\pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right)\cdot E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}\right)\middle| \begin{array}{c} S_{t_{(k)},t}=s_{t_{(k)},t} \\ ,D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)} \end{array}\right)}{\sum_{s_{t_{(k)},t}{}'} \pi_{t_{(k)},t}\left(s_{t_{(k)},t}{}'\right)\cdot E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}\right)\middle| \begin{array}{c} S_{t_{(k)},t}=s_{t_{(k)},t}{}' \\ ,D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)} \end{array}\right)}$$

Using this definition of risk-neutralized regime probabilities, it is obtained

A 3-34

$$\sum_{s_{t_{(k)},t}} \pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right) \cdot E\left(\begin{array}{c} q_{t,t+1}{}^{iI,(\Delta_C)} \\ \cdot E\left(\left(EP_{t+1}{}^{iI}+ED_{t+1}\right)^{(\nu)}\left(S_{t_{(k_t)},t}\right)\middle|\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}, S_{t_{(k_t)},t}\right)\end{array}\middle| \begin{array}{c} S_{t_{(k_t)},t}=s_{t_{(k)},t} \\ ,\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\end{array}\right)$$

$$= \frac{1}{1+r_t{}^{iI,(\Delta_C)}\left(\pi_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)} \cdot \sum_{s_{t_{(k)},t}} \theta_t\left(s_{t_{(k)},t}; \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)$$

$$\cdot E\left(\begin{array}{c} \dfrac{U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right)\middle| \begin{array}{c} S_{t_{(k_t)},t}=s_{t_{(k)},t} \\ ,D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\end{array}\right)} \\ \cdot E\left(\left(EP_{t+1}{}^{iI}+ED_{t+1}\right)^{(\nu)}\left(S_{t_{(k_t)},t}\right)\middle|\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}, S_{t_{(k_t)},t}\right)\end{array}\middle| \begin{array}{c} S_{t_{(k_t)},t}=s_{t_{(k)},t} \\ ,\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\end{array}\right)$$

<u>Step 3</u>: Simplify the conditional expectations on the right hand side of (A 3-34)

The conditional expectation $E\left(\left(EP_{t+1}{}^{iI}+ED_{t+1}\right)^{(\nu)}\left(S_{t_{(k_t)},t}\right)\middle|\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}, S_{t_{(k_t)},t}\right)$ is non-stochastic conditional on $S_{t_{(k_t)},t}=s_{t_{(k)},t}, \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}$. Hence, it can be factored out:

$$E\left(\begin{array}{c} \dfrac{U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right)\middle| \begin{array}{c} S_{t_{(k_t)},t}=s_{t_{(k)},t} \\ ,D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\end{array}\right)} \\ \cdot E\left(\left(EP_{t+1}{}^{iI}+ED_{t+1}\right)^{(\nu)}\left(S_{t_{(k_t)},t}\right)\middle|\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}, S_{t_{(k_t)},t}\right)\end{array}\middle| \begin{array}{c} S_{t_{(k_t)},t}=s_{t_{(k)},t} \\ ,\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\end{array}\right)$$

$$= E\left(\dfrac{U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{maret}(D_{t+1})\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right)\middle| \begin{array}{c} S_{t_{(k_t)},t}=s_{t_{(k)},t} \\ ,D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\end{array}\right)}\middle| \begin{array}{c} S_{t_{(k_t)},t}=s_{t_{(k)},t} \\ ,\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\end{array}\right)$$

$$\cdot E\left(\left(EP_{t+1}{}^{iI}+ED_{t+1}\right)^{(\nu)}\left(S_{t_{(k_t)},t}\right)\middle|\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}, S_{t_{(k_t)},t}\right)$$

$$= E\left(\left(EP_{t+1}{}^{iI}+ED_{t+1}\right)^{(\nu)}\left(S_{t_{(k_t)},t}\right)\middle|\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}, S_{t_{(k_t)},t}\right)$$

Note that $E\left(\dfrac{U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right)\middle| \begin{array}{c} S_{t_{(k_t)},t}=s_{t_{(k)},t} \\ ,D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\end{array}\right)}\middle| \begin{array}{c} S_{t_{(k_t)},t}=s_{t_{(k)},t} \\ ,\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\end{array}\right)$ equals one.

Putting Steps 2 and 3 together, the first term on the right-hand side of (A 3-33) reads:

$$\sum_{s_{t_{(k)},t}} \pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right) \cdot E\left(\begin{matrix} q_{t,t+1}{}^{iI,(\Delta_C)} \\ \cdot E\left(\left(EP_{t+1}{}^{iI} + ED_{t+1}\right)^{(\nu)}\left(S_{t_{(k_t)},t}\right)\middle|\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}, S_{t_{(k_t)},t}\right)\end{matrix}\middle| \begin{matrix} S_{t_{(k_t)},t} = s_{t_{(k)},t} \\ ,\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\end{matrix}\right)$$

$$= \frac{1}{1 + r_t{}^{iI,(\Delta_C)}\left(\pi_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)} \cdot \sum_{s_{t_{(k)},t}} \theta_t\left(s_{t_{(k)},t}; \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) \cdot E\left(\left(EP_{t+1}{}^{iI} + ED_{t+1}\right)^{(\nu)}\left(S_{t_{(k_t)},t}\right)\middle|\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}, S_{t_{(k_t)},t}\right)$$

<u>Step 4</u>: Evaluate the second term

$$-\sum_{s_{t_{(k)},t}} \pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right) \cdot E\left(\begin{matrix} q_{t,t+1}{}^{iI,(\Delta_C)}\left(fe_{t+1}, S_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) \\ \cdot E\left(P_{t+1}{}^{iI,(\nu)} + D_{t+1}{}^{(\nu)}\middle|\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)\end{matrix}\middle| \begin{matrix} S_{t_{(k_t)},t} = s_{t_{(k)},t} \\ ,\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\end{matrix}\right)$$

on the right-hand side of (A 3-33)

First, observe that the conditional expectation $E\left(P_{t+1}{}^{iI,(\nu)} + D_{t+1}{}^{(\nu)}\middle|\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)$ is non-stochastic conditional on $S_{t_{(k_t)},t} = s_{t_{(k)},t}, \pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}$. Hence, it can be factored out:

$$-\sum_{s_{t_{(k)},t}} \pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right) \cdot E\left(\begin{matrix} q_{t,t+1}{}^{iI,(\Delta_C)}\left(fe_{t+1}, S_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) \\ \cdot E\left(P_{t+1}{}^{iI,(\nu)} + D_{t+1}{}^{(\nu)}\middle|\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)\end{matrix}\middle| \begin{matrix} S_{t_{(k_t)},t} = s_{t_{(k)},t} \\ ,\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\end{matrix}\right)$$

$$= -\sum_{s_{t_{(k)},t}} \pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right) \cdot E\left(q_{t,t+1}{}^{iI,(\Delta_C)}\left(fe_{t+1}, S_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)\middle| \begin{matrix} S_{t_{(k_t)},t} = s_{t_{(k)},t} \\ ,\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\end{matrix}\right) \cdot E\left(P_{t+1}{}^{iI,(\nu)} + D_{t+1}{}^{(\nu)}\middle|\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)$$

Second, use the tower property of conditional expectations to write

$$E\left(P_{t+1}{}^{iI,(\nu)} + D_{t+1}{}^{(\nu)}\middle|\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right) = E\left(\left(EP_{t+1}{}^{iI} + ED_{t+1}\right)^{(\nu)}\left(S_{t_{(k_t)},t}\right)\middle|\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)$$

A3.5.1.3.2.2 Price of "Combined Risk"

Step 1: Rewrite the expectation conditional on $\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}$ as an expectation over all possible values of the unobservable regime path $S_{t_{(k_t)},t}$

$$E\left(q_{t,t+1}{}^{iI,(\Delta_C)}\left(fe_{t+1}, S_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)\cdot\Delta^{comb.risk,(v)}\middle|\pi_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)$$
$$=\sum_{s_{t_{(k)},t}}\pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right)$$
$$\cdot E\left(q_{t,t+1}{}^{iI,(\Delta_C)}\left(fe_{t+1}, S_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)\cdot\Delta^{comb.risk,(v)}\middle|\begin{matrix}S_{t_{(k_t)},t}=s_{t_{(k)},t}\\ ,\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\end{matrix}\right)$$

Step 2: Re-express the right-hand side of the previous equation with the help of risk-neutralized regime probabilities

$$E\left(q_{t,t+1}{}^{iI,(\Delta_C)}\left(fe_{t+1}, S_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)\cdot\Delta^{comb.risk,(v)}\middle|\pi_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)$$
$$=\frac{1}{1+r_t{}^{iI,(\Delta_C)}\left(\pi_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)}$$
$$\cdot\sum_{s_{t_{(k)},t}}\pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right)$$
$$\cdot E\left(\frac{U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right)\middle|\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)}\cdot\Delta^{comb.risk,(v)}\middle|\begin{matrix}S_{t_{(k_t)},t}=s_{t_{(k)},t}\\ ,\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\end{matrix}\right)$$

Factoring out $E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right)\middle|\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)$ and expanding by $E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right)\middle|\begin{matrix}S_{t_{(k_t)},t}=s_{t_{(k)},t}\\ ,D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\end{matrix}\right)$ yields

$$E\left(q_{t,t+1}{}^{iI,(\Delta_C)}\left(fe_{t+1}, S_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)\cdot\Delta^{comb.risk,(v)}\middle|\pi_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)$$
$$=\frac{1}{1+r_t{}^{iI,(\Delta_C)}\left(\pi_{t_{(k)},t}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)}$$
$$\cdot\sum_{s_{t_{(k)},t}}\theta_t\left(s_{t_{(k)},t};\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)$$
$$\cdot E\left(\frac{U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}(D_{t+1})\right)\middle|S_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)}\cdot\Delta^{comb.risk,(v)}\middle|\begin{matrix}S_{t_{(k_t)},t}=s_{t_{(k)},t}\\ ,\pi_{t_{(k_t)},t}, D_{t_{(k_t)}}{}^{(\Delta_C)}, D_t{}^{(1)}\end{matrix}\right)$$

with risk-neutralized regime path probabilities

$$\theta_t\left(s_{t_{(k)},t};\pi_{t_{(k_t)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)$$

$$\equiv\frac{\pi_{t_{(k)},t}\left(s_{t_{(k)},t}\right)\cdot E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}\right)\middle|\begin{array}{c}S_{t_{(k)},t}=s_{t_{(k)},t}\\ ,D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\end{array}\right)}{\sum_{s_{t_{(k)},t}'}\pi_{t_{(k)},t}\left(s_{t_{(k)},t}'\right)\cdot E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}\right)\middle|\begin{array}{c}S_{t_{(k)},t}=s_{t_{(k)},t}'\\ ,D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\end{array}\right)}$$

<u>Step 3</u>: Integrate the adjustment for risk

Using

$$AfR_t{}^{conditional}\left(fe_{t+1},S_{t_{(k)},t},D_{t_{(k)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)$$

$$=\frac{U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+1}{}^{market}\right)\middle|S_{t_{(k_t)},t}=s_{t_{(k)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)}$$

the price of “combined risk” (A 3-32) is obtained as follows

$$=\frac{1}{1+r_t{}^{iI,(\Delta_C)}\left(\pi_{t_{(k)},t},D_{t_{(k)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)}$$

$$\cdot\sum_{s_{t_{(k)},t}}\theta_t\left(s_{t_{(k)},t};\pi_{t_{(k_t)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)$$

$$\cdot E\left(AfR_t{}^{conditional}\left(fe_{t+1},S_{t_{(k)},t},D_{t_{(k)}}{}^{(\Delta_C)},D_t{}^{(1)}\right)\cdot\Delta^{comb.risk,(v)}\middle|\begin{array}{c}S_{t_{(k_t)},t}=s_{t_{(k)},t}\\ ,\pi_{t_{(k_t)},t},D_{t_{(k_t)}}{}^{(\Delta_C)},D_t{}^{(1)}\end{array}\right)$$

A3.5.2 Discounted Future Cash Flows Case

A3.5.2.1 Appendix to Section 4.2.1.2.4.2: Information Frequency = Cash Flow Frequency

A3.5.2.1.1 Formulation of the Problem

The problem is to show that the prices of “expectation risk” and “combined risk” read as follows:

Price of “Expectation Risk”

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$$E\left(q_{t,t+\tau}{}^{iI}\cdot\Delta^{exp.risk}\middle|\pi_t,D_t\right)$$
$$=E\left(q_{t,t+\tau}{}^{iI}\middle|\pi_t,D_t\right)$$
$$\cdot\left\{E^{rn}\left(E\left(\widehat{D}_{t+\tau}\middle|\pi_t,D_t,S_{t+\tau-1},D_{t+\tau-1}\right)\middle|\pi_t,D_t\right)\right.$$
$$\left.-E\left(E\left(\widehat{D}_{t+\tau}\middle|\pi_t,D_t,S_{t+\tau-1},D_{t+\tau-1}\right)\middle|\pi_t,D_t\right)\right\}$$

with the part of cash flows explained by aggregate cash flows abbreviated to

$$\widehat{D}_{t+\tau}\equiv E\left(D_{t+\tau}\middle|\pi_t,D_t,D_{t+\tau}{}^{market}\right)$$

with

$$q_{t,t+\tau}{}^{iI}=\frac{1}{(1+\rho)^{\tau}}\cdot\frac{U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}(D_{t+\tau})\right)}{U'\left(\frac{1}{n_I}\cdot D_t{}^{market}(D_t)\right)}$$

where E^{rn} denotes the expectation taken with respect to the risk-neutralized probability (or density) of $S_{t+\tau-1},D_{t+\tau-1}$ conditional on π_t,D_t, i.e.,

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$$\varphi^{rn,iI}(S_{t+\tau-1},D_{t+\tau-1}|\pi_t,D_t)\equiv\varphi^{iI}(S_{t+\tau-1},D_{t+\tau-1}|\pi_t,D_t)\cdot\frac{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}\right)\middle|S_{t+\tau-1},D_{t+\tau-1}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}\right)\middle|\pi_t,D_t\right)}$$

where $\varphi^{iI}(S_{t+\tau-1},D_{t+\tau-1}|\pi_t,D_t)$ denotes the joint probability (or density) of the random variables $S_{t+\tau-1},D_{t+\tau-1}$ conditional on information relevant to pricing, $z_t{}^{iI,p}=(\pi_t,D_t)$.

Price of “Combined Risk”

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$$E\left(q_{t,t+\tau}{}^{iI} \cdot \Delta^{comb.risk} \middle| \pi_t, D_t\right)$$
$$= E\left(q_{t,t+\tau}{}^{iI} \middle| \pi_t, D_t\right)$$
$$\cdot E^{rn}\left(E\left(\frac{U'\left(\frac{1}{n_I} \cdot D_{t+\tau}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I} \cdot D_{t+\tau}{}^{market}\right) \middle| S_{t+\tau-1}, D_{t+\tau-1}\right)} \cdot \Delta^{comb.risk} \middle| S_{t+\tau-1}, D_{t+\tau-1}, \pi_t, D_t \right) \middle| \pi_t, D_t \right)$$

A3.5.2.1.2 Solution

A3.5.2.1.2.1 Price of "Expectation Risk"

Using the definition of $\Delta^{exp.risk}$ in (4-73), the price of "expectation risk" reads

$$E\left(q_{t,t+\tau}{}^{iI} \cdot \Delta^{exp.risk} \middle| \pi_t, D_t\right) = E\left(q_{t,t+\tau}{}^{iI} \cdot \left\{E\left(\widehat{D}_{t+\tau} \middle| \pi_t, D_t, S_{t+\tau-1}, D_{t+\tau-1}\right) - E(D_{t+\tau}|\pi_t, D_t)\right\} \middle| \pi_t, D_t\right)$$

with

$$\widehat{D}_{t+\tau} \equiv E\left(D_{t+\tau} \middle| \pi_t, D_t, D_{t+\tau}{}^{market}\right)$$

The computation of the price of "expectation risk" is developed as follows.

Step 1: Factor out the multi-period riskless discount factor $E\left(q_{t,t+\tau}{}^{iI} \middle| \pi_t, D_t\right)$ in order to separate riskless discounting from the pricing of risk

$$E\left(q_{t,t+\tau}{}^{iI} \cdot \left\{E\left(\widehat{D}_{t+\tau} \middle| \pi_t, D_t, S_{t+\tau-1}, D_{t+\tau-1}\right) - E(D_{t+\tau}|\pi_t, D_t)\right\} \middle| \pi_t, D_t\right)$$
$$= E\left(q_{t,t+\tau}{}^{iI} \cdot E\left(\widehat{D}_{t+\tau} \middle| \pi_t, D_t, S_{t+\tau-1}, D_{t+\tau-1}\right) \middle| \pi_t, D_t\right) - E\left(q_{t,t+\tau}{}^{iI} \middle| \pi_t, D_t\right)$$
$$\cdot E(D_{t+\tau}|\pi_t, D_t)$$
$$= E\left(q_{t,t+\tau}{}^{iI} \middle| \pi_t, D_t\right)$$
$$\cdot \left\{ E\left(\frac{q_{t,t+\tau}{}^{iI}}{E\left(q_{t,t+\tau}{}^{iI} \middle| \pi_t, D_t\right)} \cdot E\left(\widehat{D}_{t+\tau} \middle| \pi_t, D_t, S_{t+\tau-1}, D_{t+\tau-1}\right) \middle| \pi_t, D_t \right) - E(D_{t+\tau}|\pi_t, D_t) \right\}$$

Step 2: Analyze and simplify the first-term in brackets

$$E\left(\frac{q_{t,t+\tau}{}^{iI}}{E\left(q_{t,t+\tau}{}^{iI} \middle| \pi_t, D_t\right)} \cdot E\left(\widehat{D}_{t+\tau} \middle| \pi_t, D_t, S_{t+\tau-1}, D_{t+\tau-1}\right) \middle| \pi_t, D_t \right)$$

Note that $\frac{q_{t,t+\tau}{}^{iI}}{E\left(q_{t,t+\tau}{}^{iI} | \pi_t, D_t\right)}$ is a multi-period adjustment for risk:

$$\frac{q_{t,t+\tau}{}^{iI}}{E\left(q_{t,t+\tau}{}^{iI} \middle| \pi_t, D_t\right)} = \frac{U'\left(\frac{1}{n_I} \cdot D_{t+\tau}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I} \cdot D_{t+\tau}{}^{market}\right) \middle| \pi_t, D_t\right)}$$

The first term in brackets obtained at the end of Step 1 therefore reads:

$$E\left(\frac{q_{t,t+\tau}{}^{iI}}{E\left(q_{t,t+\tau}{}^{iI}\middle|\pi_t, D_t\right)}\cdot E\left(\widehat{D}_{t+\tau}\middle|\pi_t, D_t, S_{t+\tau-1}, D_{t+\tau-1}\right)\middle|\pi_t, D_t\right)$$
$$= E\left(\frac{U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|\pi_t, D_t\right)}\cdot E\left(\widehat{D}_{t+\tau}\middle|\pi_t, D_t, S_{t+\tau-1}, D_{t+\tau-1}\right)\middle|\pi_t, D_t\right)$$

<u>Step 3</u>: Use the tower property of conditional expectations in order to obtain an adjustment for the risk in $S_{t+\tau-1}, D_{t+\tau-1}$ only (thus eliminating factors and residuals $fe_{t+\tau}$ which do not affect "expectation risk")

$$E\left(\frac{U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|\pi_t, D_t\right)}\cdot E\left(\widehat{D}_{t+\tau}\middle|\pi_t, D_t, S_{t+\tau-1}, D_{t+\tau-1}\right)\middle|\pi_t, D_t\right)$$
$$= E\left(E\left(\begin{array}{c}\frac{U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|\pi_t, D_t\right)}\\ \cdot E\left(\widehat{D}_{t+\tau}\middle|\pi_t, D_t, S_{t+\tau-1}, D_{t+\tau-1}\right)\end{array}\middle|\pi_t, D_t, S_{t+\tau-1}, D_{t+\tau-1}\right)\middle|\pi_t, D_t\right)$$
$$= E\left(\begin{array}{c}\frac{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|S_{t+\tau-1}, D_{t+\tau-1}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|\pi_t, D_t\right)}\\ \cdot E\left(\widehat{D}_{t+\tau}\middle|\pi_t, D_t, S_{t+\tau-1}, D_{t+\tau-1}\right)\end{array}\middle|\pi_t, D_t\right)$$

Note that

$$\frac{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|S_{t+\tau-1}, D_{t+\tau-1}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|\pi_t, D_t\right)}$$

is an adjustment for the risk that deals with $S_{t+\tau-1}, D_{t+\tau-1}$ and does not depend on factors and residuals $fe_{t+\tau}$ since factors and residual are averaged out in the numerator. Intuitively, this risk is irrelevant to the pricing of expectation risk because $\Delta^{exp.risk}$ is not affected by $fe_{t+\tau}$.

<u>Step 4</u>: Introduce the risk-neutralized probability (density) of $S_{t+\tau-1}, D_{t+\tau-1}$

If the probability (density) of $S_{t+\tau-1}, D_{t+\tau-1}$ is made explicit, the conditional expectation

$$E\left(\begin{array}{c}\frac{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|S_{t+\tau-1}, D_{t+\tau-1}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|\pi_t, D_t\right)}\\ \cdot E\left(\widehat{D}_{t+\tau}\middle|\pi_t, D_t, S_{t+\tau-1}, D_{t+\tau-1}\right)\end{array}\middle|\pi_t, D_t\right)$$

takes the following form:

$$\int \varphi^{iI}(S_{t+\tau-1}, D_{t+\tau-1}|\pi_t, D_t) \cdot \frac{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|S_{t+\tau-1}, D_{t+\tau-1}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|\pi_t, D_t\right)} \cdot E\left(\widehat{D}_{t+\tau}\middle|\pi_t, D_t, S_{t+\tau-1}, D_{t+\tau-1}\right)$$

From an economic point of view, the product of the empirical probability (density) and the adjustment for the risk of $S_{t+\tau-1}, D_{t+\tau-1}$ is a risk-neutralized probability (density) of $S_{t+\tau-1}, D_{t+\tau-1}$ and is denoted by

$$\varphi^{rn,iI}(S_{t+\tau-1}, D_{t+\tau-1}|\pi_t, D_t) \equiv \varphi^{iI}(S_{t+\tau-1}, D_{t+\tau-1}|\pi_t, D_t) \cdot \frac{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|S_{t+\tau-1}, D_{t+\tau-1}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|\pi_t, D_t\right)}$$

Thus, if E^{rn} is the expectation with respect to $\varphi^{rn,iI}$, it is obtained

$$E\left(\begin{array}{c}\frac{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|S_{t+\tau-1}, D_{t+\tau-1}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|\pi_t, D_t\right)} \\ \cdot E\left(E\left(D_{t+\tau}\middle|\pi_t, D_t, D_{t+\tau}{}^{market}\right)\middle|\pi_t, D_t, S_{t+\tau-1}, D_{t+\tau-1}\right)\end{array}\middle|\pi_t, D_t\right) = E^{rn}\left(E\left(\widehat{D}_{t+\tau}\middle|\pi_t, D_t, S_{t+\tau-1}, D_{t+\tau-1}\right)\middle|\pi_t, D_t\right)$$

By putting the Steps 1 and 4 together, the price of "expectation risk" is identified as

$$E\left(q_{t,t+\tau}{}^{iI}\cdot \Delta^{exp.risk}\middle|\pi_t, D_t\right) = E\left(q_{t,t+\tau}{}^{iI}\middle|\pi_t, D_t\right)\cdot\left\{E^{rn}\left(E\left(\widehat{D}_{t+\tau}\middle|\pi_t, D_t, S_{t+\tau-1}, D_{t+\tau-1}\right)\middle|\pi_t, D_t\right) - E(D_{t+\tau}|\pi_t, D_t)\right\}$$

<u>Step 5</u>: Transforming $E(D_{t+\tau}|\pi_t, D_t)$

By the tower property of conditional expectations, the relation between both terms in the difference in brackets can be made clear:

$$E(D_{t+\tau}|\pi_t, D_t) = E\left(E\left(D_{t+\tau}\middle|\pi_t, D_t, D_{t+\tau}{}^{market}\right)\middle|\pi_t, D_t\right) \equiv E\left(\widehat{D}_{t+\tau}\middle|\pi_t, D_t\right)$$

and

$$E\left(\widehat{D}_{t+\tau}\middle|\pi_t, D_t\right) = E\left(E\left(\widehat{D}_{t+\tau}\middle|\pi_t, D_t, S_{t+\tau-1}, D_{t+\tau-1}\right)\middle|\pi_t, D_t\right)$$

i.e., a difference of expectations of the random variable $E\left(\widehat{D}_{t+\tau}\middle|\pi_t, D_t, S_{t+\tau-1}, D_{t+\tau-1}\right)$ with respect to a risk-neutralized and empirical probability measure is taken:

$$\begin{aligned} E\left(q_{t,t+\tau}{}^{iI}\cdot \Delta^{exp.risk}\middle|\pi_t, D_t\right) &= E\left(q_{t,t+\tau}{}^{iI}\middle|\pi_t, D_t\right)\cdot\left\{E^{rn}\left(E\left(\widehat{D}_{t+\tau}\middle|\pi_t, D_t, S_{t+\tau-1}, D_{t+\tau-1}\right)\middle|\pi_t, D_t\right) - E(D_{t+\tau}|\pi_t, D_t)\right\} \\ &= E\left(q_{t,t+\tau}{}^{iI}\middle|\pi_t, D_t\right) \\ &\quad \cdot\left\{E^{rn}\left(E\left(\widehat{D}_{t+\tau}\middle|\pi_t, D_t, S_{t+\tau-1}, D_{t+\tau-1}\right)\middle|\pi_t, D_t\right)\right. \\ &\quad \left. - E\left(E\left(\widehat{D}_{t+\tau}\middle|\pi_t, D_t, S_{t+\tau-1}, D_{t+\tau-1}\right)\middle|\pi_t, D_t\right)\right\} \end{aligned}$$

Step 6: Final result

Combining the results of Steps 4 and 5 yields (4-76).

A3.5.2.1.2.2 Price of "Combined Risk"

Using the definition of $\Delta^{comb.risk}$ in (4-73), the price of "expectation risk" reads

$$E\left(q_{t,t+\tau}{}^{iI}\cdot\Delta^{comb.risk}\middle|\pi_t,D_t\right)=E\left(q_{t,t+\tau}{}^{iI}\cdot\left\{\widehat{D}_{t+\tau}-E\left(\widehat{D}_{t+\tau}\middle|\pi_t,D_t,S_{t+\tau-1},D_{t+\tau-1}\right)\right\}\middle|\pi_t,D_t\right)$$

with

$$\widehat{D}_{t+\tau}\equiv E\left(D_{t+\tau}\middle|\pi_t,D_t,D_{t+\tau}{}^{market}\right)$$

Step 1: Use the tower property of conditional expectations to separate the pricing of risk stemming form $S_{t+\tau-1},D_{t+\tau-1}$ from the pricing of risk stemming from $fe_{t+\tau}$ given $S_{t+\tau-1},D_{t+\tau-1}$

$$E\left(q_{t,t+\tau}{}^{iI}\cdot\left\{\widehat{D}_{t+\tau}-E\left(\widehat{D}_{t+\tau}\middle|\pi_t,D_t,S_{t+\tau-1},D_{t+\tau-1}\right)\right\}\middle|\pi_t,D_t\right)$$
$$=E\left(E\left(q_{t,t+\tau}{}^{iI}\cdot\begin{Bmatrix}\widehat{D}_{t+\tau}\\-E\left(\widehat{D}_{t+\tau}\middle|\pi_t,D_t,S_{t+\tau-1},D_{t+\tau-1}\right)\end{Bmatrix}\middle|\pi_t,D_t,S_{t+\tau-1},D_{t+\tau-1}\right)\middle|\pi_t,D_t\right)$$

The result of this step consists of the two summands

$$E\left(E\left(q_{t,t+\tau}{}^{iI}\cdot E\left(D_{t+\tau}\middle|\pi_t,D_t,D_{t+\tau}{}^{market}\right)\middle|\pi_t,D_t,S_{t+\tau-1},D_{t+\tau-1}\right)\middle|\pi_t,D_t\right)$$

and

$$-E\left(E\left(q_{t,t+\tau}{}^{iI}\cdot E\left(\widehat{D}_{t+\tau}\middle|\pi_t,D_t,S_{t+\tau-1},D_{t+\tau-1}\right)\middle|\pi_t,D_t,S_{t+\tau-1},D_{t+\tau-1}\right)\middle|\pi_t,D_t\right)$$
$$=-E\left(E\left(q_{t,t+\tau}{}^{iI}\middle|\pi_t,D_t,S_{t+\tau-1},D_{t+\tau-1}\right)\cdot E\left(\widehat{D}_{t+\tau}\middle|\pi_t,D_t,S_{t+\tau-1},D_{t+\tau-1}\right)\middle|\pi_t,D_t\right)$$

Step 2: Factor out the multi-period riskless discount factor $E\left(q_{t,t+\tau}{}^{iI}\middle|\pi_t,D_t\right)$ in order to separate riskless discounting from the pricing of risk

$$E\left(E\left(q_{t,t+\tau}{}^{iI}\cdot\widehat{D}_{t+\tau}\middle|\pi_t,D_t,S_{t+\tau-1},D_{t+\tau-1}\right)\middle|\pi_t,D_t\right)$$
$$-E\left(E\left(q_{t,t+\tau}{}^{iI}\middle|\pi_t,D_t,S_{t+\tau-1},D_{t+\tau-1}\right)\cdot E\left(\widehat{D}_{t+\tau}\middle|\pi_t,D_t,S_{t+\tau-1},D_{t+\tau-1}\right)\middle|\pi_t,D_t\right)$$
$$=E\left(q_{t,t+\tau}{}^{iI}\middle|\pi_t,D_t\right)$$
$$\cdot\left\{E\left(E\left(\frac{q_{t,t+\tau}{}^{iI}}{E\left(q_{t,t+\tau}{}^{iI}\middle|\pi_t,D_t\right)}\cdot\widehat{D}_{t+\tau}\middle|\pi_t,D_t,S_{t+\tau-1},D_{t+\tau-1}\right)\middle|\pi_t,D_t\right)\right.$$
$$\left.-E\left(\frac{E\left(q_{t,t+\tau}{}^{iI}\middle|\pi_t,D_t,S_{t+\tau-1},D_{t+\tau-1}\right)}{E\left(q_{t,t+\tau}{}^{iI}\middle|\pi_t,D_t\right)}\cdot E\left(\widehat{D}_{t+\tau}\middle|\pi_t,D_t,S_{t+\tau-1},D_{t+\tau-1}\right)\middle|\pi_t,D_t\right)\right\}$$

Note that the following term is the adjustment risk:

$$\frac{q_{t,t+\tau}{}^{iI}}{E\left(q_{t,t+\tau}{}^{iI}\middle|\pi_t,D_t\right)}=\frac{U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|\pi_t,D_t\right)}$$

The following term, consequently, can be interpreted as adjustment for the risk that deals with $S_{t+\tau-1}, D_{t+\tau-1}$ and does not depend on factors and residuals $fe_{t+\tau}$:

$$\frac{E\left(q_{t,t+\tau}{}^{iI}\middle|\pi_t, D_t, S_{t+\tau-1}, D_{t+\tau-1}\right)}{E\left(q_{t,t+\tau}{}^{iI}\middle|\pi_t, D_t\right)} = \frac{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|S_{t+\tau-1}, D_{t+\tau-1}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|\pi_t, D_t\right)}$$

<u>Step 3</u>: Factor out the adjustment for risk that deals with $S_{t+\tau-1}, D_{t+\tau-1}$ from the first element of the sum

$$\begin{aligned}
&E\left(q_{t,t+\tau}{}^{iI}\middle|\pi_t, D_t\right)\\
&\cdot\left\{E\left(E\left(\frac{q_{t,t+\tau}{}^{iI}}{E\left(q_{t,t+\tau}{}^{iI}\middle|\pi_t, D_t\right)}\cdot\widehat{D}_{t+\tau}\middle|\pi_t, D_t, S_{t+\tau-1}, D_{t+\tau-1}\right)\middle|\pi_t, D_t\right)\right.\\
&\left.-E\left(\frac{E\left(q_{t,t+\tau}{}^{iI}\middle|\pi_t, D_t, S_{t+\tau-1}, D_{t+\tau-1}\right)}{E\left(q_{t,t+\tau}{}^{iI}\middle|\pi_t, D_t\right)}\cdot E\left(\widehat{D}_{t+\tau}\middle|\pi_t, D_t, S_{t+\tau-1}, D_{t+\tau-1}\right)\middle|\pi_t, D_t\right)\right\}\\
&= E\left(q_{t,t+\tau}{}^{iI}\middle|\pi_t, D_t\right)\\
&\cdot\left\{E\left(\begin{array}{c}\dfrac{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|S_{t+\tau-1}, D_{t+\tau-1}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|\pi_t, D_t\right)}\\ \cdot E\left(\dfrac{U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|S_{t+\tau-1}, D_{t+\tau-1}\right)}\cdot\widehat{D}_{t+\tau}\middle|\pi_t, D_t, S_{t+\tau-1}, D_{t+\tau-1}\right)\end{array}\middle|\pi_t, D_t\right)\right.\\
&\left.-E\left(\frac{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|S_{t+\tau-1}, D_{t+\tau-1}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|\pi_t, D_t\right)}\cdot E\left(\widehat{D}_{t+\tau}\middle|\pi_t, D_t, S_{t+\tau-1}, D_{t+\tau-1}\right)\middle|\pi_t, D_t\right)\right\}
\end{aligned}$$

<u>Step 4</u>: Introduce the risk-neutralized probability (density) of $S_{t+\tau-1}, D_{t+\tau-1}$

Define the risk-neutralized probability (density) of $S_{t+\tau-1}, D_{t+\tau-1}$ by

$$\begin{aligned}
&\varphi^{rn,iI}(S_{t+\tau-1}, D_{t+\tau-1}|\pi_t, D_t)\\
&\qquad\equiv \varphi^{iI}(S_{t+\tau-1}, D_{t+\tau-1}|\pi_t, D_t)\cdot\frac{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|S_{t+\tau-1}, D_{t+\tau-1}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|\pi_t, D_t\right)}
\end{aligned}$$

By the same argument as in the case of expectation risk, the result from the previous Step 3 can be expressed through expectations with respect to $\varphi^{rn,iI}$. This yields

$$E\left(q_{t,t+\tau}{}^{iI}\cdot\Delta^{comb.risk}\middle|\pi_t,D_t\right)$$
$$=E\left(q_{t,t+\tau}{}^{iI}\middle|\pi_t,D_t\right)$$
$$\cdot\left\{E^{rn}\left(E\left(\frac{U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|S_{t+\tau-1},D_{t+\tau-1}\right)}\cdot\widehat{D}_{t+\tau}\middle|\pi_t,D_t,S_{t+\tau-1},D_{t+\tau-1}\right)\middle|\pi_t,D_t\right)\right.$$
$$\left.-E^{rn}\left(E\left(\widehat{D}_{t+\tau}\middle|\pi_t,D_t,S_{t+\tau-1},D_{t+\tau-1}\right)\middle|\pi_t,D_t\right)\right\}$$
$$=E\left(q_{t,t+\tau}{}^{iI}\middle|\pi_t,D_t\right)$$
$$\cdot\left\{E^{rn}\left(\begin{matrix}E\left(\frac{U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|S_{t+\tau-1},D_{t+\tau-1}\right)}\cdot\widehat{D}_{t+\tau}\middle|\pi_t,D_t,S_{t+\tau-1},D_{t+\tau-1}\right)\\-E\left(\widehat{D}_{t+\tau}\middle|\pi_t,D_t,S_{t+\tau-1},D_{t+\tau-1}\right)\end{matrix}\middle|\pi_t,D_t\right)\right\}$$

<u>Step 5</u>: Re-express the inner expectation from the result of the previous step through $\Delta^{comb.risk,(v)}$

Since the adjustment for risk $\frac{U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|S_{t+\tau-1},D_{t+\tau-1}\right)}$ has an expectation of one conditional on $\pi_t,D_t,S_{t+\tau-1},D_{t+\tau-1}$, the following equations hold and yield the desired result:

$$E\left(\frac{U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|S_{t+\tau-1},D_{t+\tau-1}\right)}\cdot\Delta^{comb.risk}\middle|\pi_t,D_t,S_{t+\tau-1},D_{t+\tau-1}\right)$$
$$=E\left(\begin{matrix}\frac{U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|S_{t+\tau-1},D_{t+\tau-1}\right)}\\\cdot\left\{\widehat{D}_{t+\tau}-E\left(\widehat{D}_{t+\tau}\middle|\pi_t,D_t,S_{t+\tau-1},D_{t+\tau-1}\right)\right\}\end{matrix}\middle|\pi_t,D_t,S_{t+\tau-1},D_{t+\tau-1}\right)$$
$$=E\left(\frac{U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|S_{t+\tau-1},D_{t+\tau-1}\right)}\cdot\widehat{D}_{t+\tau}\middle|\pi_t,D_t,S_{t+\tau-1},D_{t+\tau-1}\right)$$
$$-E\left(\widehat{D}_{t+\tau}\middle|\pi_t,D_t,S_{t+\tau-1},D_{t+\tau-1}\right)$$

Hence the result from Step 4 can likewise be written

$$E\left(q_{t,t+\tau}{}^{iI}\cdot\Delta^{comb.risk}\middle|\pi_t,D_t\right)$$
$$=E\left(q_{t,t+\tau}{}^{iI}\middle|\pi_t,D_t\right)$$
$$\cdot\left\{E^{rn}\left(E\left(\frac{U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t+\tau}{}^{market}\right)\middle|S_{t+\tau-1},D_{t+\tau-1}\right)}\cdot\Delta^{comb.risk}\middle|\pi_t,D_t,S_{t+\tau-1},D_{t+\tau-1}\right)\middle|\pi_t,D_t\right)\right\}$$

as was to be shown.

A3.5.2.2 Appendix to Section 4.2.2.3.4: Information Frequency ≥ Cash Flow Frequency

A3.5.2.2.1 Formulation of the Problem

Definition and price of "expectation risk" and "combined risk" for the case where information frequency is higher than or equal to cash flow frequency have been omitted in the main text because they are almost entirely analogous to the special case where information frequency equals cash flow frequency. For the sake of completeness, "expectation risk" and "combined risk" are defined and priced in this section of the appendix.

A3.5.2.2.1.1 Definition of "expectation risk" and "combined risk" in the case where information frequency is higher than or equal to cash flow frequency

In order to define "expectation risk" and "combined risk" in generalization of (4-73) for cash flows at some time $t+\tau$, two cases must be distinguished: time $t+\tau$ is a payment date of (Δ_C)-periodic cash flows (Case 1) and time $t+\tau$ is not a payment date of (Δ_C)-periodic cash flows (Case 2).
The reason why it is necessary to distinguish between both cases in the discounted cash flow case (opposed to the quasi-static case) is as follows. Cash flows are only priced insofar as they can be explained through ${D_{t+\tau}}^{market}$, but the composition of ${D_{t+\tau}}^{market}$ differs significantly in Cases 1 and 2. Obviously, ${D_{t+\tau}}^{market}$ must include (Δ_C)-periodic and (1)-periodic cash flows in Case 1 but only consists of (1)-periodic cash flows in Case 2.

<u>Case 1</u>: $t+\tau$ is a payment date of (Δ_C)-periodic cash flows, i.e.,

$$t+\tau = t_{(k')} = t_{(k_{t+\tau})}$$

for some natural number k'

For brevity, the symbols ${z_t}^{iI,(\Delta_C),p}$ is used instead of the more detailed $\pi_{t_{(k_t)},t}, {D_{t_{(k_t)}}}^{(\Delta_C)}, {D_t}^{(1)}$ and ${z_{t_{(k'-1)}}}^{cI,(\Delta_C),p}$ is short for $S_{t_{(k'-1)},t_{(k')}-1}, {D_{t_{(k'-1)}}}^{(\Delta_C)}, {D_{t_{(k')}-1-1}}^{(1)}$. In this notation, "expectation risk" and "combined risk" are defined as

A 3-35

$$\Delta^{exp.risk,(v)} = E\left(\widehat{D}_{t_{(k')}}{}^{(v)}\middle| \begin{matrix} z_t{}^{iI,(\Delta_C),p} \\ , z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p} \end{matrix}\right) - E\left(D_{t_{(k')}}{}^{(v)}\middle| z_t{}^{iI,(\Delta_C),p}\right)$$

$$v\epsilon\{1,\Delta_C\}$$

A 3-36

$$\Delta^{comb.risk,(v)} = \widehat{D}_{t_{(k')}}{}^{(v)} - E\left(\widehat{D}_{t_{(k')}}{}^{(v)}\middle| \begin{matrix} z_t{}^{iI,(\Delta_C),p} \\ , z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p} \end{matrix}\right)$$

with the part of cash flows explained by aggregate cash flows abbreviated to

$$\widehat{D}_{t_{(k')}}{}^{(v)} \equiv E\left(D_{t_{(k')}}{}^{(v)}\middle| z_t{}^{iI,(\Delta_C),p}, D_{t+\tau}{}^{market}\right)$$

with

$$v\epsilon\{1,\Delta_C\}$$

The three random elements $S_{t+\tau-1}, D_{t+\tau-1}$ and $fe_{t+\tau}$ that summarize all relevant aspects of the sources of risk are now replaced by the path of regimes $S_{t_{(k'-1)},t_{(k')}-1}$ and by the most recent cash flows (from the point of view of $t+\tau-1$) of both types of assets, $D_{t_{(k'-1)}}{}^{(\Delta_C)}, D_{t_{(k')}-1-1}{}^{(1)}$.

<u>Case 2</u>: $t+\tau$ is not a payment date of (Δ_C)-periodic cash flows, i.e.,

$$t_{(k')} < t+\tau < t_{(k'+1)}$$

In this case, there can by definition only be (1)-periodic cash flows at time $t+\tau$. “expectation risk” and “combined risk” are then defined exactly as in the special case where information frequency is equal to cash flow frequency:

A 3-37

$$\Delta^{exp.risk,(1)} = E\left(E\left(D_{t+\tau}{}^{(1)}\middle|\pi_t, D_t{}^{(1)}, D_{t+\tau}{}^{market}\right)\middle| \begin{matrix} \pi_t, D_t{}^{(1)} \\ , S_{t+\tau-1}, D_{t+\tau-1}{}^{(1)} \end{matrix}\right) - E\left(D_{t+\tau}{}^{(1)}\middle|\pi_t, D_t{}^{(1)}\right)$$

A 3-38

$$\Delta^{comb.risk,(1)} = E\left(D_{t+\tau}{}^{(1)}\middle|\pi_t, D_t{}^{(1)}, D_{t+\tau}{}^{market}\right)$$
$$- E\left(E\left(D_{t+\tau}{}^{(v)}\middle|\pi_t, D_t{}^{(1)}, D_{t+\tau}{}^{market}\right)\middle| \begin{matrix} \pi_t, D_t{}^{(1)} \\ , S_{t+\tau-1}, D_{t+\tau-1}{}^{(1)} \end{matrix}\right)$$

(observe that aggregate cash flows $D_{t+\tau}{}^{market}$ are the aggregate of (1)-periodic cash flows only)

A3.5.2.2.1.2 Price of "expectation risk" and "combined risk"

Case 1: $t+\tau$ is a payment date of (Δ_C)-periodic cash flows, i.e., $t+\tau = t_{(k')} = t_{(k_{t+\tau})}$

Price of "Expectation Risk"

A 3-39

$$E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)} \cdot \Delta^{exp.risk,(\nu)} \middle| z_t{}^{iI,(\Delta_C),p}\right)$$

$$= E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)} \middle| z_t{}^{iI,(\Delta_C),p}\right)$$

$$\cdot \left\{E^{rn,(\Delta_C)}\left(E\left(\widehat{D}_{t_{(k')}}{}^{(\nu)} \middle| z_t{}^{iI,(\Delta_C),p}, z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right) \middle| z_t{}^{iI,(\Delta_C),p}\right)\right.$$

$$\left. - E\left(E\left(\widehat{D}_{t_{(k')}}{}^{(\nu)} \middle| z_t{}^{iI,(\Delta_C),p}, z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right) \middle| z_t{}^{iI,(\Delta_C),p}\right)\right\}$$

$$\nu \epsilon \{1, \Delta_C\}$$

with the part of cash flows explained by aggregate cash flows abbreviated to

$$\widehat{D}_{t_{(k')}}{}^{(\nu)} \equiv E\left(D_{t_{(k')}}{}^{(\nu)} \middle| z_t{}^{iI,(\Delta_C),p}, D_{t+\tau}{}^{market}\right)$$

with

$$q_{t,t_{(k')}}{}^{iI,(\Delta_C)}\left(fe_{t+1,t_{(k')}}, S_{t_{(k)},t_{(k')}-1}, D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}\right)$$

$$= \frac{1}{(1+\rho)^{\tau}} \cdot \frac{U'\left(\frac{1}{n_I} \cdot D_{t_{(k')}}{}^{market}\left(D_{t_{(k)}}{}^{(\Delta_C)}, D_t{}^{(1)}, fe_{t+1,t_{(k')}}, S_{t_{(k)},t_{(k')}-1}\right)\right)}{U'\left(\frac{1}{n_I} \cdot D_t{}^{market}\right)}$$

where $E^{rn,(\Delta_C)}$ is the expectation with respect to the risk-neutralized probability (or density)

$$\varphi^{rn,iI,(\Delta_C)}\left(z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p} \middle| z_t{}^{iI,(\Delta_C),p}\right)$$

$$= \varphi^{iI,(\Delta_C)}\left(z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p} \middle| z_t{}^{iI,(\Delta_C),p}\right) \cdot \frac{E\left(U'\left(\frac{1}{n_I} \cdot D_{t_{(k')}}{}^{market}\right) \middle| z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)}{E\left(U'\left(\frac{1}{n_I} \cdot D_{t_{(k')}}{}^{market}\right) \middle| z_t{}^{iI,(\Delta_C),p}\right)}$$

Price of "Combined Risk"

A 3-40

$$E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)} \cdot \Delta^{comb.risk,(\nu)} \middle| z_t{}^{iI,(\Delta_C),p}\right)$$

$$= E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)} \middle| z_t{}^{iI,(\Delta_C),p}\right)$$

$$\cdot E^{rn,(\Delta_C)}\left(E\left(\frac{U'\left(\frac{1}{n_I} \cdot D_{t_{(k')}}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I} \cdot D_{t_{(k')}}{}^{market}\right) \middle| z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)} \cdot \Delta^{comb.risk,(\nu)} \middle| \begin{matrix} z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p} \\ , z_t{}^{iI,(\Delta_C),p} \end{matrix}\right) \middle| z_t{}^{iI,(\Delta_C),p}\right)$$

$$\nu \epsilon \{1, \Delta_C\}$$

Case 2: $t+\tau$ is not a payment date of (Δ_C)-periodic cash flows, i.e., $t_{(k')} < t+\tau < t_{(k'+1)}$

Price of "Expectation Risk"

A 3-41

$$E\left(q_{t,t+\tau}{}^{iI,(\Delta_C)} \cdot \Delta^{exp.risk} \middle| \pi_t, D_t{}^{(1)}\right) = E\left(q_{t,t+\tau}{}^{iI,(\Delta_C)} \middle| \pi_t, D_t{}^{(1)}\right) \cdot \left\{E^{rn,(1)}\left(E\left(D_{t+\tau}{}^{(1)} \middle| S_{t+\tau-1}, D_{t+\tau-1}{}^{(1)}\right) \middle| \pi_t, D_t\right) - E\left(E\left(D_{t+\tau}{}^{(1)} \middle| S_{t+\tau-1}, D_{t+\tau-1}{}^{(1)}\right) \middle| \pi_t, D_t{}^{(1)}\right)\right\}$$

with

$$q_{t,t+\tau}{}^{iI,(\Delta_C)} = \frac{1}{(1+\rho)^\tau} \cdot \frac{U'\left(\frac{1}{n_I} \cdot D_{t+\tau}{}^{market}\left(D_{t+\tau}{}^{(1)}\right)\right)}{U'\left(\frac{1}{n_I} \cdot D_t{}^{market}\left(D_t{}^{(1)}\right)\right)}$$

where $E^{rn,(1)}$ denotes the expectation taken with respect to the risk-neutralized probability (or density) of $S_{t+\tau-1}, D_{t+\tau-1}{}^{(1)}$ conditional on $\pi_t, D_t{}^{(1)}$, i.e.,

$$\varphi^{rn,iI,(1)}\left(S_{t+\tau-1}, D_{t+\tau-1}{}^{(1)} \middle| \pi_t, D_t{}^{(1)}\right) \equiv \varphi^{iI,(\Delta_C)}\left(S_{t+\tau-1}, D_{t+\tau-1}{}^{(1)} \middle| \pi_t, D_t\right) \cdot \frac{E\left(U'\left(\frac{1}{n_I} \cdot D_{t+\tau}{}^{market}\left(D_{t+\tau-1}{}^{(1)}\right)\right) \middle| S_{t+\tau-1}, D_{t+\tau-1}{}^{(1)}\right)}{E\left(U'\left(\frac{1}{n_I} \cdot D_{t+\tau}{}^{market}\left(D_{t+\tau-1}{}^{(1)}\right)\right) \middle| \pi_t, D_t{}^{(1)}\right)}$$

Price of "Combined Risk"

A 3-42

$$E\left(q_{t,t+\tau}{}^{iI,(\Delta_C)} \cdot \Delta^{comb.risk,(1)} \middle| \pi_t, D_t{}^{(1)}\right) = E\left(q_{t,t+\tau}{}^{iI} \middle| \pi_t, D_t{}^{(1)}\right) \cdot E^{rn,(1)}\left(E\left(\frac{U'\left(\frac{1}{n_I} \cdot D_{t+\tau}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I} \cdot D_{t+\tau}{}^{market}\right) \middle| S_{t+\tau-1}, D_{t+\tau-1}{}^{(1)}\right)} \cdot \Delta^{comb.risk} \middle| S_{t+\tau-1}, D_{t+\tau-1}{}^{(1)}\right) \middle| \pi_t, D_t{}^{(1)}\right)$$

A3.5.2.2.2 Solution

It suffices to consider case 1 ($t+\tau = t_{(k')}$) since case 2 corresponds exactly to the special case with information frequency equal to cash flow frequency.

A3.5.2.2.2.1.1 Price of "Expectation Risk"

Using the definition of $\Delta^{exp.risk,(\nu)}$ in (A 3-35), the price of "expectation risk" reads

$$E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)} \cdot \Delta^{exp.risk,(\nu)} \middle| z_t{}^{iI,(\Delta_C),p}\right)$$
$$= E\left(\begin{array}{c} q_{t,t_{(k')}}{}^{iI,(\Delta_C)} \\ \cdot \left\{ \begin{array}{c} E\left(\hat{D}_{t_{(k')}}{}^{(\nu)} \middle| z_t{}^{iI,(\Delta_C),p}, z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right) \\ -E\left(D_{t_{(k')}}{}^{(\nu)} \middle| z_t{}^{iI,(\Delta_C),p}\right) \end{array} \right\} \end{array} \middle| z_t{}^{iI,(\Delta_C),p} \right)$$

with

$$\hat{D}_{t_{(k')}}{}^{(\nu)} \equiv E\left(D_{t_{(k')}}{}^{(\nu)} \middle| z_t{}^{iI,(\Delta_C),p}, D_{t_{(k')}}{}^{market}\right)$$

<u>Step 1</u>: Factor out the multi-period riskless discount factor $E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)} \middle| z_t{}^{cI,(\Delta_C),p}\right)$ in order to separate riskless discounting from the pricing of risk

$$E\left(\begin{array}{c} q_{t,t_{(k')}}{}^{iI,(\Delta_C)} \\ \cdot \left\{ \begin{array}{c} E\left(\hat{D}_{t_{(k')}}{}^{(\nu)} \middle| z_t{}^{iI,(\Delta_C),p}, z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right) \\ -E\left(D_{t_{(k')}}{}^{(\nu)} \middle| z_t{}^{iI,(\Delta_C),p}\right) \end{array} \right\} \end{array} \middle| z_t{}^{iI,(\Delta_C),p} \right)$$
$$= E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)} \cdot E\left(\hat{D}_{t_{(k')}}{}^{(\nu)} \middle| z_t{}^{iI,(\Delta_C),p}, z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right) \middle| z_t{}^{iI,(\Delta_C),p}\right)$$
$$- E\left(D_{t_{(k')}}{}^{(\nu)} \middle| z_t{}^{iI,(\Delta_C),p}\right) \cdot E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)} \middle| z_t{}^{iI,(\Delta_C),p}\right)$$
$$= E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)} \middle| z_t{}^{iI,(\Delta_C),p}\right)$$
$$\cdot \left\{ E\left(\begin{array}{c} \dfrac{q_{t,t_{(k')}}{}^{iI,(\Delta_C)}}{E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)} \middle| z_t{}^{iI,(\Delta_C),p}\right)} \\ \cdot E\left(\hat{D}_{t_{(k')}}{}^{(\nu)} \middle| z_t{}^{iI,(\Delta_C),p}, z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right) \end{array} \middle| z_t{}^{iI,(\Delta_C),p} \right) \right.$$
$$\left. - E\left(D_{t_{(k')}}{}^{(\nu)} \middle| z_t{}^{iI,(\Delta_C),p}\right) \right\}$$

Step 2: Analyze and simplify the first-term in brackets

$$E\left(\left.\frac{q_{t,t_{(k')}}{}^{iI,(\Delta_C)}}{E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)}\middle|z_t{}^{iI,(\Delta_C),p}\right)\cdot E\left(\widehat{D}_{t_{(k')}}{}^{(\nu)}\middle|z_t{}^{iI,(\Delta_C),p},z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)}\right|z_t{}^{iI,(\Delta_C),p}\right)$$

Note that $\frac{q_{t,t_{(k')}}{}^{iI,(\Delta_C)}}{E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)}\middle|z_t{}^{iI,(\Delta_C),p}\right)}$ is a multi-period adjustment for risk:

$$\frac{q_{t,t_{(k')}}{}^{iI,(\Delta_C)}}{E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)}\middle|z_t{}^{iI,(\Delta_C),p}\right)}=\frac{U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)\middle|z_t{}^{iI,(\Delta_C),p}\right)}$$

The first term in brackets obtained at the end of Step 1 therefore reads:

$$E\left(\left.\frac{q_{t,t_{(k')}}{}^{iI,(\Delta_C)}}{E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)}\middle|z_t{}^{iI,(\Delta_C),p}\right)\cdot E\left(\widehat{D}_{t_{(k')}}{}^{(\nu)}\middle|z_t{}^{iI,(\Delta_C),p},z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)}\right|z_t{}^{iI,(\Delta_C),p}\right)$$

$$=E\left(\left.\frac{U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)\middle|z_t{}^{iI,(\Delta_C),p}\right)\cdot E\left(\widehat{D}_{t_{(k')}}{}^{(\nu)}\middle|z_t{}^{iI,(\Delta_C),p},z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)}\right|z_t{}^{iI,(\Delta_C),p}\right)$$

Step 3: Use the tower property of conditional expectations in order to obtain an adjustment for the risk in $z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}=\left(S_{t_{(k'-1)},t_{(k')}-1},D_{t_{(k'-1)}}{}^{(\Delta_C)},D_{t_{(k')}-1-1}{}^{(1)}\right)$ only (and thus eliminate factors and residuals $fe_{t+\tau}$ which do not affect "expectation risk")

$$E\left(\left.\frac{U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)\middle|z_t{}^{iI,(\Delta_C),p}\right)\cdot E\left(\widehat{D}_{t_{(k')}}{}^{(\nu)}\middle|z_t{}^{iI,(\Delta_C),p},z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)}\right|z_t{}^{iI,(\Delta_C),p}\right)$$

$$=E\left(E\left(\left.\frac{U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)\middle|z_t{}^{iI,(\Delta_C),p}\right)\cdot E\left(\widehat{D}_{t_{(k')}}{}^{(\nu)}\middle|z_t{}^{iI,(\Delta_C),p},z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)}\right|z_t{}^{iI,(\Delta_C),p},z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)\middle|z_t{}^{iI,(\Delta_C),p}\right)$$

$$=E\left(\left.\frac{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)\middle|z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)\middle|z_t{}^{iI,(\Delta_C),p}\right)\cdot E\left(\widehat{D}_{t_{(k')}}{}^{(\nu)}\middle|z_t{}^{iI,(\Delta_C),p},z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)}\right|z_t{}^{iI,(\Delta_C),p}\right)$$

Note that the fraction

$$\frac{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)\Big|z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)\Big|z_t{}^{iI,(\Delta_C),p}\right)}$$

is an adjustment for the risk that deals with

$$z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}=\left(S_{t_{(k'-1)},t_{(k')}-1},D_{t_{(k'-1)}}{}^{(\Delta_C)},D_{t_{(k')}-1-1}{}^{(1)}\right)$$

and does not depend on factors and residuals $fe_{t_{(k')}}$ since factors and residual are averaged out in the numerator. Intuitively, this risk is irrelevant to the pricing of expectation risk because $\Delta^{exp.risk,(v)}$ is not affected by $fe_{t_{(k')}}$.

Step 4: Introduce the risk-neutralized probability (density) of

$$z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}=\left(S_{t_{(k'-1)},t_{(k')}-1},D_{t_{(k'-1)}}{}^{(\Delta_C)},D_{t_{(k')}-1}{}^{(1)}\right)$$

If the probability (density) of $z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}$ is made explicit, the conditional expectation

$$E\left(\left.\begin{array}{c}\frac{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)\Big|z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)\Big|z_t{}^{iI,(\Delta_C),p}\right)}\\ \cdot E\left(\widehat{D}_{t_{(k')}}{}^{(v)}\Big|z_t{}^{iI,(\Delta_C),p},z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)\end{array}\right|z_t{}^{iI,(\Delta_C),p}\right)$$

takes the following form:

$$\int\varphi^{iI,(\Delta_C)}\left(z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\Big|z_t{}^{iI,(\Delta_C),p}\right)\cdot\left\{\begin{array}{c}\frac{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)\Big|z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)\Big|z_t{}^{iI,(\Delta_C),p}\right)}\\ \cdot E\left(\widehat{D}_{t_{(k')}}{}^{(v)}\Big|z_t{}^{iI,(\Delta_C),p},z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)\end{array}\right\}$$

From an economic point of view, the product of the empirical probability (density) and the adjustment for the risk of $z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}$ is a risk-neutralized probability (density) of $z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}$ and is denoted by

$$\varphi^{rn,iI,(\Delta_C)}\left(z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\Big|z_t{}^{iI,(\Delta_C),p}\right)$$
$$\equiv\varphi^{iI,(\Delta_C)}\left(z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\Big|z_t{}^{iI,(\Delta_C),p}\right)\cdot\frac{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)\Big|z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)\Big|z_t{}^{iI,(\Delta_C),p}\right)}$$

Thus, if $E^{rn,(\Delta_C)}$ is the expectation with respect to $\varphi^{rn,iI,(\Delta_C)}$, it is obtained

$$E\left(\left.\frac{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)\middle|z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)\middle|z_t{}^{iI,(\Delta_C),p}\right)} \cdot E\left(\widehat{D}_{t_{(k')}}{}^{(\nu)}\middle|z_t{}^{iI,(\Delta_C),p},z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)\right|z_t{}^{iI,(\Delta_C),p}\right)$$

$$= E^{rn,(\Delta_C)}\left(E\left(\widehat{D}_{t_{(k')}}{}^{(\nu)}\middle|z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)\middle|z_t{}^{iI,(\Delta_C),p}\right)$$

By putting the Steps 1 and 4 together, the price of “expectation risk” is identified as

$$E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)}\cdot\Delta^{exp.risk,(\nu)}\middle|z_t{}^{iI,(\Delta_C),p}\right)$$
$$= E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)}\middle|z_t{}^{iI,(\Delta_C),p}\right)$$
$$\cdot\left\{E^{rn,(\Delta_C)}\left(E\left(\widehat{D}_{t_{(k')}}{}^{(\nu)}\middle|z_t{}^{iI,(\Delta_C),p},z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)\middle|z_t{}^{iI,(\Delta_C),p}\right)\right.$$
$$\left.- E\left(D_{t_{(k')}}{}^{(\nu)}\middle|z_t{}^{iI,(\Delta_C),p}\right)\right\}$$

<u>Step 5</u>: Transforming $E\left(D_{t_{(k')}}{}^{(\nu)}\middle|z_t{}^{iI,(\Delta_C),p}\right)$

By the tower property of conditional expectations, the relation between both terms in the difference in brackets can be made clearer:

$$E\left(D_{t_{(k')}}{}^{(\nu)}\middle|z_t{}^{iI,(\Delta_C),p}\right) = E\left(E\left(D_{t_{(k')}}{}^{(\nu)}\middle|z_t{}^{iI,(\Delta_C),p},D_{t_{(k')}}{}^{market}\right)\middle|z_t{}^{iI,(\Delta_C),p}\right)$$
$$\equiv E\left(\widehat{D}_{t_{(k')}}{}^{(\nu)}\middle|z_t{}^{iI,(\Delta_C),p}\right)$$

and

$$E\left(\widehat{D}_{t_{(k')}}{}^{(\nu)}\middle|z_t{}^{iI,(\Delta_C),p}\right) = E\left(E\left(\widehat{D}_{t_{(k')}}{}^{(\nu)}\middle|z_t{}^{iI,(\Delta_C),p},z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)\middle|z_t{}^{iI,(\Delta_C),p}\right)$$

i.e., a difference of expectations of the random variable $\left(\widehat{D}_{t_{(k')}}{}^{(\nu)}\middle|z_t{}^{iI,(\Delta_C),p},z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)$ with respect to a risk-neutralized and empirical probability measure is taken:

$$E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)}\cdot\Delta^{exp.risk,(\nu)}\middle|z_t{}^{iI,(\Delta_C),p}\right)$$
$$= E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)}\middle|z_t{}^{iI,(\Delta_C),p}\right)$$
$$\cdot\left\{E^{rn,(\Delta_C)}\left(E\left(\widehat{D}_{t_{(k')}}{}^{(\nu)}\middle|z_t{}^{iI,(\Delta_C),p},z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)\middle|z_t{}^{iI,(\Delta_C),p}\right)\right.$$
$$\left.- E\left(E\left(\widehat{D}_{t_{(k')}}{}^{(\nu)}\middle|z_t{}^{iI,(\Delta_C),p},z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)\middle|z_t{}^{iI,(\Delta_C),p}\right)\right\}$$

<u>Step 6</u>: Final result

Combining the results of Steps 4 and 5 yields (A 3-39).

A3.5.2.2.2.1.2 Price of "Combined Risk"

Using the definition of $\Delta^{comb.risk,(\nu)}$ in (A 3-36), the price of "combined risk" reads

$$E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)} \cdot \Delta^{comb.risk,(\nu)} \middle| z_t{}^{iI,(\Delta_C),p}\right)$$

$$= E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)} \cdot \left\{ \begin{matrix} \widehat{D}_{t_{(k')}}{}^{(\nu)} \\ -E\left(\widehat{D}_{t_{(k')}}{}^{(\nu)} \middle| \begin{matrix} z_t{}^{iI,(\Delta_C),p} \\ , z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p} \end{matrix}\right) \end{matrix} \right\} \middle| z_t{}^{iI,(\Delta_C),p} \right)$$

with

$$\widehat{D}_{t_{(k')}}{}^{(\nu)} \equiv E\left(D_{t_{(k')}}{}^{(\nu)} \middle| z_t{}^{iI,(\Delta_C),p}, D_{t_{(k')}}{}^{market}\right)$$

Step 1: Use the tower property of conditional expectations to separate the pricing of risk stemming form $z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p} = \left(S_{t_{(k'-1)},t_{(k')}-1}, D_{t_{(k'-1)}}{}^{(\Delta_C)}, D_{t_{(k')}-1-1}{}^{(1)}\right)$ from the pricing of risk stemming from $fe_{t+\tau}$ given $z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p} = \left(S_{t_{(k'-1)},t_{(k')}-1}, D_{t_{(k'-1)}}{}^{(\Delta_C)}, D_{t_{(k')}-1-1}{}^{(1)}\right)$

$$E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)} \cdot \left\{ \begin{matrix} \widehat{D}_{t_{(k')}}{}^{(\nu)} \\ -E\left(\widehat{D}_{t_{(k')}}{}^{(\nu)} \middle| \begin{matrix} z_t{}^{iI,(\Delta_C),p} \\ , z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p} \end{matrix}\right) \end{matrix} \right\} \middle| z_t{}^{iI,(\Delta_C),p} \right)$$

$$= E\left(E\left(\begin{matrix} q_{t,t_{(k')}}{}^{iI,(\Delta_C)} \\ \cdot \left\{ \begin{matrix} \widehat{D}_{t_{(k')}}{}^{(\nu)} \\ -E\left(\widehat{D}_{t_{(k')}}{}^{(\nu)} \middle| \begin{matrix} z_t{}^{iI,(\Delta_C),p} \\ , z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p} \end{matrix}\right) \end{matrix} \right\} \end{matrix} \middle| \begin{matrix} z_t{}^{iI,(\Delta_C),p} \\ , z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p} \end{matrix} \right) \middle| z_t{}^{iI,(\Delta_C),p} \right)$$

The result of this step consists of the two summands

$$E\left(E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)} \cdot \widehat{D}_{t_{(k')}}{}^{(\nu)} \middle| \begin{matrix} z_t{}^{iI,(\Delta_C),p} \\ , z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p} \end{matrix} \right) \middle| z_t{}^{iI,(\Delta_C),p} \right)$$

and

$$-E\left(E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)} \middle| \begin{matrix} z_t{}^{iI,(\Delta_C),p} \\ , z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p} \end{matrix} \right) \cdot E\left(\widehat{D}_{t_{(k')}}{}^{(\nu)} \middle| \begin{matrix} z_t{}^{iI,(\Delta_C),p} \\ , z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p} \end{matrix} \right) \middle| z_t{}^{iI,(\Delta_C),p} \right)$$

Step 2: Factor out the multi-period riskless discount factor $E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)}\middle|z_t{}^{iI,(\Delta_C),p}\right)$ in order to separate riskless discounting from the pricing of risk

$$E\left(E\left(\begin{matrix} q_{t,t_{(k')}}{}^{iI,(\Delta_C)} \\ \cdot\left\{\begin{matrix}\widehat{D}_{t_{(k')}}{}^{(\nu)} \\ -E\left(\widehat{D}_{t_{(k')}}{}^{(\nu)}\middle| \begin{matrix} z_t{}^{iI,(\Delta_C),p} \\ ,z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\end{matrix}\right)\end{matrix}\right\}\end{matrix}\middle| \begin{matrix} z_t{}^{iI,(\Delta_C),p} \\ ,z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\end{matrix}\right)\middle| z_t{}^{iI,(\Delta_C),p}\right)$$

$$= E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)}\middle|z_t{}^{iI,(\Delta_C),p}\right)$$

$$\cdot\left\{E\left(E\left(\frac{q_{t,t_{(k')}}{}^{iI,(\Delta_C)}}{E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)}\middle|z_t{}^{iI,(\Delta_C),p}\right)}\cdot\widehat{D}_{t_{(k')}}{}^{(\nu)}\middle| \begin{matrix} z_t{}^{iI,(\Delta_C),p} \\ ,z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\end{matrix}\right)\middle| z_t{}^{iI,(\Delta_C),p}\right)\right.$$

$$\left. - E\left(\frac{E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)}\middle| \begin{matrix} z_t{}^{iI,(\Delta_C),p} \\ ,z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\end{matrix}\right)}{E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)}\middle|z_t{}^{iI,(\Delta_C),p}\right)}\cdot E\left(\widehat{D}_{t_{(k')}}{}^{(\nu)}\middle| \begin{matrix} z_t{}^{iI,(\Delta_C),p} \\ ,z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\end{matrix}\right)\middle| z_t{}^{iI,(\Delta_C),p}\right)\right\}$$

Note that the following term is the adjustment risk:

$$\frac{q_{t,t+\tau}{}^{iI}}{EE\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)}\middle|z_t{}^{iI,(\Delta_C),p}\right)} = \frac{U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)\middle|z_t{}^{iI,(\Delta_C),p}\right)}$$

The following term, consequently, can be interpreted as adjustment for the risk that deals with $z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p} = \left(S_{t_{(k'-1)},t_{(k')}-1}, D_{t_{(k'-1)}}{}^{(\Delta_C)}, D_{t_{(k')}-1-1}{}^{(1)}\right)$ and does not depend on factors and residuals $fe_{t_{(k')}}$:

$$\frac{E\left(q_{t,t+\tau}{}^{iI}\middle| \begin{matrix} z_t{}^{iI,(\Delta_C),p} \\ ,z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\end{matrix}\right)}{E\left(q_{t,t+\tau}{}^{iI}\middle|z_t{}^{iI,(\Delta_C),p}\right)} = \frac{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)\middle|z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)\middle|z_t{}^{iI,(\Delta_C),p}\right)}$$

Step 3: Factor out the adjustment for risk stemming from

$$z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p} = \left(S_{t_{(k'-1)},t_{(k')}-1}, D_{t_{(k'-1)}}{}^{(\Delta_C)}, D_{t_{(k')}-1}{}^{(1)}\right)$$

from the first element of the sum

$$E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)} \middle| z_t{}^{iI,(\Delta_C),p}\right) \cdot \left\{ E\left(E\left(\frac{U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right) \middle| z_t{}^{iI,(\Delta_C),p}\right)} \cdot \widehat{D}_{t_{(k')}}{}^{(\nu)} \middle| z_t{}^{iI,(\Delta_C),p}, z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p} \right) \middle| z_t{}^{iI,(\Delta_C),p} \right) - E\left(\frac{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right) \middle| z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right) \middle| z_t{}^{iI,(\Delta_C),p}\right)} \cdot E\left(\widehat{D}_{t_{(k')}}{}^{(\nu)} \middle| z_t{}^{iI,(\Delta_C),p}, z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right) \middle| z_t{}^{iI,(\Delta_C),p} \right) \right\}$$

$$= E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)} \middle| z_t{}^{iI,(\Delta_C),p}\right) \cdot \left\{ E\left(\frac{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right) \middle| z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right) \middle| z_t{}^{iI,(\Delta_C),p}\right)} \cdot E\left(\frac{U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right) \middle| z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)} \cdot \widehat{D}_{t_{(k')}}{}^{(\nu)} \middle| z_t{}^{iI,(\Delta_C),p}, z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p} \right) \middle| z_t{}^{iI,(\Delta_C),p} \right) - E\left(\frac{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right) \middle| z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right) \middle| z_t{}^{iI,(\Delta_C),p}\right)} \cdot E\left(\widehat{D}_{t_{(k')}}{}^{(\nu)} \middle| z_t{}^{iI,(\Delta_C),p}, z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right) \middle| z_t{}^{iI,(\Delta_C),p} \right) \right\}$$

Step 4: Introduce the risk-neutralized probability (density) of

$$z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p} = \left(S_{t_{(k'-1)},t_{(k')}-1}, D_{t_{(k'-1)}}{}^{(\Delta_C)}, D_{t_{(k')}-1-1}{}^{(1)}\right)$$

Define the risk-neutralized probability (density) of

$$z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p} = \left(S_{t_{(k'-1)},t_{(k')}-1}, D_{t_{(k'-1)}}{}^{(\Delta_C)}, D_{t_{(k')}-1-1}{}^{(1)}\right)$$

by

$$\varphi^{rn,iI,(\Delta_C)}\left(z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p} \middle| z_t{}^{iI,(\Delta_C),p}\right) \equiv \varphi^{iI,(\Delta_C)}\left(z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p} \middle| z_t{}^{iI,(\Delta_C),p}\right) \cdot \frac{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right) \middle| z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right) \middle| z_t{}^{iI,(\Delta_C),p}\right)}$$

By the same argument as in the case of expectation risk, the result from the previous Step 3 can be expressed through expectations with respect to $\varphi^{rn,iI}$. This yields

$$E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)} \middle| z_t{}^{iI,(\Delta_C),p}\right) \cdot \left\{ E^{rn,(\Delta_C)}\left(E\left(\frac{U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right) \middle| z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)} \cdot \widehat{D}_{t_{(k')}}{}^{(\nu)} \middle| z_t{}^{iI,(\Delta_C),p}, z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p} \right) \middle| z_t{}^{iI,(\Delta_C),p} \right) - E^{rn,(\Delta_C)}\left(E\left(\widehat{D}_{t_{(k')}}{}^{(\nu)} \middle| z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right) \middle| z_t{}^{iI,(\Delta_C),p}\right) \right\}$$

$$= E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)} \middle| z_t{}^{iI,(\Delta_C),p}\right) \cdot \left\{ E^{rn,(\Delta_C)}\left(\begin{array}{c} E\left(\frac{U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right) \middle| z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)} \cdot \widehat{D}_{t_{(k')}}{}^{(\nu)} \middle| z_t{}^{iI,(\Delta_C),p}, z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p} \right) \\ -E\left(\widehat{D}_{t_{(k')}}{}^{(\nu)} \middle| z_t{}^{iI,(\Delta_C),p}, z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right) \end{array} \middle| z_t{}^{iI,(\Delta_C),p} \right) \right\}$$

Step 5: Re-express the inner expectation from the result of the previous step through $\Delta^{comb.risk,(\nu)}$

Since the adjustment for risk $\frac{U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right) \middle| z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)}$ has an expectation of one conditional on $z_t{}^{iI,(\Delta_C),p}, z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}$, the following equations hold and yield the desired result:

$$E\left(\frac{U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)\middle|z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)}\cdot\Delta^{comb.risk,(\nu)}\middle|\begin{matrix}z_t{}^{iI,(\Delta_C),p}\\,z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\end{matrix}\right)$$

$$=E\left(\begin{matrix}\frac{U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)\middle|z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)}\\\cdot\left\{\widehat{D}_{t_{(k')}}{}^{(\nu)}-E\left(\widehat{D}_{t_{(k')}}{}^{(\nu)}\middle|\begin{matrix}z_t{}^{iI,(\Delta_C),p}\\,z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\end{matrix}\right)\right\}\end{matrix}\middle|\begin{matrix}z_t{}^{iI,(\Delta_C),p}\\,z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\end{matrix}\right)$$

$$=E\left(\begin{matrix}\frac{U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)\middle|z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)}\\\cdot\widehat{D}_{t_{(k')}}{}^{(\nu)}\end{matrix}\middle|\begin{matrix}z_t{}^{iI,(\Delta_C),p}\\,z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\end{matrix}\right)-E\left(\widehat{D}_{t_{(k')}}{}^{(\nu)}\middle|\begin{matrix}z_t{}^{iI,(\Delta_C),p}\\,z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\end{matrix}\right)$$

Hence the result from Step 4 can likewise be written as

$$E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)}\cdot\Delta^{comb.risk,(\nu)}\middle|z_t{}^{iI,(\Delta_C),p}\right)$$

$$=E\left(q_{t,t_{(k')}}{}^{iI,(\Delta_C)}\middle|z_t{}^{iI,(\Delta_C),p}\right)\cdot$$

$$\cdot\left\{E^{rn,(\Delta_C)}\left(E\left(\frac{U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)}{E\left(U'\left(\frac{1}{n_I}\cdot D_{t_{(k')}}{}^{market}\right)\middle|z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\right)}\cdot\Delta^{comb.risk,(\nu)}\middle|\begin{matrix}z_t{}^{iI,(\Delta_C),p}\\,z_{t_{(k'-1)}}{}^{cI,(\Delta_C),p}\end{matrix}\right)\middle|z_t{}^{iI,(\Delta_C),p}\right)\right\}$$

as was to be shown.

A3.6 Appendix to Section 4.2.3.1.2: Independence of $E\left(\pi_{t+1,s}\middle|\pi_t, D_t, D_{t+1}\right)$ of Signal Quality

A3.6.1 Formulation of the Problem

The proof that risk premia are independent of signal quality boils down to showing that the vector

A 3-43
$$E(\pi_{t+1}|\pi_t, D_t, D_{t+1})$$

with

$$\pi_{t+1} = \Pi(\pi_t, D_t, D_{t+1}, Sig_{t+1})$$

is independent of signal quality. In this context independence means that the expected value does not depend on the form of the signal function $Sig(.)$.

A3.6.2 Results

The conditional expectation $E(\pi_{t+1,s'}|\pi_t, D_t, D_{t+1})$ is the vector of conditional regime probabilities from the case without any signals (i.e., all information comes from cash flows)

A 3-44

$$E(\pi_{t+1}|\pi_t, D_t, D_{t+1}) = \Pi_{\text{no sigs}}(\pi_t, D_t, D_{t+1})$$

where $\Pi_{\text{no sigs}}(.)$ is the function that yields the recursion between conditional regime probabilities at times t and $t+1$ in the case without signals ((A 1-2) in combination with (A 1-5)).

Since the right-hand side of (A 3-44) does not depend on the function $Sig(.)$, (A 3-44) shows the independence of the conditional expectation of the function $Sig(.)$.

A3.6.3 Proof

A3.6.3.1 Idea of the Proof

Intuitively, taking the conditional expectation $E(\pi_{t+1}|\pi_t, D_t, D_{t+1})$ averages out the effect of signals Sig_{t+1} on π_{t+1}, leaving only cash flows D_{t+1} as a new source of information. This suggests that the conditional expectation $E(\pi_{t+1}|\pi_t, D_t, D_{t+1})$ simply is the vector of regime probabilities conditional on information at time t, π_t, D_t, and new information coming from cash flows, D_{t+1}. This can be proven by expressing probabilities as conditional expectations of indicator functions and the using the tower property of conditional expectations.

A3.6.3.2 Details of the Proof

Let the symbol $\Pi_s(\pi_t, D_t, D_{t+1}, Sig_{t+1})$ denote the conditional regime probability of regime s at time $t+1$. In this notation, the problem is to evaluate

A 3-45

$$E(\Pi_s(\pi_t, D_t, D_{t+1}, Sig_{t+1})|\pi_t, D_t, D_{t+1})$$
$$s = 1, \dots, K$$

with

$$D_{t+1} = D(D_t, S_t, fe_{t+1})$$
$$Sig_{t+1} = Sig(S_t, S_{t+1}, fe_{t+1}, \eta_{t+1})$$

Step 1: Use an indicator function to rewrite $\Pi_s(\pi_t, D_t, D_{t+1}, Sig_{t+1})$ as a conditional expectation:

Note that the probability of some event A can be expressed as an expectation of an indicator function:

A 3-46

$$P(A) = E(\underline{1}_A)$$

Applied to the problem at hand, it follows

A 3-47

$$\Pi_s(\pi_t, D_t, D_{t+1}, Sig_{t+1}) = \mathrm{P}(\mathrm{S}_{t+1} = \mathrm{s}|I_t, D_{t+1}, Sig_{t+1}) = \mathrm{E}(\underline{1}_{\mathrm{S}_{t+1}=\mathrm{s}}|I_t, D_{t+1}, Sig_{t+1})$$

where I_t is information available to investors at time t. Conditional regime probabilities at time t, π_t, are computed from this information.

Step 2: Plug (A 3-47) into (A 3-45) and then use the tower property of conditional expectations

$$E(\Pi_s(\pi_t, D_t, D_{t+1}, Sig_{t+1})|I_t, D_{t+1}) = E(\mathrm{E}(\underline{1}_{\mathrm{S}_{t+1}=\mathrm{s}}|I_t, D_{t+1}, Sig_{t+1})|I_t, D_{t+1}) = E(\underline{1}_{\mathrm{S}_{t+1}=\mathrm{s}}|I_t, D_{t+1})$$

Step 3: Make use of the representation of probabilities as expectations of indicator functions for a second time

$$E(\underline{1}_{\mathrm{S}_{t+1}=\mathrm{s}}|I_t, D_{t+1}) = P(\mathrm{S}_{t+1} = \mathrm{s}|I_t, D_{t+1})$$

Step 4: Combine the previous steps

$$E(\Pi_s(\pi_t, D_t, D_{t+1}, Sig_{t+1})|\pi_t, D_t, D_{t+1}) = P(\mathrm{S}_{t+1} = \mathrm{s}|I_t, D_{t+1})$$

Since s is an arbitrary regime and the probabilities $P(\mathrm{S}_{t+1} = \mathrm{s}|I_t, D_{t+1})$ can be recursively obtained from $\pi_t = \begin{pmatrix} P(S_t = 1|I_t) \\ \dots \\ P(S_t = K|I_t) \end{pmatrix}$ by the recursion for the case without signals ((A 1-2) in combination with (A 1-5)), (A 3-44) holds.

A3.7 Appendix to Section 4.3.3.1.2.1: the Price Dividend Ratio for Cash Flow Models without Lags in Growth Rates under Constant Relative Risk Aversion

A3.7.1 Formulation of the Problem

The problem is to show:

First, that complete information equilibrium asset prices for cash flow models without lags in growth rates under constant relative risk aversion exhibit the structure

A 3-48

$$P_{i,t}{}^{cI}\left(S_t, D_{i,t}, \delta_t\right) = D_{i,t} \cdot \left(\frac{P}{D}\right)_{i,t}^{cI} (S_t, \delta_t) \equiv D_{i,t} \cdot \begin{cases} \dfrac{P_{i,t}\left(S_t, D_{i,t}, \delta_t\right)}{D_{i,t}} & D_{i,t} > 0 \\ 0 & D_{i,t} = 0 \end{cases}$$

with

$$\delta_{j,t} \equiv \frac{\overline{N}_j \cdot D_{j,t}}{\sum_{\nu=1}^{n} \overline{N}_\nu \cdot D_{\nu,t}} = \frac{\overline{N}_j \cdot D_{j,t}}{D_t{}^{market}}$$

$$j = 1, \dots, n$$

where the price dividend ratio $\left(\frac{P}{D}\right)_{i,t}^{cI}(S_t, \delta_t)$ is independent of $D_{i,t}$ if $D_{i,t} > 0$ and otherwise zero.

Second, that the quasi static pricing equation (4-150) is true:

4-150

$$P_{i,t}{}^{cI}\left(S_t, D_{i,t}, \delta_t\right)$$
$$= D_{i,t} \cdot E\left(q_{t,t+1}{}^{cI}(fe_{t+1}, S_t, \delta_t) \cdot [1 + d_i(S_t, fe_{t+1})] \cdot \left\{\left(\frac{P}{D}\right)_{i,t+1}^{cI} (S_{t+1}, \delta_{t+1}) + 1\right\} \middle| S_t, \delta_t\right)$$

$$i = 1, \dots, n$$

A3.7.2 Proof

A3.7.2.1 Idea of the Proof

The proof is by induction over the remaining time horizon $T - t$, starting from the final point of time $t = T$.

A3.7.2.2 Details of the Proof

Base Case: $T-t=0$

The price dividend ratio at time T is trivially independent of $D_{i,T}$ because it is always zero:

$$\left(\frac{P}{D}\right)_{i,T}^{cI} \equiv \begin{cases} \frac{P_{i,T}(S_T, D_{i,T}, \delta_T)}{D_{i,T}} & D_{i,T} > 0 \\ 0 & D_{i,T} = 0 \end{cases} = \begin{cases} \frac{0}{D_{i,T}} & D_{i,T} > 0 \\ 0 & D_{i,T} = 0 \end{cases} = 0$$

Inductive step

Inductively assume that the price dividend ratio at time $t+1$ is a function of S_{t+1}, δ_{t+1} only, and therefore independent of $D_{i,t+1}$: $\left(\frac{P}{D}\right)_{i,t+1}^{cI} = \left(\frac{P}{D}\right)_{i,t+1}^{cI}(S_{t+1}, D_{i,t+1})$

Step 1: Use the general formula on quasi-static asset prices under complete information

Quasi-static asset prices under complete information for general cash flows read

4-41

$$P_{i,t}^{cI}(S_t, D_{i,t}, \delta_t) = E\left(q_{t,t+1}^{cI}(fe_{t+1}, S_t, D_t) \cdot \{P_{t+1}^{cI}(S_{t+1}, D_{t+1}) + D_{t+1}\} \middle| S_t, D_t\right)$$

with

4-43

$$q_{t,t+1}^{cI}(fe_{t+1}, S_t, D_t) = \frac{1}{1+\rho} \cdot \frac{U'\left(\frac{1}{n_I} \cdot D_{t+1}^{market}(D_{t+1})\right)}{U'\left(\frac{1}{n_I} \cdot D_t^{market}(D_t)\right)}$$

Step 2: Observe that the stochastic discount factor is a function of δ_t, S_t, fe_{t+1} (but not D_t)

The stochastic discount factor for cash flow models without lags in growth rates under constant relative risk aversion reads:

$$q_{t,t+1}^{cI}(fe_{t+1}, S_t, D_t) = \frac{1}{1+\rho} \cdot \{d_{t+1}^{market}(\delta_t, S_t, fe_{t+1})\}^{-\gamma}$$

with

$$d_{t+1}^{market}(\delta_t, S_t, fe_{t+1}) = \frac{D_{t+1}^{market}(D_{t+1})}{D_t^{market}(D_t)} = \sum_{v=1}^{n} \delta_{v,t} \cdot d_v(S_t, fe_{t+1})$$

Step 3: Make use of the inductive assumption and the special cash flow model

If $D_{i,t} > 0$ (otherwise the price dividend ratio is defined to be zero and (A 3-48) is correct), we have by the inductive hypothesis

$$\frac{P_{i,t+1}^{cI}(S_{t+1}, D_{t+1}) + D_{i,t+1}}{D_{i,t}} = \frac{D_{i,t+1}}{D_{i,t}} \cdot \left\{\left(\frac{P}{D}\right)_{i,t+1}^{cI}(S_{t+1}, \delta_{t+1}) + 1\right\}$$

Moreover, by definition of the cash flow model we have $\frac{D_{i,t+1}}{D_{i,t}} = 1 + d_i(S_t, fe_{t+1})$, i.e.,

A 3-49

$$\frac{P_{i,t+1}{}^{cI}(S_{t+1}, D_{t+1}) + D_{i,t+1}}{D_{i,t}} = [1 + d_i(S_t, fe_{t+1})] \cdot \left\{ \left(\frac{P}{D}\right)_{i,t+1}^{cI} (S_{t+1}, \delta_{t+1}) + 1 \right\}$$

Step 4: Plug the result from the previous step (A 3-49) into the formula for quasi-static asset prices

If $D_{i,t} > 0$, quasi-static asset prices can be tautologically rewritten as

$$\frac{P_{i,t}{}^{cI}(S_t, D_{i,t}, \delta_t)}{D_{i,t}} = E\left(\frac{1}{1+\rho} \cdot \left\{ d_{t+1}{}^{market}(\delta_t, S_t, fe_{t+1}) \right\}^{-\gamma} \cdot \left\{ \frac{P_{t+1}{}^{cI}(S_{t+1}, D_{t+1}) + D_{t+1}}{D_{i,t}} \right\} \middle| S_t, D_t \right)$$

$$= E\left(\begin{matrix} \frac{1}{1+\rho} \cdot \left\{ d_{t+1}{}^{market}(\delta_t, S_t, fe_{t+1}) \right\}^{-\gamma} \\ \cdot [1 + d_i(S_t, fe_{t+1})] \cdot \left\{ \left(\frac{P}{D}\right)_{i,t+1}^{cI} (S_{t+1}, \delta_{t+1}) + 1 \right\} \end{matrix} \middle| S_t, D_t \right)$$

Step 5: Simplify the conditioning information to obtain the final result

The right-hand side of this previous equation will be the same for all realization of D_t with the same relative dividend contributions:

$$\frac{P_{i,t}{}^{cI}(S_t, D_{i,t}, \delta_t)}{D_{i,t}} = E\left(\begin{matrix} \frac{1}{1+\rho} \cdot \left\{ d_{t+1}{}^{market}(\delta_t, S_t, fe_{t+1}) \right\}^{-\gamma} \\ \cdot [1 + d_i(S_t, fe_{t+1})] \cdot \left\{ \left(\frac{P}{D}\right)_{i,t+1}^{cI} (S_{t+1}, \delta_{t+1}) + 1 \right\} \end{matrix} \middle| S_t, \delta_t \right)$$

This inductively proves that the price dividend ratio at time is a function of S_t and δ_t (A 3-48) and also establishes (4-150).

A3.8 Appendix to Section 4.4.3: Convexity and Conditions of Convergence of the Price Dividend Ratio Function

A3.8.1 Formulation of the Problem

The problem is to analyze convergence/divergence of price dividend ratios as the remaining time horizon $T-t$ goes to infinity for model without lags in growth rates, constant relative risk aversion (CRRA) and complete information:

$$D_{i,t+1} = D_{i,t} \cdot [1 + d_i(S_t, fe_{t+1})]$$
$$i = 1, \dots, n$$

with

$$d_i(S_t, fe_{t+1}) > -1$$
$$D_{i,0} > 0$$
$$i = 1, \dots, n$$

The price dividend ratio of asset i reads (see (4-158)):

$$\left(\frac{P}{D}\right)_{i,t}^{cI} (S_t, \delta_t; T-t)$$
$$= \sum_{\tau=1}^{T-t} E\left(\frac{1}{(1+\rho)^\tau} \cdot \left\{ \sum_{\nu=1}^{n} \delta_{\nu,t} \cdot \left[1 + d_\nu{}^{+(\tau)}(S_{t,t+\tau-1}, fe_{t+1,t+\tau})\right] \right\}^{-\gamma} \cdot \left[1 + d_i{}^{+\tau}(S_{t,t+\tau-1}, fe_{t+1,t+\tau})\right] \middle| S_t, \delta_t \right)$$

with

$$\gamma > 0, \rho > 0$$

with dividend growth rates

$$d_\nu{}^{+(\tau)}(S_{t+1,t+\tau-1}, fe_{t+1,t+1+\tau}) = \prod_{\zeta=1}^{\tau} \left[1 + d_\nu(S_{t+\zeta-1}, fe_{t+1+\zeta})\right] - 1$$

with relative dividend contributions (4-149)

$$\delta_{j,t} \equiv \frac{\bar{N}_j \cdot D_{j,t}}{\sum_{\nu=1}^{n} \bar{N}_\nu \cdot D_{\nu,t}} = \frac{\bar{N}_j \cdot D_{j,t}}{D_t{}^{market}}$$
$$j = 1, \dots, n$$

Analyzing the convergence/divergence behavior leads to two problems:

The first problem is to show that the price dividend ratio is convex in relative dividend contributions for all regimes which implies that price dividend ratios are bounded by a convex combination of price dividend ratios in limit cases where all dividends are paid by one single asset:

$$\left(\frac{P}{D}\right)_{i,t}^{cI} (s, \delta; T-t) \le \sum_{j=1}^{n} \delta_j \cdot \left(\frac{P}{D}\right)_{i,t}^{cI} (s, e_j; T-t)$$

where e_j represents the j-th unit vector of dimension n (i.e., all dividends paid by asset j).

The second problem is a consequence of the first problem. Convexity in relative dividend contributions means that the analysis of convergence/divergence of price dividend ratios as the time horizon $T-t$ goes to infinity can be narrowed down to the limit cases where all dividends are paid by a single asset. For that reason, a characterization of the limiting price dividend ratios is needed that allows to examine whether these price dividend ratios diverge or converge as the remaining time horizon $T-t$ goes to infinity:

$$\left(\frac{P}{D}\right)_{i,t}^{cI}\left(S_t=s,e_j;T-t\right), j=1 s=1,\dots,K$$
$$i=1,\dots,n$$
$$j=1,\dots,n$$

A3.8.2 Proof of the Convexity of the Price Dividend Ratio in Relative Dividend Contributions (First Problem)

A3.8.2.1 Idea of the Proof

Convexity of complete information price dividend ratios in relative dividend contributions is a direct consequence of the convexity of the stochastic discount factor.

A3.8.2.2 Details of the Proof

Step 1: convexity of the stochastic discount factor

Consider the function

$$f(x)=x^{-\gamma}$$
$$x>0$$
$$\gamma>0$$

Note that the stochastic discount factor can be written as

$$\frac{1}{(1+\rho)^{\tau}}\cdot f\left(\sum\nolimits_{v=1}^{n}\delta_{v,t}\cdot\left[1+d_v{}^{+(\tau)}\left(S_{t,t+\tau-1},fe_{t+1,t+\tau}\right)\right]\right)$$

Derivatives of f are

$$f'(x)=-\gamma\cdot x^{-\gamma-1}$$
$$f''(x)=\gamma\cdot(\gamma+1)\cdot x^{-\gamma-2}$$

Since γ is restricted to be positive, the second derivative of f is likewise positive, and thus f is convex. As a consequence, the following inequality for the stochastic discount factor holds:

$$\frac{1}{(1+\rho)^{\tau}}\cdot\left\{\sum_{\nu=1}^{n}\delta_{\nu,t}\cdot\left[1+d_{\nu}{}^{+(\tau)}\left(S_{t,t+\tau-1},fe_{t+1,t+\tau}\right)\right]\right\}^{-\gamma}$$
$$\leq\frac{1}{(1+\rho)^{\tau}}\cdot\sum_{\nu=1}^{n}\delta_{\nu,t}\cdot\left\{1+d_{\nu}{}^{+(\tau)}\left(S_{t,t+\tau-1},fe_{t+1,t+\tau}\right)\right\}^{-\gamma}$$

<u>Step 2</u>: convexity of complete information price dividend ratios

Multiplying both sides of the previous inequality by the positive factor $\left[1+d_i{}^{+\tau}\left(S_{t,t+\tau-1},fe_{t+1,t+\tau}\right)\right]$ and taking expectations yields

$$E\left(\left.\begin{matrix}\frac{1}{(1+\rho)^{\tau}}\cdot\left\{\sum_{\nu=1}^{n}\delta_{\nu,t}\cdot\left[1+d_{\nu}{}^{+(\tau)}\left(S_{t,t+\tau-1},fe_{t+1,t+\tau}\right)\right]\right\}^{-\gamma}\\ \cdot\left[1+d_i{}^{+\tau}\left(S_{t,t+\tau-1},fe_{t+1,t+\tau}\right)\right]\end{matrix}\right|S_t,\delta_t\right)$$
$$\leq$$
$$\sum_{\nu=1}^{n}\delta_{\nu,t}\cdot E\left(\left.\begin{matrix}\frac{1}{(1+\rho)^{\tau}}\cdot\left\{\left[1+d_{\nu}{}^{+(\tau)}\left(S_{t,t+\tau-1},fe_{t+1,t+\tau}\right)\right]\right\}^{-\gamma}\\ \cdot\left[1+d_i{}^{+\tau}\left(S_{t,t+\tau-1},fe_{t+1,t+\tau}\right)\right]\end{matrix}\right|S_t,\delta_t\right)$$

Summing over all time indices τ finally establishes the convexity of complete information price dividend ratios:

$$\left(\frac{P}{D}\right)_{i,t}^{cI}(s,\delta;T-t)\leq\sum_{j=1}^{n}\delta_j\cdot\left(\frac{P}{D}\right)_{i,t}^{cI}\left(s,e_j;T-t\right)$$

A3.8.3 Characterization of the Limiting Price Dividend Ratios (Second Problem)

A3.8.3.1 Results

The limit case price dividend functions can be expressed as

A 3-50

$$\left(\frac{P}{D}\right)_{i,t}^{cI}\left(e_j;T-t\right)=\left\{\sum_{\tau=1}^{T-t}\frac{1}{(1+\rho)^{\tau-1}}\cdot M_{i,j}{}^{\tau-1}\right\}\cdot\frac{1}{1+\rho}\cdot h_{i,j}{}^{+(1)}$$

where $M_{i,j}$ is defined by

A 3-51

$$M_{i,j}\equiv\begin{pmatrix}h_{i,j}{}^{+1}(1)\cdot p_{11} & \dots & h_{i,j}{}^{+1}(1)\cdot p_{1K}\\ \dots & \dots & \dots\\ h_{i,j}{}^{+1}(K)\cdot p_{K1} & \dots & h_{i,j}{}^{+1}(K)\cdot p_{KK}\end{pmatrix}$$

with

$$h_{i,j}{}^{+1}(s)\equiv E\left(\left.\begin{matrix}\left\{1+d_j(S_t,fe_{t+1})\right\}^{-\gamma}\\ \cdot\left[1+d_i(S_t,fe_{t+1})\right]\end{matrix}\right|S_t=s\right)$$

If it is further assumed that the matrix $M_{i,j}$ admits an Eigenvalue decomposition

A 3-52

$$M_{i,j} = V_{i,j} \cdot D_{i,j} \cdot {V_{i,j}}^{-1}$$

$$D_{i,j} = \begin{pmatrix} \lambda_1 & & \\ & \dots & \\ & & \lambda_K \end{pmatrix}$$

where $D_{i,j}$ is a $K \times K$ diagonal matrix with the Eigenvalues of $M_{i,j}$ on the diagonal and $V_{i,j}$ is an invertible $K \times K$-matrix,

then the price dividend ratio will converge if the absolute values of the product of the Eigenvalues of $M_{i,j}$ with $\frac{1}{1+\rho}$ are strictly less than one:

A 3-53

$$\left|\lambda_s \cdot \frac{1}{1+\rho}\right| < 1, s = 1, \dots K$$

A3.8.3.2 Proof

A3.8.3.2.1 Idea of the Proof

I first show that the vector of degenerate price dividend ratios in regimes $s = 1, \dots, K$ can be expressed as a sum of products of a certain matrix, i.e., possess a simple mathematical structure. In a second step, an Eigenvalue composition of this matrix is used (provided it exists). This decomposition has the advantage that price dividend ratios can be expressed through a geometric series. But the convergence or divergence of a geometric series is easy to decide.

A3.8.3.2.2 Details of the Proof

A3.8.3.2.2.1 Complete Information Price Dividend Ratios Expressed through Powers of Matrices

In the limit case where all dividends are paid by asset j, the price dividend ratio of asset i reads:

$$\left(\frac{P}{D}\right)_{i,t}^{cI}(s, e_j; T-t) = \sum\nolimits_{\tau=1}^{T-t} \frac{1}{(1+\rho)^\tau} \cdot E\left(\begin{matrix} \left\{1 + {d_j}^{+(\tau)}(S_{t,t+\tau-1}, fe_{t+1,t+\tau})\right\}^{-\gamma} \\ \cdot \left[1 + {d_i}^{+\tau}(S_{t,t+\tau-1}, fe_{t+1,t+\tau})\right] \end{matrix} \middle| S_t = s \right)$$

For brevity, define

A 3-54

$$ {h_{i,v}}^{+\tau}(s) \equiv E\left(\begin{matrix} \left\{1 + {d_j}^{+(\tau)}(S_{t,t+\tau-1}, fe_{t+1,t+\tau})\right\}^{-\gamma} \\ \cdot \left[1 + {d_i}^{+\tau}(S_{t,t+\tau-1}, fe_{t+1,t+\tau})\right] \end{matrix} \middle| S_t = s \right)$$

In this notation, the price dividend ratio of asset i reads

A 3-55

$$\left(\frac{P}{D}\right)_{i,t}^{cI}(s,e_j;T-t)=\sum_{\tau=1}^{T-t}\frac{1}{(1+\rho)^{\tau}}\cdot {h_{i,v}}^{+\tau}(s)$$

The fact that regimes are a Markov Chain admits a recursion between the terms ${h_{i,v}}^{+\tau}(s), s = 1,\dots,K$:

Step 1: Write the τ-period quantities $\left\{1+{d_j}^{+(\tau)}(S_{t,t+\tau-1},fe_{t+1,t+\tau})\right\}^{-\gamma}$ and $1+{d_i}^{+\tau}(S_{t,t+\tau-1},fe_{t+1,t+\tau})$ as products of a single period term and a $(\tau-1)$-period term

A 3-56

$$\begin{aligned}&\left\{1+{d_j}^{+(\tau)}(S_{t,t+\tau-1},fe_{t+1,t+\tau})\right\}^{-\gamma}\\&\qquad=\left(1+d_j(S_t,fe_{t+1})\right)^{-\gamma}\cdot\left\{1+{d_j}^{+(\tau-1)}(S_{t+1,t+\tau-1},fe_{t+2,t+\tau})\right\}^{-\gamma}\end{aligned}$$

A 3-57

$$1+{d_i}^{+\tau}(S_{t,t+\tau-1},fe_{t+1,t+\tau})=\left(1+d_i(S_t,fe_{t+1})\right)\cdot\left\{1+{d_i}^{+(\tau-1)}(S_{t+1,t+\tau-1},fe_{t+2,t+\tau})\right\}$$

Step 2 a: Plug the right-hand sides of (A 3-56) and (A 3-57) into (A 3-54)

$$\begin{aligned}&{h_{i,j}}^{+\tau}(s)\\&=E\left(\begin{matrix}\left(1+d_j(S_t,fe_{t+1})\right)^{-\gamma}\cdot\left(1+d_i(S_t,fe_{t+1})\right)\\ \cdot\left\{\left\{1+{d_j}^{+(\tau-1)}(S_{t+1,t+\tau-1},fe_{t+2,t+\tau})\right\}^{-\gamma}\cdot\left\{1+{d_i}^{+(\tau-1)}(S_{t+1,t+\tau-1},fe_{t+2,t+\tau})\right\}\right\}\end{matrix}\middle|S_t=s\right)\end{aligned}$$

Step 2 b: Use the tower property of conditional expectations

$$=E\left(\begin{matrix}\left(1+d_j(S_t,fe_{t+1})\right)^{-\gamma}\cdot\left(1+d_i(S_t,fe_{t+1})\right)\\ \cdot\sum_{s'=1}^{K}p_{S_t,s'}E\left(\left\{\begin{matrix}\left\{1+{d_j}^{+(\tau-1)}(S_{t+1,t+\tau-1},fe_{t+2,t+\tau})\right\}^{-\gamma}\\ \cdot\left\{1+{d_i}^{+(\tau-1)}(S_{t+1,t+\tau-1},fe_{t+2,t+\tau})\right\}\end{matrix}\right\}\middle|S_{t+1}=s',S_t=s\right)\end{matrix}\middle|S_t=s\right)$$

Step 2 c: Use the fact that regimes are a Markov chain

$$=E\left(\begin{matrix}\left(1+d_j(S_t,fe_{t+1})\right)^{-\gamma}\cdot\left(1+d_i(S_t,fe_{t+1})\right)\\ \cdot\sum_{s'=1}^{K}p_{S_t,s'}E\left(\left\{\begin{matrix}\left\{1+{d_j}^{+(\tau-1)}(S_{t+1,t+\tau-1},fe_{t+2,t+\tau})\right\}^{-\gamma}\\ \cdot\left\{1+{d_i}^{+(\tau-1)}(S_{t+1,t+\tau-1},fe_{t+2,t+\tau})\right\}\end{matrix}\right\}\middle|S_{t+1}=s'\right)\end{matrix}\middle|S_t=s\right)$$

Step 2 d: Observe that the inner conditional expectations coincide with ${h_{i,j}}^{+(\tau-1)}$

$$=E\left(\begin{matrix}\left(1+d_j(S_t,fe_{t+1})\right)^{-\gamma}\cdot\left(1+d_i(S_t,fe_{t+1})\right)\\ \cdot\sum_{s'=1}^{K}p_{S_t,s'}{h_{i,j}}^{+(\tau-1)}(s')\end{matrix}\middle|S_t=s\right)$$

Step 2 e: Finally note that $\cdot \sum_{s'=1}^{K} p_{s,s'} h_{i,j}{}^{+(\tau-1)}(s')$ is certain conditional on $S_t = s$

$$= h_{i,j}{}^{+1}(s) \cdot \sum_{s'=1}^{K} p_{S_t,s'} h_{i,j}{}^{+(\tau-1)}(s')$$

In short, we have the following recursion:

A 3-58

$$h_{i,j}{}^{+\tau}(s) = h_{i,j}{}^{+1}(s) \cdot \sum_{s'=1}^{K} p_{s,s'} \cdot h_{i,j}{}^{+(\tau-1)}(s')$$

Step 3: Rewrite the recursion in matrix form

Defining the vector

$$h_{i,j}{}^{+(\tau)} \equiv \begin{pmatrix} h_{i,j}{}^{+(\tau)}(1) \\ \dots \\ h_{i,j}{}^{+(\tau)}(K) \end{pmatrix}$$

the recursion can be written in matrix form:

$$h_{i,j}{}^{+(\tau)} = M_{i,j} \cdot h_{i,j}{}^{+(\tau-1)}$$

with

$$M_{i,j} \equiv \begin{pmatrix} h_{i,j}{}^{+1}(1) \cdot p_{11} & \dots & h_{i,j}{}^{+1}(1) \cdot p_{1K} \\ \dots & \dots & \dots \\ h_{i,j}{}^{+1}(K) \cdot p_{K1} & \dots & h_{i,j}{}^{+1}(K) \cdot p_{KK} \end{pmatrix}$$

By iteration, $h_{i,j}{}^{+(\tau)}$ can be expressed as a function of $h_{i,j}{}^{+(1)}$:

A 3-59

$$h_{i,\nu}{}^{+(\tau)} = M_{i,j}{}^{\tau-1} \cdot h_{i,\nu}{}^{+1}$$

where $M_{i,j}{}^{\tau-1}$ is the $\tau - 1$-th power of the matrix $M_{i,j}$ and where $M_{i,j}{}^{0}$ is the $K \times K$-unity matrix.

Step 4: Plug (A 3-59) into the price dividend function (A 3-55)

Defining

$$\left(\frac{P}{D}\right)_{i,t}^{cI}(e_j; T-t) \equiv \begin{pmatrix} \left(\frac{P}{D}\right)_{i,t}^{cI}(1, e_j; T-t) \\ \dots \\ \left(\frac{P}{D}\right)_{i,t}^{cI}(K, e_j; T-t) \end{pmatrix}$$

it is obtained (A 3-50)

$$\left(\frac{P}{D}\right)_{i,t}^{cI}(e_j; T-t) = \left\{\sum_{\tau=1}^{T-t} \frac{1}{(1+\rho)^{\tau-1}} \cdot M_{i,j}{}^{\tau-1}\right\} \cdot \frac{1}{1+\rho} \cdot h_{i,j}{}^{+(1)}$$

A3.8.3.2.2.2 Characterization of Convergence Through Eigenvalues

If the matrix $M_{i,j}$ admits an Eigenvalue decomposition, convergence and divergence of the price dividend ratio $\left(\frac{P}{D}\right)_{i,t}^{cI}(e_j; T-t)$ as $T-t$ goes to infinity are readily characterized. Assume such a decomposition is possible (A 3-52):

$$M_{i,j} = V_{i,j} \cdot D_{i,j} \cdot {V_{i,j}}^{-1}$$

$$D_{i,j} = \begin{pmatrix} \lambda_1 & & \\ & \dots & \\ & & \lambda_K \end{pmatrix}$$

where $D_{i,j}$ is a $K \times K$ diagonal matrix with the Eigenvalues of $M_{i,j}$ on the diagonal and $V_{i,j}$ is an invertible $K \times K$-matrix.

This decomposition is useful because of the well-known fact that the $(\tau-1)$-th power of $M_{i,j}$ then reads

$$ {M_{i,j}}^{(\tau-1)} = V_{i,j} \cdot {D_{i,j}}^{(\tau-1)} \cdot {V_{i,j}}^{-1} = V_{i,j} \cdot \begin{pmatrix} {\lambda_1}^{(\tau-1)} & & \\ & \dots & \\ & & {\lambda_K}^{(\tau-1)} \end{pmatrix} \cdot {V_{i,j}}^{-1}$$

This, in turn, allows expressing the price dividend ratio function through the K geometric series with terms $\frac{1}{1+\rho}\lambda_1$, …,$\frac{1}{1+\rho}\lambda_K$, respectively:

$$\left(\frac{P}{D}\right)_{i,t}^{cI}(e_j; T-t)$$

$$= V_{i,j} \cdot \left\{ \sum_{\tau=1}^{T-t} \begin{pmatrix} \left\{\lambda_1 \cdot \frac{1}{1+\rho}\right\}^{\tau-1} & & \\ & \dots & \\ & & \left\{\lambda_K \cdot \frac{1}{1+\rho}\right\}^{\tau-1} \end{pmatrix} \right\} {V_{i,j}}^{-1} \cdot \left\{\frac{1}{1+\rho} \cdot {h_{i,j}}^{+(1)}\right\}$$

If $\left|\lambda_s \cdot \frac{1}{1+\rho}\right| < 1, s = 1, \dots K$, (condition (A 3-53)) then the price dividend ratios will certainly converge in each regime and can be computed as

$$\left(\frac{P}{D}\right)_{i,t}^{cI}(e_j; T-t) = V_{i,j} \cdot \begin{pmatrix} \frac{1}{1-\lambda_1 \cdot \frac{1}{1+\rho}} & & \\ & \dots & \\ & & \frac{1}{1-\lambda_K \cdot \frac{1}{1+\rho}} \end{pmatrix} \cdot {V_{i,j}}^{-1} \cdot \left\{\frac{1}{1+\rho} \cdot {h_{i,j}}^{+(1)}\right\}$$

A4 Appendix to Chapter 5

A4.1 Appendix to Section 5.3.1: Choice of the Arguments of the Risk Premium Function

The problem is to characterize the behavior of expected future regime probabilities, $E_t(\pi_{t+\tau})$ as τ goes to infinity. To that end, these expectations must be computed, and multi-period transition probabilities $P(S_{t+\tau} = s'|S_t = s)$ and their limiting behavior (as τ goes to infinity) are needed as an intermediate step. Recall that incomplete information risk premia in Chapter 5 are typically evaluated at limiting probabilities.

A4.1.1 Multi-Period Transition Probabilities

A4.1.1.1 Formulation of the Problem

The probability of a transition from regime s at time t to s' at time $t+1$ is a model input (see Section 2.3.4) and denoted by $p_{ss'}$. However, not only transition probabilities for a single period but also multi-period transition probabilities,

$$P(S_{t+\tau} = s'|S_t = s)$$

are needed for the computation of $E_t(\pi_{t+\tau})$. These multi-period probabilities are not model inputs and must be derived.

A4.1.1.2 Results

If the $K \times K$ matrix (TP) denotes the one-period transition probabilities with entries $p_{ss'}$ in row s and column s', then it is well-known (see, e.g., Norris (2009), p. 4) that the τ -period transition probability $P(S_{t+\tau} = s'|S_t = s)$ is the entry in row s and column s' of the τ-th matrix power of (TP), $(TP)^\tau$.

A4.1.1.3 Proof

For $\tau = 1$ the assertion is correct by the definition of (TP). Inductively assume that the assertion is true for $\tau - 1$. The following recursion between τ-period and $\tau - 1$-period transition probabilities holds:

$$
\begin{aligned}
P(S_{t+\tau} = s'|S_t = s) &= \sum_{s''=1}^{K} P(S_{t+\tau} = s', S_{t+\tau-1} = s''|S_t = s) \\
&= \sum_{s''=1}^{K} P(S_{t+\tau} = s'|S_{t+\tau-1} = s'', S_t = s) \cdot P(S_{t+\tau-1} = s''|S_t = s) \\
&= \sum_{s''=1}^{K} P(S_{t+\tau} = s'|S_{t+\tau-1} = s'') \cdot P(S_{t+\tau-1} = s''|S_t = s) \\
&= \sum_{s''=1}^{K} p_{s''s'} \cdot P(S_{t+\tau-1} = s''|S_t = s)
\end{aligned}
$$

By inductive assumption, $P(S_{t+\tau-1} = s''|S_t = s), s'' = 1, \dots, K$ are the entries in row s of $(TP)^{\tau-1}$. $p_{s''s'}, s'' = 1, \dots, K$ are the entries in column s' of (TP). Hence $P(S_{t+\tau} = s'|S_t = s)$ is the element in row s and column s' of the matrix product $(TP)^{\tau-1}(TP) = (TP)^{\tau}$ and the assertion is true for τ.

A4.1.2 Expectations of Future Conditional Regime Probabilities

A4.1.2.1 Formulation of the Problem

What are expected conditional regime probabilities in some future point of time $t + \tau$ conditional on the information available at time t, i.e., how can $E(\pi_{s' t+\tau}|\pi_t)$ be computed?

A4.1.2.2 Results

The expectation of future conditional regime probabilities is

$$E(\pi_{s' t+\tau}|\pi_t) = \sum_{s=1}^{K} P(S_{t+\tau} = s'|S_t = s) \cdot \pi_{s,t}$$

where the τ-period transition probabilities are found in the matrix $(TP)^{\tau}$.

A4.1.2.3 Proof

The proof follows from the tower property of conditional expectations:

$$E(\pi_{s'}{}_{t+\tau}|\pi_t) = E(P(S_{t+\tau} = s'|I_{t+\tau})|I_t) = E\left(E\left(\underline{1}_{S_{t+\tau}=s'}\middle|I_{t+\tau}\right)\middle|I_t\right) = E\left(\underline{1}_{S_{t+\tau}=s'}\middle|I_t\right)$$

$$= P(S_{t+\tau} = s'|I_t) = \sum_{s=1}^{K} P(S_{t+\tau} = s'|S_t = s) \cdot \pi_{s,t}$$

where $\underline{1}_{S_{t+\tau}=s'}$ is the indicator function for the event $\{S_{t+\tau} = s'\}$ and I_t and $I_{t+\tau}$ denote (incomplete) information at times t and $t + \tau$.

A4.1.3 Steady-State (or Limiting) Regime Probabilities[52]

A4.1.3.1 Formulation of the Problem

The problem is to characterize the behavior of τ-period transition probabilities of a finite Markov chain as τ goes to infinity.

A4.1.3.2 Results

Under certain conditions, these probabilities converge to limiting probabilities as τ goes to infinity,

$$\overline{\pi}_{s'} = lim_{\tau\to\infty} P(S_{t+\tau} = s'|S_t = s)$$

$$s', s\epsilon\{1, \dots, K\}$$

and these limiting probabilities do not depend on the state at time t, $S_t = s$.

In the context of a finite Markov chain (which is relevant here), there exist unique limiting probabilities if the Markov chain is aperiodic and irreducible. In this case, the limiting probabilities are invariant probabilities.

[52] Various terms are in use for these probabilities; Norris (2009), p. 40, uses the term "convergence to equilibrium" for the limiting behavior of probabilities of a Markov chain.

A4.1.3.3 Concepts: Invariance of Regime Probabilities; Aperiodicity and Irreducibility of a Markov Chain

A4.1.3.3.1 Invariant Regime Probabilities

Limit probabilities are closely related to so-called invariant probabilities (also called stationary probabilities, see, e.g., Norris (2009), p. 33). To motivate the latter, consider regime probabilities π_t. What are the regime probabilities for time $t+1$ if the current probabilities are π_t? This question is answered by the following identities:

$$P(S_{t+1}=s'|I_t)=\sum_{s=1}^{K}P(S_{t+1}=s',S_t=s|I_t)=\sum_{s=1}^{K}P(S_{t+1}=s'|S_t=s,I_t)\cdot P(S_t=s|I_t)$$
$$=\sum_{s=1}^{K}P(S_{t+1}=s'|S_t=s)\cdot\pi_{s,t}=\sum_{s=1}^{K}p_{ss'}\cdot\pi_{s,t}$$

Since $p_{ss'}, s=1,\dots,K$ describe the entries in column s' of the transition probability matrix (TP), $\sum_{s=1}^{K}p_{ss'}\cdot\pi_{s,t}$ simply is the product of the row vector ${\pi_t}^T$ with column s' of (TP). In matrix notation, one therefore has:

$$(P(S_{t+1}=s'|I_t)\quad\dots\quad P(S_{t+1}=s'|I_t))={\pi_t}^T(TP)$$

Current regime probabilities π_t are invariant if the regime probabilities $P(S_{t+1}=s'|I_t), s'=1,\dots,K$ coincide with π_t. Formally, regime probabilities $\hat{\pi}$ are called invariant if they satisfy the condition

$$\hat{\pi}^T=\hat{\pi}^T(TP)$$

Note that if τ-step transition probabilities $P(S_{t+\tau}=s'|I_t)$ are considered and current probabilities $\pi_t$are invariant, one has

$$\big(P(S_{t+\tau}=1|I_t),\dots,P(S_{t+\tau}=K|I_t)\big)={\pi_t}^T\cdot(TP)^\tau={\pi_t}^T$$

If regime probabilities are invariant, the probability that the regime will be s' in some future point of time is equal to the probability that the current regime is s'.

A4.1.3.3.2 Aperiodicity of a Markov Chain

To understand the relevance of "aperiodicity", first consider its opposite, a periodic regime chain, and observe why limiting probabilities cannot exist for such regime chains: The most common such example (e.g., Norris (2009), p. 40) is a two-state Markov chain with transition probabilities

$$(TP)=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

Thus if the current regime is $S_t=1$, then the state in the next period will be $S_t=2$ with certainty, and the regime in two periods will again be $S_{t+2}=1$. If the current regime is $S_t=1$ and the degenerate regime probabilities ${\pi_t}^T = (1,0)$ are considered (i.e., the regime is known), it is clear that the limit$\overline{\pi}_{s'} = lim_{\tau\to\infty} P(S_{t+\tau} = s'|S_t = s)$ does not exist.

A state s of a Markov chain is called aperiodic if there exists a τ_0 such that $P(S_{t+\tau} = s|S_t = s) > 0$ for $\tau > \tau_0$ (see Norris (2009), p. 40). This condition means that, except for a finite number τ_0 of periods, it is impossible to know with certainty that the current regime will not be the future regime at some predefined (i.e., non-stochastic) future point of time $t + \tau$. In the two-state example with alternating regimes, neither of the two regimes is aperiodic because the regime will always be regime 2 for odd time leads τ and regime 1 for even time leads τ.

The entire Markov chain (as opposed to individual regimes) is called aperiodic if all regimes are aperiodic.

A4.1.3.3.3 Irreducibility of a Markov Chain

To define the concept of “irreducibility”, an equivalence relation on the set of all regimes must be defined first. The set of regimes $s = 1, \dots, K$ can be divided into equivalence classes by the following equivalence relation: two regimes s and s' are equivalent if the probability of reaching regime s' from regime s is positive, and if the probability of reaching regime s from regime s' is also positive. If there is only one equivalence class, i.e., if it is always possible that the regime switches from any current state s into any state s' at some future point of time, the regime chain is called “irreducible” (see Norris (2009), p. 11). An example of the opposite case, a “reducible” regime chain, would be

$$\begin{pmatrix} 0.7 & 0.3 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 0.8 & 0.2 \\ 0 & 0 & 0.2 & 0.8 \end{pmatrix}$$

where it is not possible to observe regimes 3 or 4 at some future point of time if the current regime is 1 or 2, and vice versa.

A4.1.3.4 Proof

If the Markov chain is finite, aperiodic, and irreducible, then there exist unique invariant probabilities (see Häggström (2002), p. 37, Theorem 5.3).

If a Markov chain is aperiodic, irreducible, and if invariant probabilities exist, then the multi-period transition probabilities $P(S_{t+1} = s'|S_t = s), s' = 1, \dots, K$, converge to these invariant probabilities (see Norris (2009), p. 41, Theorem 1.8.3). Since only the case of a finite Markov chain is relevant to my model, aperiodicity and irreducibility are sufficient conditions.

A4.1.4 Application of Conditions for Convergence to Limiting Probabilities to the Regime Chains in Chapter 5

Aperiodicity

All examples considered in the numerical analysis (Chapter 5) are aperiodic. To see this, consider separately the three cases $p_{Draw} = 0$, and $0 < p_{Draw} < 1$, and $p_{Draw} = 1$:

If $p_{Draw} = 0$, the regime remains the same in every period, i.e., $P(S_{t+\tau} = s|S_t = s) = 1$ for all τ and all regimes s.

If $0 < p_{Draw} < 1$ and if s is an arbitrary regime at time t, there always is a positive probability that regime s also is the regime at time $t+\tau$: $P(S_{t+\tau} = s|S_t = s) > (1 - p_{Draw})^{\tau} > 0$: $(1 - p_{Draw})^{\tau}$ is the positive probability for the event that no new regime is drawn over the next τ periods, (note that $P(S_{t+\tau} = s|S_t = s)$ is the sum of the probabilities that no new regime is drawn, i.e., $(1 - p_{Draw})^{\tau}$, and the probability that a new regime $s' \neq s$ is drawn at some point of time between t and $t+\tau$ but that the regime at time $t+\tau$ again is s).

If $p_{Draw} = 1$, a regime is drawn in every period. In my specification, conditional transition probabilities are positive for all regimes. If ctp_s is the (positive) conditional transition probability for regime s, then regime s is drawn in every period until time $t+\tau$ with positive (although possibly very small) probability ${ctp_s}^{\tau}$. This implies $P(S_{t+\tau} = s|S_t = s) \geq {ctp_s}^{\tau} > 0$.

Irreducibility

There are two cases:

If $p_{Draw} = 0$, the regime chain is not irreducible (with the exception of the trivial case with only one single regime): since there are no regime switches, each regime forms its own equivalence class, and the regime chain is not irreducible (recall that irreducibility, by definition, means that there is only one equivalence class).

If $p_{Draw} > 0$, the regime is irreducible: if s and s' are any two regimes, there is a positive probability of reaching s' from regime s (and vice versa) because a drawing of regimes occurs with positive probability and conditional transition probabilities are positive for all regimes.

Conclusion Regarding Convergence to Limiting Probabilities

For the case $p_{Draw} > 0$, unique limiting probabilities exist for the models considered in Chapter 5.

For the case $p_{Draw} = 0$ (where the theorems of Appendix A4.1.3.4 are not applicable because the condition of irreducibility is not met), limiting probabilities also do exist because τ-period transition probabilities obviously are

$$P(S_{t+\tau} = s'|S_t = s) = \begin{cases} 1 & s = s' \\ 0 & s \neq s' \end{cases}$$

implying that the limiting probabilities are

$$lim_{\tau\to\infty}P(S_{t+\tau}=s'|S_t=s)=\begin{cases}1 & s=s'\\ 0 & s\neq s'\end{cases}$$

Note, however, that these probabilities depend on the current regime s and, hence, are not unique.

A4.2 Appendix to Section 5.4: Numerical Aspects

A4.2.1 Appendix to Section 5.4.3.2: Iterative Computation of Equilibrium Price Dividend Ratios

A4.2.1.1 Computation of Price Dividend Ratios for a Finite Remaining Time Horizon

A4.2.1.1.1 Formulation of the Problem

The problem is to show that complete information price dividend ratios for a remaining time horizon of n periods,

A 4-1

$$\left(\frac{P}{D}\right)_i^{(n)}(s,\delta_1)=\sum_{\tau=1}^{n}E\left(\begin{matrix}q_{t,t+\tau}{}^{cI}\left(fe_{t+1,t+\tau},S_{t+1,t+\tau-1};S_t,\delta_{1,t}\right)\\ \cdot\left[1+d_i{}^{+\tau}\left(S_{t,t+\tau-1},fe_{t+1,t+\tau}\right)\right]\end{matrix}\middle|S_t=s,\delta_{1,t}=\delta_1\right)$$

can be recursively computed by the following algorithm:

Starting from

$$\left(\frac{P}{D}\right)_i^{(0)}(s,\delta_1)\equiv 0$$

recursively define

$$\left(\frac{P}{D}\right)_i^{(n+1)}(s,\delta_1)=E\left(\begin{matrix}q_{t,t+1}{}^{cI}\left(fe_{t+1},S_t,\delta_{1,t}\right)\cdot\left[1+d_i(S_t,fe_{t+1})\right]\\ \cdot\left\{\sum_{s'=1}^{K}p_{S_ts'}\cdot\left(\frac{P}{D}\right)_i^{(n)}\left(s',\delta\left(\delta_{1,t},S_t,fe_{t+1}\right)\right)+1\right\}\end{matrix}\middle|S_t=s,\delta_{1,t}=\delta_1\right)$$

with

$$q_{t,t+1}{}^{cI}\left(fe_{t+1},S_t,\delta_{1,t}\right)=\frac{1}{1+\rho}\cdot\left\{\delta_{1,t}\cdot\left(1+d_1(S_t,fe_{t+1})\right)+\left(1-\delta_{1,t}\right)\cdot\left(1+d_2(S_t,fe_{t+1})\right)\right\}^{-\gamma}$$

A4.2.1.1.2 Proof

A4.2.1.1.2.1 Idea of the Proof

The identity (A 4-1) is shown by induction over n.

A4.2.1.1.2.2 Details of the Proof

Base case: $n = 0$

For $n = 0$, $\left(\frac{P}{D}\right)_i^{(0)}(s, \delta_1)$ is zero by definition.

Since $\sum_{\tau=1}^{0} E\left(\begin{matrix} q_{t,t+\tau}{}^{cI}\left(fe_{t+1,t+\tau}, S_{t+1,t+\tau-1}; S_t, \delta_{1,t}\right) \\ \cdot\left[1 + d_i{}^{+\tau}\left(S_{t,t+\tau-1}, fe_{t+1,t+\tau}\right)\right] \end{matrix}\middle| S_t = s, \delta_{1,t} = \delta_1\right)$ is an "empty sum" and thus zero, (A 4-1) is true for $n = 0$.

Inductive step

Inductively assume that (A 4-1) is true for n. The following steps show that (A 4-1) then also holds for $n + 1$:

Step 1: Definition of $\left(\frac{P}{D}\right)_i^{(n+1)}$

$$\left(\frac{P}{D}\right)_i^{(n+1)}(s, \delta_1) = E\left(\begin{matrix} q_{t,t+1}{}^{cI}\left(fe_{t+1}, S_t, \delta_{1,t}\right) \cdot \left[1 + d_i(S_t, fe_{t+1})\right] \\ \cdot\left\{\sum_{s'=1}^{K} p_{S_t s'} \cdot \left(\frac{P}{D}\right)_i^{(n)}\left(s', \delta\left(\delta_{1,t}, S_t, fe_{t+1}\right)\right) + 1\right\} \end{matrix}\middle| S_t = s, \delta_{1,t} = \delta_1\right)$$

Step 2: Separate the previous equation into two summands

A 4-2

$$= E\left(\begin{matrix} q_{t,t+1}{}^{cI}\left(fe_{t+1}, S_t, \delta_{1,t}\right) \cdot \left[1 + d_i(S_t, fe_{t+1})\right] \\ \cdot\left\{\sum_{s'=1}^{K} p_{S_t s'} \cdot \left(\frac{P}{D}\right)_i^{(n)}\left(s', \delta\left(\delta_{1,t}, S_t, fe_{t+1}\right)\right)\right\} \end{matrix}\middle| S_t = s, \delta_{1,t} = \delta_1\right)$$
$$+ E\left(q_{t,t+1}{}^{cI}\left(fe_{t+1}, S_t, \delta_{1,t}\right) \cdot \left[1 + d_i(S_t, fe_{t+1})\right] \middle| S_t = s, \delta_{1,t} = \delta_1\right)$$

Consider only the first term in (A 4-2) in the next few steps,

A 4-3

$$E\left(\begin{matrix} q_{t,t+1}{}^{cI}\left(fe_{t+1}, S_t, \delta_{1,t}\right) \cdot \left[1 + d_i(S_t, fe_{t+1})\right] \\ \cdot\left\{\sum_{s'=1}^{K} p_{S_t s'} \cdot \left(\frac{P}{D}\right)_i^{(n)}\left(s', \delta\left(\delta_{1,t}, S_t, fe_{t+1}\right)\right)\right\} \end{matrix}\middle| S_t = s, \delta_{1,t} = \delta_1\right)$$

Step 3: Simplify the expectation in (A 4-3)

$$E\left(\begin{matrix} q_{t,t+1}{}^{cI}\left(fe_{t+1},S_t,\delta_{1,t}\right)\cdot[1+d_i(S_t,fe_{t+1})] \\ \cdot\sum_{s'=1}^{K} p_{S_ts'}\cdot\left(\frac{P}{D}\right)_i^{(n)}\left(s',\delta\left(\delta_{1,t},S_t,fe_{t+1}\right)\right)\end{matrix}\middle| S_t=s,\delta_{1,t}=\delta_1\right)$$

$$=E\left(\begin{matrix} q_{t,t+1}{}^{cI}\left(fe_{t+1},S_t,\delta_{1,t}\right)\cdot[1+d_i(S_t,fe_{t+1})] \\ \cdot E\left(\left(\frac{P}{D}\right)_i^{(n)}\left(S_{t+1},\delta\left(\delta_{1,t},S_t,fe_{t+1}\right)\right)\middle|\begin{matrix} S_{t+1},\delta_{1,t+1} \\ S_t=s,\delta_{1,t}=\delta_1\end{matrix}\right)\end{matrix}\middle| S_t=s,\delta_{1,t}=\delta_1\right)$$

$$=E\left(E\left(\begin{matrix} q_{t,t+1}{}^{cI}\left(fe_{t+1},S_t,\delta_{1,t}\right)\cdot[1+d_i(S_t,fe_{t+1})] \\ \cdot\left(\frac{P}{D}\right)_i^{(n)}\left(S_{t+1},\delta\left(\delta_{1,t},S_t,fe_{t+1}\right)\right)\end{matrix}\middle|\begin{matrix} S_{t+1},\delta_{1,t+1} \\ S_t=s,\delta_{1,t}=\delta_1\end{matrix}\right)\middle| S_t=s,\delta_{1,t}=\delta_1\right)$$

$$=E\left(\begin{matrix} q_{t,t+1}{}^{cI}\left(fe_{t+1},S_t,\delta_{1,t}\right)\cdot[1+d_i(S_t,fe_{t+1})] \\ \cdot\left(\frac{P}{D}\right)_i^{(n)}\left(S_{t+1},\delta\left(\delta_{1,t},S_t,fe_{t+1}\right)\right)\end{matrix}\middle| S_t=s,\delta_{1,t}=\delta_1\right)$$

Step 4: Use the inductive hypothesis (i.e., (A 4-1) holds for $\left(\frac{P}{D}\right)_i^{(n)}$)

$$=E\left(\begin{matrix} q_{t,t+1}{}^{cI}\left(fe_{t+1},S_t,\delta_{1,t}\right)\cdot[1+d_i(S_t,fe_{t+1})] \\ \cdot\sum_{\tau=1}^{n} E\left(\begin{matrix} q_{t+1,t+1+\tau}{}^{cI}\left(fe_{t+2,t+1+\tau},S_{t+2,t+1+\tau-1};S_{t+1},\delta_{1,t+1}\right) \\ \cdot\left[1+d_i{}^{+\tau}\left(S_{t+1,t+1+\tau-1},fe_{t+2,t+1+\tau}\right)\right]\end{matrix}\middle| S_{t+1},\delta_{1,t+1}\right)\end{matrix}\middle| S_t=s,\delta_{1,t}=\delta_1\right)$$

Step 5: Augment the condition in the inner expectation by the (redundant) information $S_t=s,\delta_{1,t}=\delta_1,d_1,d_2$ (d_1,d_2 are the dividend growth rates of assets 1 and 2 over $[t,t+1]$)

$$=E\left(\begin{matrix} q_{t,t+1}{}^{cI}\left(fe_{t+1},S_t,\delta_{1,t}\right)\cdot[1+d_i(S_t,fe_{t+1})] \\ \cdot\sum_{\tau=1}^{n} E\left(\begin{matrix} q_{t+1,t+1+\tau}{}^{cI}\left(fe_{t+2,t+1+\tau},S_{t+2,t+1+\tau-1};S_{t+1},\delta_{1,t+1}\right) \\ \cdot\left[1+d_i{}^{+\tau}\left(S_{t+1,t+1+\tau-1},fe_{t+2,t+1+\tau}\right)\right]\end{matrix}\middle|\begin{matrix} S_{t+1},\delta_{1,t+1} \\ ,S_t=s,\delta_{1,t}=\delta_1 \\ ,d_1,d_2\end{matrix}\right)\end{matrix}\middle|\begin{matrix} S_t=s \\ ,\delta_{1,t}=\delta_1\end{matrix}\right)$$

Step 6: Use the fact that $q_{t,t+1}{}^{cI}$ and $1+d_i$ are non-stochastic conditional on the information in the inner expectation, also use $q_{t,t+1}{}^{cI}\cdot q_{t+1,t+1+\tau}{}^{cI}=q_{t,t+1+\tau}{}^{cI}$ and $[1+d_i]\left[1+d_i{}^{+\tau}\right]=\left[1+d_i{}^{+(\tau+1)}\right]$

$$=E\left(\sum_{\tau=1}^{n} E\left(\begin{matrix} q_{t,t+1+\tau}{}^{cI}\left(fe_{t+1,t+1+\tau},S_{t+1,t+1+\tau-1};S_t,\delta_{1,t}\right) \\ \cdot\left[1+d_i{}^{+(\tau+1)}\left(S_{t,t+1+\tau-1},fe_{t+1,t+1+\tau}\right)\right]\end{matrix}\middle|\begin{matrix} S_{t+1},\delta_{1,t+1} \\ ,S_t=s,\delta_{1,t}=\delta_1 \\ ,d_1,d_2\end{matrix}\right)\middle|\begin{matrix} S_t=s \\ ,\delta_{1,t}=\delta_1\end{matrix}\right)$$

Step 7: Use the tower property of conditional expectations to remove the inner conditional expectation; write expectation of a sum as the sum of expectations

$$=\sum_{\tau=1}^{n} E\left(\begin{matrix} q_{t,t+1+\tau}{}^{cI}\left(fe_{t+1,t+1+\tau},S_{t+1,t+1+\tau-1};S_t,\delta_{1,t}\right) \\ \cdot\left[1+d_i{}^{+(\tau+1)}\left(S_{t,t+1+\tau-1},fe_{t+1,t+1+\tau}\right)\right]\end{matrix}\middle|\begin{matrix} S_t=s \\ ,\delta_{1,t}=\delta_1\end{matrix}\right)$$

Step 8: Rewrite the summation index

$$= \sum_{\tau=2}^{n+1} E\left(\begin{matrix} q_{t,t+\tau}{}^{cI}\left(fe_{t+1,t+\tau}, S_{t+1,t+\tau-1}; S_t, \delta_{1,t}\right) \\ \cdot \left[1 + d_i{}^{+\tau}\left(S_{t,t+\tau-1}, fe_{t+1,t+\tau}\right)\right] \end{matrix} \middle| \begin{matrix} S_t = s \\ , \delta_{1,t} = \delta_1 \end{matrix} \right)$$

Step 9: Plug the result from Step 8 into (A 4-2)

$$\left(\frac{P}{D}\right)_i^{(n+1)} (s, \delta_1)$$

$$= E\left(\begin{matrix} q_{t,t+1}{}^{cI}\left(fe_{t+1}, S_t, \delta_{1,t}\right) \cdot \left[1 + d_i(S_t, fe_{t+1})\right] \\ \cdot \left\{ \sum_{s'=1}^{K} p_{S_t s'} \cdot \left(\frac{P}{D}\right)_i^{(n)} \left(s', \delta\left(\delta_{1,t}, S_t, fe_{t+1}\right)\right) \right\} \end{matrix} \middle| S_t = s, \delta_{1,t} = \delta_1 \right)$$

$$+ E\left(q_{t,t+1}{}^{cI}\left(fe_{t+1}, S_t, \delta_{1,t}\right) \cdot \left[1 + d_i(S_t, fe_{t+1})\right] \middle| S_t = s, \delta_{1,t} = \delta_1\right)$$

$$= \sum_{\tau=2}^{n+1} E\left(\begin{matrix} q_{t,t+\tau}{}^{cI}\left(fe_{t+1,t+\tau}, S_{t+1,t+\tau-1}; S_t, \delta_{1,t}\right) \\ \cdot \left[1 + d_i{}^{+\tau}\left(S_{t,t+\tau-1}, fe_{t+1,t+\tau}\right)\right] \end{matrix} \middle| \begin{matrix} S_t = s \\ , \delta_{1,t} = \delta_1 \end{matrix} \right)$$

$$+ E\left(q_{t,t+1}{}^{cI}\left(fe_{t+1}, S_t, \delta_{1,t}\right) \cdot \left[1 + d_i(S_t, fe_{t+1})\right] \middle| S_t = s, \delta_{1,t} = \delta_1\right)$$

$$= \sum_{\tau=1}^{n+1} E\left(\begin{matrix} q_{t,t+\tau}{}^{cI}\left(fe_{t+1,t+\tau}, S_{t+1,t+\tau-1}; S_t, \delta_{1,t}\right) \\ \cdot \left[1 + d_i{}^{+\tau}\left(S_{t,t+\tau-1}, fe_{t+1,t+\tau}\right)\right] \end{matrix} \middle| \begin{matrix} S_t = s \\ , \delta_{1,t} = \delta_1 \end{matrix} \right)$$

The identity of the first and last term in the previous equations shows that (A 4-1) holds for $n + 1$.

A4.2.1.2 Threshold for Approximate Convergence

A4.2.1.2.1 Formulation of the Problem

The limit of complete information price dividend ratios $\left(\frac{P}{D}\right)_i^{(n)} (s, \delta_1)$ (i.e., of (A 4-1)) as n goes to infinity can only be determined approximately. Hence, a stop criterion must be defined for the numerical computation.

A4.2.1.2.2 Solution

I iterate the complete information price dividend ratios $\left(\frac{P}{D}\right)_i^{(n)} (s, \delta_1)$ until a distance between $\left(\frac{P}{D}\right)_i^{(n+1)}$ and $\left(\frac{P}{D}\right)_i^{(n)}$ falls below a threshold. This distance is defined in two steps because both relative dividend contributions and regimes must be taken into account:

Step 1: Given regime s, the distance between iterations n and $n+1$ is defined as

$$dist(n,s) = \int_0^1 \left| \left(\frac{P}{D}\right)_i^{(n+1)} (s,x) - \left(\frac{P}{D}\right)_i^{(n)} (s,x) \right| dx$$

i.e., the L_1-norm of the function $\left(\frac{P}{D}\right)_i^{(n+1)} (s,.) - \left(\frac{P}{D}\right)_i^{(n)} (s,.)$ quantifies differences coming from relative dividend contributions.

Step 2: The total distance over all regimes is then defined as the maximum of these distances over all regimes, i.e.,

$$dist(n) = max\{dist(n,s), s = 1, \dots, K\}$$

L_∞-norm over the L_1-norms is taken.

The iterations stops as soon as $dist(n)$ falls below the threshold of 10^{-6}.

A4.2.2 Appendix to Section 5.4.3.2: Integration by Gaussian Quadrature

A4.2.2.1 Motivating Example and Formulation of the Problem

To illustrate integration by Gaussian quadrature, consider the evaluation of the following simple integral (which has an analytical solution) as an example:

$$E\left(\left(1 + d(s,e)\right)^{1-\gamma}\right)$$

with

$$1 + d(s, fe) = exp(\mu(s) + b(s)e)$$

where e is a univariate standard normal random variable.

For illustration purposes, assume $\gamma = 2$, $\mu(s) = 0.1$, $b(s) = 0.02$. Expressed as an integral, this expectation reads

$$E\left(\left(1 + d(s,e)\right)^{-\gamma} \cdot \left(1 + d(s,e)\right)\right) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot exp\left(-\frac{1}{2}x^2\right) \cdot exp\big((1-\gamma)\{0.1 + 0.02x\}\big)dx$$

By the assumption of normality and from the well-known identity for normal random variables X $E\big(exp(X)\big) = exp\big(E(X) + 0.5 \cdot Var(X)\big)$ the analytical solution is

$$exp\big((1-\gamma) \cdot \mu(s) + 0.5 \cdot (1-\gamma)^2 \cdot b(s)^2\big) \approx 0.905018$$

I use this integral to illustrate integration by Gaussian quadrature.

More generally, various integrals of the form

A 4-4

$$\int_a^b \omega(x) \cdot h(x) dx$$

with

$$\omega(x) = exp(-0.5 \cdot x^2)$$
$$a = -\infty$$
$$b = -\infty$$

where $h(x)$ is some continuous function, have to be evaluated numerically.

In the motivating example, $h(x) = \frac{1}{\sqrt{2\pi}} \cdot exp\big((1-\gamma)\{0.1 + 0.02e\}\big)$, but h will typically be more complicated (and not admit an analytical solution). A more relevant example is the computation of the expected return of an asset under incomplete information,

$$\sum_{s=1}^{K} \pi_{s,t} \cdot E\left([1 + d(s,e)] \cdot \frac{\sum_{s'=1}^{K} \pi_{s',t+1}\big(\pi_t, d(s,e)\big) \cdot \left(\frac{P}{D}\right)(s')}{\sum_{s''=1}^{K} \pi_{s'',t} \cdot \left(\frac{P}{D}\right)(s'')} \right)$$

where e is univariate standard normal, $\pi_{t+1,s'}\big(\pi_t, d(s,e)\big)$ is the probability of regime s' as a function of current regime probabilities π_t and dividend growth $d(s,e)$, and $\left(\frac{P}{D}\right)(k)$ is the complete information price dividend ratio in regime $k\epsilon\{1, \dots, K\}$ (assumed to be known at this stage from prior numerical computations).

In this example, one wishes to evaluate the K expectations

$$E\left([1 + d(s,e)] \cdot \frac{\sum_{s'=1}^{K} \pi_{s',t+1}\big(\pi_t, d(s,e)\big) \cdot \left(\frac{P}{D}\right)(s')}{\sum_{s''=1}^{K} \pi_{s'',t} \cdot \left(\frac{P}{D}\right)(s'')} \right)$$

with the function h_s given by

$$h_s(x) = \frac{1}{\sqrt{2\pi}} \cdot [1 + d(s,e)] \cdot \frac{\sum_{s'=1}^{K} \pi_{t+1,s'}\big(\pi_t, d(s,e)\big) \cdot \left(\frac{P}{D}\right)(s')}{\sum_{s'=1}^{K} \pi_{t,s'} \cdot \left(\frac{P}{D}\right)(s)}$$

A4.2.2.2 Solution

A4.2.2.2.1 Gaussian Quadrature (of the Hermite-type with Probabilist Weight Function)

Problems of the form (A 4-4) can be approximately computed by Gaussian quadrature. The function ω is called the “weight” function and can take various forms (the “probabilist” weight function $\omega(x) = exp(-0.5 \cdot x^2)$ is relevant in my setting). The points a and b are typically finite, but can be infinite. In combination with $a = -\infty$, $b = \infty$, $\omega(x) = exp(-0.5 \cdot x^2)$, Gaussian quadrature is referred to as Gauss-Hermite quadrature with probabilist weight function.

As in simple integration rules (such as the Newton and Cotes formulae), (A 4-4) is approximated by a sum

$$\sum_{i=1}^{n_G} w_i \cdot h(x_i)$$

where the values x_i are called "abscissae" and the values w_i are referred to as "weights", but the abscissae must neither form an equidistant partition of the interval $[a, b]$, nor does the interval $[a, b]$ have to be finite. Instead, abscissae and weights are chosen "optimally" (see Stoer/Bulirsch (2000), p. 150 for an explanation of the sense of optimality).

A4.2.2.2.2 Abscissae and Weights for Gauss-Hermite Quadrature with the Probabilist Weight Function

32 pairs of abscissae (x_i) and weight (w_i) were obtained from Burkardt for the case of Gauss-Hermite quadrature with probabilist weight function and can be found in the following table:[53]

	1	2	3	4	5	6	7	8
x_i	-7.12581391	-6.40949815	-5.81222595	-5.27555099	-4.7771645	-4.30554795	-3.85375549	-3.41716749
w_i	7.76411E-12	7.69226E-10	2.59347E-08	4.65928E-07	5.35646E-06	4.3459E-05	0.00026424	0.00125314
	9	10	11	12	13	14	15	16
x_i	-2.99249083	-2.57724954	-2.16949918	-1.76765411	-1.37037641	-0.97650046	-0.58497877	-0.19484074
w_i	0.004767375	0.014849809	0.03845272	0.083726022	0.154611703	0.24367641	0.32923447	0.38242895
	17	18	19	20	21	22	23	24
x_i	0.194840742	0.584978765	0.976500464	1.370376411	1.767654109	2.16949918	2.57724954	2.99249083
w_i	0.382428951	0.329234467	0.243676406	0.154611703	0.083726022	0.03845272	0.01484981	0.00476738
	25	26	27	28	29	30	31	32
x_i	3.417167493	3.853755485	4.305547953	4.777164504	5.275550987	5.81222595	6.40949815	7.12581391
w_i	0.001253136	0.00026424	4.3459E-05	5.35646E-06	4.65928E-07	2.5935E-08	7.6923E-10	7.7641E-12

Table A 4-1: Abscissas and weights for Gauss-Hermite Quadrature with the probabilist weight function from Burkardt

[53] The values in the table have been rounded to eight digits and are somewhat less accurate than the full digits.

A4.2.2.2.3 Illustration for the Motivating Example

Consider again the motivating example,

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot exp\left(-\frac{1}{2}x^2\right) \cdot exp\big((1-2)\{0.1 + 0.02x\}\big)dx$$

This integral is approximated by

$$\sum\nolimits_{i=1}^{32} w_i \cdot \frac{1}{\sqrt{2\pi}} \cdot exp\big((1-2)\{0.1 + 0.02x_i\}\big)$$

(with w_i and x_i found in the table, note that the term $exp\left(-\frac{1}{2}x^2\right)$ is included in the weights).

This approximation only differs from the analytical solution from the 11th digit on and is sufficiently accurate for the present context.

A4.2.3 Appendix to Section 5.5.1.2.2.1: Approximation of the Standard Deviation of the Adjustment for Risk by the Product of the Standard Deviation of Dividend Growth and the Risk Aversion Parameter γ

This section demonstrates that the standard deviation of the adjustment for risk can reasonably be approximated by the product of the standard deviation of dividend growth and the risk aversion parameter γ. The following table contains standard deviations of the adjustment for risk and the approximation for the discretized Veronesi model (both for $\sigma_D = 0.01$ and $\sigma_D = 0.1$) under incomplete information as an illustrative example. Various values for the risk aversion parameter and the level of conditional standard deviation are considered (note that p_{Draw} and ρ do not affect the standard deviation of the adjustment for risk). The exact standard deviations of the adjustment for risk can be found in the left part of the table; the approximation can be found in the right part and are based on the standard deviations of dividend growth which can be obtained by combining regime probabilities π_t, the parameter σ_D, and expectation regimes $E\left(\frac{D_{t+1}}{D_t}\middle|S_t = s\right), s = 1, \dots, K$ by the following identity:

$$Stddev(1 + d|\pi_t) = \sqrt{\sigma_D{}^2 + Var\left(E\left(\frac{D_{t+1}}{D_t}\middle|S_t\right)\middle|\pi_t\right)}$$

	exact standard deviation of the adjustment for risk (incomplete information)					approximation: product of standard deviation of dividend growth and the risk aversion parameter γ				
$\sigma_D = 0.01$		$\sigma_{means} = 0.005$	$\sigma_{means} = 0.01$	$\sigma_{means} = 0.025$	uni-form		$\sigma_{means} = 0.005$	$\sigma_{means} = 0.01$	$\sigma_{means} = 0.025$	uni-form
						Stdev Div. Growth	1.15%	1.44%	2.62%	3.32%
	$\gamma = 0.5$	0.55%	0.69%	1.25%	1.58%	$\gamma = 0.5$	0.57%	0.72%	1.31%	1.66%
	$\gamma = 1$	1.10%	1.38%	2.50%	3.16%	$\gamma = 1$	1.15%	1.44%	2.62%	3.32%
	$\gamma = 2$	2.19%	2.75%	5.00%	6.33%	$\gamma = 2$	2.29%	2.88%	5.23%	6.63%
	$\gamma = 3$	3.29%	4.13%	7.51%	9.50%	$\gamma = 3$	3.44%	4.32%	7.85%	9.95%
	$\gamma = 5$	5.49%	6.89%	12.54%	15.85%	$\gamma = 5$	5.73%	7.20%	13.08%	16.58%
$\sigma_D = 0.1$		$\sigma_{means} = 0.005$	$\sigma_{means} = 0.01$	$\sigma_{means} = 0.025$	uni-form	$\sigma_{means} = 0.005$	$\sigma_{means} = 0.01$	$\sigma_{means} = 0.025$	uni-form	$\sigma_{means} = 0.005$
						Stdev Div. Growt h	10.02%	10.05%	10.29%	10.49%
	$\gamma = 0.5$	4.76%	4.78%	4.90%	5.00%	$\gamma = 0.5$	5.01%	5.03%	5.14%	5.24%
	$\gamma = 1$	9.54%	9.58%	9.83%	10.04%	$\gamma = 1$	10.02%	10.05%	10.29%	10.49%
	$\gamma = 2$	19.21%	19.30%	19.83%	20.28%	$\gamma = 2$	20.03%	20.11%	20.58%	20.98%
	$\gamma = 3$	29.16%	29.30%	30.16%	30.89%	$\gamma = 3$	30.05%	30.16%	30.86%	31.46%
	$\gamma = 5$	50.45%	50.73%	52.50%	53.97%	$\gamma = 5$	50.08%	50.27%	51.44%	52.44%

Table A 4-2: Standard deviations of the adjustment for risk and the approximation for the discretized Veronesi model (both for $\boldsymbol{\sigma_D = 0.01}$ and $\boldsymbol{\sigma_D = 0.1}$) under incomplete information

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