Diagonalizability of Elements of a Group Algebra

Dissertation

zur Erlangung des Doktorgrades
der Naturwissenschaften
(Dr. rer. nat.)
dem Fachbereich Mathematik und Informatik
der Philipps-Universität Marburg
vorgelegt von

Hery Randriamaro
aus Antananarivo, Madagaskar

Marburg an der Lahn, April 2012

für meine Mutter

für meine geliebten verstorbenen Verwandten

Vom Fachbereich Mathematik und Informatik
der Philipps-Universität Marburg
als Dissertation angenommen am: 11. 4. 2012

Erstgutachter: Prof. Dr. Volkmar Welker
Zweitgutachter: Prof. Dr. Istvan Heckenberger

Tag der mündlichen Prüfung: 11. 4. 2012

Nomenclature

$[n]$	set of integer from 1 to $n > 0$
$[n]^*$	set of integer from 0 to $n > 0$
$\langle \{v_i\}_{i\in[j]} \rangle$	vector subspace spanned by the vectors $\{v_i\}_{i\in[j]}$
$\mathbb{C}$	complex numbers
$\mathbb{K}$	field of characteristic 0
$\mathbb{N}$	positive integers
$\mathbb{R}$	real numbers
$\mathsf{0}_n$	matrix with entry 0 of $\mathbb{K}^{n^2}$
I_n	identity matrix of $\mathbb{K}^{n^2}$
$\varnothing$	empty set
$diag\big((\mathsf{A}_i)_{i\in[k]}\big)$	diagonal block of the k matrices A_i
e	neutral element of group
G	group
$o(g)$	order of the element g of a group G
$Sp(\mathsf{A})$	spectrum of the linear map A
$tr(\mathsf{A})$	trace of the matrix A
$V_{\mathsf{A}}(\Lambda)$	multiplicity of the eigenvalue Λ of the linear map A

Danksagung

Meiner Mutter, die mich stetig dazu ermutigt hat, genug Ergebnisse für meine Doktorarbeit herauszufinden. Aus ihrer Ermutigung kam zweifellos die notwendige Kraft für meine Promotion.
Meinem Professor Volkmar Welker. Unter seiner Betreuung hat sich mein Geist entfaltet und sich meine Denkfähigkeit verbreitert. Danke für die Lehre auf ausgezeichnetem Niveau, für die Verständnis gegenüber meinen skurillsten Vermutungen. Und, über alles, danke für die Unterstützung auch in schlechten Zeiten.
Dem DAAD natürlich. Ohne seine Finanzierung hätte ich wahrscheinlich nie die Möglichkeit gehabt, diese einzigartige Promotionszeit in Deutschland zu erleben.
Meinen madagassischen Freunden, mit wem ich meine Freizeit verbracht habe. Jede Unternehmung mit ihnen war ein echtes Vergnügen.
Meinen Kollegen, die mir viel geholfen haben.
Meiner Familie für die moralische Unterstützung.

Zusammenfassung

Wir schreiben G für eine endliche Gruppe und $\mathbb{K}$ für einen Körper der Charakteristik 0 und algebraisch geschlossen. Sei $\mathfrak{f} = \sum_{g \in G} \lambda_g \, g$ ein Element der Gruppenalgebra $\mathbb{K}[G]$, mit $\lambda_g \in \mathbb{K}$. Die Abbildung $\mathsf{X}_{\mathbb{K}[G]} : \mathbb{K}[G] \to \mathbb{K}^{|G| \times |G|}$ bildet $\mathfrak{f}$ auf die Matrix $\mathsf{X}_{\mathbb{K}[G]}(\mathfrak{f}) = (\lambda_{gg'^{-1}})_{g,\, g' \in G}$ der Linksmultiplikationsaktion von $\mathfrak{f}$ auf $\mathbb{K}[G]$ ab. Unser Ziel ist es, die Eigenwerte von $\mathsf{X}_{\mathbb{K}[G]}(\mathfrak{f})$ und deren Multiplizitäten für spezielle Gruppen und für bestimmte Werte von λ_g zu bestimmen.
Wir beginnen in **Kapitel 1** mit Grundlagen der Darstellungstheorie. Insbesondere werden wir mehr über die Abbildung X_M, die $\mathfrak{f}$ zu der Matrix $\mathsf{X}_M(\mathfrak{f})$ der Linksmultiplikation von $\mathfrak{f}$ auf dem endlich dimensionalen $\mathbb{K}[G]$ Modul M abbildet, erfahren. Im selben Kapitel werden auch Grundlagen von Coxeter Gruppen dargestellt.
Dann bearbeiten wir vollständig die Fälle von abelschen Gruppen und Diedergruppen in **Kapitel 2**. Sei $\{G_i\}_{\{i \in [k]\}}$ die Menge der Konjugationsklassen von G und $g_{G_i} = \sum_{g \in G_i} g$ die Summe der Elemente von G_i. Wir schreiben $\mathbb{K}[\mathscr{C}_G]$ für die Menge der Elemente $\mathfrak{f}$ von $\mathbb{K}[G]$ derart, dass $\mathfrak{f} = \sum_{i=1}^{k} \lambda_i \, g_{G_i}$, mit $\lambda_i \in \mathbb{K}$. Bekanntlich ist $\mathbb{K}[\mathscr{C}_G]$ das Zentrum von $\mathbb{K}[G]$.
Wir beenden das Kapitel mit dem folgenden Satz, der eine direkte Konsequenz von grundlegen Aussagen der Darstellungstheorie ist.

Satz 1. *Seien* $\mathfrak{c}_G = \sum_{i=1}^{k} \lambda_i \, g_{G_i} \in \mathbb{K}[\mathscr{C}_G]$ *und* $(d_i)_{i \in [k]}$ *die Dimensionen der irreduziblen Module* $(M_i)_{i \in [k]}$ *von* $\mathbb{K}[G]$*. Dann ist die Matrix* $\mathsf{X}_{\mathbb{K}[G]}(\mathfrak{c}_G)$ *diagonalisierbar und*

$$Sp\big(\mathsf{X}_{\mathbb{K}[G]}(\mathfrak{c}_G)\big) = \Big\{ \Lambda_i = \frac{\sum_{j=1}^{k} \lambda_j |G_j| \chi_{M_i}^{G_j}}{d_i} \Big\}_{i \in [k]}$$

mit

$$\big\{ V_{\mathfrak{c}_G}(\Lambda_i) = d_i^2 \big\}_{i \in [k]}$$

worin $\chi_{M_i}^{G_j}$ *der gemeinsame Wert von* $tr\big(\mathsf{X}_{M_i}(g_j)\big)$ *für* $g_j \in G_j$ *ist.*

Wir können zum Beispiel Satz 1 benutzen, um die Eigenwerte und Multiplizitäten der Matrix der Multiplikation von

$$\mathfrak{f}_n = \sum_{\sigma \in \mathcal{S}_n} \big(\mathtt{fix}(\sigma) t + q^{\mathtt{fix}(\sigma)} \big) \, \sigma \in \mathbb{R}(t,q)[\mathscr{C}_{\mathcal{S}_n}]$$

auf $\mathbb{R}(t,q)[\mathcal{S}_n]$ zu berechnen. $\mathcal{S}_n$ ist dabei wie üblich die symmetrische Gruppe, t und q sind Variablen, und `fix` ist die Statistik

$$\mathtt{fix}(\sigma) := \#\{ i \in [n] \,|\, \sigma(i) = i \},$$

die konstant auf jeder Konjugationsklasse ist. Wir werden $\mathfrak{f}_n$ als Beispiel in **Kapitel 2** bearbeiten.
Satz 1 ist auf alle endlichen Gruppen G anwendbar. Nicht aber das folgende Theorem, das nur für endliche Coxeter Systeme (W,S) gilt. Wir werden in **Kapitel 3** sehen, dass dessen Beweis mehr Theorie voraussetzt und insbesondere Kenntnisse der Descent Algebra benötigt. Wir müssen daher zuerst die Definition der Descent Algebra $\mathbb{K}[\mathscr{D}_W]$ einer endlichen Coxeter Gruppe geben. Wir schreiben $\mathbf{C}_P$ für die Menge der Untermengen J von S derart, dass die parabolischen Untergruppen W_P und W_J von W konjugiert sind. Sei $\{\mathbf{C}_{P_i}\}_{\{i \in [p]\}}$ die Menge, die die folgenden Eigenschaften erfüllt:

$$\{\mathbf{C}_{P_i}\}_{i \in [p]} = \{\mathbf{C}_P\}_{P \subseteq S},$$

$$\mathbf{C}_{P_i} \neq \mathbf{C}_{P_j} \text{ wenn } i \neq j.$$

Seien $J_i \in \mathbf{C}_{P_i}$ und $J_j \in \mathbf{C}_{P_j}$. Wir fahren mit der Definition der Koeffizienten $a_{ij} = a_{J_i J_j J_j}$ fort, die von Solomon eingeführt wurden [21, Theorem 1]. Dann bestimmen wir eine Formel für die Koeffizienten a_{ij}. Wir schreiben $\mathbf{C}_{s_{J_i}}$ für die Konjugationsklasse des Elementes $s_{J_i} := \prod_{s\in J_i} s \in W$, mit $J_i \in \mathbf{C}_{P_i}$. Wir bemerken, dass die Ordnung der Elemente von J_i in $\prod_{s\in J_i} s$ keine Rolle spielt, da alle möglichen Permutationen der Produkte dieser Elementen zur selben Konjugationsklasse gehören. Sei $x_J := \sum_{w\in W^J} w$, wobei W^J der Menge der kürzesten Rechtsnebenklassen Vertreter der parabolischen Untergruppe W_J der Coxeter Gruppe W ist. Wir schreiben $\mathbb{K}[\mathscr{D}_W]$ für die Menge der Elemente $\mathfrak{f} = \sum_{J\subseteq S} \lambda_J x_J$ von $\mathbb{K}[W]$ mit $\lambda_J \in \mathbb{K}$. Im letzten Abschnitt des Kapitels stellen wir Schritt für Schritt den Beweis des folgenden Theorems dar.

Theorem 2. *Sei* $\mathfrak{d}_W = \sum_{J\subseteq S} \lambda_J x_J \in \mathbb{K}[\mathscr{D}_W]$. *Dann ist das Spektrum der Matrix* $\mathsf{X}_{\mathbb{K}[W]}(\mathfrak{d}_W)$

$$Sp\big(\mathsf{X}_{\mathbb{K}[W]}(\mathfrak{d}_W)\big) = \Big\{\Lambda_j = \sum_{i=1}^{p} a_{ij}\big(\sum_{K_i\in\mathbf{C}_{P_i}} \lambda_{K_i}\big)\Big\}_{j\in[p]}$$

mit

$$\big\{V_{\mathfrak{d}_W}(\Lambda_j) = |\mathbf{C}_{s_{J_j}}|\big\}_{j\in[p]}.$$

Als Beispiel haben wir das folgende Korollar, das die Eigenwerte und deren Multiplizitäten der Matrizen der Operation der Elemente von $\mathbb{K}[\mathscr{D}_{H_3}]$ auf $\mathbb{K}[H_3]$ beschreibt. Es ist wohlbekannt, dass $|H_3| = 120$ und dass der Coxeter Graph des Coxeter Systemes (H_3, S_{H_3}) gegeben ist durch:

$$H_3 := s_1 \overset{5}{\longleftrightarrow} s_2 \longleftrightarrow s_3$$

Korollar 3. *Sei* $\mathfrak{d}_{H_3} = \sum_{J\subseteq S_{H_3}} \lambda_J x_J \in \mathbb{K}[\mathscr{D}_{H_3}]$. *Dann ist das Spektrum der Matrix* $\mathsf{X}_{\mathbb{K}[H_3]}(\mathfrak{d}_{H_3})$

$$\begin{aligned}
\Lambda_1 &= 120\lambda_\varnothing + 60(\lambda_{\{s_1\}} + \lambda_{\{s_2\}} + \lambda_{\{s_3\}}) + 12\lambda_{\{s_1,s_2\}} + 20\lambda_{\{s_2,s_3\}} + 30\lambda_{\{s_1,s_3\}} + \lambda_{S_{H_3}}\\
\Lambda_2 &= 4(\lambda_{\{s_1\}} + \lambda_{\{s_2\}} + \lambda_{\{s_3\}} + \lambda_{\{s_1,s_2\}} + \lambda_{\{s_2,s_3\}} + \lambda_{\{s_1,s_3\}}) + \lambda_{S_{H_3}}\\
\Lambda_3 &= 2\lambda_{\{s_1,s_2\}} + \lambda_{S_{H_3}}\\
\Lambda_4 &= 2\lambda_{\{s_2,s_3\}} + \lambda_{S_{H_3}}\\
\Lambda_5 &= 2\lambda_{\{s_1,s_3\}} + \lambda_{S_{H_3}}\\
\Lambda_6 &= \lambda_{S_{H_3}}
\end{aligned}$$

mit

$$\begin{aligned}
V_{\mathfrak{d}_{H_3}}(\Lambda_1) &= 1\\
V_{\mathfrak{d}_{H_3}}(\Lambda_2) &= 15\\
V_{\mathfrak{d}_{H_3}}(\Lambda_3) &= 24\\
V_{\mathfrak{d}_{H_3}}(\Lambda_4) &= 20\\
V_{\mathfrak{d}_{H_3}}(\Lambda_5) &= 15\\
V_{\mathfrak{d}_{H_3}}(\Lambda_6) &= 45
\end{aligned}$$

Beweis. Wir benutzen Theorem 2 und die Werte von a_{ij} für $W = H_3$. □

Am Ende von **Kapitel 3** geben wir noch mehr Erklärungen zu der Berechnung der Eigenwerte und deren Multiplizitäten der Matrizen der Aktionen der Elemente von $\mathbb{K}[\mathscr{D}_{F_4}]$ auf $\mathbb{K}[F_4]$.

Wir möchten hier erwähnen, dass die wichtigsten und best untersuchten Descent Algebren gewiss die Descent Algebren der Coxeter Gruppen A_{n-1} bzw. symmetrischen Gruppen $\mathcal{S}_n = A_{n-1}$ sind. Es gibt viele tiefe und schöne Ergebnisse zu $\mathbb{K}[\mathscr{D}_{A_n}]$ [5]. Eine Quelle von Inspiration für uns war die Bestimmung der Eigenwerte und deren Multiplizitäten des Elementes

$$\mathfrak{t}_n = \sum_{J\subseteq S_{A_n}} q^{\mathtt{Maj}(J)}\big(\sum_{K\supseteq J}(-1)^{|K\setminus J|}x_K\big) \in \mathbb{R}(q)[\mathscr{D}_{A_n}]$$

durch Thibon [14, Theorem 56] mit Hilfe der Ergebnisse der Theorie von nicht kommutativen symmetrischen Funktionen ([10] und [15]). Dabei ist

$$\mathtt{Maj}(J) := \sum_{j \in \{i \in [n] \mid s_i \in J\}} j.$$

Die Eigenwerte von $\mathsf{X}_{\mathbb{R}(q)[\mathscr{D}_{A_n}]}(\mathfrak{t}_n)$ sind

$$\frac{(q;q)_{n+1}}{\prod_{i \geq 1}(1 - q^{\mu_i})}$$

mit Multiplizitäten

$$\frac{n!}{1^{\mu(1)}\mu(1)!2^{\mu(2)}\mu(2)!\ldots},$$

wobei $\mu = (\mu_1, \mu_2, \ldots)$ über alle Partitionen von $n+1$ läuft und $\mu(i)$ die Anzahl des Vorkommens von i in der Partition μ ist.

Unser **Kapitel 4** ist auf die Gruppenalgebra $\mathbb{K}[\mathcal{S}_n]$ konzentriert. Wir müssen zuerst eine multinomiale Version des Theorems von Perron-Frobenius [11, 8.2.11 Perron's Theorem] erarbeiten. Dann stellen wir weitere Untersuchungen mit dem Element

$$\mathfrak{d}_{\mathfrak{X}n} = \sum_{\sigma \in \mathcal{S}_n} \mathtt{des}_{\mathtt{X}}(\sigma)\, \sigma \in \mathbb{R}(X_1, \ldots, X_{n-1})[\mathcal{S}_n]$$

mit

$$\mathtt{des}_{\mathtt{X}}(\sigma) := \sum_{i \in \{k \in [n-1] \ \mid \ \sigma(k) > \sigma(k+1)\}} X_i$$

an, um das folgende Theorem zu beweisen:

Theorem 4. *Sei $n \geq 3$. Dann ist die Matrix $\mathsf{X}_{\mathbb{R}(X_1,\ldots,X_{n-1})[\mathcal{S}_n]}(\mathfrak{d}_{\mathfrak{X}n})$ diagonalisierbar und*

$$Sp\big(\mathsf{X}_{\mathbb{R}(X_1,\ldots,X_{n-1})[\mathcal{S}_n]}(\mathfrak{d}_{\mathfrak{X}n})\big) = \Big\{\frac{n!}{2}\sum_{k=1}^{n-1} X_k,\ -(n-2)!\sum_{k=1}^{n-1} X_k,\ 0\Big\}$$

mit

- $V_{\mathsf{X}_{\mathbb{R}(X_1,\ldots,X_{n-1})[\mathcal{S}_n]}(\mathfrak{d}_{\mathfrak{X}n})}\big(\frac{n!}{2}\sum_{k=1}^{n-1} X_k\big) = 1,$
- $V_{\mathsf{X}_{\mathbb{R}(X_1,\ldots,X_{n-1})[\mathcal{S}_n]}(\mathfrak{d}_{\mathfrak{X}n})}\big(-(n-2)!\sum_{k=1}^{n-1} X_k\big) = \binom{n}{2},$
- $V_{\mathsf{X}_{\mathbb{R}(X_1,\ldots,X_{n-1})[\mathcal{S}_n]}(\mathfrak{d}_{\mathfrak{X}n})}(0) = n! - \binom{n}{2} - 1.$

Wir bemerken, dass

$$\mathfrak{d}_{\mathfrak{X}n} = \sum_{J \subseteq S_{A_{n-1}}} \mathtt{Des}_{\mathtt{X}}(J)\Big(\sum_{K \supseteq J}(-1)^{|K \setminus J|} x_K\Big)$$

ein Element der Descent Algebra $\mathbb{R}(X_1, \ldots, X_{n-1})[\mathscr{D}_{A_{n-1}}]$ ist, mit

$$\mathtt{Des}_{\mathtt{X}}(J) := \sum_{i \in \{k \in [n-1] \mid s_k \in J\}} X_i.$$

Aber wir werden im selben Kapitel sehen, dass wir die Eigenwerte und deren Multiplizitäten der Matrix der Multiplikation von $\mathfrak{d}_{\mathfrak{X}n}$ auf $\mathbb{R}(X_1, \ldots, X_{n-1})[\mathcal{S}_n]$ auch mit Hilfe von elementaren mathematischen Mittel rechnen können. Als direkte Anwendung von Theorem 4 haben wir beide folgenden Korollare. Wir erinnern die beiden bekannten Statistiken von $\mathcal{S}_n$:

$$\mathtt{des}(\sigma) := \#\{k \in [n-1] \mid \sigma(k) > \sigma(k+1)\}$$

ist die Anzahl der Descents von σ,

$$\texttt{maj}(\sigma) := \sum_{i \in \{k \in [n-1] \mid \sigma(k) > \sigma(k+1)\}} i$$

ist das Major Index von σ.
Sei $\mathfrak{d}_n = \sum_{\sigma \in \mathcal{S}_n} \texttt{des}(\sigma)\, \sigma \in \mathbb{R}[\mathcal{S}_n]$ und $\mathfrak{m}_n = \sum_{\sigma \in \mathcal{S}_n} \texttt{maj}(\sigma)\, \sigma \in \mathbb{R}[\mathcal{S}_n]$.

Korollar 5. *Sei $n \geq 3$. Dann ist die Matrix $\mathsf{X}_{\mathbb{R}[\mathcal{S}_n]}(\mathfrak{d}_n)$ diagonalisierbar und*

$$Sp\big(\mathsf{X}_{\mathbb{R}[\mathcal{S}_n]}(\mathfrak{d}_n)\big) = \Big\{ \binom{n}{2}(n-1)!,\ 0,\ -(n-1)! \Big\}$$

mit

- $V_{\mathsf{X}_{\mathbb{R}[\mathcal{S}_n]}(\mathfrak{d}_n)}\big(\binom{n}{2}(n-1)!\big) = 1,$
- $V_{\mathsf{X}_{\mathbb{R}[\mathcal{S}_n]}(\mathfrak{d}_n)}\big(-(n-1)!\big) = \binom{n}{2},$
- $V_{\mathsf{X}_{\mathbb{R}[\mathcal{S}_n]}(\mathfrak{d}_n)}(0) = n! - \binom{n}{2} - 1.$

Beweis. Wir setzen $X_i = 1$ in Theorem 4. □

Korollar 6. *Sei $n \geq 3$. Dann ist die Matrix $\mathsf{X}_{\mathbb{R}[\mathcal{S}_n]}(\mathfrak{m}_n)$ diagonalisierbar und*

$$Sp\big(\mathsf{X}_{\mathbb{R}[\mathscr{S}_n]}(\mathfrak{m}_n)\big) = \Big\{ \binom{n}{2}\frac{n!}{2},\ 0,\ -\frac{n!}{2} \Big\}$$

mit

- $V_{\mathsf{X}_{\mathbb{R}[\mathcal{S}_n]}(\mathfrak{m}_n)}\big(\binom{n}{2}\frac{n!}{2}\big) = 1,$
- $V_{\mathsf{X}_{\mathbb{R}[\mathcal{S}_n]}(\mathfrak{m}_n)}\big(-\frac{n!}{2}\big) = \binom{n}{2},$
- $V_{\mathsf{X}_{\mathbb{R}[\mathcal{S}_n]}(\mathfrak{m}_n)}(0) = n! - \binom{n}{2} - 1.$

Beweis. Wir setzen $X_i = i$ in Theorem 4. □

Wir beenden unsere Untersuchungen mit dem Element

$$\mathfrak{i}\mathfrak{x}_n = \sum_{\sigma \in \mathcal{S}_n} \texttt{inv}_{\mathtt{X}}(\sigma)\, \sigma \in \mathbb{R}(X_{1,2}, \ldots, X_{n-1,n})[\mathcal{S}_n]$$

wobei

$$\texttt{inv}_{\mathtt{X}}(\sigma) := \sum_{(i,j) \in \{(k,l) \in [n-1] \times [n] \mid \sigma(k) > \sigma(l)\}} X_{i,j}$$

um dieses Theorem zu beweisen:

Theorem 7. *Sei $n \geq 4$. Dann ist die Matrix $\mathsf{X}_{\mathbb{R}(X_{1,2},\ldots,X_{n-1,n})[\mathcal{S}_n]}(\mathfrak{i}\mathfrak{x}_n)$ diagonalisierbar und*

$$Sp\big(\mathsf{X}_{\mathbb{R}(X_{1,2},\ldots,X_{n-1,n})[\mathcal{S}_n]}(\mathfrak{i}\mathfrak{x}_n)\big) = \Big\{ \frac{n!}{2} \sum_{\{(i,j) \in [n]^2 \mid i<j\}} X_{i,j},\ -(n-2)! \sum_{\{(i,j) \in [n]^2 \mid i<j\}} (j-i) X_{i,j},$$
$$-(n-3)! \sum_{\{(i,j) \in [n]^2 \mid i<j\}} \Big(n - 2(j-i)\Big) X_{i,j},\ 0 \Big\}$$

mit

- $V_{\mathsf{X}_{\mathbb{R}(X_{1,2},\dots,X_{n-1,n})[\mathcal{S}_n]}(\mathfrak{i}_{\mathfrak{X}n})}\big(\frac{n!}{2}\sum_{\{(i,j)\in[n]^2 \mid i<j\}} X_{i,j}\big) = 1,$
- $V_{\mathsf{X}_{\mathbb{R}(X_{1,2},\dots,X_{n-1,n})[\mathcal{S}_n]}(\mathfrak{i}_{\mathfrak{X}n})}\big(-(n-2)!\sum_{\{(i,j)\in[n]^2 \mid i<j\}}(j-i)X_{i,j}\big) = n-1,$
- $V_{\mathsf{X}_{\mathbb{R}(X_{1,2},\dots,X_{n-1,n})[\mathcal{S}_n]}(\mathfrak{i}_{\mathfrak{X}n})}\big(-(n-3)!\sum_{\{(i,j)\in[n]^2 \mid i<j\}}\big(n-2(j-i)\big)X_{i,j}\big) = \binom{n-1}{2},$
- $V_{\mathsf{X}_{\mathbb{R}(X_{1,2},\dots,X_{n-1,n})[\mathcal{S}_n]}(\mathfrak{i}_{\mathfrak{X}n})}(0) = n! - \binom{n}{2} - n.$

Wir bemerken, dass eine exponentielle Version von $\mathsf{X}_{\mathbb{R}(X_{1,2},\dots,X_{n-1,n})[\mathcal{S}_n]}(\mathfrak{i}_{\mathfrak{X}n})$ von Varchenko [23] untersucht wurde. Er konnte eine erstaunliche Formel für die Determinante von $\mathsf{X}_{\mathbb{R}(X_{1,2},\dots,X_{n-1,n})[\mathcal{S}_n]}\mathfrak{i}_n^{\mathfrak{X}}$ geben, wobei

$$\mathfrak{i}_n^{\mathfrak{X}} = \sum_{\sigma\in\mathcal{S}_n} \mathtt{inv}^{\mathtt{X}}(\sigma)\,\sigma \in \mathbb{R}(X_{1,2},\dots,X_{n-1,n})[\mathcal{S}_n]$$

und

$$\mathtt{inv}^{\mathtt{X}}(\sigma) := \prod_{(i,j)\in\{(k,l)\in[n-1]\times[n] \mid \sigma(k)>\sigma(l)\}} X_{i,j}.$$

Die Determinante ist

$$\prod_{L\subseteq 2^{\binom{[n]}{2}}} (1-a(L)^2)^{l(L)}$$

mit $a(L) = \prod_{i,j\in L} X_{i,j}$ und $l(L)$ ist die "Multiplizität" von L. Diese exponentielle Version motivierte Folgearbeiten in mehreren mathematischen Gebieten (e.g.[8], [24]). Als direkte Anwendung von Theorem 7 haben wir das folgende Korollar. Dieses Korollar wurde in einer unabhängigen Arbeit von Renteln bewiesen [18, Section 4.8]. Dieses Korollar ist auch ein besonderer Fall von einem Theorem von Reiner, Saliola, Welker [17, Theorem 1.4], das unsere Untersuchung auch inspiriert hat. Wir erinnern, dass

$$\mathtt{inv}(\sigma) := \#\{(k,l)\in[n-1]\times[n] \mid \sigma(k)>\sigma(l)\}$$

ist die Anzahl der Inversionen von σ.
Sei $\mathfrak{i}_n = \sum_{\sigma\in\mathcal{S}_n}\mathtt{inv}(\sigma)\,\sigma \in \mathbb{R}[\mathcal{S}_n]$.

Korollar 8. *Sei $n \geq 4$. Dann ist die Matrix $\mathsf{X}_{\mathbb{R}[\mathcal{S}_n]}(\mathfrak{i}_n)$ diagonalisierbar und*

$$Sp\big(\mathsf{X}_{\mathbb{R}[\mathcal{S}_n]}(\mathfrak{i}_n)\big) = \Big\{\frac{n!}{2}\binom{n}{2},\ -\frac{(n+1)!}{6},\ -\frac{n!}{6},\ 0\Big\}$$

mit

- $V_{\mathsf{X}_{\mathbb{R}[\mathcal{S}_n]}(\mathfrak{i}_n)}\big(\frac{n!}{2}\binom{n}{2}\big) = 1,$
- $V_{\mathsf{X}_{\mathbb{R}[\mathcal{S}_n]}(\mathfrak{i}_n)}\big(\frac{(n+1)!}{6}\big) = n-1,$
- $V_{\mathsf{X}_{\mathbb{R}[\mathcal{S}_n]}(\mathfrak{i}_n)}\big(\frac{n!}{6}\big) = \binom{n}{2},$
- $V_{\mathsf{X}_{\mathbb{R}[\mathcal{S}_n]}(\mathfrak{i}_n)}(0) = n! - \binom{n}{2} - n.$

Beweis. Wir setzen $X_{i,j} = 1$ in Theorem 7. □

Introduction

We write G for a finite group and $\mathbb{K}$ for an algebraically closed field of characteristic 0. Let $\mathfrak{f} = \sum_{g \in G} \lambda_g \, g$ be an element of the group algebra $\mathbb{K}[G]$, where $\lambda_g \in \mathbb{K}$ for $g \in G$. The map $\mathsf{X}_{\mathbb{K}[G]} : \mathbb{K}[G] \to \mathbb{K}^{|G| \times |G|}$ sends the element $\mathfrak{f}$ to the matrix $\mathsf{X}_{\mathbb{K}[G]}(\mathfrak{f}) = (\lambda_{gg'^{-1}})_{g,\, g' \in G}$ of left-multiplication action of $\mathfrak{f}$ on $\mathbb{K}[G]$. We aim to determine the eigenvalues of $\mathsf{X}_{\mathbb{K}[G]}(\mathfrak{f})$ and their multiplicities for special groups and for particular values of λ_g.

We begin in **Chapter 1** by giving some background from representation theory on question for group algebras. Especially, we will learn more about the more general map X_M which sends $\mathfrak{f}$ to the matrix $\mathsf{X}_M(\mathfrak{f})$ of the left-multiplication action of $\mathfrak{f}$ on an arbitrary finite dimensional $\mathbb{K}[G]$ module M. We will also see some background from Coxeter groups in the same chapter.

Then we give an exhaustive treatment of the case of abelian groups and dihedral groups in **Chapter 2**. Let $\{G_i\}_{\{i \in [k]\}}$ be the set of the conjugacy classes of G and $g_{G_i} = \sum_{g \in G_i} g$ be the sum of the elements of G_i. We write $\mathbb{K}[\mathscr{C}_G]$ for the set of elements of $\mathbb{K}[G]$ of the form $\sum_{i=1}^{k} \lambda_i \, g_{G_i}$, where $\lambda_i \in \mathbb{K}$. It is well known that $\mathbb{K}[\mathscr{C}_G]$ is the center of $\mathbb{K}[G]$. We finish the second chapter with the following proposition which is immediately deduced from representation theory.

Proposition 1. *Let* $\mathfrak{c}_G = \sum_{i=1}^{k} \lambda_i \, g_{G_i} \in \mathbb{K}[\mathscr{C}_G]$ *and* $(d_i)_{i \in [k]}$ *be the degrees of the irreducible modules* $(M_i)_{i \in [k]}$ *of* $\mathbb{K}[G]$*. Then the matrix* $\mathsf{X}_{\mathbb{K}[G]}(\mathfrak{c}_G)$ *is diagonalizable and*

$$Sp\big(\mathsf{X}_{\mathbb{K}[G]}(\mathfrak{c}_G)\big) = \Big\{ \Lambda_i = \frac{\sum_{j=1}^{k} \lambda_j |G_j| \chi_{M_i}^{G_j}}{d_i} \Big\}_{i \in [k]}$$

with

$$\big\{ V_{\mathfrak{c}_G}(\Lambda_i) = d_i^2 \big\}_{i \in [k]}$$

where $\chi_{M_i}^{G_j}$ *is the common value of* $tr\big(\mathsf{X}_{M_i}(g_j)\big)$, $g_j \in G_j$.

For example, we can directly apply Proposition 1 to get the eigenvalues and their multiplicities of the matrix of the action of the element

$$\mathfrak{f}_n = \sum_{\sigma \in \mathcal{S}_n} \big(\mathtt{fix}(\sigma) t + q^{\mathtt{fix}(\sigma)} \big) \, \sigma \in \mathbb{R}(t,q)[\mathscr{C}_{\mathcal{S}_n}]$$

on $\mathbb{R}(t,q)[\mathcal{S}_n]$, where t and q are varibles, $\mathcal{S}_n$ is the symmetric group, and `fix` is the well known statistic

$$\mathtt{fix}(\sigma) := \#\{ i \in [n] \,|\, \sigma(i) = i \}$$

which is invariant under conjugation. We will treat $\mathfrak{f}_n$ as an example in **Chapter 2**.

Proposition 1 holds in the generality of finite groups G. The next theorem can just be applied to finite Coxeter systems (W, S). We will see in **Chapter 3** that its proof is definitively more involved and requires knowledge of the descent algebra. We will first give the definition of the descent algebra of a finite Coxeter group. We write $\mathbf{C}_P$ for the set of subsets J of S such that the parabolic subgroups W_P und W_J of W are conjugate. Let $\{\mathbf{C}_{P_i}\}_{\{i \in [p]\}}$ be the set having the following properties:

$$\{\mathbf{C}_{P_i}\}_{i \in [p]} = \{\mathbf{C}_P\}_{P \subseteq S},$$

$$\mathbf{C}_{P_i} \neq \mathbf{C}_{P_j} \text{ if } i \neq j.$$

Let $J_i \in \mathbf{C}_{P_i}$ and $J_j \in \mathbf{C}_{P_j}$. We continue with the definition of the coefficients $a_{ij} = a_{J_i J_j J_j}$, with J_i, $J_j \subseteq S$, which were introduced by Solomon [21, Theorem 1]. Then we determine a formula for the coefficient a_{ij}. We write $\mathbf{C}_{s_{J_i}}$ for the conjugacy class of the element $s_{J_i} := \prod_{s\in J_i} s$ of W. We remark that it does not matter how the elements of J_i are ordered in $\prod_{s\in J_i} s$. Effectively, for any permutation of the ordering, $\prod_{s\in J_i} s$ belongs to the same conjugacy class. Let $x_J := \sum_{w\in W^J} w$, where W^J is the set of right coset representatives of the parabolic subgroup W_J of the Coxeter group W. We write $\mathbb{K}[\mathscr{D}_W]$ for the set of elements of $\mathbb{K}[W]$ of the form $\sum_{J\subseteq S} \lambda_J x_J$, where $\lambda_J \in \mathbb{K}$. In the last section of the third chapter, we etablish step by step the proof of the following theorem.

Theorem 2. *Let $\mathfrak{d}_W = \sum_{J\subseteq S} \lambda_J x_J \in \mathbb{K}[\mathscr{D}_W]$. Then the spectrum of the matrix $\mathsf{X}_{\mathbb{K}[W]}(\mathfrak{d}_W)$ is*

$$Sp\big(\mathsf{X}_{\mathbb{K}[W]}(\mathfrak{d}_W)\big) = \Big\{\Lambda_j = \sum_{i=1}^{p} a_{ij}\big(\sum_{K_i\in\mathbf{C}_{P_i}} \lambda_{K_i}\big)\Big\}_{j\in[p]}$$

with

$$\big\{V_{\mathfrak{d}_W}(\Lambda_j) = |\mathbf{C}_{s_{J_j}}|\big\}_{j\in[p]}.$$

As an example, we have the following corollary which gives the eigenvalues and their multiplicities of the matrices of the actions of the elements of $\mathbb{K}[\mathscr{D}_{H_3}]$ on $\mathbb{K}[H_3]$. Recall that $|H_3| = 120$ and the Coxeter graph of the Coxeter system (H_3, S_{H_3}) is

$$H_3 := s_1 \overset{5}{\longleftrightarrow} s_2 \longleftrightarrow s_3$$

Corollary 3. *Let $\mathfrak{d}_{H_3} = \sum_{J\subseteq S_{H_3}} \lambda_J x_J \in \mathbb{K}[\mathscr{D}_{H_3}]$. Then the spectrum of the matrix $\mathsf{X}_{\mathbb{K}[H_3]}(\mathfrak{d}_{H_3})$ is*

$$\begin{aligned}
\Lambda_1 &= 120\lambda_\varnothing + 60(\lambda_{\{s_1\}} + \lambda_{\{s_2\}} + \lambda_{\{s_3\}}) + 12\lambda_{\{s_1,s_2\}} + 20\lambda_{\{s_2,s_3\}} + 30\lambda_{\{s_1,s_3\}} + \lambda_{S_{H_3}}\\
\Lambda_2 &= 4(\lambda_{\{s_1\}} + \lambda_{\{s_2\}} + \lambda_{\{s_3\}} + \lambda_{\{s_1,s_2\}} + \lambda_{\{s_2,s_3\}} + \lambda_{\{s_1,s_3\}}) + \lambda_{S_{H_3}}\\
\Lambda_3 &= 2\lambda_{\{s_1,s_2\}} + \lambda_{S_{H_3}}\\
\Lambda_4 &= 2\lambda_{\{s_2,s_3\}} + \lambda_{S_{H_3}}\\
\Lambda_5 &= 2\lambda_{\{s_1,s_3\}} + \lambda_{S_{H_3}}\\
\Lambda_6 &= \lambda_{S_{H_3}}
\end{aligned}$$

with

$$\begin{aligned}
V_{\mathfrak{d}_{H_3}}(\Lambda_1) &= 1\\
V_{\mathfrak{d}_{H_3}}(\Lambda_2) &= 15\\
V_{\mathfrak{d}_{H_3}}(\Lambda_3) &= 24\\
V_{\mathfrak{d}_{H_3}}(\Lambda_4) &= 20\\
V_{\mathfrak{d}_{H_3}}(\Lambda_5) &= 15\\
V_{\mathfrak{d}_{H_3}}(\Lambda_6) &= 45
\end{aligned}$$

Proof. We use Theorem 2 and the values of a_{ij} for $W = H_3$. □

At the end of **Chapter 3**, we give more details to obtain the eigenvalues and their multiplicities of the action matrices of the elements of $\mathbb{K}[\mathscr{D}_{F_4}]$ on $\mathbb{K}[F_4]$.
Here we would like to mention that the most studied und best understood descent algebras are certainly the descent algebras of the Coxeter groups A_{n-1} repspectively the symmetric groups $\mathcal{S}_n = A_{n-1}$. Many beautiful works have been written about $\mathbb{K}[\mathscr{D}_{A_n}]$ (see e.g. [5]). An inspiration for us was the determination of the eigenvalues and their multiplicities related to the element

$$\mathfrak{t}_n = \sum_{J\subseteq S_{A_n}} q^{\mathtt{Maj}(J)}\big(\sum_{K\supseteq J}(-1)^{|K\setminus J|}x_K\big) \in \mathbb{R}(q)[\mathscr{D}_{A_n}]$$

by Thibon [14, Theorem 56] using results from the theory of noncommutative symmetric functions ([10] and [15]), where

$$\mathtt{Maj}(J) := \sum_{j \in \{i \in [n] \mid s_i \in J\}} j.$$

The eigenvalues of $\mathsf{X}_{\mathbb{R}(q)[A_n]}(\mathfrak{t}_n)$ are

$$\frac{(q;q)_{n+1}}{\prod_{i \geq 1}(1 - q^{\mu_i})}$$

with multiplicities

$$\frac{n!}{1^{\mu(1)}\mu(1)! 2^{\mu(2)}\mu(2)! \dots}$$

where $\mu = (\mu_1, \mu_2, \dots)$ varies over all the partitions of $n+1$ and $\mu(i)$ is the number of occurences of i in the partition μ.

Our **Chapter 4** more closely focuses on the group algebra $\mathbb{K}[\mathcal{S}_n]$. We first have to give a multinomial version of the theorem of Perron-Frobenius [11, 8.2.11 Perron's Theorem]. Then we begin our next diagonalizations with the element

$$\mathfrak{d}_{\mathfrak{X}_n} = \sum_{\sigma \in \mathcal{S}_n} \mathtt{des}_{\mathtt{X}}(\sigma)\, \sigma \in \mathbb{R}(X_1, \dots, X_{n-1})[\mathcal{S}_n]$$

where

$$\mathtt{des}_{\mathtt{X}}(\sigma) := \sum_{i \in \{k \in [n-1] \ \mid \ \sigma(k) > \sigma(k+1)\}} X_i$$

to get this theorem:

Theorem 4. *Let $n \geq 3$. Then the matrix $\mathsf{X}_{\mathbb{R}(X_1,\dots,X_{n-1})[\mathcal{S}_n]}(\mathfrak{d}_{\mathfrak{X}_n})$ is diagonalizable and*

$$Sp\big(\mathsf{X}_{\mathbb{R}(X_1,\dots,X_{n-1})[\mathcal{S}_n]}(\mathfrak{d}_{\mathfrak{X}_n})\big) = \Big\{\frac{n!}{2}\sum_{k=1}^{n-1} X_k,\ -(n-2)!\sum_{k=1}^{n-1} X_k,\ 0\Big\}$$

with

- $V_{\mathsf{X}_{\mathbb{R}(X_1,\dots,X_{n-1})[\mathcal{S}_n]}(\mathfrak{d}_{\mathfrak{X}_n})}\big(\frac{n!}{2}\sum_{k=1}^{n-1} X_k\big) = 1,$
- $V_{\mathsf{X}_{\mathbb{R}(X_1,\dots,X_{n-1})[\mathcal{S}_n]}(\mathfrak{d}_{\mathfrak{X}_n})}\big(-(n-2)!\sum_{k=1}^{n-1} X_k\big) = \binom{n}{2},$
- $V_{\mathsf{X}_{\mathbb{R}(X_1,\dots,X_{n-1})[\mathcal{S}_n]}(\mathfrak{d}_{\mathfrak{X}_n})}(0) = n! - \binom{n}{2} - 1.$

We remark that

$$\mathfrak{d}_{\mathfrak{X}_n} = \sum_{J \subseteq S_{A_{n-1}}} \mathtt{Des}_{\mathtt{X}}(J)\Big(\sum_{K \supseteq J} (-1)^{|K \setminus J|} x_K\Big)$$

is an element of the descent algebra $\mathbb{R}(X_1, \dots, X_{n-1})[\mathscr{D}_{A_{n-1}}]$ where

$$\mathtt{Des}_{\mathtt{X}}(J) := \sum_{i \in \{k \in [n-1] \mid s_k \in J\}} X_i.$$

We observe in the same chapter that we can get the eigenvalues and their multiplicities of the matrix of the action of $\mathfrak{d}_{\mathfrak{X}_n}$ with more elementary mathematical tools. As a direct application of Theorem 4, we have the two following corollaries. We recall two well-known statistics on $\mathcal{S}_n$:

$$\mathtt{des}(\sigma) := \#\{k \in [n-1] \mid \sigma(k) > \sigma(k+1)\}$$

is the number of the descents of σ,

$$\texttt{maj}(\sigma) := \sum_{i\in\{k\in[n-1]\ \mid\ \sigma(k)>\sigma(k+1)\}} i$$

is the major index of σ.
Let $\mathfrak{d}_n = \sum_{\sigma\in\mathcal{S}_n} \texttt{des}(\sigma)\,\sigma \in \mathbb{R}[\mathcal{S}_n]$ and $\mathfrak{m}_n = \sum_{\sigma\in\mathcal{S}_n} \texttt{maj}(\sigma)\,\sigma \in \mathbb{R}[\mathcal{S}_n]$.

Corollary 5. *Let $n \geq 3$. Then the matrix $\mathsf{X}_{\mathbb{R}[\mathcal{S}_n]}(\mathfrak{d}_n)$ is diagonalizable and*

$$Sp\big(\mathsf{X}_{\mathbb{R}[\mathcal{S}_n]}(\mathfrak{d}_n)\big) = \Big\{\binom{n}{2}(n-1)!,\ 0,\ -(n-1)!\Big\}$$

with

- $V_{\mathsf{X}_{\mathbb{R}[\mathcal{S}_n]}(\mathfrak{d}_n)}\big(\binom{n}{2}(n-1)!\big) = 1,$
- $V_{\mathsf{X}_{\mathbb{R}[\mathcal{S}_n]}(\mathfrak{d}_n)}\big(-(n-1)!\big) = \binom{n}{2},$
- $V_{\mathsf{X}_{\mathbb{R}[\mathcal{S}_n]}(\mathfrak{d}_n)}(0) = n! - \binom{n}{2} - 1.$

Proof. Set $X_i = 1$ in Theorem 4. □

Corollary 6. *Let $n \geq 3$. Then the matrix $\mathsf{X}_{\mathbb{R}[\mathscr{S}_n]}(\mathfrak{m}_n)$ is diagonalizable and*

$$Sp\big(\mathsf{X}_{\mathbb{R}[\mathscr{S}_n]}(\mathfrak{m}_n)\big) = \Big\{\binom{n}{2}\frac{n!}{2},\ 0,\ -\frac{n!}{2}\Big\}$$

with

- $V_{\mathsf{X}_{\mathbb{R}[\mathcal{S}_n]}(\mathfrak{m}_n)}\big(\binom{n}{2}\frac{n!}{2}\big) = 1,$
- $V_{\mathsf{X}_{\mathbb{R}[\mathcal{S}_n]}(\mathfrak{m}_n)}\big(-\frac{n!}{2}\big) = \binom{n}{2},$
- $V_{\mathsf{X}_{\mathbb{R}[\mathcal{S}_n]}(\mathfrak{m}_n)}(0) = n! - \binom{n}{2} - 1.$

Proof. Set $X_i = i$ in Theorem 4. □

We finish our work with the element

$$\mathfrak{i}_{\mathfrak{X}_n} = \sum_{\sigma\in\mathcal{S}_n} \texttt{inv}_{\mathtt{X}}(\sigma)\,\sigma \in \mathbb{R}(X_{1,2},\dots,X_{n-1,n})[\mathscr{S}_n]$$

where

$$\texttt{inv}_{\mathtt{X}}(\sigma) := \sum_{(i,j)\in\{(k,l)\in[n-1]\times[n]\ \mid\ \sigma(k)>\sigma(l)\}} X_{i,j}$$

to get this theorem:

Theorem 7. *Let $n \geq 4$. Then the matrix $\mathsf{X}_{\mathbb{R}(X_{1,2},\dots,X_{n-1,n})[\mathcal{S}_n]}(\mathfrak{i}_{\mathfrak{X}_n})$ is diagonalizable and*

$$Sp\big(\mathsf{X}_{\mathbb{R}(X_{1,2},\dots,X_{n-1,n})[\mathcal{S}_n]}(\mathfrak{i}_{\mathfrak{X}_n})\big) = \Big\{\frac{n!}{2}\sum_{\{(i,j)\in[n]^2\ \mid\ i<j\}} X_{i,j},\ -(n-2)!\sum_{\{(i,j)\in[n]^2\ \mid\ i<j\}}(j-i)X_{i,j},$$
$$-(n-3)!\sum_{\{(i,j)\in[n]^2\ \mid\ i<j\}}\Big(n-2(j-i)\Big)X_{i,j},\ 0\Big\}$$

with

- $V_{\mathsf{X}_{\mathbb{R}(X_{1,2},\ldots,X_{n-1,n})[\mathcal{S}_n]}(\mathfrak{i}_{\mathfrak{X}n})}\big(\frac{n!}{2}\sum_{\{(i,j)\in[n]^2 \ \mid \ i<j\}} X_{i,j}\big) = 1$,
- $V_{\mathsf{X}_{\mathbb{R}(X_{1,2},\ldots,X_{n-1,n})[\mathcal{S}_n]}(\mathfrak{i}_{\mathfrak{X}n})}\big(- (n-2)!\sum_{\{(i,j)\in[n]^2 \ \mid \ i<j\}}(j-i)X_{i,j}\big) = n-1$,
- $V_{\mathsf{X}_{\mathbb{R}(X_{1,2},\ldots,X_{n-1,n})[\mathcal{S}_n]}(\mathfrak{i}_{\mathfrak{X}n})}\big(- (n-3)!\sum_{\{(i,j)\in[n]^2 \ \mid \ i<j\}}\Big(n-2(j-i)\Big)X_{i,j}\big) = \binom{n-1}{2}$,
- $V_{\mathsf{X}_{\mathbb{R}(X_{1,2},\ldots,X_{n-1,n})[\mathcal{S}_n]}(\mathfrak{i}_{\mathfrak{X}n})}(0) = n! - \binom{n}{2} - n$.

We note that an exponential version of $\mathsf{X}_{\mathbb{R}(X_{1,2},\ldots,X_{n-1,n})[\mathcal{S}_n]}\mathfrak{i}_{\mathfrak{X}n}$ has been studied by Varchenko [23]. He was able to give a beautiful formula for the determinant of $\mathsf{X}_{\mathbb{R}(X_{1,2},\ldots,X_{n-1,n})[\mathcal{S}_n]}\mathfrak{i}_n^{\mathfrak{X}}$ where

$$\mathfrak{i}_n^{\mathfrak{X}} = \sum_{\sigma\in\mathcal{S}_n} \mathtt{inv}^{\mathtt{X}}(\sigma)\,\sigma \in \mathbb{R}(X_{1,2},\ldots,X_{n-1,n})[\mathcal{S}_n]$$

with

$$\mathtt{inv}^{\mathtt{X}}(\sigma) := \prod_{(i,j)\in\{(k,l)\in[n-1]\times[n] \ \mid \ \sigma(k)>\sigma(l)\}} X_{i,j}.$$

The determinant is

$$\prod_{L\subseteq 2^{\binom{[n]}{2}}} (1-a(L)^2)^{l(L)}$$

where $a(L) = \prod_{i,j\in L} X_{i,j}$ and $l(L)$ is the "multiplicity" of L. This exponential version has attracted considerable interest in various areas of mathematics (e.g.[8], [24]). As a direct application of the Theorem 7, we have the following corollary. This result was obtained in a recent independent work of Renteln [18, Section 4.8]. The integrality assertion of this corollary is a very special case of a theorem of Reiner, Saliola, Welker [17, Theorem 1.4] which also inspired our investigation. Recall that:

$$\mathtt{inv}(\sigma) := \#\{(k,l)\in[n-1]\times[n] \mid \sigma(k)>\sigma(l)\} \text{ is the number of inversions of } \sigma.$$

Let $\mathfrak{i}_n = \sum_{\sigma\in\mathcal{S}_n} \mathtt{inv}(\sigma)\,\sigma \in \mathbb{R}[\mathcal{S}_n]$.

Corollary 8. *Let $n \geq 4$. Then the matrix $\mathsf{X}_{\mathbb{R}[\mathcal{S}_n]}(\mathfrak{i}_n)$ is diagonalizable and*

$$Sp\big(\mathsf{X}_{\mathbb{R}[\mathcal{S}_n]}(\mathfrak{i}_n)\big) = \Big\{\frac{n!}{2}\binom{n}{2},\ -\frac{(n+1)!}{6},\ -\frac{n!}{6},\ 0\Big\}$$

with

- $V_{\mathsf{X}_{\mathbb{R}[\mathcal{S}_n]}(\mathfrak{i}_n)}\big(\frac{n!}{2}\binom{n}{2}\big) = 1$,
- $V_{\mathsf{X}_{\mathbb{R}[\mathcal{S}_n]}(\mathfrak{i}_n)}\big(\frac{(n+1)!}{6}\big) = n-1$,
- $V_{\mathsf{X}_{\mathbb{R}[\mathcal{S}_n]}(\mathfrak{i}_n)}\big(\frac{n!}{6}\big) = \binom{n}{2}$,
- $V_{\mathsf{X}_{\mathbb{R}[\mathcal{S}_n]}(\mathfrak{i}_n)}(0) = n! - \binom{n}{2} - n$.

Proof. Set $X_{i,j} = 1$ in Theorem 7. □

Contents

Chapter 1

Background

In this chapter, we present the mathematical basis which we need to comprehend the result of our work. We begin with the representation theory of the group algebra, which somehow can be seen as the source of our work because all our results are diagonalizations of representations. After defining representations of group algebras, we give some of its basic theory which we need later. Then we continue with a short presentation of finite Coxeter groups which are the groups mainly studied in this work. Especially, we give the definition of the descent of an element of a Coxeter group, the irreducible Coxeter groups and some basic results needed in subsequent chapters. Principally, **Section 1.1** is inspired from [20] and **Section 1.2** from [3].

1.1 Group Algebra Representation Theory

Throughout this work, $G = \{g_i\}_{i\in[n]}$ is a group of order n, written multiplicatively, with neutral element e, and $\mathbb{K}$ is an algebraically closed field of characteristic 0. Note that groups in this work are written multiplicatively and with neutral element e.

Definition 1.1.1. *A group algebra $\mathbb{K}[G]$ is the vector space over $\mathbb{K}$ having the n elements of G as basis with the following multiplication:*
Let $(\lambda_{g_i})_{i\in[n]},\ (\mu_{g_i})_{i\in[n]} \in \mathbb{K}^n$ and $\sum_{i=1}^n \lambda_{g_i} g_i,\ \sum_{i=1}^n \mu_{g_i} g_i \in \mathbb{K}[G]$:

$$\sum_{i=1}^n \lambda_{g_i} g_i \cdot \sum_{i=1}^n \mu_{g_i} g_i = \sum_{i=1}^n \sum_{j=1}^n \lambda_{g_i}\mu_{g_j}(g_i g_j)$$

Definition 1.1.2. *A vector subspace $M \subseteq \mathbb{K}[G]$ is a $\mathbb{K}[G]$ module if*

$$\mathbb{K}[G]\cdot M + \mathbb{K}[G]\cdot M \subseteq M.$$

The degree of a module M is its dimension as a vector space M over $\mathbb{K}$.
A module M is irreducible if there exists no module N such that $\{0\} \subsetneq N \subsetneq M$.
Let M_1, M_2 be modules of $\mathbb{K}[G]$. A module homomorphism is a linear mapping $\theta : M_1 \to M_2$ such that

$$\theta(gm_1) = g\theta(m_1)$$

for all $g \in \mathbb{K}[G]$ and $m_1 \in M_1$. A module isomorphism is a bijective module homomorphism.

Let $\{m_j\}_{j\in[d]}$ be a basis of M and $\mathfrak{f} = \sum_{i=1}^n \lambda_{g_i}\, g_i \in \mathbb{K}[G]$. We have

$$\mathfrak{f}\cdot m_j = \sum_{i=1}^d \varsigma_{ij} m_i$$

where $\varsigma_{ij} \in \mathbb{K}$. We write

$$\mathsf{X}_M(\mathfrak{f}) := (\varsigma_{ij})_{i,j\in[d]}$$

for the matrix.

Definition 1.1.3. *A matrix representation of $\mathbb{K}[G]$ on a module M of degree d is the algebra homomorphism $\mathsf{X}_M : \mathbb{K}[G] \to \mathbb{K}^{d\times d}$ where $\mathsf{X}_M(\mathfrak{f})$ is matrix of the left-multiplication of $\mathfrak{f}$ on M relative to the basis $\{m_j\}_{j\in[d]}$ of M.*

If the considered module is $\mathbb{K}[G]$ with the multiplication action, the matrix of this linear mapping relative to the standard basis $(g_i)_{i\in[n]}$ is

$$\mathsf{X}_{\mathbb{K}[G]}(\mathfrak{f}) = (\lambda_{g_i g_j^{-1}})_{i,j\in[n]}.$$

In this case, $\mathsf{X}_{\mathbb{K}[G]}(\mathfrak{f})$ is called the regular representation of $\mathfrak{f}$. It is the main representation considered in this work. We have the following result which we use in the next chapter.

Theorem 1.1.4 (Maschke's Theorem)**.** *Let $\mathfrak{f} \in \mathbb{K}[G]$ and k be the number of conjugacy classes of G. Up to isomorphism, $\mathbb{K}[G]$ has k irreducible modules M_i of degree d_i such that, on a suitable basis of $\mathbb{K}[G]$, we have*

$$\mathsf{X}_{\mathbb{K}[G]}(\mathfrak{f}) = \bigoplus_{i=1}^{k} d_i \mathsf{X}_{M_i}(\mathfrak{f})$$

where $d_i\mathsf{X}_{M_i}(\mathfrak{f}) = \overbrace{\mathsf{X}_{M_i}(\mathfrak{f}) \oplus \cdots \oplus \mathsf{X}_{M_i}(\mathfrak{f})}^{d_i}$.

This is [20, Theorem 1.5.3].
For special coefficients, we have a more refined expression of Maschke's Theorem.

Theorem 1.1.5. *Let $\mathfrak{f} \in \mathbb{K}[G]$ such that $g^{-1}\mathfrak{f}g = \mathfrak{f}$ for all $g \in G$ i.e. $\mathfrak{f}$ lies in the center of $\mathbb{K}[G]$. If k is the number of conjugacy classes of G, then on a suitable basis of $\mathbb{K}[G]$*

$$\mathsf{X}_{\mathbb{K}[G]}(\mathfrak{f}) = \bigoplus_{i=1}^{k} \lambda_i \mathsf{I}_{d_i^2}$$

where $\lambda_i \in \mathbb{K}$ and $(d_i)_{i\in[k]}$ is the degree of the nonisomorph irreducible modules $(M_i)_{i\in[k]}$.

This is [20, Theorem 1.7.9].

1.2 Coxeter Groups

Definition 1.2.1. *A finite Coxeter group is a group W with generating elements the set $S = \{s_i\}_{i\in[l]}$, which elements are subject only to relations of the form*

$$(s_i s_j)^{m(s_i,s_j)} = e$$

where $m(s_i, s_i) = 1$ and $m(s_i, s_j) = m(s_j, s_i) \geq 2$ for $i \neq j$.
The set S is called the generating system of W and the pair (W, S) a Coxeter system.

Recall that $o(w)$ is the order of the element w of W. Throughout this work, we write W for a finite Coxeter group and $(W, S) = (W, \{s_i\}_{i\in[l]})$ for a Coxeter system.

Definition 1.2.2. *The Coxeter graph of a Coxeter system (W, S) is a graph whose node set is S and whose edges are the unordered pairs $\{s_i, s_j\}$ such that $o(s_is_j) \geq 3$. The edges with $o(s_is_j) \geq 4$ are labeled by $o(s_is_j)$.*
The Coxeter system (W, S) is called irreducible if its Coxeter graph is connected.

`Example:` The Coxeter graph of the Coxeter group F_4 with generating system $S_{F_4} = \{s_i\}_{i\in[4]}$ and cardinality $|F_4| = 1152$ is:

$$F_4 := s_1 \longleftrightarrow s_2 \overset{4}{\longleftrightarrow} s_3 \longleftrightarrow s_4$$

Recall that the finite irreducible Coxeter groups are:

- A_n, $n \geq 1$, of order $(n+1)!$:

  $$s_1 \longleftrightarrow \longleftrightarrow \dashleftarrow \dashrightarrow \longleftrightarrow s_n$$

  The other more known notation of A_n is $\mathcal{S}_{n+1}$. The generating elements of the symmetric group $\mathcal{S}_{n+1}$ are the n transpositions $s_i = (i\ i+1)$, $i \in [n]$.

- B_n, $n \geq 2$, of order $2^n n!$:

  $$s_1 \overset{4}{\longleftrightarrow} \longleftrightarrow \dashleftarrow \dashrightarrow \longleftrightarrow s_n$$

- D_n, $n \geq 4$, of order $2^{n-1}n!$:

  $$s_1 \longleftrightarrow \updownarrow \longleftrightarrow \dashleftarrow \dashrightarrow \longleftrightarrow s_n$$

- E_6 of order $2^7 3^4 5$:

  $$\longleftrightarrow \longleftrightarrow \updownarrow \longleftrightarrow \longleftrightarrow$$

- E_7 of order $2^{10} 3^4 5 7$:

  $$\longleftrightarrow \longleftrightarrow \updownarrow \longleftrightarrow \longleftrightarrow \longleftrightarrow$$

- E_8 of order $2^{14} 3^5 5^2 7$:

  $$\longleftrightarrow \longleftrightarrow \updownarrow \longleftrightarrow \longleftrightarrow \longleftrightarrow \longleftrightarrow$$

- F_4 of order 1152:

  $$\longleftrightarrow \overset{4}{\longleftrightarrow} \longleftrightarrow$$

- G_2 of order 12:

  $$\overset{6}{\longleftrightarrow}$$

- H_3 of order 120:

  $$\overset{5}{\longleftrightarrow} \longleftrightarrow$$

- H_4 of order 14400:

  $$\overset{5}{\longleftrightarrow} \longleftrightarrow \longleftrightarrow$$

- $I_2(m)$, $m \geq 3$, of order $2m$:

  $$\overset{n}{\longleftrightarrow}$$

Definition 1.2.3. *Let $w \in W$ and $w \neq e$. The length of w is*

$$\mathtt{l}(w) := \min\{r \in \mathbb{N} \,|\, w = s_{i_1} \dots s_{i_r},\ (i_s)_{s\in[r]} \in [l]^r\}.$$

We set $\mathtt{l}(e) = 0$.

From now on every element w of W is considered as product of $l(w)$ elements of S. For an element σ of $\mathcal{S}_n$, $l(\sigma)$ corresponds to the number of inversions of σ [3, Proposition 1.5.2].

Definition 1.2.4. *Let $w \in W$. The set of left resp. right descent set of w is*

$$\mathtt{DES_L}(w) := \{s \in S \,|\, \mathtt{l}(sw) < \mathtt{l}(w)\} \text{ resp. } \mathtt{DES_R}(w) := \{s \in S \,|\, \mathtt{l}(ws) < \mathtt{l}(w)\}.$$

We can see in [3, Proposition 1.5.3] that for an element σ of $\mathcal{S}_n$, the set of left resp. right descents is

$$\mathtt{DES_L}(\sigma) := \{s_i \in S \,|\, \sigma^{-1}(i) > \sigma^{-1}(i+1)\} \text{ resp. } \mathtt{DES_R}(\sigma) := \{s_i \in S \,|\, \sigma(i) > \sigma(i+1)\}.$$

Let $J, K \subseteq S$. We write

$${}^{J}W^{K} := \{w \in W \,|\, \mathtt{DES_L}(w) \cap J = \mathtt{DES_R}(w) \cap K = \varnothing\}.$$

If $J = \{\varnothing\}$ resp. $K = \{\varnothing\}$, we just write W^K resp. ${}^{J}W$.

Definition 1.2.5. *Let $J \subseteq S$. A parabolic subgroup of W is a group with generating system J. We write W_J for this parabolic subgroup.*

Proposition 1.2.6. *Let $J \subseteq S$. Every $w \in W$ has a unique factorisation*

$$w = w^J \cdot w_J \text{ resp. } w = w_J \cdot {}^{J}w$$

such that $w^J \in W^J$ resp. ${}^{J}w \in {}^{J}W$, $w_J \in W_J$ and

$$\mathtt{l}(w) = \mathtt{l}(w^J) + \mathtt{l}(w_J) \text{ resp. } \mathtt{l}(w) = \mathtt{l}(w_J) + \mathtt{l}({}^{J}w).$$

This is [3, Proposition 2.4.4].

Chapter 2

Examples of Diagonalization

In this chapter, we determine eigenvalues and eigenvectors for specific examples. Recall that the matrices studied are the matrices of left-multiplication actions on group algebras by elements of the group algebras. We begin with the case of group algebras of abelian groups which is a simple exercise. We continue with group algebras of dihedral groups which are slightly more complicated than the first case. Then we finish with a study which concerns general group algebras. However, the coefficients of the element $\mathfrak{f} \in \mathbb{K}[G]$ considered must be constant for group elements from the same conjugacy class.

2.1 Examples

2.1.1 Abelian Groups

Let Z be a cyclic group generated by the element z such that $o(z) = n$. We consider the element

$$\mathfrak{z} = \sum_{j \in [n]} \lambda_{z^j} z^j \ \in \ \mathbb{C}[Z]$$

where $\lambda_{z^j} \in \mathbb{C}$.
Let r, $t \in [n]$. We have:

$$\begin{aligned}
z^r \cdot \sum_{j \in [n]} \exp(2\pi \frac{jt}{n} i) z^j &= \sum_{j \in [n]} \exp(2\pi \frac{jt}{n} i) z^{j+r} \\
&= \sum_{j \in [n]} \exp(-2\pi \frac{rt}{n} i) \exp(2\pi \frac{rt}{n} i) \exp(2\pi \frac{jt}{n} i) z^{j+r} \\
&= \sum_{j \in [n]} \exp(-2\pi \frac{rt}{n} i) \exp\left(2\pi \frac{(j+r)t}{n} i\right) z^{j+r} \\
&= \exp(-2\pi \frac{rt}{n} i) \sum_{j \in [n]} \exp\left(2\pi \frac{(j+r)t}{n} i\right) z^{j+r} \\
&= \exp(-2\pi \frac{rt}{n} i) \sum_{j \in [n]} \exp(2\pi \frac{jt}{n} i) z^j.
\end{aligned}$$

Then, by linearity, we get for $t \in [n]$:

$$\begin{aligned}
\mathfrak{z} \cdot \sum_{j \in [n]} \exp(2\pi \frac{jt}{n} i) z^j &= \sum_{r \in [n]} \lambda_{z^r} z^r \cdot \sum_{j \in [n]} \exp(2\pi \frac{jt}{n} i) z^j \\
&= \sum_{r \in [n]} \lambda_{z^r} \exp(-2\pi \frac{rt}{n} i) \sum_{j \in [n]} \exp(2\pi \frac{jt}{n} i) z^j.
\end{aligned}$$

Since the number of possible t is n, we deduce that the matrix $\mathsf{X}_{\mathbb{C}[Z]}(\mathfrak{z})$ is diagonalizable, its eigenvalues are

$$\{\sum_{r\in[n]} \lambda_{z^r} \exp(-2\pi \frac{rt}{n} i)\}_{t\in[n]}$$

generically with multiplicity 1, and its eigenvectors are

$$\{ < \sum_{j\in[n]} \exp(2\pi \frac{jt}{n} i) z^j > \}_{t\in[n]}.$$

We generalize this result to the case of any abelian group.
Let A be an abelian group with minimal generating set $\{a_h\}_{h\in[l]}$ such that $o(a_h) = n_h$. We consider the element

$$\mathfrak{a} = \sum_{(j_1,\dots,j_l)\in[n_1]\times\cdots\times[n_l]} \lambda_{\prod_{h\in[l]} a_h^{j_h}} \prod_{h\in[l]} a_h^{j_h} \ \in \ \mathbb{C}[A]$$

where $\lambda_{\prod_{h\in[l]} a_h^{j_h}} \in \mathbb{C}$.
Let $(r_1,\dots,r_l),\ (t_1,\dots,t_l) \in [n_1]\times\cdots\times[n_l]$. We have:

$$\prod_{h\in[l]} a_h^{r_h} \cdot \sum_{(j_1,\dots,j_l)\in[n_1]\times\cdots\times[n_l]} \exp(2\pi \sum_{h\in[l]} \frac{j_h t_h}{n_h} i) \prod_{h\in[l]} a_h^{j_h} = \sum_{(j_1,\dots,j_l)\in[n_1]\times\cdots\times[n_l]} \exp(2\pi \sum_{h\in[l]} \frac{j_h t_h}{n_h} i) \prod_{h\in[l]} a_h^{j_h+r_h}$$

$$= \sum_{(j_1,\dots,j_l)\in[n_1]\times\cdots\times[n_l]} \exp(-2\pi \sum_{h\in[l]} \frac{r_h t_h}{n_h} i) \exp(2\pi \sum_{h\in[l]} \frac{r_h t_h}{n_h} i) \exp(2\pi \sum_{h\in[l]} \frac{j_h t_h}{n_h} i) \prod_{h\in[l]} a_h^{j_h+r_h}$$

$$= \sum_{(j_1,\dots,j_l)\in[n_1]\times\cdots\times[n_l]} \exp(-2\pi \sum_{h\in[l]} \frac{r_h t_h}{n_h} i) \exp\big(2\pi \sum_{h\in[l]} \frac{(j_h+r_h) t_h}{n_h} i\big) \prod_{h\in[l]} a_h^{j_h+r_h}$$

$$= \exp(-2\pi \sum_{h\in[l]} \frac{r_h t_h}{n_h} i) \sum_{(j_1,\dots,j_l)\in[n_1]\times\cdots\times[n_l]} \exp\big(2\pi \sum_{h\in[l]} \frac{(j_h+r_h) t_h}{n_h} i\big) \prod_{h\in[l]} a_h^{j_h+r_h}$$

$$= \exp(-2\pi \sum_{h\in[l]} \frac{r_h t_h}{n_h} i) \sum_{(j_1,\dots,j_l)\in[n_1]\times\cdots\times[n_l]} \exp(2\pi \sum_{h\in[l]} \frac{j_h t_h}{n_h} i) \prod_{h\in[l]} a_h^{j_h}.$$

Then, by linearity, we get for $(t_1,\dots,t_l) \in [n_1]\times\cdots\times[n_l]$:

$$\mathfrak{a} \cdot \sum_{(j_1,\dots,j_l)\in[n_1]\times\cdots\times[n_l]} \exp(2\pi \sum_{h\in[l]} \frac{j_h t_h}{n_h} i) \prod_{h\in[l]} a_h^{j_h}$$

$$= \sum_{(r_1,\dots,r_l)\in[n_1]\times\cdots\times[n_l]} \lambda_{\prod_{h\in[l]} a_h^{r_h}} \prod_{h\in[l]} a_h^{r_h} \cdot \sum_{(j_1,\dots,j_l)\in[n_1]\times\cdots\times[n_l]} \exp(2\pi \sum_{h\in[l]} \frac{j_h t_h}{n_h} i) \prod_{h\in[l]} a_h^{j_h}$$

$$= \sum_{(r_1,\dots,r_l)\in[n_1]\times\cdots\times[n_l]} \lambda_{\prod_{h\in[l]} a_h^{r_h}} \exp(-2\pi \sum_{h\in[l]} \frac{r_h t_h}{n_h} i) \sum_{(j_1,\dots,j_l)\in[n_1]\times\cdots\times[n_l]} \exp(2\pi \sum_{h\in[l]} \frac{j_h t_h}{n_h} i) \prod_{h\in[l]} a_h^{j_h}.$$

Since the number of possible $(t_1,\dots,t_l)$ equals the cardinality of the group A, we deduce that the matrix $\mathsf{X}_{\mathbb{C}[A]}(\mathfrak{a})$ is diagonalizable, its eigenvalues are

$$\{ \sum_{(r_1,\dots,r_l)\in[n_1]\times\cdots\times[n_l]} \lambda_{\prod_{h\in[l]} a_h^{r_h}} \exp(-2\pi \sum_{h\in[l]} \frac{r_h t_h}{n_h} i)\}_{(t_1,\dots,t_l)\in[n_1]\times\cdots\times[n_l]}$$

with multiplicity 1, and its eigenvectors are

$$\{ < \sum_{(j_1,\dots,j_l)\in[n_1]\times\cdots\times[n_l]} \exp(2\pi \sum_{h\in[l]} \frac{j_h t_h}{n_h} i) \prod_{h\in[l]} a_h^{j_h} > \}_{(t_1,\dots,t_l)\in[n_1]\times\cdots\times[n_l]}.$$

2.1.2 Dihedral Groups

Let $m \geq 3$. We use Theorem 1.1.4 to study a more complicated example: The dihedral group which is the Coxeter group $I_2(m)$ with generating system $\{r, s\}$. Recall that $o(r) = o(s) = 2$ and $o(rs) = o(sr) = m$. For this class of example, we need the following result:

Let $P, Q \in \mathbb{C}\big[t_i \,|\, i \in [n]\big]$. The roots of the polynomial $\begin{vmatrix} P - t & Q \\ \bar{Q} & \bar{P} - t \end{vmatrix}$ in the variable t are

$$Re\,P + \sqrt{Q\bar{Q} - (Im\,P)^2} \text{ and } Re\,P - \sqrt{Q\bar{Q} - (Im\,P)^2}. \tag{2.1}$$

The matrix representations of all the irreducible modules of $\mathbb{C}[I_2(m)]$ up to isomorphism are [9, Theorem]:

- if m is odd:

$$\mathsf{X}_0 : \begin{array}{ccc} I_2(m) & \to & \mathbb{C} \\ r & \mapsto & (1) \\ s & \mapsto & (1) \end{array}, \quad \mathsf{X}_{-1} : \begin{array}{ccc} I_2(m) & \to & \mathbb{C} \\ r & \mapsto & (-1) \\ s & \mapsto & (-1) \end{array}$$

$$\text{and} \left\{ \mathsf{X}_i : \begin{array}{ccc} I_2(m) & \to & \mathbb{C}^{2\times 2} \\ r & \mapsto & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ s & \mapsto & \begin{pmatrix} 0 & \exp(-\frac{2\pi}{m}i) \\ \exp(\frac{2\pi}{m}i) & 0 \end{pmatrix} \end{array} \right\}_{i\in[\frac{m-1}{2}]},$$

- if m is even:

$$\mathsf{X}_0 : \begin{array}{ccc} I_2(m) & \to & \mathbb{C} \\ r & \mapsto & (1) \\ s & \mapsto & (1) \end{array}, \quad \mathsf{X}_{-1} : \begin{array}{ccc} I_2(m) & \to & \mathbb{C} \\ r & \mapsto & (1) \\ s & \mapsto & (-1) \end{array}, \quad \mathsf{X}_{-2} : \begin{array}{ccc} I_2(m) & \to & \mathbb{C} \\ r & \mapsto & (-1) \\ s & \mapsto & (1) \end{array},$$

$$\mathsf{X}_{-3} : \begin{array}{ccc} I_2(m) & \to & \mathbb{C} \\ r & \mapsto & (-1) \\ s & \mapsto & (-1) \end{array} \text{ and } \left\{ \mathsf{X}_i : \begin{array}{ccc} I_2(m) & \to & \mathbb{C}^{2\times 2} \\ r & \mapsto & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ s & \mapsto & \begin{pmatrix} 0 & \exp(-\frac{2\pi}{m}i) \\ \exp(\frac{2\pi}{m}i) & 0 \end{pmatrix} \end{array} \right\}_{i\in[\frac{m}{2}-1]}.$$

If m is odd, the conjugacy classes of $I_2(m)$ are [7, Theorem 1.1]: $\big(\{(rs)^{\pm j}\}\big)_{j\in[\frac{m-1}{2}]^*}$ and $\{(rs)^j r\}_{j\in[m-1]^*}$. Let λ_e, $\big(\lambda_{(rs)^j}, \lambda_{(sr)^j}\big)_{j\in[\frac{m-1}{2}]}$, $(\lambda_{(rs)^j r})_{j\in[m-1]^*} \in \mathbb{C}$. From the following table, we get the matrix of the action of the element

$$\mathsf{i}_o = \lambda_e\, e + \sum_{j\in[\frac{m-1}{2}]} \big(\lambda_{(rs)^j}\,(rs)^j + \lambda_{(sr)^j}\,(sr)^j\big) + \sum_{j\in[m-1]^*} \lambda_{(rs)^j r}\,(rs)^j r$$

on $\mathbb{C}[I_2(m)]$

	$\lambda_e\, e$	$\lambda_{(rs)^j}\,(rs)^j$	$\lambda_{(sr)^j}\,(sr)^j$
X_0	λ_e	$\lambda_{(rs)^j}$	$\lambda_{(sr)^j}$
X_{-1}	λ_e	$\lambda_{(rs)^j}$	$\lambda_{(sr)^j}$
X_i	$\begin{matrix} \lambda_e & 0 \\ 0 & \lambda_e \end{matrix}$	$\begin{matrix} \lambda_{(rs)^j}\exp(\frac{2\pi}{m}ij) & 0 \\ 0 & \lambda_{(rs)^j}\exp(-\frac{2\pi}{m}ij) \end{matrix}$	$\begin{matrix} \lambda_{(sr)^j}\exp(-\frac{2\pi}{m}ij) & 0 \\ 0 & \lambda_{(sr)^j}\exp(\frac{2\pi}{m}ij) \end{matrix}$

	$\lambda_{(rs)^j r}\,(rs)^j r$
X_0	$\lambda_{(rs)^j r}$
X_{-1}	$-\lambda_{(rs)^j r}$
X_i	$\begin{matrix} 0 & \lambda_{(rs)^j r}\ \exp(\frac{2\pi}{m} ij) \\ \lambda_{(rs)^j r}\ \exp(-\frac{2\pi}{m} ij) & 0 \end{matrix}$

where $i \in [\frac{m-1}{2}]$ and $j \in [m-1]^*$. Let

$$P_i = \lambda_e + \sum_{j\in[\frac{m-1}{2}]} \big(\lambda_{(rs)^j} \exp(\frac{2\pi}{m} ij) + \lambda_{(sr)^j} \exp(-\frac{2\pi}{m} ij)\big),$$

$$Q_i = \sum_{j\in[m-1]^*} \lambda_{(rs)^j r} \exp(\frac{2\pi}{m} ij).$$

Using this table and (2.1), we get the eigenvalues and their multiplicities for $\mathfrak{i}_o$:

Eigenvalues	Multiplicities
$\lambda_e + \sum_{j\in[\frac{m-1}{2}]} \big(\lambda_{(rs)^j} + \lambda_{(sr)^j}\big) + \sum_{j\in[m-1]^*} \lambda_{(rs)^j r}$	1
$\lambda_e + \sum_{j\in[\frac{m-1}{2}]} \big(\lambda_{(rs)^j} + \lambda_{(sr)^j}\big) - \sum_{j\in[m-1]^*} \lambda_{(rs)^j r}$	1
$\Big\{Re\, P_i + \big(Q_i\bar{Q}_i - (Im\, P_i)^2\big)^{\frac{1}{2}}\Big\}_{i\in[\frac{m-1}{2}]}$	$\{2\}_{i\in[\frac{m-1}{2}]}$
$\Big\{Re\, P_i - \big(Q_i\bar{Q}_i - (Im\, P_i)^2\big)^{\frac{1}{2}}\Big\}_{i\in[\frac{m-1}{2}]}$	$\{2\}_{i\in[\frac{m-1}{2}]}$

If m is even, the conjugacy classes of $I_2(m)$ are [7, Theorem 1.1]:

$$\big(\{(rs)^{\pm j}\}\big)_{j\in[\frac{m}{2}]^*},\ \{(rs)^{2j} r\}_{j\in[\frac{m}{2}-1]^*} \text{ and } \{(rs)^{2j+1} r\}_{j\in[\frac{m}{2}-1]^*}.$$

Let λ_e, $\lambda_{(rs)^{\frac{m}{2}}}$, $\big(\lambda_{(rs)^j},\, \lambda_{(sr)^j}\big)_{j\in[\frac{m}{2}-1]}$, $\big(\lambda_{(rs)^{2j} r},\, \lambda_{(rs)^{2j+1} r}\big)_{j\in[\frac{m}{2}-1]^*} \in \mathbb{C}$. From the following table, we get the matrix of the action of the element

$$\mathfrak{i}_e = \lambda_e\, e + \lambda_{(rs)^{\frac{m}{2}}}\, (rs)^{\frac{m}{2}} + \sum_{j\in[\frac{m}{2}-1]} \big(\lambda_{(rs)^j}\, (rs)^j + \lambda_{(sr)^j}\, (sr)^j\big)$$
$$+ \sum_{j\in[\frac{m}{2}-1]^*} \big(\lambda_{(rs)^{2j} r}\, (rs)^{2j} r + \lambda_{(rs)^{2j+1} r}\, (rs)^{2j+1} r\big)$$

on $\mathbb{C}[G]$ on $\mathbb{C}[I_2(m)]$

	$\lambda_e\, e$	$\lambda_{(rs)^{\frac{m}{2}}}\, (rs)^{\frac{m}{2}}$	$\lambda_{(rs)^j}\, (rs)^j$	$\lambda_{(sr)^j}\, (sr)^j$
X_0	λ_e	$\lambda_{(rs)^{\frac{m}{2}}}$	$\lambda_{(rs)^j}$	$\lambda_{(sr)^j}$
X_{-1}	λ_e	$(-1)^{\frac{m}{2}} \lambda_{(rs)^{\frac{m}{2}}}$	$(-1)^j \lambda_{(rs)^j}$	$(-1)^j \lambda_{(sr)^j}$
X_{-2}	λ_e	$(-1)^{\frac{m}{2}} \lambda_{(rs)^{\frac{m}{2}}}$	$(-1)^j \lambda_{(rs)^j}$	$(-1)^j \lambda_{(sr)^j}$
X_{-3}	λ_e	$\lambda_{(rs)^{\frac{m}{2}}}$	$\lambda_{(rs)^j}$	$\lambda_{(sr)^j}$
X_i	$\begin{matrix} \lambda_e & 0 \\ 0 & \lambda_e \end{matrix}$	$\begin{matrix} \lambda_{(rs)^{\frac{m}{2}}} & 0 \\ 0 & \lambda_{(rs)^{\frac{m}{2}}} \end{matrix}$	$\begin{matrix} \lambda_{(rs)^j}\ \exp(\frac{2\pi}{m} ij) & 0 \\ 0 & \lambda_{(rs)^j}\ \exp(-\frac{2\pi}{m} ij) \end{matrix}$	$\begin{matrix} \lambda_{(sr)^j}\ \exp(-\frac{2\pi}{m} ij) & 0 \\ 0 & \lambda_{(sr)^j}\ \exp(\frac{2\pi}{m} ij) \end{matrix}$

	$\lambda_{(rs)^{2j}r}\,(rs)^{2j}r$	$\lambda_{(rs)^{2j+1}r}\,(rs)^{2j+1}r$
X_0	$\lambda_{(rs)^{2j}r}$	$\lambda_{(rs)^{2j+1}r}$
X_{-1}	$\lambda_{(rs)^{2j}r}$	$-\lambda_{(rs)^{2j+1}r}$
X_{-2}	$-\lambda_{(rs)^{2j}r}$	$\lambda_{(rs)^{2j+1}r}$
X_{-3}	$-\lambda_{(rs)^{2j}r}$	$-\lambda_{(rs)^{2j+1}r}$
X_i	$\begin{matrix} 0 & \lambda_{(rs)^{2j}r}\exp(\frac{2\pi}{m}ij) \\ \lambda_{(rs)^{2j}r}\exp(-\frac{2\pi}{m}ij) & 0 \end{matrix}$	$\begin{matrix} 0 & \lambda_{(rs)^{2j+1}r}\exp(\frac{2\pi}{m}ij) \\ \lambda_{(rs)^{2j+1}r}\exp(-\frac{2\pi}{m}ij) & 0 \end{matrix}$

where $i \in [\frac{m}{2}-1]$ and $j \in [\frac{m}{2}-1]^*$. Let

$$P_i = \lambda_e + \lambda_{(rs)^{\frac{m}{2}}} + \sum_{j\in[\frac{m}{2}-1]} \big(\lambda_{(rs)^j}\exp(\frac{2\pi}{m}ij) + \lambda_{(sr)^j}\exp(-\frac{2\pi}{m}ij)\big),$$

$$Q_i = \sum_{j\in[\frac{m}{2}-1]^*} \big(\lambda_{(rs)^{2j}r} + \lambda_{(rs)^{2j+1}r}\big)\exp(\frac{2\pi}{m}ij).$$

Using this table and (2.1), we get the eigenvalues and their multiplicities for $\mathfrak{i}_e$:

Eigenvalues	Multiplicities
$\lambda_e + \lambda_{(rs)^{\frac{m}{2}}} + \sum_{j\in[\frac{m}{2}-1]}\big(\lambda_{(rs)^j} + \lambda_{(sr)^j}\big) + \sum_{j\in[\frac{m}{2}-1]^*}\big(\lambda_{(rs)^{2j}r} + \lambda_{(rs)^{2j+1}r}\big)$	1
$\lambda_e + (-1)^{\frac{m}{2}}\lambda_{(rs)^{\frac{m}{2}}} + \sum_{j\in[\frac{m}{2}-1]}(-1)^j\big(\lambda_{(rs)^j} + \lambda_{(sr)^j}\big) + \sum_{j\in[\frac{m}{2}-1]^*}\big(\lambda_{(rs)^{2j}r} - \lambda_{(rs)^{2j+1}r}\big)$	1
$\lambda_e + (-1)^{\frac{m}{2}}\lambda_{(rs)^{\frac{m}{2}}} + \sum_{j\in[\frac{m}{2}-1]}(-1)^j\big(\lambda_{(rs)^j} + \lambda_{(sr)^j}\big) + \sum_{j\in[\frac{m}{2}-1]^*}\big(-\lambda_{(rs)^{2j}r} + \lambda_{(rs)^{2j+1}r}\big)$	1
$\lambda_e + \lambda_{(rs)^{\frac{m}{2}}} + \sum_{j\in[\frac{m}{2}-1]}\big(\lambda_{(rs)^j} + \lambda_{(sr)^j}\big) - \sum_{j\in[\frac{m}{2}-1]^*}\big(\lambda_{(rs)^{2j}r} + \lambda_{(rs)^{2j+1}r}\big)$	1
$\Big\{Re\,P_i + \big(Q_i\bar{Q}_i - (Im\,P_i)^2\big)^{\frac{1}{2}}\Big\}_{i\in[\frac{m}{2}-1]}$	$\{2\}_{i\in[\frac{m}{2}-1]}$
$\Big\{Re\,P_i - \big(Q_i\bar{Q}_i - (Im\,P_i)^2\big)^{\frac{1}{2}}\Big\}_{i\in[\frac{m}{2}-1]}$	$\{2\}_{i\in[\frac{m}{2}-1]}$

Renteln could give this result unified for m odd and even [18, Theorem 16] for special coefficients of $\mathfrak{i}_o$ and $\mathfrak{i}_e$: Let $m \geq 1$ and $\mathfrak{r} = \sum_{w\in I_2(m)} \mathtt{l}(w)w \in \mathbb{R}[I_2(m)]$. The eigenvalues of the matrix $\mathsf{X}_{\mathbb{R}[I_2(m)]}(\mathfrak{r})$ are

$$\Lambda_k = \begin{cases} -\frac{1}{\sin^2(\frac{k\pi}{2m})} & \text{if } k \text{ odd and nonzero} \\ 0 & \text{if } k \text{ even and nonzero} \\ m^2 & \text{if } k = 0. \end{cases}$$

Recall that the definition of the length $\mathtt{l}(w)$ of an element w is given in Definition 1.2.3. Because of the length of the calculation, we omit to give the proof calculated from our results of this theorem.

Until now, we have seen relatively "simple" group algebras for which the eigenvalues and their multiplicities of the matrices of the actions of its elements have been completely determined. We will see that for more complicated group algebras, we can determine eigenvalues and multiplicities of matrices of actions just for special elements of those group algebras.

2.2 Center Elements

In this section, we consider the elements $\mathfrak{m} = \sum_{i=1}^{n} \lambda_{g_i} g_i$ of $\mathbb{K}[G]$ having this property:

$$\lambda_{g_i} = \lambda_{g_j} \text{ if there exists } g \in G \text{ such that } g_j = g^{-1}g_i g,$$

i.e. elements of the center of $\mathbb{K}[G]$.
Let $\mathscr{C}_G = \{G_i \,|\, i \in [k]\}$ be the set of the conjugacy classes of G. We write

$$g_{G_i} = \sum_{g \in G_i} g.$$

Let $(\lambda_i)_{i\in[k]} \in \mathbb{K}^k$. We write $\mathbb{K}[\mathscr{C}_G]$ for the set of elements of $\mathbb{K}[G]$ of the form

$$\mathfrak{c} = \sum_{i=1}^{k} \lambda_i g_{G_i}$$

which are in the center of $\mathbb{K}[G]$.
Let M be an irreducible module of $\mathbb{K}[G]$. We write $\chi_M^{G_i}$ for the common value of $tr\big(\mathsf{X}_M(g)\big)$ for all g in the conjugacy class G_i.

Proposition 2.2.1. *Let $\mathfrak{c} = \sum_{i=1}^{k} \lambda_i g_{G_i} \in \mathbb{K}[\mathscr{C}_G]$ and $(M_i)_{i\in[k]}$ be a full system of irreducible modules of $\mathbb{K}[G]$ and $(d_i)_{i\in[k]}$ be the sequence of the corresponding degrees. On a suitable basis of $\mathbb{K}[G]$, we have*

$$\mathsf{X}_{\mathbb{K}[G]}(\mathfrak{c}) = \bigoplus_{i=1}^{k} \mu_i \mathsf{I}_{d_i^2}$$

where

$$\mu_i = \frac{\sum_{j=1}^{k} \lambda_j |G_j| \chi_{M_i}^{G_j}}{d_i}$$

is an eigenvalue of $\mathfrak{c}$ which multiplicity d_i^2.

Proof. We have

$$g^{-1}\mathfrak{c}g = \sum_{i=1}^{k} \lambda_i \; g^{-1} \cdot g_{G_i} \cdot g = \sum_{i=1}^{k} \lambda_i g_{G_i} = \mathfrak{c}$$

for all $g \in G$. Then, by Theorem 1.1.5, we have $\mathsf{X}_{M_i}(c) = \mu_i \mathsf{I}_{d_i}$. The only possibility is

$$\mu_i = \frac{tr(\mathsf{X}_{M_i}(\mathfrak{c}))}{d_i} = \frac{\sum_{j=1}^{k} \lambda_j |G_j| \chi_{M_i}^{G_j}}{d_i}.$$

□

We treat

$$\mathfrak{f}_n = \sum_{\sigma \in \mathcal{S}_n} \left(\mathtt{fix}(\sigma)t + q^{\mathtt{fix}(\sigma)}\right)\sigma \in \mathbb{R}(t,q)[\mathscr{S}_n]$$

as example, where $\mathtt{fix}(\sigma) := \#\{i \in [n] \,|\, \sigma(i) = i\}$. Let $\tau \in \mathcal{S}_n$. We have

$$\tau^{-1}\mathfrak{f}_n\tau = \sum_{\sigma \in \mathcal{S}_n} \left(\mathtt{fix}(\sigma)t + q^{\mathtt{fix}(\sigma)}\right)\tau^{-1}\sigma\tau = \sum_{\sigma \in \mathcal{S}_n} \left(\mathtt{fix}(\tau^{-1}\sigma\tau)t + q^{\mathtt{fix}(\tau^{-1}\sigma\tau)}\right)\tau^{-1}\sigma\tau = \mathfrak{f}_n.$$

Then we can apply Proposition 2.2.1 to $\mathsf{X}_{\mathbb{K}[\mathcal{S}_n]}(\mathfrak{f}_n)$. Recall that the notation $\mu \vdash n$ means μ is a partition of n. For $i \in [n]$, we write $\mu(i)$ for the number of occurences of i in the partition μ of n.
Let $(G_\mu)_{\mu \vdash n}$ be the conjugacy classes of $\mathcal{S}_n$ such that

$$|G_\mu| = \frac{n!}{1^{\mu(1)}\mu(1)!2^{\mu(2)}\mu(2)!\dots n^{\mu(n)}\mu(n)!}.$$

It is known that the irreducible modules of $\mathcal{S}_n$ are the Specht modules $(S_\mu)_{\mu \vdash n}$ [20, Theorem 2.4.6] and that the degree of the Specht module S_μ is the number of standard μ-tableaux f_μ [20, Theorem 2.6.5]. So on a suitable basis of $\mathbb{K}[\mathcal{S}_n]$, we have

$$\mathsf{X}_{\mathbb{K}[\mathcal{S}_n]}(\mathfrak{f}_n) = \bigoplus_{\mu \vdash n} \frac{\sum_{\nu \vdash n} \left(\nu(1)t + q^{\nu(1)}\right)|G_\nu|\chi_{S_\mu}^{G_\nu}}{f_\mu} \mathsf{I}_{f_\mu^2}.$$

We note that it is relatively easy to get cardinalities of conjugacy classes. We have for example the cardinalities of conjugacy classes of H_3 in **Section 3.2** and those of F_4 in **Section 3.3**.

Chapter 3

Elements of the Descent Algebra

This chapter is devoted to a special subset of the group algebra of a Coxeter group which is the descent algebra of this Coxeter group. More precisely, we study the action of the elements of descent algebras by multiplication on the group algebras. We determine the eigenvalues and multiplicities of the matrices of those actions. We begin by defining the descent algebras. Then we determine a formula for special coefficients of the descent algebras. We will finally be able to calculate the eigenvalues and multiplicities in the last section.

3.1 The Descent Algebra of a Coxeter Group

Let $J \subseteq S$. We write

$$x_J := \sum_{w \in W^J} w.$$

We get the definition of the descent algebra from the following famous result of Solomon on Coxeter groups [21, Theorem 1]:

Theorem 3.1.1. *Let $J, K \subseteq S$. We have*

$$x_J x_K = \sum_{L \subseteq K} a_{JKL}\, x_L$$

where

$$a_{JKL} := \left|\{x \in {}^J W^K \mid x^{-1} W_J\, x \cap W_K = W_L\}\right|.$$

Definition 3.1.2. *The descent algebra $\mathbb{K}[\mathscr{D}_W]$ of the Coxeter group W is the vector space over $\mathbb{K}$ with the basis $\{x_J\}_{J \subseteq S}$ and the multiplication defined in Theorem 3.1.1.*

`Example:` An element $\mathfrak{d}_{A_2}$ of $\mathbb{K}[\mathscr{D}_{A_2}]$ can be written as:

$$\begin{aligned}\mathfrak{d}_{A_2} = \lambda_\varnothing&(e + s_1 + s_2 + s_1 s_2 + s_2 s_1 + s_1 s_2 s_1)\\ &+ \lambda_{\{s_1\}}(e + s_2 + s_1 s_2) + \lambda_{\{s_2\}}(e + s_1 + s_2 s_1)\\ &+ \lambda_{S_{A_2}} e\end{aligned}$$

where the Coxeter graph of A_2 is $s_1 \longleftrightarrow s_2$.

We aim to study the linear algebraic properties of $\mathsf{X}_{\mathbb{K}[W]}(\mathfrak{d})$ for an element $\mathfrak{d}$ of $\mathbb{K}[\mathscr{D}_W]$.

3.2 Special Coefficients of the Descent Algebra

In this section, we determine a formula for the values of a_{JKK}. A theorem has already been proposed in [2, Theorem 6.5] to determine those values. However this theorem seems to contain a mistake. That is why we propose a new formula which is, in fact, a slight modification of [2, Theorem 6.5].
Let $J \subseteq S$. We write $\mathbf{C}_J$ for the set of subsets L of S for which W_L is conjugate to W_J, i.e.

$$\mathbf{C}_J := \{L \subseteq S \,|\, \exists w \in W,\, w^{-1}W_J w = W_L\},$$

and N_J for the intersection of the normalizer of W_J with W^J, i.e.

$$N_J := \{w \in W \,|\, w^{-1}W_J w = W_J\} \cap W^J.$$

Let us note that N_J is a group [12, Corollary 3].
Let $w \in W$ and U, V be subgroups of W. We write

$${}^U w^V := \{x \in UwV \mid \mathtt{l}(x) \leq \mathtt{l}(uxv),\, \forall u \in U,\, \forall v \in V\}$$

for the set of minimal double coset representatives of w. If $U = \{e\}$ resp. $V = \{e\}$, we just write w^V resp. ${}^U w$.

Lemma 3.2.1. *Let $w \in W$ and $J, K \subseteq S$. Then the set ${}^{W_J}w^{W_K}$ contains a unique element.*

We can see this in [21, 1.Introduction]. In this case, we consider ${}^{W_J}w^{W_K}$ no more as a subset of W but as an element of W.
Let $K \subseteq S$ and $K' \in \mathbf{C}_K$. We write

$$C_{K'K} := \{w \in W \mid w^{-1}W_{K'}\, w = W_K\}$$

Lemma 3.2.2. *Let $c \in C_{K'K}$ and $c_{K'K}$ be a double coset representative of c:*

$$c_{K'K} \in {}^{N_{K'}W_{K'}}c^{N_K W_K}.$$

Then $c_{K'K} \in C_{K'K}$.

Proof. If $c = n'\,k'\,c_{K'K}\,n\,k$ with $n'k' \in N_{K'}W_{K'}$ and $nk \in N_K W_K$, then

$$\begin{array}{rcl} (n'\,k'\,c_{K'K}\,n\,k)^{-1}W_{K'}\; n'\,k'\,c_{K'K}\,n\,k &=& W_K, \\ (c_{K'K}\,n\,k)^{-1}W_{K'}\; c_{K'K}\,n\,k &=& W_K, \\ c_{K'K}^{-1}W_{K'}\; c_{K'K} &=& n\,k\,W_K\,(n\,k)^{-1}, \\ c_{K'K}^{-1}W_{K'}\; c_{K'K} &=& W_K. \end{array}$$

□

We fix a double coset representative $c_{K'K} \in C_{K'K}$ for the rest of the chapter.
Let $E \subseteq W$ and U, V be subgroups of W. We write ${}^U E^V$ for the set of double coset representatives of the elements of E, i.e.

$${}^U E^V := \bigcup_{w \in E} {}^U w^V.$$

Lemma 3.2.3. *Let $K \subseteq S$ and $K' \in \mathbf{C}_K$. Then*

$${}^{W_{K'}}(c_{K'K}N_K)^{W_K} = c_{K'K}N_K \text{ and } {}^{W_{K'}}(N_{K'}\,c_{K'K})^{W_K} = N_{K'}\,c_{K'K}.$$

Proof. It is clear that $(c_{K'K}N_K)^{W_K} = c_{K'K}N_K$. Effectively, since $c_{K'K}$ is a left coset representative of the group $N_K W_K$ and the group N_K is a set of left coset representatives of W_K, then $c_{K'K}N_K$ is a set of left coset representatives of W_K.
We then just have to prove that ${}^{W_{K'}}(c_{K'K}N_K) = c_{K'K}N_K$. Let $c_{K'K}\,n \in c_{K'K}N_K$ and let us suppose that $c_{K'K}\,n = k'b$ where $k' \in W_{K'}$ and $b \in {}^{K'}W$. Then $\mathtt{l}(c_{K'K}\,n) = \mathtt{l}(k'b) = \mathtt{l}(k') + \mathtt{l}(b)$ i.e.

$$\mathtt{l}(c_{K'K}\,n) \geq \mathtt{l}(b).$$

On the other hand, we have $(k')^{-1}c_{K'K}\,n = c_{K'K}\,k_1 n = c_{K'K}\,n k_2 = b$ with $k_1, k_2 \in W_K$. Then $\mathtt{l}(c_{K'K}\,nk_2) = \mathtt{l}(c_{K'K}\,n) + \mathtt{l}(k_2) = \mathtt{l}(b)$, i.e.

$$\mathtt{l}(c_{K'K}\,n) \leq \mathtt{l}(b).$$

The only possibility is then $k_2 = k_1 = k' = e$. So we get the result.
The proof for ${}^{W_{K'}}(N_{K'}\,c_{K'K})^{W_K} = N_{K'}\,c_{K'K}$ is analogous. But we provide the proof for the sake of completeness.
It is clear that ${}^{W_{K'}}(N_{K'}\,c_{K'K}) = N_{K'}\,c_{K'K}$. Effectively, since $c_{K'K}$ is a right coset representative of the group $W_{K'}N_{K'}$ and the group $N_{K'}$ is a set of right coset representatives of $W_{K'}$, then $N_{K'}c_{K'K}$ is a set of right coset representatives of $W_{K'}$.
We then just have to prove that $(N_{K'}\,c_{K'K})^{W_K} = N_{K'}\,c_{K'K}$. Let $n'\,c_{K'K} \in N_{K'}\,c_{K'K}$ and let us suppose that $n'\,c_{K'K} = bk$ where $k \in W_K$ and $b \in W^K$. Then $\mathtt{l}(n'\,c_{K'K}) = \mathtt{l}(bk) = \mathtt{l}(b) + \mathtt{l}(k)$ i.e.

$$\mathtt{l}(n'\,c_{K'K}) \geq \mathtt{l}(b).$$

On the other hand, we have $n'\,c_{K'K}\,k^{-1} = n'k'_1\,c_{K'K} = k'_2 n'\,c_{K'K} = b$ with $k'_1, k'_2 \in W_{K'}$. Then $\mathtt{l}(k'_2 n'\,c_{K'K}) = \mathtt{l}(k'_2) + \mathtt{l}(n'\,c_{K'K}) = \mathtt{l}(b)$, i.e.

$$\mathtt{l}(n'\,c_{K'K}) \leq \mathtt{l}(b).$$

The only possibility is then $k'_2 = k'_1 = k = e$. So we get the result. □

Lemma 3.2.4. *Let $K \subseteq S$ and $K' \in \mathbf{C}_K$. Then*

$$\{w \in {}^{K'}W^K \mid w^{-1}W_{K'}\,w = W_K\} = c_{K'K}N_K = N_{K'}\,c_{K'K}.$$

Proof. It is clear that $\{w \in {}^{K}W^K \mid w^{-1}W_K\,w = W_K\} = N_K$ for all $K \subseteq S$.

- The map $\phi : \{w \in {}^{K'}W^{K'} \mid w^{-1}W_{K'}\,w = W_{K'}\} \to \{w \in {}^{K'}W^K \mid w^{-1}W_{K'}\,w = W_K\}$

$$n \mapsto n\,c_{K'K}$$

  is clearly injective.

- The map $\phi' : \{w \in {}^{K'}W^K \mid w^{-1}W_{K'}\,w = W_K\} \to \{w \in {}^{K'}W^{K'} \mid w^{-1}W_{K'}\,w = W_{K'}\}$,

$$x \mapsto {}^{W_{K'}}(c_{K'K}\,x^{-1})^{W_{K'}}$$

  is injective. Effectively, $\phi'(x) = \phi'(y)$ means $c_{K'K}\,x^{-1} = u_1 c_{K'K}\,y^{-1}u_2$ with $u_1, u_2 \in W_{K'}$. Then $c_{K'K}\,x^{-1} = c_{K'K}vy^{-1}u_2$ and $x^{-1} = vy^{-1}u_2$ with $v \in W_K$. The only possibility is $v = u_2 = e$.

Then we deduce that ϕ is bijective and $\{w \in {}^{K'}W^K \mid w^{-1}W_{K'}\,w = W_K\} = N_{K'}\,c_{K'K}$.

- The map $\varphi : \{w \in {}^{K}W^K \mid w^{-1}W_K\,w = W_K\} \to \{w \in {}^{K'}W^K \mid w^{-1}W_{K'}\,w = W_K\}$

$$n \mapsto c_{K'K}\,n$$

  is clearly injective.

- The map $\varphi' : \{w \in {}^{K'}W^K \mid w^{-1}W_{K'}\, w = W_K\} \to \{w \in {}^{K}W^K \mid w^{-1}W_K\, w = W_K\}$,
$$x \mapsto {}^{W_K}(x^{-1}\, c_{K'K})^{W_K}$$
is injective. Effectively, $\varphi'(x) = \varphi'(y)$ means $x^{-1}\, c_{K'K} = u_1 y^{-1}\, c_{K'K}\, u_2$ with $u_1, u_2 \in W_K$. Then $x^{-1}\, c_{K'K} = u_1 y^{-1} v c_{K'K}$ and $x^{-1} = u_1 y^{-1} v$ with $v \in W_{K'}$. The only possibility is $u_1 = v = e$.

Then the map φ is also bijective and $\{w \in {}^{K'}W^K \mid w^{-1}W_{K'}\, w = W_K\} = c_{K'K} N_K$. $\square$

Let $J, K \subseteq S$. We write
$$\mathbf{H}_{JK} := \{K' \in \mathbf{C}_K \mid c_{K'K} = {}^{W_J}c_{K'K}\}.$$
We can now give the formula to determine a_{JKK}.

Theorem 3.2.5. *Let $J, K \subseteq S$. We have*
$$a_{JKK} = \sum_{K' \in \mathbf{H}_{JK} \cap 2^J} \frac{|N_K|}{|W_J \cap N_{K'}|}.$$

Proof. Recall that
$$a_{JKK} := \left|\{w \in {}^{J}W^K \mid W_J \cap wW_K w^{-1} = wW_K w^{-1}\}\right|$$
where $wW_K w^{-1} = W_{K'}$ and $K' \in \mathbf{C}_K \cap 2^J$.
We have $w \in c_{K'K} N_K = N_{K'}\, c_{K'K}$. But we must also have $w = {}^{W_J}w$. That means:

- On the one hand, we must have $c_{K'K} = {}^{W_J}c_{K'K}$. Otherwise ${}^{W_J}(c_{K'K} N_K) \cap C_{K'K} = \varnothing$.
- On the other hand, if $c_{K'K} = {}^{W_J}c_{K'K}$, then $w \in {}^{W_J}(N_{K'}\, c_{K'K}) = ({}^{W_J}N_{K'})\, c_{K'K}$.

Since
$$\left|{}^{W_J}N_{K'}\right| = \frac{|N_{K'}|}{|W_J \cap N_{K'}|},$$
it follows that
$$\begin{aligned} a_{JKK} &= \sum_{K' \in \mathbf{H}_{JK} \cap 2^J} \frac{|N_{K'}|}{|W_J \cap N_{K'}|} \\ &= \sum_{K' \in \mathbf{H}_{JK} \cap 2^J} \frac{|N_K|}{|W_J \cap N_{K'}|}. \end{aligned}$$
since W_K and $W_{K'}$ are conjugate. $\square$

The theorem proposed in [2, Theorem 6.5] is:

Theorem 3.2.6. *Let $J, K \subseteq S$. We have*
$$a_{JKK} = \sum_{K' \in \mathbf{C}_K \cap 2^J} \frac{|N_K|}{|W_J \cap N_{K'}|}.$$

Recall that $|H_3| = 120$ and the Coxeter graph of H_3 is:
$$s_1 \overset{5}{\longleftrightarrow} s_2 \longleftrightarrow s_3$$

We have the values of $|N_K|$ for H_3 in [12, page 79].
If we use Theorem 3.2.6, we get the following values of a_{JKK} for H_3:

	$\varnothing$	$\{s_1\}$	$\{s_1, s_2\}$	$\{s_2, s_3\}$	$\{s_1, s_3\}$	S_{H_3}
$\varnothing$	120	0	0	0	0	0
$\{s_1\}$	60	4	0	0	0	0
$\{s_1, s_2\}$	12	8	2	0	0	0
$\{s_2, s_3\}$	20	8	0	2	0	0
$\{s_1, s_3\}$	30	4	0	0	2	0
S_{H_3}	1	1	1	1	1	1

Let $\mathsf{A} = \begin{pmatrix} 120 & 0 & 0 & 0 & 0 & 0 \\ 60 & 4 & 0 & 0 & 0 & 0 \\ 12 & 8 & 2 & 0 & 0 & 0 \\ 20 & 8 & 0 & 2 & 0 & 0 \\ 30 & 4 & 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$. We will see in **Section 3.3** that $\mathsf{A}^{-1} \begin{pmatrix} 120 \\ 120 \\ 120 \\ 120 \\ 120 \\ 120 \end{pmatrix}$ gives the cardinalities of the conjugacy classes of H_3. However, we get

$$\mathsf{A}^{-1} \begin{pmatrix} 120 \\ 120 \\ 120 \\ 120 \\ 120 \\ 120 \end{pmatrix} = \begin{pmatrix} 1 \\ 15 \\ -6 \\ -10 \\ 15 \\ 105 \end{pmatrix}$$

which is absurd.
But if we use Theorem 3.2.5, the values of a_{JKK} calculated for H_3 are:

	$\varnothing$	$\{s_1\}$	$\{s_1, s_2\}$	$\{s_2, s_3\}$	$\{s_1, s_3\}$	S_{H_3}
$\varnothing$	120	0	0	0	0	0
$\{s_1\}$	60	4	0	0	0	0
$\{s_1, s_2\}$	12	4	2	0	0	0
$\{s_2, s_3\}$	20	4	0	2	0	0
$\{s_1, s_3\}$	30	4	0	0	2	0
S_{H_3}	1	1	1	1	1	1

Let $\mathsf{B} = \begin{pmatrix} 120 & 0 & 0 & 0 & 0 & 0 \\ 60 & 4 & 0 & 0 & 0 & 0 \\ 12 & 4 & 2 & 0 & 0 & 0 \\ 20 & 4 & 0 & 2 & 0 & 0 \\ 30 & 4 & 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$. Then we get the following cardinalities of conjugacy classes of H_3

$$\mathsf{B}^{-1} \begin{pmatrix} 120 \\ 120 \\ 120 \\ 120 \\ 120 \\ 120 \end{pmatrix} = \begin{pmatrix} 1 \\ 15 \\ 24 \\ 20 \\ 15 \\ 45 \end{pmatrix}$$

which gives the correct values.
Let $J' \in \mathbf{C}_J$ and $K' \in \mathbf{C}_K$. From Theorem 3.2.5, we deduce that

$$a_{JKK} = a_{J'K'K'}.$$

This can be found in [2, Theorem 6.2].

Using Theorem 3.2.5 and the values of $|N_K|$ in [12, page 74], we get the following values of a_{JKK} for the case of F_4:

	$\varnothing$	$\{s_1\}$	$\{s_4\}$	$\{s_1, s_2\}$	$\{s_2, s_3\}$	$\{s_3, s_4\}$	$\{s_1, s_4\}$	$\{s_1, s_2, s_3\}$	$\{s_2, s_3, s_4\}$	$\{s_1, s_3, s_4\}$	$\{s_1, s_2, s_4\}$	S_{F_4}
$\varnothing$	1152	0	0	0	0	0	0	0	0	0	0	0
$\{s_1\}$	576	48	0	0	0	0	0	0	0	0	0	0
$\{s_4\}$	576	0	48	0	0	0	0	0	0	0	0	0
$\{s_1, s_2\}$	192	48	0	12	0	0	0	0	0	0	0	0
$\{s_2, s_3\}$	144	24	24	0	8	0	0	0	0	0	0	0
$\{s_3, s_4\}$	192	0	48	0	0	12	0	0	0	0	0	0
$\{s_1, s_4\}$	288	24	24	0	0	0	4	0	0	0	0	0
$\{s_1, s_2, s_3\}$	24	24	6	12	4	0	0	2	0	0	0	0
$\{s_2, s_3, s_4\}$	24	6	24	0	4	12	0	0	2	0	0	0
$\{s_1, s_3, s_4\}$	96	8	24	0	0	6	4	0	0	2	0	0
$\{s_1, s_2, s_4\}$	96	24	8	6	0	0	4	0	0	0	2	0
S_{F_4}	1	1	1	1	1	1	1	1	1	1	1	1

3.3 Eigenvalues and Multiplicities

Let $\mathfrak{d} = \sum_{K \subseteq S} \lambda_K x_K \in \mathbb{K}[\mathscr{D}_W]$ and $(x_J)_{J \subseteq S}$ be the standard basis of $\mathbb{K}[\mathscr{D}_W]$. We write $\mathsf{d}_{\mathbb{K}[\mathscr{D}_W]}(\mathfrak{d})$ for the column vector of $\mathfrak{d}$ on the basis $(x_J)_{J \subseteq S}$ and $\mathsf{D}_{\mathbb{K}[\mathscr{D}_W]}(\mathfrak{d})$ for the matrix of $\mathfrak{d}$ on the basis $(x_J)_{J \subseteq S}$ i.e.

$$\mathsf{d}_{\mathbb{K}[\mathscr{D}_W]}(\mathfrak{d}) = (\lambda_K)_{K \subseteq S} \text{ and } \mathsf{D}_{\mathbb{K}[\mathscr{D}_W]}(\mathfrak{d}) = \Big(\sum_{J \subseteq S} \lambda_J a_{JKL}\Big)_{K,L \subseteq S}.$$

Let $n \geq 2$ and $(\mathfrak{d}_i)_{i \in [n]} \in \mathbb{K}[\mathscr{D}_W]^n$. We have

$$\mathsf{d}_{\mathbb{K}[\mathscr{D}_W]}\Big(\overrightarrow{\prod_{i \in [n]}} \mathfrak{d}_i\Big) = \Big(\overrightarrow{\prod_{i \in [n-1]}} \mathsf{D}_{\mathbb{K}[\mathscr{D}_W]}(\mathfrak{d}_i)\Big)\mathsf{d}_{\mathbb{K}[\mathscr{D}_W]}(\mathfrak{d}_n).$$

Recall that the noncommutative multiplication is defined in the following way:

$$\overrightarrow{\prod_{i \in [n]}} \lambda_i = \lambda_1 \lambda_2 \dots \lambda_n.$$

We write $\mathsf{D}_{\mathbb{K}[\mathscr{D}_W]}(\mathfrak{d})_{|\bullet,K}$ for the column of $\mathsf{D}_{\mathbb{K}[\mathscr{D}_W]}(\mathfrak{d})$ corresponding to the basis vector x_K, i.e.

$$\mathsf{D}_{\mathbb{K}[\mathscr{D}_W]}(\mathfrak{d})_{|\bullet,K} = \Big(\sum_{J \subseteq S} \lambda_J a_{JKL}\Big)_{L \subseteq S} = \mathsf{d}_{\mathbb{K}[\mathscr{D}_W]}(\mathfrak{d} x_K).$$

Lemma 3.3.1. *Let $\mathfrak{d} \in \mathbb{K}[\mathscr{D}_W]$. Then $\mathsf{X}_{\mathbb{K}[W]}(\mathfrak{d})$ and $\mathsf{D}_{\mathbb{K}[\mathscr{D}_W]}(\mathfrak{d})$ have the same spectrum.*

Proof. It is clear that $Sp\big(\mathsf{D}_{\mathbb{K}[\mathscr{D}_W]}(\mathfrak{d})\big) \subseteq Sp\big(\mathsf{X}_{\mathbb{K}[W]}(\mathfrak{d})\big)$.
Let $(\mu_i)_{i\in[2^l]^*} \in \mathbb{K}^{2^l}$ and $\sum_{i=0}^{2^l} \mu_i t^i$ be the characteristic polynomial of $\mathsf{D}_{\mathbb{K}[\mathscr{D}_W]}(\mathfrak{d})$ in the variable t. We have $\sum_{i=0}^{2^l} \mu_i \mathsf{D}^i_{\mathbb{K}[\mathscr{D}_W]}(\mathfrak{d}) = \mathsf{0}_{2^l}$, especially

$$\mu_0 \mathsf{I}_{2^l|\bullet,S} + \sum_{i=1}^{2^l} \mu_i \mathsf{D}^{i-1}_{\mathbb{K}[\mathscr{D}_W]}(\mathfrak{d})\mathsf{D}_{\mathbb{K}[\mathscr{D}_W]}(\mathfrak{d})_{|\bullet,S} = \mathsf{0}_{2^l|\bullet,S}.$$

Since $\mathsf{I}_{2^l|\bullet,S} = \mathsf{d}_{\mathbb{K}[\mathscr{D}_W]}(e)$ and $\mathsf{D}_{\mathbb{K}[\mathscr{D}_W]}(\mathfrak{d})_{|\bullet,S} = \mathsf{d}_{\mathbb{K}[\mathscr{D}_W]}(\mathfrak{d})$, then

$$\mu_0 \mathsf{d}_{\mathbb{K}[\mathscr{D}_W]}(e) + \sum_{i=1}^{2^l} \mu_i \mathsf{D}^{i-1}_{\mathbb{K}[\mathscr{D}_W]}(\mathfrak{d})\mathsf{d}_{\mathbb{K}[\mathscr{D}_W]}(\mathfrak{d}) = \mathsf{d}_{\mathbb{K}[\mathscr{D}_W]}(0).$$

This means $\mu_0 e + \sum_{i=1}^{2^l} \mu_i \mathfrak{d}^i = 0$ and $Sp\big(\mathsf{X}_{\mathbb{K}[W]}(\mathfrak{d})\big) \subseteq Sp\big(\mathsf{D}_{\mathbb{K}[\mathscr{D}_W]}(\mathfrak{d})\big)$. □

In the following, we need a total order $\succ$ on the subsets of S which was introduced by F. and N. Bergeron [1]: We define $\min J := \min\{i \in [l] \mid s_i \in J\}$ and assume that $\min \varnothing = l+1$. Let $J, K \subseteq S$ such that $J \neq K$.

- If $\min J > \min K$ then $J \succ K$.
- Otherwise $J \succ K$ if and only if $J \backslash \{s_{\min J}\} \succ K \backslash \{s_{\min K}\}$.

Let $\{\mathbf{C}_{P_i}\}_{i\in[p]}$ a set representative of all the $\mathbf{C}_P$, $P \subseteq S$, i.e.

$$\{\mathbf{C}_{P_i}\}_{i\in[p]} = \{\mathbf{C}_P\}_{P\subseteq S},$$
$$\mathbf{C}_{P_i} \neq \mathbf{C}_{P_j} \text{ if } i \neq j.$$

`Example:` For the case of the group F_4, we have:
$\{\mathbf{C}_{P_i}\}_{i\in[12]} = \big\{\mathbf{C}_\varnothing, \mathbf{C}_{\{s_1\}}, \mathbf{C}_{\{s_4\}}, \mathbf{C}_{\{s_1,s_2\}}, \mathbf{C}_{\{s_2,s_3\}}, \mathbf{C}_{\{s_3,s_4\}}, \mathbf{C}_{\{s_1,s_4\}}, \mathbf{C}_{\{s_1,s_2,s_3\}}, \mathbf{C}_{\{s_2,s_3,s_4\}}, \mathbf{C}_{\{s_1,s_2,s_4\}}, \mathbf{C}_{\{s_1,s_3,s_4\}}, \mathbf{C}_{S_{F_4}}\big\}$

Let J_i be the element of $\mathbf{C}_{P_i}$ such that $K_i \succ J_i$ for all $K_i \in \mathbf{C}_{P_i} \backslash \{J_i\}$. We order the conjugacy classes of parabolic subgroups $\mathbf{C}_{P_i}$ such that $J_i \succ J_j$ if $i < j$.
Let $i, j \in [p]$. We write

$$a_{ij} = a_{J_i J_j J_j}.$$

Proposition 3.3.2. *Let $\mathfrak{d} = \sum_{J\subseteq S} \lambda_J x_J \in \mathbb{K}[\mathscr{D}_W]$. Then the spectrum of the matrix $\mathsf{X}_{\mathbb{K}[W]}(\mathfrak{d})$ is*

$$Sp\big(\mathsf{X}_{\mathbb{K}[W]}(\mathfrak{d})\big) = \Big\{ \sum_{i=1}^{p} a_{ij} \big(\sum_{K_i \in \mathbf{C}_{P_i}} \lambda_{K_i} \big) \Big\}_{j\in[p]}$$

Proof. We order the basis $(x_K)_{K\subseteq S}$ of $\mathbb{K}[\mathscr{D}_W]$ according to the total order $\succ$ of F. and N. Bergeron that means we get a new ordered basis $(x_{K_i})_{i\in[2^l]}$ such that $K_i \succ K_j$ if $i < j$. We assume that the rows and columns of the matrix $\mathsf{D}_{\mathbb{K}[\mathscr{D}_W]}(\mathfrak{d})$ are ordered in increasing order by $\succ$. Using Theorem 3.1.1, we get $\mathfrak{d} \cdot x_{K_j} \in \big\langle \{x_{K_i}\}_{i\in[j]} \big\rangle$. Then the matrix of $\mathfrak{d}$ on the basis $(x_{K_i})_{i\in[2^l]}$ is an upper triangular matrix. The characteristic polynomial of this matrix in the variable t is

$$\prod_{K\subseteq S} \big((\sum_{J\subseteq S} \lambda_J a_{JKK}) - t \big).$$

and

$$Sp(\mathfrak{d}) = \{ \sum_{J\subseteq S} \lambda_J a_{JKK} \}_{K\subseteq S}.$$

Since $a_{JKK} = a_{J'KK}$ for all $J, J' \in \mathbf{C}_{P_i}$ and $a_{JKK} = a_{JK'K'}$ for all $K, K' \in \mathbf{C}_{P_j}$ according to Theorem 3.2.5, we get the eigenvalues of $\mathsf{D}_{\mathbb{K}[\mathscr{D}_W]}(\mathfrak{d})$. Using Lemma 3.3.1, we get the result. □

`Example:` The eigenvalues of $\mathsf{X}_{\mathbb{K}[W]}(\mathfrak{d}_{F_4}) = \sum_{J\subseteq S_{F_4}} \lambda_J x_J$

$$\begin{aligned}
\Lambda_1 &= 1152\lambda_\varnothing + 576(\lambda_{\{s_1\}} + \lambda_{\{s_2\}} + \lambda_{\{s_3\}} + \lambda_{\{s_4\}}) + 192(\lambda_{\{s_1,s_2\}} + \lambda_{\{s_3,s_4\}}) + 144\lambda_{\{s_2,s_3\}} \\
&\quad +288(\lambda_{\{s_1,s_3\}} + \lambda_{\{s_1,s_4\}} + \lambda_{\{s_2,s_4\}}) + 24(\lambda_{\{s_1,s_2,s_3\}} + \lambda_{\{s_2,s_3,s_4\}}) + 96(\lambda_{\{s_1,s_3,s_4\}} + \lambda_{\{s_1,s_2,s_4\}}) \\
&\quad +\lambda_{S_{F_4}} \\
\Lambda_2 &= 48(\lambda_{\{s_1\}} + \lambda_{\{s_2\}} + \lambda_{\{s_1,s_2\}}) + 24(\lambda_{\{s_2,s_3\}} + \lambda_{\{s_1,s_3\}} + \lambda_{\{s_1,s_4\}} + \lambda_{\{s_2,s_4\}} + \lambda_{\{s_1,s_2,s_3\}} + \lambda_{\{s_1,s_2,s_4\}}) \\
&\quad +6\lambda_{\{s_2,s_3,s_4\}} + 8\lambda_{\{s_1,s_3,s_4\}} + \lambda_{S_{F_4}} \\
\Lambda_3 &= 48(\lambda_{\{s_3\}} + \lambda_{\{s_4\}} + \lambda_{\{s_3,s_4\}}) + 24(\lambda_{\{s_2,s_3\}} + \lambda_{\{s_1,s_3\}} + \lambda_{\{s_1,s_4\}} + \lambda_{\{s_2,s_4\}} + \lambda_{\{s_2,s_3,s_4\}} + \lambda_{\{s_1,s_3,s_4\}}) \\
&\quad +6\lambda_{\{s_1,s_2,s_3\}} + 8\lambda_{\{s_1,s_2,s_4\}} + \lambda_{S_{F_4}} \\
\Lambda_4 &= 12\lambda_{\{s_1,s_2\}} + 12\lambda_{\{s_1,s_2,s_3\}} + 6\lambda_{\{s_1,s_2,s_4\}} + \lambda_{S_{F_4}} \\
\Lambda_5 &= 8\lambda_{\{s_2,s_3\}} + 4(\lambda_{\{s_1,s_2,s_3\}} + \lambda_{\{s_2,s_3,s_4\}}) + \lambda_{S_{F_4}} \\
\Lambda_6 &= 12(\lambda_{\{s_3,s_4\}} + \lambda_{\{s_2,s_3,s_4\}}) + 6\lambda_{\{s_1,s_3,s_4\}} + \lambda_{S_{F_4}} \\
\Lambda_7 &= 4(\lambda_{\{s_1,s_3\}} + \lambda_{\{s_1,s_4\}} + \lambda_{\{s_2,s_4\}}) + \lambda_{\{s_1,s_3,s_4\}} + \lambda_{\{s_1,s_2,s_4\}}) + \lambda_{S_{F_4}} \\
\Lambda_8 &= 2\lambda_{\{s_1,s_2,s_3\}} + \lambda_{S_{F_4}} \\
\Lambda_9 &= 2\lambda_{\{s_2,s_3,s_4\}} + \lambda_{S_{F_4}} \\
\Lambda_{10} &= 2\lambda_{\{s_1,s_3,s_4\}} + \lambda_{S_{F_4}} \\
\Lambda_{11} &= 2\lambda_{\{s_1,s_2,s_4\}} + \lambda_{S_{F_4}} \\
\Lambda_{12} &= \lambda_{S_{F_4}}
\end{aligned}$$

Let $x \in W$. We write $\mathbf{C}_x$ for the conjugacy class of x, i.e.

$$\mathbf{C}_x = \{w^{-1}xw \mid w \in W\}.$$

Let $l_i = |J_i|$ and $J_i = \{s_{j_r}\}_{r\in[l_i]}$. We write

$$s_{J_i} = \prod_{r=1}^{l_i} s_{j_r}.$$

Let $\mathfrak{d} = \sum_{J\subseteq S} \lambda_J x_J \in \mathbb{K}[\mathscr{D}_W]$ and $j \in [p]$. We write

$$\Lambda_j = \sum_{i=1}^{p} a_{ij}\big(\sum_{K_i\in\mathbf{C}_{P_i}} \lambda_{K_i}\big).$$

Proposition 3.3.3. *Let $\mathfrak{d} = \sum_{J\subseteq S} \lambda_J x_J \in \mathbb{K}[\mathscr{D}_W]$. Then the multiplicities of the elements of $Sp\big(\mathsf{X}_{\mathbb{K}[W]}(\mathfrak{d})\big)$ are*

$$V_{\mathfrak{d}}(\Lambda_i) = |\mathbf{C}_{s_{J_i}}|$$

for $i \in [p]$.

Proof. Let $\mathsf{A} = (a_{ij})_{i,j\in[p]}$, $\mathsf{v} = \big(V_{\mathfrak{d}}(\Lambda_j)\big)_{j\in[p]}$, $\mathsf{w} = (|W|)_{j\in[p]}$ and $\mathsf{c} = (|\mathbf{C}_{s_{J_j}}|)_{j\in[p]}$. At the end of the sixth section of [2], it is proved that $\mathsf{A}^{-1}\mathsf{w} = \mathsf{c}$.
Besides, we have $tr\big(\mathsf{X}_{\mathbb{K}[W]}(\mathfrak{d})\big) = |W| \sum_{J\subseteq S} \lambda_J$. Then, we have $\sum_{i=1}^{j} a_{ji} V_{\mathfrak{d}}(\Lambda_i)\lambda_{K_j} = |W|\lambda_{K_j}$ for $K_j \in \mathbf{C}_{P_j}$ i.e.

$$\sum_{i=1}^{j} a_{ji} V_{\mathfrak{d}}(\Lambda_i) = |W|.$$

In matrix form, we get $\mathsf{A}\mathsf{v} = \mathsf{w}$. Thus $\mathsf{A}^{-1}\mathsf{w} = \mathsf{v}$. □

`Example:` The multiplicities of $Sp\big(\mathsf{X}_{\mathbb{K}[W]}(\mathfrak{d}_{F_4})\big)$

$$\begin{array}{rcccc}
V_{\mathfrak{d}_{F_4}}(\Lambda_1) & = & 1 & = & |\mathbf{C}_e| \\
V_{\mathfrak{d}_{F_4}}(\Lambda_2) & = & 12 & = & |\mathbf{C}_{s_1}| \\
V_{\mathfrak{d}_{F_4}}(\Lambda_3) & = & 12 & = & |\mathbf{C}_{s_4}| \\
V_{\mathfrak{d}_{F_4}}(\Lambda_4) & = & 32 & = & |\mathbf{C}_{s_1s_2}| \\
V_{\mathfrak{d}_{F_4}}(\Lambda_5) & = & 54 & = & |\mathbf{C}_{s_2s_3}| \\
V_{\mathfrak{d}_{F_4}}(\Lambda_6) & = & 32 & = & |\mathbf{C}_{s_3s_4}| \\
V_{\mathfrak{d}_{F_4}}(\Lambda_7) & = & 72 & = & |\mathbf{C}_{s_1s_4}| \\
V_{\mathfrak{d}_{F_4}}(\Lambda_8) & = & 84 & = & |\mathbf{C}_{s_1s_2s_3}| \\
V_{\mathfrak{d}_{F_4}}(\Lambda_9) & = & 84 & = & |\mathbf{C}_{s_2s_3s_4}| \\
V_{\mathfrak{d}_{F_4}}(\Lambda_{10}) & = & 96 & = & |\mathbf{C}_{s_1s_3s_4}| \\
V_{\mathfrak{d}_{F_4}}(\Lambda_{11}) & = & 96 & = & |\mathbf{C}_{s_1s_2s_4}| \\
V_{\mathfrak{d}_{F_4}}(\Lambda_{12}) & = & 577 & = & |\mathbf{C}_{s_1s_2s_3s_4}|
\end{array}$$

We note that with the matrix relation $\mathsf{Av} = \mathsf{w}$, we can also get the cardinalities of the conjugacy classes of W.

Chapter 4

Special Elements of the Group Algebra of the Symmetric Group

In this chapter, the considered group is the Coxeter group $A_{n-1} = \mathcal{S}_n$ which is the set of permutations of the elements of $[n]$. We recall that for $\sigma \in \mathcal{S}_n$:

- its descents set is $\texttt{DES}(\sigma) := \{k \in [n-1] \mid \sigma(k) > \sigma(k+1)\}$,
- its inversions set is $\texttt{INV}(\sigma) := \{(i,j) \mid i < j,\, \sigma(i) > \sigma(j)\}$.

Our results are related to two new statistics on $\mathcal{S}_n$ that are the statistic $\texttt{des}_\texttt{X} : \mathcal{S}_n \to \mathbb{R}[X_1, \dots, X_{n-1}]$:

$$\texttt{des}_\texttt{X}(\sigma) := \sum_{i \in \texttt{DES}(\sigma)} X_i$$

and the statistic $\texttt{inv}_\texttt{X} : \mathcal{S}_n \to \mathbb{R}[X_{1,2}, \dots, X_{n-1,n}]$:

$$\texttt{inv}_\texttt{X}(\sigma) := \sum_{(i,j) \in \texttt{INV}(\sigma)} X_{i,j}$$

where $X_1, \dots, X_{n-1}$ and $X_{1,2}, \dots, X_{n-1,n}$ are variables.
Let

$$\mathfrak{d}_{\mathfrak{X}n} := \sum_{\sigma \in \mathcal{S}_n} \texttt{des}_\texttt{X}(\sigma)\, \sigma \in \mathbb{R}[X_1, \dots, X_{n-1}][\mathcal{S}_n]$$

and

$$\mathfrak{i}_{\mathfrak{X}n} := \sum_{\sigma \in \mathcal{S}_n} \texttt{inv}_\texttt{X}(\sigma)\, \sigma \in \mathbb{R}[X_{1,2}, \dots, X_{n-1,n}][\mathcal{S}_n].$$

We aim to diagonalize $\mathfrak{D}_{\mathfrak{X}n} = \mathsf{X}_{\mathbb{R}[X_1,\dots,X_{n-1}][\mathcal{S}_n]}(\mathfrak{d}_{\mathfrak{X}n})$ and $\mathfrak{I}_{\mathfrak{X}n} = \mathsf{X}_{\mathbb{R}[X_{1,2},\dots,X_{n-1,n}][\mathcal{S}_n]}(\mathfrak{i}_{\mathfrak{X}n})$. By linear algebraic calculating we get:

$(n=1)$ $Sp(\mathfrak{D}_{\mathfrak{X}1}) = \{0\}$ and $V_{\mathfrak{D}_{\mathfrak{X}1}}(0) = 1$.

$(n=2)$ $Sp(\mathfrak{D}_{\mathfrak{X}2}) = \{X_1, -X_1\}$ and $V_{\mathfrak{D}_{\mathfrak{X}2}}(X_1) = 1$, $V_{\mathfrak{D}_{\mathfrak{X}2}}(-X_1) = 1$,

and

$(n=1)$ $Sp(\mathfrak{I}_{\mathfrak{X}1}) = \{0\}$ and $V_{\mathfrak{I}_{\mathfrak{X}1}}(0) = 1$.

$(n=2)$ $Sp(\mathfrak{I}_{\mathfrak{X}2}) = \{X_{1,2}, -X_{1,2}\}$ and $V_{\mathfrak{I}_{\mathfrak{X}2}}(X_{1,2}) = 1$, $V_{\mathfrak{I}_{\mathfrak{X}2}}(-X_{1,2}) = 1$.

$(n=3)$ $Sp(\mathfrak{I}_{\mathfrak{X}3}) = \{3X_{1,2} + 3X_{1,3} + 3X_{2,3},\ -X_{1,2} - 2X_{1,3} - X_{2,3},\ -X_{1,2} + X_{1,3} - X_{2,3},\ 0\}$ and

- $V_{\mathfrak{I}_{\mathfrak{X}_3}}(3X_{1,2} + 3X_{1,3} + 3X_{2,3}) = 1$,
- $V_{\mathfrak{I}_{\mathfrak{X}_3}}(-X_{1,2} - 2X_{1,3} - X_{2,3}) = 2$,
- $V_{\mathfrak{I}_{\mathfrak{X}_3}}(-X_{1,2} + X_{1,3} - X_{2,3}) = 1$,
- $V_{\mathfrak{I}_{\mathfrak{X}_3}}(0) = 2$.

We study those matrices for arbitrary values of n. But we first have to give a multinomial version of the theorem of Perron-Frobenius [11, 8.2.11 Perron's Theorem]. Then we determine the minimal polynomials of $\mathfrak{D}_{\mathfrak{X}_n}$ and $\mathfrak{I}_{\mathfrak{X}_n}$ from which we obtain the eigenvalues. We finish by calculating the multiplicities.

4.1 Multinomial Version of the Theorem of Perron-Frobenius

For a polynomial P and a monomial M in $\mathbb{R}[X_1, \dots, X_k]$, we write $[M]P$ for the coefficient of M in P.

Proposition 4.1.1. *Let $n \geq 2$ and $\mathfrak{P}_n = \left(P_{i,j}\right)_{i,j\in[n]}$ be a matrix of polynomials $P_{i,j} \in \mathbb{R}[X_1, \dots, X_k]$ such that:*

(a) $P_{i,j} \neq 0$ and $[X_1^{i_1} \dots X_k^{i_k}]P_{i,j} \geq 0$,

(b) there is a polynomial $P_n \in \mathbb{R}[X_1, \dots, X_k]$ such that, for any $i', i'' \in [n]$,

$$\sum_{j=1}^{n} P_{i',j} = \sum_{j=1}^{n} P_{i'',j} = P_n.$$

Then $P_n \in Sp(\mathfrak{P}_n)$ and $E_{\mathfrak{P}_n}(P_n) =< \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} >$.

Proof. We prove the assertion by induction on n. With simple calculations, we get the result for $n = 2$. Now we assume that the assertion is proven for $n - 1 \geq 2$.
Let us consider the following system (i) of n equations:

$$\begin{array}{ccccccccccc} P_{1,1}x_1 & + & P_{1,2}x_2 & + & \dots & + & P_{1,n-1}x_{n-1} & + & P_{1,n}x_n & = & P_nx_1 \\ P_{2,1}x_1 & + & P_{2,2}x_2 & + & \dots & + & P_{2,n-1}x_{n-1} & + & P_{2,n}x_n & = & P_nx_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots & & \vdots \\ P_{n-1,1}x_1 & + & P_{n-1,2}x_2 & + & \dots & + & P_{n-1,n-1}x_{n-1} & + & P_{n-1,n}x_n & = & P_nx_{n-1} \\ P_{n,1}x_1 & + & P_{n,2}x_2 & + & \dots & + & P_{n,n-1}x_{n-1} & + & P_{n,n}x_n & = & P_nx_n \end{array}$$

On the last row, we have $P_{n,1}x_1 + P_{n,2}x_2 + \cdots + P_{n,n-1}x_{n-1} = (P_n - P_{n,n})x_n$. By multiplying the n-1 first rows with $(P_n - P_{n,n})$, and after by replacing $(P_n - P_{n,n})x_n$ with $P_{n,1}x_1 + P_{n,2}x_2 + \cdots + P_{n,n-1}x_{n-1}$ in the n-1 first rows, we obtain the following system (ii) of n-1 equations:

$$\begin{array}{ccccccccc} P_{1,1}(P_n - P_{n,n})x_1 & + & \dots & + & P_{1,n-1}(P_n - P_{n,n})x_{n-1} & + & P_{1,n}Q & = & P_{n-1}x_1 \\ P_{2,1}(P_n - P_{n,n})x_1 & + & \dots & + & P_{2,n-1}(P_n - P_{n,n})x_{n-1} & + & P_{2,n}Q & = & P_{n-1}x_2 \\ \vdots & & & & \vdots & & \vdots & & \vdots \\ P_{n-1,1}(P_n - P_{n,n})x_1 & + & \dots & + & P_{n-1,n-1}(P_n - P_{n,n})x_{n-1} & + & P_{n-1,n}Q & = & P_{n-1}x_{n-1} \end{array}$$

where $P_{n-1} = P_n(P_n - P_{n,n})$ and $Q = P_{n,1}x_1 + P_{n,2}x_2 + \cdots + P_{n,n-1}x_{n-1}$.
We have, for $i \in [n-1]$:

- $$\sum_{j=1}^{n-1}[x_j]\big(P_{i,1}(P_n - P_{n,n})x_1 + \cdots + P_{i,n-1}(P_n - P_{n,n})x_{n-1} + P_{i,n}Q\big)$$

$$\begin{aligned}&=\sum_{j=1}^{n-1} P_{i,j}(P_n - P_{n,n}) + \sum_{j=1}^{n-1} P_{i,n}P_{n,j}\\&=(P_n - P_{i,n})(P_n - P_{n,n}) + P_{i,n}(P_n - P_{n,n})\\&=P_{n-1}.\end{aligned}$$

- Let $j \in [n-1]$ and set

$$P'_{i,j} = [x_j]\big(P_{i,1}(P_n - P_{n,n})x_1 + \cdots + P_{i\,n-1}(P_n - P_{n,n})x_{n-1} + P_{i,n}Q\big).$$

  By (a) and the definition of $\mathfrak{P}_n$, we have that $0 \neq P_{i,j}(P_n - P_{n,n})$ is a polynomial with non-negative coefficients only. Again by (a), it follows that:

  - ▹ $P'_{i,j} = P_{i,j}(P_n - P_{n,n}) + P_{i,n}P_{n,j} \neq 0$,
  - ▹ $(P'_{i,j}\,,\, X_1^{i_1} \ldots X_k^{i_k}) \geq 0$, for $i_1, \ldots, i_k \in \mathbb{N}$.

By induction, the solution of the system (ii) is $x_1 = x_2 = \cdots = x_{n-1}$. By replacing $x_2, x_3, \ldots, x_{n-1}$ by x_1 in $P_{n,1}x_1 + P_{n,2}x_2 + \cdots + P_{n,n-1}x_{n-1} = (P_n - P_{n,n})x_n$, we have then $x_1 = x_n$. Hence the solution of this system (1) is $x_1 = x_2 = \cdots = x_n$.

Therefore, $P_n \in Sp(\mathfrak{P}_n)$ and $E_{\mathfrak{P}_n}(P_n) =< \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} >$. □

Corollary 4.1.2. *Let $n \geq 2$ and $\mathfrak{P}_n = \big(P_{i,j}\big)_{i,j\in[n]}$ be a matrix of polynomials $P_{i,j} \in \mathbb{R}[X_1, \ldots, X_k]$ such that:*

(a) $P_{i,j} \neq 0$ and $[X_1^{i_1} \ldots X_k^{i_k}]P_{i,j} \geq 0$, for $i \neq j$,

(b) $P_{i,i} = 0$, for $i \in [n]$,

(c) there is a polynomial $P_n \in \mathbb{R}[X_1, \ldots, X_k]$ such that, for any $i', i'' \in [n]$,

$$\sum_{j=1}^{n} P_{i',j} = \sum_{j=1}^{n} P_{i'',j} = P_n.$$

Then $P_n \in Sp(\mathfrak{P}_n)$ and $E_{\mathfrak{P}_n}(P_n) =< \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} >$.

Proof. The proof is obvious for $n = 2$.
Let us consider $n > 2$. We have the following system (iii) of n equations:

$$\begin{array}{ccccccccccc} 0 & + & P_{1,2}x_2 & + & \ldots & + & P_{1,n-1}x_{n-1} & + & P_{1,n}x_n & = & P_nx_1 \\ P_{2,1}x_1 & + & 0 & + & \ldots & + & P_{2,n-1}x_{n-1} & + & P_{2,n}x_n & = & P_nx_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots & & \vdots \\ P_{n-1,1}x_1 & + & P_{n-1,2}x_2 & + & \ldots & + & 0 & + & P_{n-1,n}x_n & = & P_nx_{n-1} \\ P_{n,1}x_1 & + & P_{n,2}x_2 & + & \ldots & + & P_{n,n-1}x_{n-1} & + & 0 & = & P_nx_n \end{array}$$

On the last row, we have $P_{n,1}x_1 + P_{n,2}x_2 + \cdots + P_{n,n-1}x_{n-1} = P_nx_n$. By multiplying the n-1 first rows with P_n, and after by replacing P_nx_n with $P_{n,1}x_1 + P_{n,2}x_2 + \cdots + P_{n,n-1}x_{n-1}$ in the n-1 first rows, we obtain the following system (iv) of n-1 equations:

$$\begin{array}{ccccccc} P_{1,n}P_{n,1}x_1 & + & \ldots & + & (P_nP_{1,n-1} + P_{1,n}P_{n,n-1})x_{n-1} & = & P_n^2x_1 \\ (P_nP_{2,1} + P_{2,n}P_{n,1})x_1 & + & \ldots & + & (P_nP_{2,n-1} + P_{2,n}P_{n,n-1})x_{n-1} & = & P_n^2x_2 \\ \vdots & & & & \vdots & & \vdots \\ (P_nP_{n-1,1} + P_{n-1,n}P_{n,1})x_1 & + & \ldots & + & P_{n-1,n}P_{n,n-1}x_{n-1} & = & P_n^2x_{n-1} \end{array}$$

where

$$\begin{aligned}\sum_{j=1}^{n-1}[x_j]\big(\sum_{k=1}^{n-1}[x_k](P_nP_{i,k}+P_{i,n}P_{n,k})\big) &= P_n\sum_{k=1}^{n-1}P_{i,k}+P_{i,n}\sum_{k=1}^{n-1}P_{n,k}\\ &= P_n\sum_{k=1}^{n-1}P_{i,k}+P_nP_{i,n}\\ &= P_n^2.\end{aligned}$$

Using Proposition 4.1.1 on the obtained system (iv), we get $x_1 = x_2 = \cdots = x_{n-1}$ and then the desided result. ☐

Proposition 4.1.1 can also be applied to get more general results. As example, for a matrix of polynomials $\mathfrak{P}_\mathfrak{n} = \big(P_{i,j}\big)_{i,j\in[n]}$ such that:

(a) $[X_1^{i_1}\dots X_k^{i_k}]P_{i,j} \geq 0$,

(b) $P_{i,n} \neq 0$ and $P_{n,i} \neq 0$, for $i \neq n$,

(c) there is a polynomial $P_n \in \mathbb{R}[X_1, \dots, X_k]$ such that, for any $i', i'' \in [n]$,

$$\sum_{j=1}^{n} P_{i',j} = \sum_{j=1}^{n} P_{i'',j} = P_n.$$

Then $P_n \in Sp(\mathfrak{P}_n)$ and $E_{\mathfrak{P}_n}(P_n) =< \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} >$.

Now we apply Corollary 4.1.2 to the statistics $\mathtt{des}_\mathtt{X}$ and $\mathtt{inv}_\mathtt{X}$.

Lemma 4.1.3. *For $n \geq 3$, we have*

$$\sum_{\sigma\in\mathcal{S}_n}\mathtt{des}_\mathtt{X}(\sigma) = \frac{n!}{2}\sum_{k=1}^{n-1}X_k.$$

Proof. For $k \in [n-1]$, $[X_k]\big(\sum_{\sigma\in\mathcal{S}_n}\mathtt{des}_\mathtt{X}(\sigma)\big) = \#\{\sigma \in \mathcal{S}_n \,|\, k \in \mathtt{DES}(\sigma)\}$. It is clear that the following mapping is bijective:

$$\gamma : \begin{cases} \{\sigma \in \mathcal{S}_n \,|\, k \in \mathtt{DES}(\sigma)\} & \to & \{\sigma \in \mathcal{S}_n \,|\, k \notin \mathtt{DES}(\sigma)\} \\ \tau = \begin{pmatrix} \dots & k & k+1 & \dots \\ \dots & \tau(k) & \tau(k+1) & \dots \end{pmatrix} & \mapsto & \gamma(\tau) = \begin{pmatrix} \dots & k & k+1 & \dots \\ \dots & \tau(k+1) & \tau(k) & \dots \end{pmatrix} \end{cases},$$

that means $\#\{\sigma \in \mathcal{S}_n \,|\, k \in \mathtt{DES}(\sigma)\} = \#\{\sigma \in \mathcal{S}_n \,|\, k \notin \mathtt{DES}(\sigma)\}$.
Since

$$\#\{\sigma \in \mathcal{S}_n \,|\, k \in \mathtt{DES}(\sigma)\} \;+\; \#\{\sigma \in \mathcal{S}_n \,|\, k \notin \mathtt{DES}(\sigma)\} = \#\mathcal{S}_n,$$

then

$$[X_j]\big(\sum_{\sigma\in\mathcal{S}_n}\mathtt{des}_\mathtt{X}(\sigma)\big) = \frac{n!}{2} \text{ and } \sum_{\sigma\in\mathcal{S}_n}\mathtt{des}_\mathtt{X}(\sigma) = \frac{n!}{2}\sum_{k=1}^{n-1}X_k.$$

☐

From Corollary 4.1.2 and Lemma 4.1.3, we deduce that

$$\frac{n!}{2}\sum_{k=1}^{n-1} X_k \in Sp(\mathfrak{D}_{\mathfrak{X}_n}) \text{ and } E_{\mathfrak{D}_{\mathfrak{X}_n}}(\frac{n!}{2}\sum_{k=1}^{n-1} X_k) =< \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} > . \tag{4.1}$$

Besides, from Lemma 4.1.3, we also directly get

$$\sum_{\sigma\in\mathcal{S}_n} \mathtt{des}(\sigma) = \frac{(n-1)n!}{2} \quad\text{and}\quad \sum_{\sigma\in\mathcal{S}_n} \mathtt{maj}(\sigma) = \frac{n!}{2}\binom{n}{2}$$

that are differently calculated in other books ([22, Example 2.2.5] resp. [22, Corollary 4.5.9] for example).

Lemma 4.1.4. *For $n \geq 4$, we have*

$$\sum_{\sigma\in\mathcal{S}_n} \mathtt{inv}_{\mathtt{X}}(\sigma) = \frac{n!}{2} \sum_{\{(i,j)\in[n]^2 \ | \ i<j\}} X_{i,j}.$$

Proof. For $i, j \in [n]$, $i < j$, $[X_{i,j}]\big(\sum_{\sigma\in\mathcal{S}_n} \mathtt{inv}_{\mathtt{X}}(\sigma)\big) = \#\{\sigma \in \mathcal{S}_n \,|\, (i,j) \in \mathtt{INV}(\sigma)\}$. It is clear that the following mapping is bijective:

$$\varphi : \begin{cases} \{\sigma \in \mathcal{S}_n \,|\, (i,j) \in \mathtt{INV}(\sigma)\} & \to & \{\sigma \in \mathcal{S}_n \,|\, (i,j) \notin \mathtt{INV}(\sigma)\} \\ \tau = \begin{pmatrix} \dots & i & \dots & j & \dots \\ \dots & \tau(i) & \dots & \tau(j) & \dots \end{pmatrix} & \mapsto & \varphi(\tau) = \begin{pmatrix} \dots & i & \dots & j & \dots \\ \dots & \tau(j) & \dots & \tau(i) & \dots \end{pmatrix} \end{cases},$$

that means $\#\{\sigma \in \mathcal{S}_n \,|\, (i,j) \in \mathtt{INV}(\sigma)\} = \#\{\sigma \in \mathcal{S}_n \,|\, (i,j) \notin \mathtt{INV}(\sigma)\}$.
Since

$$\#\{\sigma \in \mathcal{S}_n \,|\, (i,j) \in \mathtt{INV}(\sigma)\} \ + \ \#\{\sigma \in \mathcal{S}_n \,|\, (i,j) \notin \mathtt{INV}(\sigma)\} = \#\mathcal{S}_n,$$

then

$$[X_{i,j}]\big(\sum_{\sigma\in\mathcal{S}_n} \mathtt{inv}_{\mathtt{X}}(\sigma)\big) = \frac{n!}{2} \text{ and } \sum_{\sigma\in\mathcal{S}_n} \mathtt{inv}_{\mathtt{X}}(\sigma) = \frac{n!}{2} \sum_{\{(i,j)\in[n]^2 \ | \ i<j\}} X_{i,j}.$$

□

From Corollary 4.1.2 and Lemma 4.1.4, we deduce that

$$\frac{n!}{2} \sum_{\{(i,j)\in[n]^2 \ | \ i<j\}} X_{i,j} \in Sp(\mathfrak{I}_{\mathfrak{X}_n}) \text{ and } E_{\mathfrak{I}_{\mathfrak{X}_n}}(\frac{n!}{2} \sum_{\{(i,j)\in[n]^2 \ | \ i<j\}} X_{i,j}) =< \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} > . \tag{4.2}$$

Besides, from Lemma 4.1.4, we also directly get

$$\sum_{\sigma\in\mathcal{S}_n} \mathtt{inv}(\sigma) = \frac{n!}{2}\binom{n}{2}$$

that is differently calculated in other books ([22, Corollary 1.3.10] for example).

4.2 Minimal Polynomial of the Multinomial Descent Statistic

To determine the minimal polynomial of $\mathfrak{D}_{\mathfrak{n}}$ we need the both following lemmas:
Let $n \geq 3$ and $i, j \in [n]$. We write

$$\mathcal{S}_n^{i,j} := \{\sigma \in \mathcal{S}_n \ | \ \sigma^{-1}(j) - \sigma^{-1}(i) = 1\},$$

$$\mathcal{S}_n^{i-j} := \{\sigma \in \mathcal{S}_n \mid \sigma^{-1}(j) - \sigma^{-1}(i) > 1\}.$$

Let $n \geq 3$. We write

$$Q_n = \sum_{k=1}^{n-1} X_k.$$

Lemma 4.2.1. *Let $n \geq 3$ and $i, j \in [n]$. Then*

- *if $j > i+1$:*

$$\sum_{\sigma \in \mathcal{S}_n^{i,j}} X_{\sigma^{-1}(i)} \mathtt{des}_{\mathtt{X}}(\sigma^{-1}) = \sum_{\sigma \in \mathcal{S}_n^{j,i}} X_{\sigma^{-1}(j)} \mathtt{des}_{\mathtt{X}}(\sigma^{-1}).$$

- *if $j = i+1$:*

$$\sum_{\sigma \in \mathcal{S}_n^{i,(i+1)}} X_{\sigma^{-1}(i)} \mathtt{des}_{\mathtt{X}}(\sigma^{-1}) = \sum_{\sigma \in \mathcal{S}_n^{(i+1),i}} X_{\sigma^{-1}(i+1)} \mathtt{des}_{\mathtt{X}}(\sigma^{-1}) \ - X_i (n-2)! Q_n.$$

Proof. Let us consider the following bijective mapping:

$$\kappa_{i,j} : \left\{ \begin{array}{ccc} \mathcal{S}_n & \to & \mathcal{S}_n \\ \sigma = \begin{pmatrix} \dots & i & \dots & j & \dots \\ \dots & \sigma(i) & \dots & \sigma(j) & \dots \end{pmatrix} & \mapsto & \kappa_{i,j}(\sigma) = \begin{pmatrix} \dots & i & \dots & j & \dots \\ \dots & \sigma(j) & \dots & \sigma(i) & \dots \end{pmatrix} \end{array} \right. .$$

- Let $\sigma \in \mathcal{S}_n^{i\,j}$. Then we have the following simple facts:
  - ▷ $\sigma^{-1}(j) = \sigma^{-1}(i) + 1$,
  - ▷ $\sigma^{-1} \in \mathcal{S}_n^{\sigma^{-1}(i) - \sigma^{-1}(j)}$,
  - ▷ $\kappa_{i,j}(\sigma^{-1}) \in \mathcal{S}_n^{\sigma^{-1}(j) - \sigma^{-1}(i)}$,
  - ▷ $\mathtt{des}_{\mathtt{X}}(\sigma^{-1}) = \mathtt{des}_{\mathtt{X}}\big(\kappa_{i,j}(\sigma^{-1})\big)$,
  - ▷ $\big(\kappa_{i,j}(\sigma^{-1})\big)^{-1} \in \mathcal{S}_n^{j,i}$.

  Thus

$$\sum_{\sigma \in \mathcal{S}_n^{i,j}} X_{\sigma^{-1}(i)} \mathtt{des}_{\mathtt{X}}(\sigma^{-1}) = \sum_{\sigma \in \mathcal{S}_n^{i,j}} X_{\sigma^{-1}(i)} \mathtt{des}_{\mathtt{X}}\big(\kappa_{i,j}(\sigma^{-1})\big) = \sum_{\sigma \in \mathcal{S}_n^{j,i}} X_{\sigma^{-1}(j)} \mathtt{des}_{\mathtt{X}}(\sigma^{-1}).$$

- Let $\sigma \in \mathcal{S}_n^{i\,(i+1)}$. Again the following facts hold:
  - ▷ $\sigma^{-1} \in \mathcal{S}_n^{\sigma^{-1}(i),(\sigma^{-1}(i)+1)}$,
  - ▷ $\kappa_{i,i+1}(\sigma^{-1}) \in \mathcal{S}_n^{(\sigma^{-1}(i)+1),\sigma^{-1}(i)}$,
  - ▷ $\mathtt{des}_{\mathtt{X}}(\sigma^{-1}) = \mathtt{des}_{\mathtt{X}}\big(\kappa_{i,i+1}(\sigma^{-1})\big) - X_i$,
  - ▷ $\big(\kappa_{i,i+1}(\sigma^{-1})\big)^{-1} \in \mathcal{S}_n^{(i+1),i}$.

  Thus

$$\begin{aligned} \sum_{\sigma \in \mathcal{S}_n^{i\,(i+1)}} X_{\sigma^{-1}(i)} \mathtt{des}_{\mathtt{X}}(\sigma^{-1}) &= \sum_{\sigma \in \mathcal{S}_n^{i,(i+1)}} X_{\sigma^{-1}(i)} \Big(\mathtt{des}_{\mathtt{X}}\big(\kappa_{i,i+1}(\sigma^{-1})\big) - X_i \Big) \\ &= \sum_{\sigma \in \mathcal{S}_n^{i\,(i+1)}} X_{\sigma^{-1}(i)} \mathtt{des}_{\mathtt{X}}\big(\kappa_{i,i+1}(\sigma^{-1})\big) - \sum_{\sigma \in \mathcal{S}_n^{i,(i+1)}} X_{\sigma^{-1}(i)} X_i \\ &= \sum_{\sigma \in \mathcal{S}_n^{(i+1),i}} X_{\sigma^{-1}(i+1)} \mathtt{des}_{\mathtt{X}}(\sigma^{-1}) - X_i \sum_{k=1}^{n-1} \sum_{\substack{\sigma \in \mathcal{S}_n^{i,(i+1)} \\ \sigma(k)=i}} X_k \\ &= \sum_{\sigma \in \mathcal{S}_n^{(i+1),i}} X_{\sigma^{-1}(i+1)} \mathtt{des}_{\mathtt{X}}(\sigma^{-1}) - X_i (n-2)! Q_n. \end{aligned}$$

□

Let $n \geq 3$. We write ι for the identity permutation of $\mathcal{S}_n$ and define:

$$\begin{array}{rccc} \Pi_n : & \mathcal{S}_n & \to & \mathbb{R}[X_1, \ldots, X_{n-1}] \\ & \tau & \mapsto & \sum_{\sigma \in \mathcal{S}_n} \mathtt{des}_{\mathtt{X}}(\tau\sigma)\mathtt{des}_{\mathtt{X}}(\sigma^{-1}) \end{array}$$

Lemma 4.2.2. *Let $n \geq 3$ and $\tau \in \mathcal{S}_n$. Then*

$$\Pi_n(\tau) = \Pi_n(\iota) - \mathtt{des}_{\mathtt{X}}(\tau)(n-2)!Q_n.$$

Proof. Every permutation $\tau \in \mathcal{S}_n$ can be written as $\tau = \tau_k \ldots \tau_2\tau_1\iota$ for some $\tau_i \in \{(j\ j{+}1) \mid 1 \leq j \leq n{-}1\}$. We consider the following mapping:

$$\phi : \left\{ \begin{array}{ccc} \mathcal{S}_n & \to & \mathbb{R}[X_1, \ldots, X_{n-1}] \\ \tau = \tau_k \ldots \tau_2\tau_1\iota & \mapsto & \Big(\mathtt{des}_{\mathtt{X}}(\tau) - \mathtt{des}_{\mathtt{X}}(\tau_{k-1} \ldots \tau_2\tau_1\iota)\Big) + \phi\Big(\tau_{k-1} \ldots \tau_2\tau_1\iota\Big) \end{array} \right. ,$$

$$\text{where } \phi(\iota) = 0 \text{ and } \phi\Big((j\ j+1)\Big) = X_j.$$

We are going to prove that $\Pi_n(\tau) = \Pi_n(\iota) - \phi(\tau)(n-2)!Q_n$.
The case $k = 0$ and $\tau = \iota$ is trivial.
For $k = 1$ and $\tau = (j\ j+1)$ we have:

$$\begin{aligned} \Pi_n\big((j\ j+1)\big) = & \sum_{\sigma \in \mathcal{S}_n^{j\,-\,(j+1)}} \mathtt{des}_{\mathtt{X}}\Big((j\ j+1)\sigma\Big)\mathtt{des}_{\mathtt{X}}\Big(\sigma^{-1}\Big) + \sum_{\sigma \in \mathcal{S}_n^{(j+1)\,-\,j}} \mathtt{des}_{\mathtt{X}}\Big((j\ j+1)\sigma\Big)\mathtt{des}_{\mathtt{X}}\Big(\sigma^{-1}\Big) \\ & + \sum_{\sigma \in \mathcal{S}_n^{j,(j+1)}} \mathtt{des}_{\mathtt{X}}\Big((j\ j+1)\sigma\Big)\mathtt{des}_{\mathtt{X}}\Big(\sigma^{-1}\Big) + \sum_{\sigma \in \mathcal{S}_n^{(j+1),j}} \mathtt{des}_{\mathtt{X}}\Big((j\ j+1)\sigma\Big)\mathtt{des}_{\mathtt{X}}\Big(\sigma^{-1}\Big) \\ = & \sum_{\sigma \in \mathcal{S}_n^{j\,-\,(j+1)}} \mathtt{des}_{\mathtt{X}}(\sigma)\mathtt{des}_{\mathtt{X}}(\sigma^{-1}) + \sum_{\sigma \in \mathcal{S}_n^{(j+1)\,-\,j}} \mathtt{des}_{\mathtt{X}}(\sigma)\mathtt{des}_{\mathtt{X}}(\sigma^{-1}) \\ & + \sum_{\sigma \in \mathcal{S}_n^{j,(j+1)}} \Big(\mathtt{des}_{\mathtt{X}}(\sigma) + X_{\sigma^{-1}(j)}\Big)\mathtt{des}_{\mathtt{X}}(\sigma^{-1}) + \sum_{\sigma \in \mathcal{S}_n^{(j+1),j}} \Big(\mathtt{des}_{\mathtt{X}}(\sigma) - X_{\sigma^{-1}(j+1)}\Big)\mathtt{des}_{\mathtt{X}}(\sigma^{-1}) \\ = & \Pi_n(\iota) + \sum_{\sigma \in \mathcal{S}_n^{j,(j+1)}} X_{\sigma^{-1}(j)}\mathtt{des}_{\mathtt{X}}(\sigma^{-1}) - \sum_{\sigma \in \mathcal{S}_n^{(j+1),j}} X_{\sigma^{-1}(j+1)}\mathtt{des}_{\mathtt{X}}(\sigma^{-1}) \\ = & \Pi_n(\iota) + \sum_{\sigma \in \mathcal{S}_n^{(j+1),j}} X_{\sigma^{-1}(j+1)}\mathtt{des}_{\mathtt{X}}(\sigma^{-1}) - X_j(n-2)!Q_n - \sum_{\sigma \in \mathcal{S}_n^{(j+1),j}} X_{\sigma^{-1}(j+1)}\mathtt{des}_{\mathtt{X}}(\sigma^{-1}) \\ = & \Pi_n(\iota) - \phi\Big((j\ j+1)\Big)(n-2)!Q_n. \end{aligned}$$

Now we assume that the assertion is proven for $k - 1 \geq 1$.

Let $k \geq 2$ and $\tau = \tau_k \dots \tau_2\tau_1\iota = (j\ j+1)\tau'$, where $\tau_k = (j\ j+1)$ and $\tau' = \tau_{k-1}\dots\tau_2\tau_1\iota$:

$$\begin{aligned}
\Pi_n\big((j\ j+1)\tau'\big) = &\sum_{\{\sigma\in\mathcal{S}_n \mid \tau'\sigma\in\mathcal{S}_n^{j\,-\,(j+1)}\}} \mathtt{des}_\mathtt{X}\Big((j\ j+1)\tau'\sigma\Big)\mathtt{des}_\mathtt{X}\Big(\sigma^{-1}\Big) + \sum_{\{\sigma\in\mathcal{S}_n \mid \tau'\sigma\in\mathcal{S}_n^{(j+1)\,-\,j}\}} \mathtt{des}_\mathtt{X}\Big((j\ j+1)\tau'\sigma\Big)\mathtt{des}_\mathtt{X}\Big(\sigma^{-1}\Big)\\
&+ \sum_{\{\sigma\in\mathcal{S}_n \mid \tau'\sigma\in\mathcal{S}_n^{j,(j+1)}\}} \mathtt{des}_\mathtt{X}\Big((j\ j+1)\tau'\sigma\Big)\mathtt{des}_\mathtt{X}\Big(\sigma^{-1}\Big) + \sum_{\{\sigma\in\mathcal{S}_n \mid \tau'\sigma\in\mathcal{S}_n^{(j+1),j}\}} \mathtt{des}_\mathtt{X}\Big((j\ j+1)\tau'\sigma\Big)\mathtt{des}_\mathtt{X}\Big(\sigma^{-1}\Big)\\
= &\sum_{\{\sigma\in\mathcal{S}_n \mid \tau'\sigma\in\mathcal{S}_n^{j\,-\,(j+1)}\}} \mathtt{des}_\mathtt{X}(\tau'\sigma)\mathtt{des}_\mathtt{X}(\sigma^{-1}) + \sum_{\{\sigma\in\mathcal{S}_n \mid \tau'\sigma\in\mathcal{S}_n^{(j+1)\,-\,j}\}} \mathtt{des}_\mathtt{X}(\tau'\sigma)\mathtt{des}_\mathtt{X}(\sigma^{-1})\\
&+ \sum_{\{\sigma\in\mathcal{S}_n \mid \tau'\sigma\in\mathcal{S}_n^{j,(j+1)}\}} \Big(\mathtt{des}_\mathtt{X}(\tau'\sigma) + X_{\sigma^{-1}\tau'^{-1}(j)}\Big)\mathtt{des}_\mathtt{X}(\sigma^{-1})\\
&+ \sum_{\{\sigma\in\mathcal{S}_n \mid \tau'\sigma\in\mathcal{S}_n^{(j+1),j}\}} \Big(\mathtt{des}_\mathtt{X}(\tau'\sigma) - X_{\sigma^{-1}\tau'^{-1}(j+1)}\Big)\mathtt{des}_\mathtt{X}(\sigma^{-1})\\
=&\Pi_n(\tau') + \sum_{\{\sigma\in\mathcal{S}_n \mid \tau'\sigma\in\mathcal{S}_n^{j,(j+1)}\}} X_{\sigma^{-1}\tau'^{-1}(j)}\mathtt{des}_\mathtt{X}(\sigma^{-1}) - \sum_{\{\sigma\in\mathcal{S}_n \mid \tau'\sigma\in\mathcal{S}_n^{(j+1),j}\}} X_{\sigma^{-1}\tau'^{-1}(j+1)}\mathtt{des}_\mathtt{X}(\sigma^{-1})\\
=&\Pi_n(\tau') + \sum_{\sigma\in\mathcal{S}_n^{\tau'^{-1}(j),\tau'^{-1}(j+1)}} X_{\sigma^{-1}\tau'^{-1}(j)}\mathtt{des}_\mathtt{X}(\sigma^{-1}) - \sum_{\sigma\in\mathcal{S}_n^{\tau'^{-1}(j+1),\tau'^{-1}(j)}} X_{\sigma^{-1}\tau'^{-1}(j+1)}\mathtt{des}_\mathtt{X}(\sigma^{-1}).
\end{aligned}$$

- If $\tau' \in \mathcal{S}_n^{j\,-\,(j+1)}$ i.e. $\tau'^{-1}(j+1) - \tau'^{-1}(j) > 1$ and $\mathtt{des}_\mathtt{X}\Big((j\ j+1)\tau'\Big) = \mathtt{des}_\mathtt{X}(\tau')$, then

$$\sum_{\sigma\in\mathcal{S}_n^{\tau'^{-1}(j),\tau'^{-1}(j+1)}} X_{\sigma^{-1}\tau'^{-1}(j)}\mathtt{des}_\mathtt{X}(\sigma^{-1}) = \sum_{\sigma\in\mathcal{S}_n^{\tau'^{-1}(j+1),\tau'^{-1}(j)}} X_{\sigma^{-1}\tau'^{-1}(j+1)}\mathtt{des}_\mathtt{X}(\sigma^{-1})$$

  and

$$\Pi_n\big((j\ j+1)\tau'\big) = \Pi_n(\tau') = \Pi_n(\iota) - \phi\Big((j\ j+1)\tau'\Big)(n-2)!Q_n.$$

- If $\tau' \in \mathcal{S}_n^{(j+1)\,-\,j}$ i.e. $\tau'^{-1}(j) - \tau'^{-1}(j+1) > 1$ and $\mathtt{des}_\mathtt{X}\Big((j\ j+1)\tau'\Big) = \mathtt{des}_\mathtt{X}(\tau')$, then

$$\sum_{\sigma\in\mathcal{S}_n^{\tau'^{-1}(j),\tau'^{-1}(j+1)}} X_{\sigma^{-1}\tau'^{-1}(j)}\mathtt{des}_\mathtt{X}(\sigma^{-1}) = \sum_{\sigma\in\mathcal{S}_n^{\tau'^{-1}(j+1),\tau'^{-1}(j)}} X_{\sigma^{-1}\tau'^{-1}(j+1)}\mathtt{des}_\mathtt{X}(\sigma^{-1})$$

  and, once again, we have

$$\Pi_n\big((j\ j+1)\tau'\big) = \Pi_n(\tau') = \Pi_n(\iota) - \phi\Big((j\ j+1)\tau'\Big)(n-2)!Q_n.$$

- If $\tau' \in \mathcal{S}_n^{j,(j+1)}$ i.e. $\tau'^{-1}(j+1) - \tau'^{-1}(j) = 1$ and $\mathtt{des}_\mathtt{X}\Big((j\ j+1)\tau'\Big) = \mathtt{des}_\mathtt{X}(\tau') + X_{\tau'^{-1}(j)}$, then

$$\sum_{\sigma\in\mathcal{S}_n^{\tau'^{-1}(j),\tau'^{-1}(j+1)}} X_{\sigma^{-1}\tau'^{-1}(j)}\mathtt{des}_\mathtt{X}(\sigma^{-1}) = \sum_{\sigma\in\mathcal{S}_n^{\tau'^{-1}(j+1),\tau'^{-1}(j)}} X_{\sigma^{-1}\tau'^{-1}(j+1)}\mathtt{des}_\mathtt{X}(\sigma^{-1})\ - X_{\tau'^{-1}(j)}(n-2)!Q_n$$

  and

$$\Pi_n\big((j\ j+1)\tau'\big) = \Pi_n\big(\tau'\big) - X_{\tau'^{-1}(j)}(n-2)!Q_n = \Pi_n(\iota) - \phi\Big((j\ j+1)\tau'\Big)(n-2)!Q_n.$$

- If $\tau' \in \mathcal{S}_n^{(j+1),j}$ i.e. $\tau'^{-1}(j) - \tau'^{-1}(j+1) = 1$ and $\mathtt{des}_\mathtt{X}\Big((j\ j+1)\tau'\Big) = \mathtt{des}_\mathtt{X}(\tau') - X_{\tau'^{-1}(j+1)}$, then

$$\sum_{\sigma\in\mathcal{S}_n^{\tau'^{-1}(j),\tau'^{-1}(j+1)}} X_{\sigma^{-1}\tau'^{-1}(j)}\mathtt{des}_\mathtt{X}(\sigma^{-1}) = \sum_{\sigma\in\mathcal{S}_n^{\tau'^{-1}(j+1),\tau'^{-1}(j)}} X_{\sigma^{-1}\tau'^{-1}(j+1)}\mathtt{des}_\mathtt{X}(\sigma^{-1})\ + X_{\tau'^{-1}(j+1)}(n-2)!Q_n$$

  and

$$\Pi_n\big((j\ j+1)\tau'\big) = \Pi_n\big(\tau'\big) + X_{\tau'^{-1}(j)}(n-2)!Q_n = \Pi_n(\iota) - \phi\Big((j\ j+1)\tau'\Big)(n-2)!Q_n.$$

Finally, we have

$$\begin{aligned}\Pi_n(\tau) &= \Pi_n(\iota) - \phi(\tau)(n-2)!Q_n \\ &= \Pi_n(\iota) - \Big(\Big(\mathtt{des}_{\mathtt{X}}(\tau) - \mathtt{des}_{\mathtt{X}}(\tau_{k-1}\dots\tau_2\tau_1\iota)\Big) + \phi\Big(\tau_{k-1}\dots\tau_2\tau_1\iota\Big)\Big)(n-2)!Q_n \\ &= \Pi_n(\iota) - \Big(\mathtt{des}_{\mathtt{X}}(\tau) - \mathtt{des}_{\mathtt{X}}(\iota)\Big)(n-2)!Q_n \\ &= \Pi_n(\iota) - \mathtt{des}_{\mathtt{X}}(\tau)(n-2)!Q_n.\end{aligned}$$

□

Now we are in position to determine the minimal polynomial of $\mathfrak{D}_{\mathfrak{X}_n}$.

Proposition 4.2.3. *Let $n \geq 3$. The minimal polynomial of $\mathfrak{D}_{\mathfrak{X}_n}$ is:*

$$X\big(X - \frac{n!}{2}Q_n\big)\big(X + (n-2)!Q_n\big).$$

Proof. Using Lemma 4.1.3 and Lemma 4.2.2, we get:

$$\Big(\mathfrak{D}_{\mathfrak{X}_n} - \frac{n!}{2}Q_n\mathrm{I}_{n!}\Big)\mathfrak{D}_{\mathfrak{X}_n}\Big(\mathfrak{D}_{\mathfrak{X}_n} + (n-2)!Q_n\mathrm{I}_{n!}\Big) = \Big(\mathfrak{D}_{\mathfrak{X}_n} - \frac{n!}{2}Q_n\mathrm{I}_{n!}\Big)\Big(\Pi_n(\iota)\Big)_{\pi,\tau\in\mathcal{S}_n} = \Big(0\Big)_{\pi,\tau\in\mathcal{S}_n}.$$

Hence the minimal polynomial of $\mathfrak{D}_{\mathfrak{X}_n}$ divides $X\big(X - \frac{n!}{2}Q_n\big)\big(X + (n-2)!Q_n\big)$.
Since

$$\Big((\mathfrak{D}_{\mathfrak{X}_n} - \frac{n!}{2}Q_n\mathrm{I}_{n!})\mathfrak{D}_{\mathfrak{X}_n}\Big)_{\pi,\pi} = \Pi_n \text{ resp. } \Big(\mathfrak{D}_{\mathfrak{X}_n}\big(\mathfrak{D}_{\mathfrak{X}_n} + (n-2)!Q_n\mathrm{I}_{n!}\big)\Big)_{\pi,\pi} = \Pi_n,$$

then $(X - \frac{n!}{2}Q_n)X$ resp. $X\big(X + (n-2)!Q_n\big)$ is not the minimal polynomial of $\mathfrak{D}_{\mathfrak{X}_n}$.
We have

$$\Big(\big(\mathfrak{D}_{\mathfrak{X}_n} - \frac{n!}{2}Q_n\mathrm{I}_{n!}\big)\big(\mathfrak{D}_{\mathfrak{X}_n} + (n-2)!Q_n\mathrm{I}_{n!}\big)\Big)_{\pi,\iota} = \Pi_n - \mathtt{des}_{\mathtt{X}}(\pi)\frac{n!}{2}Q_n.$$

Since there are $\pi,\tau \in \mathcal{S}_n$ such that $\mathtt{des}_{\mathtt{X}}(\pi) \neq \mathtt{des}_{\mathtt{X}}(\tau)$, then

$$\Big(\big(\mathfrak{D}_{\mathfrak{X}_n} - \frac{n!}{2}Q_n\mathrm{I}_{n!}\big)\big(\mathfrak{D}_{\mathfrak{X}_n} + (n-2)!Q_n\mathrm{I}_{n!}\big)\Big)_{\pi,\tau\in\mathcal{S}_n} \neq (0)_{\pi,\tau\in\mathcal{S}_n},$$

and $\big(X - \frac{n!}{2}Q_n\big)\big(X + (n-2)!Q_n\big)$ is not the minimal polynomial of $\mathfrak{D}_{\mathfrak{X}_n}$.
We conclude that the minimal polynomial of $\mathfrak{D}_{\mathfrak{X}_n}$ is $X\big(X - \frac{n!}{2}Q_n\big)\big(X + (n-2)!Q_n\big)$. □

4.3 Minimal Polynomial of the Multinomial Inversion Statistic

To determine the minimal polynomial of $\mathfrak{I}_{\mathfrak{X}_n}$ we need to prove several multinomial formulas:
Let $n \geq 4$ and $i_1, j_1, i_2, j_2, a, b \in [n]$. We write:

$$\mathcal{S}_n^{i_1\,j_1} := \{\sigma \in \mathcal{S}_n \mid \sigma(i_1) > \sigma(j_1)\},$$

$$\mathcal{S}_n^{i_1\,j_1}{}_{|\bar{i_2}=a,\bar{j_2}=b} := \{\sigma \in \mathcal{S}_n^{i_1\,j_1} \mid \sigma^{-1}(i_2) = a,\ \sigma^{-1}(j_2) = b\}.$$

Lemma 4.3.1. *Let $n \geq 4$ and $i_1, j_1, i_2, j_2, a, b \in [n]$, with $i_1 < j_1$, $i_2 < j_2$. We have:*

- *if $\{a, b\} = \{i_1, j_1\}$:*
$$\#\mathcal{S}_n^{i_1\,j_1}{}_{|\bar{i_2}=j_1,\bar{j_2}=i_1} = \#\mathcal{S}_n^{j_1\,i_1}{}_{|\bar{i_2}=i_1,\bar{j_2}=j_1} = (n-2)!.$$
- *if $a, b \notin \{i_1, j_1\}$:*
$$\#\mathcal{S}_n^{i_1\,j_1}{}_{|\bar{i_2}=a,\bar{j_2}=b} = \#\mathcal{S}_n^{j_1\,i_1}{}_{|\bar{i_2}=a,\bar{j_2}=b} = \frac{(n-2)!}{2}.$$

- *if* $a = i_1$ *and* $b \notin \{i_1, j_1\}$*:*

$$\begin{aligned} \#\mathcal{S}_n^{i_1\,j_1}{}_{|\bar{i_2}=i_1,\bar{j_2}=b} &= (i_2-1)(n-3)!, \\ \#\mathcal{S}_n^{j_1\,i_1}{}_{|\bar{i_2}=i_1,\bar{j_2}=b} &= (n-i_2-1)(n-3)!. \end{aligned}$$

- *if* $a \notin \{i_1, j_1\}$ *and* $b = i_1$*:*

$$\begin{aligned} \#\mathcal{S}_n^{i_1\,j_1}{}_{|\bar{i_2}=a,\bar{j_2}=i_1} &= (j_2-2)(n-3)!, \\ \#\mathcal{S}_n^{j_1\,i_1}{}_{|\bar{i_2}=a,\bar{j_2}=i_1} &= (n-j_2)(n-3)!. \end{aligned}$$

- *if* $a = j_1$ *and* $b \notin \{i_1, j_1\}$*:*

$$\begin{aligned} \#\mathcal{S}_n^{i_1\,j_1}{}_{|\bar{i_2}=j_1,\bar{j_2}=b} &= (n-i_2-1)(n-3)!, \\ \#\mathcal{S}_n^{j_1\,i_1}{}_{|\bar{i_2}=j_1,\bar{j_2}=b} &= (i_2-1)(n-3)!. \end{aligned}$$

- *if* $a \notin \{i_1, j_1\}$ *and* $b = j_1$*:*

$$\begin{aligned} \#\mathcal{S}_n^{i_1\,j_1}{}_{|\bar{i_2}=a,\bar{j_2}=j_1} &= (n-j_2)(n-3)!, \\ \#\mathcal{S}_n^{j_1\,i_1}{}_{|\bar{i_2}=a,\bar{j_2}=j_1} &= (j_2-2)(n-3)!. \end{aligned}$$

Proof. Cardinality calculating. □

Let $n \geq 4$ and $i, j \in [n]$. We write

$${}^{i\,j}\mathcal{S}_n := \{\sigma \in \mathcal{S}_n \mid \sigma^{-1}(i) > \sigma^{-1}(j)\},$$

$$\chi_{i,j} = j - i - 1.$$

Lemma 4.3.2. *Let* $n \geq 4$ *and* $i_1, j_1, i_2, j_2 \in [n]$ *with* $i_1 < j_1$ *and* $i_2 < j_2$*. Then:*

(a)

$$\#^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{i_2\,j_2} = (n-3)!\chi_{i_1,j_1}\chi_{i_2,j_2} + (n-2)!\frac{n^2-n+2}{4},$$

(b)

$$\#^{i_i\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{j_2\,i_2} = (n-2)!\frac{n^2-n-2}{4} - (n-3)!\chi_{i_1,j_1}\chi_{i_2,j_2}.$$

Proof. (a) We have the following equations:

$$\begin{aligned} \#\{\sigma \in \mathcal{S}_n \mid \sigma(i_2) = j_1, \sigma(j_2) = i_1\} &= (n-2)!, \\ \#\{\sigma \in \mathcal{S}_n \mid \sigma(i_2) = j_1, \sigma(j_2) \neq i_1, \sigma^{-1}(j_1) < \sigma^{-1}(i_1), \sigma(i_2) > \sigma(j_2)\} &= (j_1-2)(n-i_2-1)(n-3)!, \\ \#\{\sigma \in \mathcal{S}_n \mid \sigma(i_2) \neq j_1, \sigma(j_2) = i_1, \sigma^{-1}(j_1) < \sigma^{-1}(i_1), \sigma(i_2) > \sigma(j_2)\} &= (n-i_1-1)(j_2-2)(n-3)!, \\ \#\{\sigma \in \mathcal{S}_n \mid \sigma(j_2) = j_1, \sigma^{-1}(j_1) < \sigma^{-1}(i_1), \sigma(i_2) > \sigma(j_2)\} &= (n-j_1)(n-j_2)(n-3)!, \\ \#\{\sigma \in \mathcal{S}_n \mid \sigma(i_2) = i_1, \sigma^{-1}(j_1) < \sigma^{-1}(i_1), \sigma(i_2) > \sigma(j_2)\} &= (i_1-1)(i_2-1)(n-3)!, \\ \#\{\sigma \in \mathcal{S}_n \mid \sigma(i_2) \notin \{i_1, j_1\}, \sigma(j_2) \notin \{i_1, j_1\}, \sigma^{-1}(j_1) < \sigma^{-1}(i_1), \sigma(i_2) > \sigma(j_2)\} &= \binom{n-2}{2}\binom{n-2}{2}(n-4)!. \end{aligned}$$

Then we deduce

$$\begin{aligned}
\#^{i_1 j_1}\mathcal{S}_n \cap \mathcal{S}_n^{i_2 j_2} &= \#\{\sigma \in \mathcal{S}_n \mid \sigma(i_2) = j_1, \sigma(j_2) = i_1\} \\
&\quad + \#\{\sigma \in \mathcal{S}_n \mid \sigma(i_2) = j_1, \sigma(j_2) \neq i_1, \sigma^{-1}(j_1) < \sigma^{-1}(i_1), \sigma(i_2) > \sigma(j_2)\} \\
&\quad + \#\{\sigma \in \mathcal{S}_n \mid \sigma(i_2) \neq j_1, \sigma(j_2) = i_1, \sigma^{-1}(j_1) < \sigma^{-1}(i_1), \sigma(i_2) > \sigma(j_2)\} \\
&\quad + \#\{\sigma \in \mathcal{S}_n \mid \sigma(j_2) = j_1, \sigma^{-1}(j_1) < \sigma^{-1}(i_1), \sigma(i_2) > \sigma(j_2)\} \\
&\quad + \#\{\sigma \in \mathcal{S}_n \mid \sigma(i_2) = i_1, \sigma^{-1}(j_1) < \sigma^{-1}(i_1), \sigma(i_2) > \sigma(j_2)\} \\
&\quad + \#\{\sigma \in \mathcal{S}_n \mid \sigma(i_2) \notin \{i_1, j_1\}, \sigma(j_2) \notin \{i_1, j_1\}, \sigma^{-1}(j_1) < \sigma^{-1}(i_1), \sigma(i_2) > \sigma(j_2)\} \\
&= (n-3)!\Big((i_1 - j_1 + 1)(i_2 - j_2 + 1) + \frac{n^3 - 3n^2 + 4n - 4}{4}\Big).
\end{aligned}$$

(b) We have the following equations:

$$\begin{aligned}
\#\{\sigma \in \mathcal{S}_n \mid \sigma(i_2) = i_1, \sigma^{-1}(j_1) < \sigma^{-1}(i_1), \sigma(i_2) < \sigma(j_2)\} &= (n - i_1 - 1)(i_2 - 1)(n-3)!, \\
\#\{\sigma \in \mathcal{S}_n \mid \sigma(j_2) = j_1, \sigma^{-1}(j_1) < \sigma^{-1}(i_1), \sigma(i_2) < \sigma(j_2)\} &= (j_1 - 2)(n - j_2)(n-3)!, \\
\#\{\sigma \in \mathcal{S}_n \mid \sigma(i_2) \neq j_1, \sigma(j_2) = i_1, \sigma^{-1}(j_1) < \sigma^{-1}(i_1), \sigma(i_2) < \sigma(j_2)\} &= (i_1 - 1)(j_2 - 2)(n-3)!, \\
\#\{\sigma \in \mathcal{S}_n \mid \sigma(i_2) = j_1, \sigma(j_2) \neq i_1, \sigma^{-1}(j_1) < \sigma^{-1}(i_1), \sigma(i_2) < \sigma(j_2)\} &= (n - j_1)(n - i_2 - 1)(n-3)!, \\
\#\{\sigma \in \mathcal{S}_n \mid \sigma(i_2) \notin \{i_1, j_1\}, \sigma(j_2) \notin \{i_1, j_1\}, \sigma^{-1}(j_1) < \sigma^{-1}(i_1), \sigma(i_2) < \sigma(j_2)\} &= \binom{n-2}{2}\binom{n-2}{2}(n-4)!.
\end{aligned}$$

Then we deduce

$$\begin{aligned}
\#^{i_i j_1}\mathcal{S}_n \cap \mathcal{S}_n^{j_2 i_2} &= \#\{\sigma \in \mathcal{S}_n \mid \sigma(i_2) = i_1, \sigma^{-1}(j_1) < \sigma^{-1}(i_1), \sigma(i_2) < \sigma(j_2)\} \\
&\quad + \#\{\sigma \in \mathcal{S}_n \mid \sigma(j_2) = j_1, \sigma^{-1}(j_1) < \sigma^{-1}(i_1), \sigma(i_2) < \sigma(j_2)\} \\
&\quad + \#\{\sigma \in \mathcal{S}_n \mid \sigma(i_2) \neq j_1, \sigma(j_2) = i_1, \sigma^{-1}(j_1) < \sigma^{-1}(i_1), \sigma(i_2) < \sigma(j_2)\} \\
&\quad + \#\{\sigma \in \mathcal{S}_n \mid \sigma(i_2) = j_1, \sigma(j_2) \neq i_1, \sigma^{-1}(j_1) < \sigma^{-1}(i_1), \sigma(i_2) < \sigma(j_2)\} \\
&\quad + \#\{\sigma \in \mathcal{S}_n \mid \sigma(i_2) \notin \{i_1, j_1\}, \sigma(j_2) \notin \{i_1, j_1\}, \sigma^{-1}(j_1) < \sigma^{-1}(i_1), \sigma(i_2) < \sigma(j_2)\} \\
&= (n-3)!\Big((i_1 - j_1 + 1)(-i_2 + j_2 - 1) + \frac{(n+1)(n-2)^2}{4}\Big).
\end{aligned}$$

□

Let $n \geq 4$. We define:

$$\begin{array}{rccc}
\mathtt{f}_2: & \mathcal{S}_n & \to & \mathbb{R}[X_{1,2}, \ldots, X_{n-1,n}] \\
& \pi & \mapsto & \sum_{\sigma \in \mathcal{S}_n} \mathtt{inv}_{\mathtt{X}}(\sigma^{-1})\mathtt{inv}_{\mathtt{X}}(\sigma\pi)
\end{array}$$

Lemma 4.3.3. *Let $n \geq 4$. Then:*

(a) For $(i_1, j_1) = (i_2, j_2)$:

- *If $\pi(i_1) < \pi(j_1)$:*

$$[X_{i_1,j_1}^2]\mathtt{f}_2(\pi) = (n-3)!\chi_{i_1,j_1}\chi_{\pi(i_1),\pi(j_1)} + (n-2)!\frac{n^2 - n + 2}{4}.$$

- *If $\pi(i_1) > \pi(j_1)$:*

$$[X_{i_1,j_1}^2]\mathtt{f}_2(\pi) = (n-2)!\frac{n^2 - n - 2}{4} - (n-3)!\chi_{i_1,j_1}\chi_{\pi(j_1),\pi(i_1)}.$$

(b) For $(i_1, j_1) \neq (i_2, j_2)$:

▷ *If $\pi(i_1) < \pi(j_1)$ and $\pi(i_2) < \pi(j_2)$:*

$$[X_{i_1,j_1}X_{i_2,j_2}]\mathtt{f}_2(\pi) = (n-3)!\big(\chi_{i_1,j_1}\chi_{\pi(i_2),\pi(j_2)} + \chi_{\pi(i_1),\pi(j_1)}\chi_{i_2,j_2}\big) + (n-2)!\frac{n^2-n+2}{2}.$$

▷ *If $\pi(i_1) < \pi(j_1)$ and $\pi(i_2) > \pi(j_2)$:*

$$[X_{i_1,j_1}X_{i_2,j_2}]\mathtt{f}_2(\pi) = (n-3)!\big(\chi_{\pi(i_1),\pi(j_1)}\chi_{i_2,j_2} - \chi_{i_1,j_1}\chi_{\pi(j_2),\pi(i_2)}\big) + \frac{n!}{2}.$$

▷ *If $\pi(i_1) > \pi(j_1)$ and $\pi(i_2) > \pi(j_2)$:*

$$[X_{i_1,j_1}X_{i_2,j_2}]\mathtt{f}_2(\pi) = (n-2)!\frac{n^2-n-2}{2} - (n-3)!\big(\chi_{i_1,j_1}\chi_{\pi(j_2),\pi(i_2)} + \chi_{\pi(j_1),\pi(i_1)}\chi_{i_2,j_2}\big).$$

Proof. (a) For $(i_1, j_1) = (i_2, j_2)$:

$$[X^2_{i_1,j_1}]\mathtt{f}_2(\pi) = \#^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{\pi(i_1)\,\pi(j_1)}.$$

(b) For $(i_1, j_1) \neq (i_2, j_2)$:

$$[X_{i_1,j_1}X_{i_2,j_2}]\mathtt{f}_2(\pi) = \#^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{\pi(i_2)\,\pi(j_2)} + \#^{i_2\,j_2}\mathcal{S}_n \cap \mathcal{S}_n^{\pi(i_1)\,\pi(j_1)}.$$

□

Lemma 4.3.4. *Let $n \geq 4$ and $i_1, j_1, i_2, j_2, i_3, j_3 \in [n]$ with $i_1 < j_1$, $i_2 < j_2$ and $i_3 < j_3$. Then:*

(a)

$$\sum_{a,b\in[n]} \#^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{a\,b} \times \mathcal{S}_n^{i_3\,j_3}{}_{|\bar{i_2}=a,\bar{j_2}=b} = \frac{n!^2}{8} - \frac{(n-2)!^2}{2} - (n-4)(n-3)!^2\chi_{i_1,j_1}\chi_{i_2,j_2}\chi_{i_3,j_3}$$
$$-(n-3)!(n-2)!\Big(\chi_{i_1,j_1}\chi_{i_2,j_2} + \chi_{i_1,j_1}\chi_{i_3,j_3} + \chi_{i_2,j_2}\chi_{i_3,j_3}\Big).$$

(b)

$$\sum_{a,b\in[n]} \#^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{a\,b} \times \mathcal{S}_n^{j_3\,i_3}{}_{|\bar{i_2}=a,\bar{j_2}=b} = \frac{n!^2}{8} + \frac{(n-2)!^2}{2} + (n-4)(n-3)!^2\chi_{i_1,j_1}\chi_{i_2,j_2}\chi_{i_3,j_3}$$
$$+(n-3)!(n-2)!\Big(\chi_{i_1,j_1}\chi_{i_2,j_2} + \chi_{i_1,j_1}\chi_{i_3,j_3} + \chi_{i_2,j_2}\chi_{i_3,j_3}\Big).$$

Proof. We use Lemma 4.3.1 and Lemma 4.3.2.

(a) We have the following equations:

$$\#^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{j_3\,i_3} \times \mathcal{S}_n^{i_3\,j_3}{}_{|\bar{i_2}=j_3,\bar{j_2}=i_3} = (n-2)!^2\frac{n^2-n-2}{4} - (n-3)!(n-2)!\chi_{i_1,j_1}\chi_{i_3,j_3},$$

$$\#\bigcup_{a,b\notin\{i_3,j_3\}} {}^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{a\,b} \times \mathcal{S}_n^{i_3\,j_3}{}_{|\bar{i_2}=a,\bar{j_2}=b} = \binom{n-2}{2}\frac{(n-2)!n!}{4},$$

$$\#\bigcup_{b<i_3} {}^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{i_3\,b} \times \mathcal{S}_n^{i_3\,j_3}{}_{|\bar{i_2}=i_3,\bar{j_2}=b} = (i_2-1)(i_3-1)(n-3)!(n-2)!\frac{n^2-n-2}{4}$$
$$-(i_2-1)\binom{i_3-1}{2}(n-3)!^2\chi_{i_1,j_1},$$

$$\#\bigcup_{\substack{b>i_3\\ b\neq j_3}} {}^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{i_3\,b} \times \mathcal{S}_n^{i_3\,j_3}{}_{|\bar{i_2}=i_3,\bar{j_2}=b} = (i_2-1)\binom{n-i_3}{2}(n-3)!^2\chi_{i_1,j_1} - (i_2-1)(n-3)!^2\chi_{i_1,j_1}\chi_{i_3,j_3}$$
$$+(i_2-1)(n-i_3-1)(n-3)!(n-2)!\frac{n^2-n+2}{4},$$

$$\# \bigcup_{\substack{b<j_3\\ b\neq i_3}} {}^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{i_3\,b} \times \mathcal{S}_n^{i_3\,j_3}{}_{|\bar{i_2}=i_3,\bar{j_2}=b} = (n-i_2-1)(j_3-2)(n-3)!(n-2)!\frac{n^2-n-2}{4} - (n-i_2-1)\binom{j_3-1}{2}(n-3)!^2\chi_{i_1,j_1} + (n-i_2-1)(n-3)!^2\chi_{i_1,j_1}\chi_{i_3,j_3},$$

$$\# \bigcup_{b>j_3} {}^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{j_3\,b} \times \mathcal{S}_n^{i_3\,j_3}{}_{|\bar{i_2}=j_3,\bar{j_2}=b} = (n-i_2-1)\binom{n-j_3}{2}(n-3)!^2\chi_{i_1,j_1} + (n-i_2-1)(n-j_3)(n-3)!(n-2)!\frac{n^2-n+2}{4},$$

$$\# \bigcup_{a<i_3} {}^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{a\,i_3} \times \mathcal{S}_n^{i_3\,j_3}{}_{|\bar{i_2}=a,\bar{j_2}=i_3} = (j_2-2)\binom{i_3-1}{2}(n-3)!^2\chi_{i_1,j_1} + (j_2-2)(i_3-1)(n-3)!(n-2)!\frac{n^2-n+2}{4},$$

$$\# \bigcup_{\substack{a>i_3\\ a\neq j_3}} {}^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{a\,i_3} \times \mathcal{S}_n^{i_3\,j_3}{}_{|\bar{i_2}=a,\bar{j_2}=i_3} = (j_2-2)(n-i_3-1)(n-3)!(n-2)!\frac{n^2-n-2}{4} - (j_2-2)\binom{n-i_3}{2}(n-3)!^2\chi_{i_1,j_1} + (j_2-2)(n-3)!^2\chi_{i_1,j_1}\chi_{i_3,j_3},$$

$$\# \bigcup_{\substack{a<j_3\\ a\neq i_3}} {}^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{a\,j_3} \times \mathcal{S}_n^{i_3\,j_3}{}_{|\bar{i_2}=a,\bar{j_2}=j_3} = (n-j_2)\binom{j_3-1}{2}(n-3)!^2\chi_{i_1,j_1} - (n-j_2)(n-3)!^2\chi_{i_1,j_1}\chi_{i_3,j_3} + (n-j_2)(j_3-2)(n-3)!(n-2)!\frac{n^2-n+2}{4},$$

$$\# \bigcup_{a>j_3} {}^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{a\,j_3} \times \mathcal{S}_n^{i_3\,j_3}{}_{|\bar{i_2}=a,\bar{j_2}=j_3} = (n-j_2)(n-j_3)(n-3)!(n-2)!\frac{n^2-n-2}{4} - (n-j_2)\binom{n-j_3}{2}(n-3)!^2\chi_{i_1,j_1}.$$

Then we deduce

$$\sum_{a,b\in[n]} {}^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{a\,b} \times \mathcal{S}_n^{i_3\,j_3}{}_{|\bar{i_2}=a,\bar{j_2}=b}$$

$$\begin{aligned}
&= \#{}^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{j_3\,i_3} \times \mathcal{S}_n^{i_3\,j_3}{}_{|\bar{i_2}=j_3,\bar{j_2}=i_3} + \# \bigcup_{a,b\notin\{i_3,j_3\}} {}^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{a\,b} \times \mathcal{S}_n^{i_3\,j_3}{}_{|\bar{i_2}=a,\bar{j_2}=b}\\
&\quad + \# \bigcup_{b<i_3} {}^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{i_3\,b} \times \mathcal{S}_n^{i_3\,j_3}{}_{|\bar{i_2}=i_3,\bar{j_2}=b} + \# \bigcup_{\substack{b>i_3\\ b\neq j_3}} {}^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{i_3\,b} \times \mathcal{S}_n^{i_3\,j_3}{}_{|\bar{i_2}=i_3,\bar{j_2}=b}\\
&\quad + \# \bigcup_{\substack{b<j_3\\ b\neq i_3}} {}^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{i_3\,b} \times \mathcal{S}_n^{i_3\,j_3}{}_{|\bar{i_2}=i_3,\bar{j_2}=b} + \# \bigcup_{b>j_3} {}^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{j_3\,b} \times \mathcal{S}_n^{i_3\,j_3}{}_{|\bar{i_2}=j_3,\bar{j_2}=b}\\
&\quad + \# \bigcup_{a<i_3} {}^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{a\,i_3} \times \mathcal{S}_n^{i_3\,j_3}{}_{|\bar{i_2}=a,\bar{j_2}=i_3} + \# \bigcup_{\substack{a>i_3\\ a\neq j_3}} {}^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{a\,i_3} \times \mathcal{S}_n^{i_3\,j_3}{}_{|\bar{i_2}=a,\bar{j_2}=i_3}\\
&\quad + \# \bigcup_{\substack{a<j_3\\ a\neq i_3}} {}^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{a\,j_3} \times \mathcal{S}_n^{i_3\,j_3}{}_{|\bar{i_2}=a,\bar{j_2}=j_3} + \# \bigcup_{a>j_3} {}^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{a\,j_3} \times \mathcal{S}_n^{i_3\,j_3}{}_{|\bar{i_2}=a,\bar{j_2}=j_3}\\
&= \frac{n!^2}{8} - \frac{(n-2)!^2}{2} - (n-4)(n-3)!^2\chi_{i_1,j_1}\chi_{i_2,j_2}\chi_{i_3,j_3}\\
&\quad - (n-3)!(n-2)!\Big(\chi_{i_1,j_1}\chi_{i_2,j_2} + \chi_{i_1,j_1}\chi_{i_3,j_3} + \chi_{i_2,j_2}\chi_{i_3,j_3}\Big).
\end{aligned}$$

(b) We have the following equations:

$$\#^{i_1\, j_1}\mathcal{S}_n \cap \mathcal{S}_n^{i_3\, j_3} \times \mathcal{S}_n^{j_3\, i_3}{}_{|\bar{i_2}=i_3,\bar{j_2}=j_3} = (n-3)!(n-2)!\chi_{i_1,j_1}\chi_{i_3,j_3} + (n-2)!^2\frac{n^2-n+2}{4}$$

$$\# \bigcup_{a,b\notin\{i_3,j_3\}} {}^{i_1\, j_1}\mathcal{S}_n \cap \mathcal{S}_n^{a\, b} \times \mathcal{S}_n^{j_3\, i_3}{}_{|\bar{i_2}=a,\bar{j_2}=b} = \binom{n-2}{2}\frac{(n-2)!n!}{4}$$

$$\# \bigcup_{b<i_3} {}^{i_1\, j_1}\mathcal{S}_n \cap \mathcal{S}_n^{i_3\, b} \times \mathcal{S}_n^{j_3\, i_3}{}_{|\bar{i_2}=i_3,\bar{j_2}=b} = (n-i_2-1)(i_3-1)(n-3)!(n-2)!\frac{n^2-n-2}{4} - (n-i_2-1)\binom{i_3-1}{2}(n-3)!^2\chi_{i_1,j_1}$$

$$\# \bigcup_{\substack{b>i_3\\ b\neq j_3}} {}^{i_1\, j_1}\mathcal{S}_n \cap \mathcal{S}_n^{i_3\, b} \times \mathcal{S}_n^{j_3\, i_3}{}_{|\bar{i_2}=i_3,\bar{j_2}=b} = (n-i_2-1)\binom{n-i_3}{2}(n-3)!^2\chi_{i_1,j_1} - (n-i_2-1)(n-3)!^2\chi_{i_1,j_1}\chi_{i_3,j_3} + (n-i_2-1)(n-i_3-1)(n-3)!(n-2)!\frac{n^2-n+2}{4}$$

$$\# \bigcup_{\substack{b<j_3\\ b\neq i_3}} {}^{i_1\, j_1}\mathcal{S}_n \cap \mathcal{S}_n^{i_3\, b} \times \mathcal{S}_n^{j_3\, i_3}{}_{|\bar{i_2}=i_3,\bar{j_2}=b} = (i_2-1)(j_3-2)(n-3)!(n-2)!\frac{n^2-n-2}{4} - (i_2-1)\binom{j_3-1}{2}(n-3)!^2\chi_{i_1,j_1} + (i_2-1)(n-3)!^2\chi_{i_1,j_1}\chi_{i_3,j_3}$$

$$\# \bigcup_{b>j_3} {}^{i_1\, j_1}\mathcal{S}_n \cap \mathcal{S}_n^{j_3\, b} \times \mathcal{S}_n^{j_3\, i_3}{}_{|\bar{i_2}=j_3,\bar{j_2}=b} = (i_2-1)\binom{n-j_3}{2}(n-3)!^2\chi_{i_1,j_1} + (i_2-1)(n-j_3)(n-3)!(n-2)!\frac{n^2-n+2}{4}$$

$$\# \bigcup_{a<i_3} {}^{i_1\, j_1}\mathcal{S}_n \cap \mathcal{S}_n^{a\, i_3} \times \mathcal{S}_n^{j_3\, i_3}{}_{|\bar{i_2}=a,\bar{j_2}=i_3} = (n-j_2)\binom{i_3-1}{2}(n-3)!^2\chi_{i_1,j_1} + (n-j_2)(i_3-1)(n-3)!(n-2)!\frac{n^2-n+2}{4}$$

$$\# \bigcup_{\substack{a>i_3\\ a\neq j_3}} {}^{i_1\, j_1}\mathcal{S}_n \cap \mathcal{S}_n^{a\, i_3} \times \mathcal{S}_n^{j_3\, i_3}{}_{|\bar{i_2}=a,\bar{j_2}=i_3} = (n-j_2)(n-i_3-1)(n-3)!(n-2)!\frac{n^2-n-2}{4} - (n-j_2)\binom{n-i_3}{2}(n-3)!^2\chi_{i_1,j_1}$$

$$\# \bigcup_{\substack{a<j_3\\ a\neq i_3}} {}^{i_1\, j_1}\mathcal{S}_n \cap \mathcal{S}_n^{a\, j_3} \times \mathcal{S}_n^{j_3\, i_3}{}_{|\bar{i_2}=a,\bar{j_2}=j_3} = (j_2-2)\binom{j_3-1}{2}(n-3)!^2\chi_{i_1,j_1} - (j_2-2)(n-3)!^2\chi_{i_1,j_1}\chi_{i_3,j_3} + (j_2-2)(j_3-2)(n-3)!(n-2)!\frac{n^2-n+2}{4}$$

$$\# \bigcup_{a>j_3} {}^{i_1\, j_1}\mathcal{S}_n \cap \mathcal{S}_n^{a\, j_3} \times \mathcal{S}_n^{j_3\, i_3}{}_{|\bar{i_2}=a,\bar{j_2}=j_3} = (j_2-2)(n-j_3)(n-3)!(n-2)!\frac{n^2-n-2}{4} - (j_2-2)\binom{n-j_3}{2}(n-3)!^2\chi_{i_1,j_1}$$

Then we deduce

$$\sum_{a,b\in[n]} {}^{i_1\, j_1}\mathcal{S}_n \cap \mathcal{S}_n^{a\, b} \times \mathcal{S}_n^{j_3\, i_3}{}_{|\bar{i_2}=a,\bar{j_2}=b}$$

$$= \#^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{i_3\,j_3} \times \mathcal{S}_n^{j_3\,i_3}{}_{|\bar{i_2}=i_3,\bar{j_2}=j_3} + \# \bigcup_{a,b\notin\{i_3,j_3\}} {}^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{a\,b} \times \mathcal{S}_n^{j_3\,i_3}{}_{|\bar{i_2}=a,\bar{j_2}=b}$$
$$+ \# \bigcup_{b<i_3} {}^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{i_3\,b} \times \mathcal{S}_n^{j_3\,i_3}{}_{|\bar{i_2}=i_3,\bar{j_2}=b} + \# \bigcup_{\substack{b>i_3\\ b\neq j_3}} {}^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{i_3\,b} \times \mathcal{S}_n^{j_3\,i_3}{}_{|\bar{i_2}=i_3,\bar{j_2}=b}$$
$$+ \# \bigcup_{\substack{b<j_3\\ b\neq i_3}} {}^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{i_3\,b} \times \mathcal{S}_n^{j_3\,i_3}{}_{|\bar{i_2}=i_3,\bar{j_2}=b} + \# \bigcup_{b>j_3} {}^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{j_3\,b} \times \mathcal{S}_n^{j_3\,i_3}{}_{|\bar{i_2}=j_3,\bar{j_2}=b}$$
$$+ \# \bigcup_{a<i_3} {}^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{a\,i_3} \times \mathcal{S}_n^{j_3\,i_3}{}_{|\bar{i_2}=a,\bar{j_2}=i_3} + \# \bigcup_{\substack{a>i_3\\ a\neq j_3}} {}^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{a\,i_3} \times \mathcal{S}_n^{j_3\,i_3}{}_{|\bar{i_2}=a,\bar{j_2}=i_3}$$
$$+ \# \bigcup_{\substack{a<j_3\\ a\neq i_3}} {}^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{a\,j_3} \times \mathcal{S}_n^{j_3\,i_3}{}_{|\bar{i_2}=a,\bar{j_2}=j_3} + \# \bigcup_{a>j_3} {}^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{a\,j_3} \times \mathcal{S}_n^{j_3\,i_3}{}_{|\bar{i_2}=a,\bar{j_2}=j_3}$$
$$= \frac{n!^2}{8} + \frac{(n-2)!^2}{2} + (n-4)(n-3)!^2\chi_{i_1,j_1}\chi_{i_2,j_2}\chi_{i_3,j_3}$$
$$+ (n-3)!(n-2)!\Big(\chi_{i_1,j_1}\chi_{i_2,j_2} + \chi_{i_1,j_1}\chi_{i_3,j_3} + \chi_{i_2,j_2}\chi_{i_3,j_3}\Big)$$

□

Let $n \geq 4$. We define:

$$\begin{array}{cccc} \mathtt{f_3}: & \mathcal{S}_n & \to & \mathbb{R}[X_{1,2},\ldots,X_{n-1,n}] \\ & \pi & \mapsto & \sum_{\sigma,\tau\in\mathcal{S}_n} \mathtt{inv_X}(\sigma^{-1})\mathtt{inv_X}(\sigma\tau^{-1})\mathtt{inv_X}(\tau\pi). \end{array}$$

Lemma 4.3.5. *Let $n \geq 4$. Then:*

(a) For $(i_1,j_1) = (i_2,j_2) = (i_3,j_3)$:

- *If $\pi(i_1) < \pi(j_1)$:*

$$[X^3_{i_1,j_1}]\mathtt{f_3}(\pi) = \frac{n!^2}{8} - \frac{(n-2)!^2}{2} - (n-4)(n-3)!^2\chi^2_{i_1,j_1}\chi_{\pi(i_1),\pi(j_1)}$$
$$-(n-3)!(n-2)!\chi^2_{i_1,j_1} - 2(n-3)!(n-2)!\chi_{i_1,j_1}\chi_{\pi(i_1),\pi(j_1)}.$$

- *If $\pi(i_1) > \pi(j_1)$:*

$$[X^3_{i_1,j_1}]\mathtt{f_3}(\pi) = \frac{n!^2}{8} + \frac{(n-2)!^2}{2} + (n-4)(n-3)!^2\chi^2_{i_1,j_1}\chi_{\pi(j_1),\pi(i_1)}$$
$$+(n-3)!(n-2)!\chi^2_{i_1,j_1} + 2(n-3)!(n-2)!\chi_{i_1,j_1}\chi_{\pi(j_1),\pi(i_1)}.$$

(b) For $(i_1,j_1) = (i_2,j_2) \neq (i_3,j_3)$:

- *If $\pi(i_1) < \pi(j_1)$ and $\pi(i_3) < \pi(j_3)$:*

$$[X^2_{i_1,j_1}X_{i_3,j_3}]\mathtt{f_3}(\pi) = \frac{3n!^2}{8} - \frac{3(n-2)!^2}{2} - (n-4)(n-3)!^2\big(\chi^2_{i_1,j_1}\chi_{\pi(i_3),\pi(j_3)} + 2\chi_{i_1,j_1}\chi_{i_3,j_3}\chi_{\pi(i_1),\pi(j_1)}\big)$$
$$-(n-3)!(n-2)!\big(\chi^2_{i_1,j_1} + 2\chi_{i_1,j_1}\chi_{\pi(i_3),\pi(j_3)} + 2\chi_{i_1,j_1}\chi_{i_3,j_3} + 2\chi_{i_1,j_1}\chi_{\pi(i_1),\pi(j_1)} + 2\chi_{i_3,j_3}\chi_{\pi(i_1),\pi(j_1)}\big).$$

- *If $\pi(i_1) < \pi(j_1)$ and $\pi(i_3) > \pi(j_3)$:*

$$[X^2_{i_1,j_1}X_{i_3,j_3}]\mathtt{f_3}(\pi) = \frac{3n!^2}{8} - \frac{(n-2)!^2}{2} + (n-4)(n-3)!^2\big(\chi^2_{i_1,j_1}\chi_{\pi(j_3),\pi(i_3)} - 2\chi_{i_1,j_1}\chi_{i_3,j_3}\chi_{\pi(i_1),\pi(j_1)}\big)$$
$$+(n-3)!(n-2)!\big(\chi^2_{i_1,j_1} + 2\chi_{i_1,j_1}\chi_{\pi(j_3),\pi(i_3)} - 2\chi_{i_1,j_1}\chi_{i_3,j_3} - 2\chi_{i_1,j_1}\chi_{\pi(i_1),\pi(j_1)} - 2\chi_{i_3,j_3}\chi_{\pi(i_1),\pi(j_1)}\big).$$

▷ *If $\pi(i_1) > \pi(j_1)$ and $\pi(i_3) < \pi(j_3)$:*

$$[X^2_{i_1,j_1}X_{i_3,j_3}]\mathtt{f_3}(\pi) = \frac{3n!^2}{8} + \frac{(n-2)!^2}{2} - (n-4)(n-3)!^2\big(\chi^2_{i_1,j_1}\chi_{\pi(i_3),\pi(j_3)} - 2\chi_{i_1,j_1}\chi_{i_3,j_3}\chi_{\pi(j_1),\pi(i_1)}\big)$$

$$-(n-3)!(n-2)!\big(\chi^2_{i_1,j_1} + 2\chi_{i_1,j_1}\chi_{\pi(i_3),\pi(j_3)} - 2\chi_{i_1,j_1}\chi_{i_3,j_3} - 2\chi_{i_1,j_1}\chi_{\pi(j_1),\pi(i_1)} - 2\chi_{i_3,j_3}\chi_{\pi(j_1),\pi(i_1)}\big).$$

▷ *If $\pi(i_1) > \pi(j_1)$ and $\pi(i_3) > \pi(j_3)$:*

$$[X^2_{i_1,j_1}X_{i_3,j_3}\mathtt{f_3}(\pi) = \frac{3n!^2}{8} + \frac{3(n-2)!^2}{2} + (n-4)(n-3)!^2\big(\chi^2_{i_1,j_1}\chi_{\pi(j_3),\pi(i_3)} + 2\chi_{i_1,j_1}\chi_{i_3,j_3}\chi_{\pi(j_1),\pi(i_1)}\big)$$

$$+(n-3)!(n-2)!\big(\chi^2_{i_1,j_1} + 2\chi_{i_1,j_1}\chi_{\pi(j_3),\pi(i_3)} + 2\chi_{i_1,j_1}\chi_{i_3,j_3} + 2\chi_{i_1,j_1}\chi_{\pi(j_1),\pi(i_1)} + 2\chi_{i_3,j_3}\chi_{\pi(j_1),\pi(i_1)}\big).$$

(c) For $(i_1,j_1) \neq (i_2,j_2) \neq (i_3,j_3)$:

▷ *If $\pi(i_1) < \pi(j_1)$, $\pi(i_2) < \pi(j_2)$ and $\pi(i_3) < \pi(j_3)$:*

$$[X_{i_1,j_1}X_{i_2,j_2}X_{i_3,j_3}]\mathtt{f_3}(\pi) = \frac{3n!^2}{4} - 3(n-2)!^2$$

$$-2(n-4)(n-3)!^2\big(\chi_{i_1,j_1}\chi_{i_2,j_2}\chi_{\pi(i_3),\pi(j_3)} + \chi_{i_1,j_1}\chi_{\pi(i_2),\pi(j_2)}\chi_{i_3,j_3} + \chi_{\pi(i_1),\pi(j_1)}\chi_{i_2,j_2}\chi_{i_3,j_3}\big)$$

$$-2(n-3)!(n-2)!\big(\chi_{i_1,j_1}\chi_{i_2,j_2} + \chi_{i_1,j_1}\chi_{\pi(i_3),\pi(j_3)} + \chi_{i_2,j_2}\chi_{\pi(i_3),\pi(j_3)}$$

$$+\chi_{i_1,j_1}\chi_{i_3,j_3} + \chi_{i_1,j_1}\chi_{\pi(i_2),\pi(j_2)} + \chi_{i_3,j_3}\chi_{\pi(i_2),\pi(j_2)} + \chi_{i_2,j_2}\chi_{i_3,j_3} + \chi_{i_2,j_2}\chi_{\pi(i_1),\pi(j_1)} + \chi_{i_3,j_3}\chi_{\pi(i_1),\pi(j_1)}\big).$$

▷ *If $\pi(i_1) < \pi(j_1)$, $\pi(i_2) < \pi(j_2)$ and $\pi(i_3) > \pi(j_3)$:*

$$[X_{i_1,j_1}X_{i_2,j_2}X_{i_3,j_3}]\mathtt{f_3}(\pi) = \frac{3n!^2}{4} - (n-2)!^2$$

$$+2(n-4)(n-3)!^2\big(\chi_{i_1,j_1}\chi_{i_2,j_2}\chi_{\pi(j_3),\pi(i_3)} - \chi_{i_1,j_1}\chi_{\pi(i_2),\pi(j_2)}\chi_{i_3,j_3} - \chi_{\pi(i_1),\pi(j_1)}\chi_{i_2,j_2}\chi_{i_3,j_3}\big)$$

$$+2(n-3)!(n-2)!\big(\chi_{i_1,j_1}\chi_{i_2,j_2} + \chi_{i_1,j_1}\chi_{\pi(j_3),\pi(i_3)} + \chi_{i_2,j_2}\chi_{\pi(j_3),\pi(i_3)}$$

$$-\chi_{i_1,j_1}\chi_{i_3,j_3} - \chi_{i_1,j_1}\chi_{\pi(i_2),\pi(j_2)} - \chi_{i_3,j_3}\chi_{\pi(i_2),\pi(j_2)} - \chi_{i_2,j_2}\chi_{i_3,j_3} - \chi_{i_2,j_2}\chi_{\pi(i_1),\pi(j_1)} - \chi_{i_3,j_3}\chi_{\pi(i_1),\pi(j_1)}\big).$$

▷ *If $\pi(i_1) < \pi(j_1)$, $\pi(i_2) > \pi(j_2)$ and $\pi(i_3) > \pi(j_3)$:*

$$[X_{i_1,j_1}X_{i_2,j_2}X_{i_3,j_3}]\mathtt{f_3}(\pi) = \frac{3n!^2}{4} + (n-2)!^2$$

$$+2(n-4)(n-3)!^2\big(\chi_{i_1,j_1}\chi_{i_2,j_2}\chi_{\pi(j_3),\pi(i_3)} + \chi_{i_1,j_1}\chi_{\pi(j_2),\pi(i_2)}\chi_{i_3,j_3} - \chi_{\pi(i_1),\pi(j_1)}\chi_{i_2,j_2}\chi_{i_3,j_3}\big)$$

$$+2(n-3)!(n-2)!\big(\chi_{i_1,j_1}\chi_{i_2,j_2} + \chi_{i_1,j_1}\chi_{\pi(j_3),\pi(i_3)} + \chi_{i_2,j_2}\chi_{\pi(j_3),\pi(i_3)}$$

$$+\chi_{i_1,j_1}\chi_{i_3,j_3} + \chi_{i_1,j_1}\chi_{\pi(j_2),\pi(i_2)} + \chi_{i_3,j_3}\chi_{\pi(j_2),\pi(i_2)} - \chi_{i_2,j_2}\chi_{i_3,j_3} - \chi_{i_2,j_2}\chi_{\pi(i_1),\pi(j_1)} - \chi_{i_3,j_3}\chi_{\pi(i_1),\pi(j_1)}\big).$$

▷ *If $\pi(i_1) > \pi(j_1)$, $\pi(i_2) > \pi(j_2)$ and $\pi(i_3) > \pi(j_3)$:*

$$[X_{i_1,j_1}X_{i_2,j_2}X_{i_3,j_3}]\mathtt{f_3}(\pi) = \frac{3n!^2}{4} + 3(n-2)!^2$$

$$+2(n-4)(n-3)!^2\big(\chi_{i_1,j_1}\chi_{i_2,j_2}\chi_{\pi(j_3),\pi(i_3)} + \chi_{i_1,j_1}\chi_{\pi(j_2),\pi(i_2)}\chi_{i_3,j_3} + \chi_{\pi(j_1),\pi(i_1)}\chi_{i_2,j_2}\chi_{i_3,j_3}\big)$$

$$+2(n-3)!(n-2)!\big(\chi_{i_1,j_1}\chi_{i_2,j_2} + \chi_{i_1,j_1}\chi_{\pi(j_3),\pi(i_3)} + \chi_{i_2,j_2}\chi_{\pi(j_3),\pi(i_3)}$$

$$+\chi_{i_1,j_1}\chi_{i_3,j_3} + \chi_{i_1,j_1}\chi_{\pi(j_2),\pi(i_2)} + \chi_{i_3,j_3}\chi_{\pi(j_2),\pi(i_2)} + \chi_{i_2,j_2}\chi_{i_3,j_3} + \chi_{i_2,j_2}\chi_{\pi(j_1),\pi(i_1)} + \chi_{i_3,j_3}\chi_{\pi(j_1),\pi(i_1)}\big).$$

Proof. (a) For $(i_1,j_1) = (i_2,j_2) = (i_3,j_3)$:

$$[X^3_{i_1,j_1}]\mathtt{f_3}(\pi) = \sum_{a,b\in[n]} \#^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{a\,b} \times \mathcal{S}_n^{\pi(i_1)\,\pi(j_1)}{}_{|\bar{i_1}=a,\bar{j_1}=b}.$$

(b) For $(i_1, j_1) = (i_2, j_2) \neq (i_3, j_3)$:

$$[X_{i_1,j_1}^2 X_{i_3,j_3}]\mathtt{f_3}(\pi) = \sum_{a,b\in[n]} \#^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{a\,b} \times \mathcal{S}_n^{\pi(i_3)\,\pi(j_3)}{}_{|\bar{i_1}=a,\bar{j_1}=b}$$

$$+ \sum_{a,b\in[n]} \#^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{a\,b} \times \mathcal{S}_n^{\pi(i_1)\,\pi(j_1)}{}_{|\bar{i_3}=a,\bar{j_3}=b} + \sum_{a,b\in[n]} \#^{i_3\,j_3}\mathcal{S}_n \cap \mathcal{S}_n^{a\,b} \times \mathcal{S}_n^{\pi(i_1)\,\pi(j_1)}{}_{|\bar{i_1}=a,\bar{j_1}=b}.$$

(c) For $(i_1, j_1) \neq (i_3, j_3) \neq (i_2, j_2)$:

$$[X_{i_1,j_1} X_{i_2,j_2} X_{i_3,j_3}]\mathtt{f_3}(\pi) = \sum_{a,b\in[n]} \#^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{a\,b} \times \mathcal{S}_n^{\pi(i_3)\,\pi(j_3)}{}_{|\bar{i_2}=a,\bar{j_2}=b}$$

$$+ \sum_{a,b\in[n]} \#^{i_2\,j_2}\mathcal{S}_n \cap \mathcal{S}_n^{a\,b} \times \mathcal{S}_n^{\pi(i_3)\,\pi(j_3)}{}_{|\bar{i_1}=a,\bar{j_1}=b} + \sum_{a,b\in[n]} \#^{i_1\,j_1}\mathcal{S}_n \cap \mathcal{S}_n^{a\,b} \times \mathcal{S}_n^{\pi(i_2)\,\pi(j_2)}{}_{|\bar{i_3}=a,\bar{j_3}=b}$$

$$+ \sum_{a,b\in[n]} \#^{i_3\,j_3}\mathcal{S}_n \cap \mathcal{S}_n^{a\,b} \times \mathcal{S}_n^{\pi(i_2)\,\pi(j_2)}{}_{|\bar{i_1}=a,\bar{j_1}=b} + \sum_{a,b\in[n]} \#^{i_2\,j_2}\mathcal{S}_n \cap \mathcal{S}_n^{a\,b} \times \mathcal{S}_n^{\pi(i_1)\,\pi(j_1)}{}_{|\bar{i_3}=a,\bar{j_3}=b}$$

$$+ \sum_{a,b\in[n]} \#^{i_3\,j_3}\mathcal{S}_n \cap \mathcal{S}_n^{a\,b} \times \mathcal{S}_n^{\pi(i_1)\,\pi(j_1)}{}_{|\bar{i_2}=a,\bar{j_2}=b}.$$

□

Let $n \geq 4$. We write

$$\Lambda_n = (n-2)! \sum_{\{(i,j)\in[n]^2 \;|\; i<j\}} (j-i)X_{i,j},$$

$$\Delta_n = (n-3)! \sum_{\{(i,j)\in[n]^2 \;|\; i<j\}} \Big(n - 2(j-i)\Big)X_{i,j}.$$

Lemma 4.3.6. *Let $n \geq 4$. Then:*

(a)

$$[X_{i,j}](\Lambda_n + \Delta_n) = 2(n-2)! + (n-4)(n-3)!\chi_{i,j}.$$

(b)

$$[X_{i_1,j_1} X_{i_2,j_2}](\Lambda_n \Delta_n) = (n-3)!(n-2)! \frac{\big(2(n-2) + (n-4)\chi_{i_1,j_1} + (n-4)\chi_{i_2,j_2} - 4\chi_{i_1,j_1}\chi_{i_2,j_2}\big)}{1 + \delta_{(i_1,j_1),(i_2,j_2)}}$$

Proof. Coefficient calculating. □

Let $n \geq 4$. We define:

$$\begin{array}{rccc} \mathtt{f}: & \mathcal{S}_n & \to & \mathbb{R}[X_{1,2}, \ldots, X_{n-1,n}] \\ & \pi & \mapsto & \Lambda_n \Delta_n \mathtt{inv_X}(\pi) + \big(\Lambda_n + \Delta_n\big)\mathtt{f_2}(\pi) + \mathtt{f_3}(\pi) \end{array}$$

Lemma 4.3.7. *Let $n \geq 4$ and $\pi \in \mathcal{S}_n$. Then:*

$$\mathtt{f}(\pi) = \mathtt{f}(e) = \big(\Lambda_n + \Delta_n\big)\mathtt{f_2}(e) + \mathtt{f_3}(e).$$

Proof. Using Lemma 4.3.3, Lemma 4.3.5 and Lemma 4.3.6, we prove that:

- for $(i_1, j_1) = (i_2, j_2) = (i_3, j_3)$:

$$\begin{aligned}[X_{i_1,j_1}^3]\mathtt{f}(e) &= [X_{i_1,j_1}^3]\mathtt{f}(\pi) \\ &= \frac{n!^2}{8} + (n-2)!^2\frac{n^2-n+1}{2} \\ &\quad + (n-4)(n-3)!(n-2)!\frac{n^2-n+2}{4}\chi_{i_1,j_1} - (n-3)!(n-2)!\chi_{i_1,j_1}^2,\end{aligned}$$

- for $(i_1, j_1) = (i_2, j_2) \neq (i_3, j_3)$:

$$\begin{aligned}[X_{i_1,j_1}^2 X_{i_3,j_3}]\mathtt{f}(e) &= [X_{i_1,j_1}^2 X_{i_3,j_3}]\mathtt{f}(\pi) \\ &= \frac{3n!^2}{8} + \frac{3(n-2)!n!}{2} + \frac{3(n-2)!^2}{2} \\ &\quad + (n-4)(n-3)!(n-2)!\frac{n^2-n+2}{4}\big(2\chi_{i_1,j_1} + \chi_{i_3,j_3}\big) \\ &\quad - (n-3)!(n-2)!\big(\chi_{i_1,j_1}^2 + 2\chi_{i_1,j_1}\chi_{i_3,j_3}\big),\end{aligned}$$

- for $(i_1, j_1) \neq (i_2, j_2) \neq (i_3, j_3) \neq (i_1, j_1)$:

$$\begin{aligned}[X_{i_1,j_1} X_{i_2,j_2} X_{i_3,j_3}]\mathtt{f}(e) &= [X_{i_1,j_1} X_{i_2,j_2} X_{i_3,j_3}]\mathtt{f}(\pi) \\ &= \frac{3n!^2}{4} + 3(n-2)!n! + 3(n-2)!^2 \\ &\quad + (n-4)(n-3)!(n-2)!\frac{n^2-n+2}{2}\big(\chi_{i_1,j_1} + \chi_{i_2,j_2} + \chi_{i_3,j_3}\big) \\ &\quad - 2(n-3)!(n-2)!\big(\chi_{i_1,j_1}\chi_{i_2,j_2} + \chi_{i_1,j_1}\chi_{i_3,j_3} + \chi_{i_2,j_2}\chi_{i_3,j_3}\big).\end{aligned}$$

□

We are now in position to determine the minimal polynomial of $\mathfrak{I}_{\mathfrak{X}_n}$.
Let $n \geq 4$. We write

$$\Omega_n = \frac{n!}{2} \sum_{\{(i,j)\in[n]^2 \ | \ i<j\}} X_{i,j}.$$

Proposition 4.3.8. *Let $n \geq 4$. The minimal polynomial of $\mathfrak{I}_{\mathfrak{X}_n}$ is:*

$$X\big(X + \Lambda_n\big)\big(X + \Delta_n\big)\big(X - \Omega_n\big).$$

Proof. Using Lemma 4.1.4 and Lemma 4.3.7, we get:

$$\Big(\sum_{\sigma\in\mathcal{S}_n} \mathtt{inv}_{\mathtt{X}}(\sigma)\sigma + \Lambda_n e\Big)\Big(\sum_{\sigma\in\mathcal{S}_n} \mathtt{inv}_{\mathtt{X}}(\sigma)\sigma + \Delta_n e\Big) \sum_{\sigma\in\mathcal{S}_n} \mathtt{inv}_{\mathtt{X}}(\sigma)\sigma$$

$$
\begin{aligned}
&= \sum_{\sigma\in\mathcal{S}_n} \Big(\sum_{\substack{\sigma_1,\sigma_2,\sigma_3\in\mathcal{S}_n\\ \sigma_1\sigma_2\sigma_3=\sigma}} \mathtt{inv}_{\mathtt{X}}(\sigma_1)\mathtt{inv}_{\mathtt{X}}(\sigma_2)\mathtt{inv}_{\mathtt{X}}(\sigma_3)\Big)\sigma \\
&\quad + \sum_{\sigma\in\mathcal{S}_n} \Big((\Lambda_n+\Delta_n) \sum_{\substack{\sigma_1,\sigma_2\in\mathcal{S}_n\\ \sigma_1\sigma_2=\sigma}} \mathtt{inv}_{\mathtt{X}}(\sigma_1)\mathtt{inv}_{\mathtt{X}}(\sigma_2)\Big)\sigma + \sum_{\sigma\in\mathcal{S}_n\setminus\{e\}} \Lambda_n\Delta_n\mathtt{inv}_{\mathtt{X}}(\sigma)\sigma \\
&= \sum_{\sigma\in\mathcal{S}_n} \Big(\sum_{\sigma_1,\sigma_2\in\mathcal{S}_n} \mathtt{inv}_{\mathtt{X}}(\sigma_1^{-1})\mathtt{inv}_{\mathtt{X}}(\sigma_1\sigma_2^{-1})\mathtt{inv}_{\mathtt{X}}(\sigma_2\sigma)\Big)\sigma \\
&\quad + \sum_{\sigma\in\mathcal{S}_n} \Big((\Lambda_n+\Delta_n) \sum_{\sigma_1\in\mathcal{S}_n} \mathtt{inv}_{\mathtt{X}}(\sigma_1^{-1})\mathtt{inv}_{\mathtt{X}}(\sigma_1\sigma)\Big)\sigma + \sum_{\sigma\in\mathcal{S}_n\setminus\{e\}} \Lambda_n\Delta_n\mathtt{inv}_{\mathtt{X}}(\sigma)\sigma \\
&= \sum_{\sigma\in\mathcal{S}_n} \mathtt{f}(\sigma)\sigma \\
&= \mathtt{f}(e)\sum_{\sigma\in\mathcal{S}_n}\sigma.
\end{aligned}
$$

Then:

$$
\begin{aligned}
&\Big(\sum_{\sigma\in\mathcal{S}_n} \mathtt{inv}_{\mathtt{X}}(\sigma)\sigma + \Lambda_n e\Big)\Big(\sum_{\sigma\in\mathcal{S}_n} \mathtt{inv}_{\mathtt{X}}(\sigma)\sigma + \Delta_n e\Big)\sum_{\sigma\in\mathcal{S}_n} \mathtt{inv}_{\mathtt{X}}(\sigma)\sigma\Big(\sum_{\sigma\in\mathcal{S}_n} \mathtt{inv}_{\mathtt{X}}(\sigma)\sigma - \Omega_n e\Big) \\
&\qquad = \mathtt{f}(e)\sum_{\sigma\in\mathcal{S}_n}\sigma\Big(\sum_{\sigma\in\mathcal{S}_n} \mathtt{inv}_{\mathtt{X}}(\sigma)\sigma - \Omega_n e\Big) \\
&\qquad = 0.
\end{aligned}
$$

Hence the minimal polynomial of $\mathfrak{I}_{\mathfrak{X}n}$ divides $X\big(X+\Lambda_n\big)\big(X+\Delta_n\big)\big(X-\Omega_n\big)$.
It is clear that the minimal polynomial of $\mathfrak{I}_{\mathfrak{X}n}$ does not divide $X\big(X+\Lambda_n\big)\big(X+\Delta_n\big)$.
We have:

$$
\begin{aligned}
&[X_{1,4}^3]\bigg([e]\Big(\Big(\sum_{\sigma\in\mathcal{S}_n} \mathtt{inv}_{\mathtt{X}}(\sigma)\sigma + \Lambda_n e\Big)\Big(\sum_{\sigma\in\mathcal{S}_n} \mathtt{inv}_{\mathtt{X}}(\sigma)\sigma - \Omega_n e\Big)\sum_{\sigma\in\mathcal{S}_n} \mathtt{inv}_{\mathtt{X}}(\sigma)\sigma\Big)\bigg) \\
&\qquad = (n-3)!^2\frac{n^4-8n^3+22n^2-36n+44}{2} \neq 0.
\end{aligned}
$$

Then the minimal polynomial of $\mathfrak{I}_{\mathfrak{X}n}$ does not divide $X\big(X+\Lambda_n\big)\big(X-\Omega_n\big)$.
We have:

$$
\begin{aligned}
&[X_{1,3}^3]\bigg([e]\Big(\Big(\sum_{\sigma\in\mathcal{S}_n} \mathtt{inv}_{\mathtt{X}}(\sigma)\sigma + \Delta_n e\Big)\Big(\sum_{\sigma\in\mathcal{S}_n} \mathtt{inv}_{\mathtt{X}}(\sigma)\sigma - \Omega_n e\Big)\sum_{\sigma\in\mathcal{S}_n} \mathtt{inv}_{\mathtt{X}}(\sigma)\sigma\Big)\bigg) \\
&\qquad = -4(n-3)!(n-2)! - (n-3)!n! \neq 0.
\end{aligned}
$$

Then the minimal polynomial of $\mathfrak{I}_{\mathfrak{X}n}$ does not divide $X\big(X+\Delta_n\big)\big(X-\Omega_n\big)$.
We have:

$$
\begin{aligned}
&[X_{1,3}^3]\bigg([e]\Big(\Big(\sum_{\sigma\in\mathcal{S}_n} \mathtt{inv}_{\mathtt{X}}(\sigma)\sigma + \Lambda_n e\Big)\Big(\sum_{\sigma\in\mathcal{S}_n} \mathtt{inv}_{\mathtt{X}}(\sigma)\sigma + \Delta_n e\Big)\Big(\sum_{\sigma\in\mathcal{S}_n} \mathtt{inv}_{\mathtt{X}}(\sigma)\sigma - \Omega_n e\Big)\Big)\bigg) \\
&\qquad = \frac{(n-2)!^2}{2} + \frac{(n-2)!n!}{2} - \frac{(n-2)!^2n!}{2} \neq 0.
\end{aligned}
$$

Then the minimal polynomial of $\mathfrak{I}_{\mathfrak{X}n}$ does not divide $\big(X+\Lambda_n\big)\big(X+\Delta_n\big)\big(X-\Omega_n\big)$. □

4.4 Multiplicities

We begin with the multiplicities of $\mathfrak{D}_{\mathfrak{X}_n}$.
From Proposition 4.2.3, we deduce that $\mathfrak{D}_{\mathfrak{X}_n}$ is diagonalizable and:

$$Sp(\mathfrak{D}_{\mathfrak{X}_n}) = \{\frac{n!}{2}Q_n,\ -(n-2)!Q_n,\ 0\}.$$

From the result (4.1) of Section 4.2, we get

$$V_{\mathfrak{D}_{\mathfrak{X}_n}}(\frac{n!}{2}Q_n) = 1$$

.
The trace of $\mathfrak{D}_{\mathfrak{X}_n}$ is 0. Then:

$$\frac{n!}{2}Q_n\, V_{\mathfrak{D}_{\mathfrak{X}_n}}(\frac{n!}{2}Q_n) - (n-2)!Q_n\, V_{\mathfrak{D}_{\mathfrak{X}_n}}\big(-(n-2)!Q_n\big) + 0\, V_{\mathfrak{D}_{\mathfrak{X}_n}}(0) = 0$$

$$V_{\mathfrak{D}_{\mathfrak{X}_n}}\big(-(n-2)!Q_n\big) = \binom{n}{2}.$$

The dimension of $\mathfrak{D}_{\mathfrak{X}_n}$ is $n!$. Then:

$$V_{\mathfrak{D}_{\mathfrak{X}_n}}(\frac{n!}{2}Q_n) + V_{\mathfrak{D}_{\mathfrak{X}_n}}\big(-(n-2)!Q_n\big) + V_{\mathfrak{D}_{\mathfrak{X}_n}}(0) = n!$$

$$V_{\mathfrak{D}_{\mathfrak{X}_n}}(0) = n! - \binom{n}{2} - 1.$$

We finish with the multiplicities of $\mathfrak{I}_{\mathfrak{X}_n}$.
From Proposition 4.3.8, we deduce that $\mathfrak{I}_{\mathfrak{X}_n}$ is diagonalizable and:

$$Sp(\mathfrak{I}_{\mathfrak{X}_n}) = \{\Omega_n, -\Lambda_n, -\Delta_n, 0\}.$$

From the result (4.2) of Section 4.3, we get

$$V_{\mathfrak{I}_{\mathfrak{X}_n}}(\Omega_n) = 1$$

.
The trace of $\mathfrak{I}_{\mathfrak{X}_n}$ is 0. Then:

- $$V_{\mathfrak{I}_{\mathfrak{X}_n}}(\Omega_n)\,[X_{1,2}]\Omega_n + V_{\mathfrak{I}_{\mathfrak{X}_n}}(\Lambda_n)\,[X_{1,2}]\Lambda_n + V_{\mathfrak{I}_{\mathfrak{X}_n}}(\Delta_n)\,[X_{1,2}]\Delta_n + V_{\mathfrak{I}_{\mathfrak{X}_n}}(0)\,[X_{1,2}]0 = 0,$$
  $$V_{\mathfrak{I}_{\mathfrak{X}_n}}(\Lambda_n) + V_{\mathfrak{I}_{\mathfrak{X}_n}}(\Delta_n) = \binom{n}{2}.$$

- $$V_{\mathfrak{I}_{\mathfrak{X}_n}}(\Omega_n)\,[X_{1,n}]\Omega_n + V_{\mathfrak{I}_{\mathfrak{X}_n}}(\Lambda_n)\,[X_{1,n}]\Lambda_n + V_{\mathfrak{I}_{\mathfrak{X}_n}}(\Delta_n)\,[X_{1,n}]\Delta_n + V_{\mathfrak{I}_{\mathfrak{X}_n}}(0)\,[X_{1,n}]0 = 0,$$
  $$(n-1)V_{\mathfrak{I}_{\mathfrak{X}_n}}(\Lambda_n) - V_{\mathfrak{I}_{\mathfrak{X}_n}}(\Delta_n) = \binom{n}{2}.$$

So we deduce:

$$V_{\mathfrak{I}_{\mathfrak{X}_n}}(\Lambda_n) = n-1 \text{ and } V_{\mathfrak{I}_{\mathfrak{X}_n}}(\Delta_n) = \binom{n-1}{2}.$$

The dimension of $\mathfrak{I}_{\mathfrak{X}_n}$ is $n!$. Then:

$$V_{\mathfrak{I}_{\mathfrak{X}_n}}(\Omega_n) + V_{\mathfrak{I}_{\mathfrak{X}_n}}(\Lambda_n) + V_{\mathfrak{I}_{\mathfrak{X}_n}}(\Delta_n) + V_{\mathfrak{I}_{\mathfrak{X}_n}}(0) = n!,$$

$$V_{\mathfrak{I}_{\mathfrak{X}_n}}(0) = n! - \binom{n}{2} - 1.$$

Bibliography

[1] F. Bergeron, and N. Bergeron, *Symbolic Manipulation for the Study of the Descent Algebra of Finite Coxeter Groups*, Journal of Symbolic Computation **14** (1992) 127-139

[2] F. Bergeron, N. Bergeron, R. B. Howlett, and D. E. Taylor, *A Decomposition of the Descent Algebra of a Finite Coxeter Group*, Journal of Algebraic Combinatorics **1** (1992) 23-44

[3] A. Björner, and F. Brenti, *Combinatorics of Coxeter Groups*, Graduate Texts in Mathematics (2004)

[4] D. Blessenohl, C. Hohlweg, and M. Schocker, *A Symmetry of the Descent Algebra of a Finite Coxeter Group*, Advances in Mathematics **193** (2005) 416-437

[5] D. Blessenohl, and M. Schocker, *Noncommutative Character Theory of the Symmetric Group*, Imperial College Press (2005)

[6] B. Brink, and R. Howlett, *Normalizers of Parabolic Subgroups in Coxeter Group*, Inventiones Mathematicae **136** (1999) 323-351

[7] K. Conrad, *Dihedral Groups II*,
http://www.math.uconn.edu/ kconrad/blurbs/grouptheory/dihedral2.pdf

[8] G. Denham, and Ph. Hanlon, *Some Algebraic Properties of the Schechtman-Varchenko Bilinear Form*, MSRI Publications **38** (1999) 149-176

[9] B. Galin, *Classification of the Irreducible Representations of the Dihedral Group* D_{2n},
http://www.bens.ws/papers/representationDihedral.pdf (2007)

[10] I. M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. S. Retakh, and J.-Y. Thibon, *Noncommutative Symmetric Functions*, Advances in Mathematics **112** (1995) 218-348

[11] R. Horn, and C. Johnson, *Matrix Analysis*, Cambridge University Press (1985)

[12] R. Howlett, *Normalizers of Parabolic Subgroups of Reflection Groups*, Journal of the London Mathematical Society (2) **21** (1980) 62-80

[13] J. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge Studies in Advanced Mathematics (1990)

[14] C. Krattenthaler, *Advanced Determinant Calculus*, Séminaire Lotharingien de Combinatoire **42** (1999) Article B42q, 67 pp

[15] D. Krob, B. Leclerc, and J.-Y. Thibon, *Noncommutative Symmetric Functions II: Transformations of Alphabets*, International Journal of Algebra and Computation **7** (1997) 181-264

[16] H. Randriamaro, *Diagonalization of the Matrices of the Multinomial Descent and Multinomial Inversion Statistics on the Symmetric Group*, http://arxiv.org/abs/1104.4099 (2011)

[17] V. Reiner, F. Saliola, and V. Welker, *Spectra of Symmetrized Shuffling operators*, http://arxiv.org/abs/1102.2460 (2011)

[18] P. Renteln, *The Distance Spectra of Cayley Graphs of Coxeter Groups*, Discrete Mathematics **311** (2011) 738-755

[19] C. Reutenauer, *Free Lie Algebras*, Oxford Science Publications (1993)

[20] B. Sagan, *The Symmetric Group Representations, Combinatorial Algorithms and Symmetric Functions*, Graduate Texts in Mathematics (2001)

[21] L. Solomon, *A Mackey Formula in the Group Ring of a Coxeter Group*, Journal of Algebra **41** (1976) 255-268

[22] R. Stanley, *Enumerative Combinatorics*, Cambridge University Press (1997)

[23] A. Varchenko, *Bilinear Form of Real Configuration of Hyperplanes*, Advances in Mathematics **97** (1993) 110-144

[24] D. Zagier, *Realizability of a Model in Infinite Statistics*, Communications in Mathematical Physics **147** (1992) 199-210

Index of Notation

Index