Minimal CW-Complexes for Complements of Reflection Arrangements of Type A_{n-1} and B_n

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1 Overview

An *arrangement of hyperplanes* (or just an *arrangement*) $\mathcal{A}$ is a finite collection of linear subspaces of codimension 1 in a finite dimensional vector space. Each hyperplane H is the kernel of a linear function α_H, which is unique up to a constant.
When the underlying field is $\mathbb{R}$ there arise quite natural questions which have been studied in detail over the last century. The problem of counting regions formed by an arbitrary arrangement of n lines in the plane already occurred in the late 19th century. The general research on the properties of complex hyperplane arrangements started in the late 1960's with the groundbreaking work of Arnold and Brieskorn.
Even though these objects are easily defined, they yield nice and deep results. The study of arrangements represents an interesting interface of diverse fields of mathematics, such as algebra, algebraic geometry, topology and combinatorics.
In this work we examine combinatorial properties of the complements of certain classical hyperplane arrangements.

$\mathcal{A}_{n-1}^{\mathbb{R}}$ denotes the braid arrangement in $\mathbb{R}^n$, consisting of the hyperplanes $H_{i,j} := \{x \in \mathbb{R}^n \mid x_i = x_j\}$, for $1 \leq i < j \leq n$.

$\mathcal{B}_n^{\mathbb{R}}$ denotes the arrangement in $\mathbb{R}^n$ which in addition to the hyperplanes $H_{i,j}$ of the braid arrangement consists of the hyperplanes
$H_{i,-j} := \{x \in \mathbb{R}^n \mid x_i = -x_j\}$, for $1 \leq i < j \leq n$ and the coordinate-hyperplanes $H_i := \{x \in \mathbb{R}^n \mid x_i = 0\}$, for $i = 1, \ldots, n$.

A *complexification* of a real hyperplane arrangement in $\mathbb{R}^n$ is defined to be the hyperplane arrangement in $\mathbb{C}^n$ which is defined by the same linear forms.
We omit the index $\mathbb{C}$ and denote by $\mathcal{A}_{n-1}$ and $\mathcal{B}_n$ the complexifications of the real arrangements $\mathcal{A}_{n-1}^{\mathbb{R}}$ and $\mathcal{B}_n^{\mathbb{R}}$, respectively. The notation is chosen according to the respective reflection groups of type A_{n-1} and B_n.
For an arrangement of hyperplanes $\mathcal{A}$ we denote by $M(\mathcal{A})$ the complement of the union of all hyperplanes of $\mathcal{A}$. The complements $M(\mathcal{A}_{n-1})$ and $M(\mathcal{B}_n)$ of the complexifications of the two arrangements above are the objects of our study.
The topology of such complements have been the subject of studies since the early 1970's. The development started in 1972, when P. Deligne proved that the complement of a complexified arrangement is $K(\pi, 1)$ when the chambers of the subdivision of $\mathbb{R}^n$ induced by the hyperplanes are simplicial cones [7].

With regard to this thesis one result of M. Salvetti from 1987 is of great importance. He proved that the complement of a complexified real hyperplane arrangement is homotopy equivalent to a regular CW-complex [18].
Since the groups $H^i(X^i, X^{i-1})$ of the cellular cochain complex of a CW-complex X are free abelian with basis in one-to-one correspondence with the i-cells of X, we call a CW-complex *minimal* if its number of cells of dimension i equals the rank of the cohomology group $H^i(X, \mathbb{Q})$.
Taking the regular CW-complexes, which are based on Salvetti's work, as a starting point, we derive minimal CW-complexes $\Gamma_{\mathcal{A}_{n-1}}$ and $\Gamma_{\mathcal{B}_n}$ for the complements $M(\mathcal{A}_{n-1}) \subset \mathbb{C}^n$ and $M(\mathcal{B}_n) \subset \mathbb{C}^n$ of the complexifications of the two arrangements above. Hence, we deduce CW-complexes which are homotopy equivalent to $M(\mathcal{A}_{n-1})$ or $M(\mathcal{B}_n)$ and which have a minimal number of cells.
In order to decrease the number of cells, discrete Morse Theory provides our basis tool. It was developed by R. Forman in the late 1990's. Discrete Morse Theory allows to decimate the number of cells of a regular CW-complex without changing its homotopy type.
Parallel to our work, a general approach to finding a CW-complex homotopic to the complement of an arrangement using discrete Morse theory was developed in [19]. Our approach is different for the cases studied and leads to a much more explicit description than the statement in [19].

It is well known that the rank of the cohomology groups $H^i(M(\mathcal{A}_{n-1}), \mathbb{Q})$ and $H^i(M(\mathcal{B}_n), \mathbb{Q})$ of the complements $M(\mathcal{A}_{n-1})$ and $M(\mathcal{B}_n)$ equals the number of elements of length i in the underlying reflection groups S_n and S_n^B, respectively [1]. Here, S_n is the symmetric group and S_n^B is the group of signed permutations, consisting of all bijections ω of the set $[\pm n] := \{1, \ldots, n, -n, \ldots, -1\}$ onto itself, such that $\omega(-a) = -\omega(a)$ for all $a \in [\pm n]$. Indeed, the numbers of cells of the minimal complexes $\Gamma_{\mathcal{A}_{n-1}}$ and $\Gamma_{\mathcal{B}_n}$ are equal to the numbers of elements in S_n and S_n^B, respectively.
The *cell-order* of a CW-complex X is defined to be the order relation on the cells of X with $\sigma \leq \tau$ for two cells σ, τ of X if and only if the closure of σ is contained in the closure of τ. The poset of all cells of X ordered in this way is called the *face poset* of X.
A main part of this thesis is devoted to the cell-orders of the minimal CW-complexes. In case of the complex $\Gamma_{\mathcal{A}_{n-1}}$ the face poset turns out to have a concise description.
The combinatorics of the face poset of $\Gamma_{\mathcal{B}_n}$ seems to be too complicated to be described through a concise and explicit rule. Thus we formulate a description in terms of mechanisms which allow to construct the cells $\mathfrak{B}$ with $\mathfrak{A} < \mathfrak{B}$ from a given cell $\mathfrak{A}$. Even though this description is relatively compact, there

is still a lot of combinatorics included that has yet to be discovered.

This thesis is organized as follows:

In Section 2 we provide the mathematical background. We start Section 2 by introducing the real reflection groups A_{n-1} and B_n. Afterwards we briefly present the main definitions concerning CW-complexes. For an in-depth overview of the theory of CW-complexes we refer to [13]. After a brief introduction to hyperplane arrangements, we give a short summary of the construction of Salvetti's complex, which is based on the work of Björner and Ziegler [4].

Section 3 is an introduction to discrete Morse Theory. After a presentation of Forman's approach we give a reformulation of the theory in terms of acyclic matchings, which for our purpose is more applicable. Indeed, a large part of this thesis is concerned with finding appropriate matchings.

We deduce a minimal CW-complex for $M(\mathcal{A}_{n-1})$ in Section 4. For this we define a representation of the cells of the initial complex in terms of certain partitions of $[n] := \{1, \ldots, n\}$ and adapt the original cell-order to the new representations. Afterwards the number of cells is decreased by defining an appropriate matching and applying the methods of discrete Morse Theory to the initial complex. The resulting minimal complex $\Gamma_{\mathcal{A}_{n-1}}$ has as many cells as elements of the symmetric group S_n.
At the end of Section 4 we examine the cell-order of $\Gamma_{\mathcal{A}_{n-1}}$ and present a description. Finally we present the face poset of the minimal CW-complexes $\Gamma_{\mathcal{A}_2}$ and $\Gamma_{\mathcal{A}_3}$.

In Section 5 we construct a minimal CW-complex for $M(\mathcal{B}_n)$. Compared to the $\mathcal{A}_{n-1}$-case, this requires much more effort. We define a representation of the cells of the initial complex in terms of symmetric partitions of $[\pm n] := \{1, \ldots, n, -n, \ldots, -1\}$ and adapt the original cell-order to the new representations. Afterwards, we apply the methods provided by discrete Morse Theory twice, in order to decimate the number of cells. Hence, we define two matchings and we prove that after the removal of the cells of the first matching, the methods are still applicable to the second matching.
The minimal CW-complex $\Gamma_{\mathcal{B}_n}$ has as many cells as elements of the group of signed permutations S_n^B. The remainder of Section 5 is needed to specify a description of the cell-order of $\Gamma_{\mathcal{B}_n}$. We give a counterexample showing that, in contrast to the complex $\Gamma_{\mathcal{A}_{n-1}}$, the relations $\mathfrak{A} < \mathfrak{B}$ of cells of $\Gamma_{\mathcal{B}_n}$

in general have no representation as a chain of facets

$$\mathfrak{A} \lessdot \mathfrak{A}_1 \lessdot \cdots\cdots \lessdot \mathfrak{A}_m \lessdot \mathfrak{B} \quad .$$

Due to the complexity we derive a description of the cell-order in terms of mechanisms, which can be applied to the partition corresponding to a cell. Therefore, a main part of the section is concerned with the translation of the structure of the face poset of $\Gamma_{\mathcal{B}_n}$ into mechanisms.

In Section 6 we discuss the relations $\mathfrak{A} \lessdot \mathfrak{B}$, i.e. $\mathfrak{A} < \mathfrak{B}$ and $\dim(\mathfrak{B}) = \dim(\mathfrak{A}) + 1$, in detail. We present a description of all cells $\mathfrak{B}$, such that $\mathfrak{A} \lessdot \mathfrak{B}$. This description is given in terms of algorithms which can be applied to the partition corresponding to a cell of $\Gamma_{\mathcal{B}_n}$ and allow to determine the cells $\mathfrak{B}$ with $\mathfrak{A} \lessdot \mathfrak{B}$ from $\mathfrak{A}$ effectively. It provides an insight to the structural details of $\Gamma_{\mathcal{B}_n}$ but also to its complexity. We present some examples at the end of Section 6 which illustrate that compared to their complicated formulation, these algorithms are easily applicable.

Section 7 is a translation of this overview into German.

2 Preliminaries

2.1 The reflection groups A_{n-1} and B_n

As a start of this thesis we provide a short description of the real reflection groups A_{n-1} and B_n. Since we do not need the details of the theory of finite reflection groups, we briefly list the facts concerning these two special cases. For a deeper insight into the theory we refer to [11] and [3].

Let V be a finite dimensional real vector space endowed with a positive definite symmetric bilinear-form $\langle \cdot, \cdot \rangle$. A *reflection* in V is a linear function s_α which sends some nonzero vector α to its negative while fixing pointwise the hyperplane H_α which is orthogonal to α. It is easy to see that s_α can be written as follows:

$$s_\alpha(\gamma) = \gamma - \frac{2\,\langle \gamma, \alpha \rangle}{\langle \alpha, \alpha \rangle}\,\alpha\ .$$

The bilinearity of $\langle \cdot, \cdot \rangle$ implies that s_α is an orthogonal transformation, i.e. $\langle s_\alpha(\gamma), s_\alpha(\mu) \rangle = \langle \gamma, \mu \rangle$.
A *finite real reflection group* is a finite subgroup of $O(V)$ generated by reflections.

Example 2.1. $(A_{n-1}, n \geq 2)$: This reflection group is the symmetric group S_n, i.e. the group of all permutations of $[n] := \{1, \ldots, n\}$. It can be thought of as a subgroup of $O(\mathbb{R}^n)$ by assigning to each transposition $(ij) \in S_n$ the reflection $s_{e_j - e_i}$. Then S_n acts on $\mathbb{R}^n$ by permuting the basis vectors $e_1, \ldots, e_n$. Since S_n is generated by transpositions, it is a reflection group.
The set of fixed points of the action of S_n on $\mathbb{R}^n$ equals the line which is spanned by $e_1 + \cdots + e_n$. Furthermore it leaves stable the orthogonal complement which consists of the points with coordinates summing up to 0. Thus S_n also acts on an $(n-1)$-dimensional vector space. This accounts for the subscript $n-1$.

Example 2.2. $(B_n, n \geq 2)$: Let the symmetric group S_n act on $\mathbb{R}^n$ as above. Define additional reflections s_{e_i} for $i = 1, \ldots, n$ and $s_{e_i + e_j}$ for $1 \leq i < j \leq n$. These reflections together with the reflections $s_{e_j - e_i}$ generate the reflection group B_n. It can be considered as the group of signed permutations S_n^B which is the group of all bijections ω of the set $[\pm n] := \{-n, \ldots, -1, 1, \ldots, n\}$, such that $\omega(-a) = -\omega(a)$ for all $a \in [\pm n]$. The number of elements of S_n^B equals $2^n n!$.

2.2 CW-complexes

In this section we provide a short outline of the main definitions and some important facts concerning CW-complexes. For more details see [13].

Let B^n be the unit ball in $\mathbb{R}^n$.

Definition 2.3 (CW-complex)**.** A *CW-complex* is a space X constructed in the following way:

(1) Start with a set X^0, equipped with the discrete topology, whose points are regarded as 0-cells of X.

(2) Inductively, form the n-skeleton X^n from X^{n-1} by attaching n-cells σ^n_α via continuous maps $\varphi_\alpha : S^{n-1} \to X^{n-1}$. This means that X^n is the quotient space of the disjoint union $X^{n-1} \biguplus_\alpha B^n_\alpha$ of X^{n-1} with a collection of n-balls B^n_α under the identifications $x \sim \varphi_\alpha(x)$ for $x \in \partial B^n_\alpha$. Thus, as a set, $X^n = X^{n-1} \biguplus_\alpha \sigma^n_\alpha$ where each σ^n_α is an open n-ball.

(3) One can either stop this inductive process at a finite stage, setting $X = X^n$ for some $n < \infty$, or one can continue indefinitely, setting $X = \bigcup_n X^n$. In the latter case X is given the weak topology: A set $A \subset X$ is open (or closed) if and only if $A \cap X^n$ is open (or closed) in X^n for each n.

All CW-complexes in this paper are finite.
A CW-complex for which all the attaching maps φ_α are homeomorphisms is called a *regular CW-complex*.

Example 2.4. (compare Figure 1, page 8) The n-ball B^n has a CW-structure with just three cells σ^0, σ^{n-1} and σ^n where the $(n-1)$-cell is attached by the constant map $S^{n-2} \to \sigma^0$ obtaining an $(n-1)$-sphere. The cell σ^n is then attached by the identity map sending an element $x \in \partial B^n = S^{n-1}$ to itself.

Each cell σ^n_α in a cell complex X has a *characteristic map* $\Phi_\alpha : D^n_\alpha \to X$ which extends the attaching map φ_α and is a homeomorphism from the interior of B^n_α onto σ^n_α. One can take Φ_α to be the composition $B^n_\alpha \hookrightarrow X^{n-1} \biguplus_\alpha B^n_\alpha \to X^n \hookrightarrow X$ where the middle map is the quotient map defining X^n. A *subcomplex* of a cell complex X is a closed subspace $A \subset X$ that is a union of cells of X. Since A is closed, the image of the characteristic map of each cell in A is contained in A. In particular the image of the attaching map of each cell in A is contained in A. Thus A is itself a cell complex.

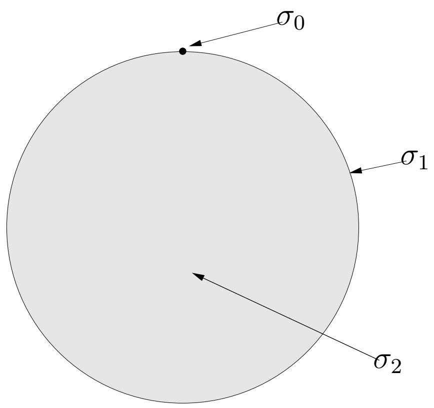


Figure 1: CW-decomposition of the 2-ball B^2

There is a natural way of defining a partial order on a CW-complex X: For two cells $\sigma_\alpha, \sigma_\beta$ we set $\sigma_\alpha \leq \sigma_\beta$ if and only if the closed cell $\overline{\sigma}_\alpha$ is a subset of the closed cell $\overline{\sigma}_\beta$. In this case we say that σ_α is a *face* of σ_β. We say that σ_α is a *facet* of σ_β if $\sigma_\alpha \neq \sigma_\beta$ and for each cell σ_γ of X the inclusion $\sigma_\alpha \leq \sigma_\gamma \leq \sigma_\beta$ implies $\sigma_\gamma \in \{\sigma_\alpha, \sigma_\beta\}$.
Conversely we say that σ_β is an upper neighbor of σ_α if $\sigma_\alpha < \sigma_\beta$ and if there is no cell strictly between them.
For a pair of cells $\sigma_\alpha < \sigma_\beta$ we denote the fact that $\dim(\sigma_\beta) = \dim(\sigma_\alpha) + 1$ by $\sigma_\alpha \lessdot \sigma_\beta$.
The poset of the cells of X ordered by $\leq$ is called the *face poset* of X.

2.2.1 Cellular Homology

Cellular Homology provides the main tool for computing singular homology groups of a CW-complex.

Let X be a CW-complex and let X^n denote the n-skeleton of X.

Lemma 2.5. *If X is a CW-complex, then:*

(a) $H_k(X^n, X^{n-1})$ *is zero for $k \neq n$ and is free abelian for $k = n$, with basis in one-to-one correspondence with the n-cells of X.*

(b) $H_k(X^n) = 0$ *for $k > n$. In particular, if X is finite dimensional then $H_k(X) = 0$ for $k > dimX$.*

(c) *The inclusion $i : X^n \hookrightarrow X$ induces an isomorphism $i_* : H_k(X^n) \to H_k(X)$ if $k < n$.*

We can now arrange pieces of the long exact sequences of the pairs (X^{n+1}, X^n), (X^n, X^{n-1}) and (X^{n-1}, X^{n-2}) into the following diagram:

$$\begin{tikzcd}
 & & & & 0 & & \\
 & 0 \arrow[dr] & & H_n(X^{n+1}) \simeq H_n(x) \arrow[ur] & & & \\
 & & H_n(X^n) \arrow[ur] \arrow[dr, "j_n^*"] & & & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_{n+1}^*"] \arrow[rr, "d_{n+1}"] & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n^*"'] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \cdots \\
 & & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}^*"'] & & \\
 & & & 0 \arrow[ur] & & &
\end{tikzcd}$$

where we define d_n to be the composition $j_{n-1}^* \partial_n^*$. Clearly $d_n d_{n+1}$ is zero since this composition includes the maps ∂_{n+1}^* and ∂_n^*. Therefore, the horizontal row in the diagram is a chain complex, called the *cellular chain complex* of X which is denoted by $C_*^{CW}(X)$. The homology groups of the cellular chain complex are called *cellular homology groups* of X. We denote them $H_n^{CW}(X)$.

Theorem 2.6. $H_n^{CW}(X) = H_n(X)$

An easy consequence of this is that $H_n(X) = 0$ if X is a CW-complex with no n-cells.

By dualization of the cellular chain complex one obtains the *cellular cochain complex* which defines the *cellular cohomology*.

2.3 Hyperplane Arrangements

Definition 2.7. Let $\mathbb{K}$ be a field and let $V_{\mathbb{K}}$ be a vector space of dimension n over $\mathbb{K}$. A *hyperplane* H in $V_{\mathbb{K}}$ is an affine subspace of dimension $(n-1)$. A *hyperplane arrangement* $\mathcal{A} = (\mathcal{A}_{\mathbb{K}}, V_{\mathbb{K}})$ over $V := V_{\mathbb{K}}$ is a finite set of hyperplanes in $V_{\mathbb{K}}$.

Each hyperplane H is the kernel of a linear function α_H which is unique up to a constant. In the following we refer to an arrangement of hyperplanes simply as an *arrangement*.

Definition 2.8. For α_H, $H \in A$ a linear function which defines H the product $Q(A) := \prod\limits_{H \in A} \alpha_H$ is called a *defining polynomial* of A.

Definition 2.9. The *variety* of an arrangement A over V is defined by $N(A) := \bigcup\limits_{H \in A} H = \{v \in V \mid Q(A)(v) = 0\}$.
The *complement* of an arrangement A over V is defined by $M(A) := V \backslash N(A)$.

Definition 2.10. Let $A := \{H_1, \ldots, H_l\}$ be an arrangement in $\mathbb{R}^n$, given by linear forms $\alpha_{H_1}, \ldots, \alpha_{H_l}$. The *complexified arrangement* $A^{\mathbb{C}} := \{H_1^{\mathbb{C}}, \ldots, H_l^{\mathbb{C}}\}$ in $\mathbb{C}^n$ is given by the same forms, i.e. $H_i^{\mathbb{C}} = \ker_{\mathbb{C}^n}(\alpha_{H_i})$.

A *reflection arrangement* in $\mathbb{R}^n$ is an arrangement consisting of hyperplanes which are the reflecting hyperplanes of a finite real reflection group, i.e. a group which is generated by reflections. In this work we study the structure of complexifications of classical reflection arrangements. In particular we investigate the following arrangements:

- $\mathcal{A}_{n-1} := (\mathcal{A}_{\mathbb{C}}, \mathbb{C}^n)$ is the arrangement consisting of the hyperplanes $H_{i,j} := \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i = z_j\}$, for $1 \le i < j \le n$.
  Due to its connection to braid theory it is called the *braid arrangement*.

- $\mathcal{B}_n := (\mathcal{B}_{\mathbb{C}}, \mathbb{C}^n)$ is the arrangement which contains all hyperplanes of $\mathcal{A}_{n-1}$ and in addition to the hyperplanes $H_{i,j}$, contains the hyperplanes $H_{i,-j} := \{z = (z_1, \ldots, z_n) \mid z_i = -z_j\}$, for $1 \le i < j \le n$ and the coordinate-hyperplanes $H_i := \{z = (z_1, \ldots, z_n) \mid z_i = 0\}$, for $i = 1, \ldots, n$.

Let $[n] := \{1, \ldots, n\}$ and $[\pm n] := \{-n, \ldots, -1, 1, \ldots, n\}$.
In Section 2.1 we learned that the reflections corresponding to the hyperplanes $H_{i,j}$ are the linear maps transposing the vectors e_i and e_j of the standard basis $e_1, \ldots, e_n$ while fixing the remaining basis vectors. The underlying

reflection group of $\mathcal{A}_{n-1}$, therefore, is the symmetric group S_n.

The reflections corresponding to the hyperplanes $H_{i,-j}$ are the linear maps sending e_i to $-e_j$ and e_j to $-e_i$ while fixing the remaining basis vectors of $e_1, \dots, e_n$. The reflections corresponding to the hyperplanes H_i are the linear maps sending e_i to $-e_i$ and fixing the remaining basis vectors. Hence, the underlying reflection group of $\mathcal{B}_n$ is the group of signed permutations S_n^B consisting of all bijections ω of the set $[\pm n]$ to itself such that $\omega(-a) = -\omega(a)$ for all $a \in [\pm n]$.

Let π be an element of S_n or S_n^B. The *length* of π (denoted $l(\pi)$) is the minimal number r such that π can be written as a product of reflections $\pi = \pi_1 \cdots \pi_r$ with $\pi_i \in S_n$ and $\pi_i \in S_n^B$, respectively, for $i = 1, \dots, r$.

Example 2.11.
Let $\pi = (142) \in S_n$. $l(\pi) = 2$ since $\pi = (14)(12)$.
Let $\pi = (14-2) \in S_n^B$. $l(\pi) = 2$ since $\pi = \pi_1\pi_2$ with $\pi_1 = (14)(-1-4), \pi_2 = (1-2)(-12)$.

The following theorem is due to Arnold [1]. With regard to this thesis it is of crucial importance. It gives a lower bound for the number of cells of a CW-complex being homotopy equivalent to $M(\mathcal{A}_{n-1})$ or $M(\mathcal{B}_n)$.

Theorem 2.12. *The rank of the cohomology groups $H^i(M(\mathcal{A}_{n-1}), \mathbb{Q})$ and $H^i(M(\mathcal{B}_n), \mathbb{Q})$ are equal to the number of permutations of length i in their underlying reflection group.*

Since the coefficients of the Poincaré Polynomial are equal to the Betti numbers of the complements, we can state the following fact:

Corollary 2.13. *The Poincaré Polynomial of $M(\mathcal{A}_n)$ is*

$$Poin(M(\mathcal{A}_n)) = (1+t)(1+2t)\cdots(1+(n-1)t) \ .$$

The Poincaré Polynomial of $M(\mathcal{B}_n)$ is

$$Poin(M(\mathcal{B}_n)) = (1+t)(1+3t)\dots(1+(2n-1)t) \ .$$

For details on the topology of the complement of an arrangement see [16]. A computation of $Poin(M(\mathcal{B}_n))$ can be found in [11].

The following fact was first proved by Salvetti [18].

Theorem 2.14. *Let $\mathcal{A}$ be a complexified real hyperplane arrangement in $\mathbb{C}^n$. Then there exists a regular CW complex $\Gamma^{\mathcal{A}}$ that is homotopy equivalent to the complement $M(\mathcal{A}) \subset \mathbb{C}^n$.*

We give a sketch of the construction of $\Gamma^{\mathcal{A}}$ which is based on [4].
With each complex number we associate a *complex sign*

$$s^{(1)}(x+iy) = \begin{cases} i & \text{if } y > 0, \\ j & \text{if } y < 0, \\ + & \text{if } y = 0 \text{ and } x > 0, \\ - & \text{if } y = 0 \text{ and } x < 0, \\ 0 & \text{if } y = x = 0. \end{cases}$$

For an arrangement of hyperplanes $\mathcal{A} = \{H_1, \ldots, H_m\}$ in $\mathbb{C}^n$ and fix linear forms α_x with $H_x = \ker(\alpha_x)$, for $x = 1, \ldots, m$, the position of each point $z \in \mathbb{C}^n$ with respect to the hyperplanes in $\mathcal{A}$ can be encoded by assigning to it the following complex sign vector:

$$s^{(1)}_{\mathcal{A}}(z) := \left(s^{(1)}(\alpha_1(z)), \ldots, s^{(1)}(\alpha_m(z))\right) \in \{i, j, +, -, 0\}^m .$$

$s^{(1)}_{\mathcal{A}}(\mathbb{C}^n)$ can be turned into a poset by ordering the sign vectors componentwise according to the paradigm

$$\begin{tikzcd} i \arrow[d, dash] \arrow[drr, dash] & & j \arrow[d, dash] \arrow[dll, dash] \\ + \arrow[dr, dash] & & - \arrow[dl, dash] \\ & 0 & \end{tikzcd}$$

The points in $\mathbb{C}^n$ having the same sign vectors form relative open convex cones.
If $\mathcal{A} := \{H_1, \ldots, H_m\} \subset \mathbb{C}^n$ is an essential arrangement (i.e. $\bigcap\limits_{i=1}^{m} H_i = \{0\}$), these cones form what is called a *combinatorial stratification* [BZ, Theorem 2.5.]:

Definition 2.15. A *combinatorial stratification* K of a complex arrangement $\mathcal{A}$ in $\mathbb{C}^n$ is a partition of $\mathbb{R}^{2n} \cong \mathbb{C}^n$ into finitely many subsets (called *strata*) which have the following properties:

(i) the strata are relative-open convex cones

(ii) the intersection of the strata with the unit sphere S^{2n-1} in $\mathbb{C}^n$ are the open cells of a regular CW-decomposition Γ_K of S^{2n-1}

(iii) every hyperplane $H \in \mathcal{A}$ is a union of strata, that is, every $H \cap S^{2n-1}$ is a subcomplex of Γ_K.

The intersections of these cones with the unit sphere S^{2n-1} induce a regular cell decomposition of S^{2n-1} with face poset $K_{\mathcal{A}}^{(1)}$.
This face poset is, after adding a minimal element $\hat{0}$, isomorphic to the set of closures of strata, ordered by containment. From ([4], Theorem 2.5.) it follows that the augmented face poset is isomorphic to the poset of sign vectors $s_{\mathcal{A}}^{(1)}(\mathbb{C}^n)$ corresponding to $\mathcal{A}$.
Let $K_{comp}^{(1)}$ be the poset consisting of all sign vectors which have no 0-component. From standard homotopy arguments it follows that the opposite poset K_{comp}^{op} of $K_{comp}^{(1)}$ yields a face poset of a regular CW-complex $\Gamma^{\mathcal{A}}$, having the homotopy type of the complement $M(\mathcal{A})$, (see [4], Theorem 3.5.). Here, the opposite poset P^{op} is obtained from a poset P by reversing its order-relation. Therefore K_{comp}^{op} can be obtained by reversing the order of $K_{comp}^{(1)}$. In fact $\Gamma^{\mathcal{A}}$ is a subcomplex of the CW-complex dual to the CW-complex which belongs to the face poset $K_{\mathcal{A}}^{(1)}$.

3 Discrete Morse Theory

In this section we give an introduction to discrete Morse Theory which can be seen as a combinatorial version of Morse Theory. It provides the basis tool for this thesis. It allows starting from a regular CW-complex to construct a homotopy equivalent CW-complex with fewer cells.

Definition 3.1. Let X be a CW-complex, $\sigma^{n-1}, \tau^n \in X^{(*)}$ cells in X of dimension $n-1$ resp. n such that σ^{n-1} is a face of τ^n.

Let

$$\Phi_\sigma : B^{n-1} \to X \text{ and } \Phi_\tau : B^n \to X$$

be the characteristic maps of σ^{n-1} and τ^n.
We say that σ^{n-1} is a *regular face* of τ^n if:

(i) $\Phi_\tau : \Phi_\tau^{-1}(\sigma^{n-1}) \to \sigma^{n-1}$ is a homeomorphism

(ii) $\overline{\Phi_\tau^{-1}(\sigma^{n-1})}$ is a closed $(n-1)$-ball.

Clearly if X is regular it follows that for all cover relations $\sigma < \tau$ with σ a facet of τ of cells of X, σ a regular facet of τ.

Next we define the notion of a discrete Morse function. The existence of such a function guarantees that we can decimate the number of cells of a given regular CW-complex without changing its homotopy type.

Definition 3.2. (discrete Morse function)
A *discrete Morse function* on the CW-complex X is a function

$$f : X^{(*)} \to \mathbb{R}$$

such that

1. for every cell $\sigma \in X^{(*)}$

$$|\{\tau \in X^{(*)} \mid \tau \text{ is a facet of } \sigma,\ f(\tau) \geq f(\sigma)\}| \leq 1$$

$$|\{\tau \in X^{(*)} \mid \sigma \text{ is a facet of } \tau,\ f(\tau) \leq f(\sigma)\}| \leq 1$$

   and

2. if $\sigma,\ \tau \in X^{(*)}$ are cells in X with σ a facet of τ and such that
$$f(\sigma) \geq f(\tau)\ ,$$
then σ is a regular facet of τ.

Morse functions can be regarded as functions that increase with dimension up to one exception locally.

Definition 3.3. For $c \in \mathbb{R}$ we define
$$M(c) = \bigcup_{\substack{\sigma\in X^{(*)}\\ f(\sigma\leq c)}} \bigcup_{\tau\leq\sigma} \tau\ .$$

M(c) is the union of all cells with values of f smaller than c together with all their faces.
We say σ is a *free face* of τ if $\sigma^{(p)} < \tau^{(p+1)}$ are two cells of a CW-complex X and

(i) σ is a regular face of τ.

(ii) σ is not a face of any other cell.

If σ is a free face of τ we can simply deform X by pulling σ along τ onto its boundary without changing the homotopy type of X. We denote this operation by $X \searrow Y$ where $Y := X - (\sigma \cup \tau)$ and call it an elementary collapse. Precisely this operation defines a deformation retraction.
Moreover, we write $X \searrow Y$ for two CW-complexes X and Y if X *collapses onto* Y, i.e. X can be transformed into Y by a finite sequence of elementary collapses.

Definition 3.4. (critical cells)
Let X be a finite CW-complex, f a discrete Morse function on X. A cell σ of X is called f-*critical*, if
$$|\{\tau \in X^{(*)} \mid \tau \text{ is a facet of } \sigma,\ f(\tau) \geq f(\sigma)\}| = 0$$
$$|\{\tau \in X^{(*)} \mid \sigma \text{ is a facet of } \tau,\ f(\tau) \leq f(\sigma)\}| = 0$$
We set
$$X^{(*)}_{crit}(f) := \{\sigma \in X^{(*)} | \sigma \text{ is } f\text{-critical}\}\ .$$
For every dimension i, the number of f-critical cells in dimension i is called the *Morse number* $m_i(f)$.

A main result of Formans work is:

Theorem 3.5. *Let X be a finite CW-complex and f a discrete Morse function on X. Then X is homotopy equivalent to a CW-complex with exactly $m_i(f)$ cells of dimension i.*

A detailed proof can be found in [8].

In the following we give a short outline of the main parts of the proof:
Given a CW-complex X and a discrete Morse function $\tilde{f}$ on $X^{(*)}$ one shows that there exists a discrete Morse function f with the same critical cells such that:

(i) For every pair of cells σ and τ, such that τ is contained in the smallest subcomplex which contains σ, and τ not a face of σ, it follows that $f(\tau) \leq f(\sigma)$.

(ii) If $\tau^{(p+r+1)} > \sigma^p$ and $f(\tau) < f(\sigma)$ for some p and $r \geq 0$, then there exists a cell $\tilde{\tau}^{p+1}$ with $\tilde{\tau} > \sigma$ and $f(\tilde{\tau}) \leq f(\tau)$.

With these two conditions one can proof the following two theorems which yield Theorem 3.5:

Theorem 3.6. *If $a < b$ are real numbers such that $[a, b]$ contains no critical values of f, then $M(b) \searrow M(a)$*

Theorem 3.7. *Let $\sigma^{(p)}$ be the only critical cell of dimension p with $f(\sigma) \in [a, b]$. Then $M(b)$ is homotopy equivalent to $M(a) \bigcup\limits_{\partial e^p} e^p$.*

One can think of this complex as being constructed step by step.
Assume there is a non-critical p-cell σ of X with a $(p-1)$-cell τ being a regular face of σ, such that $f(\tau) =: b > a := f(\sigma)$.
It follows that τ is also contained in $M(a)$ although its value of f is larger than a. By the construction of the Morse function f it follows that τ is a free face of σ in the subcomplex $M(a)$ and therefore σ and τ can be collapsed onto the boundary of σ without changing the homotopy type of $M(a)$. The complete proof follows by induction.

3.1 Matchings

For our purpose we reformulate discrete Morse theory in terms of acyclic matchings. For this we follow the work of Chari [6].

Definition 3.8. Let X be a CW-complex. Consider the directed graph G_X on $X^{(*)}$ whose set E_X of edges is given by

$$E_X := \{\tau \to \sigma \mid \sigma \text{ ist facet von } \tau\} \ .$$

We call $G_X = (X^{(*)}, E_X)$ the *cell graph* of X. Note that reversing the edges of the cell graph yields an illustration of the face-poset of a complex. We use both notions parallel.

Definition 3.9. (acyclic matching) Let X be a CW-complex and $G_X = (X^{(*)}, E_X)$ its cell graph. Let $A \subset E_X$ be a subset of edges $\tau \to \sigma \in E_X$ such that σ is a regular face of τ for all $\tau \to \sigma \in E_X$.

1. We denote by $G_X^A = (X^{(*)}, E_X^A)$ the induced graph with set of edges

$$E_X^A := (E_X \setminus A) \cup \{\sigma \to \tau \mid \tau \to \sigma \in A\}$$

   that is built from G_X by reversing the direction of all edges $\tau \to \sigma$ of A.
   We call an edge $\sigma \to \tau \in E_X^A$ an A-*edge* if for its reversed edge there is $\tau \to \sigma \in A$.

2. We call A a *matching* on X if each cell $\sigma \in X^{(*)}$ occurs in at most one edge of A.

3. We call A an *acyclic matching* on X if A is a matching and if the induced graph G_X^A is *acyclic*, i.e. it contains no directed cycle.

4. A cell of X is called A-*critical* if it does not occur in any edge $\tau \to \sigma \in A$.

5. We set

$$X^{(*)}_{\text{crit}}(A) := \{\sigma \in X^{(*)} \mid \sigma \text{ is } A\text{-critical}\} \ .$$

6. We denote by $\tilde{G}_X^A = (X^{(*)}, \tilde{E}_X^A)$ the induced graph with edge set

$$\tilde{E}_X^A := E_X \cup \{\sigma \to \tau \mid \tau \to \sigma \in A\}$$

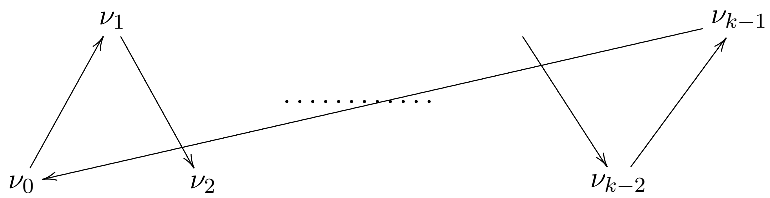
   that is built from G_X by adding for each edge $\tau \to \sigma \in A$ its reversed edge $\sigma \to \tau$.

Lemma 3.10. *Let X be a CW-complex, A an acyclic matching on X and $A' \subset A$. Then A' is an acyclic matching on X.*

Proof. Suppose

$$\gamma = \nu_0 \to \cdots \to \nu_k,\ \nu_k = \nu_0$$

is a cycle in A'. Since A' is a matching there cannot be two consecutive edges $\nu_i \to \nu_{i+1}$ and $\nu_{i+1} \to \nu_{i+2}$ contained in A'. Without loss of generality it follows that γ has the form



with edges pointing downwards contained in A'. The edges pointing upwards cannot be contained in A since A is a matching. Thus, γ must be also a cycle in G_X^A which is a contradiction, and therefore, A' is acyclic. $\square$

In the following we want to explain the connection between acyclic matchings and discrete Morse functions.

Definition 3.11. Let X be a CW-complex, $G_X = (X^{(*)}, E_X)$ its cell graph and $f : X^{(*)} \to \mathbb{R}$ a discrete Morse function on X. We set

$$A_f := \{\tau \to \sigma \in E_X \mid \sigma \text{ is a face of } \tau, f(\tau) \leq \sigma\} .$$

We call A_f the acyclic matching on X corresponding to f.

Indeed A_f defines the right acyclic matching:

Lemma 3.12. *Let X be a CW-complex, $f : X^{(*)} \to \mathbb{R}$ a discrete Morse function on X. Then*

1. *A_f is an acyclic matching.*
2. *$X^{(*)}_{crit}(A_f) = X^{(*)}_{crit}(f)$.*

A proof can be found in [8].

Lemma 3.13. *Let X be a CW-complex, A an acyclic matching on X. Then the only cycles in the graph $\tilde{G}_X^A$ consist of exactly one edge $e \in A$ and its reversed edge.*

Proof. If γ is a cycle in $\tilde{G}_X^A$ it must contain one edge $e \in A$ and its reversed edge. If not, γ would also be a cycle in $\tilde{G}_X^{A'}$ for some $A' \subset A$. This is a contradiction to Lemma 3.10.
The two edges divide γ into two cycles. Thus, γ consists only of pairs of oppositely directed edges. Since A is a matching, γ must in fact consist of only one such pair. $\square$

Definition 3.14. (**matching poset** $(A^{(*)}, \prec_A)$) Let X be a CW-complex, A an acyclic matching on X.

1. Let
$$A^{(*)} := A \cup X_{crit}^{(*)}(A) \ .$$
For $a, b \in A^{(*)}$ we set $a \prec_A b \Leftrightarrow$ there exists a path in $\tilde{G}_X^A$ from b to a. And if $a = \tau \to \sigma \in A$ the path in $\tilde{G}_X^A$ can go to either σ or τ. Conversely if $b = \tau \to \sigma \in A$ the path can start either form σ or τ. We call the partially ordered set $(A^{(*)}, \prec_A)$ the *matching poset* of A. It follows from the last Lemma that it is well defined.

2. We call the function
$$\begin{aligned} gr_A : X^{(*)} &\to A^{(*)} \\ \sigma &\mapsto \begin{cases} \sigma & \text{if } \sigma \in X_{crit}^{(*)} \\ \tau' \to \tau & \text{if } \sigma \in \{\tau, \tau'\} \text{ and } \tau' \to \tau \in A \end{cases} \end{aligned}$$
the *A-universal-grading* of X.

Lemma 3.15. *Let X be a CW-complex and A an acyclic matching on X. Then there exists a discrete Morse function on X such that*
$$A = A_f \ .$$

Proof. For a linear extension $\prec$ of $\prec_A$ let
$$f^{(*)} : A^{(*)} \to \mathbb{R}$$
be strictly order preserving, i.e for $a, b \in A^{(*)}$ we have
$$a \prec b \Rightarrow f^{(*)}(a) < f^{(*)}(b) \ .$$
We define a discrete Morse function $f : X^{(*)} \to \mathbb{R}$ by
$$f(\sigma) = \begin{cases} f^{(*)}(a) & \text{if } \sigma = a \in X_{crit}^{(*)} \\ f^{(*)}(a) & \text{if } a = \tau \to \nu \in A \text{ and } \sigma \in \{\tau, \nu\} \ . \end{cases}$$
In fact f defines a discrete Morse function on X with $A_f = A$. $\square$

We can now reformulate Theorem 3.5:

Theorem 3.16. *Let X be a finite CW-complex, A an acyclic matching on X. Then there is a CW-complex X_A whose i-cells are in one-to-one correspondence with the A-critical i-cells of X such that X_A is homotopy equivalent to X.*

Thus in practice, we delete all cells and edges corresponding to an acyclic matching A of the cell graph of X. The resulting complex consisting of the remaining cells of X (i.e. the critical cells) is equipped with new edges which are inherited from the paths in $\tilde{G}_X^A$.

4 $\mathcal{A}_{n-1}$

In this section we construct a CW-complex $\Gamma_{\mathcal{A}_{n-1}}$ that is homotopy equivalent to the complement $M(\mathcal{A}_{n-1})$ of the complexified braid arrangement in $\mathbb{C}^n$ and which has a minimal number of cells. The number of cells will be equal to the number of elements in S_n.
Let $\Gamma^{\mathcal{A}_{n-1}}$ denote the initial complex which we have introduced in Section 2.3 and which is homotopy equivalent to $M(\mathcal{A}_{n-1})$. At the beginning we construct a representation of this complex, which uses fully ordered set-partitions as representatives of the cells. The information carried by the different signs of a sign vector, as it was used for the original description of $\Gamma^{\mathcal{A}_{n-1}}$, is then encoded in the order of the blocks of the partition.

4.1 Structure

For a set Ω, an ordered set-partition of Ω is a k-tuple $\mathfrak{C} = (C_1, \dots, C_k)$ of sets C_i such that

(i) $C_i \neq \emptyset$, for $1 \leq i \leq k$

(ii) $C_i \cap C_j = \emptyset$, for $1 \leq i < j \leq k$

(iii) $C_1 \cup \dots \cup C_k = \Omega$.

A fully ordered set-partition is a pair $(\mathfrak{C}, \prec_{\mathfrak{C}})$ where $\mathfrak{C} = (C_1, \dots, C_k)$ is an ordered set-partition and $\prec_{\mathfrak{C}} = (\prec_1, \dots, \prec_k)$ is a k-tuple of linear orders $\prec_i$ on C_i for $1 \leq i \leq k$.

For the sake of simplicity we sometimes present a set C_i of the partition $\mathfrak{C}$ as $C_i = (c_1, \dots, c_n)$ and thereby we set $c_k \succ_i c_l$ for $k < l$.
For illustration of fully ordered set-partitions we use the *reduced notation*. The reduced notation of the fully ordered set-partition $\mathfrak{C} = (C_1, \dots, C_m)$ with $C_i := \{c^i_1, \dots, c^i_{n_i}\}$ and $c^i_1 \succ_i c^i_2 \succ_i \dots \succ_i c^i_{n_i}$ (i.e. $C_i = (c^i_1, \dots, c^i_{n_i})$) is

$$|c^1_1 \dots c^1_{n_1}|c^2_1 \dots c^2_{n,2}| \dots \dots |c^n_1 \dots c^n_{m_n}| \quad .$$

A *part* $(c_j, c_{j+1}, \dots, c_{j+r})$ of a set $|c_1 \dots c_n|$ is a subset of consecutive elements $c_j, c_{j+1}, \dots, c_{j+r}$ with $1 \leq j \leq j + r \leq n$. In order to simplify notations we consider a part as the sequence of its elements too. In this case we write $c_j c_{j+1} \dots c_{j+r}$. Thus we use both terms parallel depending on what is adequate in the special situation. A *left part* of the set $|c_1 \dots c_n|$ is a part

$(c_1, c_2, \dots, c_j)$ or just $c_1c_2\dots c_j$, for $1 \leq j \leq n$. A *right part* is defined analogous.
When there is no danger of confusion we denote a fully ordered set-partition simply by $\mathfrak{C}$. We also omit the index $\mathfrak{C}$ of $\prec_{\mathfrak{C}}$ and just write $\prec$ when it is clear from the context that $\prec = \prec_{\mathfrak{C}}$.

A doubly ordered set-partition of $[n] := \{1, \dots, n\}$ is an r-tuple $\mathfrak{p} = (\mathfrak{C}_1, \dots, \mathfrak{C}_r)$ of fully ordered set-partitions $\mathfrak{C}_i$, such that

(i) $\mathfrak{C}_i$ is a fully ordered set-partition of the set $C_i \subseteq [n]$.

(ii) $(C_1, \dots, C_r)$ is an ordered set-partition of $[n]$.

To each doubly ordered set-partition $\mathfrak{p} = (\mathfrak{C}_1, \dots, \mathfrak{C}_r)$ we associate a set of points in $\mathbb{C}^n$. An n-tuple $(z_1, \dots, z_n) \in \mathbb{C}^n$ is associated to the doubly ordered set-partition $\mathfrak{p}$ if and only if

(i) If $n \in C_i$ and $m \in C_j$ for some $1 \leq i < j \leq r$, then $\mathrm{Im}(z_n) < \mathrm{Im}(z_m)$

(ii) If $n, m \in C_i$ for some $1 \leq i \leq r$ and $\mathfrak{C}_i = (C_{i,1}, \dots, C_{i,k_i})$, then

  (a) $\mathrm{Im}(z_n) = \mathrm{Im}(z_m)$
  (b) If $n \in C_{i,s}$, $m \in C_{i,t}$ for $1 \leq s < t \leq k_i$, then $\mathrm{Re}(z_n) < \mathrm{Re}(z_m)$
  (c) If $n, m \in C_{i,s}$ for some $1 \leq s \leq k_i$, then $\mathrm{Re}(z_n) = \mathrm{Re}(z_m)$.

The union of all sets that are associated to doubly ordered set-partitions $\mathfrak{p} = (\mathfrak{C}_1, \dots, \mathfrak{C}_r)$ of $[n]$, such that each $\mathfrak{C}_i$ is an ordered set-partition of a set C_i into singletons is the set-theoretic complement $M(\mathcal{A}_{n-1})$ of the complexified braid arrangement, i.e. of the union of all hyperplanes $H_{i,j} = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i = z_j\}$.
Indeed these sets belong to the stratification which equals the one described by Björner and Ziegler in [4] corresponding to the braid arrangement. Precisely, these sets are the strata corresponding to the face poset $K^{(1)}_{\mathcal{A}_{n-1}}$ which are not contained in any of the hyperplanes. Therefore, we can identify each doubly ordered set-partition $\mathfrak{p} = (\mathfrak{C}_1, \dots, \mathfrak{C}_r)$ of $[n]$, such that each $\mathfrak{C}_i$ is an ordered set-partition of a set C_i into singletons, with its corresponding cell $c_{\mathfrak{p}}$ in the cell-complex corresponding to the face poset $K^{(1)}_{\mathcal{A}_{n-1}}$, as it was introduced in Section 2.3.
In the dual cell-complex we denote by $d_{\mathfrak{p}}$ the cell dual to the cell $c_{\mathfrak{p}}$. It follows that the cells $d_{\mathfrak{p}}$ of the dual cell complex, for which $\mathfrak{p} = (\mathfrak{C}_1, \dots, \mathfrak{C}_r)$ is a doubly ordered set-partition of $[n]$, such that each $\mathfrak{C}_i$ is an ordered set-partition of a set C_i into singletons, are the cells of the regular subcomplex

$\Gamma^{\mathcal{A}_{n-1}}$, which is homotopy equivalent to $M(\mathcal{A}_{n-1})$.
Clearly, the fully ordered set partitions $(\mathfrak{C}, \prec)$ where $\mathfrak{C}$ is an ordered set-partition into singletons can be identified with the linear order they impose on the ground set. Thus, from now on we identify the doubly ordered set-partitions $\mathfrak{p} = (\mathfrak{C}_1, \ldots, \mathfrak{C}_r)$ of $[n]$, for which each $\mathfrak{C}_i$ is a fully ordered set-partition with singleton blocks with fully ordered set-partitions $(\mathfrak{C}, \prec)$.

We still have to translate the cell-order of $\Gamma^{\mathcal{A}_{n-1}}$ to the set of fully ordered set-partitions of $[n]$.
The following order can be easily deduced from the order of sign vectors, as it was defined in Section 2.3:

For two fully ordered set-partitions $\mathfrak{p} = (\mathfrak{C}, \prec)$ and $\mathfrak{q} = (\mathfrak{C}', \prec')$ we set $\mathfrak{p} \leq \mathfrak{q}$ if and only if:

(O1) If $\mathfrak{C} = (C_1, \ldots, C_k)$ and $\mathfrak{C}' = (C'_1, \ldots, C'_l)$ then for each $1 \leq i \leq k$ there is a $1 \leq j \leq l$, such that $C_i \subseteq C'_j$

(O2) If $i_1 < i_2$ and $C_{i_1} \subset C'_{j_1}$, $C_{i_2} \subset C'_{j_2}$ it follows that $j_1 \leq j_2$

(O3) If $C_i \subseteq C'_j$ for some $1 \leq i \leq k$ and $1 \leq j \leq l$ then the restriction of $\prec'_j$ to C_i coincides with $\prec_i$.

The second condition implies that if $C_{i_1} \subset C'_j$ and $C_{i_2} \subset C'_j$ for $i_1 < i_2$ it follows that $C_l \subset C'_j$, for all $i_1 < l < i_2$.

It follows that for two fully ordered set-partitions $\mathfrak{p} = (\mathfrak{C}, \prec)$ and $\mathfrak{q} = (\mathfrak{C}', \prec')$ of $[n]$, the cell $d_{\mathfrak{p}}$ lies in the boundary of $d_{\mathfrak{q}}$ if and only if $\mathfrak{p} \leq \mathfrak{q}$ as fully ordered set-partitions. In particular, the cell $d_{\mathfrak{p}}$ is of codimension 1 in the boundary of $c_{\mathfrak{p}}$ if and only if there are exactly two blocks of $\mathfrak{C}$ that are merged in $\mathfrak{C}'$.

Via this order preserving bijection from the set of cells $d_{\mathfrak{p}}$ to the poset of fully ordered set-partitions we from now on identify the cell $d_{\mathfrak{p}}$ and the fully ordered set-partition $\mathfrak{p}$.

Example 4.1.

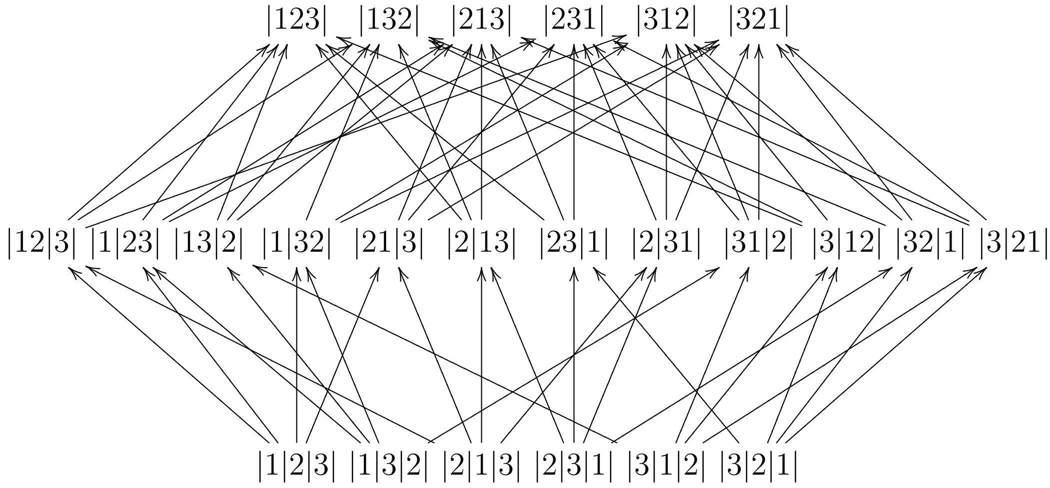


Figure 2: Face poset of the initial complex $\Gamma^{\mathcal{A}_2}$ for $M(\mathcal{A}_2)$

Let $(\mathfrak{C}, \prec)$ with $\mathfrak{C} = (C_1, \ldots . C_k)$ and $\prec = (\prec_1, \ldots, \prec_k)$ be a fully ordered set-partition. We call $\mathfrak{C}$ *standard* if

(i) $\max(C_1) < \max(C_2) < \cdots < \max(C_k)$ and

(ii) $a \preceq_i \max C_i$ for all $a \in C_i$.

Clearly the set of standard fully ordered set-partitions of $[n]$ with k blocks is in bijection with the set of permutations in S_n with k cycles.
We know from Section 2.3 that the rank of the group $H^k(M(\mathcal{A}_{n-1}), \mathbb{Q})$ equals the number of permutations in S_n which can be written minimally as a product of k reflections, i.e. cycles of the form (ij).

4.2 Matching

We want to apply the methods of discrete Morse Theory in order to minimize the initial cell-complex $\Gamma^{\mathcal{A}_{n-1}}$, such that the critical cells are exactly the standard fully ordered set-partitions of $[n]$. Therefore, it follows that the corresponding cell complex has a minimal number of cells in each dimension.

Consider a fully ordered set partition $(\mathfrak{C}, \prec)$ which is not standard. Let $\mathfrak{C} = (C_1, \ldots, C_k)$ and $\prec = (\prec_1, \ldots, \prec_k)$. Since $\mathfrak{C}$ is not standard there must exist a minimal $i \in \{1, \ldots, k\}$, such that either:

(1) $\max(C_i) > \max(C_{i+1})$ and there exists no $a \in C_i$, such that $\max(C_i) \prec_i a$

or

(2) there exists an $a \in C_i$, such that $\max(C_i) \prec_i a$.

We call i the *critical index* of $\mathfrak{C}$ and say that i is of type (1) and type (2), respectively.

Let M be the set of all cover relations $\mathfrak{C} < \mathfrak{C}'$ of fully ordered set-partitions such that:

(i) $\mathfrak{C}$ and $\mathfrak{C}'$ are not standard

(ii) The critical index i of $\mathfrak{C}$ is of type (1)

(iii) If $\mathfrak{C} = (\mathfrak{C}, \prec)$ and $\mathfrak{C}' = (\mathfrak{C}', \prec')$ then

  (a) $C'_j = C_j$ for $1 \leq j \leq i-1$, $C'_i = C_i \cup C_{i+1}$ and $C'_j = C_{j+1}$ for $i+1 \leq j \leq k-1$

  (b) $\prec'_j = \prec_j$ for $1 \leq j \leq i-1$ and $\prec'_j = \prec'_{j+1}$ for $i+1 \leq j \leq k-1$

  (c) The restrictions of $\prec'_i$ to C_i or C_{i+1} coincide with $\prec_i$ and $\prec_{i+1}$ respectively

  (d) $u \succ'_i v$ for all $u \in C_{i+1}$ and $v \in C_i$.

We call $(C_i, \prec_i, C_{i+1}, \prec_{i+1})$ the *type (1) area* of $\mathfrak{C}$, $(C'_i, \prec'_i)$ the *type (2) area* of $\mathfrak{C}'$ and we call $\mathfrak{C}$ of type (1) (or a type (1) cell) and $\mathfrak{C}'$ of type (2) (or a type (2) cell).
We can identify M with the set of all edges of the poset of fully ordered set-partitions ending at an element $\mathfrak{C}'$ of type (2) and starting at the partition $\mathfrak{C}$ which is formed by turning the type (2) area of $\mathfrak{C}'$ into the corresponding type (1) area. In other words M is the set of all edges contained in the face poset of $\Gamma^{\mathcal{A}_{n-1}}$ starting at a type (1) cell and ending at its corresponding type (2) cell.

Example 4.2.

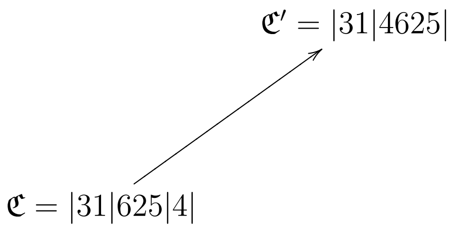
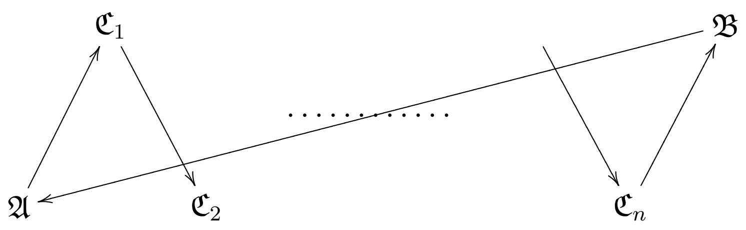


Figure 3: Type (1) cell $\mathfrak{C}$ and corresponding type (2) cell $\mathfrak{C}'$ in $\Gamma^{\mathcal{A}_{n-1}}$.

Lemma 4.3. *M is a matching.*

Proof. If $\mathfrak{C} < \mathfrak{C}'$ is an edge of M then $\mathfrak{C}$ is of type (1). Since any upper element in a cover relation in M must be of type (2) it follows that $\mathfrak{C}$ is not the upper part in a cover relation in M. Since $\mathfrak{C}'$ is of type (2) it cannot be the lower part in a cover relation in M.
On the other hand, if the type (2) area of $\mathfrak{C}'$ is splitted into a type (1) area, then there is a unique way of doing so. Hence, there is no $\overline{\mathfrak{C}} \neq \mathfrak{C}$ for which the edge from $\overline{\mathfrak{C}}$ to $\mathfrak{C}'$ can be in M. Due to uniqueness of the pairs $\mathfrak{C}, \mathfrak{C}'$ with $\mathfrak{C}$ being the type (1) cell corresponding to the type (2) cell $\mathfrak{C}'$, it also follows that there exists no $\overline{\mathfrak{C}}' \neq \mathfrak{C}'$ with $\mathfrak{C} < \overline{\mathfrak{C}}'$. □

In order to show that M is an acyclic matching we consider the graph G_Γ^M which can be obtained from the face poset of $\Gamma^{\mathcal{A}_{n-1}}$ by reversing all edges of M.
Let us consider a directed cycle in G_Γ^M



with $\mathfrak{A}$ of type (1) and $\mathfrak{B}$ of type (2). Hence, the arrows pointing upwards correspond to the order of fully ordered set-partitions or, in other words, the arrows pointing upwards indicate inclusions of the corresponding cells.
We refer to a sequence of edges which starts at a cell $\mathfrak{A}$ and ends at a cell

$\mathfrak{B}$ with $\dim(\mathfrak{B}) = \dim(\mathfrak{A}) + 1$ and which passes only over type (1) and type (2) cells between $\mathfrak{A}$ and $\mathfrak{B}$ as a *type (1) - type (2) sequence*.

Let A and $B_1, \ldots, B_l$ be sets. In the following we use the phrase *"A is merged to $B_1, \ldots, B_l$"* if the elements of A are inserted into these sets.

Consider a partition $\mathfrak{C} := (C_1 \ldots, C_k)$ which is not critical of type (2) together with a $c \in C_i$ for $i \in \{1, \ldots, k\}$. Assume there is a path in G_Γ^M from $\mathfrak{C}$ ending at a partition $\widetilde{\mathfrak{C}} := (\widetilde{C}_1, \ldots, \widetilde{C}_l)$.
The following statement indicates that the corresponding maximum of a set containing c cannot be decreased, i.e. for $j \in \{1, \ldots, l\}$ and $c \in \widetilde{C}_j$ there must be $\max(\widetilde{C}_j) \geq \max(C_i)$.

Lemma 4.4. (Lemma A)
Let $\mathfrak{C} = (C_1, \ldots, C_k)$ be a partition in G_Γ^M.
Let $i \in \{1, \ldots, k\}$ and $C_i := (d_1, \ldots, d_t, \max(C_i), c_1, \ldots, c_r)$. Consider the right part $P := (\max(C_i), c_1, \ldots, c_r)$ of C_i.
There exists no path in G_Γ^M from $\mathfrak{C}$ to a partition $\widetilde{\mathfrak{C}} = (\widetilde{C}_1, \ldots, \widetilde{C}_k)$, such that there exists a $j \in \{1, \ldots, k\}$ with:

(i) $(\widetilde{C}_j \cap P) \neq \emptyset$

(ii) $\max(\widetilde{C}_j) < \max(C_i)$.

Proof. For such a set $\widetilde{C}_j$ it follows that $\max(C_i) \notin \widetilde{C}_j$. Thus the part P must be splitted. Since $\max(C_i) \succ_i a$ for all $a \in P$, P cannot be splitted directly into two parts by a transition to a corresponding type (1) area, in case P is a proper subset of C_i and therefore C_i a type (2) area.
Thus, the only way to split the part P within a path is merging a corresponding set $\overline{C}_i$ which contains P with a set C in order to create a type (2) partition $\mathfrak{D}$, such that the corresponding type (2) area splits in between the elements of P at the transition to the corresponding type (1) partition. Without loss of generality we may assume that $\overline{C}_i = C_i$ and $C = C_x$, for $x = i - 1$ or $x = i + 1$.
In order to split P there must be $\max(C_x) > \max(C_i)$ since otherwise it follows that in the corresponding type (1) cell $\mathfrak{B} := ((B_1, \ldots, B_k), \succ_\mathfrak{B})$, there is P completely contained in B_i for $x = i + 1$, resp. B_{i-1} for $x = i - 1$.
Let $\max(C_x) > \max(C_i)$. Clearly we must position $\max(C_x)$ to the right of $\max(C_i)$, i.e. there is $\max(C_i) \succ_\mathfrak{D} \max(C_x)$, where by $\prec_\mathfrak{D}$ we mean the order of the corresponding set which results from merging C_i and C_x. This results in $\max(C_i)$ and $\max(C_x)$ being the maxima with respect to $\prec_B$ of the sets B_x and B_i, respectively.

It follows that it is not possible to reduce the maxima corresponding to the elements of P. $\square$

Note that compared to Lemma A the index k in the partition $\widetilde{\mathfrak{C}}$ is changed to m in the following lemma.

Lemma 4.5. (Lemma B)
Let $\mathfrak{C} = (C_1, \ldots, C_k)$ a partition of G_Γ^M and $i \in \{1, \ldots, k\}$ with $\max(C_i) \succ_i c$ for all $c \in C_i$.
Then there exists no path in G_Γ^M from $\mathfrak{C}$ to a partition $\widetilde{\mathfrak{C}} = (\widetilde{C}_1, \ldots, \widetilde{C}_m)$, such that there exists an $j \in \{1, \ldots, m\}$ with:

(i) $(\widetilde{C}_j \cap C_i) \neq \emptyset$

(ii) $\max(\widetilde{C}_j) < \max(C_i)$.

Proof. Since merging sets does not reduce maxima, this statement is a direct corollary of Lemma A. $\square$

Lemma 4.6. (Lemma C)
Let $\mathfrak{A} := (A_1, \ldots, A_n)$ be a partition in G_Γ^M with $\max(A_i) > \max(A_{i+1})$.
Then there exists no path in G_Γ^M from $\mathfrak{A}$ to a partition $\mathfrak{B} := (B_1, \ldots, B_m)$, such that

$$\max(A_{i+1}) = \max(B_s) \text{ and } \max(A_i) = \max(B_t), \text{ for } s < t\ .$$

Proof. Within a path from $\mathfrak{A}$ to $\mathfrak{B}$ in G_Γ^M it follows that $\max(A_{i+1})$ has to "pass" $\max(A_i)$. Hence both of them must be inserted into a type (2) area $(C, \prec_C)$ with $\max(A_{i+1}) \prec_C \max(A_i)$.
Since $\max(A_i) > \max(A_{i+1})$, there must be a $c \in C$ with $\max(A_{i+1}) \prec_C c \prec_C \max(A_i)$ such that $c > \max(A_i)$. Otherwise C would not be a type (2) area.
After passing to the corresponding type (1) partition, $\max(A_{i+1})$ would be contained in a set D with $\max(D) \geq c \succ_D d$ for all $d \in D$ and the assertion follows from Lemma B. $\square$

Lemma 4.7. *M is acyclic.*

Proof. Consider the directed cycle

$$\begin{tikzcd} & \mathfrak{C}_1 \arrow[dr] & & \cdots\cdots \arrow[dr] & & \mathfrak{B} \arrow[dlllll] \\ \mathfrak{A} \arrow[ur] & & \mathfrak{C}_2 & & \mathfrak{C}_n \arrow[ur] & \end{tikzcd}$$

Clearly, $\mathfrak{C}_1$ is not of type (1).
Let $\mathfrak{A} = (C_1, \ldots, C_k)$ with $(C_l, \prec_l, C_{l+1}, \prec_{l+1})$ the type (1) area of $\mathfrak{A}$.

Case 1:
There is a type (2) area in $\mathfrak{C}_1 = (C_1^1, \ldots, C_{k-1}^1)$ with critical index smaller then l.
It can be produced only by merging two sets C_i and C_{i+1} with $i < l$ obtaining a new set C_i^1 with $\max(C_{i+1}) \prec_i^1 c$ for a $c \in C_i$. If there is also a $\tilde{c} \in C_i$ with $\tilde{c} \prec_i^1 \max(C_{i+1})$ there is $\tilde{c} \prec_i^2 \max(C_{i+1})$ in the corresponding type (1) area $(C_i^2, \prec_i^2, C_{i+1}^2, \prec_{i+1}^2)$ of $\mathfrak{C}_2 = (C_1^2, \ldots, C_k^2)$. Lemma A implies that we cannot recover C_i.
If $\max(C_{i+1}) \prec_i^1 c$ for all $c \in C_i$, there is $C_i^2 = C_{i+1}$ and $C_{i+1}^2 = C_i$. Therefore we have just reduced the critical index what, indeed, cannot be repeated indefinitely.

Case 2:
The type (1) area of $\mathfrak{A}$ is eliminated by merging either C_l and C_{l+1} (case A) or C_{l+1} and C_{l+2} (case B) provided that $l < k-2$, obtaining a new sets C_l^1 and C_{l+1}^1 of $\mathfrak{C}_1$ respectively.

In case A, $c \prec_l^1 \tilde{c}$ for all $c \in C_l$ and for all $\tilde{c} \in C_{l+1}$ is not allowed since merging in this way corresponds exactly to an edge of M. If on the other hand there is at least one $\tilde{c} \in C_{l+1}$ with $\tilde{c} \prec_l^1 \max(C_l)$, it follows that $\tilde{c} \prec_l^2 \max(C_l)$. Lemma A implies that C_{l+1} cannot be recovered.

Assuming Case B, it follows that there exists a $\hat{c} \in C_{l+2}$ with $\hat{c} > \max(C_l)$. Otherwise $\mathfrak{C}_1$ would be of type (1). If $\tilde{c} \prec_{l+1}^1 \max(C_{l+2})$ for a $\tilde{c} \in C_{l+1}$, this implies $\max(C_{l+2}) \succ_{i+1}^2 \tilde{c}$ in $\mathfrak{C}_2$. Again Lemma A implies that C_{l+1} cannot be recovered.
If otherwise $\tilde{c} \succ_l^1 \max(C_{l+2})$ for all $\tilde{c} \in C_{l+1}$, let the type (2) area C_{l+1}^1 of $\mathfrak{C}_1$ be divided into its corresponding type (1) area $(C_{l+1}^2, \prec_{l+1}^2, C_{l+2}^2, \prec_{l+2}^2)$ in $\mathfrak{C}_2$. Here, there are no elements of C_{l+1} contained in C_{l+1}^2. All elements of C_{l+1} are contained in C_{l+2}^2. Lemma C implies that C_{l+1} cannot be recovered. $\square$

Since M is an acyclic matching we can reduce the cell complex $\Gamma^{\mathcal{A}_{n-1}}$ to a complex homotopy equivalent to the complement of the braid arrangement with cells the standard fully ordered set-partitions (see Theorem 3.16). We denote this minimal complex by $\Gamma_{\mathcal{A}_{n-1}}$.

Theorem 4.8. $\Gamma_{\mathcal{A}_{n-1}}$ *is homotopy equivalent to the complement* $M(\mathcal{A}_{n-1})$ *of the complexified braid arrangement.*

$$\Gamma_{\mathcal{A}_{n-1}} \simeq M(\mathcal{A}_{n-1})$$

The cells of $\Gamma_{\mathcal{A}_{n-1}}$ *are in one-to-one correspondence with the elements of* S_n.

It remains to determine the resulting cell-order of $\Gamma_{\mathcal{A}_{n-1}}$.

4.3 Order

For two cells $\mathfrak{A}$ and $\mathfrak{B}$ of $\Gamma_{\mathcal{A}_{n-1}}$ with $\dim(\mathfrak{B}) = \dim(\mathfrak{A}) + 1$ there is $\mathfrak{A} \lessdot \mathfrak{B}$ if there exists a path

$$\begin{tikzcd}
 & \mathfrak{C}_1 \arrow[dr] & & \cdots\cdots\cdots\cdots \arrow[dr] & & \mathfrak{B} \\
\mathfrak{A} \arrow[ur] & & \mathfrak{C}_2 & & \mathfrak{C}_n \arrow[ur] &
\end{tikzcd}$$

in G_Γ^M.
Let $\mathfrak{A} := (C_1, \ldots, C_n)$ and let $C_i = |c_1^i \ldots c_{r_i}^i|$, $i \in \{1, \ldots, n\}$.
In order to obtain $\mathfrak{C}_1$, two sets C_l and C_{l+1} of $\mathfrak{A}$ are merged creating the type (2) area C_l^1 of $\mathfrak{C}_1$. For the corresponding type (1) area $(C_l^2, \prec_l^2, C_{l+1}^2 \prec_{l+1}^2)$ of $\mathfrak{C}_2$ it follows that C_{l+1}^2 is a left part of C_l. The elements of the corresponding right part of C_l are contained in C_l^2 and $\max(C_{l+1})$ is maximal with respect to $\prec_l^2$. The relations $\prec_l^2$ and $\prec_{l+1}^2$ respect the order-relations $\prec_l$ and $\prec_{l+1}$.
In the next step we have to eliminate the type (1) area of $\mathfrak{C}_2$ since otherwise the sequence cannot end at a standard cell of dimension $\dim(\mathfrak{A}) + 1$.
Thus, we can either merge C_l^2 and C_{l+1}^2, or C_{l+1}^2 and C_{l+2}^2.
Note that we can merge C_l^2 and C_{l+1}^2 in such a way that after passing to the corresponding type (1) cell only one element of C_{l+1}^2 has been transposed. In this way we are able to insert a right part of C_l in arbitrary order into C_{l+1} to the right of $\max(C_{l+1})$.

We have to proceed until C_l is completely merged to the sets $C_{l+1}, \dots, C_n$ in the same way as described. The result of this type (1) - type (2) sequence is the standard cell $\mathfrak{B}$.

From the discussion above, we can directly deduce a formal description of the cover relations $\mathfrak{A} <\!\cdot\, \mathfrak{B}$ of $\Gamma_{\mathcal{A}_{n-1}}$:

Proposition 4.9. *There is* $\mathfrak{A} = (A_1, \dots, A_n) <\!\cdot\, \mathfrak{B} = (B_1, \dots, B_{n-1})$ *for two cells of* $\Gamma_{\mathcal{A}_n}$ *with* $\dim(\mathfrak{B}) = \dim(\mathfrak{A}) + 1$ *if and only if:*

(P1) There exists a set A_l *in* $\mathfrak{A}$ *and a partition* $A_l = A_1^l \cup \dots \cup A_r^l$ *of* A_l*, such that* $a_p \prec_l a_s$ *for* $a_p \in A_i^l, a_s \in A_j^l$ *and* $i < j$*.*

(P2) There exist r *indices* $i_1 < \dots < i_r$ *with* $n - 1 > i_j \geq l, j \in \{1, \dots, r\}$*, such that* B_{i_j} *is created from* A_{i_j+1} *by inserting the elements of* A_j^l *in arbitrary order, such that* $a \preceq_{i_j} \max(A_{i_j+1})$ *for all* $a \in B_{i_j}$*, i.e. right of the corresponding maximum in the reduced notation.*

(P3) $B_s = A_s$ *for* $s < l$ *and* $B_s = A_{s+1}$ *for* $s \geq l$ *and* $s \notin \{i_1, \dots, i_r\}$*.*

In fact, an iteration of this process yields the cover relations $\mathfrak{A} < \mathfrak{B}$.

Proposition 4.10. (Order)
Let $\mathfrak{A} = (A_1, \dots, A_n)$ *and* $\mathfrak{B} = (B_1, \dots, B_{n-m})$ *be two cells of* $\Gamma_{\mathcal{A}_{n-1}}$ *with* $\dim(\mathfrak{B}) = \dim(\mathfrak{A}) + m,\ n > m > 1$*. Then:*
$\mathfrak{A}$ *is a face of* $\mathfrak{B}$ *if and only if* $\mathfrak{B}$ *can be generated from* $\mathfrak{A}$ *by an iteration of the process described by (P1)-(P3).*

Proof. It follows from Lemma B that

$$\{\max(B_i) | i = 1, \dots, n - m\} \subset \{\max(A_i) | i = 1, \dots, n\}.$$

No pair of sets of $\mathfrak{A}$ containing maxima of the sets of $\mathfrak{B}$ can be merged. This can be seen as follows: Let D be a set resulting from merging two sets $|a_i \dots|$ and $|a_j \dots|$, such that a_i is the maximum of A_i and a_j is the maximum of A_j and $j > i$.
If in D there is $a_i \prec_D a_j$, Lemma B implies that there is no path in G_Γ^M leading to a partition which contains a set with maximum a_i.
If on the other hand $a_i \succ_D a_j$ in D then a_i and a_j are either divided by inserting an element d with $d > a_j$ between a_i and a_j in D to produce a type (2) area, or D itself becomes a type (2) area.
In the first case Lemma B implies that there is no path leading to a partition with a_j as a maximum.

In the second case Lemma C implies that there is no path leading to a standard cell with a_i as a maximum.
Hence there must be m sets $A_{i_1}, \ldots, A_{i_m},\ 1 \leq i_j < n,\ j = 1, \ldots, m$ of the partition $\mathfrak{A} = (A_1, \ldots, A_n)$, each merged to the sets of $\mathfrak{A}$ which stay to their right and which contain maxima of the sets of $\mathfrak{B}$.

We find that A_{i_m} can be merged to the sets $A_{i_m+1}, \ldots, A_n$ only by using a type (1) - type (2) sequence which passes over 2 dimensions as described in the mechanism preceding this theorem, since these sets are not merged to other sets.
Note that using such a type (1) - type (2) sequence we can insert elements of the sets $A_{i_1}, \ldots, A_{i_{m-1}}$ into A_{i_m} (and thereby change the relative positions of their elements). Afterwards these elements can be inserted into the sets $A_{i_m+1}, \ldots, A_x$ using A_{i_m}, where A_x is the rightmost set of $\mathfrak{A}$ in which elements of A_{i_m} are inserted. But we lose this effect, if we merge A_{i_m} with one of the sets $A_{i_m+1}, \ldots, A_n$ first, producing a type (2) area D.
This follows from the fact that there cannot be produced a type (2) area of the form $A := |x_1 \ldots x_p \max(A_{i_m}) y_1 \ldots y_q|$ with $\max(A_{i_m}) > x_i$ and $\max(A_{i_m}) > y_j$ for all $1 \leq i \leq p$, $1 \leq j \leq q$ after passing to the type (1) area corresponding to D.
This can be seen as follows: After passing to this corresponding type (1) area of D there are sets to the left of the set containing $\max(A_{i_m})$ which contain larger maxima.
Altogether it follows that we can first do all merging-processes where just the elements of $A_{i_1}, \ldots, A_{i_{m-1}}$ are involved in, until they are completely merged to the sets $\{A_1, \ldots, A_n\} \setminus \{A_{i_1}, \ldots, A_{i_{m-1}}\}$. Since at this point we have a standard partition the proposition follows by induction. □

Corollary 4.11. *Assume the setting of Proposition 4.10.*
The sets $\{A_{i_1}, \ldots, A_{i_m}\}$ can be merged successively, from the left to the right beginning with A_{i_1}, completely to the sets to their right according to (P1), (P2) and (P3).

Corollary 4.12. *Each cell α of $\Gamma_{A_{n-1}}$ equals an element of a chain*

$$\alpha^0 <\cdot\ \alpha^1 <\cdot\ \alpha^2 <\cdot\ \alpha^3 <\cdot\ \cdots <\cdot\ \alpha^{n-2} <\cdot\ \alpha^{n-1}$$

with $\dim(\alpha^i) = i$.

4.4 Examples

Example 4.13.

$|321|54|6|7|8|$ represents a face of $|624|715|83|$ and also a face of $|5214|8367|$.

Example 4.14.

The following complex is the minimal complex to the initial complex $\Gamma^{\mathcal{A}_2}$ presented in Figure 4.1.

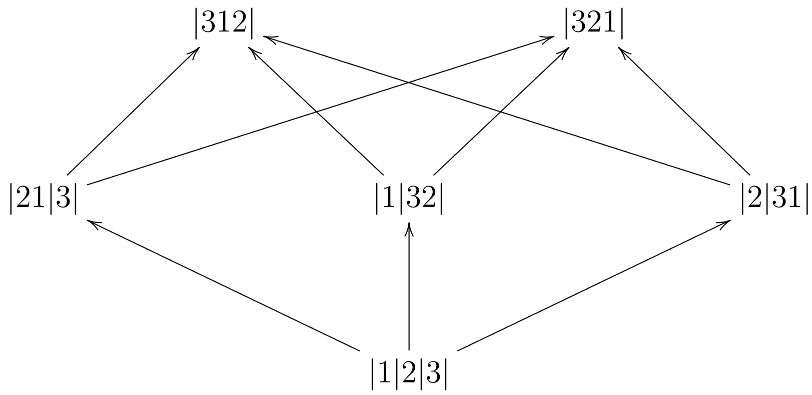


Figure 4: Face poset of the minimal complex $\Gamma_{\mathcal{A}_2}$ of $M_{\mathcal{A}_2}$

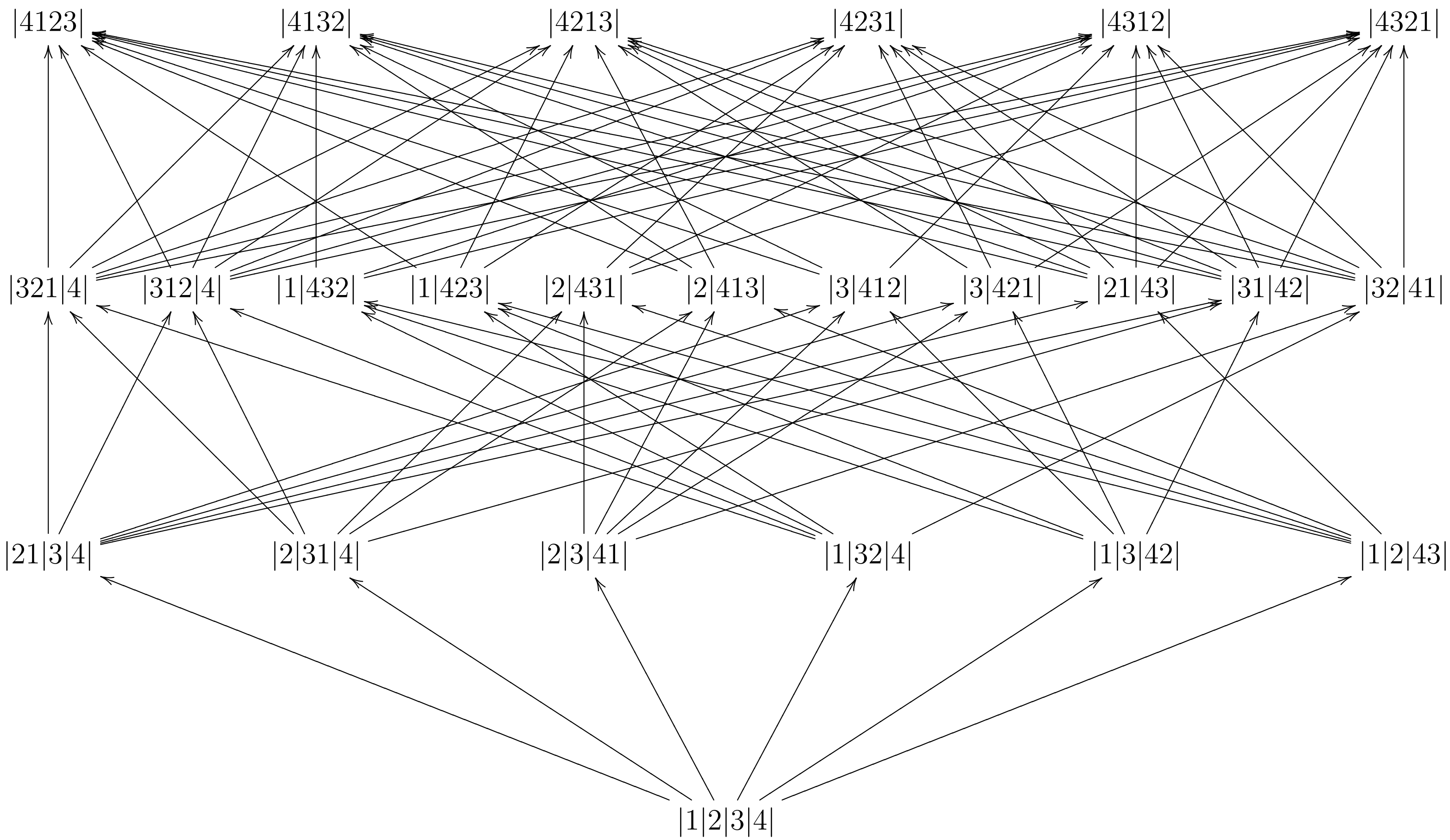


Figure 5: Face poset of the minimal complex $\Gamma_{\mathcal{A}_3}$ of $M(\mathcal{A}_3)$

Example 4.15.

Representing each cell as an element a of S_n with ordering of its maximal cycles:

$$a = (a_{1,1} \ldots a_{1,n_1})(a_{2,1} \ldots a_{2,n_2}) \ldots (a_{m,1} \ldots a_{m,n_m})$$

with $a_{i,1} > a_{i,j}$, for all i and $a_{i,1} < a_{j,1}$, for all $j > i$ we obtain the order:

$a \lessdot b \Longleftrightarrow$ There exists a cycle $z = (a_{j,1} \ldots a_{j,n_j})$ in a and an ordered partition

$$p_1 + \ldots + p_l = n_j, \;\; 1 \leq l \leq n_j, \;\; p_i \geq 1 \text{ for all } i \in \{1, \ldots, l\} \;,$$

such that

$$b = (a_{1,1} \ldots) \ldots (a_{j-1,1} \ldots)\widehat{(a_{j,1} \ldots)}(a_{j,2} \ldots)$$
$$\ldots \widehat{(a_{k_1,1} \ldots)}\widetilde{k_1} \ldots \widehat{(a_{k_l,1} \ldots)}\widetilde{k_l} \ldots (a_{m,1} \ldots) \;,$$

where $\widehat{\cdot}$ means that the corresponding cycle is deleted from the permutation and the cycle $\widetilde{k_s}$, $1 \leq s \leq l$ can be constructed from $(a_{k_s,1} \ldots)$ by inserting right to $a_{k_s,1}$ the elements

$$a_{j,(n_j - (\sum_1^{s-1} p_i) - p_s + 1)}, \ldots, a_{j,n_j - (\sum_1^{s-1} p_i)}$$

in arbitrary order.

5 $\mathcal{B}_n$

Let $\mathcal{B}_n$ be the complexified arrangement as introduced in Section 2.3. In addition to the hyperplanes $H_{i,j}$ of the braid arrangement it is equipped with the hyperplanes $H_{i,-j}$ and the coordinate hyperplanes H_i. The underlying reflection group is the group of signed permutations S_n^B.
In this section we construct a CW-complex $\Gamma_{\mathcal{B}_n}$ that is homotopy equivalent to the complement of $\mathcal{B}_n$ with minimal number of cells. Analogous to the $\mathcal{A}_{n-1}$-case the number of cells of $\Gamma_{\mathcal{B}_n}$ is equal to the number of elements of the group S_n^B.
Let $\Gamma^{\mathcal{B}_n}$ denote the initial complex which we have introduced in Section 2.3 and which is homotopy equivalent to $M(\mathcal{B}_n)$. At the beginning we extend the notion of fully ordered set-partitions to symmetric fully ordered set-partitions which serve as representatives of the cells of $\Gamma^{\mathcal{B}_n}$.

5.1 Structure

Let $\mathfrak{C} = (C_1, \dots, C_k)$, $k \in \{1, \dots, 2n\}$ be an ordered set-partition of the set $[\pm n] := \{-n, \dots, -1, 1, \dots, n\}$. We call $\mathfrak{C}$ *symmetric* if:

- ▷ For k odd, $\mathfrak{C}$ has the form $\mathfrak{C} = (C_1, \dots C_{s-1}, C_s, -C_{s-1}, \dots, -C_1)$, with $s = \frac{k+1}{2}$ and $|C_s|$ even, such that $c \in C_s \Rightarrow -c \in C_s$.
- ▷ For k even, $\mathfrak{C}$ has the form $\mathfrak{C} = (C_1, \dots, C_s, -C_s, \dots, -C_1)$, $s = \frac{k}{2}$.

Here $-C_i = \{-c | c \in C_i\}$ is the *negative set* of C_i, for $1 \le i \le \frac{k}{2}$.

Let $\mathfrak{p} = (\mathfrak{C}, \prec_{\mathfrak{C}})$ with $\mathfrak{C} = (C_1, \dots, C_k)$ for $k \in \{1, \dots, 2n\}$ and $\prec_{\mathfrak{C}} = (\prec_1, \dots, \prec_k)$ be a fully ordered set-partition of the set $[\pm n]$.
We call $\mathfrak{C}$ *symmetric* if:

(i) $\mathfrak{C} = (C_1, \dots, C_k)$ is a symmetric set-partition.

(ii) The linear orderings of the sets $-C_i$, $1 \le i \le \lfloor \frac{k}{2} \rfloor$ are given by $\overline{\prec_i}$, which is defined by

$$c_1 \prec_i c_2 \prec_i \dots \prec_i c_l \Longleftrightarrow: -c_l \overline{\prec_i} - c_{l-1} \overline{\prec_i} \dots \overline{\prec_i} - c_1.$$

(iii) If k is odd, then for $s = \frac{k+1}{2}$ there is $|C_s| = r$, for r even and $1 \le r \le 2n$. The order $\prec_s$ has the form $c_1 \prec_s \dots \prec_s c_{\frac{r}{2}} \prec_s -c_{\frac{r}{2}} \prec_s \dots \prec_s -c_1$ where $C_s = \{c_1, \dots, c_{\frac{r}{2}}, -c_{\frac{r}{2}}, \dots, -c_1\}$.

For k odd we call the set $C_{\frac{k+1}{2}}$ the *central-set* of the partition. We define a function $s : P_{odd} \to \mathbb{N}$ from the set P_{odd} of all symmetric fully ordered set-partitions with odd number of sets to $\mathbb{N}$, which assigns to each partition $\mathfrak{p} = (\mathfrak{C}, \prec_{\mathfrak{C}}) \in P_{odd}$ the subscript $s(\mathfrak{p})$ of its central-set. In the following we just write C_s instead of $C_{s(\mathfrak{p})}$ for the central-set of the partition $\mathfrak{p}$.

A set together with a linear ordering $(C, \prec)$ is called symmetric if it fulfills the conditions in (iii), for a central-set of a symmetric fully ordered set-partition.
We sometimes refer to a symmetric fully ordered set-partition $\mathfrak{p} = (\mathfrak{C}, \prec_{\mathfrak{C}})$ just as $\mathfrak{C}$. We will also skip the subscript $\mathfrak{C}$ of $\prec_{\mathfrak{C}}$, when the partition is clear from the context.

Let $\mathfrak{p} = (\mathfrak{C}_1, \ldots, \mathfrak{C}_k)$, $k \in \{1, \ldots, 2n\}$ be a doubly ordered set-partition of the set $[\pm n]$ with $\mathfrak{C}_i$ a fully ordered set-partition of the set C_i, for $i = 1, \ldots, k$. We call $\mathfrak{p}$ *symmetric* if

(i) $(C_1, \ldots, C_k)$ is a symmetric ordered set-partition.

(ii) For k even, $\mathfrak{p}$ has the form $\mathfrak{C} = (\mathfrak{C}_1, \ldots, \mathfrak{C}_l, -\mathfrak{C}_l, \ldots, -\mathfrak{C}_1)$, for $l = \frac{k}{2}$.

(iii) For k odd, $\mathfrak{p}$ has the form $\mathfrak{p} = (\mathfrak{C}_1, \ldots, \mathfrak{C}_{s-1}, \mathfrak{C}_s, -\mathfrak{C}_{s-1}, \ldots, -\mathfrak{C}_1)$, for $s = \frac{k+1}{2}$, where $\mathfrak{C}_s$ is a symmetric fully ordered set-partition.

To each symmetric doubly ordered set-partition $\mathfrak{p} = (\mathfrak{C}_1, \ldots, \mathfrak{C}_k)$ of $[\pm n]$ we associate a set of points in $\mathbb{C}^n$.
Let $\mathfrak{C}_i$ be a fully ordered set-partition of C_i, for $i = 1, \ldots, k$. An n-tuple $(z_1, \ldots, z_n)$ is associated to $\mathfrak{C}$ if and only if

(C1) If $r \in C_i$ and $s \in C_j$ for some $r, s \in [\pm n], r \neq s$ and $1 \leq i < j \leq k$, then $\mathrm{Im}(z_r) < \mathrm{Im}(z_s)$, where for $l < 0$ we set $z_l = -z_{-l}$

(C2) If $r, s \in C_i$ for some $1 \leq i \leq k$ and $\mathfrak{C}_i = (C_{i,1}, \ldots, C_{i,k_i})$, then

 (a) $\mathrm{Im}(z_r) = \mathrm{Im}(z_s)$
 (b) If $r \in C_{i,p}$, $s \in C_{i,q}$, for $1 \leq p < q \leq k_i$, then $\mathrm{Re}(z_r) < \mathrm{Re}(z_s)$
 (c) If $r, s \in C_{i,t}$, for some $1 \leq t \leq k_i$, then $\mathrm{Re}(z_r) = \mathrm{Re}(z_s)$.

Lemma 5.1. *The union of all sets which are associated to symmetric doubly ordered set-partitions $\mathfrak{p} = (\mathfrak{C}_1, \ldots, \mathfrak{C}_k)$ of $[\pm n]$, such that each $\mathfrak{C}_i$ is an ordered set-partition of a set C_i into singletons, is the set-theoretic complement $M(\mathcal{B}_n)$ of the complexified arrangement $\mathcal{B}_n$.*

Proof. Let $p = (z_1, \ldots, z_n)$ be contained in the complement $M(\mathcal{B}_n)$ of the arrangement $\mathcal{B}_n$, i.e. $z_i \neq z_j$ for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$, and $z_i \neq 0$ for all $i \in \{1, \ldots, n\}$.
We extend $(z_1, \ldots, z_n)$ to a $2n-$tuple $(z_{-n}, \ldots, z_{-1}, z_1, \ldots, z_n)$ by setting $z_i = -z_{-i}$. Thus there is $z_r \neq z_s$ for all $r, s \in [\pm n]$.
It follows that for the corresponding doubly ordered set-partition $\mathfrak{p} = (\mathfrak{C}_1, \ldots, \mathfrak{C}_k)$ of p, each $\mathfrak{C}_i$ is partitioned into singleton blocks.

Conversely consider a symmetric doubly ordered set-partition $\mathfrak{p} = (\mathfrak{C}_1, \ldots, \mathfrak{C}_k)$ of $[\pm n]$, such that each $\mathfrak{C}_i$ is an ordered set-partition into singleton blocks. Because of (C2)(c), each point $p = (p_1, \ldots, p_n)$ associated to the partition $\mathfrak{p}$ has different entries, i.e. $p_i \neq p_j$ for all $1 \leq i < j \leq n$.
Furthermore we have no coordinate of p equal to zero, since $z_i = 0$ for some $i \in \{1, \ldots, n\}$ implies $z_i = z_{-i}$. $\square$

In fact, the sets associated to symmetric doubly ordered set-partitions $\mathfrak{p} = (\mathfrak{C}_1, \ldots, \mathfrak{C}_k)$ of $[\pm n]$, such that each $\mathfrak{C}_i$ is an ordered set-partition of a set C_i into singletons, belong to the stratification of $\mathbb{C}^n$ which equals the one described by Björner and Ziegler in [4] corresponding to the arrangement $\mathcal{B}_n$. Precisely, these sets are the strata corresponding to the face poset $K^{(1)}_{\mathcal{B}_n}$ which are not contained in any of the hyperplanes.
Thus we can identify these doubly ordered set-partition with its corresponding cell $c_\mathfrak{p}$ of the cell-complex with face poset $K^{(1)}_{\mathcal{B}_n}$, as it was introduced in Section 2.3.
In the dual complex we denote by $d_\mathfrak{p}$ the cell dual to the cell $c_\mathfrak{p}$. Thus the cells $d_\mathfrak{p}$ with $\mathfrak{p} = (\mathfrak{C}_1, \ldots, \mathfrak{C}_k)$, such that each $\mathfrak{C}_i$ is an ordered set-partition of a set C_i into singletons, are the cells of a regular subcomplex $\Gamma^{\mathcal{B}_n}$ which is homotopy equivalent to the complement $M(\mathcal{B}_n)$.

Analogous to the last section we can identify symmetric doubly ordered set-partitions $\mathfrak{p} = (\mathfrak{C}_1, \ldots, \mathfrak{C}_k)$ of $[\pm n]$, for which each $\mathfrak{C}_i$ is an ordered set-partition with singleton blocks with symmetric fully ordered set-partitions $(\mathfrak{C}, \prec)$.

For simplicity, we sometimes present a set C_i of the partition $\mathfrak{C}$ as $C_i = (c_1, \ldots, c_n)$ and thereby we set $c_k \succ_i c_l$ for $k < l$.

The *reduced notation* of $\mathfrak{C} := (C_1, \ldots, C_k, C_s, -C_k, \ldots, -C_1)$ is

$$|c_1^1 \ldots c_{n_1}^1| \ldots |c_1^k \ldots c_{n_k}^k|c_1^s \ldots -c_1^s|-c_{n_k}^k \ldots -c_1^k| \ldots |-c_{n_1}^1 \ldots -c_1^1|$$

analogous to the last section.

We still have to translate the cell-order of the complex $\Gamma^{\mathcal{B}_n}$ to the set of symmetric fully ordered set-partitions.

For two symmetric fully ordered set-partitions $\mathfrak{p} = (\mathfrak{C}, \prec)$ and $\mathfrak{q} = (\mathfrak{C}', \prec')$. we set $\mathfrak{p} <\cdot\, \mathfrak{q}$ if and only if:

▷ If $\mathfrak{C} = (C_1, \ldots, C_k)$ then $\mathfrak{C}' = (C_1', \ldots, C_l')$ with $l \in \{k-1, k-2\}$, and for each $1 \leq i \leq k$ there is a $1 \leq j \leq l$ with $C_i \subset C_j'$ such that:

(i) The sets of $\mathfrak{C}$ and $\mathfrak{C}'$ are equal except for two pairs of consecutive sets C_i, C_{i+1} and C_{k-i+1}, C_{k-i} for some $i \in \{1, \ldots, \frac{k}{2}\}$. Each of the pairs is combined to a single set in $\mathfrak{C}'$.

(ii) If $C_i \subset C_j'$ for some $1 \leq i \leq k$ and $1 \leq j \leq l$ then the restriction of $\prec_j'$ to C_i coincides with $\prec_i$.

Of course, in (i) there can be $C_{i+1} = C_{k-i+1}$ equal to the central-set C_s if it exists. In this case the sets directly left and directly right of C_s are merged to it.

Note that when joining with or producing a central-set this must be done in a symmetric way.

It can be easily deduced from the order of sign vectors (see Section 2.3), that for two symmetric fully ordered set-partitions $\mathfrak{p} = (\mathfrak{C}, \prec)$ and $\mathfrak{q} = (\mathfrak{C}', \prec')$ of $[\pm n]$ the cell $d_\mathfrak{p}$ lies in the boundary of $d_\mathfrak{q}$ if and only if $\mathfrak{p} < \mathfrak{q}$ as symmetric fully ordered set-partitions.

Via this order preserving bijection from the set of cells $d_\mathfrak{p}$ and the poset of symmetric fully ordered set-partitions we from now identify the cell $d_\mathfrak{p}$ and $\mathfrak{p}$.

Example 5.2.

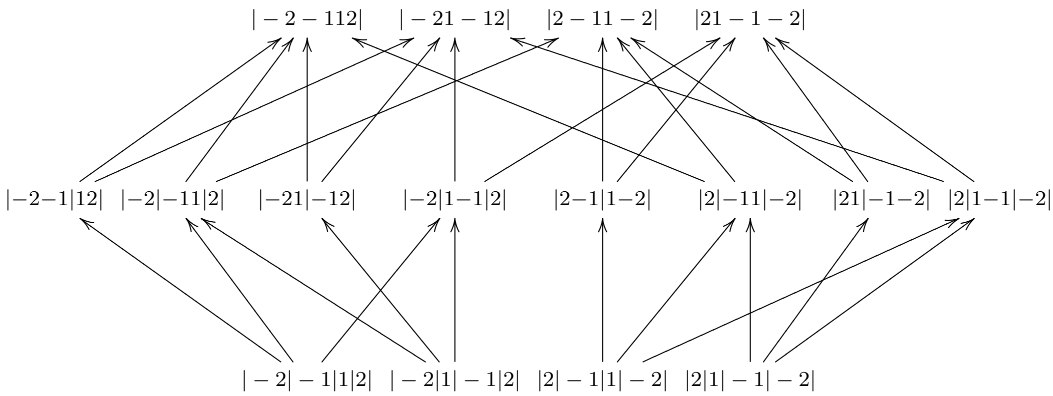


Figure 6: Face poset of the initial complex $\Gamma^{\mathcal{B}_2}$ for $M(\mathcal{B}_2)$

Analog to the $\mathcal{A}_{n-1}$-case we require a subset of the symmetric fully ordered set-partitions, such that the elements of this subset serve as representatives of the cells for the minimal complex of the arrangement.

We call a symmetric fully ordered set-partition $\mathfrak{C} = ((C_1, \ldots, C_k), \prec)$ *standard* if and only if

▷ For k even and $\mathfrak{C} = (C_1, \ldots, C_l, -C_l, \ldots, -C_1)$ with $l = \frac{k}{2}$ there is:

(S1) $\max(C_1) < \max(C_2) < \cdots < \max(C_l)$

(S2) $a \preceq_i \max(C_i)$, for all $a \in C_i$ and for $i = 1, \ldots, l$

(S3) For a pair C_i, C_{k-i+1}, $i = 1, \ldots, l$, there is $\max(C_i \cup C_{k-i+1}) \in C_{k-i+1}$.

▷ For k odd and $\mathfrak{C} = (C_1, \ldots, C_s, \ldots, C_k)$ with central set C_s, $s = \frac{k+1}{2}$ there is:

(S4) $\mathfrak{C} = (C_1, \ldots, C_{s-1}, C_{s+1}, \ldots, C_k)$ is standard.

(S5) There exists a symmetric fully ordered set-partition $\mathfrak{D} = (D_1, \ldots, D_l)$ of the central-set $(C_s, \prec_s)$ with l even, such that:

(a) $\max(D_1) < \max(D_2) < \cdots < \max(D_{\frac{l}{2}})$

(b) $a \preceq_i^{\mathfrak{D}} \max(D_i)$, for all $a \in D_i$ and $i = 1, \ldots, \frac{l}{2}$

(c) $\max(D_i \cup D_{l-i+1}) \in D_i$, for $i = 1, \ldots, \frac{l}{2}$

(d) The restriction of $\prec_s$ coincides with $\prec_i^{\mathfrak{D}}$, for $i = 1, \ldots, l$.

(e) For $c_i \in D_i$, $c_j \in D_j$ and $i < j$ there is $c_i \succ_s c_j$.

For further investigations we refer to a symmetric fully ordered set-partition as described in (S5) as a *good partition*.

For the definition of a matching and for the well-defined assignment of an element $b \in S_n^B$ to each standard partition we need the following fact.

Lemma 5.3. *For each central-set $C_s = (c_1, \ldots, c_k, -c_k, \ldots, -c_1)$ of a standard symmetric fully ordered set-partition there exists a unique good partition.*

Proof. Conditions (S5)(d) and (S5)(e) imply that the elements of $C_s := (c_1, \ldots, c_r, -c_r, \ldots, -c_1)$ cannot change places in $\mathfrak{D} = (D_1, \ldots, D_l)$ in the following sense: Let $D_i := (d_1^i, \ldots, d_{n_i}^i)$, then

$$(d_1^1, \ldots, d_{n_1}^1, d_1^2, \ldots, d_{n_2}^2, \ldots\ldots, d_1^l, \ldots, d_{n_l}^l) = (c_1, \ldots, c_r, -c_r, \ldots, -c_1)$$

as symmetric ordered sets.
It follows from (S5)(a), (S5)(b) and (S5)(c) that a good partition of C_s can uniquely be defined as follows:
Starting with D_1 with maximum c_1, the maximum d_i of D_i is the next element right of the maximum of D_{i-1} that is larger than d_i.
The elements of C_s between $\max(D_i)$ and $\max(D_{i+1})$ are inserted into D_i preserving the order of C_s. □

Example 5.4. Let $C_s := |2{-}1435{-}5{-}3{-}41{-}2|$. The unique good partition of C_s is $|2{-}1|43|5|{-}5|{-}3{-}4|1{-}2|$.

Lemma 5.5. *The set of standard symmetric fully ordered set-partitions of $[\pm n]$ is in bijection with the set of signed permutations S_n^B.*

Proof. A bijection can be defined as follows:
Let $\mathfrak{C} = (C_1, \ldots, C_k)$ be a standard symmetric fully ordered set-partition which in case k is odd, contains a central set C_s, $s = \frac{k+1}{2}$.
Then we assign to $\mathfrak{C}$ the signed permutation τ, such that:

- ▷ For each pair $C_i, -C_i$ with $i = 1, \ldots, \lfloor \frac{k}{2} \rfloor$ and $C_i = (c_1^i, \ldots, c_{n_i}^i)$ (i.e. $c_1^i \succ_i \cdots \succ_i c_{n_i}^i$), τ contains the cycle $(c_1^i \ldots c_{n_i}^i)(-c_1^i \ldots -c_{n_i}^i)$.

▷ For the central set C_s together with a good partition

$$C_s = (C_{s,1}, \dots, C_{s,r}, -C_{s,r}, \dots, -C_{s,1})$$

the permutation τ contains for each pair $C_{s,i}, -C_{s,i}$ with $C_{s,i} = (c_{i,1}, \dots, c_{i,l_i})$ the cycle $(c_{i,1} \dots c_{i,l_i} - c_{i,1} \dots - c_{i,l_i})$.

□

A *part* $(c_1, \dots, c_n)$ of a set C_i of a symmetric fully ordered set-partition is a subset of consecutive elements $c_1, c_2, \dots, c_n \in C_i$, i.e. there is no element between c_i and c_{i+1} in C_i, for $i := 1, \dots, n-1$.
In order to simplify notations we also consider a part as the sequence of its elements an write $c_1 \dots c_n$ in this case. Thus we use both terms parallel depending on what is more adequate in the special situation.
For a set $C_i = (c^i_1, \dots, c^i_{n_i})$ a *left part* of C_i is a part $(c^i_1, c^i_2, \dots, c^i_k)$, for $k \in \{1, \dots, n\}$. We define a *right part* of C_i to be a part $(c^i_l, c^i_{l+1}, \dots, c^i_{n_i})$, for $l \in \{1, \dots, n_i\}$.
For a symmetric fully ordered set-partition $\mathfrak{C}$ with central-set $C_s = (c_1, \dots, c_l, -c_l, \dots, -c_1)$ and $c_i \in [\pm n]$, for $i = 1, \dots, l$ we define a symmetric ordered set $\overline{C}$ to be an *inner part* of C_s if it has the form $\overline{C} = (c_j, \dots, c_l, -c_l, \dots, -c_j)$, for $j \in \{1, \dots, l\}$.

We want to fix the following property:

Lemma 5.6. *Let $\overline{C} := (c_1, \dots, c_l, -c_l, \dots, -c_1)$ be a symmetric ordered set. Consider the ground-set $X := \overline{C} \cup \{y_1, \dots, y_n, -y_n, \dots, -y_1\}$.*
There exists a symmetric ordered set C_s with ground-set X, such that

1. *$\overline{C}$ is an inner part of C_s*

2. *There exists a good partition of C_s*

if and only if:

There exists no $c_i \in \overline{C}$, $i \in \{1, \dots, l\}$ such that:

(i) *c_i is negative*

(ii) *$|c_i| > |c_k|$ for $1 \le k \le i-1$*

(iii) *$|c_i| > |y_r|$ for $1 \le r \le n$.*

Proof. Assume there exits such a c_i. By (S5)(c) it must be a non-maximal element in one of the sets of the good partition of C_s which contains a positive maximum m with $m > |c_i|$.
Because of (S5)(b), (S5)(d) and (S5)(e) there must be $c_i \prec m$, for $\prec$ the order´-relation of C_s. Obviously, from (ii) and (iii), it follows that there does not exist a candidate for m.
The other implication is obvious. □

Of course, a non-standard symmetric fully ordered set-partition violates at least one of the conditions for a standard partition. In the following lemma we modify these conditions in such a way, that for every non-standard partition there is exclusively one condition violated. This allows to classify the non-standard partitions according to these conditions.

Lemma 5.7. *Let $(\mathfrak{C}, \prec)$ a symmetric fully ordered set partition which is not standard. Let $\mathfrak{C} = (C_1, \ldots, C_k)$ and $\prec = (\prec_1, \ldots, \prec_k)$. For k odd let C_s, $s = \frac{k+1}{2}$ be the central-set of $\mathfrak{C}$. Then one of the following two conditions holds:*

(A) There exists a minimal index i such that either

(1) $\max(C_i) > \max(C_{i+1})$ and $i \in \{1, \ldots, \lfloor \frac{k}{2} \rfloor - 1\}$ and $c \preceq_i \max(C_i)$ for all $c \in C_i$

or

(2) there exists a $c \in C_i$, such that $c \succ_i \max(C_i)$ and $i \in \{1, \ldots, \lfloor \frac{k}{2} \rfloor\}$.

(B) Either

(3) there exists a maximal index $i \in \{1, \ldots, \lfloor \frac{k}{2} \rfloor\}$, such that

- *$\max(C_i \cup C_{k-i+1}) \in C_i$*
- *There exists a symmetric ordered set S with ground-set $C_s \cup C_i \cup C_{k-i+1}$, such that*
  - *(i) There exists a good partition of S*
  - *(ii) C_s is an inner part of S*
- *Condition (A) does not hold.*

or

(4)
- *There does not exist a symmetric ordered set S with ground-set $C_s \cup C_i \cup C_{k-i+1}$, such that*

(i) There exists a good partition of S

(ii) C_s is an inner part of S

Here, C_i is the set with maximal index i, such that $\max(C_i \cup C_{k-i+1}) \in C_i$. Thereby we set C_i and C_{k-i+1} to be empty if there is no such index i.

- *Condition (A) does not hold.*

Proof. It follows from the definition of a standard symmetric fully ordered set-partition that for a non-standard partition one of the following conditions hold:

(1) There exists an $i \in \{1, \dots, \lfloor \frac{k}{2} \rfloor - 1\}$ such that $\max(C_i) > \max(C_{i+1})$.

(2) There exists an $i \in \{1, \dots, \lfloor \frac{k}{2} \rfloor\}$ and an $a \in C_i$ such that $a \succ_i \max(C_i)$.

(3) There exists an $i \in \{1, \dots, \lfloor \frac{k}{2} \rfloor\}$ such that $\max(C_i \cup C_{k-i+1}) \in C_{k-i+1}$.

(4) There does not exist a good partition of the central-set C_s.

Conditions (1) and (2) imply (A). Conditions (3) and (4) imply (B). Since we do not want an overlapping of the conditions (A) and (B) we define that in case of (B) condition (A) is not allowed to hold. □

Since the sub cases of (A) and (B) exclude each other, we can assign to each non-standard partition a unique type according to the cases (1), (2), (3) and (4).

We call the index i postulated in (A) or (B) of Lemma 5.7 the critical index of $\mathfrak{C}$ and say that i is of type (1), (2) or (3) according to the cases above.
For a type (3) cell $\mathfrak{D} := (D_1, \dots, D_n)$ with critical index i we call the pair of sets $(D_i, -D_i)$ the *critical pair* of $\mathfrak{D}$.
In general we say that $(D_j, -D_j)$ is a type (3) pair if $\max(D_j \cup -D_j) \in D_j$. We say the critical index is of type (4) if conditions (4) holds. We say a partition is of type (1), (2), (3) or (4) if the critical index is of type (1), (2), (3) or (4) respectively. A cell is of type (1), (2), (3) or (4) if its corresponding partition is of type (1), (2), (3) or (4) respectively.

A pair of elements $(c_j, -c_j)$ of the central-set $C_s = (c_1, \dots, c_k, -c_k, \dots, -c_1)$ of a cell $\mathfrak{C}$, such that there does not exist a good partition of C_s, is called *critical* if j is the largest index, such that $(c_j, \dots, c_k, -c_k, \dots, -c_j)$ is not an inner part of a good partition of a symmetric ordered set with ground-set C_s.

In this case we call the inner part $(c_{j+1}, \ldots, c_k, -c_k, \ldots, -c_{j+1})$ a *maximal inner part.* For convenience we sometimes just say c_j is critical or $-c_j$ is critical.

Example 5.8. $C_s := |{-4}{-5}{-3}1{-2}2{-1}354|$. The maximal inner part of C_s is $-31-22-13$.

5.2 Matching 1

Our first aim is to eliminate pairs of type (1) cells and corresponding type (2) cells in the complex $\Gamma^{\mathcal{B}_n}$ by defining a suitable matching similar to the one we defined for the braid arrangement. In fact the following construction can be considered as a $\mathcal{B}_n$-analogon of the $\mathcal{A}_{n-1}$-case.

Let M_1 be the set of all cover relations $\mathfrak{q} < \mathfrak{p}$ of symmetric fully ordered set-partitions of $[\pm n]$ such that:

(M1) $\mathfrak{q}$ and $\mathfrak{p}$ are not standard.

(M2) The critical index i of $\mathfrak{q}$ is of type (1).

(M3) If $\mathfrak{q} = (\mathfrak{C}, \prec)$ and $\mathfrak{p} = (\mathfrak{C}', \prec')$ then

(a) $C'_j = C_j$ for $1 \leq j \leq i-1$
$C'_i = C_i \cup C_{i+1}$ and $C'_{k-i-1} = C_{k-i} \cup C_{k-i+1}$
$C'_j = C_{j+1}$ for $i+1 \leq j \leq k-i-2$
$C'_j = C_{j+2}$ for $k-i \leq j \leq k-2$

(b) $\prec'_j = \prec_j$ for $1 \leq j \leq i-1$
$\prec'_j = \prec_{j+1}$ for $i+1 \leq j \leq k-i-2$
$\prec'_j = \prec_{j+2}$ for $k-i \leq j \leq k-2$

(c) The restriction of $\prec'_i$ to C_i and C_{i+1} coincides with $\prec_i$ and $\prec_{i+1}$. There is $u \succ'_i v$, for all $u \in C_{i+1}$ and $v \in C_i$.

The restriction of $\prec'_{k-i-1}$ to C_{k-i} and C_{k-i+1} coincides with $\prec_{k-1}$ and $\prec_{k-i+1}$. There is $u \prec'_{k-i-1} v$, for all $u \in C_{k-i+1}$ and $v \in C_{k-i}$.

Remark 5.9. *The symmetric fully ordered set-partition $\mathfrak{p}$ defined in (M3) is of type (2) with critical index i.*

Proof. Since $\mathfrak{q}$ is of type (1) with critical index i there is $\max(C_i) > \max(C_{i+1})$. Since in $\mathfrak{p}$, $\max(C_i)$ is not the largest element of C'_i with respect to $\prec'_i$ it follows that $\mathfrak{p}$ is of type (2) with critical index i. $\square$

In M_1 we matched each type (1) partition $\mathfrak{q}$ with a type (2) partition $\mathfrak{p}$ and vice versa. From now on we will refer to $\mathfrak{p}$ as the *corresponding type (2) cell* (resp. the *corresponding type (2) partition*) of $\mathfrak{q}$ and we refer to $\mathfrak{q}$ as the *corresponding type (1) cell* (resp. the *corresponding type (1) partition*) of $\mathfrak{p}$.

$$\begin{array}{c} \mathfrak{p} = ((-1,-2,3,-4),(-5,5),(4,-3,2,1)) \\ \uparrow \\ \mathfrak{q} = ((3,-4),(-1,-2),(-5,5),(2,1),(4,-3)) \end{array}$$

Figure 7: Type (1) cell $\mathfrak{q}$ and corresponding type (2) cell $\mathfrak{p}$ in $\Gamma^{\mathcal{B}_n}$

Lemma 5.10. *M_1 is acyclic.*

Proof. The proof is an analogon of the proof of Lemma 4.7. $\square$

Let H denote the complex resulting from the initial complex $\Gamma^{\mathcal{B}_n}$ after the removal of the cells of M_1, i.e. it consists of the M_1-critical cells of $\Gamma^{\mathcal{B}_n}$. By H we denote also its face poset. Let $G_{M_1} := G_{\Gamma}^{M_1}$ be the graph which results from the graph of the initial complex $\Gamma^{\mathcal{B}_n}$ by reversing the edges of M_1.
Another fact from the $\mathcal{A}_{n-1}$-case which we can apply here is that after removing the cells of the matching M_1, the resulting complex is in addition to the relations inherited from the initial complex, equipped with the following relations:

Lemma 5.11. *Let $\mathfrak{q} = (\mathfrak{C}, \prec)$ and $\mathfrak{p} = (\mathfrak{C}', \prec')$ with $\mathfrak{C} = (C_1, \ldots, C_n)$ and $\mathfrak{C}' = (C'_1, \ldots, C'_m)$, for $m \in \{n-1, n-2\}$, be two cells in H.*
Then there is $\mathfrak{q} \lessdot \mathfrak{p}$ if and only if:

(i) There exists a set C_i, $i \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$ in $\mathfrak{C}$ and an ordered partition $(C^i_1, \ldots, C^i_{n_i})$ of C_i, such that $c_r \succ_i c_s$, for $c_r \in C^i_l$ and $c_s \in C^i_h$ and $l < h$.

(ii) There exist n_i indices $i_1 < \cdots < i_{n_i}$ with $i \leq i_j \leq \lceil \frac{m}{2} \rceil$, $j \in \{1, \ldots, n_i\}$, such that C'_{i_j} is created from C_{i_j+1} by inserting the elements of C^i_j in

arbitrary order, such that $c \prec'_{i_j} \max(C_{i_j+1})$*, for all* $c \in C'_{i_j}$*. Except for* m *odd and* $i_{n_i} = \lfloor \frac{m}{2} \rfloor + 1$ *(i.e.* $C'_{i_{n_i}}$ *equals the central-set of* $\mathfrak{C}'$*).*
In this case $C^i_{n_i}$ *is inserted into the central-set of* $\mathfrak{C}$ *in a symmetric way and such that the order of* $C^i_{n_i}$ *is respected.*

(iii) $C_r = C'_r$ *for* $r < i$*,* $C_{r+1} = C'_r$ *for* $r \geq i$ *and* $r \notin \{i_1, \ldots, i_{n_i}\}$*.*

The following remark is a less formal but a more concise version of Lemma 5.11:

Remark 5.12. *Assume the setting of Lemma 5.11.*
Let $\mathfrak{C} := (C_1, \ldots, C_k, C_s, -C_k, \ldots, -C_1)$ *with optional central-set* C_s*. There is* $\mathfrak{q} \lessdot \mathfrak{p}$ *if and only if the partition* $\mathfrak{C}'$ *can be obtained in the following way:*

(i) There exits a set C_i *of* $\mathfrak{C} := (C_1, \ldots, C_k, C_s, -C_k, \ldots, -C_1)$*, such that a left part* L_i *of* C_i *is merged to the central-set, respecting the order of* L_i*.*

(ii) The remaining right part $R_i := (r_1, \ldots, r_l)$ *is merged to the sets* $C_{i+1}, \ldots, C_k$ *to the right of the corresponding maxima such that:*

If r_x *is inserted into* A_{l_1} *and* r_y *into* A_{l_2}*, for* $x < y$*, it follows that* $l_2 \leq l_1$*.*

$$\mathfrak{C}' = ((-1, -6, -7), (2, -3), (-5, 4, -4, 5), (3, -2), (7, 6, 1))$$

$$\uparrow$$

$$\mathfrak{C} = ((-5, -7, -6), (-1), (2, -3), (4, -4), (3, -2), (1), (6, 7, 5))$$

Figure 8: Two cells $\mathfrak{C}$ and $\mathfrak{C}'$ with $\mathfrak{C} \lessdot \mathfrak{C}'$ in the CW-complex H.

Remark 5.13. *It can be deduced from Theorem 4.10 that every relation* $\mathfrak{q} < \mathfrak{p}$ *of cells with* $\dim(\mathfrak{q}) = \dim(\mathfrak{p} + l)$ *in* H *can be splitted into a chain*

$$\mathfrak{q} \lessdot \mathfrak{p}_1 \lessdot \mathfrak{p}_2 \lessdot \cdots\cdots \mathfrak{p}_{l-1} \lessdot \mathfrak{p} \quad .$$

5.3 Matching 2

In the following we define a second matching which allows to eliminate all cells of type (3) and type (4) in H.
Before we apply the methods of discrete Morse Theory to the matching, we ensure that for each pair of cells $\mathfrak{A} <\cdot \mathfrak{B}$ of the matching, $\mathfrak{A}$ is a regular face of $\mathfrak{B}$, (compare Definition 3.1).

Let M_2 be the set of all cover relations $\mathfrak{q} < \mathfrak{p}$ in H such that:

(M1) $\mathfrak{q} = ((C_1, \ldots, C_k), \prec)$ is of type (3) with critical index i

(M2) $\mathfrak{p} = ((C'_1, \ldots, C'_l), \prec')$ with $l \in \{k-1, k-2\}$ and central-set C'_s, $s = \frac{l+1}{2}$ is of type (4)

(M3) Let $\overline{C}$ be the central-set of $\mathfrak{q}$ if it exists. Then:

(a) $C_j = C'_j$ for $1 \le j \le i-1$
$C'_s = \overline{C} \cup C_i \cup C_{k-i+1}$
$C_{i+1+j} = C'_{i+j}$ for $0 \le j \le \lfloor \frac{k}{2} \rfloor - 1 - i$

(b) $\prec_j = \prec'_j$ for $1 \le j \le i-1$
$\prec_{i+1+j} = \prec'_{i+j}$ for $0 \le j \le \lfloor \frac{k}{2} \rfloor - 1 - i$

(c) The restrictions of $\prec'_s$ to C_i and C_{k-i+1} coincide with $\prec_i$ and $\prec_{k-i+1}$, respectively.
If $\overline{C}$ exists the restriction of $\prec'_s$ to $\overline{C}$ coincides with $\prec_{\overline{C}}$.

(d) There is $u \prec'_s v \prec'_s w$, for all $u \in C_i, v \in \overline{C}, w \in C_{k-i+1}$.

The existence of such a cover relation can be seen as follows:
Consider $\mathfrak{q}$ in G_{M_1}. We insert the elements C_i into C_{i+1} without changing the orderings of C_i and C_{i+1} producing a new cell $\mathfrak{C}_1 = (C_1, \ldots, \widetilde{C}_i, C_{i+2}, \ldots, C_k)$, such that $a \tilde{\prec}_i b$, for all $a \in C_{i+1}$ and $b \in C_i$, i.e. the elements of C_i are inserted to the left of the elements of C_{i+1}, respecting the individual orderings. It follows that the cell $\mathfrak{C}_1$ is of type (2) with critical index i.
In the corresponding type (1) cell the sets C_i and C_{i+1} have exchanged their positions. Proceeding with this process we can move C_i rightwards and we end up merging C_i and $-C_i$ to the central-set as described in (M3).
This results in a type (4) cell, since in the new central-set C'_s there is $\max(C_i \cup -C_i) \prec'_s -\max(C_i \cup -C_i)$ due to (d).
Since $\mathfrak{q}$ is not of type (2) there is $\max(C_i) = \max(C_i \cup -C_i)$ the largest element of C_i with respect to $\prec_i$. By symmetry, there is $-\max(C_i)$ the element with maximal absolute value in $-C_i$.

Consequently all $c \in C'_s$ with $c \succ'_s -\max(C_i \cup -C_i)$ have smaller absolute values than $-\max(C_i \cup -C_i)$ and therefore there cannot exist a good partition of C'_s due to Lemma 5.6.

For a pair $\mathfrak{q} < \mathfrak{p}$ as above we refer to $\mathfrak{q}$ as the corresponding type (3) cell (or partition) of $\mathfrak{p}$ and, conversely, we refer to $\mathfrak{p}$ as the corresponding type (4) cell (or partition) of $\mathfrak{q}$.

$$
\begin{tikzcd}
((2,-3),(-1,1),(3,-2)) \\
((1),(2,-3),(3,-2),(-1)) \arrow[u]
\end{tikzcd}
$$

Figure 9: Type (3) and corresponding type (4) cell in H.

The following fact follows directly from Lemma 5.6 and Lemma 5.7:

Lemma 5.14. *For the pair of cells as described by (M1) - (M3) above, there is* $\max(C_i)$ *a critical element of the corresponding type (4) cell.*

Let us consider a path, from a type (3) cell $\mathfrak{A}$ to its corresponding type (4) cell $\mathfrak{B}$ in G_{M_1}. Such a path can be illustrated by the following sequence, since it can clearly pass only over two dimensions.

$$
\begin{tikzcd}
 & \mathfrak{C}_1 \arrow[dr] & & \cdots\cdots\cdots & \mathfrak{C}_{n-1} \arrow[dr] & & \mathfrak{B} \\
\mathfrak{A} \arrow[ur] & & \mathfrak{C}_2 & & & \mathfrak{C}_n \arrow[ur] &
\end{tikzcd}
$$

It can be deduced from a simple adaption of Lemma 4.4 that within such a sequence, there exists no other possibility as the one described above to move the set C_i, with i the critical index of $\mathfrak{A}$ rightwards, while leaving the other sets as they are. For convenience we refer to a sequence of edges in G_{M_1} as above, such that

- the sequence passes only over two dimensions,
- $\mathfrak{A}$ and $\mathfrak{B}$ are standard, of type (3) or of type (4),
- all other cells are of type (1) and type (2), respectively,

as a *type (1) - type (2)* sequence. These sequences describe the merging-processes mentioned in Remark 5.12(ii).

We conclude:

Lemma 5.15. *There is a unique path between a type (3) cell $\mathfrak{A}$ and its corresponding type (4) cell $\mathfrak{B}$ in G_{M_1}.*

Note that as a consequence of that, it follows that also all sub-paths of the sequence in G_{M_1}, as shown above, are unique.

In order to apply the methods of discrete Morse Theory to H and the matching M_2 we need the following property:

Lemma 5.16. *A type (3) cell $\mathfrak{A}$ is a regular face of its corresponding type (4) cell $\mathfrak{B}$ after removal of all type (1) - type (2) pairs with dimension of the type (1) cells larger ot equal to the dimension of $\mathfrak{A}$.*

Proof. Let $\dim(\mathfrak{A}) = n$. If the preceding sequence consists only of $\mathfrak{A}$ and $\mathfrak{B}$ there is $\overline{\Phi_{\mathfrak{B}}^{-1}(\mathfrak{A})}$ a closed n-ball, where $\Phi_{\mathfrak{B}}$ is the characteristic map of $\mathfrak{B}$ corresponding to the complex H. This follows since there is a unique path between $\mathfrak{A}$ and $\mathfrak{B}$ in G_{M_1} by Lemma 5.15. Consequently, there cannot exist deformations of $\partial(\mathfrak{B}) \setminus \overline{\mathfrak{A}}$ onto $\mathfrak{A}$ by removing the type (1) - type (2) pairs. Clearly $\Phi_{\mathfrak{B}}$ is a homeomorphism on $\Phi_{\mathfrak{B}}^{-1}(\mathfrak{A})$ since our initial-complex is regular.
For a general sequence $\mathfrak{A} \to \mathfrak{B}$ as above we show that $\mathfrak{A}$ is a regular face of $\mathfrak{C}_3$ after removal of the cells $\mathfrak{C}_1$, $\mathfrak{C}_2$ and all other pairs of type (1) and type (2) cells of the given dimesnions. Then the assertion follows by induction.
In Section 3 we have explained that removing the edge $\mathfrak{C}_2 \to \mathfrak{C}_1$ corresponds to a deformation retraction of the free face $\mathfrak{C}_2$ along $\mathfrak{C}_1$ onto the remaining boundary $\partial(\mathfrak{C}_1) \setminus \overline{\mathfrak{C}}_2$. Since all faces of $\mathfrak{C}_1$ are regular faces, this is a deformation of a closed n-ball onto a closed n-ball which keeps their common boundary S^{n-1} and maps $\overline{\mathfrak{C}_2}$ homeomorphically onto the remaining faces of $\mathfrak{C}_1$.
From Lemma 5.15 it follows, that the path from $\mathfrak{A}$ to $\mathfrak{C}_3$ is unique. Thus it follows, that $\overline{\tilde{\Phi}_{\mathfrak{C}_3}^{-1}(\mathfrak{A})}$ is a closed n-ball, where $\tilde{\Phi}_{\mathfrak{C}_3}$ is the characteristic map of $\mathfrak{C}_3$ belonging to the complex which results from removing $\mathfrak{C}_1$, $\mathfrak{C}_2$ and all

other pairs of corresponding type (1) and type (2) cells which are not included in the sequence and with dimensions large enough. Consequently $\mathfrak{A}$ is a regular face of $\mathfrak{C}_3$ in this complex and here we have the initial situation with a shorter sequence. $\square$

Let $H_{M_2} := G_H^{M_2}$ be the graph which results from H by reversing the edges of M_2.

Lemma 5.17. *Let $\mathfrak{A} = (A_1, \ldots, -A_1)$ be of type (3) with critical index i and $a := \max(A_i)$. Assume after inserting A_i, the central-set has the following form:*

$$(\underbrace{\ldots}_{R}\underbrace{a \ldots - a}_{T}\underbrace{\ldots}_{-R}) \, .$$

Then T is an inner part of a symmetric ordered set S with ground set $R \cup -R \cup T$, such that there exists a good partition of S.

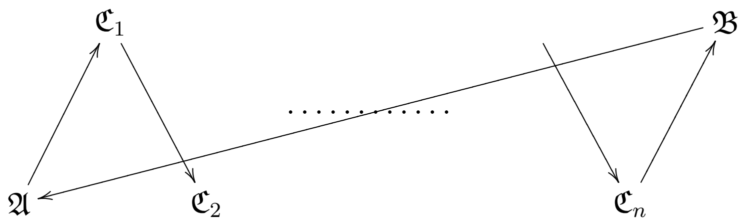
Proof. Assume that T is not such an inner part. From Lemma 5.6 it follows that there exists an element $t > 0$ in the right half of T with $|t|$ larger than the absolute values of all elements to the right of it. Therefore $t \notin A_i \cup -A_i$. Furthermore t cannot be contained in the central-set of $\mathfrak{A}$, since there it must stay in the right half what in turn implies that $\mathfrak{A}$ is of type (4). This yields a contradiction. $\square$

Lemma 5.18. *Let $\mathfrak{A} := (A_1, \ldots, A_k, A_s, -A_k, \ldots, -A_1)$ be a cell in H_{M_2} with optional central-set A_s. Assume there is a set A_i, $i \leq k$, such that $(A_i, -A_i)$ is not a type (3) pair, and such that A_i is merged to the other sets obtaining an upper neighbor $\mathfrak{B}$ of $\mathfrak{A}$ in H_{M_2}.*
It follows that there exists no path in H_{M_2} leading from $\mathfrak{B}$ to a cell which contains the set A_i in the left half of the partition, i.e. A_i cannot be recovered.

Proof. Assume an element $a \in A_i$ is inserted into A_j for $j \in \{i+1, \ldots, k\}$, using an appropriate type (1) - type (2) sequence, which clearly just corresponds to an edge in H_{M_2}.
It follows from Lemma 4.4 that by type (1) - type (2) sequences, the maximum of the set which contains a cannot be reduced.
We conclude that a left part of A_j containing a must be merged to the central-set.
As a consequence, all elements of A_i must be merged to the central-set in order to recover A_i.
We consider the element m of maximal absolute value in A_i. Since A_i does

not belong to a type (3) pair it follows that $m < 0$. It is clear that m can only be removed from the central-set using a transition to a corresponding type (3) cell. It follows that after this transition, for the corresponding critical pair $(C, -C)$ with $m \in C$, there is m staying right of a positive element that has a larger absolute value than m and therefore a larger absolute value than $\max(A_i)$.
It follows that A_i cannot be recovered, such that it stays in the left half of the partition. $\square$

Lemma 5.19. *Assume there is a directed cycle*



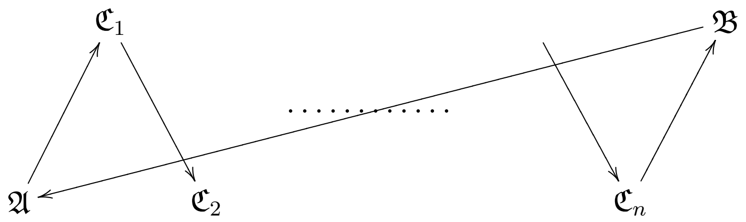
in H_{M_2}, of type (3) and type (4) cells with $\mathfrak{A} = (A_1, \ldots, -A_1)$ of type (3) with critical index i and central-set A_s. Then, except for transitions of indices, only the sets A_i, $-A_i$ and A_s can be changed along the path, i.e. apart from their indices all other sets of $\mathfrak{A}$ do not change in the directed cycle and also do not change their relative positions.

Proof. Of course there cannot be two left parts of sets of $\mathfrak{A}$ merged to the central-set at once, i.e. without passing to a corresponding type (3) cell after inserting the first set, since this yields a path passing over more then two dimensions.
From Lemma 5.18 it follows, that the sets $A_{i+1}, \ldots, A_k$ are not merged to other sets within the cycle.
From Lemma 5.18 it also follows, that the sets of $A_1, \ldots, A_{i-1}$ which do not belong to type (3) pairs cannot be merged to the other sets.
The sets among $A_1, \ldots, A_{i-1}$ corresponding to type (3) pairs cannot be merged in order to produce a type (4) cell $\mathfrak{C}_1$, since the elements of these sets have smaller absolute values than $\max(A_i)$.
From an analogous argumentation as in the proof of Lemma 5.18 it follows, that no element of A_i is merged to the sets $A_{i+1}, \ldots, A_k$.
Thus A_i is inserted into A_s as a whole set.
Since we cannot merge two sets to the central-set at once, the corresponding type (3) cell $\mathfrak{C}_2$ has critical index r larger than the indices of the sets among $A_1, \ldots, A_{i-1}$ belonging to type (3) pairs. Inductively it follows that only

the sets $A_i, -A_i$ and the central-set A_s are involved in the merging-process corresponding to the directed cycle. $\square$

Lemma 5.20. *M_2 is acyclic.*

Proof. Assume there is a directed cycle in H_{M_2}:



with $\mathfrak{A} = (A_1, \ldots, A_n)$ of type (3) and critical index i and $\mathfrak{B}$ of type (4) with central-set $\mathcal{B}_s$.

From Lemma 5.19 it follows that the only sets which are changed within the directed cycle are $A_i, -A_i$ and the central-set A_s. Thus we only have to take in account these sets.
Let $a := \max(A_i)$.
Assume a is positioned in the left half of the central-set $C^1_{s_1}$ of $\mathfrak{C}_1$, i.e. $C^1_{s_1}$ has the form

$$C^1_{s_1} = (\underbrace{\ldots}_{R}\underbrace{\ldots a \ldots - a \ldots}_{T}\underbrace{\ldots}_{-R})$$

with T the maximal inner part of $C^1_{s_1}$. The existence of such a part follows from Lemma 5.17. It follows that the corresponding type (3) cell $\mathfrak{C}_2$ has critical index $j \leq i$. Here we use that there cannot be a positive element larger than a right of $-a$ having a larger absolute value than all elements to its right in $C^1_{s_1}$, since this element then must have caused $\mathfrak{A}$ to be of type (4).
It follows from Lemma 5.19 that we cannot split the inner part $a \ldots -a$ which implies that we cannot recover A_i.

Assume a is positioned in the right half of the central-set $C^1_{s_1}$. It follows that there must be an element c right of a in the central-set with $c \notin A_i \cup -A_i$ since $\mathfrak{C}_1$ is not the corresponding type (4) cell of $\mathfrak{A}$.

If

$$C^1_{s_1} = (\underbrace{\ldots}_{R}\underbrace{\ldots - a \ldots a \ldots}_{T}\underbrace{\ldots}_{-R})$$

with a maximal inner part T of $C^1_{s_1}$, it follows that $\mathfrak{C}_2$ has critical index $j \leq i$. This follows, since otherwise there must exist an element $t > 0$ in $-R$ with $t > a$, which then must have caused $\mathfrak{A}$ to be of type (4).
Again it follows from Lemma 5.19 that we can not split the inner part $-a \dots a$ and we cannot recover A_i.

The last option is

$$C^1_{s_1} = (\underbrace{\dots - a}_{R}\underbrace{\dots\dots\dots}_{T}\underbrace{a \dots}_{-R})$$

with T the maximal inner part of $C^1_{s_1}$ as above. Clearly T is not smaller, since there is an obvious good partition of the elements of $C^1_{s_1}$ with T an inner part of it.
Hence, for $\mathfrak{C}_2$ with critical index j it follows that $C^2_j = (\underbrace{a \dots}_{-R})$, since again there cannot be an element $t > 0$ with $t > a$ staying right of a. Consequently there is $c \in C^2_j = (\underbrace{a \dots}_{-R})$.
We conclude that in any of the three cases it is not possible to recover A_i. $\square$

Let $\Gamma_{\mathcal{B}_n}$ denote the face poset of the initial complex after removing M_1 and M_2 with the methods of discrete Morse Theory. Since this will not lead to confusion we will also denote the corresponding CW-complex by $\Gamma_{\mathcal{B}_n}$.
Let us now formulate the main result of the preceding work. It follows from Theorem 3.16.

Theorem 5.21. $\Gamma_{\mathcal{B}_n}$ *is homotopy equivalent to the complement* $M(\mathcal{B}_n)$*.*

$$\Gamma_{\mathcal{B}_n} \simeq M(\mathcal{B}_n)$$

The cells of $\Gamma_{\mathcal{B}_n}$ *are in one-to-one correspondence with the elements of* S^B_n*.*

5.4 Describing Paths

For a standard cell $\mathfrak{A}$ we have already determined the standard cells $\mathfrak{B}$ with $\mathfrak{A} < \mathfrak{B}$, such that a path between $\mathfrak{A}$ and $\mathfrak{B}$ either exists in the graph of the initial complex $\Gamma^{\mathcal{B}_n}$ or is a result of reversing the edges of M_1. In the latter case the path occurs in G_{M_1}. Next we want to examine the cover relations $\mathfrak{A} < \mathfrak{B}$ in general. Thus the path from $\mathfrak{A}$ to $\mathfrak{B}$ can also occur in H_{M_2}.
In order to describe the combinatorics along a path in H_{M_2}, we can assign to each path in H_{M_2} a new path, such that the edges of the new path can be interpreted as mechanisms which can be applied to the partitions.

5.4.1 Basic Idea

Let us start our investigation with a standard cell

$$\mathfrak{A} = (C_1, \dots C_k, C_s, -C_k, \dots, -C_1)$$

with optional central-set C_s, $s = k + 1$, in H_{M_2}. Assume there is an edge leading to a type (4) cell $\mathfrak{C}$:

$$\begin{tikzcd} & \mathfrak{C} \\ \mathfrak{A} \arrow[ur] & \end{tikzcd}$$

Here the edge is either contained in the initial complex $\Gamma^{\mathcal{B}_n}$ or it first occurs in H.
In the first case the sets C_k and $-C_k$ are merged to the central-set.
In the second case the edge is a result of a path in G_{M_1}

$$\begin{tikzcd} & \mathfrak{C}_1 \arrow[dr] & & & \mathfrak{C}_{n-1} \arrow[dr] & & \mathfrak{C} \\ \mathfrak{A} \arrow[ur] & & \mathfrak{C}_2 & \cdots\cdots\cdots & & \mathfrak{C}_n \arrow[ur] & \end{tikzcd}$$

where the cells $\mathfrak{C}_i$, $i = 1, \dots, n$ are of type (1) and type (2) respectively, i.e. it is the result of a type (1) - type (2) sequence.
Since $\mathfrak{C}_n := (C_1^n, \dots, C_k^n, C_s^n, -C_k^n, \dots, -C_1^n)$ (which contains a central-set C_s^n if and only if $\mathfrak{A}$ does) is of type (1) and $\mathfrak{C}$ of type (4), the sets C_k^n and

$-C_k^n$ are merged to the central-set of $\mathfrak{C}_n$ in order to produce the central-set of $\mathfrak{C}$. Here we used, that the central-set of $\mathfrak{C}_n$ does not contain a critical element.
It follows that $\mathfrak{C}_n$ has critical index $k-1$ since otherwise the type (1) status does not vanish by merging C_k^n to the central-set.
It also follows that $\max(C_k^n) \succeq_k^n a$ for all $a \in C_k^n$, because C_k^n must be a left part of one of the sets $C_1, \ldots, C_{k-1}$ of the standard set $\mathfrak{A}$. This can be seen directly from the type (1) - type (2) sequence.
Thus in $\mathfrak{C}$ either a left part of a single set $C_i \in \{C_1, \ldots, C_{k-1}\}$ or C_k is merged to the central-set. In the first case the remaining right part of the set C_i is partitioned and merged to the sets $C_{i+1}, \ldots, C_k$ according to Remark 5.12(ii).

In order to examine paths in H_{M_2} which pass only over two dimensions we sometimes only analyze the central-set, the critical pairs of the type (3) cells and the two sets of the initial cell (which is standard) of which a left part is merged to the central-set at the beginning of the path.
We do this, when the merging-processes left of the central-set are not important.
Therefore for a type (3) partition

$$\mathfrak{C} = (C_1, \ldots, C_k, C_s, -C_k, \ldots, -C_1)$$

with optional central-set C_s we only record the triple of sets $C_i, C_s, -C_i$ where i is the critical index of $\mathfrak{C}$. Hence the reduced notation of $\mathfrak{C}$ is

$$|c_1^i \ldots c_l^i | c_1^s, \ldots, c_r^s - c_r^s \ldots - c_1^s | - c_l^i \ldots - c_1^i|$$

where $C_i = (c_1^i, \ldots, c_l^i)$ and $C_s = (c_1^s, \ldots, c_r^s, -c_r^s, \ldots, -c_1^s)$.
By an arrow $\mathfrak{A} \nearrow \mathfrak{B}$ we indicate that $\mathfrak{A} \lessdot \mathfrak{B}$, i.e. $\mathfrak{A} < \mathfrak{B}$ and $\dim(\mathfrak{B}) = \dim(\mathfrak{A}) + 1$.
Similarly an arrow $\mathfrak{B} \searrow \mathfrak{A}$ indicates that $\mathfrak{B}$ is of type (4) and $\mathfrak{A}$ is its corresponding type (3) cell. In other words this arrow represents an edge of H contained in M_2, which is reversed H_{M_2}.

Example 5.22.

Consider a standard cell $\mathfrak{A} = (C_1, \ldots C_k, C_s, -C_k, \ldots, -C_1)$ and an edge in H_{M_2} to a type (4) cell $\mathfrak{C}$.

$$\begin{tikzcd} & \mathfrak{C} \\ \mathfrak{A} \arrow[ur] & \end{tikzcd}$$

Thus a left part L of the set C_i is merged to the central-set and there is either $\max(L \cup -L) \in L$ or $\max(L \cup -L) \in -L$.
We exemplify the second case.

We extend the path in order to derive a path that leads to a standard cell. For simplicity we assume that $\mathfrak{A}$ contains no central-set.
The pair $C_i, -C_i$ is:

$$|\underbrace{\dots}_{-T_r} -c_{m,1} \underbrace{\dots}_{-\widetilde{T}} | \underbrace{\dots}_{\widetilde{T}} c_{m,1} \underbrace{\dots}_{T_r}| \,.$$

where $c_{m,1} := \max(-C_i)$.
Next we insert the two sets into the central-set without mixing its elements, which yields the central-set:

$$|\underbrace{\dots}_{-T_r} -c_{m,1} \underbrace{\dots}_{-\widetilde{T}} \underbrace{\dots}_{\widetilde{T}} c_{m,1} \underbrace{\dots}_{T_r}| \,.$$

Alternatively we could have inserted only a left part of the set

$$C_i = |\underbrace{\dots}_{-T_r} -c_{m,1} \underbrace{\dots}_{-\widetilde{T}}|$$

and due to symmetry a right part of the set $-C_i$, while the remaining right part of C_i is merged to the sets left of the central-set with larger index than i according to Remark 5.12(ii).
Clearly the resulting cell is of type (4). The corresponding type (3) cell is:

$$|c_{m,1} \underbrace{\dots}_{T_r} | \underbrace{\dots}_{-\widetilde{T}} \underbrace{\dots}_{\widetilde{T}} | \underbrace{\dots}_{-T_r} -c_{m,1}| \,.$$

We write this sequence as follows:

$$\begin{aligned} &|\underbrace{\dots}_{-T_r} -c_{m,1} \underbrace{\dots}_{-\widetilde{T}} | \underbrace{\dots}_{\widetilde{T}} c_{m,1} \underbrace{\dots}_{T_r}| \\ \nearrow\ &|\underbrace{\dots}_{-T_r} -c_{m,1} \underbrace{\dots}_{-\widetilde{T}} \underbrace{\dots}_{\widetilde{T}} c_{m,1} \underbrace{\dots}_{T_r}| \\ \searrow\ &|c_{m,1} \underbrace{\dots}_{T_r} | \underbrace{\dots}_{-\widetilde{T}} \underbrace{\dots}_{\widetilde{T}} | \underbrace{\dots}_{-T_r} -c_{m,1}| \,. \end{aligned}$$

Let $\widetilde{T} = t_1 \dots t_n$. The following sequence demonstrates how we can position the elements of $\widetilde{T}$ in arbitrary order to the right of $c_{m,1}$.

$$\begin{aligned}
&\nearrow |\underbrace{\ldots}_{-T_r} -t_n - c_{m,1} - t_{n-1} \ldots - t_1 t_1 \ldots t_{n-1} c_{m,1} t_n \underbrace{\ldots}_{T_r}| \\
&\searrow |c_{m,1} t_n \underbrace{\ldots}_{T_r}| - t_{n-1} \ldots - t_1 t_1 \ldots t_{n-1}| \underbrace{\ldots}_{-T_r} -t_n - c_{m,1}| \\
&\nearrow |\underbrace{\ldots}_{-T_r} -t_{n-1} - t_n - c_{m,1} - t_{n-2} \ldots - t_1 t_1 \ldots t_{n-2} c_{m,1} t_n t_{n-1} \underbrace{\ldots}_{T_r}| \\
&\searrow |c_{m,1} t_n t_{n-1} \underbrace{\ldots}_{T_r}| - t_{n-2} \ldots - t_1 t_1 \ldots t_{n-2}| \underbrace{\ldots}_{-T_r} -t_{n-1} - t_n - c_{m,1}| \,.
\end{aligned}$$

Note that instead of inserting the elements t_n and t_{n-1} directly to the right of $c_{m,1}$, we could have inserted them anywhere right of $c_{m,1}$.
Furthermore in each step starting with an $\nearrow$ we could have merged a right part of T_r to the sets left of the central-set.

We can insert $c_{m,1}$ to the left of $-c_{m,1}$ in the central-set which exemplarily yields the sequence:
$\nearrow| \underbrace{\ldots}_{-T_r} c_{m,1} t_n t_{n-1} - t_{n-2} \ldots - t_1 t_1 \ldots t_{n-2} - t_{n-1} - t_n - c_{m,1} \underbrace{\ldots}_{T_r}|$
$\searrow |c_{m,2} \underbrace{\ldots}_{T_r^2}| \underbrace{\ldots}_{-T_r^1} c_{m,1} t_n t_{n-1} - t_{n-2} \ldots - t_1$

$$t_1 \ldots t_{n-2} - t_{n-1} - t_n - c_{m,1} \underbrace{\ldots}_{T_r^1}| \underbrace{\ldots}_{-T_r^2} -c_{m,2}|$$

where we set $T_r =: \underbrace{\ldots}_{T_r^1} c_{m,2} \underbrace{\ldots}_{T_r^2}$. In this example $c_{m,2}$ is the maximum of $T_r \cup -T_r$.

5.4.2 Mechanisms

In this section we give a precise description of the mechanisms which we have illustrated at the end of the last section.
We reduce the combinatorics of the face poset of H_{M_2} to three mechanisms, which in our opinion are a reasonable choice in order to demonstrate the integral structural details on one hand and to allow a simple description on the other hand.
Even though these mechanisms yield a simplification of the structure of H_{M_2}, the real benefit is that we are able to describe the cell-order of H_{M_2} at all.

Still these mechanisms have a rather general character, but they allow to reduce the number of paths being considered. In the subsequent sections, we refer to these mechanisms in order to prove some details of the cell-order of H_{M_2}.

Consider the standard cell

$$\mathfrak{A} = (C_1, \ldots, C_k, C_s, -C_k, \ldots, -C_1)$$

with optional central-set C_s, $s := k + 1$.
Assume we want to merge a left part L of the set C_i, $i \in \{1, \ldots, k\}$ and its negative part $-L$ of $-C_i$ to the central-set. Whereas we merge the remaining right part of C_i to the sets $C_{i+1}, \ldots, C_k$ and the remaining left part of $-C_i$ to the sets $-C_k, \ldots, -C_{i+1}$.
For the pair $L, -L$ there is either $\max(L \cup -L) \in L$ or $\max(L \cup -L) \in -L$. We will see that this makes an essential difference.

We need the following definitions:
For a given standard symmetric fully ordered set-partition $\mathfrak{A}$ with central-set C_s we define the *leading coefficients* of $C_s = (c_1, \ldots, c_l, -c_l, \ldots, -c_1)$ to be the positive elements $c_i \in C_s$ with $i \in \{1, \ldots, l\}$ and $c_i > |c_j|$ for all $j < i$.

Let $C \in \{L, -L\}$ be the set which contains $\max(L \cup -L)$. An *external maximum* of C or an *external maximum* of the pair $(L, -L)$ is a positive element of C which has a larger absolute value than all elements to its right in C. Similarly we define the external maxima of a part of a set C_i of $\mathfrak{A}$. An external maximum t is a right (left) external maximum of an element y of C if it stays right (left) of it in C.

An useful fact is:

Remark 5.23. *Let $C_r = (c_1, \ldots, c_l)$, $r \in \{1, \ldots, k\}$ be a set of a symmetric fully ordered set-partition $\mathfrak{C} = (C_1, \ldots, C_k, C_s, -C_k, \ldots, -C_1)$ with optional central-set C_s, $s = k + 1$ which is either standard or of type (3).*
Then, if not all elements of $-C_r$ are positive it follows that $-c_1$ is the element with maximal absolute value among the negative elements of $-C_r$.

Proof. If not all elements of C_r are negative, the maximal element with respect to $\prec_r$ is the positive element with maximal absolute value of the positive elements. The assertion follows by symmetry. $\square$

We conclude that in $-L$ all external maxima have larger absolute values than all negative elements of $-L$.

Mechanism 0:

Let $\mathfrak{A} = (C_1, \ldots, C_k, C_s, -C_k, \ldots, -C_1)$ with optional central-set C_s, $s = k+1$. Let $\mathfrak{A}$ be standard or of type (3).
Let L be a left part of C_i and let $X \in \{L, -L\}$ contain $\max(L \cup -L)$.
We merge the remaining right part of C_i according to Remark 5.12(ii).
Let

$$X := \underbrace{\ldots}_{T_0} c_{m,1} \underbrace{\ldots}_{T_1} c_{m,2} \underbrace{\ldots}_{T_2} \ldots\ldots c_{m,r} \underbrace{\ldots}_{T_r}$$

with external maxima $c_{m,i}$, $i = 1, \ldots, r$. Clearly each of the parts T_i, $i = 0, \ldots, r$ may be empty.
Let $C_s := |\underbrace{\ldots\ldots}_{M}|$.
We merge the elements of T_0 and $-T_0$ to the central-set in the following way. The order of the two parts must be kept but the relative positions of two elements from different parts are arbitrary.

$$\nearrow |\underbrace{\ldots}_{-T_r} -c_{m,r} \ldots\ldots - c_{m,2} \underbrace{\ldots}_{-T_1} -c_{m,1} \underbrace{\ldots\ldots}_{\widetilde{M}} c_{m,1} \underbrace{\ldots}_{T_1} c_{m,2} \ldots\ldots c_{m,r} \underbrace{\ldots}_{T_r}| \ .$$

Here, $\widetilde{M}$ equals the part M, except for the elements of T_0 and $-T_0$ being inserted.
In case the latter cell is of type (4) we pass to the corresponding type (3) cell:

$$\searrow |c_{m,1} \underbrace{\ldots}_{T_1} c_{m,2} \ldots\ldots c_{m,r} \underbrace{\ldots}_{T_r}|\underbrace{\ldots\ldots}_{\widetilde{M}}|\underbrace{\ldots}_{-T_r} -c_{m,r} \ldots\ldots - c_{m,2} \underbrace{\ldots}_{-T_1} -c_{m,1}| \ .$$

Clearly, it is excluded to merge the left set of the critical pair of a type (3) cell to the indicated position in the central-set without inserting an element of it into the sets left of the central-set, because the resulting cell would be equal to its corresponding type (4) cell.

This mechanism allows for elements of a left part of T_0 to change from one half of the partition to the other. We also refer to such a change of sides as a *sign change*.
Repeated application allows to merge a right part of X arbitrarily to the sets $C_j, \ldots, C_k$ with $\max(C_j) > c_{m,1}$ and $\max(C_{j-1}) < c_{m,1}$, provided that $c_{m,1}$ is

larger or equal to the largest element of the type (3) pairs.

Mechanism 1:

Let $\mathfrak{A} = (C_1, \ldots, C_k, C_s, -C_k, \ldots, -C_1)$ with optional central-set C_s, $s = k+1$.
Let $C_i \in \{C_1, \ldots, C_k\}$, such that $\max(C_i \cup -C_i) = c_{m,1} \in C_i$ and such that $\mathfrak{A}$ is of type (3) with critical index equal to i.
Let C_i be of the following form

$$C_i := |c_{m,1} \underbrace{\ldots}_{T_1} c_{m,2} \underbrace{\ldots}_{T_2} \ldots\ldots c_{m,r} \underbrace{\ldots}_{T_r}|$$

with external maxima $c_{m,i}$, $i = 1, \ldots, r$.
Let

$$C_s := |\underbrace{\ldots}_{-F} x \ldots\ldots - x \underbrace{\ldots}_{F}|$$

be the central-set with x the first positive element from the left which is larger than $c_{m,1}$. It follows that x is a leading coefficient and that there exists a good partition of C_s, such that $x \ldots\ldots - x$ is an inner part. Let $F := f_1 \ldots f_n$.

We can merge the elements of F arbitrarily to the right of $c_{m,1}$ by the following mechanism:

$$\nearrow |\underbrace{\ldots}_{-T_r} -c_{m,r} \ldots\ldots - c_{m,2} \underbrace{\ldots}_{-T_1} -f_n - c_{m,1} - f_{n-1} \ldots - f_1 x \ldots$$

$$\ldots - x f_1 \ldots f_{n-1} c_{m,1} f_n \underbrace{\ldots}_{T_1} c_{m,2} \ldots\ldots c_{m,r} \underbrace{\ldots}_{T_r}|.$$

The corresponding type (3) cell is:

$$\searrow |c_{m,1} f_n \underbrace{\ldots}_{T_1} c_{m,2} \ldots\ldots c_{m,r} \underbrace{\ldots}_{T_r}| - f_{n-1} \ldots - f_1 x \ldots$$

$$\ldots - x f_1 \ldots f_{n-1}| \underbrace{\ldots}_{-T_r} -c_{m,r} \ldots\ldots - c_{m,2} \underbrace{\ldots}_{-T_1} -f_n - c_{m,1}| \ (\star).$$

Note that we could have inserted f_n anywhere to the right of $c_{m,1}$.
We can proceed until F is completely merged arbitrarily to the right of $c_{m,1}$.

In the following we refer to the last step (marked by ($\star$)) as a *breakup* of the central-set at $c_{m,1}$.
In general, we refer to the transition of a central-set $C_s := |\ldots - a \underbrace{\ldots}_{B} a \ldots|$ of a type (4) cell to its corresponding type (3) cell $|a\ldots|\underbrace{\ldots}_{B}|\ldots - a|$ as a *breakup* of C_s at a and we say that the central-set *breaks up* at a.

In case that a is an element of $C_i \cup -C_i$ the set $|a\ldots|$ is called a *fragment* of C_i.
In the above sequence $|c_{m,1} f_n \underbrace{\ldots}_{T_1} c_{m,2} \ldots\ldots c_{m,r} \underbrace{\ldots}_{T_r}|$ is a *fragment* of C_i.

Mechanism 1 allows to manipulate the relative positions of elements systematically, without inserting them into the sets $\{C_1, \ldots, C_k\}$ directly.

Mechanism 2:

Let $\mathfrak{A} = (C_1, \ldots, C_k, C_s, -C_k, \ldots, -C_1)$ with optional central-set C_s, $s = k+1$.
Let $\mathfrak{A}$ be of type (3) or standard and let $C_i \in \{C_1, \ldots, C_k\}$.
Let L be a left part of C_i and let $X \in \{L, -L\}$ contain $\max(L \cup -L)$.
We insert the elements of the remaining right part of C_i according to Remark 5.12(ii).
Let

$$X := \underbrace{\ldots}_{T_0} c_{m,1} \underbrace{\ldots}_{T_1} c_{m,2} \underbrace{\ldots}_{T_2} \ldots\ldots c_{m,r} \underbrace{\ldots}_{T_r}$$

with external maxima $c_{m,i}$, $i = 1, \ldots, r$.
We can merge X to the central-set, such that if the new central-set contains a critical pair $(-c, c)$, then $c \neq c_{m,1}$.
We insert X into the central-set, such that we have either the situation

$$|\ldots c_{m,1} \ldots - c_{m,1} \ldots|$$

or the situation

$$|\ldots x \ldots - c_{m,1} \ldots c_{m,1} \ldots - x \ldots| \,.$$

In the latter case $x > 0$ is the first element from the left in the central-set which is larger than $c_{m,1}$.
If the resulting cell is of type (4) we pass to the corresponding type (3) cell.

Mechanism 3:

Let $\mathfrak{C} = (C_1, \ldots, C_k, C_s, -C_k, \ldots, -C_1)$ be of type (3) with optional central-set C_s, $s = k+1$, and let $(C_i, -C_i)$ be a type (3) pair, such that

(i) $i > 1$

(ii) There exists a set $C_j \in \{C_1, \ldots, C_{i-1}\}$ such that $\max\left\{|x| \,\middle|\, x \in C_j\right\} > \max(C_i)$.

We can eliminate the type (3) status of the pair $(C_i, -C_i)$. For this, we insert an element $e < 0$ of a set C_j which meets (ii) into C_i, such that $|e| > \max(C_i)$, using a suitable type (1) - type (2) sequence. The remaining left part of C_j is merged according to one of the mechanisms above.

Obviously, Mechanism 3 can be either described by Mechanism 0 or Mechanism 2, but we want to refer to this special case later. Mechanism 3 provides the possibility to merge several sets within a path that only passes over two dimensions.

Example 5.24.

Let

$$-L := abc_{m,1}\underbrace{\ldots}_{T_1}\ldots\ldots c_{m,r}\underbrace{\ldots}_{T_r}$$

be a right part of the set $-C_i$ of a standard partition $\mathfrak{C}$.
Let $c_{m,1} := \max(L \cup -L)$ and let

$$C_s := |-f_2 - f_1 x \ldots\ldots -x f_1 f_2|$$

be the central-set with x the first element of C_s from the left which is larger than $c_{m,1}$.
We use Mechanism 0 to insert a and b into the central-set:

$$\nearrow |\underbrace{\ldots}_{-T_r} -c_{m,r}\ldots\ldots\underbrace{\ldots}_{-T_1} -c_{m,1}-f_2-f_1-bxa\ldots$$

$$\ldots -a-xbf_1f_2c_{m,1}\underbrace{\ldots}_{T_1}\ldots\ldots c_{m,r}\underbrace{\ldots}_{T_r}|$$

$$\searrow |c_{m,1}\underbrace{\ldots}_{T_1}\ldots\ldots c_{m,r}\underbrace{\ldots}_{T_r}|{-f_2}{-f_1}{-bxa}\ldots$$

$$\ldots{-a}{-xbf_1f_2}|\underbrace{\ldots}_{-T_r}{-c_{m,r}}\ldots\ldots\underbrace{\ldots}_{-T_1}{-c_{m,1}}|$$

We use Mechanism 1 to break up the elements f_1 and f_2:

$$\nearrow |\underbrace{\ldots}_{-T_r}{-c_{m,r}}\ldots\ldots\underbrace{\ldots}_{-T_1}{-f_2}{-c_{m,1}}{-f_1}{-bxa}\ldots$$

$$\ldots{-a}{-xbf_1c_{m,1}f_2}\underbrace{\ldots}_{T_1}\ldots\ldots c_{m,r}\underbrace{\ldots}_{T_r}|$$

$$\searrow |c_{m,1}f_2\underbrace{\ldots}_{T_1}\ldots\ldots c_{m,r}\underbrace{\ldots}_{T_r}|{-f_1}{-bxa}\ldots$$

$$\ldots{-a}{-xbf_1}|\underbrace{\ldots}_{-T_r}{-c_{m,r}}\ldots\ldots\underbrace{\ldots}_{-T_1}{-f_2}{-c_{m,1}}|$$

$$\nearrow \ldots\ldots \searrow |c_{m,1}f_2bf_1\underbrace{\ldots}_{T_1}\ldots\ldots c_{m,r}\underbrace{\ldots}_{T_r}|xa\ldots$$

$$\ldots{-a}{-x}|\underbrace{\ldots}_{-T_r}{-c_{m,r}}\ldots\ldots\underbrace{\ldots}_{-T_1}{-f_1}{-b}{-f_2}{-c_{m,1}}|$$

We use Mechanism 2 for insertion of $c_{m,1}$:

$$\nearrow |\ldots\ldots{-c_{m,2}}\ldots{-f_1}{-bxa}\ldots c_{m,1}\ldots{-f_2}\ldots$$

$$\ldots f_2\ldots{-c_{m,1}}\ldots{-a}{-xbf_1}\ldots c_{m,2}\ldots\ldots|$$

The corresponding type (3) cell depends on the values b, f_1 and $c_{m,2}$. Assuming that $c_{m,2}$ is the largest among the positive elements of b, f_1 and $c_{m,2}$, the corresponding type (3) cell is:

$$\searrow |c_{m,2}\ldots\ldots|\ldots{-f_1}{-bxa}\ldots c_{m,1}\ldots{-f_2}\ldots$$

$$\ldots f_2\ldots{-c_{m,1}}\ldots{-a}{-xbf_1}\ldots|\ldots\ldots{-c_{m,2}}|$$

Note that in each step of the example, such that elements are inserted to the central-set, i.e. the steps beginning with an $\nearrow$ (except for Mechanism 1), we could have merged a right part of the set staying left in the triple to the sets left of the central-set.

In this situation there are two ways to obtain a standard cell with dimension equal to $\dim(\mathfrak{C}) + 1$. The first way is to proceed in the same way until all elements of $-L$ are merged to the other sets using the mechanisms above.
The second way is to use Mechanism 3, i.e. to insert a negative element of a set C_j which stays left of the fragment $|c_{m,2} \ldots\ldots|$ into $|c_{m,2} \ldots\ldots|$, such that this negative element has a larger absolute value than $c_{m,2}$.

The following proposition is a consequence of the preceding discussion.

Proposition 5.25. *For each pair $\mathfrak{A} < \mathfrak{B}$ in $\Gamma_{\mathcal{B}_n}$ there exists a path in H_{M_2}, starting at $\mathfrak{A}$ and ending at $\mathfrak{B}$, such that this path can be described in terms of Mechanism 0 - Mechanism 2. Thus, we can assign one of the mechanisms to each edge of the path.*

Proof. By transitions to the corresponding type (3) cells, we can realize that the mechanisms do not start with a type (4) cell.
In fact there is no mechanism which allows to merge a set or a fragment L to the central-set such that its maximum $c_{m,1} := \max(L \cup -L)$ stays in between the positions stated by Mechanism 0 and Mechanism 2.
If such an insertion of L yields a type (4) cell, we can use Mechanism 1 and afterwards Mechanism 0 instead, in order to obtain this situation.
If the resulting cell is not of type (4), it is not obvious that we are allowed to position L with Mechanism 0 instead. It follows from the results of Section 6 that positioning L with Mechanism 0 is the optimal way because in doing so we avoid the occurrence of unnecessary so-called *unfreeness*.
Note that an insertion of the left set of the critical pair of a type (3) cell according to Mechanism 0, without inserting an element into the sets left of the central-set is not allowed. We have already mentioned this fact in the definition of Mechanism 0. It is easy to see that this exception does not cause any problems, because such an insertion can be compensated by type (1) - type (2) sequences.
We can describe all other merging processes by a transition from a path to a finer path (i.e. to a path which consists of more edges), such that the new path can be described in terms of Mechanism 0 - Mechanism 2. $\square$

Note that solely the process of merging a set C_i completely to the sets left of the central-set using a type (1) - type (2) sequence can be described by two mechanisms (0 and 2) at the same time.
There is only little space to specialize these mechanisms, such that they still can be used to describe the structure of $\Gamma_{\mathcal{B}_n}$. Anyway, this yields not to a simpler description of its cell order.

5.5 Order

From Proposition 5.25 we deduce a basic description of the cover relations $\mathfrak{A} \lessdot \mathfrak{B}$ of $\Gamma_{\mathcal{B}_n}$, i.e. with $\mathfrak{A} < \mathfrak{B}$ and $\dim(\mathfrak{B}) = \dim(\mathfrak{A}) + 1$.

Proposition 5.26. *Let $\mathfrak{A}$ and $\mathfrak{B}$ cells of $\Gamma_{\mathcal{B}_n}$. Let*

$$\mathfrak{A} = (C_1, \ldots, C_k, C_s, -C_k, \ldots, -C_1)$$

with optional central-set C_s, $s := k + 1$.
Then $\mathfrak{A} \lessdot \mathfrak{B}$ if and only if $\mathfrak{B}$ is the result of the following construction:

> *Choose a set $C_i \in \{C_1, \ldots, C_k\}$ and merge it to the other sets using Mechanism 0 - Mechanism 2 until either*
>
> *(i) $\mathfrak{B}$ is obtained*
>
> *or*
>
> *(ii) Start again with another set with the precondition of using Mechanism 3 in order to eliminate the present type (3) pair.*

We observe that, as a difference to the $\mathcal{A}_{n-1}$-case, elements of several sets can be inserted into other sets in order to construct a cell $\mathfrak{B}$ with $\mathfrak{A} \lessdot \mathfrak{B}$.

Indeed we lack a method for testing $\mathfrak{A} \lessdot \mathfrak{B}$ that is more efficient than exhaustive check through all alternatives given in Proposition 5.26.
In the subsequent section we give an explicit description of the relations $\mathfrak{A} \lessdot \mathfrak{B}$ in $\Gamma_{\mathcal{B}_n}$ that closes this gap.

Next we examine the general case of a cover relation $\mathfrak{A} < \mathfrak{B}$ with $\mathfrak{A}, \mathfrak{B}$ standard.

Let $\mathfrak{C} := (C_1, \ldots, C_k, C_s, -C_k, \ldots, -C_1)$ be a symmetric fully ordered set-partition with optional central-set C_s, $s = k + 1$. When we manipulate a set $C_i \in \{C_1, \ldots, C_k\}$ of $\mathfrak{C}$ by inserting or removing elements from it, we keep referring to it as C_i as long as its elements are not completely inserted into other sets and as long as there is no danger of confusion.

Example 5.27.

Let

$$\mathfrak{A} = |4(-8)|5(-9)123|(10)(-14)|(12)(-13)|$$

and

$$\mathfrak{B} = |(10)(-9)(-14)8(-4)|(12)(-13)5132|$$

be two cells of $\Gamma_{\mathcal{B}_n}$, where only the sets are displayed which are involved in the merging- process.

The following sequence shows that indeed $\mathfrak{A}$ is a face of $\mathfrak{B}$:

$$\begin{gathered}
\nearrow |4(-8)5(-9)123|(10)(-14)|(12)(-13)| \\
\searrow |5(-9)123|4(-8)|(10)(-14)|(12)(-13)| \\
\vdots \\
\nearrow |5(-9)123|(10)(-14)|(12)(-13)|4(-8)8(-4)| \\
\searrow |5(-9)123|8(-4)|(10)(-141)|(12)(-13)| \\
\nearrow |5(-9)8(-4)123|(10)(-14)|(12)(-13)| \\
\searrow |8(-4)123|5(-9)|(10)(-14)|(12)(-13)| \\
\vdots \\
\nearrow |8(-4)123|(10)(-9)(-14)|(12)(-13)5| \\
\vdots \\
\nearrow |(10)(-9)(-14)|(12)(-13)5132|4(-8)8(-4)| \\
\searrow |8(-4)|(10)(-9)(-14)|(12)(-13)5132| \\
\nearrow |(10)(-9)(-14)8(-4)|(12)(-13)5132|
\end{gathered}$$

Here we have merged the sets $|4(-8)|$ and $|5(-9)123|$ to the other sets. We cannot merge $|8(-4)|$ to $|(10)(-14)|$ at first, since we then would not be able to change the relative positions of -9 to the block 123.
Merging $|5(-9)123|$ first is also not an option since -9 stays left of the block 123 and therefore, must be inserted into a set with equal or larger index than the set containing the block 123.
Last, merging $|5(-9)123|$ into the central-set would result in -9 either remaining in the central-set or moving to the right half of the partition after a breakup.

Example 5.28.

Consider the standard cell

$$|3(-7)2|5(-8)|6(-9)4|(10)(-11)|(12)(-13)|\ .$$

Again we display only the sets which are involved in the merging-process. Assume we want to merge the sets $|3(-7)2|$ and $|6(-9)4|$ to the other sets. In this example we have to decide whether we first want to merge $|3(-7)2|$ to the central-set in order to change the relative positions of -9 and 4 using 7 as described in the following sequence:

$$\nearrow\searrow\cdots\nearrow|5(-8)|6(-9)4|(10)(-11)|(12)2(-13)|3(-7)7(-3)|$$

$$\searrow|5(-8)|6(-9)4|7(-3)|(10)(-11)|(12)2(-13)|$$
$$\nearrow|5(-8)|6(-9)7(-3)4|(10)(-11)|(12)2(-13)|$$
$$\searrow|5(-8)|7(-3)4|6(-9)|(10)(-11)|(12)2(-13)||$$
$$\nearrow\searrow\cdots\nearrow|5(-8)|7(-3)4|(10)(-11)(-9)|(12)2(-13)|6(-6)|$$
$$\nearrow\searrow\cdots\nearrow|5(-8)|(10)(-11)(-9)|(12)26(-13)4|673(-3)(-7)(-6)|$$

or whether we merge $|6(-9)4|$ to the central-set first in order to change the relative positions of -7 and 2 using 4:

$$\nearrow\searrow\cdots\nearrow|3(-7)2|5(-8)|(10)(-11)|(12)(-13)|(-4)9(-6)6(-9)4|$$

$$\searrow|3(-7)2|4|5(-8)|(10)(-11)|(12)(-13)|9(-6)6(-9)|$$
$$\nearrow\searrow\cdots\nearrow|42|5(-8)(-7)|(10)3(-11)|(12)(-13)|9(-6)6(-9)|$$
$$\nearrow\searrow\cdots\nearrow|5(-8)(-7)|(10)3(-11)|(12)(-13)24|9(-6)6(-9)|$$

The preceding examples illustrate an important fact. The order of merging sets has a crucial influence. From Example 5.28 we deduce that there exists no unique order of merging sets to other sets, which in a way is optimal, as there was in the $\mathcal{A}_{n-1}-$case. There we merged successively from the left to the right each set completely to the other sets and in doing so we were able to construct all possible upper neighbors. Clearly, two standard cells $\mathfrak{B}_1, \mathfrak{B}_2$ with identical central-sets and $\mathfrak{A} < \mathfrak{B}_1$, $\mathfrak{A} < \mathfrak{B}_2$ can be different.

The increase of complexity is accompanied by a lack of structure: The maxima of the sets in the right half of the partition can stay in arbitrary order from the left to the right. In other words, there is no relation between the relative positions of the sets and their elements of maximal absolute value.

In fact, each of the cover relations presented in Example 5.27 and Example 5.28 can be written as a sequence $\mathfrak{A}_1 <\cdot \mathfrak{A}_2 <\cdot \mathfrak{A}_3$ of standard cells. Of course, the sequences shown above do not pass a standard cell. However, the modifications of the sequences of the examples above, such that each of them passes a standard cell are not obvious. Since it is not easy to find a counterexample and because even such non-basic sequences yield a representation as a chain of standard cells one might think of that being true in general. The following example shows that in fact not all relations $\mathfrak{A} < \mathfrak{B}$ of $\Gamma_{\mathcal{B}_n}$ have representations as a chain of standard cells

$$\mathfrak{A} <\cdot \mathfrak{A}_1 <\cdot \mathfrak{A}_2 <\cdot \cdots\cdots <\cdot \mathfrak{B} .$$

Example 5.29.

Let

$$\mathfrak{A} := |-9|41-3-7|5-6| \text{ and } \mathfrak{B} := |5-6-7|4-3-19-913-4| ,$$

displaying only the sets involved in the merging-process. $\mathfrak{A}$ is a face of $\mathfrak{B}$ as can be seen by the following sequence:

$$\nearrow\searrow\nearrow\ |-9|5-6-7|3-1-441-3|$$
$$\nearrow\searrow\nearrow\ |5-6-7|-93-1-441-39|$$
$$\searrow\ |5-6-7|9|3-1-441-3|$$
$$\nearrow\searrow \ldots \searrow\ |5-6-7|94-31|$$
$$\nearrow\searrow \ldots \nearrow\ |5-6-7|4-3-19-913-4|$$

From the special configuration of $\mathfrak{A}$ it follows that there exists no path in H_{M_2} connecting $\mathfrak{A}$ and $\mathfrak{B}$ which passes a standard cell.
The relative order of 1 and -3 must be manipulated using 9. Since -3 stays left of 9 in $\mathfrak{B}$ it must either be inserted to the central-set with a fragment or set $|4\ldots|$ using Mechanism 2, or there must be a sign change from -3 to 3 in order to position -3.

The first option is displayed in the sequence above. Here 9 must be used to manipulate the relative positions of 4, 1 and -3. Thereby, changing the sign of -9 first and using Mechanism 0 afterwards in order to manipulate the positions of the elements of the block $41{-}3$ using a type (1) - type (2) sequence does not yield to a standard cell within the sequence. This follows, because after insertion of the fragment $|94{-}31|$ into the central-set using Mechanism 2, it breaks up at 4.
The second option implies also a change of sides of 1 to -1 and 4 to -4. But then -1 can only be positioned to the right of 9.

Of course the above sequence cannot be described in terms of Mechanism 0 - Mechanism 2. Using Mechanism 0 with $|9|$ first and afterwards inserting $|41{-}3|$ to the fragment $|9|$ using Mechanism 0 again, yields a sequence which can be described in terms of the mechanisms.

It seems complicated, to give an explicit description of the effects occurring in Example 5.29. It can be seen easily, that twisting 1 and -1 or inserting a pair $8, -8$ into the central-set will change the situation of the example in such a way, that $\mathfrak{A} < \mathfrak{B}$ has a representation as a chain of standard cells. Thus it does not only depend on the shape of the sets which are merged to the others, whether we can or cannot construct a chain of standard cells. It does also depend on the shape of all sets of $\mathfrak{B}$ and on the order of merging sets.
Because the last example is in a way exotic, a description of its configuration together with the results of the subsequent section eventually could yield a method for a successive determination of the cell order of $\Gamma_{\mathcal{B}_n}$.

We have shown that unlike to the $\mathcal{A}_{n-1}$-case we cannot revert to the relations $\mathfrak{A} <\!\cdot\, \mathfrak{B}$ for a general description of the order of $\Gamma_{\mathcal{B}_n}$. We combine the preceding results in the subsequent reformulation of Proposition 5.25.

We say that a set A_i of a standard $\mathfrak{A} := (A_1, \dots, A_k)$ is *active* with respect to a standard cell $\mathfrak{B}$ with $\mathfrak{A} < \mathfrak{B}$ if there exists a path in H_{M_2} from $\mathfrak{A}$ to $\mathfrak{B}$ which can be described in terms of Mechanism 0 - Mechanism 2, such that the elements of A_i are positioned using these mechanisms.
Roughly speaking, the active sets of $\mathfrak{A}$ are the sets which must be moved to construct $\mathfrak{B}$. Clearly, the active sets do not depend on the choice of such a path.

Proposition 5.30. *Let* $\mathfrak{A} := (A_1, \ldots, A_k, A_s, -A_k, \ldots, -A_1)$ *with optional central-set* A_s *and* $\mathfrak{B}$ *be two cells of* $\Gamma_{\mathcal{B}_n}$ *with* $\dim(\mathfrak{B}) = \dim(\mathfrak{A}) + m$*. Then the following statements are equivalent:*

(i) $\mathfrak{A}$ *is a face of* $\mathfrak{B}$*.*

(ii) There exist $m+d$ *active sets* $A_{i_1}, \ldots, A_{i_{m+d}}$, $1 \leq i_1 \leq i_2 \leq \cdots \leq i_{m+d} \leq k$*, that are completely merged to the other sets using Mechanism 0 - Mechanism 2 and the resulting cell is* $\mathfrak{B}$*.*

Thereby Mechanism 3 is used at least d *times.*

Recall the complex $\Gamma^{\mathcal{B}_2}$ of Figure 5.1. The following complex is the minimal complex to $\Gamma^{\mathcal{B}_2}$.

Example 5.31.

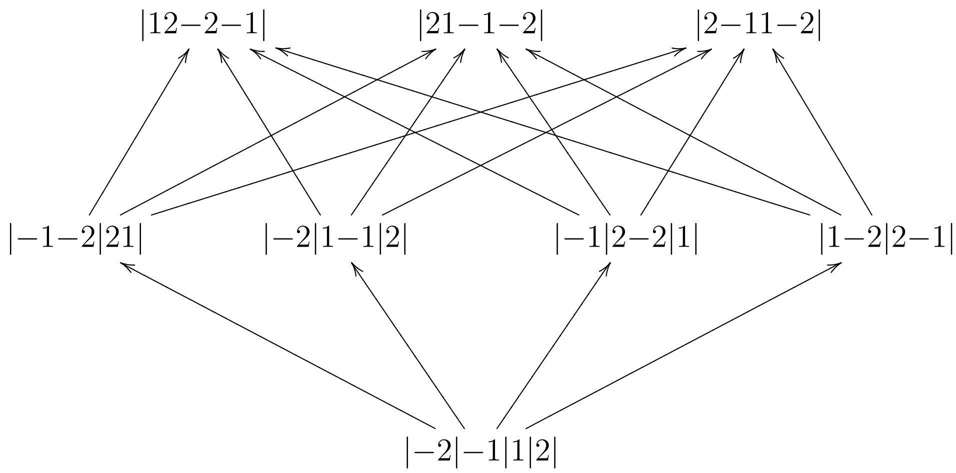


Figure 10: Face poset of the minimal complex $\Gamma_{\mathcal{B}_2}$ of $M(\mathcal{B}_2)$

6 Details

In the last section we learned how to decide for a pair of cells $\mathfrak{A}, \mathfrak{B}$ of $\Gamma_{\mathcal{B}_n}$ whether $\mathfrak{A} < \mathfrak{B}$ theoretically. In practice, especially in higher dimensions, we are barely able to prove whether there is $\mathfrak{A} <\cdot \mathfrak{B}$, i.e. $\mathfrak{A} < \mathfrak{B}$ and $\dim(\mathfrak{B}) = \dim(\mathfrak{A}) + 1$.
In this section we discuss the process of inserting elements into the central-set in detail. In doing so we gain a deeper understanding of the structure of $\Gamma_{\mathcal{B}_n}$.
Because of the complexity we present the details in terms of algorithms which provide an efficient tool to determine all cells $\mathfrak{B}$ with $\mathfrak{A} <\cdot \mathfrak{B}$ from a given cell $\mathfrak{A}$ without a recourse to the preceding constructions.
The results allow to formulate an explicit version of Proposition 5.26.

6.1 Definitions

Let $\mathfrak{A} = (C_1, \ldots, C_k, C_s, -C_k, \ldots, -C_1)$ be a standard cell with optional central-set C_s, $s = k+1$. In order to obtain the corresponding partition of a cell $\mathfrak{B}$ of $\Gamma_{\mathcal{B}_n}$ with $\mathfrak{A} <\cdot \mathfrak{B}$ there must be some active sets which are merged to the other sets according to Mechanism 0 - Mechanism 2. We confine ourselves to the case of only one active set $C_i \in \{C_1, \ldots, C_k\}$ of $\mathfrak{A}$ which is merged to the sets

$$C_1, \ldots, C_{i-1}, C_{i+1}, \ldots, C_s, \ldots -C_{i+1}, -C_{i-1}, \ldots, -C_1$$

using Mechanism 0 - Mechanism 2. In order to avoid an overflow of formalism we present the concept for the case of more active sets in the examples at the end of this section.
Even the special case of one active set requires a huge amount of conditions and case differentiation. But it already points out the crucial points and demonstrates the difficulties in finding cells $\mathfrak{B}$ with $\mathfrak{A} < \mathfrak{B}$.

We consider the left part L of C_i which is first merged to the central-set, whereas the remaining right part R of C_i is merged to the sets $C_{i+1}, \ldots, C_k$ according to Remark 5.12(ii).

Although we manipulate the left part L of C_i and the sets $C_1, \ldots, C_{i-1}, C_{i+1}, \ldots, C_s$ during our investigations we keep referring to them as L and $C_1, \ldots, C_{i-1}, C_{i+1}, \ldots, C_s$, respectively, in order to simplify notations.

There are two cases to take in account:

$$\max(L \cup -L) \in L \text{ or } \max(L \cup -L) \in -L \ .$$

Since $\mathfrak{A}$ is standard there is $\max(C_i \cup -C_i) \in -C_i$.
For $i = k$ there does not exist a right part R as mentioned above. Thus, regarding C_k, there is $L = C_k$ and $R = \emptyset$.

Let

$$-C_i := |\dots m_1 \dots m_2 \dots\dots m_l \underbrace{\dots - m}_{K}|$$

where $m_1, \dots, m_l$ are the external maxima of $-C_i$ and m is the maximum of C_i.
We observe that the condition $\max(L \cup -L) \in L$ implies that the right part R, that is merged to the sets $C_{i+1}, \dots, C_k$ in the first step, contains the elements $-m_1, \dots, -m_l$. Consequently L is a left part of $-K$ for $i < k$. We recover the details concerning the right part R in Section 6.2.3.

We need the following definitions:

Let $\max(L \cup -L) \in X \in \{L, -L\}$ and let

$$X := \underbrace{\dots}_{T_0} c_{m,1} \underbrace{\dots}_{T_1} c_{m,2} \dots\dots c_{m,r} \underbrace{\dots}_{T_r}$$

where $c_{m,1}, \dots, c_{m,r}$ are the external maxima of X. If $X = L$, it follows that the left part T_0 is empty.
Let

$$C_s := |\underbrace{\dots}_{-F} x \underbrace{\dots}_{-M} \underbrace{\dots}_{M} -x \underbrace{\dots}_{F}|$$

where x is the first positive element from the left of C_s which is larger than $c_{m,1}$.
Since not all elements of $X \cup F$ must be merged to the central-set we divide $X \cup F$ into the three sets M, M_l and M_r. Here, M contains all elements which are be merged to the central-set. M_l contains the elements of $X \cup F$ which are be merged to the sets $C_1, \dots, C_{i-1}, C_{i+1} \dots, C_k$, and M_r the elements which are merged to the negatives of these sets.

Thus, if $\max(L \cup -L) \in L$ we can write

$$L \cup F = M \uplus M_l$$

and if $\max(L \cup -L) \in -L$ we can write

$$-L \cup F = M \uplus M_l \uplus M_r \, .$$

Clearly there exist restrictions on such partitions. We recover these restrictions later.

For $\mathfrak{A} = (C_1, \dots, C_k, C_s, -C_k, \dots, -C_1)$, let $Y = f_1 \dots f_n$ be a part of the set C_i, for an $i \in \{1, \dots, k\}$. We define the *critical elements* of Y by the following recursive definition:

1. An element $t < 0$ of Y is critical, if in Y there is no positive element x of Y with $x > |t|$ staying left or right of t.
2. An element $t < 0$ of Y is critical if

   (i) for each $y > 0$ to the right of t with $y > |t|$ there exists a critical element f between t and y

   and

   (ii) for each $x > 0$ to the left of t with $x > |t|$ there exists a critical element f between t and x.

We call a connected part *collapsible* if it contains no critical element.

Example 6.1.
Let $Y := 3{-}2{-}6{-}1{-}75$. The critical elements are $-6, -1, -7$.

Example 6.2.
Let $Y := 6{-}7{-}235$. The only critical element is -7.

To point out the relevance of the critical elements we present the following property:

Remark 6.3. *Regard the part Y as the right half of a central-set (i.e. $C_s := |-YY|$). Let $t < 0$ be a critical element of Y. Then:*

(i) There exists a good partition of the inner part $-t \dots\dots t$.

(ii) Assuming the cell containing C_s is of type (4), t cannot be broken up using Mechanism 1 together with a positive element to its right in Y.

6.2 Algorithms

Let
$$\mathfrak{A} := (C_1, \ldots, C_k, C_s, -C_k, \ldots, -C_1)$$
be a standard symmetric fully ordered set-partition with optional central-set $C_s,\ s = k+1$.
We provide the algorithmic description which allows to construct cells $\mathfrak{B}$ with $\mathfrak{A} \lessdot \mathfrak{B}$ from $\mathfrak{A}$. In this section, we make frequent use of the mechanisms, which we have introduced in the preceding section.
Subsequent to this section we give a few examples which show that despite of the complexity the algorithms can be easily applied.

With regard to Proposition 5.26 we formulate the following fact.

Proposition 6.4. *Let $\mathfrak{A} := (C_1, \ldots, C_k, C_s. - C_k, \ldots, -C_1)$ be a standard symmetric fully ordered set-partition with optional central-set $C_s,\ s = k+1$. Then the following statements are equivalent.*

- *There is $\mathfrak{A} \lessdot \mathfrak{B}$ for a cell $\mathfrak{B}$ in $\Gamma_{\mathcal{B}_n}$, such that concerning Mechanism 0 - Mechanism 2, there exists just one active set in $\mathfrak{A}$.*
- *$\mathfrak{B}$ can be constructed from $\mathfrak{A}$ in the following way:*
  - *(i) Choose a set $C_i,\ i \in \{1, \ldots, k\}$ and a partition of C_i into a right part R and a left part L.*
  - *(ii) Merge the elements of $R := r_1 \ldots r_m$ to the sets $C_{i+1}, \ldots, C_k$ arbitrarily to the right of the corresponding maxima such that:*

    If r_x is inserted into C_{l_1} and r_y into C_{l_2} for $x < y$ then $l_2 \leq l_1$.
  - *(iii) For the remaining left part L there is either $\max(L \cup -L) \in L$ or $\max(L \cup -L) \in -L$.*
    In the first case proceed with Algorithm 1 whereas in the second case proceed with Algorithm 2.

Proof. This proposition presents just a frame. The main work is included in the two algorithms which are formulated further down and which are proved separately. (i) and (ii) have been discussed already.

□

Since for $i = k$ there does not exist a right part R of C_k as described above, we must use Algorithm 2 to merge C_k.

Let us investigate the case $\max(L \cup -L) \in L$ first.

6.2.1 Algorithm 1

Let $\mathfrak{A} := (C_1, \ldots, C_k, C_s, -C_k, \ldots, -C_1)$ be a standard cell with optional central-set C_s, $s = k+1$. Let L be a left part of one of the sets $C_1, \ldots, C_{k-1}$. Let

$$L := c_{m,1} \underbrace{\ldots}_{T_1} c_{m,2} \underbrace{\ldots}_{T_3} c_{m,3} \cdots\cdots c_{m,r} \underbrace{\ldots}_{T_r} ,$$

and

$$C_s := | \underbrace{\ldots}_{-F} x \underbrace{\ldots}_{-M} \underbrace{\ldots}_{M} -x \underbrace{\ldots}_{F} | ,$$

where $c_{m,1}, \ldots, c_{m,r}$ are the external maxima of L and x is the first positive element from the left of C_s which is larger than $c_{m,1}$.
It follows that $\max(L \cup -L) = c_{m,1}$.
We divide each part T_i into $T_i = T_i^1 T_i^2$ where T_i^1 is the minimal left part of T_i which contains the negative elements with larger absolute value than $c_{m,i+1}$. From Remark 5.23 it follows that this is not necessary when $\max(L \cup -L) \in -L$. It follows that $T_i^1 = t_{i,1}^1 \ldots t_{i,n_i}^1$ with $t_{i,n_i}^1 < -c_{m,i+1} < 0$.

We require the following settings:

Let $T_0 := F$.

Let $X := \emptyset$, $Y := \emptyset$.

Into X we insert certain elements of $L \cup F$ which are not external maxima. We call the elements of X *free*. The largest positive element that will be inserted into X within Algorithm 1 is called the *joker*. The elements of X can be merged righwards in the corresponding fragments using Mechanism 1.
Into Y we insert parts $y \underbrace{\ldots}_{U}$ of $L \cup F$, such that y causes a breakup in Algorithm 1. For each part $y \underbrace{\ldots}_{U}$ contained in Y we call the part U an *unfree part* and its elements *unfree* elements. For each unfree part there is $U \subset M$. The relative positions of the elements of an unfree part cannot be manipulated. We explain this fact subsequent to Algorithm 1.
The elements of X and parts in Y are inserted into the central-set at the end of Algorithm 1.

Algorithm:

We formulate Algorithm 1 such that it can be applied to certain type (3) cells too. This is necessary since we refer back to Algorithm 1 in Algorithm 2. Thus we require the definition of the following condition:

Condition 1:

> To a given $x > 0$, there exists a set C_j, $j \in \{1, \ldots, i-1, i+1, \ldots, k\}$ with $\max(C_j) > x$.

1. Decide whether $c_{m,1} \in M$ or, provided it meets Condition 1, $c_{m,1} \in M_l$.

- Let $c_{m,1} \in M_l$.

  It follows that all elements to its right are contained in M_l too. Thus the elements of L must be merged to the sets $C_{i+1}, \ldots, C_k$ in arbitrary order to the right of the corresponding maxima.

  In the following, by an *insertion with* $x > 0$, we mean that the corresponding elements, which are supposed to be inserted, must be inserted to the sets $C_j, \ldots, C_k$ with $\max(C_j) > x$ and $\max(C_{j-1}) < x$ in arbitrary order to the right of the corresponding maxima. More precisely, the elements are contained in a fragment $|x \ldots|$ and must be inserted using Mechanism 0.

  Decide whether $F \subset M$ or whether there is a critical element of F contained in M_l.

  ▷ The first case implies that F just keeps its position in the central-set.

  ▷ In the second case let f be the leftmost critical element of F which is contained in M_l.
  It follows that the right part $f \ldots$ of F is contained in M_l. It must be inserted into the sets $C_{i+1}, \ldots, C_k$ with $c_{m,1}$ too.

  The remaining left part of F eventually contains external maxima in which case it is of the form

$$\ldots t_1 \ldots t_r \underbrace{\ldots}_{F'} \underbrace{e_1 \ldots e_2 \ldots \ldots e_n \ldots}_{L'}$$

with $t_1, \ldots, t_r$ the critical elements and $e_1, \ldots, e_n$ the external maxima.
Here we have the initial situation with F' instead of F.
Therefore we partition and insert the remaining elements by starting the algorithm again with the corresponding type (3) cell

$$|e_1 \ldots e_2 \ldots\ldots e_n \ldots | \underbrace{\ldots}_{-F'} -t_r \ldots$$
$$\ldots t_r \underbrace{\ldots}_{F'} | \ldots - e_n \ldots\ldots - e_2 \ldots - e_1| \quad .$$

Then F and L are exchanged by F' and L', respectively.

- Let $c_{m,1} \in M$.

  Divide the positive elements of F into M and M_l such that for each $x \in F$ with $x > 0$ and $x \in M_l$ Condition 1 holds.
  Insert the elements of F into X.

  Let c_0 be the largest positive element of F which is contained in M. Define freezone$_0$ to be the right part $l_1 \ldots l_n$ of L with l_1 the leftmost element such that $l_1, \ldots, l_n$ have smaller absolute values than c_0.

2. $T_i \longrightarrow T_{i+1}$

Proceed with the appropriate case:

(A)

(A_1) If $i + 1 = r$ and $T_{i+1} \not\subset$ freezone$_i$ continue with ($\star$).

(A_2) If $i + 1 \neq r$, $T_{i+1} \not\subset$ freezone$_i$ and $c_{m,i+2} \in$ freezone$_i$ continue with ($\star$).

(B) If $T_{i+1} \subset$ freezone$_i$ then:

Decide whether there is a left part $K := k_1 \ldots k_h$ of T_{i+1} being unfree. Here k_h is a critical element of K.

Insert $c_{m,i+1} \underbrace{\ldots}_{K}$ (as a part) into Y.

Let T be the remaining right part of L which consists of all elements staying right of K.

Divide the positive elements of T into M and M_l such that for each $x \in T$ with $x > 0$ and $x \in M_l$ Condition 1 holds.
Insert the elements of T into X.

Partition the negative elements of X into M and M_l such that for each $t < 0$ with $t \in X \cap M_l$ the following condition holds:

Condition 2:

There exists an $x > |t|$ such that either:

- ▷ $x \in X \cap M$, and x meets Condition 1
- ▷ $x \in X \cap M_l$
- ▷ x is the maximum of a part in Y staying right of t in L, such that $x \in M$ and x meets Condition 1
- ▷ x is the maximum of a part in Y staying right of t in L, such that $x \in M_l$.

Consider the absolute values of the maxima of the parts in Y. In other words, consider the absolute values of the elements staying leftmost in the parts contained in Y.

Regard the absolute values of the elements of X and of the maxima of the parts in Y.
Beginning with the maximal absolute value, insert the parts of Y and the elements of X with decreasing absolute value into the central-set and into the sets left of the central-set, respectively, in the following way:

▷ Insert the elements of X according to Lemma 6.11, Lemma 6.13 and Lemma 6.14.

▷ The parts of Y which are contained in M_l consist only of one element y by construction. Thus insert these elements to the sets left of the central-set.

▷ Insert the parts contained in Y whose elements are contained in M into the central-set while preserving the relative positions of their elements and such that:

(i) For each maximum y of a part which is contained in M, there exists an element $t < 0$ right of y in the central-set with $|t| \geq y$.

(ii) Each maximum y of a part in Y must *respect unfreeness*, i.e. y must stay right of all unfree parts U such that:
   1. U stays left of y in L.
   2. The rightmost critical element of U has smaller absolute value than y.

(iii) For each unfree part $U := t_1 \dots t_k$ contained in the parts of Y there must exist an element $t < 0$ right of it in the central-set or being equal to t_k such that:
   1. $|t|$ is larger than all maxima y of parts in Y which stay right of the unfree part U in L and with $y \in M$.
   2. $|t|$ is larger than all positive elements of $X \cap M$.

(C) If $i + 1 \neq r$ and $c_{m,i+2} \notin \text{freezone}_i$ then:

Decide whether T^1_{i+1} is unfree or, provided that each critical element t of T^1_{i+1} with $t \in M_l$ meets Condition 3, whether T^1_{i+1} is not unfree.

Condition 3:

There exists an element $x > 0$, such that either

(i) $x = c_{m,i+1}$ and $c_{m,i+1}$ meets Condition 1, or

(ii) $x \in X \cap M_l$ and x is larger than all absolute values of the elements of the right part $t \dots \dots$ of L.

▷ In the second case it follows that all elements right of and including the leftmost critical element f of T^1_{i+1} with $f \in M_l$ and $|f| > c_i$ are contained in M_l.
Without loss of generality we may assume that the situation matches case 2 of $(\star)$ below, with $T := T^1_{i+1}$ and freezone_i replaced by $T^2_{i+1} c_{m,i+2} \dots \dots$.
Continue with $(\star)$.

▷ In the first case decide whether $c_{m,i+2} \in M$ or, provided that $c_{m,i+2}$ meets Condition 1, whether $c_{m,i+2} \in M_l$.
   → In the first case decide whether there is a left part $U := t_1 \dots t_v$ of T^2_{i+1} with $t_v < 0$ critical in U, such that $T^1_{i+1} U$ is unfree. Insert the part $c_{m,i+1} T^1_{i+1} U$ into Y.

Divide the positive elements of the remaining part P of T_{i+1}^2 into M and M_l, such that each $x > 0$ with $x \in P \cap M_l$ meets Condition 1.

Insert the elements of P into X.

Let c_{i+1} be the largest positive element of $X \cap M$ and let freezone$_{i+1}$ be the right part $l_1 \dots l_n$ of L with l_1 the leftmost element such that $l_1, \dots, l_n$ have smaller absolute value than c_{i+1}.

Continue with step 2.

→ In the second case the elements right of $c_{m,i+2}$ are contained in M_l and must be inserted into the sets left of the central-set with $c_{m,i+2}$.

Insert $c_{m,i+2}$ as a part into Y.

Start from 2. again with $c_{m,i+1} \underbrace{\dots}_{T_{i+1}}$ and transfer X and Y. In doing so we equip the fragment $P := c_{m,i+1} \underbrace{\dots}_{T_{i+1}}$ with new labels. That is from $c_{m,i+1} \underbrace{\dots}_{T_{i+1}}$ we switch to the new labellings

$$\underbrace{c_{m,i+1} \underbrace{\dots}_{T_{i+1}} c_{m,i+2} \underbrace{\dots}_{T_{i+2}} \dots\dots c_{m,r} \underbrace{\dots}_{T_r}}_{P} \quad ,$$

where the $c_{m,i}$ are the external maxima of P.

($\star$) We have the situation that T_{i+1} is not contained in freezone$_i$ and either $i + 1 = r$ or $c_{m,i+2} \in$ freezone$_i$.

Thus we have the situation

$$c_{m,i+1} \underbrace{\dots f}_{T} \underbrace{\dots}_{\text{freezone}_i}$$

with f a critical element of T.

Case 1: If $T \subset M$ it follows that T must be unfree, i.e. it must be contained in an unfree part in Y.

Decide whether there is left part $\tilde{T} := t_1 \dots t_h$ of freezone$_i$ such that $\underbrace{\dots}_{T}\underbrace{\dots}_{\tilde{T}}$ is unfree. Here t_h is a critical element of $\tilde{T}$.

Insert $c_{m,i+1}\underbrace{\dots}_{T}\underbrace{\dots}_{\tilde{T}}$ as a part into Y.

Partition the positive elements of the remaining right part of freezone$_i$ as done above and insert its elements into X.

Insert the elements and parts of X and Y as described in (B).

Case 2: $T \not\subset M$ and each critical element of T which is contained in M_l meets Condition 3.

Let

$$T := \dots f_1 \dots f_{j-1}\underbrace{\dots}_{G} f_j \dots f_n$$

with $f_1, \dots, f_n$, $f_n = f$ the critical elements that have a larger absolute value than c_i. Let f_j be the leftmost among them that is contained in M_l.
It follows that the part $\dots f_1 \dots f_{j-1}$ must be contained in an unfree part to the right of $c_{m,i+1}$ and that the part right of G is contained in M_l.

Let $\tilde{c} > 0$ be the largest element of $X \cap M_l$ which is smaller than $c_{m,i+1}$.

- ▷ If there does not exist such an element $\tilde{c}$, or if $\tilde{c}$ is smaller than f_j, merge all elements to the sets left of the central set with $c_{m,i+1}$.
- ▷ If not, let f_w be the leftmost element among $f_j, \dots, f_n$ with $|f_w| > \tilde{c}$.

  Let $y_1, \dots, y_h \leq \tilde{c}$, $h \in \{1, \dots, w-j\}$ be an increasing sequence of positive elements of $X \cap M_l$ such that:

  (i) y_1 is the smallest of these elements with $y_1 > |f_j|$.

  (ii) For $f_{n_{i+1}}$ the first element among $f_j, \dots, f_{w-1}$ from the left that has a larger absolute value than y_i, there is y_{i+1} the smallest positive element of $X \cap M_l \setminus \{y_1, \dots, y_i\}$ with $y_{i+1} > |f_{n_{i+1}}|$.

  Insert the parts between f_{n_i} and $f_{n_{i+1}}$ including f_{n_i} to the sets left of the central-set with y_i.

Merge the part $f_w \ldots f_n$ to the sets left of the central-set with $c_{m,i+1}$.

We have the situation

$$c_{m,i+1} \ldots f_1 \ldots f_{j-1} \underbrace{\ldots}_{G} .$$

Start again from with 2. ($T_i \to T_{i+1}$) and transfer X, Y and the elements which are already chosen to be in M resp. M_l. Before starting at 2. again, we must, as in (C), equip the part $c_{m,i+1} \ldots f_1 \ldots f_{j-1} \underbrace{\ldots}_{G}$ with a new labeling.

We assume the situation of Algorithm 1 for the following lemmas. The sets X and Y denote the sets as they were used in Algorithm 1.

Consider the following preliminary considerations:
It can be seen from Algorithm 1 that the elements contained in X are elements which can freely be positioned in the fragments of C_i using Mechanism 1. Let $P := yy_1 \ldots y_l$ be a part in Y, such that $P \subset M$. It follows that y_l is critical in P. From Algorithm 1 (condition (B)(ii) and Lemma 6.11) it follows that P is positioned in the central-set in such a way that:

- There exists no positive free element $x \in X \cap M$ to the left of y_l in the central-set with $x > |y_l|$.
- There exists no maximum $\overline{y}$ of a part in Y which stays right of P in L and with $\overline{y} > |y_l|$, such that $\overline{y}$ stays left of y_l in the central-set.
  $\triangleright$ $\overline{y}$ can be an external maximum of L or a free element which causes a breakup in Algorithm 1.

It follows that these elements cannot be used to position $y_1 \ldots y_l$ using Mechanism 2. The elements of other parts in Y can also not be used to position the elements $y_1, \ldots, y_l$, what can be shown by induction. Therefore $y_1, \ldots, y_l$ are positioned using y with Mechanism 2. As a consequence, $y_1 \ldots y_l$ cannot be manipulated using Mechanism 1, since:

1. There exists no positive free element $x \in X \cap M$ which is larger than y.

2. Each maximum $\overline{y}$ of a part in Y staying right of y in L is smaller than y.

That is, in order to manipulate the relative positions of the elements $y_1, \ldots, y_l$, the maximum y of the corresponding part in Y must be merged to the central-set using Mechanism 2, such that for the critical pair $(-t, t)$ (if it exists) there is $t < y$. It follows that the elements $y_1, \ldots y_l$ cannot be inserted to the indicated positions with y anymore. This yields a contradiction. Thus the elements $y_1, \ldots, y_l$ of P can only be positioned in the central-set taking their former relative positions staying right of y.
We call the parts which stay right of the maxima of the parts in Y *unfree*. An element of C_i is *unfree* if it is contained in an unfree part.
Condition (iii) in (B) is required since otherwise it would not be possible to insert the free positive elements of $X \cap M$ without getting into conflict with the conditions for the unfree parts. Inductively it follows, that positioning an unfree part U as required in (i)-(iii) is always possible.

The following examples will reveal the notion of unfreeness. Note that we only display the two parts $L, -L$ and the central-set C_s in the form $L|C_s|-L$.

Example 6.5.

$$|71-24-63-36-42-1-7|$$

$$71-2-3|6-44-6|32-1-7$$

The unfree elements are -1 and -2.

Example 6.6.

$$|41-29-38-77-83-92-1-4|$$

$$9-7841-2-3|32-1-4-87-9$$

The unfree elements are 1, -2 and -3.

Example 6.7.

$$|41{-}29{-}38{-}77{-}83{-}92{-}1{-}4|$$

$$97841{-}2{-}3|32{-}1{-}4{-}8{-}7{-}9$$

Although we have only twisted 7 and -7 compared to the last example, there exists no unfree element, because the part $41{-}2{-}3$ can be manipulated using 7.

Lemma 6.8. *Assume the setting of Algorithm 1. Let $c_{m,i} \in M_l$ for an $i \in \{1, \ldots, r\}$.*
If there does not exist a free positive element $m \in M \cap X$ which is larger than $c_{m,i}$, (i.e. $c_{m,i} \notin freezone_i$ for all i) it follows that all elements to the right of $c_{m,i}$ in L are contained in M_l.

Proof. Let $c_1 \ldots c_l$ be the elements staying right of $c_{m,i}$ in L at the beginning. Note that it does not make any difference in the argumentation if some elements with smaller absolute value than $c_{m,i}$ are inserted into the part $c_{m,i}c_1 \ldots c_l$ right of $c_{m,i}$. Thus by abuse of notation we write $c_{m,i}c_1 \ldots c_l$ for this part although, during the algorithm, there may have been inserted elements of smaller absolute value than $c_{m,i}$.

$c_{m,i}$ must be merged to the sets left of the central-set.
Clearly in order to insert $c_{m,i}$ into these sets it is necessary to produce a breakup at an element c such that $c_{m,i}$ stays right of c or is equal to c. Afterwards $c_{m,i}$ can be inserted using Mechanism 0 or Mechanism 2.
Since $c_{m,i}$ is an external maximum, we can in fact insert $c_{m,i}$ into the sets $C_j, \ldots, C_k$ left of the central-set with $\max(C_j) > c_{m,i}$ and $\max(C_{j-1}) < c_{m,i}$, i.e. we may assume $c = c_{m,i}$. But all elements which stay right of $c_{m,i}$ in the fragment L after this breakup must be merged to these sets too. This follows from the mechanisms.
Thus if $c_1, \ldots, c_l$ stay right of $c_{m,i}$ in L at this point, it follows that $c_1, \ldots, c_l$ are contained in M_l.
If $c_1, \ldots, c_l$ do not all stay right of $c_{m,1}$ at this point there must have been a breakup in between the part $c_{m,i}c_1 \ldots c_l$, i.e. it is divided into a right part breaking up with a fragment and a left part which keeps staying in the central-set. Such a breakup can be caused in two ways only:

• The first way is that the part $c_{m,i}c_1 \dots c_l$ is positioned in the central-set in such a way, that a left part keeps staying in the central-set while a right part $c_j \dots c_l$ breaks up at c_j, i.e. c_j is critical. But this implies $c_{m,i} \in M$ what follows from Remark 5.6 since $c_{m,i} > c_j$.

• The second way is to split the part by merging a free positive element t with $t > c_{m,i}$ in between the elements of the part in the central-set, which by the assumption is contained in M_l.
This just yields the situation with two fragments $|c_{m,i}c_1 \dots c_j|$ and $|tc_{j+1} \dots c_l|$. This process cannot be continued indefinitely and we conclude that the elements staying right of $c_{m,i}$ are contained in M_l. □

Remark 6.9. *From the proof of Lemma 6.8 it follows that the sets* $C_j, \dots, C_k$ *with* $\max(C_j) > c_{m,i}$ *and* $\max(C_{j-1}) < c_{m,i}$ *as mentioned in the proof, are the only sets into which the elements* $c_1, \dots, c_l$ *can be inserted.*

Lemma 6.10. *Assume the setting of Algorithm 1. Let* $f \in M_l$ *be a critical element of a part* T^1_{i+1} *such that* $f \notin freezone_i$ *for all* i*.*
Then all elements to the right of f *are contained in* M_l*.*

Proof. The argumentation is analog to the proof of Lemma 6.8 □

In this situation, for $c_{m,i+1} \in M$, the positions in the sets left of the central-set of the elements right of the critical element $f \in M_l$ may vary from the positions indicated by Remark 6.9. Assume there exists a free positive element $\tilde{c} \in X \cap M_l$ which is smaller than $c_{m,i+1}$ but larger than the absolute values of the elements right of and including f. It could be used together with Mechanism 1 to insert elements into sets which stay more left than the ones indicated by Remark 6.9. In addition this depends on the maxima of these sets. We formalize this situation in Algorithm 1, ($\star$), Case 2.

The proofs of the following lemmas describe how we implicitly deal with the elements of X in Algorithm 1.
Assume the setting of Algorithm 1. We say that an element $x > 0,\ x \in X \cap M$ *respects unfreeness*, if in the central-set of the upper neighbor $\mathfrak{B}$ it stays right of all unfree parts U, such that the rightmost critical element of U (which is simply the rightmost element of U) has smaller absolute value than x.

Lemma 6.11. *Assume the setting of Algorithm 1:*

- *Each $x \in X \cap M$ with $x > 0$ can be placed arbitrarily into the central-set, respecting unfreeness and such that right of it in the central-set there exists a negative element t with $|t| \geq x$.*
- *Each $\overline{x} > 0$ with $\overline{x} \in X \cap M_l$ can be placed into the sets $C_j, \ldots, C_k$ with $\max(C_j) > \overline{x}$ and $\max(C_{j-1}) < \overline{x}$, right to the corresponding maxima.*

Proof. Let

$$L := c_{m,1} \underbrace{\ldots}_{T_1} c_{m,2} \ldots \ldots c_{m,r} \underbrace{\ldots}_{R_1} \quad .$$

Since x and $\overline{x}$ are contained in X it follows from Algorithm 1 that one of the following conditions holds:

- There exists a positive element y staying right of x resp. $\overline{x}$, such that
  (i) y is the maximum of a part in Y
  (ii) y has larger absolute value than x resp. $\overline{x}$
  (iii) y is contained in M.
- There is $x \in \text{freezone}_i$ resp. $\overline{x} \in \text{freezone}_i$.

Therefore either y or the joker can be used together with Mechanism 1 to place x or $\overline{x}$ rightmost in L but left of all elements of M_l which have a larger absolute value than x resp. $\overline{x}$.
Additionally we place these free positive elements with increasing absolute value from the left to the right. In doing so we avoid the appearance of new interdependencies.
It follows from Lemma 6.13 and Lemma 6.14 that the free negative elements are placed in such a way that the positive free elements cause breakups and therefore can be placed as asserted using Mechanism 2. Here Condition 1 guarantees that there exists a set C with $\max(C) > \overline{x}$ left of the central-set.
Clearly there must exist a negative element t with $|t| \geq x$ staying right of x in the central-set for each positive element $x \in X \cap M$ since otherwise there does not exist a good partition of the central-set. $\square$

Example 6.12. Consider the triple

$$|8{-}136\overline{5}\overline{7}\overline{2}| \underbrace{-\overline{4} \ldots\ldots \overline{4}}_{\text{central-set}} |{-}\overline{2}{-}\overline{7}{-}\overline{5}{-}6{-}31{-}8|$$

where an overline on top of a number indicates that the corresponding element is contained in M_l or M_r respectively.
Consider the free element $\overline{4}$ and 3.
We use Mechanism 1 and Mechanism 2 to insert 8 into the central-set and place $\overline{4}$ rightmost in L but left of the elements of M_l with larger absolute value:

$$\nearrow\searrow \ldots \nearrow |{-\overline{2}}{-\overline{7}}{-\overline{5}}{-\overline{4}}{-6}{-31}\ldots 8 \ldots {-8} \ldots {-136}\overline{4572}| \;\; .$$

The corresponding type (3) cell is:

$$\searrow |\overline{72}|{-\overline{5}}{-\overline{4}}{-6}{-31}\ldots 8 \ldots {-8} \ldots {-136}\overline{45}|{-\overline{2}}{-\overline{7}}| \; .$$

We insert $\overline{7}$ and $\overline{2}$ into the sets left of the central-set using Mechanism 2:

$$\nearrow |{-\overline{5}}{-\overline{4}}{-6}{-31}\ldots 8 \ldots {-8} \ldots {-136}\overline{45}| \; .$$

The corresponding type (3) cell is:

$$\searrow |6\overline{45}|{-31}\ldots 8 \ldots {-8} \ldots {-13}|{-\overline{5}}{-\overline{4}}{-6}| \; .$$

We use again Mechanism 1 to place 3 rightwards in L but left of all elements of M_l with larger absolute value. Afterwards we insert 6 into the central-set using Mechanism 2:

$$\ldots \nearrow |{-\overline{2}}{-\overline{7}}{-\overline{5}}{-\overline{4}}{-31}\ldots 8 \ldots {-6} \ldots 6 \ldots {-8} \ldots {-13}\overline{4572}| \; .$$

Here we could have inserted the pair $1, -1$ to the central-set such that -1 stays anywhere right to 6.
We can see now that successively the free positive elements $\overline{7}, \overline{5}$, $\overline{4}$ and 3 brake up and can be placed as asserted.

Lemma 6.13. *Assume the setting of Algorithm 1. Each negative element t of $X \cap M$ can be placed in the following two ways:*

- *Arbitrarily right of any maximum $y \in M$ of a part in Y which stays right of t in L and which has larger absolute value than t.*
- *Arbitrarily right of any $x \in X \cap M$ with larger absolute value than t.*

Proof. Just as in the proof of Lemma 6.11 it follows from $t \in X$ by Algorithm 1 that one of the following conditions holds:

(i) There exists an $y \in M$ to the right of t which is a maximum of a part in Y which can be used together with Mechanism 1 to insert t rightwards into L.

(ii) t is contained in freezone$_i$ for some i. In this case c_i can be used together with Mechanism 1 to insert t rightwards into L.

Thus we can position t to the right of the maximum y. We can also place it right to any positive element x of $X \cap M$ with $x > |t|$ which in turn is placed to the right in L as indicated by Lemma 6.11. □

Lemma 6.14. *Assume the setting of Algorithm 1.*
At the time of insertion in Algorithm 1 each negative element t of $X \cap M_l$ must be inserted into the sets $C_j, \ldots, C_k$ left of the central-set with $\max(C_j) > x$ and $\max(C_{j-1}) < x$, arbitrarily right of the corresponding maxima, for each x with x either

(i) the smallest maximum y of the parts contained in Y which meets Condition 1, which stays right of t in L and which has a larger absolute value than t,

(ii) the smallest positive element of $X \cap M$ with $x > |t|$ and which meets Condition 1,

(iii) the smallest positive element of $X \cap M_l$ with $x > |t|$.

Proof. The first two statements can be proved analogously to Lemma 6.13. The requirement of Condition 1 is obvious.
Thereby observe that for $t \in T_{i+1}$ and $c_{m,i+2} \in M_l, c_{m,i+2} \notin freezone_i$ (case (C)) the external maximum $c_{m,i+2}$ cannot be used to position t even though it is a maximum of a part in Y which stays right of T and with $c_{m,i+2} > |t|$. This follows from $t \in X$. However the assertion is true. This follows from the fact that t is free. Therefore there must exist an x which meets (ii) such that x is smaller than $c_{m,i+2}$, because $c_{m,i+2} \notin freezone_i$.

Since $t \in M_l$ it is allowed to merge t to the right of a positive element $x \in X \cap M_l$ in L. This implies (iii). □

Lemma 6.15. *There cannot be any other positions for the elements of X than the positions indicated by Lemma 6.11, Lemma 6.13 and Lemma 6.14.*

Proof. The statement is clear for $x > 0$.
For $x < 0$ note that there is always needed a positive element of larger absolute value than x in order to position x in the central-set or in the sets left of it. This positive element is the maximum of a fragment of L which contains x.

Since $x \in X$ it follows that x is free. Thus we do not take in account positions right of external maxima which stay left of x in L. The remaining possible positions are included in Lemma 6.13 and Lemma 6.14. $\square$

Proof of correctness of Algorithm 1

Condition 1 - 3 just guarantee the existence of appropriate sets left of the central-set, such that the relevant elements can be inserted into them. These adjustments for an application in Algorithm 2 will become clear in the following discussion.
We do not discuss these conditions further and we confine ourselves to the case of a standard partition with L a left part of a set $C_i \in \{C_1, \ldots, C_{k-1}\}$ such that $\max(L \cup -L) \in L$.

Let $C_i := LR$ be a partition of C_i into a left part L and a right part R as described above and such that $\max(L \cup -L) \in L$. Assume C_i is merged to the other sets of $\mathfrak{A}$ using Mechanism 0 - Mechanism 2 and assume that C_i is the only active set. We have to prove that the standard cells $\mathfrak{B}$ with $\mathfrak{A} <\cdot \mathfrak{B}$ are exactly the ones which can be constructed by Algorithm 1.

- If $c_{m,1} \in M_l$ it follows from Lemma 6.8 that all elements right of $c_{m,1}$ are contained in M_l. L must be merged to the right of $-x$ in C_s, because otherwise the pair $c_{m,1}, -c_{m,1}$ would remain in the central-set. After inserting L to the right half of C_s it breaks up.
  In the corresponding type (3) cell, L is positioned between C_{i-1} and C_{i+1}. Now the elements right of $c_{m,1}$ can be inserted successively into the sets $C_{i+1}, \ldots, C_k$ by an iteration of Mechanism 0.
  It is not possible to insert the elements staying right of $c_{m,1}$ into a set C_j with $j < i$ by Remark 6.9.

  It is clear that F keeps its position in the central-set in case it is contained in M, because all positive elements which are large enough for being used for a breakup are contained in M_l. Since F is the right part of a good partition of the central-set it does not break up by itself.

  Thus, if there exists an element x of F which is contained in M_l this element cannot be separated from F without using Mechanism 1 together with $c_{m,1}$. It suffices to break up from the right all critical elements

of F (i.e. a right part of F) until an element of F becomes an external maximum which can be used to break up x. Since the rightmost element of F is a critical element it follows that $x \in M_l$ implies the existence of a critical element contained in M_l.
If there is a critical element of F contained in M_l the part right of the leftmost critical element f of this kind is contained in M_l and must be merged to the sets $C_{i+1}, \ldots, C_k$ of $\mathfrak{A}$ with $c_{m,1}$. This can be seen as follows: Because f is critical, the only way to insert it into one of the sets $C_1, \ldots, C_k$ is to do a breakup with $c_{m,1}$ to the left of f using Mechanism 1. Thus in the corresponding type (3) cell f stays right of $c_{m,1}$.

- If $c_{m,1} \in M$ we can use $c_{m,1}$ together with Mechanism 1 to merge the elements of F to the positions indicated by the lemmas above. Since $c_{m,1} \in M$, F can be divided arbitrarily into M and M_l. We can in fact merge F in its previous order to the right of $c_{m,1}$ and insert it together with $c_{m,1}$ to the central-set to its previous position using Mechanism 2. The elements of F can be arbitrarily positioned to the right of $c_{m,1}$ in L and are therefore inserted into X.

We proceed with the next part T_{i+1}.

We distinguish 4 cases:

1. $(i+1 = r \ \wedge \ T_{i+1} \not\subset \text{freezone}_i)$
2. $(i+1 \neq r \ \wedge \ T_{i+1} \not\subset \text{freezone}_i \ \wedge \ c_{m,i+2} \in \text{freezone}_i)$
3. $(i+1 \neq r \ \wedge \ c_{m,i+2} \notin \text{freezone}_i)$
4. $(T_{i+1} \subset \text{freezone}_i)$

- The first two cases are included in (A) and lead to $(\star)$.
- If $T_{i+1} \subset \text{freezone}_i$ we can completely rearrange $\text{freezone}_{\text{i}}$ with c_i using Mechanism 1 and we can arbitrarily partition its elements. It follows that c_i equals the joker.

  In Y we have inserted parts P consisting of a positive element y together with an optional unfree part to their right. Each of these positive elements y is maximal in its part and acts as external maxima of the fragments of L in Algorithm 1. They are neither contained in freezone_i

for all i nor positioned right of an external maximum contained in M_l. In other words the maxima of the parts in Y are exactly the elements which are not contained in freezone$_i$ for all i and where the central-sets of the type (4) cells break up.
Inserting parts of Y into the central-set can be done by using Mechanism 2.

We insert into the central-set with decreasing absolute value since the position of a free element x of $X \cap M$ or a part $y \ldots$ in Y depends on the elements with larger absolute value than x resp. y. Of course, this is the natural way of insertion anyway.

While inserting, we have to avoid that the maxima of Y do not cause the central-set to break up again. Therefore there must exist a negative element to their right in the central-set with larger absolute value.

We must also ensure that the unfree parts which we have already merged to the central-set keep being unfree.

Last we have to guarantee that it is possible at all, to merge the remaining elements to the central-set without getting into conflict with unfreeness.
We do this by the restrictions given in the algorithm. As already mentioned, from an inductive argument it follows that we can insert the parts of Y in such a way.

The elements of X can be placed according to the lemmas above.

- Let $(i + 1 \neq r \wedge c_{m,i+2} \notin \text{freezone}_i)$.

  If T^1_{i+1} is unfree there is either $c_{m,i+2} \in M$ or $c_{m,i+2} \in M_l$:

  If $c_{m,i+2} \in M$ we just insert $c_{m,1}$ together with an unfree left part of T_{i+1} into Y. Implicitly we insert this part into the central-set using Mechanism 2.
  The remaining right part of T_{i+1} can be partitioned arbitrarily and merged to the right of $c_{m,i+2}$ using Mechanism 1 together with a fragment $|c_{m,i+2} \ldots|$, as indicated by the lemmas above. Thus we insert its elements into X.

  If $c_{m,i+2} \in M_l$ it follows from Lemma 6.8 that all elements to its right are contained in M_l, because it is not contained in freezone$_i$. The remaining elements are partitioned by starting from 2. again.

If T^1_{i+1} is not unfree there must exist an element $g \in T^1_{i+1}$ with $g \in M_l$. This implies the existence of a critical element of T^1_{i+1} being contained in M_l. This follows from the fact that $c_{m,i+2} \notin \text{freezone}_i$ which implies that there does not exist a free positive element which is contained in M, which can be used to manipulate the order of T^1_{i+1} using Mechanism 1.

Therefore we can either

(a) use a breakup with $c_{m,i+1}$ to the left of g, in order to insert g into the sets left of the central-set. This implies that all elements to the right of g are contained in M_l.

(b) Or we can use a positive free element of $X \cap M_l$ which is larger than c_i in order to break up a right part of T^1_{i+1}.

In both cases the rightmost element f of T^1_{i+1} is a critical element with larger absolute value than c_i which is contained in M_l.
It follows from Lemma 6.10 that all elements right of T^1_{i+1} are contained in M_l. As mentioned in Algorithm 1 we can proceed as described in Case 2 of $(\star)$.

- It remains to prove $(\star)$:

  We have the situation

$$c_{m,i+1} \underbrace{\dots f}_{T} \underbrace{\dots}_{\text{freezone}_i}$$

with f a critical element of T, since for the case ($i+1 = r$ and $T_r \not\subset \text{freezone}_i$), as well as for the case ($i+1 \neq r$, $c_{m,i+2} \in \text{freezone}_i$, $T_{i+1} \not\subset \text{freezone}_i$) there must exist a critical element which has a larger absolute value than c_i.

If $T \subset M$ it follows that T is unfree since there is no positive element contained in $X \cap M$, which is large enough to split T, i.e. to cause a breakup in between the elements of T. A left part of freezone_i may be unfree too.
The remaining elements of freezone_i can be partitioned arbitrarily. Therefore we insert them into X.

In Case 2 of $(\star)$ it follows that all elements right of the part G are contained in M_l since there is no positive element contained in $X \cap M$ which is large enough to split the part T.
The f_i cannot cause breakups since they are critical.
Thus we can either merge a positive free element of $X \cap M_l$ to T which is large enough to cause a breakup. Or we use $c_{m,i+1}$ to insert the critical elements into the sets left of C_s.

We consider the largest positive element of $X \cap M_l$ which is smaller than $c_{m,i+1}$, since larger elements of $X \cap M_l$ must have been inserted to the sets left of the central-set already.

If $\tilde{c}_i < |f_j|$ the first option, of using a positive free element contained in $X \cap M_l$ for a breakup, is not possible.

If $\tilde{c}_i > |f_j|$ an increasing sequence of elements $y_1, \dots, y_h$ of $X \cap M_l$ is defined, such that we can break up the parts between the critical elements f_j and f_n using these elements.
$y_1, \dots, y_h$ are smaller or equal to $\tilde{c}_i$. Each $y_l, l \in \{1, \dots, h\}$ is merged to the left of a corresponding part of critical elements which have smaller absolute values than y_l using Mechanism 1.
For the right part $f_\omega \dots f_n$ of T there exists no such element and therefore it must be merged to the sets left of the central-set with $c_{m,i+1}$.
The y_l, $l = 1, \dots, h$ are chosen minimal such that successively from the right to the left, parts containing critical elements of smaller absolute values break up with them. This guarantees that in the corresponding type (3) cell the fragment L is positioned leftmost.

The elements $y_1, \dots y_h$ have no effect to further iterations since they are larger than c_i and c_i is already larger than all negative elements of G.
It is clear that the part $\dots f_1 \dots f_{j-1}$ is unfree.

$\square$

6.2.2 Algorithm 2

Let $\mathfrak{A} := (C_1, \dots, C_k, C_s, -C_k, \dots, -C_1)$ be a standard symmetric fully ordered set-partition with optional central-set C_s, $s = k + 1$.

In this section we examine the case $\max(L \cup -L) \in -L$, where L is a left part of the set $C_i \in \{C_1, \dots, C_k\}$.
In contrast to the last case in

$$-L := \underbrace{\dots}_{T_0} c_{m,1} \underbrace{\dots}_{T_1} c_{m,2} \dots\dots c_{m,r} \underbrace{\dots}_{R_1}$$

there exists a part T_0. The elements of T_0 and $-T_0$ can change sides via Mechanism 0 what makes this case more complicated. On the other hand from Remark 5.23 it follows that several of the case differentiations of Algorithm 1 can be skipped.

Algorithm 2

We start with the triple $(L, C_s, -L)$ in the following form:
Let

$$C_s := |\underbrace{\dots}_{-F} x \underbrace{\dots}_{-M} \underbrace{\dots}_{M} -x \underbrace{\dots}_{F}| \, ,$$

and

$$-L := \underbrace{\dots}_{T_0} c_{m,1} \underbrace{\dots}_{T_1} c_{m,2} \dots\dots c_{m,r} \underbrace{\dots}_{T_r} \, ,$$

where $c_{m,1}, \dots, c_{m,r}$ are the external maxima of $-L$ and x is the first positive element from the left in C_s which is larger than $c_{m,1}$.
It follows that $\max(L \cup -L) = c_{m,1}$.
Again, although we remove or insert elements we keep referring to the sets and fragments of $\mathfrak{A}$ and L as $C_1, \dots, C_k$, C_s and L, respectively.

Partition T_0 into $T_0 := T_0^1 T_0^2 T_0^3$, such that Mechanism 0 can be applied in order to insert T_0^2 into the central-set and change the signs of T_0^1. Thus, we have the following sequence:

$$\begin{aligned}
&|\dots\dots - c_{m,1} \underbrace{\dots}_{-T_0^3} \underbrace{\dots}_{-T_0^2} \underbrace{\dots}_{-T_0^1} | \underbrace{\dots\dots}_{C_s} | \underbrace{\dots}_{T_0^1} \underbrace{\dots}_{T_0^2} \underbrace{\dots}_{T_0^3} c_{m,1} \dots\dots | \\
&\nearrow |\dots\dots - c_{m,1} \underbrace{\dots}_{-\overline{T}} \underbrace{\dots\dots}_{\overline{C}_s} \underbrace{\dots}_{\overline{T}} c_{m,1} \dots\dots | \\
&\searrow |c_{m,1} \dots\dots | \underbrace{\dots}_{-\overline{T}} \underbrace{\dots\dots}_{\overline{C}_s} \underbrace{\dots}_{\overline{T}} | \dots\dots - c_{m,1}|.
\end{aligned}$$

Here $\overline{C}_s$ results from C_s by merging T_0^2 and $-T_0^2$ to C_s while keeping their orders. $\overline{T}$ results from T_0^3 by inserting the elements of $-T_0^1$ into it while keeping their relative positions.

Now we are in position to apply Algorithm 1 to the resulting type (3) cell, i.e. we start Algorithm 1 with with L a left part of $|c_{m,1}\ldots\ldots|$.

It is possible to give a formal decision-process similar to Algorithm 1. Despite of little changes in the settings compared to Algorithm 1, this turns out to be rather complicated. The reason for this is the implication that if a sign of one element is changed using Mechanism 0, all elements to its left in T_0 have to change signs too. Thus Algorithm 2 as it is formulated above seems to be a reasonable way to deal with the case $\max(L \cup -L) \in -L$. We give some specifications below.

Even though we have partitioned T_0 into a triple of sets $T_0^1 T_0^2 T_0^3$ there is no need for the part T_0^2, i.e. the elements of T_0^2 can always be inserted into T_0^1 or T_0^3.

Lemma 6.16. *Let* $\max(L \cup -L) \in -L$ *and let*

$$-L := \underbrace{\ldots}_{T_0} c_{m,1} \underbrace{\ldots}_{T_1} c_{m,2} \ldots\ldots c_{m,r} \underbrace{\ldots}_{T_r} \quad .$$

Let $T_0 := T_0^1 T_0^2 T_0^3$ *be the partition of* T_0 *as introduced above.*
For the construction of a standard cell $\mathfrak{B}$ *with* $\mathfrak{A} <\cdot\, \mathfrak{B}$ *it suffices to partition* T_0 *in such a way, that* T_0^2 *is empty.*

Proof. Assume $T_0^2 := q_1 \ldots\ldots q_2$ is merged to the central-set using Mechanism 0.

$\triangleright$ In case $c_{m,1} \in M$ the preceding sequence shows that the positive elements of $-T_0^1$ and T_0^3 can be considered as free elements. Each negative element of the two parts can be merged to the central-set according to Lemma 6.13. Assume we start Algorithm 1 with the configuration $|c_{m,1}\ldots|$ and $(-T_0^1)T_0^2T_0^3$ contained in F. If all elements of T_0^2 stay in positions according to Lemma 6.11 and Lemma 6.13 with respect to Algorithm 1 and this configuration, there is no need for the part T_0^2. Clearly we can position all positive and possibly some of the negative elements to their desired position. It remains to consider the negative elements which cannot be positioned as desired, when we start with this configuration. Roughly speaking, with respect to this configuration, in the central-set of $\mathfrak{B}$, each of these negative elements t of T_0^2 stays left of all positive free elements with larger absolute value than t and left of all maxima of the parts contained in Y, having a larger absolute value than t. If t stays in such a position we say that t is in *l-position* with respect to T_0^1.

Let $q < 0$ be the first element from the right in T_0^2 which stays in l-position. We include the left part $P := \ldots q$ of T_0^2 in T_0^1. Thus the negative part $-P = -q \ldots$ is positioned of $-x$ in the central-set. Since q stays in l-position it follows that all positive elements of P have smaller absolute value than q. It follows that P has the form

$$P = \ldots q^1 \ldots q^2 \ldots \ldots q^m \ldots q$$

where $q^1, \ldots, q^m, q$ are the negative elements of P in l-position.
If we include $-P$ in $-T_0^1$, the negatives of the q^i become free positive elements. Each of the q^i has no positive element to its left in P which has a larger absolute value than q^i. It follows by symmetry that in $-P$ there exists no negative element which has no positive element of larger absolute value to its left in $-P$.
Therefore, we can successively insert the part $-P$ at its desired position at the point when all free elements are merged to the central-set in Algorithm 1. Due to symmetry the elements of P have exactly the same positions compared to an insertion with T_0^2 at the beginning.
By induction the same is true for the remaining right part of T_0^2.

$\triangleright$ In case $c_{m,1} \in M_l$, let q be the element of T_0^2 with maximal absolute value.

- If $q > 0$ leave the right part $q \ldots q_2$ of T_0^2 in T_0^3 and replace the remaining left part $P := q_1 \ldots$ by $-P = \ldots - q_1$ by inserting it into T_0^1. It follows that we can position q, such that it causes a breakup and the right part $q \ldots q_2$ can be merged to the central-set as desired.

  We can either use Mechanism 0 to merge $-P$ directly to the right of the part $q \ldots$ or we can use Mechanism 0 to place $-P$ to the left of q. In the latter case Mechanism 1 can be used together with q to place the elements of $-P$ to the right of the part $q \ldots$.

  We conclude that we can merge the whole part $q \ldots q_2 \ldots -q_1$ to the central-set, such that due to symmetry the part T_0^2 is positioned at its desired position.

- If $q < 0$ we replace the left part $q_1 \ldots q$ by $-q \ldots - q_1$.
  As above, it follows that $-q$ can be positioned according to Lemma 6.11. We can merge the part $-q \ldots - q_1$ and due to symmetry the part $q_1 \ldots q$ into the central-set as desired.
  Inductively the same follows for the remaining right part.

$\square$

Compared to Algorithm 1, we modify the procedure slightly. For the case $c_{m.1} \in M$ we insert the elements of $-L$ and F with respect to the current configuration of T_0. Thus, we encode the current configuration of T_0 by a set T. At the beginning we set $T := T_0$. We insert the elements of $-L \cup F$ with decreasing absolute value according to T and to Algorithm 1.
When an element t of T_0 is inserted, we can decide whether we want to execute a sign change before or not. Afterwards we insert it according to Algorithm 1.
It follows that the position of t can be achieved either only after a sign change, only without sign change or the sign has no influence on the position of t.
If t can only be positioned with a prior sign change, we delete the left part $P := \ldots t$ from T and insert the elements of $-P$ into X, where X denotes the set, as it was used in Algorithm 1.
If the position of t can only be achieved without a sign change, we delete the right part $P := t \ldots$ from T and insert its elements into X.
If the position of t can be achieved in both ways, i.e. with or without a sign change, we leave it in T.
As a consequence of this construction, we also consider the positives of the negative elements in T as potential candidates in order to position negative elements according to Lemma 6.13 and Lemma 6.14. Clearly, this eventually leads to a new configuration of T.
The difficulty is to deal with the case of an element t that is contained in T. If t is supposed to be positioned with an element p of $T \cup -T$, we have to pay attention to the interdependencies of the sign changes. The simplest way to solve this problem is just to decide whether one wants to insert t with or without sign change. A more complicated way, that we do not discuss further, is to decide on the position of t first and record the interdependencies of the sign changes afterwards. For a simple example, the insertion-process is demonstrated in Example 6.19. A simple case differentiation can be used to deal with the case $c_{m,1} \in M_l$.

6.2.3 Generalizations

In general, there may exist more than one active set. It is not difficult to adapt Algorithm 1 and Algorithm 2, such that they can be applied in the general case.
Mechanism 3 can be seen as a start of Algorithm 1 or Algorithm 2, with the precondition of eliminating the type (3) pair. In principle, every time we insert an element $x \in M_l$ into the sets left of the central-set, we may use

Mechanism 3 in order to involve another set. Thus we can change Condition 1 of Algorithm 1 in the following way:

▷ There exists a set C_j, $j \in \{1, \dots, i-1, i+1, \dots, k\}$ such that
$\max\{|c| \mid c \in C_j\} > x$.

Rather than going into detail here, we refer to the examples at the end of the next section.

Let $\mathfrak{A} := (C_1, \dots, C_k, C_s, -C_k, \dots, -C_1)$ be a cell of $\Gamma_{\mathcal{B}_n}$ with optional central-set C_s, $s = k+1$. Assume we want to merge the set C_i to the other sets.
Up to now we have not examined the right part R of C_i which is merged to the sets $C_{i+1}, \dots, C_k$ before the corresponding left part L is merged to the central-set C_s. We close this gap now.
Let

$$C_i := |m \dots -m_1 \dots -m_2 \dots\dots -m_l \dots|$$

with $-m_1, \dots, -m_l$ the negative elements with larger absolute value than m. If all elements m_i are contained in M_r it follows that the right part $-m_1 \dots -m_l \dots$ is contained in R. We set L to be equal to the remaining left part $m \dots$ in this case.

For t the leftmost element of $-C_i$ which is contained in $M \cup M_l$ it follows that the right part $t \dots - m_j \dots - m$ is contained in $-L$.
For the corresponding left part of $-C_i$ there is a maximal right part Q containing only elements m_i which are smaller than $m_j, \dots, m_1, m$. It follows that the part left of Q is contained in $-R$.
When we merge Q together with L to the central-set it follows from Algorithm 2 that we do a sign change $Q \to -Q \subset M_l$.
Depending on the partition $L = M \cup M_l \cup M_r$ and the positions of the elements of the rest of L we can decide whether a left part of Q remains in $-R$ or not.
That is, when we insert the elements of L with decreasing absolute value according to Lemma 6.11, Lemma 6.13 and Lemma 6.14 we can decide whether we insert the elements of Q with L or position them according to their positions in an additional right part of $-R$.

6.3 Examples

In this section we give some short examples in order to demonstrate the methods of the last sections.

Example 6.17.

Let $n = 4$.
Consider the 3-cell

$$\mathfrak{A} := |1{-}3{-}2|4{-}4|23{-}1| \quad .$$

Let $1, 2, 3 \in M$. The only external maximum is 3.
Using Algorithm 1, we can determine all upper neighbors of $\mathfrak{A}$ with this setting:

-1 unfree:

$|43{-}12{-}21{-}3{-}4|$, $|43{-}1{-}221{-}3{-}4|$, $|43{-}2{-}112{-}3{-}4|$, $|4{-}23{-}11{-}32{-}4|$
$|43{-}21{-}12{-}3{-}4|$, $|4{-}231{-}1{-}32{-}4|$
$|4{-}213{-}3{-}12{-}4|$

$|4{-}21{-}33{-}124|$

$|3{-}14{-}41{-}3|$, $|3{-}124{-}4-21{-}3|$, $|3{-}142{-}2{-}41{-}3|$, $|3{-}14{-}22{-}41{-}3|$,
$|3{-}1{-}24{-}421{-}3|$, $|3{-}2{-}14{-}412{-}3|$

-1 free:

$|2{-}134{-}4{-}31{-}2|$, $|23{-}14{-}4{-}13{-}2|$, $|234{-}11{-}4{-}3{-}2|$, $|2341{-}1{-}4{-}3{-}2|$,
$|2314{-}4{-}1{-}3{-}2|$, $|2134{-}4{-}3{-}1{-}2|$, $|1234{-}4{-}3{-}2{-}1|$
$|32{-}14{-}41{-}2{-}3|$, $|324{-}11{-}4{-}2{-}3|$, $|3241{-}1{-}4{-}2{-}3|$, $|3214{-}4{-}1{-}2{-}3|$,
$|3124{-}4{-}2{-}1{-}3|$, $|1324{-}4{-}2{-}3{-}1|$
$|342{-}11{-}2{-}4{-}3|$, $|3421{-}1{-}2{-}4{-}3|$, $|3412{-}2{-}1{-}4{-}3|$, $|3142{-}2{-}4{-}1{-}3|$,
$|1342{-}2{-}4{-}3{-}1|$

$|2{-}143{-}3{-}41{-}2|$, $|24{-}13{-}31{-}4{-}2|$, $|243{-}11{-}3{-}4{-}2|$, $|2431{-}1{-}3{-}4{-}2|$,
$|2413{-}3{-}1{-}4{-}2|$, $|2143{-}3{-}4{-}1{-}2|$, $|1243{-}3{-}4{-}2{-}1|$
$|42{-}13{-}31{-}2{-}4|$, $|423{-}11{-}3{-}2{-}4|$, $|4231{-}1{-}3{-}2{-}4|$, $|4213{-}3{-}1{-}2{-}4|$,
$|4123{-}3{-}2{-}1{-}4|$, $|1423{-}3{-}2{-}4{-}1|$
$|432{-}11{-}2{-}3{-}4|$, $|4321{-}1{-}2{-}3{-}4|$, $|4312{-}2{-}1{-}3{-}4|$, $|4132{-}2{-}3{-}1{-}4|$,
$|1432{-}2{-}3{-}4{-}1|$
$|431{-}22{-}1{-}3{-}4|$, $|413{-}22{-}3{-}1{-}4|$, $|143{-}22{-}3{-}4{-}1|$

$|41{-}23{-}32{-}1{-}4|$, $|14{-}23{-}32{-}4{-}1|$

$|2{-}14{-}33{-}41{-}2|$, $|24{-}1{-}331{-}4{-}2|$, $|24{-}3{-}113{-}4{-}2|$, $|24{-}31{-}13{-}4{-}2|$, $|241{-}33{-}1{-}4{-}2|$, $|214{-}33{-}4{-}1{-}2|$, $|124{-}33{-}4{-}2{-}1|$
$|42{-}1{-}331{-}2{-}4|$, $|42{-}3{-}113{-}2{-}4|$, $|42{-}31{-}13{-}2{-}4|$, $|421{-}33{-}1{-}2{-}4|$, $|412{-}33{-}2{-}1{-}4|$, $|142{-}33{-}2{-}4{-}1|$
$|4{-}32{-}11{-}23{-}4|$, $|4{-}321{-}1{-}23{-}4|$, $|4-312{-}2{-}13{-}4|$, $|41{-}32{-}23{-}1{-}4|$, $|14{-}32{-}23{-}4{-}1|$
$|4{-}31{-}22{-}13{-}4|$, $|41{-}3{-}223{-}1{-}4|$, $|14{-}3{-}223{-}4{-}1|$
$|41{-}2{-}332{-}1{-}4|$, $|14{-}2{-}332{-}4{-}1|$

Example 6.18.

Consider the cell

$$\mathfrak{A} := |6{-}75{-}4{-}8{-}32|19{-}9{-}1|{-}2384{-}57{-}6|\ .$$

The cell

$$|618{-}592{-}37{-}44{-}73{-}2{-}95{-}8{-}1{-}6|$$

is not an upper neighbor of $\mathfrak{A}$ since 4 must be positioned with the unfree part $4{-}5$ to the right of 8.

Checking with Algorithm 2, we find that

$$|247{-}5{-}619{-}8{-}338{-}9{-}165{-}7{-}4{-}2|$$

is an upper neighbor of $\mathfrak{A}$. The only unfree element is -6.

Example 6.19.

This example demonstrates Algorithm 2.
Let $-L := 3{-}6{-}7491{-}5{-}8$ be a right part of a set $-A_i$ of a standard cell $\mathfrak{A} := (A_1, \dots, A_k, A_s, -A_k, \dots, -A_1)$ that we want to merge to the central-set $A_s := |2{-}2|$. Let $T := 3{-}6{-}74$ encode the configuration of T_0.
Let $9 \in M$.
Insertion of 9: $|9{-}9|$.
Let $-8 \in M_l$ (We assume that this is possible).
Insertion of -8 to the sets left of the central-set with 9.
Let $-7 \in M$.

Insertion of -7 with 9: $|97{-}7{-}9|$. The position is allowed with or without sign change of -7.
Let $-6 \in M$.
Insertion of -6: $|697{-}7{-}9{-}6|$. This position is allowed with or without sign change of -6.
Let $-5 \in M$.
Insertion of -5 as a free element with 6: $|6{-}597{-}7{-}95{-}6|$. Therefore we do a sign change of -6 to 6 what implies a sign change of 3 to -3. Therefore we have $T := -74$.
Let $4 \in M_r$.
Insertion of -4 left of the central-set.
This sign change is allowed since the position of -7 can also be achieved with 8. It follows that T is empty.
Let $-3 \in M$.
Insertion of -3 with 6: $|6{-}5{-}397{-}7{-}935{-}6|$
Let $1 \in M$.
Insertion of 1: $|16{-}5{-}397{-}7{-}935{-}6{-}1|$.

The following examples demonstrate the procedure in the case of more than one active set.

Example 6.20.

Let $\mathfrak{A} := |1{-}6|7{-}3{-}9{-}42|5{-}5|{-}2493{-}7|6{-}1|$.
We want to create the partition of a cell $\mathfrak{B}$ with $\dim(\mathfrak{B}) = \dim(\mathfrak{A}) + 1$.
Let $L := 7{-}3{-}9{-}42$. The only external maximum of $-L$ is 9. Let $T := -24$.
Let $9 \in M$.
Insertion of 9: $|1{-}6|9{-}9|6{-}1|$.
Let $-7 \in M$. -7 is unfree because there is no free positive element with larger absolute value. It follows that 3 is unfree.
Insertion of -7 together with unfree part: $|1{-}6|7{-}39{-}93{-}7|6{-}1|$.
Let $-5 \in M$.
Insertion of -5 with 9: $|1{-}6|7{-}39{-}55{-}93{-}7|6{-}1|$.
Let $4 \in M_l$. This is possible because of the existence of $|1{-}6|$. $T := -2$.
Insertion of 4 using Mechanism 3: $|4{-}6|7{-}39{-}55{-}93{-}7|6{-}4|$.
New left part is 1.
Let $-2 \in M$.
Insertion of -2 after sign change: $|1{-}6|7{-}2{-}359{-}9 - 532{-}7|6{-}1|$. T is empty.
Insertion of 1: $|4{-}6|7{-}2{-}359{-}11{-}9 - 532{-}7|6{-}4|$.

Example 6.21.

Let $\mathfrak{A} := |{-2}{-3}|1{-7}{-9}|6{-8}45|{-5}{-4}8{-6}|97{-1}|32|$ with all sets active.
We want to create an upper neighbor $\mathfrak{B}$ with $\dim(\mathfrak{B}) = \dim(\mathfrak{A}) + 1$.
There must be one set to start with. Let us start with $|6{-8}45|$.
The only external maximum is 8. $8 \in M_l$ is allowed, although 8 does not meet Condition 1, because of the existence of -9 within the set $|1{-7}{-9}|$.
We set $8 \in M$.
Insertion of 8: $|8{-8}|$.
Let $-6 \in M$, $-5 \in M_r$. $-5 \in M_r$ requires the existence of $|1{-7}{-9}|$.
Insertion of -6: $|68{-8}{-6}|$.
Insertion of -5 with prior sign change and Mechanism 3: $|5{-9}|68{-8}{-6}|9{-5}|$.
New left part is $1{-7}$.
Let $7 \in M$.
Insertion of 7: $|5{-9}|68{-7}7{-8}{-6}|9{-5}|$.
Let $-4 \in M_l$.
Insertion of -4 with 5: $|5{-9}{-4}|68{-7}7{-8}{-6}|49{-5}|$.
Let $1 \in M_l$.
Insertion of 1 with Mechanism 3: $|1{-3}|5{-9}{-4}|68{-7}7{-8}{-6}|49{-5}|3{-1}|$.
New left part is -2.
Let $-2 \in M_l$.
Insertion of -2: $|1{-3}{-2}|5{-9}{-4}|68{-7}7{-8}{-6}|49{-5}|23{-1}|$.

7 Deutsche Zusammenfassung

Ein Arrangement von Hyperebenen $\mathcal{A}$ (kurz: Arrangement) besteht aus einer endlichen Menge von linearen Unterräumen der Kodimension 1 in einem endlichdimensionalen Vektorraum. Jede dieser Hyperebenen H ist der Kern einer linearen Abbildung α_H, welche bis auf Konstanten eindeutig ist.
Betrachtet man ein Arrangement im $\mathbb{R}^n$, so ergeben sich natürliche Fragestellungen. Schon im späten 19ten Jahrhundert stellte man sich die Frage nach der Anzahl der Kammern, die durch n Geraden in der Ebene entstehen. Die eigentliche Untersuchung von komplexen Hyperebenen Arrangements begann in den späten 1960ern Jahren durch wegweisende Arbeiten von Arnold und Brieskorn.
Obwohl die Objekte leicht zu definieren sind, führen sie dennoch zu tiefen Resultaten. Zudem stellt das Gebiet eine interessante Schnittstelle verschiedener mathematischer Disziplinen, wie etwa Algebra, algebraischer Geometrie, Topologie und Kombinatorik dar.
In dieser Arbeit untersuchen wir kombinatorische Eigenschaften von Komplementen klassischer Arrangements.

$\mathcal{A}_{n-1}^{\mathbb{R}}$ bezeichnet das Braid-Arrangement im $\mathbb{R}^n$. Es besteht aus den Hyperebenen $H_{i,j} := \{x \in \mathbb{R}^n \mid x_i = x_j\}$ für $1 \leq i < j \leq n$.

$\mathcal{B}_n^{\mathbb{R}}$ bezeichnet das Arrangement im $\mathbb{R}^n$, dass zusätzlich zu den Hyperebenen $H_{i,j}$ des Braid-Arrangements, die Hyperebenen $H_{i,-j} := \{x \in \mathbb{R}^n \mid x_i = -x_j\}$, $1 \leq i < j \leq n$, und die Koordinaten-Hyperebenen $H_i := \{x \in \mathbb{R}^n \mid x_i = 0\}$, $i = 1, \ldots, n$, enthält.

Als Komplexifizierung eines reellen Arrangments im $\mathbb{R}^n$ versteht man das Arrangement im $\mathbb{C}^n$, welches durch dieselben definierenden Linearformen gegeben ist.
Wir verzichten auf den Index $\mathbb{C}$ und bezeichnen mit $\mathcal{A}_{n-1}$ bzw. $\mathcal{B}_n$ die Komplexifizierungen der beiden Arrangements, die wir oben definiert haben. Die Notation ist in Anlehnung an die zugehörigen Spiegelungsgruppen A_{n-1} und B_n gewählt.
Für ein Hyperebenen Arrangement $\mathcal{A}$ definieren wir das Komplement $M(\mathcal{A})$ von $\mathcal{A}$ als das Komplement der Vereinigung aller Hyperebenen in $\mathcal{A}$. In dieser Arbeit untersuchen wir die Komplemente $M(\mathcal{A}_{n-1}) \subset \mathbb{C}^n$ und $M(\mathcal{B}_n) \subset \mathbb{C}^n$ der Komplexifizierungen der beiden obigen reellen Arrangements.
Die topologischen Eigenschaften solcher Komplemente werden seit den frühen 1970er Jahren untersucht. P. Deligne zeigte 1972, dass das Komplement eines komplexifizierten Arrangements $K(\pi, 1)$ ist, falls die Regionen, die durch die

Hyperebenen im $\mathbb{R}^n$ entstehen simpliziale Kegel sind [7].
Wichtig für diese Arbeit ist die Tatsache, dass das Komplement eines komplexifizierten reellen Arrangements homotopie-äquivalent zu einem regulären CW-Komplex ist. Dies wurde 1987 von M. Salvetti gezeigt [18]. Die Gruppen $H^i(X^i, X^{i-1})$ des zellulären Ko-Kettenkomplexes eines CW-Komplexes X sind frei abelsch, mit den i-Zellen von X als Basis. Daher nennen wir einen CW-Komplex *minimal*, falls die Anzahl der i-Zellen genau dem Rang der Gruppe $H^i(X, \mathbb{Q})$ entspricht.
Ausgehend von regulären CW-Komplexen, wie sie Salvetti beschreibt, konstruieren wir minimale Komplexe $\Gamma_{\mathcal{A}_{n-1}}$ und $\Gamma_{\mathcal{B}_n}$ für die Komplemente $M(\mathcal{A}_{n-1})$ und $M(\mathcal{B}_n)$. Dass heißt, wir konstruieren jeweils einen CW-Komplex, der homotopie-äquivalent zu $M(\mathcal{A}_{n-1})$ bzw. $M(\mathcal{B}_n)$ ist und der eine minimale Anzahl von Zellen besitzt.
Hierzu benutzen wir die Methoden der diskreten Morse Theorie. Diese wurde in den späten 1990er Jahren von R. Forman entwickelt [8]. Mit diesen Methoden kann man die Anzahl der Zellen eines regulären CW-Komplexes verkleinern, ohne seinen Homotopie-Typ zu verändern.
Parallel zu unserer Arbeit wurde ein allgemeiner Ansatz untersucht, wie man CW-Komplexe mit Hilfe diskreter Morse Theorie findet, welche homotopie-äquivalent zum Komplement eines gegebenen Arrangements sind [19]. Unser Ansatz unterscheidet sich von jenem und führt, im Falle unserer Beispiele, zu detailierteren Resultaten.

Es ist bekannt, dass die Erzeuger der Kohomologie-Gruppen der Komplemente $M(\mathcal{A}_{n-1})$ und $M(\mathcal{B}_{n-1})$ den Elementen der zugehörigen Spiegelungsgruppen S_n und S_n^B entsprechen [1]. Hierbei ist S_n die symmetrische Gruppe und S_n^B die Gruppe der *signed permutations*. Diese besteht aus den Permuationen der Menge $[\pm n] := \{1, \ldots, n, -n, \ldots, -1\}$, so dass $\omega(-a) = -\omega(a)$ für alle $a \in [\pm n]$.
Tatsächlich ist die Anzahl der Zellen der minimalen Komplexe $\Gamma_{\mathcal{A}_{n-1}}$ und $\Gamma_{\mathcal{B}_n}$ gleich der Anzahl der Elemente der Gruppen S_n bzw. S_n^B.
Die Zell-Ordnung eines CW-Komplexes X ist definiert durch die Ordnung auf den Zellen von X, mit $\sigma \leq \tau$ für zwei Zellen σ und τ von X, genau dann, wenn der Abschluss von σ im Abschluss von τ enthalten ist. Die partiell geordnete Menge der auf diese Weise geordneten Zelle von X heißt *face poset* von X.
Ein großer Teil der Arbeit befasst sich mit der Zell-Ordnung der beiden minimalen Komplexe.
Im Falle des Komplexes $\Gamma_{\mathcal{A}_{n-1}}$ lässt sich eine prägnante Beschreibung der Ordnung herleiten.
Die Zellordnung des Komplexes $\Gamma_{\mathcal{B}_n}$ scheint zu kompliziert, um ebenso präg-

nant beschrieben zu werden. Daher verfolgen wir den Ansatz, diese mit Hilfe bestimmter Mechanismen zu beschreiben, welche auf die Zellen des Komplexes angewendet werden können, um neue Zellen zu erzeugen.

Diese Arbeit ist wie folgt gegliedert:

In Kapitel 2 beschäftigen wir uns mit dem mathematischen Hintergrund. Zu Beginn definieren wir die Spiegelungsgruppen A_{n-1} und B_n. Danach präsentieren wir die wichtigsten Definitionen zu CW-Komplexen. Für einen tieferen Einblick in die Materie verweisen wir auf [13]. Nach einer kurzen Einführung zu Hyperebenen Arrangements beschreiben wir kurz den von Salvetti eingeführten Komplex. Wir halten uns dabei an die Arbeit von Björner und Ziegler [4].

In Kapitel 3 führen wir diskrete Morse Theorie ein. Nach Formans Original-Ansatz präsentieren wir eine Formulierung, die auf azyklischen Matchings basiert, und für uns praktischer ist. In der Tat wird die Suche nach geeigneten Matchings einen großen Teil der Arbeit ausmachen.

Wir leiten einen minimalen Komplex für $M(\mathcal{A}_{n-1})$ in Kapitel 4 her. Wir entwickeln eine Darstellung der Zellen des Initial-Komplexes, basierend auf Partitionen der Menge $[n]$ und formulieren die Zell-Ordnung entsprechend um. Danach minimieren wir die Anzahl der Zellen, in dem wir ein Matching definieren und diskrete Morse Theorie anwenden.
Der resultierende minimale Komplex $\Gamma_{\mathcal{A}_{n-1}}$ besitzt soviele Zellen, wie Elemente der symmetrischen Gruppe S_n. Am Ende des Kapitels untersuchen wir die Zell-Ordnung dieses Komplexes und beschreiben diese. Danach präsentieren wir die Beispiele $\Gamma_{\mathcal{A}_2}$ und $\Gamma_{\mathcal{A}_3}$.

In Kapitel 5 konstruieren wir den minimalen Komplex $\Gamma_{\mathcal{B}_n}$ zu $M(\mathcal{B}_n)$. Im Vergleich zu Kapitel 4 erweist sich dies als ungleich komplizierter. Wir entwickeln eine Darstellung der Zellen basierend auf symmetrischen Partitionen der Menge $[\pm n]$ und formuliern die Zell-Ordnung entsprechend. Danach reduzieren wir die Anzahl der Zellen des Komplexes zweimal, indem wir geeignete Matchings definieren. Hierbei muss darauf geachtet werden, dass nach Entfernen der Zellen des ersten Matchings die Methodik noch anwendbar ist.
Der minimale CW-Komplex $\Gamma_{\mathcal{B}_n}$ besitzt soviele Zellen wie Elemente in der Gruppe S_n^B. Im weiteren Verlauf von Kapitel 5 untersuchen wir die Zell-Ordnung dieses Komplexes.
Wir geben ein Gegenbeispiel an, das zeigt, dass im Gegensatz zum Kom-

plex $\Gamma_{\mathcal{A}_{n-1}}$, die Relationen $\mathfrak{A} < \mathfrak{B}$ von Zellen von $\Gamma_{\mathcal{B}_n}$ im Allgemeinen keine Darstellung als Ketten von *facets*

$$\mathfrak{A} <\!\cdot\ \mathfrak{A}_1 <\!\cdot\ \cdots\cdots <\!\cdot\ \mathfrak{A}_m <\!\cdot\ \mathfrak{B}$$

besitzt.
Aufgrund der Komplexität entwickeln wir eine Beschreibung, die auf Mechanismen basiert, welche die Partitionen manipulieren, welche zu den Zellen gehören. Daher wird der Großteil des Kapitels benötigt, um die Struktur der face poset von $\Gamma_{\mathcal{B}_n}$ in diese Mechanismen zu übersetzen.

Im darauffolgenden Kapitel diskutieren wir die Relationen $\mathfrak{A} <\!\cdot\ \mathfrak{B}$, d.h. $\mathfrak{A} < \mathfrak{B}$ und $\dim(\mathfrak{B}) = \dim(\mathfrak{A}) + 1$, im Detail. Wir präsentieren eine Beschreibung aller Zellen $\mathfrak{B}$, so dass $\mathfrak{A} <\!\cdot\ \mathfrak{B}$. Diese Bescheibung basiert auf Algorithmen, die auf die zu den Zellen gehörenden Partitionen angewendet werden können. Das Kapitel gibt einen Einblick in die Struktur von $\Gamma_{\mathcal{B}_n}$ aber auch in deren Komplexität. Am Ende präsentieren wir einige Beispiele. Jene verdeutlichen, dass trotz der komplizierten Formulierung der Algorithmen, diese relativ einfach anzuwenden sind.

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