Cusp forms, Spanning sets and Super Symmetry

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Contents

Introduction

Automorphic and cusp forms on a complex bounded symmetric domain B are a classical field of research in mathematics, which famous mathematicians have have been occupied with, for example H. POINCARÉ, A. BOREL, W. L. BAILY Jr., H. MAASS, M. KOECHER and I. SATAKE . Let us give a definition:

Suppose $B \subset \mathbb{C}^n$ is a bounded symmetric domain and G a semisimple LIE group of Hermitian type acting transitively and holomorphically on B , in general $G = \mathrm{Aut}_1(B)$ will be the 1-component of the automorphism group $\mathrm{Aut}(B)$ of B . Let $j \in \mathcal{C}^\infty(G \times B)^{\mathbb{C}}$ be a cocycle, holomorphic in the second entry. In general $j\,(g, \diamondsuit) = \det g'$ for all $g \in G$ if $G = \mathrm{Aut}_1(B)$. Let $k \in \mathbb{Z}$ and $\Gamma \sqsubset G$ be a discrete subgroup. Then a function $f \in \mathcal{O}(B)$ is called an automorphic form of weight k with respect to Γ if and only if $f = f|_\gamma$ for all $\gamma \in \Gamma$, where $f|_g\,(\mathbf{Z}) := f\,(g\mathbf{Z})\,j\,(g, \mathbf{Z})^k$ for all $\mathbf{Z} \in B$ and $\gamma \in \Gamma$. The function f is called a cusp form of weight k with respect to Γ if and only if f is in addition square-integrable over $\Gamma \backslash B$ in a certain sense, see section 1.2 .

Automorphic and cusp forms play a fundamental role in representation theory of semisimple LIE groups of Hermitian type, they have various applications to number theory, especially in the simplest case where B is the unit disc in $\mathbb{C}$, biholomorphic to the upper half plane H via a CAYLEY transform, $G = SL(2, \mathbb{R})$ acting on H via MÖBIUS transformations and $\Gamma \sqsubset SL(2, \mathbb{Z})$ of finite index. Also for mathematical physics cusp forms are of some interest since the space $S_k(\Gamma)$ of cusp forms is a quantization space of the space $\Gamma \backslash B$ treated as the phase space of a physical system. In this concept one obtains the classical limit by taking $k \rightsquigarrow \infty$.

The starting point of the research presented in this thesis have been two articles by Svetlana KATOK and Tatyana FOTH , namely

- FOTH, Tatyana and KATOK, Svetlana: Spanning sets for automorphic forms and dynamics of the frame flow on complex hyperbolic spaces, [5] ,

- KATOK, Svetlana: Livshitz theorem for the unitary frame flow, [11] .

In these articles FOTH and KATOK construct spanning sets for the space of cusp forms on a complex bounded symmetric domain B of rank 1 , which by classification is (biholomorphic to) the unit ball of some $\mathbb{C}^n$, $n \in \mathbb{N}$, and $\Gamma \sqsubset G = \mathrm{Aut}_1(B)$ is discrete such that $\mathrm{vol}\, \Gamma\backslash G < \infty$, $\Gamma\backslash G$ not necessarily compact. They use a new geometric approach, whose main ingredient is the concept of a hyperbolic (or ANOSOV) diffeomorphism resp. flow on a Riemannian manifold and an appropriate version of the ANOSOV closing lemma. This concept originally comes from the theory of dynamical systems, see for example in [10] . Roughly speaking a flow $(\varphi_t)_{t\in\mathbb{R}}$ on a Riemannian manifold M is called hyperbolic if there exists an orthogonal and $(\varphi_t)_{t\in\mathbb{R}}$-stable splitting $TM = T^+ \oplus T^- \oplus T^0$ of the tangent bundle TM such that the differential of the flow $(\varphi_t)_{t\in\mathbb{R}}$ is uniformly expanding on T^+ , uniformly contracting on T^- and isometric on T^0 , and finally T^0 is one-dimensional generated by $\partial_t\varphi_t$. In this situation the ANOSOV closing lemma says that given an 'almost' closed orbit of the flow $(\varphi_t)_{t\in\mathbb{R}}$ there exists a closed orbit nearby. Indeed given a complex bounded symmetric domain B of rank 1 , $G = \mathrm{Aut}_1(B)$ is a semisimple LIE group of real rank 1 , and the root space decomposition of its LIE algebra $\mathfrak{g}$ with respect to a CARTAN subalgebra $\mathfrak{a} \sqsubset \mathfrak{g}$ shows that the geodesic flow $(\varphi_t)_{t\in\mathbb{R}}$ on the unit tangent bundle $S(B)$, which is at the same time the left-invariant flow on $S(B)$ generated by $\mathfrak{a} \simeq \mathbb{R}$, is hyperbolic.

The purpose of the research presented in this thesis now is to generalize FOTH's and KATOK's approach in two directions: the higher rank case and the case of super automorphic and super cusp forms on a bounded symmetric super domain.

In chapter 1 we treat the generalization to the higher rank case. It is well known that the theory of complex bounded symmetric domains is closely related to the theory of semisimple LIE groups of Hermitian type and also to the theory of Hermitian JORDAN triple systems, see for example [13] . If G is a semisimple LIE group of Hermitian type then the quotient G/K , where K denotes a maximal compact subgroup of G , can be realized as a complex bounded symmetric domain B such that G is a covering of $\mathrm{Aut}_1(B)$. On the other hand there exists a one-to-one correspondence between complex bounded symmetric domains B and Hermitian JORDAN triple systems Z such that B is realized as the unit ball in Z . Hence there exist equivalent classifications of complex bounded symmetric domains, semisimple LIE groups of Hermitian type and

Hermitian JORDAN triple systems. A classification of bounded symmetric domains can be found for example in section 1.5 of [16] . In this thesis the classification does not play a fundamental role, but the general theory of semisimple LIE groups and Hermitian JORDAN triple systems does, in particular when clarifying the correspondence between MFTG (maximally flat and totally geodesic) submanifolds of B , maximal split Abelian subgroups of G (which are in one-to-one correspondence with CARTAN subalgebras of $\mathfrak{g}$ via $\exp_G$) and frames in the corresponding JORDAN triple system. This is treated in section 1.1 . Let q be the rank of B . Then by definition MFTG submanifolds of B are q-dimensional, and they are the natural generalizations of geodesics in the rank 1 case. Also a CARTAN subalgebra of $\mathfrak{g}$ now is q-dimensional, and so the geodesic flow generalizes to a q-dimensional multiflow $(\varphi_{\mathbf{t}})_{\mathbf{t}\in\mathbf{R}^q}$ on $S(B)$, the frame bundle on B .

In generalizing KATOK's and FOTH's approach there are two major steps:

(i) On the geometric-dynamical side one has to generalize the notion of hyperbolic flows and the ANOSOV closing lemma.

(ii) On the analytic-arithmetic side one has to prove and apply an appropriate version of SATAKE's theorem, which says that under certain conditions and with respect to a certain measure on $\Gamma\backslash B$ the space of cusp forms is the intersection of the space of automorphic forms with the space $L^r(\Gamma\backslash B)$ for all $r \in [1,\infty]$ and $k \gg 0$.

In this thesis we present a solution of part (i) generalizing the theory to *partially* hyperbolic flows. Concerning part (ii) , as expected, there are major difficulties. The main problem is that so far we are not able to handle the FOURIER expansion of an automorphic form at a cusp of $\Gamma\backslash B$ in the higher rank case, which would lead to an appropriate version of SATAKE's theorem and a growth condition of a cusp form at cusps. However we obtain a result for discrete subgroups $\Gamma \sqsubset G$ such that $\Gamma\backslash G$ is compact and hence there are no cusps. Clearly this is an area where more research is needed.

In the second part of the thesis we treat a generalization to super automorphic forms, where our approach is more successful. For doing so it is necessary to develop the theory of super manifolds first. This is done in chapter 2 . Of course the general theory of ($\mathbb{Z}_2$-) graded structures and super manifolds is already well established, see for example [4] . It has first been developed by F. A. BEREZIN as a mathematical method for describing super symmetry in physics of elementary particles. However even for mathematicians the elegance within the theory of super manifolds is really amazing and satisfying. Roughly speaking a real super manifold is an object which has a pair $(p, q) \in \mathbb{N}^2$ as dimension, p being the even and q being the odd dimension. Characteristic of a supermanifold $\mathcal{M}$ of dimension (p, q) is:

(i) it has a so-called body $M = \mathcal{M}^{\#}$, which is an ordinary p-dimensional $\mathcal{C}^\infty$-manifold,

(ii) we have a graded algebra $\mathcal{D}(\mathcal{M})$ of 'functions' on $\mathcal{M}$, which are the global sections of a sheaf $\mathcal{S}$ on M locally isomorphic to $\mathcal{C}^\infty_M \otimes \Lambda(\mathbb{R}^q)$, and finally

(iii) there is a body map $^{\#} : \mathcal{S} \to \mathcal{C}^\infty_M$ being a unital graded algebra epimorphism.

For the application to super automorphic forms we develop the concept of *parametrisation*, where the 'parameters' are odd elements of some Grassmann algebra $\mathcal{P} := \Lambda(\mathbb{R}^n)$. It turns out that this concept, which seems to be new in the theory of super manifolds, has far reaching applications. The original purpose for doing so is the following: For the definition of the space of super automorphic or super cusp forms we need something like a discrete subgroup of a super Lie group $\mathcal{G}$ acting on a complex bounded symmetric super domain $\mathcal{B}$. But an ordinary discrete subgroup of $\mathcal{G}$ is nothing but a discrete subgroup of the body $G = \mathcal{G}^{\#}$ of $\mathcal{G}$, which is an ordinary real $\mathcal{C}^\infty$-Lie group acting on the body $B = \mathcal{B}^{\#}$ of $\mathcal{B}$. On the other hand considering parametrized discrete subgroups Υ of $\mathcal{G}$ gives a much wider class of discrete sub super Lie groups of $\mathcal{G}$ not necessarily restricted to the body G . It turns out that even within the theory of super manifolds, especially in the theory of super Lie groups, the new concept of parametrization is very useful. In particular the idea of parametrized super points of super manifolds gives nice interpretations of the definition of super embeddings and super projections between super manifolds, see for example lemma 2.27 in section 2.2 . The same holds for the multiplication and inversion super morphisms on super Lie groups, see section 2.4 . Parametrized super points of a super manifold separate points on the graded algebra $\mathcal{D}(\mathcal{M})$ of super functions on $\mathcal{M}$, more precisely if $f \in \mathcal{D}(\mathcal{M})$ such that $f(\Xi) = 0$ for all parametrized super points Ξ of $\mathcal{M}$ then $f = 0$. And so in some sense parametrized super points are the analogon to ordinary points of $\mathcal{C}^\infty$-manifolds.

Most surprising when dealing with parametrisation within the theory of super manifolds is the fact that parametrization even makes sense if there are no odd dimensions at all and so we deal with classical **non**-super objects.

The category of ordinary open subsets of all $\mathbb{R}^p$, $p \in \mathbb{N}$, together with $\mathcal{P}$-super morphisms is a proper extension of the category of open subsets of all $\mathbb{R}^p$ together with $\mathcal{C}^\infty$-maps. In other words given $U \underset{\text{open}}{\subset} \mathbb{R}^p$ and $V \underset{\text{open}}{\subset} \mathbb{R}^r$ there are $\mathcal{P}$-super morphisms from U to V which are **not** ordinary $\mathcal{C}^\infty$-maps! Also the subcategory of all $\mathcal{P}$-super manifolds having dimension $(p,0)$, $p \in \mathbb{N}$, together with $\mathcal{P}$-super morphisms contains the category of $\mathcal{C}^\infty$-manifolds together with $\mathcal{C}^\infty$-maps as a proper subcategory. In other words there exist $\mathcal{P}$-super manifolds $\mathcal{M}$ of dimension $(p,0)$, $p \in \mathbb{N}$, which are **not** ordinary $\mathcal{C}^\infty$- manifolds. However in the case $\mathcal{P} = \mathbb{R}$ (the **non**-parametrized case) the subcategory of all $\mathcal{P}$-super manifolds having dimension $(p,0)$, $p \in \mathbb{N}$, together with $\mathcal{P}$-super morphisms is equal to the category of $\mathcal{C}^\infty$-manifolds together with $\mathcal{C}^\infty$-maps, and an $\mathbb{R}$-super morphism between open sets $U \subset \mathbb{R}^p$ and $V \subset \mathbb{R}^r$ is nothing but an ordinary $\mathcal{C}^\infty$-map.

Another result, which seems to be new, about super manifolds is the following: Given an odd complex dimension represented by an odd complex coordinate function ζ it is indeed possible to split this single complex odd dimension into two real odd dimensions represented by the real odd coordinate functions

$$\xi = \mathrm{Re}\ \zeta := \frac{\zeta - i\overline{\zeta}}{2} \text{ and } \eta = \mathrm{Im}\ \zeta := \frac{-i\zeta + \overline{\zeta}}{2} .$$

Hence a complex (p,q)-dimensional ($\mathcal{P}$-) super manifold is at the same time a real $(2p,2q)$-dimensional ($\mathcal{P}$-) super manifold, and we obtain a functor from the category of holomorphic ($\mathcal{P}$-) super manifolds together with holomorphic ($\mathcal{P}$-) super morphisms to the category of real ($\mathcal{P}$-) super manifolds together with ($\mathcal{P}$-) super morphisms forgetting about the 'complex structure'.

For a discussion of super automorphic and super cusp forms we restrict ourselves to the case of the super special pseudo unitary group $sSU(p,q|r)$, $p,q,r \in \mathbb{N}$, acting on the super matrix ball $B^{p,q|r}$ which is the complex bounded symmetric super domain of dimension (pq,qr) with the full matrix ball $B^{p,q} \subset \mathbb{C}^{p\times q}$ as body. So far there seems to be no classification of super complex bounded symmetric doimains although we know some basic examples, see for example in chapter IV of [3] . In this context the reader perhaps is missing the notion of super integration, see for example in [4]. In super integration there is indeed an analogon for the change of variables formula, but there are still open problems constructing fundamental domains for the quotient $\Upsilon\backslash\mathcal{G}$, which is a $\mathcal{P}$-super manifold, Υ being a discrete $\mathcal{P}$-subgroup of $\mathcal{G}$.

However in the case of a **non**-parametrized discrete subgroup $\Gamma = \Upsilon$ of $\mathcal{G}$, which is simply an ordinary discrete subgroup of the body $G = \mathcal{G}^{\#}$ of $\mathcal{G}$, we are able to define the space of super cusp forms $sS_k(\Gamma)$ of weight k as a HILBERT space containing all super automorphic forms of weight k with respect to Γ which are square-integrable in a certain sense.

As the main result of this thesis we succeed to generalize FOTH's and KATOK's method for rank $q = 1$ and either $\Gamma\backslash G$ compact or $p \geq 2$ and $\operatorname{vol} \Gamma\backslash G < \infty$. In this case we construct a spanning set for the space of super cusp forms under the additional assumption that the right translation with the maximal split Abelian subgroup $A \sqsubset G$ is topologically transitive on $\Gamma\backslash G$, which is satisfied by 'almost all' discrete subgroups $\Gamma \sqsubset G$.
As the major step in the proof, we are able to prove a super analogon for SATAKE's theorem using FOURIER expansion of super automorphic forms at cusps after transforming the situation to the unbounded realization $\mathcal{H}$ of $\mathcal{B}$ via a CAYLEY transform.

By the way the calculations in chapter 3 when dealing with super automorphic and super cusp forms with respect to **non**-parametrized discrete subgroups Γ in the case $q = 1$ are equivalent to the notion of 'twisted' automorphic resp. cusp forms, and so chapter 3 shows in particular how to extend FOTH's and KATOK's approach to such 'twisted' automorphic and cusp forms. By 'twisted' automorphic resp. cusp forms we mean the following:

> Let V be a finite-dimensional unitary vector space over $\mathbb{C}$ and $\chi : \Gamma \to U(V)$ a homomorphism. Then $f \in \mathcal{O}(B) \otimes V$ is called a twisted automorphic form of weight k with respect to Γ and χ if and only if $f|_\gamma = \chi(\gamma) f$. f is called a twisted automorphic form of weight k with respect to Γ and χ if and only if it is in addition square integrable in a certain sense.

For discrete *parametrized* subgroups Υ of $\mathcal{G}$ we obtain partial results. The space $sM_k(\Upsilon)$ of automorphic forms of weight k with respect to Υ is a graded $\mathcal{P}^{\mathbb{C}}$-module, and in the general case it is not clear how to define the space of cusp forms for such Υ as a graded $\mathcal{P}^{\mathbb{C}}$-submodule of $sM_k(\Upsilon)$ since by the reasons desribed above there is no concept of square integrability on $\mathcal{D}(\Upsilon\backslash\mathcal{G})$. However in some special cases we can give some ideas how to define the space $sS_k(\Upsilon)$ of super cusp forms, **not** as a Hilbert space, and how to obtain spanning sets of $sS_k(\Upsilon)$ over $\mathcal{P}^{\mathbb{C}}$. Hereby we treat a parametrized discrete subgroup Υ of $\mathcal{G}$ as a perturbation of its body $\Gamma = \Upsilon^{\#}$ and so the space $sS_k(\Upsilon)$ of super cusp forms as a perturbation of $sS_k(\Gamma) \boxtimes \mathcal{P}^{\mathbb{C}}$. Hence the idea is first to find a spanning set $(\varphi_\lambda)_{\lambda\in\Lambda}$ for $sS_k(\Gamma)$ and then to deform the elements φ_λ to super cusp forms $\psi_\lambda \in sS_k(\Upsilon)$, $\lambda \in \Lambda$, which then under certain conditions will give a spanning set for $sS_k(\Upsilon)$ over $\mathcal{P}^{\mathbb{C}}$. Again notice that even in the case where Υ is a parametrized discrete subgroup of $\mathcal{G} = G = sSU(p,q|0) = SU(p,q)$, the classical case where there are no odd dimensions, the definition of the space $sS_k(\Upsilon)$ of super cuspforms is a non-trivial problem, not to mention the problem of constructing spanning sets for $sS_k(\Upsilon)$. For a general concept of super cusp forms for parametrized subgroups further research is needed.

Finally, the last chapter, chapter 4 , of this thesis deals with another aspect of super manifolds, namely the *pointwise realization* of super open sets in contrast to chapter 2 , where we introduce super open sets as ringed spaces. It turns out that given a real super open set $U^{|q}$ the graded algebra $\mathcal{D}\left(U^{|q}\right)$ belonging to $U^{|q}$ is nothing but the (reduced) graded algebra of continuous and partially differentiable functions on the set $U^{|q}$ (which is now really a set of points) . Surprisingly this is at the same time the (reduced) graded algebra of all arbitrarily often diffentiable functions on $U^{|q}$, see theorem 4.8 in section 4.1 , and this gives a hint why super theory is a generalization only of $\mathcal{C}^\infty$-structures while there is no super analogon to $\mathcal{C}^k$-structures, $k \in \mathbb{N}$. This is not directly related to super autorphic forms, but could be of potential value when studying the fine structure of fundamental domains for parametrized discrete subgroups.

Here for short the dependence amoung the 4 chapters of this thesis:

$$\begin{tikzcd} \boxed{1} \arrow[dr] & & \boxed{2} \arrow[dl] \arrow[r, dashed] & \boxed{4} \\ & \boxed{3} & & \end{tikzcd} .$$

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List of symbols

$\mathbb{N}$	$:= \{0, 1, 2, 3, \dots\}$, the set of natural numbers,
$\mathbb{Z}$	the ring of integer numbers,
$\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$	the fields of rational, real resp. complex numbers,
$\mathcal{C}$, $\mathcal{C}^\infty$, $\mathcal{O}$	the sheaves of continuous resp. infinitely often differentiable functions with values in $\mathbb{R}$ resp. the sheaf of holomorphic functions with values in $\mathbb{C}$,
$\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{n}$, $\mathfrak{a}$	the LIE algebras of the real LIE groups G , K , N , A ,
$\mathrm{Aut}_1(B)$	the 1-component of the group of automorphisms of the domain $B \subset \mathbb{C}^n$,
$\{\ ,\ {}^*,\ \}$	$: Z \times Z \times Z \to Z$, the JORDAN triple product on the JORDAN triple system Z , section 1.1 ,
$S(B)$	the frame bundle over the bounded symmetric domain B , section 1.1 ,
Ad_g , ad_ξ	$: \mathfrak{g} \to \mathfrak{g}$, the adjoint representations of G resp. $\mathfrak{g}$ on $\mathfrak{g}$,
$Z = Z_1(\mathbf{c}) \oplus Z_{\frac{1}{2}}(\mathbf{c}) \oplus Z_0(\mathbf{c})$	the PIERCE decomposition of the JORDAN triple system Z with respect to the tripotent $\mathbf{c} \in Z$, section 1.1 ,
$Z(Q)$, $N(Q)$	the centralizer resp. normalizer of the set Q ,
$B^{p,q}$	$:= \{\mathbf{Z} \in \mathbb{C}^{p\times q} \mid \mathbf{Z}^*\mathbf{Z} \ll 1\}$ the full complex matrix ball,
$SU(p,q)$	$:= \{g \in SL(p+q, \mathbb{C}) \mid g^* J g = J\}$, the special pseudo unitary group, $J := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{matrix} \}p \\ \}q \end{matrix}$,
$\mathfrak{g} = \bigoplus_{\alpha\in\Phi} \mathfrak{g}^\alpha$	the root space decomposition of the LIE algebra $\mathfrak{g}$, section 1.1 ,

$\widetilde{f}$	$\in \mathbb{C}^G$ resp. $\in \mathcal{D}(\mathcal{G})$ the 'lift' of $f \in \mathbb{C}^B$ resp. $f \in \mathcal{D}(\mathcal{B})$ to the Lie group G resp. super Lie group $\mathcal{G}$, section 1.2 resp. 3.1 ,
$L_k^r(\Gamma\backslash B)$	$:= \left\{ f \in \mathbb{C}^B \,\middle|\, \widetilde{f} \in L^r(\Gamma\backslash G) \right\}$, section 1.2 ,
$L_k^s(\Gamma\backslash \mathcal{B})$	$:= \left\{ f \in \mathcal{D}(\mathcal{B}) \,\middle|\, \left\lvert \widetilde{f}' \right\rvert \in L^s(\Gamma\backslash G) \right\}$, section 3.1 ,
$\Delta(\mathbf{Z}, \mathbf{W})$	$\in \mathcal{O}(B) \otimes \overline{\mathcal{O}(B)}$ the Jordan triple determinant on the complex bounded symmetric domain B , section 1.2 ,
$M_k(\Gamma)$, $S_k(\Gamma)$	the space of automorphic resp. cusp forms of weight k with respect to the discrete subgroup $\Gamma \sqsubset G$, section 1.2 ,
$TM = T^0 \oplus T^+ \oplus T^-$	the splitting of the tangent space of a manifold M with respect to a partially hyperbolic diffeomorphism resp. flow on M , section 1.3 ,
$\dot{v}$	$\in \mathbb{Z}_2$ the parity of the homogeneous element v of a graded vector space , see section 2.1 ,
$\wp(n)$	the power set of $\{1, \dots, n\}$,
$\Lambda(V)$	the exterior (Grassmann) algebra over the vector space V ,
$a^\#$, $\mathcal{M}^\#$, $f^\#$	the body of an element $a \in \Lambda(K^n)$, a super manifold $\mathcal{M}$ resp. a function $f \in \mathcal{D}(\mathcal{M})$ on a super manifold $\mathcal{M}$, see sections 2.1 and 2.2 ,
$\mathcal{A}^{(p\vert q)\times(r\vert s)}$	the graded vector space of $(p\vert q)\times(r\vert s)$ - matrices over the graded algebra $\mathcal{A}$, section 2.1 ,
$GL(p\vert q, \mathcal{A})$	the group of invertible even $(p\vert q) \times (p\vert q)$ - matrices over the graded algebra $\mathcal{A}$, section 2.1 ,
$\mathcal{A}^\times$	the set of invertible elements of the unital algebra $\mathcal{A}$,
$s\mathrm{tr}\, X$, $\mathrm{Ber}\ g$	the supertrace of a matrix $X \in \mathcal{A}^{(p\vert q)\times(p\vert q)}$ resp. the Berezinian of a matrix $g \in GL(p\vert q, \mathcal{A})$, section 2.1 ,
$\mathcal{A} \boxtimes \mathcal{B}$	the graded tensor product of the graded algebras $\mathcal{A}$ and $\mathcal{B}$, section 2.1 ,

$\mathcal{D}\left(\diamondsuit^{\vert q}\right)_M$, $\mathcal{D}\left(\diamondsuit^{\vert q,\overline{q}}\right)_M$	the sheaf $\mathcal{C}^\infty_M \otimes \Lambda\left(\mathbb{R}^q\right)$ on the real $\mathcal{C}^\infty$-manifold M resp. the sheaf $\left(\mathcal{C}^\infty_M\right)^{\mathbb{C}} \otimes \Lambda\left(\mathbb{C}^q\right) \boxtimes \Lambda\left(\mathbb{C}^q\right)$ on the holomorphic manifold M , section 2.2 resp. 2.3 ,
$U^{\vert q}$, $U^{\vert q,\overline{q}}$	the real resp. complex (p,q)-dimensional super open set with body $U \underset{\text{open}}{\subset} \mathbb{R}^p$ resp. $U \underset{\text{open}}{\subset} \mathbb{C}^p$, see sections 2.2 and 4.1 resp. 2.3 and 4.2 ,
$D(\varphi,\Phi)$	the super Jacobian of the $\mathcal{P}$-super morphism (φ,Φ) between two super open sets, section 2.2 ,
Γ^∞_E	the sheaf of $\mathcal{C}^\infty$-sections from a real $\mathcal{C}^\infty$-manifold into the $\mathcal{C}^\infty$-vectorbundle E over M ,
$\mathcal{M} \times_{\mathcal{P}} \mathcal{N}$	the $\mathcal{P}$-cross product between two $\mathcal{P}$-super manifolds $\mathcal{M}$ and $\mathcal{N}$, section 2.2 ,
$\mathcal{O}\left(\diamondsuit^{\vert q,\overline{q}}\right)_M$	the sheaf $\mathcal{O}_M \otimes \Lambda\left(\mathbb{C}^q\right)$ on the holomorphic manifold M , section 2.3 ,
$s\mathrm{GL}(n\vert r)$	the $\left(n^2+r^2, 2nr\right)$-dimensional holomorphic super LIE group $\left(\mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(r,\mathbb{C})\right)^{\vert 2nr,\overline{2nr}}$, section 3.1 ,
$sSU(p,q\vert r)$	$\hookrightarrow s\mathrm{GL}(p+q\vert r)$ the super special pseudo unitary super LIE group, section 3.1 ,
$B^{p,q\vert r}$	$:= \left(B^{p,q}\right)^{\vert qr,\overline{qr}}$, the complex (pq,qr)-dimensional super open set with body $B^{p,q} \underset{\text{open}}{\subset} \mathbb{C}^{p\times q}$,
$sM_k(\Upsilon)$, $sS_k(\Upsilon)$	the spaces of super automorphic resp. super cusp forms of weight k with respect to the discrete $\mathcal{P}$-subgroup Υ of $\mathcal{G}$, see sections 3.1 and 3.4 ,
$\widetilde{f}'$	$\in \mathcal{C}^\infty(G) \otimes \Lambda\left(\mathbb{C}^{r\times s}\right)$, the alternative 'lift' of a function $f \in \mathcal{D}(\mathcal{B})$ to the LIE group $G = \mathcal{G}^\#$, section 3.1 ,
$\Delta'(\mathbf{z},\mathbf{w})$	$:= z_1 + \overline{w_1} - \mathbf{w}_2^* \mathbf{z}_2 \in \mathcal{O}(H) \otimes \overline{\mathcal{O}(H)}$, the JORDAN triple determinant on the unbounded realization H of the unit ball $B \subset \mathbb{C}^n$, section 3.2 ,
$\mathrm{tr}_I D$	$:= \sum_{j\in I} d_j$, $I \in \wp(r)$, $D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_r \end{pmatrix}$,
$f^{\#'}$, $\deg' f$	the relative body resp. the relative degree of a function $f \in \mathcal{D}(\mathcal{M}) \boxtimes \mathcal{P}$, $\mathcal{M}$ being a real super manifold, see section 3.4 ,

$\wp$	$:= \{I \subset \mathbb{Z} \text{ finite}\}$,
$\Lambda^{\mathbb{K}}$	the graded algebra $\mathbb{K}^{\wp}$, $\mathbb{K}$ being $\mathbb{R}$ or $\mathbb{C}$, see the beginning of chapter 4 ,
$\widehat{f}$	$\in \mathcal{C}\left(U^{\mid 0}\right)$ resp. $\in \mathcal{C}\left(U^{\mid 0,\overline{0}}\right)$ the extension of a function $f \in \mathcal{C}^{\infty}(U)$ to $U^{\mid 0}$, $U \underset{\text{open}}{\subset} \mathbb{R}^p$ resp. to $U^{\mid 0,\overline{0}}$, $U \underset{\text{open}}{\subset} \mathbb{C}^p$, section 4.1 resp. 4.2 .

Chapter 1

Automorphic and cusp forms in the higher rank case

1.1 The geometry of a bounded symmetric domain

Let us first recall some well known basic facts about bounded symmetric domains, see for example in [1], [2], [9], [13] , [15] and [16] . Let $B \subset \mathbb{C}^n$ be a bounded symmetric domain. Then by classification we may assume without loss of generality that $\mathbf{0} \in B$ and B is circled around $\mathbf{0}$ and convex. Let $G := \mathrm{Aut}_1(B)$ be the identity component of $\mathrm{Aut}(B)$. By a well known theorem of H. Cartan we know that G is a semisimple non-compact Lie group of Hermitian type which acts transitively on B . Let

$$K := \{g \in G \,|\, g\mathbf{0} = \mathbf{0}\}$$

be the stabilizer group of $\mathbf{0}$. Then K is a maximal compact subgroup of G , and

$$B \simeq G/K$$

as a real analytic manifold. According to the Cartan decomposition we can split the Lie algebra $\mathfrak{g}$ of G , which is precisely the Lie algebra of completely integrable vectorfields on B , as

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}\,,$$

where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ with respect to the Killing form of $\mathfrak{g}$. The Killing form is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$, $exp_G : \mathfrak{p} \hookrightarrow G$ is an injection, but $\mathfrak{p}$ is not a sub Lie algebra of $\mathfrak{g}$, more precisely $[\mathfrak{k}, \mathfrak{k}], [\mathfrak{p}, \mathfrak{p}] \sqsubset \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{p}] \sqsubset \mathfrak{p}$.

Furthermore by classification we know that $Z := \mathbb{C}^n$ can be written as a Hermitian JORDAN triple system such that B is the unit ball in Z in the following sense: Let $\{\ ,\ ^*,\ \}$ denote the JORDAN triple product on Z , which is by definition $\mathbb{C}$-linear and commutative in the outer variables and $\mathbb{C}$-antilinear in the second variable. Then for all $\mathbf{Z}, \mathbf{W} \in Z$ we have a linear operator $\{Z, W^*, \diamondsuit\} : Z \to Z$, and it turns out that

$$\langle \mathbf{Z}, \mathbf{W} \rangle := \operatorname{tr}\{\mathbf{Z}, \mathbf{W}^*, \diamondsuit\}$$

for all $\mathbf{Z}, \mathbf{W} \in Z$ gives a scalar product on Z , and $\{Z, W^*, \diamondsuit\}^* = \{W, Z^*, \diamondsuit\}$ for all $\mathbf{Z}, \mathbf{W} \in Z$ with respect to $\langle\ ,\ \rangle$. So for all $\mathbf{Z} \in Z$ the operator $\{\mathbf{Z}, \mathbf{Z}^*, \diamondsuit\}$ is self ajoint and positive semi-definite. Finally

$$B = \{\, \mathbf{Z} \in Z \mid \{\mathbf{Z}, \mathbf{Z}^*, \diamondsuit\} \ll 1 \,\} \ .$$

Z is uniquely determined up to isomorphism by B , K is the automorphism group of Z , it can be shown that each automorphism of B belonging to K extends uniquely to a linear (!) automorphism of Z and so it is unitary with respect to $\langle\ ,\ \rangle$. Therefore there exists a unique G-invariant Hermitian metric on B which is $\langle\ ,\ \rangle$ at $\mathbf{0} \in B$, it is called the BERGMAN metric. We have canonical isomorphisms $\mathfrak{p} \simeq T_{\mathbf{0}}B$ as real verctorspaces and
$T_{\mathbf{Z}}B \simeq T_{\mathbf{0}}B = Z$ as complex vector spaces for all $\mathbf{Z} \in B$, and this fact turns $\mathfrak{p}$ and $T_{\mathbf{Z}}B$ into a JORDAN triple product and at the same time into a Hilbert space. Recall that the latter isomorphisms are not the identity although $T_{\mathbf{Z}} = \mathbb{C}^n$ for all $\mathbf{Z} \in B$. The first isomorphism is the consequence of a more subtle construction. As a bounded symmetric domain B has a so-called compact dual X which is a compact symmetric analytic manifold such that $Z \subset X$ is open and dense. The automorphism group of X is $G^{\mathbb{C}}$ which has $\mathfrak{g}^{\mathbb{C}}$ as LIE algebra. We have two embeddings of Z into $\mathfrak{g}^{\mathbb{C}}$, which is the LIE algebra of all completely integrable vectorfields on X . The first one is the identity, each $\mathbf{Z}_0 \in Z$ identified with the constant vectorfield equal to $\mathbf{Z}_0$, it is $\mathbb{C}$-linear, and the second is given by

$$\widetilde{\ } : Z \hookrightarrow \mathfrak{g}^{\mathbb{C}} \ , \ \mathbf{Z}_0 \mapsto \widetilde{\mathbf{Z}_0} \, ,$$

where $\widetilde{\mathbf{Z}_0}\,(\mathbf{Z}) := \{\mathbf{Z}, \mathbf{Z}_0^*, \mathbf{Z}\}$ for all $\mathbf{Z} \in Z$, which is clearly $\mathbb{C}$-antilinear. The images of both embeddings are commutative sub LIE algebras of $\mathfrak{g}^{\mathbb{C}}$, and a straight forward calculation shows that

$$\left[\mathbf{Z}, \widetilde{\mathbf{W}}\right] = 2\,\{\mathbf{Z}, \mathbf{W}^*, \diamondsuit\}$$

for all $\mathbf{Z}, \mathbf{W} \in Z$. The isomorphism $Z \xrightarrow{\sim} \mathfrak{p}$ is precisely the diagonal
$\mathbf{Z} \mapsto \mathbf{Z} - \widetilde{\mathbf{Z}}$. Via this isomorphism Re $\langle\ ,\ \rangle$ on Z coincides with the KILLING

form on $\mathfrak{p}$ up to a positive constant. Via the isomorphisms $\mathfrak{p} \simeq Z \simeq T_{\mathbf{0}} B$ as real vector spaces the adjoint representation of K on $\mathfrak{p}$ corresponds to the action of K on Z as automorphism group and to the action of K on $T_{\mathbf{0}}B$ via the differential. Especially

$$\mathrm{Ad}_k \left(\widetilde{\mathbf{Z}}\right) = \widetilde{k\mathbf{Z}} .$$

A fundamental concept in the theory of symmetric domains is that of the rank, which we will define in terms of Z , G and the geometry of B .

An element $\mathbf{c}$ of the JORDAN triple system Z is called a tripotent if and only if $\{\mathbf{c}, \mathbf{c}^*, \mathbf{c}\} = \mathbf{c}$. Associated to a tripotent $\mathbf{c} \in Z$ we have the PEIRCE decomposition

$$Z = Z_1(\mathbf{c}) \oplus Z_{\frac{1}{2}}(\mathbf{c}) \oplus Z_0(\mathbf{c}) \ ,$$

as a $\mathbb{C}$-vectorspace, where $Z_\alpha(\mathbf{c})$ is the α-eigenspace of the linear operator $\{\mathbf{c}, \mathbf{c}^*, \diamondsuit\}$ on Z , $\alpha = 1, \frac{1}{2}, 0$.

Definition 1.1 *Two tripotents $\mathbf{c}, \mathbf{c}'$ are called orthogonal if and only if one of the following equivalent conditions is fulfilled:*

$\{i\}$ $\mathbf{c} \in Z_0(\mathbf{c}')$,

$\{ii\}$ $\mathbf{c}' \in Z_0(\mathbf{c})$,

$\{iii\}$ $\{\mathbf{c}, \mathbf{c}'^*, \diamondsuit\} = 0$.

It turns out that a sum of two orthogonal tripotents is again a tripotent, and a tripotent $\mathbf{c} \neq \mathbf{0}$ is called primitive if and only if $\mathbf{c}$ cannot be written as a sum of two orthogonal tripotents $\neq \mathbf{0}$. Finally a maximal tuple $(\mathbf{e}_1, \ldots, \mathbf{e}_q) \in Z^q$ of primitive and pairwise orthogonal tripotents is called a frame in Z . Every tripotent $\mathbf{c}$ can be written as a sum of primitive, pairwise orthogonal tripotents, and the number of summands only depends on $\mathbf{c}$ and is called the rank $\mathrm{rk}\, \mathbf{c}$ of the tripotent $\mathbf{c}$. $\mathrm{rk}\, \mathbf{c} = 0$ if and only if $\mathbf{c} = \mathbf{0}$, and $\mathrm{rk}\, \mathbf{c} = 1$ if and only if $\mathbf{c}$ is primitive if and only if $Z_1(\mathbf{c}) = \mathbb{C}\, \mathbf{c}$ for all tripotents $\mathbf{c} \in Z$. Finally every $\mathbf{Z} \in Z$ can be uniquely written as

$$\mathbf{Z} = \sum_{j=1}^{r} \lambda_j \mathbf{c}_j \tag{1.1}$$

with pairwise orthogonal non-zero tripotents $\mathbf{c}_1, \ldots, \mathbf{c}_r \in Z$ and $0 < \lambda_1 < \cdots < \lambda_r$. Then all $\mathbf{c}_1, \ldots, \mathbf{c}_r \in Z$ are linear combinations of odd powers of $\mathbf{Z}$, and $\mathbf{Z} \in B$ if and only if $\lambda_r < 1$. We call 1.1 the spectral decomposition of $\mathbf{Z}$.

A subgroup $A \sqsubset G$ is called split Abelian if and only if $A = \exp \mathfrak{a}$ where the LIE algebra $\mathfrak{a} \sqsubset \mathfrak{g}$ of A is a commutative sub LIE algebra (a so-called CARTAN subalgebra if it is maximal) of $\mathfrak{p}$. Then of course exp is an isomorphism from $\mathfrak{a}$ to A , so A is non-compact Abelian isomorphic to some $\mathbb{R}^q$.

Definition 1.2

(i) Let Z be a JORDAN *triple system. Then the rank of Z is the common length of all frames in Z .*

(iii) Let G be a semisimple LIE *group of Hermitian type. Then the real rank of G is the common dimension of all maximal split Abelian subgroups of G .*

(ii) Let $B \subset \mathbb{C}^n$ be a bounded symmetric domain. Then the rank of B is the common dimension of all (real and connected) maximal flat and totally geodesic (MFTG for short) submanifolds of B .

It turns out that in the case where B is the unit ball of the Hermitian JORDAN triple system Z and G is the identity component of Aut$B$ the ranks of Z , G and B coincide. A following theorem clarifies the situation, but before we have to handle these constructions in the reducible case:

Definition 1.3

(i) B is called reducible if there exist bounded symmetric domains B_1 , B_2 such that $B \simeq B_1 \times B_2$. Otherwise B is called irreducible.

(ii) the JORDAN *triple system Z is called simple if and only if $\{\mathbf{Z}, \diamondsuit, \mathbf{W}\} \neq 0$ for all $\mathbf{Z}, \mathbf{W} \in Z \setminus \{\mathbf{0}\}$ and Z has no non-trivial ideals.*

Theorem 1.4

(i) B is irreducible if and only if G is a simple LIE *group if and only if Z is simple.*

(ii) If B is reducible then there exist irreducible symmetric bounded domains $B_1, \ldots, B_s$ such that $B = B_1 \times \cdots \times B_s$. Then

$G = G_1 \times \cdots \times G_s$ where $G_1, \ldots, G_s$ are the identity components of Aut$B_1, \ldots,$ AutB_s *resp. ,*

$B_1, \ldots, B_s$ are (isomorphic to) the unit balls of JORDAN *triple systems $Z_1, \ldots, Z_s$ resp. such that $Z = Z_1 \oplus \cdots \oplus Z_s$,*

$K = K_1 \times \cdots \times K_s$ where $K_1, \ldots, K_s$ are the stabilizer groups of $\mathbf{0} \in B_1, \ldots, B_s$ resp. ,

the maximal split Abelian subgroups of G are precisely $A_1 \times \cdots \times A_s$ where $A_1, \ldots, A_s$ are maximal split Abelian subgroups of $G_1, \ldots, G_s$ resp. ,

the MFTG submanifolds of B are precisely $Q_1 \times \cdots \times Q_s$ where

$Q_1, \ldots, Q_s$ are MFTG submanifolds of $B_1, \ldots, B_s$ resp. , and finally
the frames in Z up to order are precisely $\left(\mathbf{e}_1^{(1)}, \ldots, \mathbf{e}_{q_1}^{(1)}, \ldots, \mathbf{e}_1^{(s)}, \ldots, \mathbf{e}_{q_s}^{(s)}\right)$ where $\left(\mathbf{e}_1^{(1)}, \ldots, \mathbf{e}_{q_1}^{(1)}\right)$, ..., $\left(\mathbf{e}_1^{(s)}, \ldots, \mathbf{e}_{q_s}^{(s)}\right)$ are frames in $Z_1, \ldots, Z_s$ resp. .

Proof: (i) B is irreducible if and only if Z is simple, this is the main result of section 4.11 in [13] , and B is irreducible if and only if G is simple, this result can be found for example in [1] section 11. 4 . □
(ii) Since $B \subset \mathbb{C}^n$ and n is finite, clearly B can be written as a product of finitely many irreducible symmetric domains.
$G = G_1 \times \cdots \times G_s$ by iterated application of proposition II in chapter 5 of [14] , which tells us that given two bounded domains $D_1, D_2 \in \mathbb{C}^n$ then any $f \in \mathrm{Aut}_1 (D_1 \times D_2)$ is of the form $f(\mathbf{z}_1, \mathbf{z}_2) = f_1(\mathbf{z}_1) f_2(\mathbf{z}_2)$ with $f_1 \in \mathrm{Aut}_1(D_1)$ and $f_2 \in \mathrm{Aut}_1(D_2)$.
Let $Z_1, \ldots, Z_s$ be the JORDAN triple systems belonging to $B_1, \ldots, B_s$ resp. .
Then by an easy calculation one sees that B is the unit ball of $Z_1 \oplus \cdots \oplus Z_s$.
Trivially $K := K_1 \times \cdots \times K_s$ is the stabilizer of $\mathbf{0} \in B$.
The rest can be easily shown by projecting on each factor resp. summand, since we have $\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_s$,

$$T_{\mathbf{Z}} B = T_{\mathbf{Z}_1} B_1 \oplus \cdots \oplus T_{\mathbf{Z}_s} B_s$$

for each $\mathbf{Z} = (\mathbf{Z}_1, \ldots, \mathbf{Z}_s) \in B$, $Z = Z_1 \oplus \cdots \oplus Z_s$ as orthogonal splittings, and if $\mathbf{c} \in Z$ is a primitive tripotent, then there exists an $i \in \{1, \ldots, s\}$ such that $\mathbf{c} \in Z_i$. □

Theorem 1.5

(i) If Q is an MFTG submanifold of B , then gQ is again an MFTG submanifold of B for all $g \in G$, and there always exists $g \in G$ such that $\mathbf{0} \in gQ$.

(ii) Conversely if Q and Q' are two MFTG submanifolds of B , $\mathbf{Z} \in Q$ and $\mathbf{Z}' \in Q'$ then there exists $g \in G$ such that $g\mathbf{Z} = \mathbf{Z}'$ and $gQ = Q'$.

(iii) If A is a maximal split Abelian subgroup of G and $k \in K$ then kAk^{-1} is again a maximal split Abelian subgroup of G , conversely if A and A' are two maximal split Abelian subgroups of G then there exists $k \in K$ such that $A' = kAk^{-1}$.

(iv) If $(\mathbf{e}_1, \ldots, \mathbf{e}_q)$ is a frame in Z and $k \in K$ then $(k\mathbf{e}_1, \ldots, k\mathbf{e}_q)$ is again a frame in Z . Conversely if B is irreducible (equivalently if Z is simple) and $(\mathbf{e}_1, \ldots, \mathbf{e}_q)$ and $\left(\mathbf{e}'_1, \ldots, \mathbf{e}'_q\right)$ are two frames in Z then there exists $k \in K$ such that $\mathbf{e}'_j = k\mathbf{e}_j$, $j = 1, \ldots, q$.

(v) If $(\mathbf{e}_1, \ldots, \mathbf{e}_q)$ *is a frame in* Z *then the image of the isometric embedding*

$$\mathbb{R}^q \hookrightarrow B\,,\ \mathbf{t} \mapsto \exp_B \left(\sum_{j=1}^{q} t_j\, \mathbf{e}_j \right) = \sum_{j=1}^{q} \tanh\, t_j\, \mathbf{e}_j$$

is an MFTG submanifold of B *containing* $\mathbf{0}$ *. Conversely if* Q *is an MFTG submanifold of* B *containing* $\mathbf{0}$ *then there exists a frame* $(\mathbf{e}_1, \ldots, \mathbf{e}_q)$ *in* Z *such that* Q *is the image of the isometric embedding*

$$\mathbb{R}^q \hookrightarrow B\,,\ \mathbf{t} \mapsto \exp_B \left(\sum_{j=1}^{q} t_j\, \mathbf{e}_j \right) = \sum_{j=1}^{q} \tanh\, t_j\, \mathbf{e}_j\,.$$

(vi) If $(\mathbf{e}_1, \ldots, \mathbf{e}_q)$ *is a frame then the image* A *of the* Lie *group embedding*

$$\mathbb{R}^q \hookrightarrow G\,,\ \mathbf{t} \mapsto a_{\mathbf{t}} := \exp_G \left(\sum_{j=1}^{q} t_j\ (\mathbf{e}_j - \widetilde{\mathbf{e}_j}) \right)$$

is a maximal split Abelian subgroup of G *. Conversely if* A *is a maximal split Abelian subgroup of* G *then there exists a frame* $(\mathbf{e}_1, \ldots, \mathbf{e}_q)$ *such that* A *is the image of the* Lie *group embedding*

$$\mathbb{R}^q \hookrightarrow G\,,\ \mathbf{t} \mapsto \exp_G \left(\sum_{j=1}^{q} t_j\ (\mathbf{e}_j - \widetilde{\mathbf{e}_j}) \right)\,.$$

(vii) Let $(\mathbf{e}_1, \ldots, \mathbf{e}_q)$ *be a frame ,* Q *be the MFTG submanifold containing* $\mathbf{0}$ *defined by the frame via (v) , and* A *be the maximal split Abelian subgroup defined by the frame via (vi) . Then for all* $\mathbf{t} \in \mathbb{R}^q$ *we have*

$$a_{\mathbf{t}}\, \mathbf{0} = \sum_{j=1}^{q} \tanh\, t_j\, \mathbf{e}_j \in Q\,,$$

and so A *acts simply transitively on* Q *. The stabilizer* $M := Z(Q)$ *of* Q *in* G *is precisely the stabilizer of the frame* $(\mathbf{e}_1, \ldots, \mathbf{e}_q)$ *and at the same time the centralizer* $Z_K(A)$ *of* A *in* K *. The normalizer* $N(Q)$ *of* Q *in* G *is precisely* $AN_K(A)$ *.*

For proving (v) and (vi) of theorem 1.5 we need a technical lemma.

Lemma 1.6 *Let* $(\mathbf{e}_1, \ldots, \mathbf{e}_q)$ *be a frame in* Z *and* $\mathbf{Z} \in Z$ *such that*

$$\{\mathbf{e}_j, \mathbf{Z}^*, \mathbf{e}_j\} = \left\{\mathbf{Z}, \mathbf{e}_j^*, \mathbf{e}_j\right\}$$

for all $j = 1, \ldots, q$ *. Then* $\mathbf{Z} \in \mathbb{R}\mathbf{e}_1 + \cdots + \mathbb{R}\mathbf{e}_q$ *.*

Proof: We apply theorem 3.14 of [13] to the frame $(\mathbf{e}_1, \ldots, \mathbf{e}_q)$, which says that given pairwise orthogonal tripotents $\mathbf{c}_1, \ldots, \mathbf{c}_r \in Z$ then

$$Z = \bigoplus_{0 \le i \le n} Z_{ij}$$

(the so-called joint PEIRCE decomposition), where

$$Z_{ij} := Z_{ji} := \begin{cases} Z_1(\mathbf{c}_i) & \text{if} \quad i = j \neq 0 \\ Z_{\frac{1}{2}}(\mathbf{c}_i) \cap Z_{\frac{1}{2}}(\mathbf{c}_j) & \text{if} \quad 0 \neq i \neq j \neq 0 \\ Z_{\frac{1}{2}}(\mathbf{c}_i) \cap \bigcap_{j=1,\ldots,r\ ,\ j \neq i} V_0(\mathbf{c}_j) & \text{if} \quad i \neq 0 = j \\ Z_0(\mathbf{c}_1) \cap \cdots \cap Z_0(\mathbf{c}_r) & \text{if} \quad i = j = 0 \end{cases}$$

for all $i, j \in \{0, \ldots, r\}$, the PEIRCE decomposition of $\mathbf{c}_I := \sum_{i \in I} \mathbf{c}_i$ is given by

$$\begin{aligned} Z_1(\mathbf{c}_I) &= \sum_{i,j \in I} Z_{ij} \\ Z_{\frac{1}{2}}(\mathbf{c}_I) &= \sum_{i \in I, j \notin I} Z_{ij} \\ Z_0(\mathbf{c}_I) &= \sum_{i,j \notin I} Z_{ij} \end{aligned}$$

for all $I \subset \{1, \ldots, r\}$, and we have the multiplication rule

$$\left\{Z_{ij}, Z_{jk}^*, Z_{kl}\right\} \subset Z_{il} \tag{1.2}$$

for all $i, j, k, l \in \{0, \ldots, r\}$, and all other types of products are zero.

Since all $\mathbf{e}_j \in Z_{jj}$, $j = 1, \ldots, q$, by property 1.2 we see that all Z_{kl} , $k, l = 0, \ldots, q$, are invariant under all $\{\mathbf{e}_j, \diamondsuit^*, \mathbf{e}_j\}$ and $\left\{\mathbf{e}_j, \mathbf{e}_j^*, \diamondsuit\right\}$, $j = 1, \ldots, q$, and so without loss of generality we may assume that $\mathbf{Z} \in Z_{kl}$ for some $k, l \in \{0, \ldots, q\}$.
Assume $k \neq l$ and without loss of generality $k \neq 0$. Then $\mathbf{Z} \in Z_{\frac{1}{2}}(\mathbf{e}_k)$, and so by 1.2

$$\mathbf{Z} = 2\{\mathbf{Z}, \mathbf{e}_k^*, \mathbf{e}_k\} = 2\{\mathbf{e}_k, \mathbf{Z}^*, \mathbf{e}_k\} = \mathbf{0}\,.$$

Now assume $k = l = 0$. Then by 1.2 all odd powers of $\mathbf{Z}$ are contained in Z_{00} , and so $\mathbf{Z}$ can be written as $\mathbf{Z} = \sum_{s=1}^{r} \lambda_s \mathbf{c}_s$ where $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$, and $\mathbf{c}_1, \ldots, \mathbf{c}_r \in Z_{00}$ are pairwise orthogonal non-zero tripotents, which we

can assume to be primitive since Z_{00} is a sub JORDAN triple system of Z by 1.2 . So $(\mathbf{e}_1, \ldots, \mathbf{e}_q, \mathbf{c}_1, \ldots, \mathbf{c}_r)$ is a tuple of primitive pairwise orthogonal tripotents, but since $(\mathbf{e}_1, \ldots, \mathbf{e}_q)$ is already a frame, we see that $r = 0$ and so $\mathbf{Z} = \mathbf{0}$.
Finally assume $k = l \in \{1, \ldots, q\}$. Then $\mathbf{Z} \in V_1(\mathbf{e}_k) = \mathbb{C}\mathbf{e}_k$. So let $\mathbf{Z} = \lambda\mathbf{e}_k$ with an appropriate $\lambda \in \mathbb{C}$. Then

$$\overline{\lambda}\mathbf{e}_k = \{\mathbf{e}_k, (\lambda\mathbf{e}_k)^*, \mathbf{e}_k\} = \{\lambda\mathbf{e}_k, \mathbf{e}_k^*, \mathbf{e}_k\} = \lambda\mathbf{e}_k \,,$$

and so $\lambda \in \mathbb{R}$. □

Proof of theorem 1.5 : (i) trivial since G acts isometrically and transitively on B .
(ii) This is precisely theorem 6.2 in chapter V of [9] . □
(iii) The first statement is trivial, the second is the group theoretic version of lemma 6.3 of [9] , which says that given two maximal Abelian subspaces $\mathfrak{a}$ and $\mathfrak{a}'$ of $\mathfrak{p}$ there exists $k \in K$ such that $\mathfrak{a}'$ is the image of $\mathfrak{a}$ under the adjoint representation of k . □
(iv) The first statement is trivial since K is the automorphism group of Z . The second is theorem 5.9 of [13] . □
(v) By 4.5 and corollary 4.8 of [13]

$$\exp_B : Z \simeq T_{\mathbf{0}}B \to B \;,\; \mathbf{Z} \mapsto \tanh \mathbf{Z}$$

is a real analytic diffeomorphism, so $\mathbb{R} \hookrightarrow B$, $t \mapsto \tanh\,(t\mathbf{Z})$ is the geodesic through $\mathbf{0}$ in direction of $\mathbf{Z}$ and at the same time the integral curve through $\mathbf{0}$ to the vector field $\mathbf{Z} - \widetilde{\mathbf{Z}}$ for all $\mathbf{Z} \in Z \simeq T_{\mathbf{0}}$. Now let $(\mathbf{e}_1, \ldots, \mathbf{e}_q)$ be a frame in Z . Then we have a unitary embedding

$$\mathbb{R}^q \hookrightarrow \mathbf{Z} \;,\; \mathbf{t} \mapsto \mathbf{Z}_{\mathbf{t}} := t_1\mathbf{e}_1 + \cdots + t_q\mathbf{e}_q \,,$$

and since all the vector fields $\mathbf{Z}_{\mathbf{t}} - \widetilde{\mathbf{Z}_{\mathbf{t}}}$ commute, together with $\exp_B$ this unitary embedding leads to an isometric embedding

$$\mathbb{R}^q \hookrightarrow B \;,\; \mathbf{t} \mapsto \exp_B\left(\sum_{j=1}^{q} t_j\,\mathbf{e}_j\right) = \sum_{j=1}^{q} \tanh\, t_j\,\mathbf{e}_j \,,$$

whose image Q_0 is a flat and totally geodesic submanifold of B . To see that it is maximal assume there exists a connected, flat and totally geodesic submanifold Q of B such that $Q_0 \subset Q$. Then $T_{\mathbf{0}}Q_0 \sqsubset T_{\mathbf{0}}Q$, and
$T_{\mathbf{0}}Q_0 = T_{\mathbf{0}}Q$ implies $Q_0 = Q$. Let us assume $\mathbf{Z} \in T_{\mathbf{0}}Q$. Then since Q is flat and totally geodesic, for all $\mathbf{t} \in \mathbb{R}^q$ and $\mathbf{u} \in]-1, 1[^q$

$$\mathbf{0} = \left[\mathbf{Z} - \widetilde{\mathbf{Z}}, \mathbf{Z}_{\mathbf{t}} - \widetilde{\mathbf{Z}_{\mathbf{t}}}\right](\mathbf{Z}_{\mathbf{u}}) = 2\left(\{\mathbf{Z}_{\mathbf{t}}, \mathbf{Z}^*, \mathbf{Z}_{\mathbf{u}}\} - \{\mathbf{Z}, \mathbf{Z}_{\mathbf{t}}^*, \mathbf{Z}_{\mathbf{u}}\}\right) ,$$

especially $\{\mathbf{e}_j, \mathbf{Z}^*, \mathbf{e}_j\} = \left\{\mathbf{Z}, \mathbf{e}_j^*, \mathbf{e}_j\right\}$ for all $j = 1, \ldots, q$. So by lemma 1.6 $\mathbf{Z} \in \mathbb{R}\mathbf{e}_1 + \ldots \mathbb{R}\mathbf{e}_q = T_0 Q_0$.
Conversely let Q be an MFTG submanifold of B containing $\mathbf{0}$. Then by (ii) there exists $k \in K$ such that $kQ_0 = Q$, $(k\mathbf{e}_1, \ldots, k\mathbf{e}_q)$ is again a frame in Z by (iv) , and an easy calculation shows that Q is the image of the isometric embedding

$$\mathbb{R}^q \hookrightarrow B \ , \ \mathbf{t} \mapsto k \exp_B \left(\sum_{j=1}^{q} t_j \, \mathbf{e}_j \right) = \sum_{j=1}^{q} \tanh \, t_j \, k\mathbf{e}_j \, . \square$$

(vi) Let $(\mathbf{e}_1, \ldots, \mathbf{e}_q)$ be a frame in Z . Then clearly all $\mathbf{e}_j - \widetilde{\mathbf{e}_j} \in \mathfrak{p}$, $j = 1, \ldots, q$, commute, and so we have a Lie group embedding

$$\mathbb{R}^q \hookrightarrow G \, , \, \mathbf{t} \mapsto a_{\mathbf{t}} := \exp_G \left(\sum_{j=1}^{q} t_j \ (\mathbf{e}_j - \widetilde{\mathbf{e}_j}) \right) \ ,$$

whose image is a split Abelian subgroup A_0 of G . To see that it is maximal assume there exists $\mathbf{Z} \in Z$ such that $\mathbf{Z} - \widetilde{\mathbf{Z}} \in \mathfrak{p}$ commutes with all $\mathbf{e}_j - \widetilde{\mathbf{e}_j} \in \mathfrak{p}$, $j = 1, \ldots, q$. This implies

$$0 = \left[\mathbf{Z} - \widetilde{\mathbf{Z}}, \mathbf{e}_j - \widetilde{\mathbf{e}_j}\right] = 2 \left(\{\mathbf{e}_j, \mathbf{Z}^*, \diamondsuit\} - \left\{\mathbf{Z}, \mathbf{e}_j^*, \diamondsuit\right\}\right) \ ,$$

and so again by lemma 1.6 $\mathbf{Z} \in \mathbb{R}\mathbf{e}_1 + \ldots \mathbb{R}\mathbf{e}_q$, therefore $\mathbf{Z} - \widetilde{\mathbf{Z}}$ already belongs to the Lie algebra of A_0 .
Conversely let A be a maximal split Abelian subgroup of G . Then by (iii) there exists $k \in K$ such that $kA_0 = A$, and by (iv) $(k\mathbf{e}_1, \ldots, k\mathbf{e}_q)$ is again a frame in Z . Finally A is the image of the Lie group embedding

$$\mathbb{R}^q \hookrightarrow G \, , \, \mathbf{t} \mapsto k a_{\mathbf{t}} k^{-1} = \exp_G \left(\sum_{j=1}^{q} t_j \ \left(k\mathbf{e}_j - \widetilde{k\mathbf{e}_j}\right) \right) \ . \square$$

(vii) The first statement is a trivial consequence of the fact that $\mathbb{R} \hookrightarrow B$, $t \mapsto \tanh \ (t\mathbf{Z})$ is the geodesic through $\mathbf{0}$ in direction of $\mathbf{Z}$ and at the same time the integral curve through $\mathbf{0}$ to the vector field $\mathbf{Z} - \widetilde{\mathbf{Z}}$ for all $\mathbf{Z} \in Z \simeq T_{\mathbf{0}}$.
Let us now prove that $Z(Q) = Z_K(\mathbf{e}_1, \ldots, \mathbf{e}_q) = Z_K(A)$:

'$Z(Q) \sqsubset Z_K(\mathbf{e}_1, \ldots, \mathbf{e}_q)$' : Let $w \in G$ such that $w|_Q$ is the identity. Then especially $w \in K$, and w acts identically on $T_{\mathbf{0}}Q = \mathbb{R}\mathbf{e}_1 + \cdots + \mathbb{R}\mathbf{e}_q$.

'$Z_K(\mathbf{e}_1, \ldots, \mathbf{e}_q) \sqsubset Z_K(A)$' : Let $w \in K$ stabilize $\mathbf{e}_1, \ldots, \mathbf{e}_q$. Then, as we have already seen, w stabilizes also all $\mathbf{e}_j - \widetilde{\mathbf{e}_j} \in \mathfrak{g}$, $j = 1, \ldots, q$, which span the Lie algebra of A .

'$Z_K(A) \sqsubset Z(Q)$' : Let $w \in K$ stabilize all elements of A . Then $w\mathbf{0} = \mathbf{0}$, and so w stabilizes all elements of $Q = A\mathbf{0}$.

Finally let us show that $N(Q) = AN_K(A)$:

'$\sqsubset$' : Let $g \in G$ such that $gQ = Q$. Then $g\mathbf{0} \in Q$, and so there exists $a \in A$ such that $g\mathbf{0} = a\mathbf{0}$. So $g = an$ with an appropriate $n \in N_K(Q)$. We see that n normalizes $\mathbb{R}\mathbf{e}_1 + \cdots + \mathbb{R}\mathbf{e}_q$, and so $\mathfrak{a} = \mathbb{R}\left(\mathbf{e}_1 - \widetilde{\mathbf{e}_1}\right) + \cdots + \mathbb{R}\left(\mathbf{e}_1 - \widetilde{\mathbf{e}_1}\right)$ as well.

'$\sqsupset$' : Let $a \in A$ and $n \in N_K(A)$. Then

$$anQ = anA\mathbf{0} = aAn\mathbf{0} = A\mathbf{0} = Q\,.$$

$\square$

Let us denote the rank of Z and B by q and fix a frame $(\mathbf{e}_1, \ldots, \mathbf{e}_q)$ in $Z \simeq \mathfrak{p}$. Then we have a standard MFTG surface

$$Q := \exp_B\left(\bigoplus_{j=1}^{q} \mathbb{R}\mathbf{e}_j\right) = \sum_{j=1}^{q} \left]-1,1\right[\,\mathbf{e}_j$$

and a standard maximal Abelian subgroup of G being the image of the LIE group embedding

$$\mathbb{R}^q \hookrightarrow G\,,\; \mathbf{t} \mapsto a_{\mathbf{t}} := \exp_G\left(\sum_{j=1}^{q} t_j\,\left(\mathbf{e}_j - \widetilde{\mathbf{e}_j}\right)\right)\,.$$

By M we denote $Z(Q)$. The decomposition $B = B_1 \times \cdots \times B_s$ into irreducible factors of B leads to a corresponding decomposition of A and so finally to the decomposition

$$\mathbb{R}^q = \mathbb{R}^{q_1} \oplus \cdots \oplus \mathbb{R}^{q_s}\,,$$

where $q_1, \ldots, q_s$ are the ranks of $B_1, \ldots B_s$ resp. . This decomposition of $\mathbb{R}^q$ is called the decomposition into irreducible summands.

Let us consider the full matrix ball

$$B^{p,q} := \left\{\mathbf{Z} \in \mathbb{C}^{p\times q} \,\middle|\, \mathbf{Z}^*\mathbf{Z} \ll 1\right\}$$

as an example. $G := SU\,(p,q)$ acts transitively on B from the left by fractional linear (MÖBIUS) transformations, and one can show that G is a finite covering of the identity component of $\mathrm{Aut}B$.

$$K = S\left(U\left(p\right) \times U\left(q\right)\right) \sqsubset G$$

is the stabilizer group of $\mathbf{Z} = \mathbf{0}$. B is irreducible, and the JORDAN triple product on $\mathbb{C}^{p\times q}$ associated to B is

$$\{\mathbf{Z}, \mathbf{W}^*, \mathbf{U}\} := \frac{1}{2}\left(\mathbf{Z}\mathbf{W}^*\mathbf{U} + \mathbf{U}\mathbf{W}^*\mathbf{Z}\right) .$$

$\mathfrak{p}$ is the image of the $\mathbb{C}$-vectorspace embedding

$$Z \hookrightarrow \mathfrak{g} \ , \ \mathbf{Z} \mapsto \left(\begin{array}{c|c} 0 & \mathbf{Z} \\ \hline \mathbf{Z}^* & 0 \end{array}\right) ,$$

and the BERGMAN metric on B at $\mathbf{0} \in B$ coincides with the euclidian one up to a positive constant. Z and so B are of rank q , since the standard frame of Z is $(\mathbf{e}_1, \ldots, \mathbf{e}_q)$, where

$$\mathbf{e}_j := \begin{array}{c} \left(\begin{array}{ccccccc} 0 & & & & & & \\ & \ddots & & & & & 0 \\ & & 0 & & & & \\ & & & 1 & & & \\ & & & & 0 & & \\ & 0 & & & & \ddots & \\ & & & & & & 0 \\ \hline & & & 0 & & & \end{array}\right) \end{array} \begin{array}{c} \\ \\ \\ \leftarrow j \\ \\ \\ \\ \\ \end{array} \quad \in \mathbb{C}^{p\times q} .$$

Any other tripotent $\mathbf{c}$ in Z can be written as $\mathbf{c} = k \sum_{j=1}^{\mathrm{rk}\,\mathbf{c}} \mathbf{e}_j$ with an appropriate element $k \in K$, since Z is ireducible. The PEIRCE decomposition to the tripotent $\mathbf{c} := \sum_{j=1}^{r} \mathbf{e}_j$ is

$$\begin{aligned} Z_1\left(\mathbf{c}\right) &= \left\{ \left(\begin{array}{c|c} \mathbf{z} & 0 \\ \hline 0 & 0 \end{array}\right) \,\middle|\, \mathbf{z} \in \mathbb{C}^{r\times r} \right\} , \\ Z_{\frac{1}{2}}\left(\mathbf{c}\right) &= \left\{ \left(\begin{array}{c|c} 0 & \mathbf{w}_1 \\ \hline \mathbf{w}_2 & 0 \end{array}\right) \,\middle|\, \mathbf{w}_1 \in \mathbb{C}^{r\times(q-r)}, \mathbf{w}_2 \in \mathbb{C}^{(p-r)\times r} \right\} , \\ Z_0\left(\mathbf{c}\right) &= \left\{ \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathbf{u} \end{array}\right) \,\middle|\, \mathbf{u} \in \mathbb{C}^{(p-r)\times(q-r)} \right\} \sqsubset \mathbb{C}^{p\times q} . \end{aligned}$$

The standard maximal Abelian subgroup A of G is the image of the LIE group embedding

$$\mathbb{R}^q \hookrightarrow G\,,$$

$$\mathbf{t} \mapsto a_{\mathbf{t}} := \left(\begin{array}{ccc|c|ccc}
\cosh t_1 & & 0 & & \sinh t_1 & & 0 \\
& \ddots & & 0 & & \ddots & \\
0 & & \cosh t_q & & 0 & & \sinh t_q \\
\hline
& 0 & & 1 & & 0 & \\
\hline
\sinh t_1 & & 0 & & \cosh t_1 & & 0 \\
& \ddots & & 0 & & \ddots & \\
0 & & \sinh t_q & & 0 & & \cosh t_q
\end{array}\right),$$

all the MFTG surfaces can be transformed by an element $g \in G$ to the standard MFTG submanifold $Q := A\mathbf{0}$, and

$$\mathbb{R}^q \xrightarrow{\sim} Q\,,\ \mathbf{t} \mapsto a_{\mathbf{t}}\mathbf{0} = \left(\begin{array}{ccc}
\tanh t_1 & & 0 \\
& \ddots & \\
0 & & \tanh t_q \\
\hline
& 0 &
\end{array}\right)$$

is an isometry between $\mathbb{R}^q$ and the standard MFTG submanifold Q . The centralizer M of A in K is the subgroup of K of all

$$\left(\begin{array}{ccc|c|ccc}
\varepsilon_1 & & 0 & & & & \\
& \ddots & & 0 & & 0 & \\
0 & & \varepsilon_q & & & & \\
\hline
& 0 & & u & & & \\
\hline
& & & & \varepsilon_1 & & 0 \\
& 0 & & & & \ddots & \\
& & & & 0 & & \varepsilon_q
\end{array}\right),$$

where $\varepsilon \in U(1)^q$ and $u \in U(p-q)$ such that $\varepsilon_1^2 \cdots \varepsilon_q^2 \det u = 1$.

Now let us return to the general case. On G we have an analytic multiflow $(\varphi_{\mathbf{t}})_{\mathbf{t}\in\mathbb{R}^q}$ given by the right translation by elements of A :

$$\varphi_{\mathbf{t}} : G \to G\,,\ g \mapsto g a_{\mathbf{t}}\,.$$

Since the multiflow $(\varphi_{\mathbf{t}})_{\mathbf{t}\in\mathbb{R}^q}$ commutes with right translations by elements of M it canonically projects down to the quotient G/M of G , where it has a nice but more complicated geometric interpretation as the so-called frame flow: On B we have the frame bundle

$$\begin{aligned} S(B) &:= \{(\mathbf{Z}, \xi_1, \ldots, \xi_q) \mid \mathbf{Z} \in B \text{ and } (\xi_1, \ldots, \xi_q) \text{ is a frame in } T_{\mathbf{Z}}B\} \\ &\simeq G/M \end{aligned}$$

since G acts transitively on $S(B)$, and M is the stabilizer of the standard frame sitting at $\mathbf{0}$. Let $(\overline{\varphi}_{\mathbf{t}})_{\mathbf{t}\in\mathbb{R}^q}$ be the projection of $(\varphi_{\mathbf{t}})_{\mathbf{t}\in\mathbb{R}^q}$ down to $S(B)$. Then for any $(\mathbf{Z}, \xi_0, \ldots, \xi_q) \in S(B)$ and $\mathbf{t} \in \mathbb{R}^q$ we obtain the image point $\varphi_{\mathbf{t}}(\mathbf{Z}, \xi_0, \ldots, \xi_q) = \left(\mathbf{Z}', \xi_0', \ldots, \xi_q'\right) \in S(B)$ by the following procedure: To $(\mathbf{Z}, \xi_0, \ldots, \xi_q) \in S(B)$ there exists a unique MFTG submanifold Q of B containing $\mathbf{Z}$ and having the linear span of $\xi_0, \ldots, \xi_q$ as tangent space at $\mathbf{Z}$. Following Q starting in $\mathbf{Z}$ and walking t_j-far in the direction of ξ_j , $j = 1, \ldots, q$, we reach the point $\mathbf{Z}'$, and finally the frame $\left(\xi_0', \ldots, \xi_q'\right) \in TB(\mathbf{Z}')^q$ of primitive pairwise orthogonal tripotents is given by parallel transport of $(\xi_0, \ldots, \xi_q)$ along Q . Hereby the result is independent of the choice of the curve joining $\mathbf{Z}$ and $\mathbf{Z}'$, since Q is flat (isometric to $\mathbb{R}^q$)! So the frame flow generalizes the geodesic flow on the unit tangent bundle $S(B)$ in the $q = 1$ case.

Since all right translations on G are left invariant, the differential of the flow $(\varphi_{\mathbf{t}})_{\mathbf{t}\in\mathbb{R}^q}$ corresponds to the adjoint representation of A on the Lie algebra $\mathfrak{g}$ of G via the identification of all tangentspaces T_gG , $g \in G$, with $\mathfrak{g}$ by left translation. So let us decompose the adjoint representation of A :

$$\mathfrak{g} = \bigoplus_{\alpha\in\Phi} \mathfrak{g}^\alpha ,$$

where for all $\alpha \in (\mathbb{R}^q)^*$

$$\mathfrak{g}^\alpha := \left\{\xi \in \mathfrak{g} \;\middle|\; \mathrm{Ad}_{a_{\mathbf{t}}}(\xi) = e^{\alpha\mathbf{t}}\xi \right\}$$

and

$$\Phi := \{\, \alpha \in (\mathbb{R}^q)^* \mid g^\alpha \neq 0\} \ .$$

Then Φ is called the root system of G , it is clearly always finite. For all $\alpha, \beta \in \Phi$ we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ if $\alpha + \beta \in \Phi$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$ otherwise. In [13] proposition 9.19 we have an explicit description of Φ :

Theorem 1.7

(i) $\mathbf{0} \in \Phi$ *, and* $g^{\mathbf{0}} = \mathfrak{a} + \mathfrak{m}$ *is the* Lie *algebra of* AM *.*

(ii) If B *is irreducible then*

$$\Phi = \{\mathbf{0}\} \cup \{\, \pm 2\mathbf{e}_i^* \mid i = 1, \ldots, q\} \cup \left\{\, \pm\mathbf{e}_i^* \pm \mathbf{e}_j^* \;\middle|\; i, j = 1, \ldots, q, i \neq j \right\}$$

if B is of tube type,

$$\Phi = \{\mathbf{0}\} \cup \{\pm\mathbf{e}_i^* \mid i = 1, \ldots, q\} \cup \{\pm 2\mathbf{e}_i^* \mid i = 1, \ldots, q\} \\ \cup \left\{ \pm\mathbf{e}_i^* \pm \mathbf{e}_j^* \,\middle|\, i, j = 1, \ldots, q, i \neq j \right\}$$

if B is not of tube type.

(iii) The spaces $\mathfrak{g}^{2\mathbf{e}_j^*}$ are always one-dimensional.

Let $\mathbf{v} \in \mathbb{R}^q$. Then $\mathbf{v}$ is called regular if and only if $\alpha\mathbf{v} \neq 0$ for all $\alpha \in \Phi \setminus \{\mathbf{0}\}$. The connected components of the open and dense subset of all regular $\mathbf{v} \in \mathbb{R}^q$ are called the WEYL chambers of G . If B is irreducible then $\mathbf{v}$ is regular if and only if all $0, |v_1|, \ldots, |v_q|$ are pairwise different, and we have a standard WEYL chamber

$$\{\mathbf{v} \in \mathbb{R}^q \mid 0 < v_1 < \cdots < v_q\} \ .$$

Fix a regular $\mathbf{v}_0 \in \mathbb{R}^q$. Then

$$\mathfrak{n} := \bigoplus_{\alpha\in\Phi\,,\,\alpha\mathbf{v}_0>0} \mathfrak{g}^\alpha \,,$$

which actually only depends on the WEYL chamber containing $\mathbf{v}$, is a sub LIE algebra of $\mathfrak{g}$ such that $[\mathfrak{a}, \mathfrak{n}] \sqsubset \mathfrak{n}$, and the corresponding subgroup $N := \exp_G \mathfrak{n}$ of G is a nilpotent sub LIE group of G . Note that $\mathfrak{n}$ and $\mathfrak{a}$ are perpendicular with respect to the KILLING form on $\mathfrak{g}$, and $\mathfrak{n} \cap \mathfrak{p} = \mathfrak{n} \cap \mathfrak{k} = \{0\}$ since the KILLING form is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$, but for $\alpha, \beta \in \Phi$, $\xi \in \mathfrak{g}^\alpha$ and $\eta \in \mathfrak{g}^\beta$ we have $(\xi, \eta) \neq 0$ only if $\beta = -\alpha$, and so $(\eta, \eta) = 0$ for all $\eta \in \mathfrak{n}$. Finally we have the so-called IWASAWA decomposition

$$G = KAN = NAK = ANK \,.$$

Definition 1.8 (loxodromic elements resp. subgroups of G) *Let $a \in G$, and let $\Gamma_0 \sqsubset G$ be a discrete subgroup.*

(i) a is called loxodromic if and only if there exists $g \in G$ such that $a \in gAMg^{-1}$.

(ii) If a is loxodromic, it is called regular if and only if $a = ga_{\mathbf{t}}wg^{-1}$ with $\mathbf{t} \in \mathbb{R}^q$ regular.

(iii) Γ_0 is called loxodromic if and only if there exists $g \in G$ such that $\Gamma_0 \sqsubset gAMg^{-1}$ is a lattice, and hence cocompact.

So any loxodromic element $a \in G$ can be written in the form $a = ga_{\mathbf{v}}wg^{-1}$ for some $g \in G$, $\mathbf{v} \in \mathbb{R}^q$ and $w \in M$, and for all loxodromic $\Gamma_0 \sqsubset G$ and $g \in G$ such that $\Gamma_0 \sqsubset gAMg^{-1}$ is a lattice we see that

$$\phi : \Gamma_0 \to \mathbb{R}^q \, , \; \gamma = ga_{\mathbf{t}}wg^{-1} \mapsto \mathbf{t}$$

is a group homomorphism, $\mathrm{Ker}\phi = \Gamma_0 \cap \left(gMg^{-1}\right)$ is finite, and

$$\Lambda := \mathrm{Im}\;\phi \sqsubset \mathbb{R}^q$$

is a lattice, hence cocompact. The next theorem should clarify in which way the elements $g \in G$, $\mathbf{v} \in \mathbb{R}^q$, the group homomorphism Φ , its kernel and its image Λ are uniquely determined by a resp. Γ_0 . Especially we will see that (ii) of definition 1.8 is independent of the choice of g .

Lemma 1.9

$$N(A) = AN_K(A) = N(AM) \sqsubset N(M)\,.$$

Proof: '$N(A) \sqsubset AN_K(A)$' : Let $g \in N(A)$. Then by the Iwasawa decomposition we can write $g = ank$ with appropriate $k \in K$, $n \in N$ and $a \in A$. But since clearly $A \sqsubset N(A)$ we may assume $a = 1$ without loss of generality. Let $\zeta \in \mathfrak{a}$ and $\zeta' := \mathrm{Ad}_g(\zeta) = \mathrm{Ad}_k\left(\mathrm{Ad}_n(\zeta)\right) \in \mathfrak{a}$. Then on one hand we have $\mathrm{Ad}_{k^{-1}}(\zeta') \in \mathfrak{p}$, and on the other hand

$$\mathrm{Ad}_{k^{-1}}(\zeta') = \mathrm{Ad}_n(\zeta) = \zeta + \eta$$

with an appropriate $\eta \in \mathfrak{n}$. So $\eta \in \mathfrak{n} \cap \mathfrak{p}$ and therefore $\eta = 0$. So n commutes with all $a \in A$, but this is only possible if $n = 1$ since conjugation by $a_{-\mathbf{v}_0}$ is a contraction on N with respect to the left invariant metric on G .

'$AN_K(A) \sqsubset N(AM)$' is trivial since M is the centralizer of A in K .

'$N(AM) \sqsubset N(M)$' : Since $M = Z_K(A)$, all the root spaces of A in $\mathfrak{g}$ are invariant under M , and since M is a compact group we see that the adjoint representation of M on $\mathfrak{g}$ is unitary on each of them with respect to an appropriately chosen scalar product. So $g \in N(AM)$ permutes the root spaces, and the adjoint representation of gMg^{-1} on $\mathfrak{g}$ is unitary on all root spaces with respect to an appropriate scalar product. Now let $w \in M$. Then there exist $\mathbf{t} \in \mathbb{R}^q$ and $w' \in M$ such that

$$gwg^{-1} = a_{\mathbf{t}}w'\,. \tag{1.3}$$

Looking for example at the spaces $\mathfrak{g}^{2\mathbf{e}_j^*}$, $j = 1, \ldots q$, shows that equation 1.3 is possible only if $\mathbf{t} = \mathbf{0}$, and this means $gwg^{-1} = w' \in M$.

'$N(AM) \sqsubset N(A)$' : Let $g \in N(AM)$. Then $g \in N(M)$ as we have already seen. Since $\mathfrak{a}$ is the unique orthogonal complement to $\mathfrak{m}$ with respect to the KILLING form restricted to $\mathfrak{a} + \mathfrak{m}$ and the adjoint representation of G on $\mathfrak{g}$ respects the KILLING form, it follows immediately that even $g \in N(A)$. □

Clearly M is a normal subgroup of $N_K(A)$, and the so-called WEYL group $W := N_K(A)/M$ acts on A via conjugation, and so on $\mathbb{R}^q$ as well, and this action is simply transitive on the WEYL chambers, and so it is isomorphic to the subgroup of $GL(q, \mathbb{R})$ permuting the components within each irreducible summand of $\mathbb{R}^q$ and changing signs of the components, hence isomorphic to

$$\mathfrak{S}(q_1) \times \cdots \times \mathfrak{S}(q_s) \times \{\pm 1\}^q ,$$

see for example in [15] , Lecture 2 .

Definition 1.10 *We call two vectors $\mathbf{v}$ and $\mathbf{v}' \in \mathbb{R}^q$ WEYL equivalent if and only if the corresponding elements $a_{\mathbf{v}}$ and $a_{\mathbf{v}'}$ of A are conjugated by an element of the WEYL group, in other words $\mathbf{v}$ and $\mathbf{v}'$ are equal up to permuting the components within each irreducible summand of $\mathbb{R}^q$ and changing signs.*

Clearly regularity is invariant under WEYL equivalence.

Theorem 1.11

(i) Let $a \in G$ be loxodromic, $g \in G$, $w \in M$ and $\mathbf{v} \in \mathbb{R}^q$ be regular such that $a = g a_{\mathbf{v}} w g^{-1}$. Then g is uniquely determined up to right translation by elements of $AN_K(A)$, and $\mathbf{v}$ is uniquely determined up to WEYL equivalence.

(ii) Let $\Gamma_0 \sqsubset G$ be loxodromic, $g \in G$ such that $\Gamma_0 \sqsubset gAMg^{-1}$,

$$\phi : \Gamma_0 \to \mathbb{R}^q$$

the corresponding group homomorphism and $\Lambda := \mathrm{Im}\, \phi \sqsubset \mathbb{R}^q$ the corresponding lattice. Then again g is uniquely determined up to right translation by elements of $AN_K(A)$, ϕ and so Λ are uniquely determined up to WEYL equivalence, and $\mathrm{Ker}\phi = \Gamma_0 \cap gMg^{-1}$ is independent of g .

Proof: (i) Let $g' \in G$, $w' \in M$ and $\mathbf{v}' \in \mathbb{R}$ such that also $a = g' a_{\mathbf{v}'} w' g'^{-1}$. Then $a_{\mathbf{v}} w = \left(g^{-1} g'\right) a_{\mathbf{v}'} w' \left(g^{-1} g'\right)$. Since $\mathbf{v} \in \mathbb{R}^q$ is regular, $\mathfrak{a} + \mathfrak{m}$ is the largest subspace of $\mathfrak{g}$ on which the conjugation with $a_{\mathbf{v}} w$ is orthogonal with respect to an appropiate scalar product. So conjugation with $g^{-1}g'$ stabilizes $\mathfrak{a}+\mathfrak{m}$. This implies $g^{-1}g' \in N(AM) = AN_K(A)$ by lemma 1.9 , and so $\mathbf{v}'$ is in the image of $\mathbf{v}$ under the WEYL group W . □

(ii) This is a trivial consequence of (i) , because in the lattice Λ we always find regular elements. □

In both cases the groups $gAMg^{-1}$, gAg^{-1} and gMg^{-1} are independent of the choice of g .

Now let $\Gamma \sqsubset G$ be a discrete subgroup of G . Then geometrically the maximal loxodromic subgroups of Γ correspond to closed MFTG submanifolds on $\Gamma\backslash B$. Hereby a subset $R \subset \Gamma\backslash B$ is called a closed MFTG submanifold, if and only if it is the image of an MFTG submanifold $R' \in B$ under the canonical projection, and the composition $\mathbb{R}^q \xrightarrow{\sim} R' \to R$ of the canonical isometry and the canonical projection factors through a lattice of $\mathbb{R}^q$.

Theorem 1.12 *There is a one to one correspondence*

$$\{R \subset \Gamma\backslash B \textit{ closed MFTG surface }\}$$
$$\leftrightarrow \{\Gamma_0 \sqsubset \Gamma \textit{ max. lox. subgroup of } \Gamma\}/ \textit{ conjugation by elements of } \Gamma$$

given by

$$\begin{array}{rcl} R & \mapsto & \Gamma \cap \left(gAMg^{-1}\right) \textit{ , where } g \in G \textit{ such that } R = \Gamma\backslash(gQ)\,, \\ \Gamma\backslash(gQ) & \leftarrow\!\shortmid & \Gamma_0 \textit{ , where } g \in G \textit{ such that } g^{-1}\Gamma_0 g \sqsubset AM\,. \end{array}$$

Proof: Let us first check that both mappings are well-defined:

Let $g, g' \in G$ such that $\Gamma\backslash(g'Q) = \Gamma\backslash(gQ)$ and define

$$\Gamma_0 := \Gamma \cap \left(gAMg^{-1}\right)\,,$$

which is clearly a maximal loxodromic subgroup of Γ . Then since Q is connected and Γ discrete there exists $\gamma \in \Gamma$ such that $\gamma g'Q = gQ$, and so $n := g'^{-1}\gamma^{-1}g \in N(Q) = N(AM)$. Therefore we have

$$\begin{aligned} \Gamma \cap \left(g'AMg'^{-1}\right) &= \Gamma \cap \left(\gamma^{-1}gn^{-1}AMng^{-1}\gamma\right) \\ &= \gamma^{-1}\left(\Gamma \cap \left(gAMg^{-1}\right)\right)\gamma \\ &= \gamma^{-1}\Gamma_0\gamma\,. \end{aligned}$$

Now let Γ_0 be a maximal loxodromic subgroup of Γ , let $g, g' \in G$ such that $g^{-1}\Gamma_0 g, g'^{-1}\Gamma_0 g' \sqsubset AM$ and $\gamma \in \Gamma$. Then by theorem 1.11 $g' = gn$ with an appropriate $n \in AN_K(A) = N(Q)$. Then we have $g'Q = gnQ = gQ$, and so the result is independent of the choice of g .

Clearly $\Gamma_0' := \gamma^{-1}\Gamma_0\gamma$ is again a maximal loxodromic subgroup of Γ , and

$$\Gamma_0' \sqsubset \left(\gamma^{-1}g\right) AM \left(\gamma^{-1}g\right)^{-1} ,$$

and so finally

$$\Gamma \backslash \left(\left(\gamma^{-1}g\right) Q\right) = \Gamma \backslash (gQ) .$$

Clearly both mappings are inverse to each other. □

The boundary of B can easily be described in terms of tripotents of Z . Let $\mathbf{c} \in Z$ be a tripotent. Then both $Z_1(\mathbf{c})$ and $Z_0(\mathbf{c})$ are sub Jordan triple systems of Z , and so $B_{\mathbf{c}} := B \cap Z_0(\mathbf{c})$ is the unit ball in $Z_0(\mathbf{c})$. We see that $B_{\mathbf{c}}$ is itself a bounded symmetric domain of lower dimension and rank $q - \operatorname{rk} \mathbf{c}$. The tripotents in $Z_0(\mathbf{c})$ are precisely the tripotents in Z orthogonal to $\mathbf{c}$.
A subset F of $\overline{B}$ is called a face of $\overline{B}$ if and only if it is closed and convex and fulfills the extremality condition: If $\mathbf{a}, \mathbf{b} \in \overline{B}$ such that

$$]\, \mathbf{a}, \mathbf{b}\, [\, \cap F \neq \emptyset$$

then $[\mathbf{a}, \mathbf{b}] \subset F$.

Theorem 1.13

(i) Let $\mathbf{c} \in \mathbb{Z}$ be a tripotent. Then $F_{\mathbf{c}} := \mathbf{c} + \overline{B}_{\mathbf{c}}$ is a face of $\overline{B}$.

(ii) Let F be a face of $\overline{B}$. Then there exists a unique tripotent of Z such that $F = \mathbf{c} + \overline{B} \cap Z_0(\mathbf{c})$.

Proof: Up to uniqueness in (ii) this is theorem 1.5.47 of [16] . To see uniqueness of $\mathbf{c}$ in (ii) let $\mathbf{c}'$ be another tripotent such that $F = \mathbf{c}' + \overline{B} \cap Z_0(\mathbf{c}')$. Then $\mathbf{c}' - \mathbf{c}$ and $\mathbf{c}$ are orthogonal, and therefore $\mathbf{c}' - \mathbf{c}$ is a tripotent in $B_{\mathbf{c}}$, and so it must be $\mathbf{0}$. □

For all $j \in \{0, \ldots, q\}$ let S_j be the set of all tripotents $\mathbf{c}$ of Z of rank j . Then S_j is a closed submanifold of Z , $S_0 = \{\mathbf{0}\}$, and since the (relative to $\mathbf{c} + Z_0(\mathbf{c})$) open faces $\mathbf{c} + B_{\mathbf{c}}$ are pairwise disjoint, we have a partition

$$\partial B = \bigcup_{j=1,\dots,j} \bigcup_{\mathbf{c}\in S_j} B_{\mathbf{c}}\,.$$

If $\mathbf{c} = \mathbf{0}$ then $B_{\mathbf{c}} = B$, and this is the only open face of $\overline{B}$ which is not contained in ∂B . If $\mathbf{c} \in S_q$ then $Z_0 = \{\mathbf{0}\}$, and so the associated face is $\{\mathbf{c}\}$ itself. We see that S_q is the submanifold of extremepoints of B , and it is called the SHILOV boundary of B .

To each tripotent $\mathbf{c} \in Z$ we have a partial CAYLEY transformation

$$R_{\mathbf{c}} := \exp\left(\frac{\pi}{4}\left(\mathbf{c} + \widetilde{\mathbf{c}}\right)\right) \in G^{\mathbb{C}},$$

which maps biholomorphically B onto the unbounded symmetric domain $H_{\mathbf{c}} := R_{\mathbf{c}}(B) \subset Z$. For describing $H_{\mathbf{c}}$, which is a SIEGEL domain of type III , we need some more information about the PEIRCE decomposition $Z = Z_1\left(\mathbf{c}\right) \oplus Z_{\frac{1}{2}}\left(\mathbf{c}\right) \oplus Z_0\left(\mathbf{c}\right)$, see for example chapter 10 and section 3.13 in [13] .

$Z_1\left(\mathbf{c}\right)$ is a Hermitian JORDAN algebra with product given by

$$\mathbf{ZW} = \{\mathbf{Z}, \mathbf{c}^*, \mathbf{W}\}$$

for all $\mathbf{Z}, \mathbf{W} \in Z_1\left(\mathbf{c}\right)$. $Z_1\left(\mathbf{c}\right)$ has the unit element $\mathbf{c}$ and the involution * given by

$$\mathbf{Z}^* := \{\mathbf{c}, \mathbf{Z}^*, \mathbf{c}\}$$

for all $\mathbf{Z} \in Z_1\left(\mathbf{c}\right)$. The set

$$\mathcal{A} := \{\mathbf{Z} \in Z_1\left(\mathbf{c}\right) \mid \mathbf{Z}^* = \mathbf{Z}\ \}$$

of with respect to * real points of $Z_1\left(\mathbf{c}\right)$ is a formally real sub-$\mathbb{R}$-JORDAN-algebra of $Z_1\left(\mathbf{c}\right)$, and $Z_1\left(\mathbf{c}\right) = \mathcal{A} \oplus i\mathcal{A}$. Let Y be the positive cone of $\mathcal{A}$, this means

$$Y := \left\{\mathbf{Z}^2 \,|\, \mathbf{Z} \in \mathcal{A} \setminus \{\mathbf{0}\}\right\}\,.$$

Define

$$F : Z_{\frac{1}{2}}\left(\mathbf{c}\right) \times Z_{\frac{1}{2}}\left(\mathbf{c}\right) \to Z_1\left(\mathbf{c}\right)\,,\; F\left(\mathbf{V}, \mathbf{W}\right) := \{\mathbf{V}, \mathbf{W}^*, \mathbf{c}\}\,.$$

Then F is $\mathbb{C}$-linear in the first and $\mathbb{C}$-antilinear in the second variable, it is Hermitian with respect to * and positive definite with respect to Y . For all $\mathbf{Z} \in Z_0\left(\mathbf{c}\right)$ we define

$$\varphi_{\mathbf{Z}} : Z_{\frac{1}{2}}(\mathbf{c}) \to Z_{\frac{1}{2}}(\mathbf{c}) \, , \, \mathbf{W} \mapsto \{\mathbf{c}, \mathbf{W}^*, \mathbf{Z}\} \, ,$$

which is $\mathbb{C}$-antilinear and selfadjoint with respect to F . We have

$$0 \neq F(\mathbf{W}, \mathbf{W}) - F(\varphi_{\mathbf{Z}}(\mathbf{W}), \varphi_{\mathbf{Z}}(\mathbf{W})) \in \overline{Y} \setminus \{\mathbf{0}\} \, ,$$

where $\overline{Y}$ here denotes the topological closure of Y in $\mathcal{A}$, therefore $\mathrm{Id} + \varphi_{\mathbf{Z}} \in \mathrm{GL}_{\mathbb{R}}\left(Z_{\frac{1}{2}}\right)$ for each $\mathbf{Z} \in Z_1(\mathbf{c})$, and so for all $\mathbf{Z} \in Z_1(\mathbf{c})$ we can define

$$F_{\mathbf{Z}} : Z_{\frac{1}{2}}(\mathbf{c}) \times Z_{\frac{1}{2}}(\mathbf{c}) \to Z_1(\mathbf{c}) \, , \, F_{\mathbf{Z}}(\mathbf{V}, \mathbf{W}^*) := F_{\mathbf{Z}}\left(\mathbf{V}, (\mathrm{Id} + \varphi_{\mathbf{Z}})^{-1}(\mathbf{W})^*\right) \, ,$$

which is $\mathbb{C}$-linear in the first but in general only $\mathbb{R}$-linear in the second variable.

Finally we have

Theorem 1.14

$$\begin{aligned} H_{\mathbf{c}} \;&=\; \left\{ \mathbf{Z}_1 + \mathbf{Z}_{\frac{1}{2}} + \mathbf{Z}_0 \,\middle|\, \mathbf{Z}_1 \in Z_1(\mathbf{c}) \, , \, \mathbf{Z}_{\frac{1}{2}} \in Z_{\frac{1}{2}}(\mathbf{c}) \, , \, \mathbf{Z}_0 \in B_{\mathbf{c}} \, , \right. \\ &\qquad \left. \mathrm{Re}\left(\mathbf{Z}_1 - \frac{1}{2} F_{\mathbf{Z}_0}\left(\mathbf{Z}_{\frac{1}{2}}, \mathbf{Z}_{\frac{1}{2}}\right)\right) \in Y \right\} . \end{aligned}$$

Proof: This is precisely theorem 10.8 of [13] . □

We see that there is a canonical embedding $i\mathcal{A} \hookrightarrow \mathrm{Aut}_0(D_{\mathbf{c}}) = R_{\mathbf{c}} G R_{\mathbf{c}}^{-1}$, $i\mathcal{A}$ acting on $H_{\mathbf{c}}$ via translation.

1.2 The space of cusp forms on a bounded symmetric domain

Let $k \in \mathbb{Z}$ be a fixed integer and let

$$j : G \times B \to \mathbb{C} \, , \, (g, \mathbf{Z}) \mapsto \det g'(\mathbf{Z}) \, .$$

Then clearly j fulfills the cocycle property $j(gh, \mathbf{Z}) = j(g, h\mathbf{Z}) \, j(h, \mathbf{Z})$, and so on $B^{\mathbb{C}}$ we have a right action of G :

$$|_g : \mathbb{C}^B \to \mathbb{C}^B \, , \, f \mapsto f|_g$$

for all $g \in G$ where

$$f|_g(\mathbf{Z}) := f(g\mathbf{Z}) \, j(g, \mathbf{Z})^k$$

for all $\mathbf{Z} \in B$. This action is clearly holomorphic in the sense that if $f \in \mathcal{O}(B)$ then again $f|_g \in \mathcal{O}(B)$ for all $g \in G$, and we have a lift

$$\widetilde{\ } : \mathbb{C}^B \hookrightarrow \mathbb{C}^G \, , \; f \mapsto \widetilde{f}$$

where

$$\widetilde{f}(g) := f\left(g\mathbf{0}\right) j\left(g, \mathbf{0}\right)^k = f|_g\left(\mathbf{0}\right)$$

for all $f \in \mathbb{C}^B$ and $g \in G$, which is $\mathcal{C}^\infty$ in the sense that if $f \in \mathcal{C}^\infty(B)^\mathbb{C}$ then again $\widetilde{f} \in \mathcal{C}^\infty(G)^\mathbb{C}$. The right action $|_g$ on $\mathbb{C}^B$ lifted to $\mathbb{C}^G$ is simply the left translation, more precisely

$$\begin{array}{ccc} \mathbb{C}^G & \overset{(g\diamondsuit)}{\longrightarrow} & \mathbb{C}^G \\ \uparrow\sim & \% & \uparrow\sim \\ \mathbb{C}^B & \underset{|_g}{\longrightarrow} & \mathbb{C}^B \end{array}$$

for all $g \in G$.

On G we always use the left invariant Haar measure, which is at the same time the right invariant Haar measure since G is semisimple, hence unimodular. Let Γ be a discrete subgroup of G . Then we define a 'scalar product'

$$(f, h)_\Gamma := \int_{\Gamma\backslash G} \overline{\widetilde{f}}\,\widetilde{h}$$

for all $f, g \in \mathbb{C}^B$ such that $\overline{\widetilde{f}}\,\widetilde{h} \in L^1\left(\Gamma\backslash G\right)$ and for all $r \in]0, \infty]$

$$L_k^r\left(\Gamma\backslash B\right) := \left\{ f \in \mathbb{C}^B \,\middle|\, \widetilde{f} \text{ left-}\Gamma\text{-invariant and } \in L^r\left(\Gamma\backslash G\right) \right\} .$$

Then clearly especially all $(\ ,\ ) := (\ ,\ )_{\{1\}}$ and all $L_k^r(B)$ are invariant under the action $|_g$, $g \in G$.

Definition 1.15 (automorphic resp. cusp forms on B)

(i) Let $f \in \mathcal{O}(B)$. f is called an automorphic form for Γ of weight k if and only if $\widetilde{f}$ is left-Γ-invariant or equivalently $f = f|_\gamma$ for all $\gamma \in \Gamma$. The $\mathbb{C}$-vector space of all automorphic forms for Γ of weight k is denoted by $M_k(\Gamma)$.

(ii) Let $f \in M_k(\Gamma)$. f is called a cusp form for Γ of weight k if and only if $f \in L_k^2\left(\Gamma\backslash B\right)$ .The $\mathbb{C}$-vector space of all cusp forms for Γ of weight k is denoted by $S_k(\Gamma) := M_k(\Gamma) \cap L_k^2\left(\Gamma\backslash B\right) = \mathcal{O}(B) \cap L_k^2\left(\Gamma\backslash B\right)$.

Let $\Delta : Z \times Z \to \mathbb{C}$ be the Jordan triple determinant of Z and P be the genus of B , see for example in 1.5 and 2.9 of [16] . Then Δ has the following properties:

(i) $\Delta(\mathbf{0},\diamondsuit) = 1$,

(ii) Δ is a sesqui polynomial, holomorphic in the first and antiholomorphic in the second variable,

(iii) $\Delta(\mathbf{Z},\mathbf{W}) = \overline{\Delta(\mathbf{W},\mathbf{Z})}$ for all $\mathbf{Z},\mathbf{W} \in Z$ and $\Delta(\mathbf{Z},\mathbf{Z}) > 0$ for all $\mathbf{Z} \in B$,

(iv) $|j(g,\mathbf{0})| = \Delta(g\mathbf{0},g\mathbf{0})^{\frac{P}{2}}$ for all $g \in G$,

(v) $\Delta(g\mathbf{Z},g\mathbf{W})^P = \Delta(\mathbf{Z},\mathbf{W})^P\, j(g,\mathbf{Z})\,\overline{j(g,\mathbf{W})}$ for all $g \in G$ and $\mathbf{Z},\mathbf{W} \in B$,

(vi) $\int_B \Delta(\mathbf{Z},\mathbf{Z})^\lambda\, dV_{\text{Leb}} < \infty$ if and only if $\lambda > -1$, and finally

(vii) if $Z = Z_1 \oplus \cdots \oplus Z_s$ is the decomposition of Z into simple summands then for all $\mathbf{Z},\mathbf{W} \in Z$

$$\Delta(\mathbf{Z},\mathbf{W}) = \Delta_1(\mathbf{Z}_1,\mathbf{W}_1)\cdots\Delta_s(\mathbf{Z}_s,\mathbf{W}_s) ,$$

where $\Delta_1,\ldots,\Delta_s$ are the Jordan triple determinants of the Jordan triple systems $Z_1,\ldots,Z_q$ resp. .

By (iv) and (v) we have the G-invariant volume element $\Delta(\mathbf{Z},\mathbf{Z})^{-P}dV_{\text{Leb}}$ on B . So for all $r \in]\,0,\infty\,]$ and $f \in \mathbb{C}^B$ such that $\widetilde{f} \in \mathbb{C}^G$ is left-Γ-invariant we have $f \in L^r_k(\Gamma\backslash G)$ if and only if

$$|f|\,\Delta(\mathbf{Z},\mathbf{Z})^{\frac{kP}{2}} \in L^r(\Gamma\backslash B)$$

with respect to the measure $\Delta(\mathbf{Z},\mathbf{Z})^{-P}\,dV_{\text{Leb}}$ on B , and for all $f,g \in \mathbb{C}^B$ such that $\widetilde{f}g \in L^1(\Gamma\backslash G)$

$$(f,h)_\Gamma := \int_{\Gamma\backslash G} \overline{\widetilde{f}}\,\widetilde{h} \equiv \int_{\Gamma\backslash B} \overline{f}\,h\Delta(\mathbf{Z},\mathbf{Z})^{(k-1)P}\,dV_{\text{Leb}}\,.$$

Clearly $S_k(\Gamma)$ is the subspace of all $f \in M_k(\Gamma)$ such that $(f,f)_\Gamma < \infty$. Since convergence with respect to $(\ ,\ )_\Gamma$ implies compact convergence, we see that $(S_k(\Gamma),(\ ,\ )_\Gamma)$ is a Hilbert space.

Now fix a discrete subgroup $\Gamma \sqsubset G$.

A famous theorem by I. Satake says that under certain conditions there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and $r \in [1,\infty]$

$$S_k(\Gamma) = M_k(\Gamma) \cap L^r_k(\Gamma\backslash B)\,.$$

In many cases SATAKE's theorem holds, trivially if $\Gamma\backslash G$ is compact, but for example also in the case where B is (biholomorphic) to the unit ball of some $\mathbb{C}^n$, $n \geq 2$, (so rank $q = 1$) and vol $\Gamma\backslash G < \infty$. This can be shown by a calculation similar to that of section 3.2 . If SATAKE's theorem holds and vol $\Gamma\backslash G < \infty$ then $S_k(\Gamma)$ is finite dimensional for all $k \geq k_0$ since we have the following lemma, lemma 12 of [1] section 10. 2 :

> Let (X, μ) be a locally compact measure space, where μ is a positive measure such that $\mu(X) < \infty$. Let $\mathcal{F}$ be a closed subspace of $L^2(X, \mu)$ which is contained in $L^\infty(X, \mu)$. Then
>
> $$\dim \mathcal{F} < \infty .$$

Merely the only way to construct automorphic forms for Γ is by relative POINCARÉ series. Let $\Gamma' \sqsubset \Gamma$ be a subgroup and $f \in M_k(\Gamma')$. Then the relative POINCARÉ series

$$\sum_{\gamma \in \Gamma'\backslash\Gamma} f|_\gamma$$

defines a function in $M_k(\Gamma)$ provided that the convergence is 'good enough'. Recall that the summation is independent of the choice of a fundamental set of $\Gamma'\backslash\Gamma$ to be summated over since already $f \in M_k(\Gamma')$.

Theorem 1.16 (convergence of relative POINCARÉ series) *Let $\Gamma' \sqsubset \Gamma$ be a subgroup and*

$$f \in M_k\left(\Gamma'\right) \cap L^1_k\left(\Gamma'\backslash B\right) .$$

Then

$$\Phi := \sum_{\gamma \in \Gamma'\backslash\Gamma} f|_\gamma \text{ and } \widetilde{\Phi} := \sum_{\gamma \in \Gamma'\backslash\Gamma} \widetilde{f}\,(\gamma\diamondsuit)$$

converge absolutely and uniformly on compact subsets of B resp. G ,

$$\Phi \in M_k\left(\Gamma\right) \cap L^1_k\left(\Gamma\backslash B\right) ,$$

$\widetilde{\Phi}$ is the lift of Φ to G , and for all $\varphi \in M_k\left(\Gamma\right) \cap L^\infty_k\left(\Gamma\backslash B\right)$ we have

$$\left(\Phi, \varphi\right)_\Gamma = \left(f, \varphi\right)_{\Gamma'} .$$

Proof: Let $g_0 \in G$ and $L \subset G$ be a compact neighbourhood of g_0 in G such that $\gamma L \cap L = \emptyset$ for all $\gamma \in \Gamma \setminus \{1\}$. Since the canonical projection $\pi : G \to G/K \simeq B$ is open $\pi(L)$ is a compact neighbourhood of $g_0\mathbf{0}$. So by the mean value property of holomorphic functions and since j is continuous and nowhere zero, there exists a neighbourhood $U \subset L$ of g_0 in G and $C \in \mathbb{R}$ such that for all $h \in \mathcal{O}(B)$ and $g \in U$

$$\left|\widetilde{h}(g)\right| \leq C \int_L \left|\widetilde{h}\right| .$$

So for all $g \in U$

$$\begin{aligned}
\sum_{\gamma \in \Gamma' \backslash \Gamma} \left|\widetilde{f}(\gamma g)\right| &= \sum_{\gamma \in \Gamma' \backslash \Gamma} \left|\widetilde{f|_\gamma}(g)\right| \\
&\leq C \sum_{\gamma \in \Gamma' \backslash \Gamma} \int_L \left|\widetilde{f|_\gamma}\right| \\
&= C \sum_{\gamma \in \Gamma' \backslash \Gamma} \int_L \left|\widetilde{f}(\gamma \Diamond)\right| \\
&\leq C \int_{\Gamma' \backslash G} \left|\widetilde{f}\right| < \infty .
\end{aligned}$$

Since j is continuous and G is a Lie group we see that Φ and $\widetilde{\Phi}$ converge absolutely and uniformly on compact subsets of B resp. G , and $\widetilde{\Phi}$ is the lift of Φ to G . So clearly $\Phi \in M_k(\Gamma)$.

$$\int_{\Gamma \backslash G} \left|\widetilde{\Phi}\right| \leq \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma' \backslash \Gamma} \left|\widetilde{f}(\gamma \Diamond)\right| = \int_{\Gamma' \backslash G} \left|\widetilde{f}\right| < \infty ,$$

and so $\Phi \in L_k^1(\Gamma \backslash B)$. Now let $\varphi \in L_k^\infty(\Gamma \backslash B)$. Then $\widetilde{f}\widetilde{\varphi} \in L^1(\Gamma' \backslash G)$, and so

$$(\Phi, \varphi)_\Gamma = \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma' \backslash \Gamma} \overline{\widetilde{f}(\gamma \Diamond)} \widetilde{\varphi} = \int_{\Gamma' \backslash G} \overline{\widetilde{f}} \widetilde{\varphi} = (f, \varphi)_{\Gamma'} . \square$$

Let $\mathbf{W} \in B$. Since $S_k(\Gamma)$ is a Hilbert space and the evaluation

$$S_k(\Gamma) \to \mathbb{C} , \ \varphi \mapsto \varphi(\mathbf{W})$$

is a continuous linear form on $S_k(\Gamma)$, there exists a unique $\Phi_{\mathbf{W}} \in S_k(\Gamma)$ such that

$$\varphi(\mathbf{W}) = (\Phi_{\mathbf{W}}, \varphi)$$

for all $\varphi \in S_k(\Gamma)$. The following theorem in combination with Satake's theorem and theorem 1.16 gives an idea how to get the 'reproducing kernel' $\Phi_{\mathbf{W}}$ for $S_k(\Gamma)$.

Theorem 1.17 *Let $k \geq 2$. Then for all $\mathbf{W} \in B$*

$$\Delta\left(\diamondsuit, \mathbf{W}\right)^{-k} \in L^1_k(B)\,,$$

and for all $f \in \mathcal{O}(B) \cap L^\infty_k(B)$ we have

$$\left(\Delta\left(\diamondsuit, \mathbf{W}\right)^{-k}, f\right) \equiv f\left(\mathbf{W}\right)\,,$$

where $\equiv$ denotes equality up to a constant $\neq 0$ independent of $\mathbf{W}$ and f .

Proof: First treat the case $\mathbf{W} = \mathbf{0}$. Then $\Delta\left(\diamondsuit, \mathbf{0}\right)^{-kP} = 1$,

$$\int_G \left|\widetilde{1}\right| = \int_G \left|j\left(g, \mathbf{0}\right)^k\right| \equiv \int_B \Delta\left(\mathbf{Z}, \mathbf{Z}\right)^{\left(\frac{k}{2}-1\right)P} < \infty\,,$$

and for all $f \in \mathcal{O}(B) \cap L^\infty_k(B)$

$$(1, f) \equiv \int_B f \Delta\left(\mathbf{Z}, \mathbf{Z}\right)^{(k-1)P} dV_{\text{Leb}} \equiv f\left(\mathbf{0}\right)\,,$$

since $f \in \mathcal{O}(B)$ and $\Delta\left(\mathbf{Z}, \mathbf{Z}\right)$ and B are invariant under the circle group $U(1) \hookrightarrow K$.
Now let $\mathbf{W} \in B$ be arbitrary. Then there exists $g \in G$ such that $\mathbf{W} = g\mathbf{0}$, and so

$$\begin{aligned}\Delta\left(\diamondsuit, \mathbf{W}\right)^{-kP} = \Delta\left(\diamondsuit, g\mathbf{0}\right)^{-kP} = j\left(g^{-1}, \diamondsuit\right)^k \overline{j\left(g, \mathbf{0}\right)}^{-k} = \overline{j\left(g, \mathbf{0}\right)}^{-k} 1|_{g^{-1}} \\ \in L^1_k(B)\,.\end{aligned}$$

Let $f \in \mathcal{O}(B) \cap L^\infty_k(B)$. Then

$$\begin{aligned}\left(\Delta\left(\diamondsuit, \mathbf{W}\right)^{-kP}, f\right) &= \left(\overline{j\left(g, \mathbf{0}\right)}^{-k} 1|_{g^{-1}}, f\right) \\ &= j\left(g, \mathbf{0}\right)^{-k} \left(1, f|_g\right) \\ &\equiv j\left(g, \mathbf{0}\right)^{-k} f|_g\left(\mathbf{0}\right) \\ &= f\left(\mathbf{W}\right)\,.\,\Box\end{aligned}$$

Corollary 1.18 *Assume* Satake*' theorem holds, and let $k \geq \max\left(2, k_0\right)$. Then for all $\mathbf{W} \in B$*

$$\Phi_{\mathbf{W}} \equiv \sum_{\gamma \in \Gamma} \Delta\left(\diamondsuit, \mathbf{W}\right)^{-kP}\Big|_\gamma\,,$$

where $\equiv$ denotes equality up to a constant $\neq 0$ independent of $\mathbf{W}$ and Γ .

By the way a simple calculation shows that $\overline{\Phi_{\mathbf{W}}(\mathbf{Z})} = \Phi_{\mathbf{Z}}(\mathbf{W})$ for all $\mathbf{Z}, \mathbf{W} \in B$ and so $\Phi_{\mathbf{W}}(\mathbf{Z})$ is holomorphic in $\mathbf{Z}$ and antiholomorphic in $\mathbf{W}$.

Let us consider again the full matrix ball

$$B := \{\mathbf{Z} \in \mathbb{C}^{p\times q} | \mathbf{Z}^*\mathbf{Z} \gg 1\}$$

and the group $G := SU(p,q)$ acting on B as an example. The genus of B is $P := p + q$, and for all

$$g = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \begin{array}{l} \}p \\ \}q \end{array} \in G$$

and $\mathbf{Z} \in B$ we have

$$j(g, \mathbf{Z}) = \det(C\mathbf{Z} + D)^{-P} ,$$

although in general one defines $j(g, \mathbf{Z}) := \det(C\mathbf{Z} + D)^{-1}$ since this is already a cocylce, but it is not well-defined on $\mathrm{Aut}_1(B)$ which is the quotient of G by its centre. Finally we have

$$\Delta(\mathbf{Z}, \mathbf{W}) := \det(1 - \mathbf{W}^*\mathbf{Z})$$

for all $\mathbf{Z}, \mathbf{W} \in Z = \mathbb{C}^{p\times q}$.

1.3 An Anosov type result for the frame flow

Hyperbolic (or Anosov) diffeomorphisms and hyperbolic (or Anosov) flows on manifolds have been delt with for example in [10] . A diffeomorphism or a flow being hyperbolic implies a rich structure of periodic orbits. Roughly speaking a diffeomorphism φ on a manifold W is called hyperbolic if there exists a Riemannian metric on W and a φ-invariant splitting of the tangent bundle $TM = T^+ \oplus T^-$ such that φ is expanding on T^+ and contracting on T^- both with a global constant C . The famous Anosov closing lemma (theorem 6.4.15 in [10]) says that for a hyperbolic diffeomorphism φ on W given an ε-closed orbit of φ there exists a closed orbit ε-nearby. Here we have to deal with partially hyperbolic diffeomorphisms and flows, and we can state a partial Anosov closing lemma (see theorem 1.21) for them. For our purposes it is enough to restrict ourselves to the $\mathcal{C}^\infty$-case.
Let W be a smooth Riemannian manifold and φ a $\mathcal{C}^\infty$-diffeomorphism of W .

Definition 1.19 (partially hyperbolic diffeomorphism) *Let $C > 1$. φ is called partially hyperbolic with constant C if and only if there exists an orthogonal $D\varphi$ (and therefore $D\varphi^{-1}$) -invariant $\mathcal{C}^\infty$-splitting*

$$TW = T^0 \oplus T^+ \oplus T^-$$

of the tangent bundle TW such that $T^0 \oplus T^+$, $T^0 \oplus T^-$, T^0 , T^+ and T^- are closed under the commutator, $D\varphi|_{T^0}$ is an isometry, $||D\varphi|_{T^-}|| \leq \frac{1}{C}$ and $\left|\left|D\varphi^{-1}|_{T^+}\right|\right| \leq \frac{1}{C}$.

φ being partially hyperbolic, $T^0 \oplus T^+$, $T^0 \oplus T^-$, T^0 , T^+ and T^- give rise to $\mathcal{C}^\infty$-foliations on W . Let us denote the distances along the $T^0 \oplus T^+$- , T^0- , T^+- respectively T^--leaves by $d^{0,+}$, d^0 , d^+ respectively d^- . Then clearly for any two points $a, b \in W$ belonging to the same T^--leaf the points $\varphi(a)$ and $\varphi(b)$ again belong to the same T^--leaf and $d^-\left(\varphi(a), \varphi(b)\right) \leq \frac{1}{C} d^-(a, b)$, and for two points $c, d \in W$ belonging to the same T^+-leaf the points $\varphi(c)$ and $\varphi(d)$ resp. the points $\varphi^{-1}(c)$ and $\varphi^{-1}(d)$ again belong to the same T^+-leaf and $d^+\left(\varphi^{-1}(c), \varphi^{-1}(d)\right) \leq \frac{1}{C} d^+(c, d)$.
$T^+ \oplus T^-$ in general is not closed under the commutator.

Definition 1.20 *Let $TW = T^0 \oplus T^+ \oplus T^-$ be an orthogonal $\mathcal{C}^\infty$-splitting of the tangent bundle TW of W such that $T^0 \oplus T^+$, T^0 , T^+ and T^- are closed under the commutator, $C' \geq 1$ and $U \subset W$. U is called C'-rectangular (with respect to the splitting $TW = T^0 \oplus T^+ \oplus T^-$) if and only if for all $y, z \in U$*

- $\{i\}$ *there exists a unique intersection point $a \in U$ of the $T^0 \oplus T^+$-leaf containing y and the T^--leaf containing z and a unique intersection point $b \in U$ of the $T^0 \oplus T^+$-leaf containing z and the T^--leaf containing y ,*

$$d^{0,+}(y, a), d^-(y, b), d^-(z, a), d^{0,+}(z, b) \leq C' d(y, z) ,$$

and

$$\frac{1}{C'} d^{0,+}(z, b) \leq d^{0,+}(y, a) \leq C' d^{0,+}(z, b) ,$$
$$\frac{1}{C'} d^-(z, a) \leq d^-(y, b) \leq C' d^-(z, a) ,$$

see figure 1.1 .

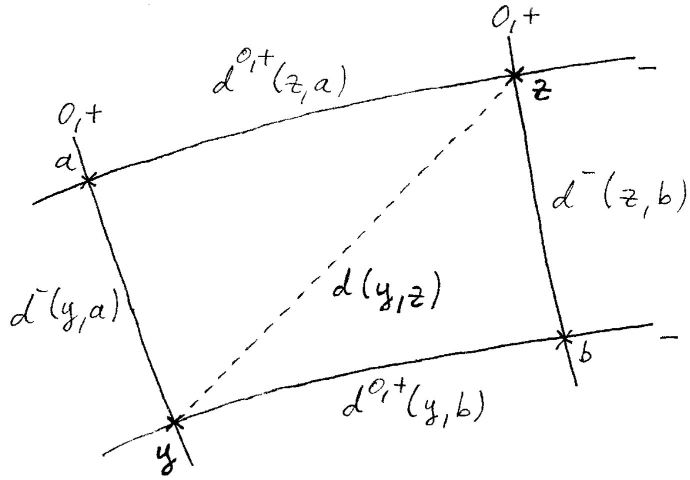

Figure 1.1: intersection points in {i} .

{*ii*} *if y and z belong to same $T^0 \oplus T^+$-leaf there exists a unique intersection point $c \in U$ of the T^0-leaf containing y and the T^+-leaf containing z and a unique intersection point $d \in U$ of the T^0-leaf containing z and the T^+-leaf containing y ,*

$$d^0(y,c), d^+(y,d), d^+(z,c), d^0(z,d) \leq C' d^{0,+}(y,z) \ ,$$

and

$$\frac{1}{C'} d^0(z,d) \leq d^0(y,c) \leq C' d^0(z,d) \ ,$$
$$\frac{1}{C'} d^+(z,c) \leq d^+(y,d) \leq C' d^+(z,c) \ ,$$

see figure 1.2 .

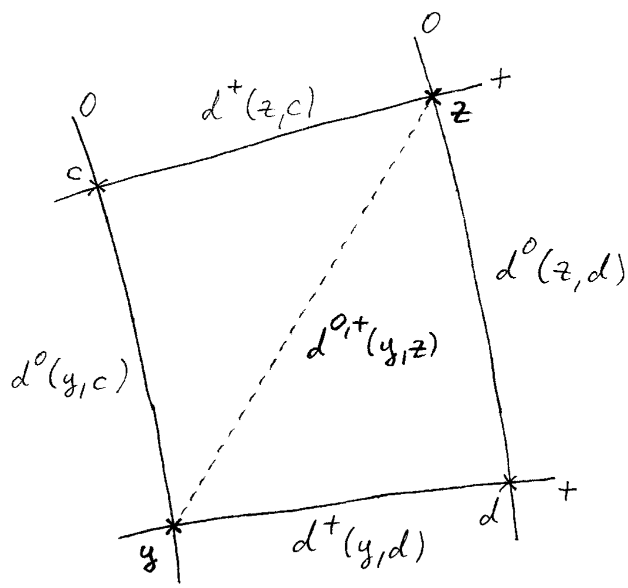

Figure 1.2: intersection points in {ii} .

Since the splitting $TW = T^0 \oplus T^+ \oplus T^-$ is orthogonal and C^∞ we see that for all $x \in W$ and $C' > 1$ there exists a C'-rectangular neighbourhood of x .

Theorem 1.21 (partial Anosov **closing lemma)** *Let φ be partially hyperbolic with constant C , let $x \in W$, $C' \in]\,1, C\,[$ and $\delta > 0$ such that $\overline{U_\delta(x)}$ is contained in a C'-rectangular subset $U \subset W$.*
If $d(x, \varphi(x)) \le \delta \frac{1-\frac{C'}{C}}{C'^2+1}$ then there exist $y, z \in U$ such that

(i) x and y belong to the same T^--leaf and

$$d^-(x,y) \le \frac{C'}{1-\frac{C'}{C}} d(x,\varphi(x)) \ ,$$

(ii) y and $\varphi(y)$ belong to the same $T^0 \oplus T^+$-leaf and

$$d^{0,+}(y,\varphi(y)) \le C'^2 d(x,\varphi(x)) \ ,$$

(iii) y and z belong to the same T^+-leaf and

$$d^+(\varphi(y),\varphi(z)) \le \frac{C'^3}{1-\frac{C'}{C}} d(x,\varphi(x)) \ ,$$

(iv) z and $\varphi(z)$ belong to the same T^0-leaf and

$$d^0(z,\varphi(z)) \le \mathcal{C}'^4 d(x,\varphi(x)) .$$

Proof: Let U be C'-rectangular neighbourhood of x in W and $\delta > 0$ such that $\overline{U_\delta(x)} \subset U$. Suppose $\varepsilon := d(x,\varphi(x)) \le \delta \frac{1-\frac{C'}{C}}{C'^2+1}$.

Step I **Show that there exists a point $y \in U$ such that y and x belong to the same T^--leaf, y and $\varphi(y)$ belong to the same $T^0 \oplus T^+$-leaf,**

$$d(\varphi(y),x) \le \frac{\varepsilon}{1-\frac{C'}{C}} ,$$
$$d^-(y,x) \le \varepsilon \frac{C'}{1-\frac{C'}{C}}$$

and

$$d^{0,+}(y,\varphi(y)) \le \varepsilon C'^2 .$$

We inductively construct points $x_n \in U$, $n \in \mathbb{N}$, such that $x_0 := x$ and for all $n \in \mathbb{N}$

(i) x_n and x belong to the same T^--leaf,

(ii) x_n and $\varphi(x_{n-1})$ belong to the same $T^0 \oplus T^+$-leaf if $n \ge 1$,

(iii) $d^-(x_n,x_{n-1}) \le \varepsilon C' \left(\frac{C'}{C}\right)^{n-1}$ if $n \ge 1$,

(iv) $d^-(\varphi(x_n),\varphi(x_{n-1})) \le \varepsilon \left(\frac{C'}{C}\right)^{n}$ if $n \ge 1$,

(v)
$$d(\varphi(x_n),x) \le \varepsilon \sum_{k=0}^{n} \left(\frac{C'}{C}\right)^k ,$$

see figure 1.3 . $x_0 = x$ clearly fulfills (i) and (v) . Now let us assume $n \in \mathbb{N}$ and $x_n \in U$ fulfills (i) - (v) . Since by (v)

$$d(\varphi(x_n),x) \le \frac{\varepsilon}{1-\frac{C'}{C}} \le \delta ,$$

even $\varphi(x_n) \in U$, and so by {i} of definition 1.20 there exists a unique intersection point in U of the T^--leaf containing x_n and the $T^0 \oplus T^+$-leaf containing $\varphi(x_n)$, which we define to be x_{n+1} . Then clearly (i) and (ii) are fulfilled. $d^-(x_1,x_0) \le C'd(x,\varphi(x)) = C'\varepsilon$ follows from {i} . If $n \ge 1$ then

$\varphi(x_{n-1})$ is the unique intersection point of the T^--leaf containing $\varphi(x_n)$ and the $T^0 \oplus T^+$-leaf containing x_n , and so by {i} and (iv) we obtain

$$d^-(x_{n+1}, x_n) \leq C' d^-(\varphi(x_n), \varphi(x_{n-1})) \leq \varepsilon C' \left(\frac{C'}{C}\right)^n .$$

Since x_{n+1} and x_n belong to the same T^--leaf, we get

$$d^-(\varphi(x_{n+1}), \varphi(x_n)) \leq \frac{1}{C} d^-(x_n, x_{n+1}) \leq \varepsilon \left(\frac{C'}{C}\right)^{n+1} ,$$

and this is (iv) . Finally by (iv) and (v) we see that

$$\begin{aligned} d(\varphi(x_{n+1}), x) &\leq d^-(\varphi(x_{n+1}), \varphi(x_n)) + d(\varphi(x_n), x) \\ &\leq \varepsilon \left(\frac{C'}{C}\right)^{n+1} + \varepsilon \sum_{k=0}^{n} \left(\frac{C'}{C}\right)^k = \varepsilon \sum_{k=0}^{n+1} \left(\frac{C'}{C}\right)^k . \end{aligned}$$

By (iii) we see that $(x_n)_{n \in \mathbb{N}}$ is a CAUCHY sequence in U . Let $y := \lim_{n \to \infty} x_n \in U$. Then y and x belong to the same T^--leaf , and by (iii) we see that

$$d^-(x, y) \leq \varepsilon C' \sum_{k=0}^{\infty} \left(\frac{C'}{C}\right)^k \leq \varepsilon \frac{C'}{1 - \frac{C'}{C}} .$$

y and $\varphi(y)$ belong to the same $T^0 \oplus T^+$-leaf, and

$$d(\varphi(y), x) \leq \varepsilon \sum_{n=0}^{\infty} \left(\frac{C'}{C}\right)^k = \frac{\varepsilon}{1 - \frac{C'}{C}} \leq \delta .$$

So $\varphi(y) \in U$. Finally y is the unique intersection point in U of the T^--leaf containing x_1 and the $T^0 \oplus T^+$-leaf containing $\varphi(y)$, and $\varphi(x)$ is the unique intersection point in U of the T^--leaf containing $\varphi(y)$ and the $T^0 \oplus T^+$-leaf containing x_1 . So by {i}

$$d^{0,+}(y, \varphi(y)) \leq C' d(x_1, \varphi(x)) \leq \varepsilon C'^2 .$$

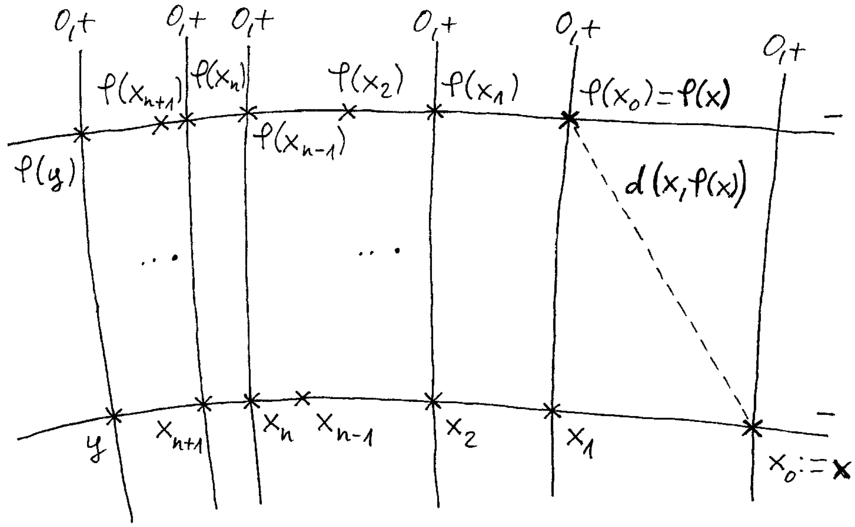

Figure 1.3: constructing the CAUCHY sequence $(x_n)_{n\in\mathbb{N}}$ in step I .

Step II **Show that there exists a point $z \in U$ such that z and y belong to the same T^+-leaf, z and $\varphi(z)$ belong to the same T^0-leaf,**

$$d^+\left(\varphi(y),\varphi(z)\right) \le \varepsilon \frac{C'^3}{1-\frac{C'}{C}}$$

and

$$d^0\left(z,\varphi(z)\right) \le \varepsilon C'^4 .$$

We inductively construct points $y'_n \in U$, $n \in \mathbb{N}$, such that $y'_0 := \varphi(y)$ and for all $n \in \mathbb{N}$

(i) y'_n and $\varphi(y)$ belong to the same T^+-leaf,

(ii) y'_n and $\varphi^{-1}\left(y'_{n-1}\right)$ belong to the same T^0-leaf if $n \ge 1$,

(iii) $d^+\left(y'_n, y'_{n-1}\right) \le \varepsilon C'^3 \left(\frac{C'}{C}\right)^{n-1}$ if $n \ge 1$,

(iv) $d^+\left(\varphi^{-1}\left(y'_n\right), \varphi^{-1}\left(y'_{n-1}\right)\right) \le \varepsilon C'^2 \left(\frac{C'}{C}\right)^{n}$ if $n \ge 1$,

(v)
$$d\left(\varphi^{-1}\left(y'_n\right), x\right) \le \varepsilon \left(C'^2 \sum_{k=0}^{n} \left(\frac{C'}{C}\right)^k + \frac{1}{1-\frac{C'}{C}} \right) ,$$

see figure 1.4 . $y'_0 = \varphi(y)$ clearly fulfills (i) and (v) . Now let us assume $n \in \mathbb{N}$ and $y_n \in W$ fulfills (i) - (v) . Since by (v)

$$d\left(\varphi^{-1}\left(y'_n\right), x\right) \le \varepsilon \frac{C'^2+1}{1-\frac{C'}{C}} \le \delta ,$$

we see that again even $\varphi^{-1}\left(y'_n\right) \in U$. By (i) since the splitting of the tangent bundle is $D\varphi^{-1}$-invariant , $\varphi^{-1}\left(y'_n\right)$ and y belong to the same T^+-leaf, and so $\varphi^{-1}\left(y'_n\right)$ and y'_n belong to the same $T^0 \oplus T^+$-leaf. So by {ii} of definition 1.20 there exists a unique intersection point of the T^+-leaf containing y'_n and the T^0-leaf containing $\varphi^{-1}\left(y'_n\right)$, which we define to be y'_{n+1} . Then clearly (i) and (ii) are fulfilled. $d^+\left(y'_1, y'_0\right) \le \varepsilon C'^3$ follows from {ii} since $d\left(y, \varphi(y)\right) \le \varepsilon C'^2$. If $n \ge 1$ then $\varphi^{-1}\left(y'_{n-1}\right)$ is the unique intersection point of the T^+-leaf containing $\varphi^{-1}\left(y'_n\right)$ and the T^0-leaf containing y'_n , and so by {ii} and (iv) we have

$$d^+\left(y'_{n+1}, y'_n\right) \le C' d^+\left(\varphi^{-1}\left(y'_n\right), \varphi^{-1}\left(y'_{n-1}\right)\right) \le \varepsilon C'^3 \left(\frac{C'}{C}\right)^n .$$

Since y'_{n+1} and y'_n belong to the same T^+-leaf, we get

$$d^+\left(\varphi^{-1}\left(y'_{n+1}\right), \varphi^{-1}\left(y'_n\right)\right) \le \frac{1}{C} d^+\left(y'_{n+1}, y'_n\right) \le \varepsilon C'^2 \left(\frac{C'}{C}\right)^{n+1} ,$$

and this is (iv) . Finally by (iv) and (v) we obtain

$$\begin{aligned} d\left(\varphi^{-1}\left(y'_{n+1}\right), x\right) &\le d\left(\varphi^{-1}\left(y'_{n+1}\right), \varphi^{-1}\left(y'_n\right)\right) + d\left(\varphi^{-1}\left(y'_n\right), x\right) \\ &\le \varepsilon C'^2 \left(\frac{C'}{C}\right)^{n+1} + \varepsilon\left(C'^2 \sum_{k=0}^{n} \left(\frac{C'}{C}\right)^k + \frac{1}{1-\frac{C'}{C}}\right) \\ &= \varepsilon\left(C'^2 \sum_{k=0}^{n+1} \left(\frac{C'}{C}\right)^k + \frac{1}{1-\frac{C'}{C}}\right) . \end{aligned}$$

By (iii) we see that $\left(y'_n\right)_{n\in\mathbb{N}}$ is a CAUCHY sequence in U . Let $z' := \lim_{n\to\infty} y'_n \in U$ and $z := \varphi^{-1}(z')$. Then z' and $\varphi(y)$ belong to the same T^+-leaf, so again z and y belong to the same T^+-leaf, and by (iii)

$$d^+\left(\varphi(y), \varphi(z)\right) \le \varepsilon C'^3 \sum_{k=0}^{\infty} \left(\frac{C'}{C}\right)^k = \varepsilon \frac{C'^3}{1-\frac{C'}{C}} .$$

z and $\varphi(z)$ belong to the same T^0-leaf, and by (v)

$$d\left(z, x\right) \le \varepsilon \frac{C'^2+1}{1-\frac{C'}{C}} \le \delta .$$

So $z \in U$. Finally $\varphi(z)$ is the unique intersection point in U of the T^+-leaf containing y'_1 and the T^0-leaf containing z , and y is the unique intersection point in U of the T^+-leaf containing z and the T^0-leaf containing y_1' . So by {ii}

$$d\left(z,\varphi(z)\right) \leq d^0\left(z,\varphi(z)\right) \leq C' d^0\left(y,y_1'\right) \leq C'^2 d\left(y,\varphi(y)\right) \leq C'^4 \varepsilon \,.\square$$

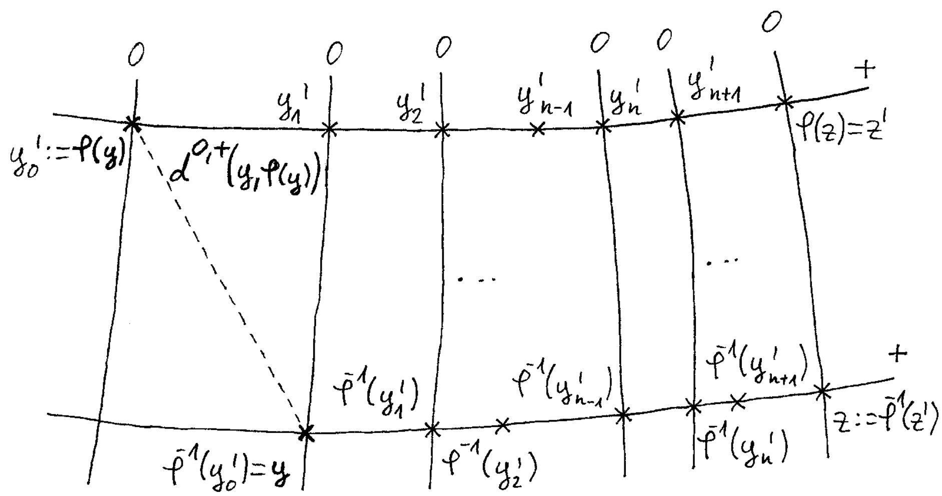

Figure 1.4: constructing the CAUCHY sequence $(y_n')_{n\in\mathbb{N}}$ in step II .

Now let $(\varphi_t)_{t\in\mathbb{R}}$ be a $\mathcal{C}^\infty$-flow on W , which means that $t \mapsto \varphi_t$ is a homomorphism of $\mathbb{R}$ into the group of diffeomorphisms of W and the map $\mathbb{R} \times W \to W, (t,x) \mapsto \varphi_t(x)$ is smooth.

Definition 1.22 (partially hyperbolic flow) *Let $C > 0$. The flow $(\varphi_t)_{t\in\mathbb{R}}$ is called partially hyperbolic with constant C if and only if there exists an orthogonal $D\varphi_t$-invariant $\mathcal{C}^\infty$-splitting*

$$TW = T^0 \oplus T^+ \oplus T^-$$

of the tangent bundle TW such that $T^0 \oplus T^+$, $T^0 \oplus T^-$, T^0 , T^+ and T^- are closed under the commutator, $D\varphi_t|_{T^0}$ is an isometry, $||D\varphi_t|_{T^-}|| \leq e^{-Ct}$ and $||D\varphi_{-t}|_{T^+}|| \leq e^{-Ct}$ for all $t > 0$, and T_0 contains the generator $\partial_t \varphi_t|_{t=0}$ of the flow.

If the flow $(\varphi_t)_{t\in\mathbb{R}}$ is partially hyperbolic with constant C , then for all $t > 0$ clearly φ_t is a diffeomorphism of W which is hyperbolic with constant $e^{Ct} > 1$ and corresponding splitting $TW = T^0 \oplus T^+ \oplus T^-$.

Now let us return to the LIE group G . Let $\mathbf{v} \in \mathbb{R}^q$. Then we can restrict the multiflow $(\varphi_{\mathbf{t}})_{\mathbf{t}\in\mathbb{R}^q}$ to the 'diagonal flow' $(\varphi_{\tau\mathbf{v}})_{\tau\in\mathbb{R}}$, which is simply the right translation by the group

$$A_{\mathbf{v}} := \{\, a_{\tau\mathbf{v}} \mid \tau \in \mathbb{R}\} \ .$$

Choose a left invariant metric on G such that all $\mathfrak{g}^\alpha$, $\alpha \in \Phi \setminus \{\mathbf{0}\}$, $\mathfrak{a}$ and $\mathfrak{m}$ are pairwise orthogonal and the isomorphism $\mathbb{R}^q \simeq A \subset G$ is even isometric. Let $\mathbf{v} \in \mathbb{R}^q$. Then since the 'diagonal flow' $(\varphi_{\tau\mathbf{v}})_{\tau\in\mathbb{R}}$ commutes with left translations it is partially hyperbolic, as one sees immediately in the root space decomposition of $\mathfrak{g}$, see theorem 1.7 , and after rescaling $\mathbf{v}$ we may assume the constant of hyperbolicity to be equal to 1 . The corresponding splitting of the tangent bundle of G is the unique left invariant splitting such that

$$T_1G = \mathfrak{g} = \underbrace{\bigoplus_{\alpha\in\Phi\,,\,\alpha\mathbf{v}=0} \mathfrak{g}^\alpha}_{T_1^0:=} \oplus \underbrace{\bigoplus_{\alpha\in\Phi\,,\,\alpha\mathbf{v}>0} \mathfrak{g}^\alpha}_{T_1^-:=} \oplus \underbrace{\bigoplus_{\alpha\in\Phi\,,\,\alpha\mathbf{v}<0} \mathfrak{g}^\alpha}_{T_1^+:=} \ .$$

Indeed $T^0 \oplus T^+$, $T^0 \oplus T^-$, T^0 , T^+ and T^- are closed under the commutator since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ if $\alpha + \beta \in \Phi$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$ otherwise for all $\alpha, \beta \in \Phi$. So we can apply the partial ANOSOV closing lemma, theorem 1.21 , which here is really convenient since G acts transitively and isometrically on itself by left translations. Before we do so we need two little lemmas. Recall that $T_1^0 = \mathfrak{a}+\mathfrak{m}$ if and only if $\mathbf{v}$ is regular.

For $L \subset G$ compact, $T, \varepsilon > 0$ define

$$M_{L,T} := \left\{ g a_{\mathbf{t}} g^{-1} \,\middle|\, g \in L, \mathbf{t} \in \mathbb{R}^q \text{ such that } |\mathbf{t}| \leq T \right\}$$

and

$$N_{L,T,\varepsilon} := \{ g \in G \mid \mathrm{dist}\,(g, M_{L,T}) \leq \varepsilon \} \ .$$

Lemma 1.23 *For all $L \subset G$ compact there exist $T_0, \varepsilon_0 > 0$ such that $\Gamma \cap N_{L,T_0,\varepsilon_0} = \{1\}$.*

Proof: Let $L \subset G$ be compact and $T > 0$. Then $M_{L,T}$ is compact, and so there exists $\varepsilon > 0$ such that $N_{L,T,\varepsilon}$ is again compact. Since Γ is discrete, $\Gamma \cap N_{L,T,\varepsilon}$ is finite. Clearly for all T, T', ε and $\varepsilon' > 0$ if $T \leq T'$ and $\varepsilon \leq \varepsilon'$ then $N_{L,T,\varepsilon} \subset N_{L,T',\varepsilon'}$. Finally we have

$$\bigcap_{T,\varepsilon>0} N_{T,\varepsilon} = \{1\} \, ,$$

and so the claim follows. $\square$

Lemma 1.24 *For all $L \subset G$ compact there exists a constant $c \geq 1$ such that for all $g \in L$ and $a, b \in G$*

$$\frac{1}{c} d(ag, bg) \leq d(a,b) \leq c d(ag, bg) \ .$$

Proof: The conjugation map

$$C : G \times \mathfrak{g} \to \mathfrak{g} \, , \, (g, \xi) \mapsto \mathrm{Ad}_g(\xi)$$

clearly is linear with respect to ξ and continuous with respect to g . So if $L \subset G$ is compact then there exists $c \geq 1$ such that $||C(g, \diamondsuit)||$, $\left|\left|C\left(g^{-1}, \diamondsuit\right)\right|\right| \leq c$ for all $g \in L$. □

Theorem 1.25 *Let $\mathbf{v} \in \mathbb{R}^q$ be regular such that the flow $\varphi_{\tau \mathbf{v}}$, $\tau \in \mathbb{R}$ is hyperbolic with constant 1 .*

(i) For all $T_1 > 0$ there exist $C_1 \geq 1$ and $\varepsilon_1 > 0$ such that for all $x \in G$, $\gamma \in \Gamma$ and $T \geq T_1$ if

$$\varepsilon := d(\gamma x, x a_{T\mathbf{v}}) \leq \varepsilon_1$$

then there exist $z \in G$, $w \in M$ and $\mathbf{t} \in \mathbb{R}^q$ regular such that $\gamma z = z a_{\mathbf{t}} w$, $d((\mathbf{t}, w), (T\mathbf{v}, 1)) \leq C_1 \varepsilon$, and for all $\tau \in [0, T]$

$$d(x a_{\tau \mathbf{v}}, z a_{\tau \mathbf{v}}) \leq C_1 \varepsilon \left(e^{-\tau} + e^{-(T-\tau)}\right) .$$

(ii) For all $L \subset G$ compact there exists $\varepsilon_2 > 0$ such that for all $x \in L$, $\gamma \in \Gamma$ and $T \in [0, T_0]$, $T_0 > 0$ given by lemma 1.23 , if

$$\varepsilon := d(\gamma x, x a_{T\mathbf{v}}) \leq \varepsilon_2$$

then $\gamma = 1$ and $T \leq 2\varepsilon$.

Proof: (i) Let $T_1 > 0$ and define

$$C_1 := \max\left(\frac{e^{\frac{3}{2}T_1}}{1 - e^{-\frac{T_1}{2}}}, e^{2T_1}\right) \geq 1 \, .$$

Let $TG = T^0 \oplus T^+ \oplus T^-$ be the splitting corresponding to the flow $(\varphi_{\tau \mathbf{v}})_{\tau \in \mathbf{R}}$ on G . Define $C' := e^{\frac{T_1}{2}}$, let U be a C'-rectangular neighbourhood of $1 \in G$ and let $\delta > 0$ such that $\overline{U_\delta(1)} \subset U$. Then by the left invariance of the splitting and the metric on G we see that gU is a C'-rectangular neighbourhood of g and $\overline{U_\delta(g)} = g\overline{U_\delta(1)} \subset gU$ for all $g \in G$. Since $\mathbf{v}$ is regular there exists $\varepsilon' > 0$ such that if $T \geq T_1$ and $\mathbf{t} \in \mathbb{R}^q$ such that $||\mathbf{t} - T\mathbf{v}||_2 \leq \varepsilon' C_1$ then $\mathbf{t}$ is regular.

Define

$$\varepsilon_1 := \min\left(\delta\frac{1-e^{-\frac{T_1}{2}}}{e^{T_1}+1}, \varepsilon'\right) > 0\,.$$

Let $x \in L$, $\gamma \in \Gamma$, $T \geq T_1$ and

$$\varepsilon := d\left(\gamma x, x a_{T\mathbf{v}}\right) \leq \varepsilon_1\,.$$

Then $\varphi : G \to G$, $g \mapsto \gamma^{-1} g a_{T\mathbf{v}}$ is a partially hyperbolic diffeomorphism with constant $e^{T_1} > 1$ and the same splitting $TG = T^0 \oplus T^+ \oplus T^-$ as the one of the flow $(\varphi_{\tau\mathbf{v}})_{\tau\in\mathbb{R}}$ on G . Then since

$$\varepsilon \leq \delta\frac{1-e^{-\frac{T_1}{2}}}{e^{T_1}+1} = \delta\frac{1-C'e^{-T_1}}{C'^2+1}$$

the partial Anosov closing lemma, theorem 1.21 , tells us that there exist $y, z \in G$ such that

(i) x and y belong to the same T^--leaf and

$$d^-\left(x,y\right) \leq \varepsilon\frac{C'}{1-\frac{C'}{C}}\,,$$

(iii) y and z belong to the same T^+-leaf and

$$d^+\left(y a_{T\mathbf{v}}, z a_{T\mathbf{v}}\right) \leq \varepsilon\frac{C'^3}{1-\frac{C'}{C}}\,,$$

(iv) γz and $z a_{T\mathbf{v}}$ belong to the same T^0-leaf and

$$d^0\left(\gamma z, z a_{T\mathbf{v}}\right) \leq \varepsilon C'^4\,.$$

In (iii) and (iv) we already used that the metric and the flow $(\varphi_{\tau\mathbf{v}})_{\tau\in\mathbb{R}}$ on G are left invariant. So by (iv) and since the T^0-leaf containing $z a_{T\mathbf{v}}$ is zAM , there exist $w \in M$ and $\mathbf{t} \in \mathbb{R}^q$ such that $\gamma z = z a_{\mathbf{t}} w$. So

$$d^0\left(a_{\mathbf{t}-T\mathbf{v}} w, 1\right) \leq \varepsilon C'^4\,,$$

and so, since $AM \simeq \mathbb{R}^q \times M$ isometrically, we see that

$$d\left(\left(\mathbf{t}, w\right), \left(T\mathbf{v}, 1\right)\right) \leq \varepsilon C'^4 = \varepsilon e^{2T_1} \leq \varepsilon C_1\,.$$

Especially $\|\mathbf{t} - T\mathbf{v}\|_2 \leq \varepsilon' C_1$, and so $\mathbf{t}$ regular.

Now let $\tau \in [0, T]$. Then since x and y belong to the same T^--leaf the same is true for $x a_{\tau\mathbf{v}}$ and $y a_{\tau\mathbf{v}}$, and

$$d^-\left(xa_{\tau\mathbf{v}}, ya_{\tau\mathbf{v}}\right) \le d^-\left(x,y\right)e^{-\tau} \le \varepsilon\frac{C'}{1-\frac{C'}{C}}e^{-\tau} \le \varepsilon C_1 e^{-\tau}\,.$$

Since y and z belong to the same T^+-leaf the same is true for $ya_{\tau\mathbf{v}}$ and $za_{\tau\mathbf{v}}$, and

$$\begin{aligned} d^+\left(ya_{\tau\mathbf{v}}, za_{\tau\mathbf{v}}\right) &\le d^+\left(ya_{T\mathbf{v}}, za_{T\mathbf{v}}\right)e^{-(T-\tau)} \\ &\le \varepsilon\frac{C'^3}{1-\frac{C'}{C}}e^{-(T-\tau)} \le \varepsilon C_1 e^{-(T-\tau)}\,. \end{aligned}$$

Combining these two inequalities we obtain

$$d\left(xa_{\tau\mathbf{v}}, za_{\tau\mathbf{v}}\right) \le \varepsilon C_1\left(e^{-\tau}+e^{-(T-\tau)}\right)\,.$$

(ii) Let $L \subset G$ be compact and let $c \ge 1$ be given by lemma 1.24 . Let $\varepsilon_0 > 0$ be given by lemma 1.23 and define

$$\varepsilon_2 := \frac{\varepsilon_0}{c} > 0\,.$$

Let $x \in L$, $\gamma \in \Gamma$, $T \in [\,0, T_0\,]$ and

$$\varepsilon := d\left(\gamma x, xa_{T\mathbf{v}}\right) \le \varepsilon_2\,.$$

Then since $x \in L$ we get

$$d\left(\gamma, xa_{T\mathbf{v}}x^{-1}\right) \le c\varepsilon \le \varepsilon_0$$

and so $\gamma \in \Gamma \cap N_{L,T_0,\varepsilon_0}$. This implies $\gamma = 1$ and so $d\left(1, a_{T\mathbf{v}}\right) = \varepsilon$ and therefore $T \le 2\varepsilon$. $\square$

1.4 A spanning set for the space of cusp forms

Assume $\Gamma \sqsubset G$ discrete and $k_0 \in \mathbb{N}$ such that SATAKE's theorem holds, more precisely

$$S_k(\Gamma) = M_k(\Gamma) \cap L^r_k\left(\Gamma\backslash G\right)$$

for all $k \ge k_0$ and $r \in [\,1,\infty\,]$. Let $\mathbf{v} \in \mathbb{R}^q$ be regular such that $\varphi_{\mathbf{v}}$ is partially hyperbolic with constant 1 , $C > 0$, and let Φ_C be the subset of $\left(\mathbb{R}^q\right)^*$ of all $\mathbf{l} \in \left(\mathbb{R}^q\right)^*$ such that there exists $\mathbf{v}' \in \mathbb{R}^q$ WEYL equivalent to $\mathbf{v}$ with $|\mathbf{l}\mathbf{v}'| < C$.

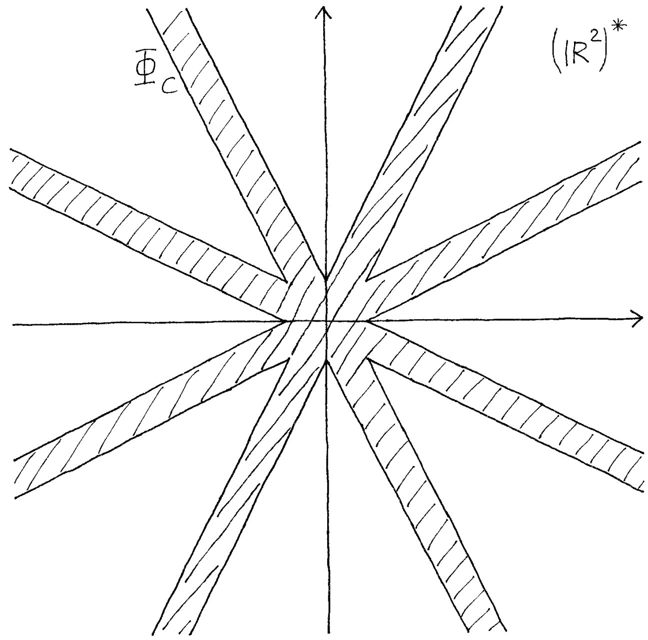

Figure 1.5: Φ_C in the case B irreducible of rank $q := 2$ and $\mathbf{v} := \binom{1}{2}$.

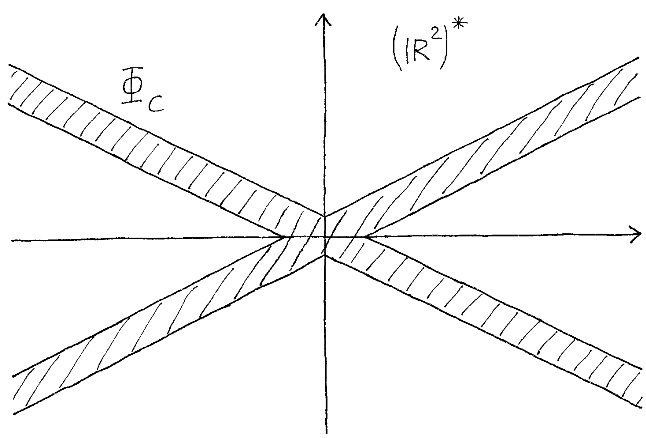

Figure 1.6: Φ_C in the case $B = B_1 \times B_2$ reducible of rank $q := 2$ and $\mathbf{v} := \binom{1}{2}$.

Let us consider a maximal loxodromic subgroup $\Gamma_0 \sqsubset \Gamma$. Let $g \in G$ such that $\Gamma_0 \sqsubset gAMg^{-1}$.

Definition 1.26 Γ_0 *is called k-admissible if and only if $j(w)^k = 1$ for all $w \in g^{-1}\Gamma_0 g \cap M$.*

Recall that $j := j(\diamondsuit, \mathbf{Z})|_K$, $\mathbf{Z} \in Z$, is independent of $\mathbf{Z}$ and a character of K , since K is compact. Let us now check that this definition is independent of the choice of $g \in G$.

> By theorem 1.11 (ii) it suffices to show that M and $j|_M$ are independent of conjugation by elements of $AN_K(A)$. So let $a_{\mathbf{T}}n \in AN_K(A)$, $\mathbf{T} \in \mathbb{R}^q$ and $n \in N_K(A)$, and $w \in M$. Then $N_K(A) \sqsubset N_K(M)$ since M centralizes A . And so
>
> $$(a_{\mathbf{T}}n)^{-1} w (a_{\mathbf{T}}n) = n^{-1}a_{-\mathbf{T}}wa_{\mathbf{T}}n^{-1} = nwn \in M$$
>
> and
>
> $$j\left((a_{\mathbf{T}}n)^{-1} w (a_{\mathbf{T}}n)\right) = j\left(k'^{-1}wn\right) = j(w) .$$

By the same reason if $\Gamma_0 \sqsubset \Gamma$ is loxodromic and $\gamma \in \Gamma$ then Γ_0 is k-admissible if and only if $\gamma\Gamma_0\gamma^{-1}$ is k-admissible.

Proposition 1.27 *If Γ_0 is k-admissible, then there exists $\chi \in (\mathbb{R}^q)^*$ such that for all $\mathbf{t} \in \mathbb{R}^q$ and $w \in M$ if $ga_{\mathbf{t}}wg^{-1} \in \Gamma_0$ then $j(w)^k = e^{2\pi i\chi\mathbf{t}}$. Having g fixed, χ is unique up to Λ^* . Otherwise χ is unique up to Λ^* and* Weyl *equivalence.*

Proof: Since $AM \simeq \mathbb{R}^q \times M$, having g fixed we get a well-defined character

$$\mu : \Gamma \to U(1) , \gamma = ga_{\mathbf{t}}wg^{-1} \mapsto j(w)^k ,$$

$\mathbf{t} \in \mathbb{R}^q$, $w \in M$, which is independent of the choice of g , as we have seen above. On the other hand we have a group homomorphism $\phi : \Gamma_0 \to \mathbb{R}^q$ with image $\phi(\Gamma_0) = \Lambda$ and kernel $\ker\phi = \Gamma_0 \cap gMg^{-1}$. Γ_0 being k-admissible then implies precisely that $\ker\phi \subset \ker\mu$ and so there exists a unique character $\mu' : \Lambda \to U(1)$ such that $\mu = \mu' \circ \Phi$, and this means

$$j(w)^k = \mu'(\mathbf{t})$$

for all $\mathbf{t} \in \mathbb{R}^q$ and $w \in M$ such that $ga_{\mathbf{t}}wg^{-1} \in \Gamma_0$. Since $\Lambda \sqsubset \mathbb{R}^q$ is a lattice we can write $\mu' = e^{2\pi i\chi\diamondsuit}$ with an up to Λ^* unique $\chi \in (\mathbb{R}^q)^*$. Since ϕ without fixing $g \in G$ is uniquely determined up to Weyl equivalence, we have the desired uniqueness for $\chi*$. □

To each $\Gamma_0 \sqsubset \Gamma$ loxodromic there is a torus $\mathbb{T} := \Gamma_0 \backslash gAM$ belonging to Γ_0 . $\mathbb{T}$ is independent of g up to right translation with an element of the Weyl group $W = N_K(A)/M$. Let us check it.

Let $g, g' \in G$. Then by theorem 1.11 (ii) there exists $\mathbf{T} \in \mathbb{R}^q$ and $n \in N_K(A) = N_K(AM)$ such that $g' = g a_{\mathbf{T}} n$. So

$$g'AM = g a_{\mathbf{T}} n AM = g a_{\mathbf{T}} AMn = gAMn .$$

We see that $g'AM$ only depends on the class $Mn = nM \in W$ of n .

Let $f \in S_k(\Gamma)$. Then $\widetilde{f} \in \mathcal{C}^\infty(\Gamma \backslash G)^{\mathbb{C}}$. Define $h \in \mathcal{C}^\infty(\mathbb{R}^q \times M)^{\mathbb{C}}$ as

$$h(\mathbf{t}, w) := \widetilde{f}(g a_{\mathbf{t}} w)$$

'screening up' the values of $\widetilde{f}$ on $\mathbb{T}$. Clearly $h(\mathbf{t}, w) = j(w)^k h(\mathbf{t}, 1)$ for all $(\mathbf{t}, w) \in \mathbb{R}^q \times M$.

Lemma 1.28 *If Γ_0 is not k-admissible then $h = 0$.*

Proof: Let $w \in g^{-1}\Gamma_0 g \cap M$ such that $j(w)^k \neq 1$. Then $gwg^{-1} \in \Gamma$, and so for all $\mathbf{t} \in \mathbb{R}^q$ we have

$$h(\mathbf{t}, 1) = \widetilde{f}\left(gwg^{-1} g a_{\mathbf{t}} w_0^{-1}\right) = \widetilde{f}\left(g a_{\mathbf{t}} w^{-1}\right) = h(\mathbf{t}, 1)\, j(w)^{-k} ,$$

and this implies $h = 0$. $\square$

From now on assume Γ_0 to be k-admissible.

Theorem 1.29 (Fourier expansion of h)

(i) $h(\mathbf{t} + \mathbf{T}, w) = h(\mathbf{t}, w)\, e^{-2\pi i \chi \mathbf{T}}$ for all $(\mathbf{t}, w) \in \mathbb{R}^q \times M$ and $\mathbf{T} \in \Lambda$, and there exist unique $b_{\mathbf{l}} \in \mathbb{C}$, $\mathbf{l} \in \Lambda^* - \chi$, such that

$$h(\mathbf{t}, w) = j(w)^k \sum_{\mathbf{l} \in \Lambda^* - \chi} b_{\mathbf{l}} e^{2\pi i \mathbf{l} \mathbf{t}}$$

for all $(\mathbf{t}, w) \in \mathbb{R}^q \times M$, where the sum converges uniformly in all derivatives.

(ii) If $b_{\mathbf{l}} = 0$ for all $\mathbf{l} \in \Phi_C$ then for all $\mathbf{v}' \in \mathbb{R}^q$ Weyl equivalent to $\mathbf{v}$ there exists $H_{\mathbf{v}'} \in \mathcal{C}^\infty(\mathbb{R}^q \times M)^{\mathbb{C}}$ uniformly Lipshitz continuous with a Lipshitz constant $C_2 \geq 0$ independent of Γ_0 and $\mathbf{v}'$ such that

$$h = \partial_{\mathbf{v}'} H_{\mathbf{v}'} ,$$

$$H_{\mathbf{v}'}(\mathbf{t}, w) = j(w)^k H_{\mathbf{v}'}(\mathbf{t}, 1)$$

and

$$H_{\mathbf{v}'}(\mathbf{t}+\mathbf{T},w) = H_{\mathbf{v}'}(\mathbf{t},w)\, e^{-2\pi i\chi\mathbf{T}}$$

for all $(\mathbf{t},w) \in \mathbb{R}^q \times M$ *and* $\mathbf{T} \in \Lambda$.

Proof: (i) Let $\mathbf{t} \in \mathbb{R}^q$ and $\mathbf{T} \in \Lambda$. Then there exists $w \in M$ such that $ga_{\mathbf{T}}wg^{-1} \in \Gamma_0 \sqsubset \Gamma$. So

$$\begin{aligned} h(\mathbf{t}+\mathbf{T},1) &= \widetilde{f}^{(k)}\left(ga_{\mathbf{T}}wg^{-1}ga_{\mathbf{t}}w^{-1}\right) \\ &= \widetilde{f}^{(k)}(ga_{\mathbf{t}})\, j(w)^{-k} = h(\mathbf{t},1)\, e^{-2\pi i\chi\mathbf{T}} . \end{aligned}$$

The rest follows by standard FOURIER expansion. □

For proving (ii) we need a lemma, which we will deduce from the ordinary reverse BERNSTEIN inequality, see for example theorem 8.4 in chapter I of [12] ($\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ here):

> Let $\mathcal{B}$ be a homogeneous BANACH space on $\mathbb{T}$, and $m > 0$ an integer. There exists a constant C_m such that if
>
> $$f = \sum_{|j|\geq n} a_j e^{ijt}$$
>
> is m-times differentiable and $f^{(m)} \in \mathcal{B}$, then $f \in \mathcal{B}$ and
>
> $$||f||_{\mathcal{B}} \leq C_m\, |n|^{-m} \left|\left|f^{(m)}\right|\right|_{\mathcal{B}} .$$
>
> For even m we obtain $C_m = m+1$; for odd m we can take $C_m = 12m$.

Lemma 1.30 (generalization of the reverse BERNSTEIN inequality) *Let* $\Lambda \sqsubset \mathbb{R}^q$ *be a lattice,* $\chi \in (\mathbb{R}^q)^*$, $\mathbf{v}' \in \mathbb{R}^q$ *and* $C > 0$. *Let* $\mathcal{S}$ *be the space of all convergent* FOURIER *series*

$$s = \sum_{\mathbf{l}\in(\Lambda^*-\chi)\,,\,|\mathbf{l}\mathbf{v}'|\geq C} s_{\mathbf{l}} e^{2\pi i\mathbf{l}\diamondsuit} \in \mathcal{C}^\infty(\mathbb{R}^q)^{\mathbb{C}} ,$$

all $s_{\mathbf{l}} \in \mathbb{C}$. *Then*

$$\widehat{\ }: \mathcal{S} \to \mathcal{S}\,,\; s = \sum_{\mathbf{l}\in(\Lambda^*-\chi)\,,\,|\mathbf{l}\mathbf{v}'|\geq C} s_{\mathbf{l}} e^{2\pi i\mathbf{l}\diamondsuit} \mapsto \widehat{s} := \sum_{\mathbf{l}\in(\Lambda^*-\chi)\,,\,|\mathbf{l}\mathbf{v}'|\geq C} \frac{s_{\mathbf{l}}}{2\pi i\mathbf{l}\mathbf{v}'} e^{2\pi i\mathbf{l}\diamondsuit}$$

is a well-defined linear map, and $||\widehat{s}||_\infty \leq \frac{6}{\pi C}\, ||s||_\infty$ *for all* $s \in \mathcal{S}$.

Proof: Of course for all $s \in \mathcal{S}$ the FOURIER series converges uniformly in all derivatives. For checking that $\widehat{}$ is well-defined observe that given a function

$$s = \sum_{\mathbf{l} \in (\Lambda^* - \chi)\,,\, |\mathbf{l}\mathbf{v}'| \geq C} s_{\mathbf{l}} e^{2\pi i \mathbf{l} \diamondsuit} \in \mathcal{S}\,,$$

we have $|\mathbf{l}|^n s_{\mathbf{l}} \rightsquigarrow 0$ for all $n \in \mathbb{N}$ if $\mathbf{l} \rightsquigarrow \infty$ and so again

$$|\mathbf{l}|^n \frac{s_{\mathbf{l}}}{2\pi i \mathbf{l}\mathbf{v}'} \rightsquigarrow 0$$

for all $n \in \mathbb{N}$ since $|\mathbf{l}\mathbf{v}'| \geq C > 0$ for all $\mathbf{l}$ that occur in the sum, and so

$$\sum_{\mathbf{l} \in (\Lambda^* - \chi)\,,\, |\mathbf{l}\mathbf{v}'| \geq C} \frac{s_{\mathbf{l}}}{2\pi i \mathbf{l}\mathbf{v}'} e^{2\pi i \mathbf{l} \diamondsuit}$$

again converges uniformly in all derivatives to a function $\widehat{s} \in \mathcal{C}^{\infty}\left(\mathbb{R}^q\right)^{\mathbb{C}}$.

Now let $\mathcal{S}_0$ be the subspace of $\mathcal{S}$ of all

$$s = \sum_{\mathbf{l} \in M} s_{\mathbf{l}} e^{2\pi i \mathbf{l} \diamondsuit}$$

where $M \subset (\Lambda^* - \chi)$ is finite such that $\mathbf{l}\mathbf{v}' \geq C$ for all $\mathbf{l} \in M$ and all $s_{\mathbf{l}} \in \mathbb{C}$. Then since $\mathcal{S}_0$ is dense in $\mathcal{S}$ with respect to $\|\ \|_{\infty}$ it suffices to show the desired estimate for all $s \in \mathcal{S}_0$. So let

$$s = \sum_{\mathbf{l} \in M} s_{\mathbf{l}} e^{2\pi i \mathbf{l} \diamondsuit} \in \mathcal{S}_0\,,$$

$M \subset (\Lambda^* - \chi)$ finite such that $\mathbf{l}\mathbf{v}' \geq C$ for all $\mathbf{l} \in M$ and all $s_{\mathbf{l}} \in \mathbb{C}$. For all $\xi \in \left(\mathbb{R}^q\right)^*$ and $\mathbf{w} \in \mathbb{R}^q$ let

$$C\left(\xi, \mathbf{w}\right) := \operatorname{dist}\left(\left(M + \chi - \xi\right)\mathbf{w}, 0\right)\,,$$

which is clearly continuous with respect to $(\xi, \mathbf{w}) \in \left(\mathbb{R}^q\right)^* \times \mathbb{R}^q$, and

$$U := \left\{ (\xi, \mathbf{w}) \in \left(\mathbb{R}^q\right)^* \times \mathbb{R}^q \mid C\left(\xi, \mathbf{w}\right) > 0 \right\}\,,$$

which is clearly a neighbourhood of $(\chi, \mathbf{v}')$ in $\left(\mathbb{R}^q\right)^* \times \mathbb{R}^q$ since $C\left(\chi, \mathbf{v}'\right) \geq C > 0$. For all $\xi \in \left(\mathbb{R}^q\right)^*$ define

$$s_{\xi} := e^{2\pi i (\chi - \xi) \diamondsuit} s = \sum_{\mathbf{l} \in M} s_{\mathbf{l}} e^{2\pi i (\mathbf{l} + \chi - \xi) \diamondsuit} \in \mathcal{C}^{\infty}\left(\mathbb{R}^q\right)^{\mathbb{C}}\,,$$

and for all $(\xi, \mathbf{w}) \in U$

$$S_{\xi, \mathbf{w}} := \sum_{\mathbf{l} \in M} \frac{s_{\mathbf{l}}}{2\pi i \left(\mathbf{l} + \chi - \xi\right)\mathbf{w}} e^{2\pi i (\mathbf{l} + \chi - \xi) \diamondsuit} \in \mathcal{C}^{\infty}\left(\mathbb{R}^q\right)^{\mathbb{C}}\,.$$

Clearly $||s_\xi||_\infty = ||s||_\infty$ for all $\xi \in (\mathbb{R}^q)^*$, $s_\chi = s$ and $S_{\chi,\mathbf{v}'} = \widehat{s}$. We will prove that

$$||S_{\xi,\mathbf{w}}||_\infty \leq \frac{6}{\pi C(\xi,\mathbf{w})} ||s||_\infty \tag{1.4}$$

for all $(\xi, \mathbf{w}) \in U$, then taking $(\xi, \mathbf{w}) := (\chi, \mathbf{v}')$ gives the desired estimate.

The right hand side of 1.4 is clearly continuous with respect to $(\xi, \mathbf{w}) \in U$, but also the left hand side. To see this observe that for any $\mathbf{T} \in \Lambda$

$$S_{\xi,\mathbf{w}}(\diamondsuit + \mathbf{T}) = e^{-2\pi i \xi \mathbf{T}} S_{\xi,\mathbf{w}} ,$$

and so $||S_{\xi,\mathbf{w}}||_\infty = ||S_{\xi,\mathbf{w}}||_{\infty,\mathcal{F}}$, where $\mathcal{F}$ is a fundamental domain for $\mathbb{R}^q/\Lambda$ which can be chosen to be compact, and that if $(\xi_n, \mathbf{w}_n) \rightsquigarrow (\xi, \mathbf{w})$ in U then $S_{\xi_n,\mathbf{w}_n} \rightsquigarrow S_{\xi,\mathbf{w}}$ uniformly on any compact subset of $\mathbb{R}^q$.

So since $U \cap (\mathbb{Q}\Lambda^* \times \mathbb{Q}\Lambda) \underset{\text{dense}}{\subset} U$ it suffices to show the inequality for all $(\xi, \mathbf{w}) \in U \cap (\mathbb{Q}\Lambda^* \times \mathbb{Q}\Lambda)$. So let
$(\xi, \mathbf{w}) \in U \cap (\mathbb{Q}\Lambda^* \times \mathbb{Q}\Lambda)$. Then there exists $n \in \mathbb{N} \setminus \{0\}$ such that $(n\xi, n\mathbf{w}) \in \Lambda^* \times \Lambda$. Let $\mathbf{t} \in \mathbb{R}^q$ be arbitrary, and define

$$\widetilde{s} := s_\xi \left(\frac{n^2}{2\pi} \diamondsuit \mathbf{w} + \mathbf{t} \right) = \sum_{\mathbf{l} \in M} s_{\mathbf{l}}\, e^{2\pi i (\mathbf{l}+\chi-\xi)\mathbf{t}}\, e^{i(\mathbf{l}+\chi-\xi)n^2 \mathbf{w} \diamondsuit}$$

and

$$\widetilde{S} := \frac{2\pi}{n^2} S_{\xi,\mathbf{w}} \left(\frac{n^2}{2\pi} \diamondsuit \mathbf{w} + \mathbf{t} \right) = \sum_{\mathbf{l} \in M} \frac{s_{\mathbf{l}}\, e^{2\pi i (\mathbf{l}+\chi-\xi)\mathbf{t}}}{i\,(\mathbf{l}+\chi-\xi)\, n^2 \mathbf{w}}\, e^{i(\mathbf{l}+\chi-\xi)n^2 \mathbf{w} \diamondsuit}$$
$$\in \mathcal{C}^\infty (\mathbb{R}/2\pi i \mathbb{Z})^{\mathbb{C}}$$

since $(\mathbf{l} + \chi - \xi)\, n^2 \mathbf{w} \in 2\pi\mathbb{Z}$ for all $\mathbf{l} \in M$. For all $\mathbf{l} \in M$ we get $\left|(\mathbf{l}+\chi-\xi)\, n^2\mathbf{w}\right| \geq n^2 C(\xi, \mathbf{w}) > 0$. And therefore the reverse Bernstein inequality with $m := 1$ and $\mathcal{B} := \left(\mathcal{C}(\mathbb{T})^{\mathbb{C}}, ||\ ||_\infty\right)$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ here, gives us

$$\left|\left|\widetilde{S}\right|\right|_\infty \leq \frac{12}{n^2 C(\xi, \mathbf{v}')} ||\widetilde{s}||_\infty .$$

In particular

$$|S_{\xi,\mathbf{w}}(\mathbf{t})| = \frac{n^2}{2\pi}\left|\widetilde{S}(0)\right| \leq \frac{6}{\pi C(\xi,\mathbf{w})}\left|\left|\widetilde{s}\right|\right|_\infty \leq \frac{6}{\pi C(\xi,\mathbf{w})}\left|\left|s\right|\right|_\infty .$$

Since $\mathbf{t} \in \mathbb{R}^q$ has been arbitrary, we have the desired estimate.

$\square$

Clearly $s = \partial_{\mathbf{v}'}\widehat{s}$ for all $s \in \mathcal{S}$, $\mathcal{S}$ is invariant under taking partial derivatives and $\widehat{\ }$ commutes with taking partial derivatives.

Proof of theorem 1.29 (ii) : Since differentiation along a direction of the flow is a left invariant differential operator , $f \in \mathcal{O}(B)$ and $\widetilde{f} \in L^\infty(G)$ imply that there exists a constant $C' > 0$ independent of Γ_0 such that $||f||_\infty \leq C'$ and $\left|\left|\partial_{t_j} f(\diamondsuit a_{\mathbf{t}})\big|_{\mathbf{t}=\mathbf{0}}\right|\right|_\infty \leq C'$ for all $j = 1, \ldots, q$.
Now let $b_{\mathbf{l}} = 0$ for all $\mathbf{l} \in \Phi_C$, and let $\mathbf{v}' \in \mathbb{R}^q$ be Weyl equivalent to $\mathbf{v}$. Then since $|\mathbf{l}\mathbf{v}'| \geq C$ for all $\mathbf{l} \in (\Lambda^* - \chi) \setminus \Phi_C$ we can apply lemma 1.30 to $h(\diamondsuit, 1)$ and all $\partial_j h(\diamondsuit, 1)$, $j = 1, \ldots, q$, and so we can define
$H_{\mathbf{v}'} \in \mathcal{C}^\infty(\mathbb{R}^q \times M)^{\mathbb{C}}$ as

$$H_{\mathbf{v}'}(\mathbf{t}, w) := j(w)^k \widehat{h(\diamondsuit, 1)}(\mathbf{t})$$

for all $(\mathbf{t}, w) \in \mathbb{R}^q \times M$. By lemma 1.30

$$\left|\left|\widehat{h(\diamondsuit, 1)}\right|\right|_\infty \leq \frac{6}{\pi C}\left|\left|h(\diamondsuit, 1)\right|\right|_\infty \leq \frac{6C'}{\pi C}$$

and similarly

$$\left|\left|\partial_j \widehat{h(\diamondsuit, 1)}\right|\right|_\infty \leq \frac{6}{\pi C}\left|\left|\partial_{t_j} f(\diamondsuit a_{\mathbf{t}})\big|_{\mathbf{t}=\mathbf{0}}\right|\right|_\infty \leq \frac{6C'}{\pi C} .$$

Since j is smooth on the compact set M , j^k itself is uniformly Lipshitz continuous on M with a Lipshitz constant C'' independent of Γ_0 . So we see that H is uniformly Lipshitz continuous with Lipshitz constant $C_2 := (C''+1)\frac{6C'}{\pi C} \geq 0$ independent of Γ_0 , and the rest is trivial. $\square$

Let $\mathbf{l} \in \Lambda^* - \chi$. Since $S_k(\Gamma)$ is a Hilbert space and $S_k(\Gamma) \to \mathbb{C}$, $f \mapsto b_{\mathbf{l}}$ is linear and continuous there exists exactly one $\varphi_{\Gamma_0,\mathbf{l}} \in S_k(\Gamma)$ such that $b_{\mathbf{l}} = (\varphi_{\Gamma_0,\mathbf{l}}, f)_\Gamma$ for all $f \in S_k(\Gamma)$.
Clearly, having g fixed, the family

$$\{\varphi_{\Gamma_0,\mathbf{l}}\}_{\mathbf{l} \in (\Lambda^* - \chi) \cap \Phi_C}$$

is independent of the choice of χ , but it is even independent of the choice of $g \in G$ up to permutation and multiplication with constants in $U(1)$ and invariant under conjugating Γ_0 with elements of Γ . Let us check it.

Let $g' \in G$ be another element such that $\Gamma_0 \sqsubset g' AM g'^{-1}$. Then by theorem 1.11 (ii) there exist $\mathbf{T} \in \mathbb{R}^q$ and $n \in N_K(A)$ such that $g' = g a_{\mathbf{T}} n$. Let $h' \in \mathcal{C}^\infty (\mathbb{R}^q \times M)^{\mathbb{C}}$ be given by $h'(\mathbf{t}, w) := \widetilde{f}(g' a_{\mathbf{t}} w)$ for all $\mathbf{t} \in \mathbb{R}^q$ and $w \in M$. Then we have

$$
\begin{aligned}
h'(\mathbf{t}, 1) &= \widetilde{f}(g a_{\mathbf{T}} n a_{\mathbf{t}}) \\
&= \widetilde{f}(g a_{\mathbf{T}} a_{\mathbf{t}'} n) \\
&= j(n)^k \widetilde{f}(g a_{\mathbf{t}'+\mathbf{T}}) \\
&= j(n)^k h\left(\mathbf{t}' + \mathbf{T}, 1\right) ,
\end{aligned}
$$

where we obtain $\mathbf{t}' \in \mathbb{R}^q$ by transforming $\mathbf{t}$ with the element $nM \in W$. So if we decompose

$$
h'(\mathbf{t}, w) = j(w)^k \sum_{\mathbf{l} \in (\Lambda')^* - \chi'} b'_{\mathbf{l}} e^{2\pi i \mathbf{l}\mathbf{t}}
$$

for all $(\mathbf{t}, w) \in \mathbb{R}^q \times M$, all $b'_{\mathbf{l}} \in \mathbb{C}$, $\mathbf{l} \in (\Lambda')^* - \chi'$, then

$$
\begin{aligned}
\sum_{\mathbf{l} \in (\Lambda')^* - \chi'} b'_{\mathbf{l}} e^{2\pi i \mathbf{l}\mathbf{t}} &= h'(\mathbf{t}, 1) = j(n)^k h\left(\mathbf{t}' + \mathbf{T}, 1\right) \\
&= j(n)^k \sum_{\mathbf{l} \in \Lambda^* - \chi} b_{\mathbf{l}} e^{2\pi i \mathbf{l}(\mathbf{t}'+\mathbf{T})} .
\end{aligned}
$$

We see that $b'_{\mathbf{l}'} = j(n)^k e^{2\pi i \mathbf{l}\mathbf{T}} b_{\mathbf{l}}$ and so if we define $\varphi'_{\mathbf{l}'} := j(n)^{-k} e^{-2\pi i \mathbf{l}\mathbf{T}} \varphi_{\Gamma_0, \mathbf{l}}$ for all $\mathbf{l} \in \Lambda^* - \chi$ then $b'_{\mathbf{l}} = (\varphi'_{\mathbf{l}}, f)$ for all $f \in S_k(\Gamma)$ and $\mathbf{l} \in (\Lambda')^* - \chi'$, where we obtain Λ' , $\chi' \in (\mathbb{R}^q)^*$ and $\mathbf{l}' \in (\Lambda')^* - \chi'$ by transforming Λ , χ resp. $\mathbf{l}$ with the element $nM \in W$. Clearly Φ_C itself is invariant under the Weyl group W .

Now let $\gamma \in \Gamma$ and $\Gamma'_0 := \gamma \Gamma_0 \gamma^{-1}$. Then clearly $\Gamma'_0 \sqsubset \gamma g A W (\gamma g)^{-1}$, and so, if we define $h' \in \mathcal{C}^\infty (\mathbb{R}^q \times M)^{\mathbb{C}}$ by $h'(\mathbf{t}, w) := \widetilde{f}(\gamma g a_{\mathbf{t}} w)$ for all $\mathbf{t} \in \mathbb{R}^q$ and $w \in M$, then we obtain

$$
h'(\mathbf{t}, w) = \widetilde{f}(\gamma g a_{\mathbf{t}} w) = h(\mathbf{t}, w)
$$

by the left-Γ-invariance of $\widetilde{f}$.

For the rest of the chapter we simply write $\mathbf{l} \in \Phi_C$ instead of $\mathbf{l} \in (\Lambda^* - \chi) \cap \Phi_C$. In the end we will compute $\varphi_{\Gamma_0, \mathbf{l}}$ as a relative

POINCARÉ series.

Now we are able to formulate the main goal of this chapter. Let Ω be a fundamental set for all k-admissible maximal loxodromic subgroups of Γ modulo conjugation by elements of Γ .

Theorem 1.31 (spanning set for $S_k(\Gamma)$) *Assume*

{*i*} $\Gamma \sqsubset G$ *irreducible, which means that the projection on each simple factor* G_i , $i = 1, \ldots, s$, *of* G *is dense in* G_i ,

{*ii*} $\Gamma\backslash G$ *is compact,*

{*iii*} *if* $\gamma \in G$ *regular loxodromic, then there exists a loxodromic subgroup* Γ_0 *of* Γ *such that* $\gamma \in \Gamma_0$, *and*

{*iv*} $\mathbf{v} \in \{\pm 1\}^q$ *and therefore* $B = B_1 \times \cdots \times B_q$ *where* B_j *are bounded symmetric domains of rank* 1 .

Then

$$\{\varphi_{\Gamma_0,\mathbf{l}} \,|\, \Gamma_0 \in \Omega, \mathbf{l} \in \Phi_C\}$$

is a spanning set for $S_k(\Gamma)$.

Clearly condition {ii} implies SATAKE's theorem and $\dim S_k(\Gamma) < \infty$, see section 1.2 . We conjecture that SATAKE's theorem and theorem 1.31 remain true even if we replace condition {ii} by the weaker condition of $\Gamma \sqsubset G$ being a lattice (discrete such that $\mathrm{vol}\, \Gamma\backslash G < \infty$) under the additional assumption that $\dim_{\mathbb{C}} B_j \geq 2$, $j = 1, \ldots, q$, using a calculation similar to that of section 3.2 and a generalized version of theorem 0.6 of [6] giving a nice 'fundamental domain' for $\Gamma\backslash G$, see the proof of theorem 3.13 in section 3.2 . In the rank $q = 1$ case the conjecture is true, this is Katok's and Foth's result, see [11] .

For proving theorem 1.31 we need some tools:

Theorem 1.32

(i) There exists a unique LIE *algebra embedding*

$$\rho : \bigoplus_{j=1}^{q} \mathfrak{sl}(2, \mathbb{C}) \hookrightarrow \mathfrak{g}^{\mathbb{C}}$$

such that

$$\rho\left(0,\ldots,0,\underset{\substack{\uparrow\\ j}}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}},0\ldots,0\right)=\mathbf{e}_j$$

and

$$\rho\left(0,\ldots,0,\underset{\substack{\uparrow\\ j}}{\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}},0\ldots,0\right)=\mathbf{e}_j$$

for all $j=1,\ldots,q$.

(ii) The preimage of $\mathfrak{g}$ *under* ρ *is* $\bigoplus_{j=1}^{q}\mathfrak{s}u(1,1)$ *, and the preimage of* $\mathfrak{k}$ *is* $\bigoplus_{j=1}^{q}\mathfrak{s}\left(\mathfrak{u}(1)\oplus\mathfrak{u}(1)\right)\simeq\bigoplus_{j=1}^{q}\mathfrak{u}(1)$. ρ *lifts to a* Lie *group homomorphism* $\widetilde{\rho}:SL(2,\mathbb{C})^q\to G^{\mathbb{C}}$ *such that* $\widetilde{\rho}\left(SU(1,1)^q\right)\sqsubset G$.

Proof: (i) By lemma 9.7 of [13] and its proof for any tripotent $\mathbf{c}$ of Z there exists a unique Lie algebra homomorphism $\rho':\mathfrak{s}l(2,\mathbb{C})\to\mathfrak{g}$ such that

$$\rho'\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}=\mathbf{c}\text{ and }\rho'\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}=\widetilde{\mathbf{c}}\,.$$

Since $\mathbf{e}_1,\ldots,\mathbf{e}_q$ are pairwise orthogonal tripotents we get

$$\left[\mathbf{e}_i,\widetilde{\mathbf{e}_j}\right]=-2\left\{\mathbf{e}_i,\mathbf{e}_j^*,\diamondsuit\right\}=0$$

if $i\neq j$, and

$$\left[\mathbf{e}_i,\widetilde{\mathbf{e}_i}\right](\mathbf{e}_j)=-2\left\{\mathbf{e}_i,\mathbf{e}_i^*,\mathbf{e}_j\right\}=-2\delta_{ij}\mathbf{e}_i$$

for all $i,j=1,\ldots,q$. That implies that $\left[\mathbf{e}_j,\widetilde{\mathbf{e}_j}\right]$, $j=1,\ldots,q$, are linearly independent, and so ρ is indeed an embedding. □
(ii) This follows again by lemma 9.7 of [13] , which says that the preimage of $\mathfrak{g}$ under ρ' is $\mathfrak{s}u(1,1)$, and the preimage of $\mathfrak{k}$ is $\mathfrak{s}\left(\mathfrak{u}(1)\oplus\mathfrak{u}(1)\right)$. The last statement is trivial since $SL(2,\mathbb{C})^q$ is simply connected. □

Let us now identify the elements of $\mathfrak{g}$ with the corresponding left invariant differential operators, they are defined on a dense subset of $L^2\left(\Gamma\backslash G\right)$, and define

$$\mathcal{D}_j \;:=\; \rho\left(0,\ldots,0,\underset{\substack{\uparrow\\ j}}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}},0\ldots,0\right) = \mathbf{e}_j - \widetilde{\mathbf{e}_j} \in \mathfrak{a}\,,$$

$$\mathcal{D}'_j \;:=\; \rho\left(0,\ldots,0,\underset{\substack{\uparrow\\ j}}{\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}},0\ldots,0\right) = i\left(\mathbf{e}_j + \widetilde{\mathbf{e}_j}\right) \in \mathfrak{p} \quad \text{and}$$

$$\phi_j \;:=\; \rho\left(0,\ldots,0,\underset{\substack{\uparrow\\ j}}{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}},0,\ldots,0\right) \in \mathfrak{k}\,,$$

$j = 1,\ldots,q$. We see that $\mathcal{D}_1,\ldots,\mathcal{D}_q$ span the LIE algebra $\mathfrak{a}$ of A , and so as left-invariant differential operators they generate the multiflow $\varphi_{\mathbf{t}}$. By theorem 1.32 the $\mathbb{R}$-linear span of all $\mathcal{D}_j, \mathcal{D}'_j, \phi_j$ is the $3q$-dimensional sub LIE algebra $\rho\left(\mathfrak{su}(1,1)^q\right)$ of $\mathfrak{g}$, and we have the following commutation relations:

$$[\phi_j, \mathcal{D}_j] = 2\mathcal{D}'_j\,,\ \left[\phi_j, \mathcal{D}'_j\right] = -2\mathcal{D}_j \text{ and } \left[\mathcal{D}_j, \mathcal{D}'_j\right] = -2\phi_j$$

for all $j = 1,\ldots,q$, and all the other commutators are 0 . $\phi_1,\ldots,\phi_q$ generate a subgroup of K , and again by theorem 1.32 we have a LIE group homomorphism

$$\begin{aligned}(\mathbb{R}/2\pi\mathbb{Z})^q \to K\,,\ \mathbf{t} \mapsto {} & \exp\left(t_1\phi_1 + \cdots + t_q\phi_q\right) \\ & = \widetilde{\rho}\left(\begin{pmatrix} e^{it_1} & 0 \\ 0 & e^{-it_1} \end{pmatrix},\ldots,\begin{pmatrix} e^{it_q} & 0 \\ 0 & e^{-it_q} \end{pmatrix}\right)\,.\end{aligned}$$

Now define

$$\mathcal{D}^+_j := \frac{1}{2}\left(\mathcal{D}_j - i\mathcal{D}'_j\right)\,,\ \mathcal{D}^-_j := \frac{1}{2}\left(\mathcal{D}_j + i\mathcal{D}'_j\right) \text{ and } \Psi_j := -i\phi_j\,,$$

$j = 1,\ldots,q$, as left invariant differential operators on G . Then clearly

$$\left[\Psi_j, \mathcal{D}^+_j\right] = 2\mathcal{D}^+_j, \left[\Psi_j, \mathcal{D}^-_j\right] = -2\mathcal{D}^-_j \text{ and } \left[\mathcal{D}^+_j, \mathcal{D}^-_j\right] = \Psi_j\,,$$

$j = 1, \ldots, q$, and all the other commutators are 0 . Define

$$\mathcal{D}_{\mathbf{v}}^{+} := \sum_{j=1}^{q} v_j \mathcal{D}_j^{+}, \mathcal{D}_{\mathbf{v}}^{-} := \sum_{j=1}^{q} v_j \mathcal{D}_j^{-} \text{ and } \Psi := \sum_{j=1}^{q} \Psi_j \in \mathfrak{g}^{\mathbb{C}} .$$

Then again we have the commutation relations

$$\left[\Psi, \mathcal{D}_{\mathbf{v}}^{+}\right] = 2\mathcal{D}_{\mathbf{v}}^{+} \text{ and } \left[\Psi, \mathcal{D}_{\mathbf{v}}^{-}\right] = -2\mathcal{D}_{\mathbf{v}}^{-} ,$$

and **if $\mathbf{v} \in \{\pm 1\}^q$ then also $[\mathcal{D}_{\mathbf{v}}^{+}, \mathcal{D}_{\mathbf{v}}^{-}] = \Psi$.** Since the left invariant measure on G is at the same time the right invariant measure, we see that all $\xi \in \mathfrak{g}$ are skew self adjoint on $L^2(\Gamma \backslash G)$. So in particular

$$\left(\mathcal{D}_j^{+}\right)^* = -\mathcal{D}_j^{-}, \left(\mathcal{D}_j^{-}\right)^* = -\mathcal{D}_j^{+} \text{ and } \Psi_j^* = \Psi_j$$

for all $j = 1, \ldots, q$, and so

$$\left(\mathcal{D}_{\mathbf{v}}^{+}\right)^* = -\mathcal{D}_{\mathbf{v}}^{-} , \left(\mathcal{D}_{\mathbf{v}}^{-}\right)^* = -\mathcal{D}_{\mathbf{v}}^{+} \text{ and } \Psi^* = \Psi .$$

By standard FOURIER analysis we see that

$$L^2(\Gamma \backslash G) = \widehat{\bigoplus_{\mathbf{m} \in \mathbb{Z}^q}} H_{\mathbf{m}} = \widehat{\bigoplus_{\nu \in \mathbb{Z}}} H_{\nu}$$

where

$$H_{\mathbf{m}} := \left\{ F \in L^2(\Gamma \backslash G) \cap \bigcap_{j=1}^{q} \text{ domain } \Psi_j \;\middle|\; \Psi_j F = m_j F \text{ for all } j = 1, \ldots, q \right\}$$

for all $\mathbf{m} \in \mathbb{Z}^q$,

$$\begin{aligned} H_{\nu} &:= \widehat{\bigoplus_{\mathbf{m} \in \mathbb{Z}^q, \sum_{j=1}^{q} m_j = \nu}} H_{\mathbf{m}} \\ &= \left\{ F \in L^2(\Gamma \backslash G) \cap \text{ domain } \Psi \;\middle|\; \Psi F = \nu F \right\} \end{aligned}$$

for all $\nu \in \mathbb{Z}$, and both sums are orthogonal. Since $f \in S_k(\Gamma)$ we have $\widetilde{f} \in H_{(k,\ldots,k)} \sqsubset H_{qk}$. By a simple calculation we get

$$\mathcal{D}_j^{+}\left(H_{\mathbf{m}} \cap \text{ domain } \mathcal{D}_j^{+}\right) \subset H_{\mathbf{m}+2e_j} \text{ and } \mathcal{D}_j^{-}\left(H_{\mathbf{m}} \cap \text{ domain } \mathcal{D}_j^{-}\right) \subset H_{\mathbf{m}-2e_j}$$

for all $j = 1, \ldots, q$, $\mathbf{m} \in \mathbb{Z}^q$, and so

$$\mathcal{D}_{\mathbf{v}}^{+}\left(H_{\nu} \cap \text{ domain } \mathcal{D}_{\mathbf{v}}^{+}\right) \subset H_{\nu+2} \text{ and } \mathcal{D}_{\mathbf{v}}^{-}\left(H_{\nu} \cap \text{ domain } \mathcal{D}_{\mathbf{v}}^{-}\right) \subset H_{\nu-2}$$

for all $\nu \in \mathbb{Z}$.

Lemma 1.33 *For all $j = 1, \ldots, q$ and $h \in \mathcal{O}(B)$*

$$\mathcal{D}_j^- \widetilde{h} = 0\,.$$

Proof: Let $g \in G$. Then $h|_g \in \mathcal{O}(B)$ and $\widetilde{h}\,(g\diamondsuit) = \widetilde{h|_g}$. So

$$\mathcal{D}_j^- \widetilde{h}(g) = \mathcal{D}_j^- \left(\widetilde{h}\,(g\diamondsuit)\right)(1) = \partial_{\overline{z}} \left(h|_g \left(z\mathbf{e}_j\right)\right)\big|_{z=0} = 0\,.\,\square$$

Lemma 1.34 *Let $f \in S_k(\Gamma)$. Then $\widetilde{f}$ is uniformly* Lipshitz *continuous.*

Proof: Since on G we use a left invariant metric it suffices to show that there exists a constant $c \geq 0$ such that for all $g \in G$ and $\xi \in \mathfrak{g}$ with $||\xi||_2 \leq 1$

$$\left|\xi \widetilde{f}(g)\right| \leq c\,.$$

Then c is a Lipshitz constant for $\widetilde{f}$. So choose an orthonormal basis $(\xi_1, \ldots, \xi_N)$ of $\mathfrak{g}$ and a compact neighbourhood L of $\mathbf{0}$ in B . Then by Cauchy's integral formula there exist $C', C'' \geq 0$ such that for all $h \in \mathcal{O}(B) \cap L_k^\infty(B)$ and $n \in \{1, \ldots, N\}$

$$\left|\left(\xi \widetilde{h}\right)(1)\right| \leq C' \int_L |h| \leq C' \mathrm{vol}\, L\, ||h||_{\infty, L} \leq C'' \mathrm{vol}\, L \left|\left|\widetilde{h}\right|\right|_\infty ,$$

and since $\mathfrak{g} \to \mathbb{C}\,,\, \xi \mapsto \left(\xi \widetilde{h}\right)(1)$ is linear we obtain

$$\left|\left(\xi \widetilde{h}\right)(1)\right| \leq N C'' \mathrm{vol}\, L \left|\left|\widetilde{h}\right|\right|_\infty$$

for general $\xi \in \mathfrak{g}$ with $||\xi||_2 \leq 1$. Now let $g \in G$. Then again $f|_g \in \mathcal{O}(B)$, $\widetilde{f}\,(g\diamondsuit) = \widetilde{f|_g}$, and by Satake's theorem, f and so $f|_g \in L_k^\infty(B)$. So

$$\left|\xi \widetilde{f}(g)\right| = \left|\left(\xi \widetilde{f}\,(g\diamondsuit)\right)(1)\right| \leq N C'' \mathrm{vol}\, L \left|\left|\widetilde{f}\,(g\diamondsuit)\right|\right|_\infty \leq N C'' \mathrm{vol}\, L \left|\left|\widetilde{f}\right|\right|_\infty ,$$

and we can define $c := N C'' \mathrm{vol}\, L \left|\left|\widetilde{f}\right|\right|_\infty$. $\square$

Lemma 1.35 *There exists $g_0 \in G$ such that*

$$\Gamma g_0 A_{\mathbf{v}} \underset{\text{dense}}{\subset} G\,.$$

Proof: This is a direct consequence of Moore's ergodicity theorem, see for example theorem 2.2.6 in [18] :

> Let $G = \prod G_i$ be a (finite) product of connected non-compact simple Lie groups with finite center. Let $\Gamma \subset G$ be an irreducible lattice. If $H \subset G$ is a closed subgroup and H is not compact, then H is ergodic on G/Γ .

Hereby 'H is ergodic on G/Γ' means that every measurable H-invariant subset of G/Γ is either null or conull, and proposition 2.1.7 in [18] :

> Suppose S is a second countable topological space , that G acts continuously, and that a quasi-invariant μ is positive on open sets. If the action is properly ergodic then for almost every $s \in S$, orbit(s) is a dense null set.

Hereby 'properly ergodic' means that there is no conull orbit. $\square$

Proof of theorem 1.31 : Let $f \in S_k(\Gamma)$ such that $(\varphi_{\Gamma_0,\mathbf{l}}, f) = 0$ for all $\varphi_{\Gamma_0,\mathbf{l}}$, Γ_0 k-admissible loxodromic subgroup of Γ , $\mathbf{l} \in \Phi_C$. We will show that $f = 0$ in several steps.

Lemma 1.36 *There exists $F \in \mathcal{C}(\Gamma\backslash G)^{\mathbb{C}}$ uniformly* LIPSHITZ *continuous on compact sets and differentiable along the diagonal flow $\varphi_{\tau\mathbf{v}}$ such that $f = \partial_\tau F(\diamondsuit a_{\tau\mathbf{v}})|_{\tau=0} = \mathcal{D}_{\mathbf{v}} F$.*

Proof: Let $g_0 \in G$ be given by lemma 1.35 . Define $s \in \mathcal{C}^\infty(\mathbb{R})^{\mathbb{C}}$ by

$$s(t) := \int_0^t \widetilde{f}(g_0 a_{\tau\mathbf{v}})\, d\tau$$

for all $t \in \mathbb{R}$.

Step I **Show that for all $L \subset G$ compact there exist constants $C_3 \geq 0$ and $\varepsilon_3 > 0$ such that for all $t \in \mathbb{R}$, $T \geq 0$ and $\gamma \in \Gamma$ if $g_0 a_{t\mathbf{v}} \in L$ and**

$$\varepsilon := d\left(\gamma g_0 a_{t\mathbf{v}}, g_0 a_{(t+T)\mathbf{v}}\right) \leq \varepsilon_3$$

then $|s(t) - s(t+T)| \leq C_3 \varepsilon$.

Let $L \subset G$ be compact, $T_0 > 0$ be given by lemma 1.23 and $C_1 \geq 1$ and ε_1 be given by theorem 1.25 (i) with $T_1 := T_0$. Define $C_3 := \max\left(C_1(C_2 + 2c), 2\left\|\widetilde{f}\right\|_\infty\right) \geq 0$, where $C_2 \geq 0$ is the LIPSHITZ constant from theorem 1.29 (ii) and $c \geq 0$ is the LIPSHITZ constant of $\widetilde{f}$. Define $\varepsilon_3 := \min\left(\varepsilon_1, \varepsilon_2, \frac{T_0}{2C_1}\right) > 0$, where $\varepsilon_2 > 0$ is given by theorem 1.25 (ii) .

Let $t \in \mathbb{R}$, $T \geq 0$ and $\gamma \in \Gamma$ such that $g_0 a_{t\mathbf{v}} \in L$ and $\varepsilon := d\left(\gamma g_0 a_{t\mathbf{v}}, g_0 a'_{(t+T)\mathbf{v}}\right) \leq \varepsilon'$.

First assume $T \geq T_0$. Then by theorem 1.25 (i) since $\varepsilon \leq \varepsilon_1$ there exist $g \in G$, $w_0 \in M$ and $\mathbf{t}_0 \in \mathbb{R}^q$ regular such that $\gamma g = g a_{\mathbf{t}_0} w_0$, $d((\mathbf{t}_0, w_0), (T\mathbf{v}, 1)) \leq C_1 \varepsilon$, and for all $\tau \in [0, T]$

$$d\left(g_0 a_{(t+\tau)\mathbf{v}}, g a_{\tau\mathbf{v}}\right) \leq C_1\varepsilon\left(e^{-\tau}+e^{-(T-\tau)}\right) .$$

We get

$$s(t+T)-s(t)=\underbrace{\int_0^T \widetilde{f}\left(g a_{\tau\mathbf{v}}\right) d\tau}_{I_1:=}+\underbrace{\int_0^T\left(\widetilde{f}\left(g_0 a_{(t+\tau)\mathbf{v}}\right)-\widetilde{f}\left(g a_{\tau\mathbf{v}}\right)\right) d\tau}_{I_2:=},$$

$$\begin{aligned}
|I_2| &\leq \int_0^T\left|\widetilde{f}\left(g_0 a_{(t+\tau)\mathbf{v}}\right)-\widetilde{f}\left(g a_{\tau\mathbf{v}}\right)\right| d\tau \\
&\leq c\int_0^T d\left(g_0 a_{(t+\tau)\mathbf{v}}, g a_{\tau\mathbf{v}}\right) d\tau \\
&\leq cC_1\varepsilon\int_0^T\left(e^{-\tau}+e^{-(T-\tau)}\right) d\tau \\
&\leq 2cC_1\varepsilon .
\end{aligned}$$

By our assumption there exists a maximal loxodromic subgroup Γ_0 of Γ such that $\gamma\in\Gamma_0$ and, since theorem 1.11 tells us that $g\in G$ is already determined by γ up to right translation with elements of $AN_K(A)$, even $\Gamma_0 \sqsubset gAWg^{-1}$. We define $h\in\mathcal{C}^\infty\left(\mathbb{R}^q\times M\right)^{\mathbb{C}}$ as $h(\mathbf{t},w):=\widetilde{f}\left(g a_{\mathbf{t}} w\right)$ for all $\mathbf{t}\in\mathbb{R}^q$ and $w\in M$. Then

$$I_1=\int_0^T h(\tau\mathbf{v},1)\, d\tau .$$

If Γ_0 is not k-admissible then $I_1=0$ by lemma 1.28 and the claim follows. If Γ_0 is k-admissible then we can apply theorem 1.29 (i) and, since f is perpendicular to all $\varphi_{\Gamma_0,\mathbf{l}}$, $\mathbf{l}\in\Phi_C$, even 1.29 (ii) , and so

$$\begin{aligned}
|I_1| &= \left|H(T\mathbf{v},1)-H(\mathbf{0},1)\right| \\
&= \left|H(T\mathbf{v},1)-H(\mathbf{t}_0,w_0)\right| \\
&\leq C_2 d\left((T\mathbf{v},1),(\mathbf{t}_0,w_0)\right) \\
&\leq C_1C_2\varepsilon ,
\end{aligned}$$

where we used the fact that $j\left(w_0\right)^k=e^{2\pi i\chi\mathbf{t}_0}$ since $\gamma\in\Gamma_0$, and so

$$H(\mathbf{t}_0,w_0)=H(\mathbf{0},1)\, j(w_0)^k e^{-2\pi i\chi\mathbf{t}_0}=H(\mathbf{0},1) ,$$

and the claim follows again.
Now assume $T\leq T_0$. Then by theorem 1.25 (ii) since $\varepsilon\leq\varepsilon_0$ we obtain $T\leq 2\varepsilon$ and so

$$|s(t+T)-s(t)| = \left|\int_0^T \widetilde{f}\left(g_0 a_{(t+\tau)\mathbf{v}}\right) d\tau\right| \leq 2\varepsilon \left|\left|\widetilde{f}\right|\right|_\infty .$$

Step II **Show that there exists a unique $F \in \mathcal{C}\left(\Gamma\backslash G\right)^{\mathbb{C}}$ uniformly** LIPSHITZ **continuous on compact sets such that for all $t \in \mathbb{R}$**

$$s(t) = F\left(g_0 a_{t\mathbf{v}}\right) .$$

By step I for all $L \subset \Gamma\backslash G$ compact with $L^\circ \underset{\text{dense}}{\subset} L$ there exists a unique $F_L \in \mathcal{C}\left(\Gamma\backslash G\right)^{\mathbb{C}}$ uniformly LIPSHITZ continuous such that for all $t \in \mathbb{R}$ if $\Gamma g_0 a_{t\mathbf{v}} \in L$ then $s(t) = F_L\left(\Gamma g_0 a_{t\mathbf{v}}\right)$. So we see that there exists a unique $F \in \mathcal{C}\left(\Gamma\backslash G\right)^{\mathbb{C}}$ such that $F|_L = F_L$ for all $L \subset \Gamma\backslash G$ compact with $L^\circ \underset{\text{dense}}{\subset} L$.

Step III **Show that F is differentiable along the diagonal flow, and for all $g \in G$**

$$\partial_\tau F\left(g a_{\tau\mathbf{v}}\right)|_{\tau=0} = \widetilde{f}(g) .$$

Let $g \in G$. It suffices to show that for all $T \in \mathbb{R}$

$$\int_0^T \widetilde{f}\left(g a_{\tau\mathbf{v}}\right) d\tau = F\left(g a_{T\mathbf{v}}\right) - F(g) .$$

If $g = g_0 a_{t\mathbf{v}}$ for some $t \in \mathbb{R}$ then it is clear by construction. For general $g \in G$ since $\Gamma g_0 A_{\mathbf{v}} \underset{\text{dense}}{\subset} G$ there exists $\left(\gamma_n, t_n\right)_{n\in\mathbb{N}} \in \left(\Gamma \times \mathbb{R}\right)^{\mathbb{N}}$ such that

$$\lim_{n\to\infty} \gamma_n g_0 a_{t_n\mathbf{v}} = g ,$$

and so by lemma 1.24

$$\lim_{n\to\infty} \gamma_n g_0 a_{(\tau+t_n)\mathbf{v}} = g a_{\tau\mathbf{v}}$$

compact in $\tau \in \mathbb{R}$, finally $\widetilde{f}$ is uniformly LIPSHITZ continuous. Therefore we can interchange integration and taking limit $n \rightsquigarrow \infty$:

$$\begin{aligned}
\int_0^T \widetilde{f}\left(g a_{\tau\mathbf{v}}\right) d\tau &= \lim_{n\to\infty} \int_0^T \widetilde{f}\left(\gamma_n g_0 a_{(\tau+t_n)\mathbf{v}}\right) d\tau \\
&= \lim_{n\to\infty} \left(F\left(\gamma_n g_0 a_{(T+t_n)\mathbf{v}}\right) - F\left(\gamma_n g_0 a_{t_n\mathbf{v}}\right)\right) \\
&= F\left(g a_{T\mathbf{v}}\right) - F(g) . \square
\end{aligned}$$

Lemma 1.37

(i) For all $L \subset G$ compact there exists $\varepsilon_4 > 0$ such that for all $g, h \in L$ if g and h belong to the same T^--leaf and $d^-(g,h) \le \varepsilon_4$ then

$$\lim_{t\to\infty} \left(F\left(ga_{t\mathbf{v}}\right) - F\left(ha_{t\mathbf{v}}\right)\right) = 0\,,$$

and if g and h belong to the same T^+-leaf and $d^+(g,h) \le \varepsilon_4$ then

$$\lim_{t\to-\infty} \left(F\left(ga_{t\mathbf{v}}\right) - F\left(ha_{t\mathbf{v}}\right)\right) = 0\,.$$

(ii) F is continuously differentiable along T^-- and T^+-leafs, more precisely if $\rho : I \to G$ is a continuously differentiable curve in a T^--leaf then

$$\partial_t \left(F\circ\rho\right)(t) = -\int_0^\infty \partial_t \widetilde{f}\left(\rho(t)a_{\tau\mathbf{v}}\right) d\tau\,,$$

and if $\rho : I \to G$ is a continuously differentiable curve in a T^+-leaf then

$$\partial_t \left(F\circ\rho\right)(t) = \int_{-\infty}^0 \partial_t \widetilde{f}\left(\rho(t)a_{\tau\mathbf{v}}\right) d\tau\,.$$

Proof: (i) Let $L \subset G$ be compact, and let $L' \subset G$ be a compact neighbourhood of L . Let $T_0 > 0$ be given by lemma 1.23 and $\varepsilon_2 > 0$ by theorem 1.25 (ii) both with respect to L' . Define

$$\varepsilon_4 := \frac{1}{3}\min\left(\varepsilon_1, \varepsilon_2, \frac{T_0}{2C_1}\right) > 0\,,$$

where $\varepsilon_1 > 0$ and $C_1 \ge 1$ are given by theorem 1.25 (i) with $T_1 := T_0$. Let $\delta_0 > 0$ such that $\overline{U_{\delta_0}(L)} \subset L'$ and let

$$\delta \in \;]\,0, \min\left(\delta_0, \varepsilon_4\right)[\;.$$

Let $g, h \in L$ in the same T^--leaf such that $\varepsilon := d^-(g,h) \le \varepsilon_4$. Fix some $T' > 0$. Since $\Gamma g_0 A_{\mathbf{v}} \underset{\text{dense}}{\subset} G$ there exist $\gamma_g, \gamma_h \in \Gamma$ and $t_g, t_h \in \mathbb{R}$ such that

$$d\left(ga_{t\mathbf{v}}, \gamma_g g_0 a_{(t_g+t)\mathbf{v}}\right), d\left(ha_{t\mathbf{v}}, \gamma_h g_0 a_{(t_h+t)\mathbf{v}}\right) \le \delta$$

for all $t \in [\,0, T'\,]$, and so especially $\gamma_g g_0 a_{t_g\mathbf{v}}, \gamma_h g_0 a_{t_h\mathbf{v}} \in L'$. We show that for all $t \in [\,0, T'\,]$

$$\left|F\left(\gamma_g g_0 a_{(t_g+t)\mathbf{v}}\right) - F\left(\gamma_h g_0 a_{(t_h+t)\mathbf{v}}\right)\right| \le C_3'\left(\varepsilon e^{-t} + 2\delta\right)$$

with the same constant $C_3' \ge 0$ as in step I of the proof of lemma 1.36 with respect to L' .

Without loss of generality we may assume $T := t_h - t_g \geq 0$. Define $\gamma := \gamma_g \gamma_h^{-1}$. Then for all $t \in [0, T']$

$$d\left(\gamma \gamma_g g_0 a_{(t_g+t)\mathbf{v}}, \gamma_g g_0 a_{(t_g+t+T)\mathbf{v}}\right) \leq \varepsilon e^{-t} + 2\delta$$

by the left invariance of the metric on G .

First assume $T \geq T_0$ and fix $t \in [0, T']$. Then by theorem 1.25 (i) since $\varepsilon e^{-t} + 2\delta \leq \varepsilon + 2\delta \leq \min\left(\varepsilon_1, \frac{T_0}{2C_1}\right)$ there exist $z \in G$, $\mathbf{t}_0 \in \mathbb{R}^q$ and $w \in M$ such that $\gamma z = z a_{\mathbf{t}_0} w$,

$$d\left((\mathbf{t}_0, w), (T\mathbf{v}, 1)\right) \leq C_1\left(2\delta + \varepsilon e^{-t}\right),$$

and for all $\tau \in [0, T]$

$$d\left(\gamma_g g_0 a_{(t_g+t+\tau)\mathbf{v}}, z a_{\tau\mathbf{v}}\right) \leq C_1\left(\varepsilon e^{-t} + 2\delta\right)\left(e^{-\tau} + e^{-(T-\tau)}\right).$$

And so by the same calculations as in the proof of lemma 1.36 we get the estimate

$$\left|F\left(\gamma_g g_0 a_{(t_g+t)\mathbf{v}}\right) - F\left(\gamma_g g_0 a_{(t_h+t)\mathbf{v}}\right)\right| \leq C_3'\left(\varepsilon e^{-t} + 2\delta\right).$$

Now assume $T \leq T_0$. Then by theorem 1.25 (ii) since $\gamma_g g_0 a_{t_g\mathbf{v}} \in L'$ and $\varepsilon + 2\delta \leq \varepsilon_2$ we get $\gamma = 1$ and so by the left invariance of the metric on G

$$d\left(1, a_{T\mathbf{v}}\right) \leq \varepsilon e^{-T'} + 2\delta .$$

We see that $T \leq 2\left(\varepsilon e^{-T'} + 2\delta\right)$. So as in the proof of lemma 1.36

$$\begin{aligned} \left|F\left(\gamma_g g_0 a_{(t_g+t)\mathbf{v}}\right) - F\left(\gamma_g g_0 a_{(t_h+t)\mathbf{v}}\right)\right| &\leq 2\left\|\tilde{f}\right\|_\infty\left(\varepsilon e^{-T'} + 2\delta\right) \\ &\leq C_3'\left(\varepsilon e^{-t} + 2\delta\right). \end{aligned}$$

Since F is left-Γ-invariant we have the desired estimate.

Now let us take the limit $\delta \rightsquigarrow 0$. Then $\gamma_g g_0 a_{t_g\mathbf{v}} \rightsquigarrow g$ and $\gamma_h g_0 a_{t_h\mathbf{v}} \rightsquigarrow h$, so since F is continuous

$$|F(ga_{t\mathbf{v}}) - F(ha_{t\mathbf{v}})| \leq C_3' \varepsilon e^{-t}$$

for all $t \in [0, T']$, and since $T' > 0$ has been arbitrary, we obtain this estimate for all $t \geq 0$ and so $\lim_{t\to\infty} F(ga_{t\mathbf{v}}) - F(ha_{t\mathbf{v}}) = 0$. By similar calculations we can prove $\lim_{t\to-\infty} F(ga_{t\mathbf{v}}) - F(ha_{t\mathbf{v}}) = 0$ if g and h belong to the same T^+-leaf and $d^+(g,h) \leq \varepsilon_4$. □

(ii) Let $\rho : I \to G$ be a continuously differentiable curve in a T^--leaf, and let $t_0, t_1 \in I$, $t_1 > t_0$. It suffices to show that

$$F(\rho(t_1)) - F(\rho(t_0)) = -\int_{t_0}^{t_1} \int_0^\infty \partial_t \widetilde{f}(\rho(t) a_{\tau\mathbf{v}})\, d\tau dt\, .$$

Let $C' \geq 0$ such that $||\partial_t \rho(t)|| \leq C'$ for all $t \in [t_0, t_1]$. Then since ρ lies in a T^--leaf we have $||\partial_t(\rho(t) a_{\tau\mathbf{v}})|| \leq C' e^{-\tau}$ and so

$$\partial_t \widetilde{f}(\rho(t) a_{\tau\mathbf{v}}) \leq c C' e^{-\tau}$$

for all $\tau \geq 0$ and $t \in [t_0, t_1]$ where $c \geq 0$ is the LIPSHITZ constant of $\widetilde{f}$. So the double integral on the right side is absolutely convergent and so we can interchange the order of integration:

$$\begin{aligned}
\int_{t_0}^{t_1} \int_0^\infty \partial_t \widetilde{f}(\rho(t) a_{\tau\mathbf{v}})\, d\tau dt &= \int_0^\infty \int_{t_0}^{t_1} \partial_t \widetilde{f}(\rho(t) a_{\tau\mathbf{v}})\, dt d\tau \\
&= \int_0^\infty \left(\widetilde{f}(\rho(t_1) a_{\tau\mathbf{v}}) - \widetilde{f}(\rho(t_0) a_{\tau\mathbf{v}})\right) d\tau \\
&= \lim_{T\to\infty} \left(F(\rho(t_1) a_{T\mathbf{v}}) - F(\rho(t_0) a_{T\mathbf{v}})\right) \\
&\quad - F(\rho(t_1)) + F(\rho(t_0))\, .
\end{aligned}$$

Now let $L \subset G$ be compact such that $\rho([t_1, t_2]) \subset L$ and let $\varepsilon_4 > 0$ as in (i) . Without loss of generality we may assume that $d^-(\rho(t_0), \rho(t_1)) \leq \varepsilon_4$. Then

$$\lim_{T\to\infty} \left(F(\rho(t_1) a_{T\mathbf{v}}) - F(\rho(t_0) a_{T\mathbf{v}})\right) = 0$$

by (i) . By similar calculations we can also prove

$$\partial_t (F \circ \rho)(t) = \int_{-\infty}^0 \partial_t \widetilde{f}(\rho(t) a_{\tau\mathbf{v}})\, d\tau$$

in the case when $\rho : I \to G$ is a continuously differentiable curve in a T^+-leaf. □

Lemma 1.38

(i) $F \in L^2(\Gamma\backslash G)$,

(ii) $\xi F \in L^2(\Gamma\backslash G)$ *for all* $\xi \in \mathbb{R}\mathcal{D}_{\mathbf{v}} \oplus \mathfrak{g} \cap (T^+ \oplus T^-)$.

Proof: Since by conditiion {ii} we assume $\Gamma\backslash G$ to be compact, the assertions are trivial. In the case where condition {ii} is replaced by the weaker condition of vol $\Gamma\backslash G < \infty$, (i) must be a calculation similar to that of the proof of theorem 3.24 in the super case, section 3.3 , using a fundamental domain of Γ similar to the one described in theorem 3.13 in section 3.2 resp. in theorem 0.6 of [6] (both in the rank 1 case) and a FOURIER decomposition similar to that given by theorem 3.15 in section 3.2 . But up to now we not able to handle these things.

(ii) goes through even in the case where vol $\Gamma\backslash G < \infty$ and SATAKE's theorem holds: Since $\partial_\tau F(\diamondsuit a_{\tau\mathbf{v}})|_{\tau=0} = \widetilde{f} \in L^2(\Gamma\backslash G)$ and vol $(\Gamma\backslash G) < \infty$ it suffices to show that ξF is bounded for all $\alpha \in \Phi\backslash\{\mathbf{0}\}$ and $\xi \in \mathfrak{g}^\alpha$. So let $\alpha \in \Phi\backslash\{\mathbf{0}\}$ and $\xi \in \mathfrak{g}^\alpha$. Then since $\mathbf{v} \in \mathbb{R}^q$ is regular we have $\alpha\mathbf{v} \neq 0$. First assume $\alpha\mathbf{v} > 0$. Then since we assume $(\varphi_{\tau\mathbf{v}})_{\tau\in\mathbb{R}}$ to be hyperbolic of constant 1 we even know $\alpha\mathbf{v} \geq 1$. Clearly $\xi \in T^-$ and so there exists a continuously differential curve $\rho : I \to G$ contained in the T^--leaf containing 1 such that $0 \in I$, $\rho(0) = 1$ and $\partial_t\rho(t)|_{t=0} = \xi$. Let $g \in G$. Then by lemma 1.37 (ii) we have

$$\begin{aligned} (\xi F)(g) &= \partial_t F(g\rho(t))|_{t=0} \\ &= -\int_0^\infty \partial_t \widetilde{f}(g\rho(t)a_{\tau\mathbf{v}})\Big|_{t=0} d\tau \\ &= -\int_0^\infty \partial_t \widetilde{f}(ga_{\tau\mathbf{v}}a_{-\tau\mathbf{v}}\rho(t)a_{\tau\mathbf{v}})\Big|_{t=0} d\tau \\ &= -\int_0^\infty \left(\left(\mathrm{Ad}_{a_{-\tau\mathbf{v}}}(\xi)\right)\widetilde{f}\right)(ga_{\tau\mathbf{v}})\, d\tau \\ &= -\int_0^\infty e^{-\alpha\mathbf{v}\tau}\left(\xi\widetilde{f}\right)(ga_{\tau\mathbf{v}})\, d\tau\, , \end{aligned}$$

So

$$|(\xi F)(g)| \leq c\,||\xi||_2 < \infty$$

where c is the LIPSHITZ constant of $\widetilde{f}$. The case $\alpha\mathbf{v} < 0$ is done similarly. □

So by the FOURIER decomposition described above we have

$$F = \sum_{\nu\in\mathbb{Z}} F_\nu\, ,$$

where $F_\nu \in H_\nu$ for all $\nu \in \mathbb{Z}$. $\mathcal{D}_{\mathbf{v}} = \mathcal{D}_{\mathbf{v}}^{+} + \mathcal{D}_{\mathbf{v}}^{-}$, and a simple calculation shows that $\mathcal{D}_{\mathbf{v}}^{+}$ and $\mathcal{D}_{\mathbf{v}}^{-} \in \mathbb{R}\mathcal{D}_{\mathbf{v}} \oplus \mathfrak{g} \cap (T^+ \oplus T^-)$, and so $\mathcal{D}_{\mathbf{v}}^{+}F, \mathcal{D}_{\mathbf{v}}^{-}F \in L^2(\Gamma\backslash G)$ by lemma 1.38 . So we get the FOURIER decomposition of $\widetilde{f}$ as

$$\widetilde{f} = \mathcal{D}_{\mathbf{v}}F = \sum_{\nu\in\mathbb{Z}} \left(\mathcal{D}_{\mathbf{v}}^{+}F_{\nu-2} + \mathcal{D}_{\mathbf{v}}^{-}F_{\nu+2}\right)$$

with $\mathcal{D}_{\mathbf{v}}^{+}F_{\nu-2} + \mathcal{D}_{\mathbf{v}}^{-}F_{\nu+2} \in H_\nu$ for all $\nu \in \mathbb{Z}$. But $\widetilde{f} \in H_{qk}$, and so

$$\mathcal{D}_{\mathbf{v}}^{+}F_{\nu-2} + \mathcal{D}_{\mathbf{v}}^{-}F_{\nu+2} = \begin{cases} \widetilde{f} & \text{if } \nu = qk \\ 0 & \text{otherwise} \end{cases} .$$

Lemma 1.39 $F_\nu = 0$ *for* $\nu \in \mathbb{N}_{\geq qk}$.

Proof: similar to the argument of GUILLEMIN and KAZHDAN in [8] . By the commutation relations of D_j^+ , D_j^- and Ψ_j , $j = 1, \ldots, q$, **and since** $\mathbf{v} \in \mathbb{R}\{\pm 1\}^q$ **,** we get for all $m \in \mathbb{Z}$

$$\left|\left|\mathcal{D}_{\mathbf{v}}^{+}F_m\right|\right|_2^2 = \left|\left|\mathcal{D}_{\mathbf{v}}^{-}F_m\right|\right|_2^2 + m\,||F_m||_2^2 , \tag{1.5}$$

and for all $m \in \mathbb{N}_{\geq qk+1}$ we have $\mathcal{D}_{\mathbf{v}}^{+}F_{m-2} + \mathcal{D}_{\mathbf{v}}^{-}F_{m+2} = 0$ and so

$$\left|\left|\mathcal{D}_{\mathbf{v}}^{-}F_{m+2}\right|\right|_2 = \left|\left|\mathcal{D}_{\mathbf{v}}^{+}F_{m-2}\right|\right|_2 .$$

Now let $\nu \in \mathbb{N}_{\geq qk}$. We will prove that

$$\left|\left|\mathcal{D}_{\mathbf{v}}^{+}F_{\nu+4l}\right|\right|_2 \geq ||F_\nu||_2$$

for all $l \in \mathbb{N}$ by induction on l :

> If $l = 0$ then the inequality is clear by 1.5 . So let us assume that the inequality is true for some $l \in \mathbb{N}$. Then again by 1.5 we have

$$\left|\left|\mathcal{D}_{\mathbf{v}}^{+}F_{\nu+4l+4}\right|\right|_2^2 \geq \left|\left|\mathcal{D}_{\mathbf{v}}^{-}F_{\nu+4l+4}\right|\right|_2^2 = \left|\left|\mathcal{D}_{\mathbf{v}}^{+}F_{\nu+4l}\right|\right|_2^2 \geq ||F_\nu||_2^2 .$$

On the other hand $\mathcal{D}_{\mathbf{v}}^{+}F \in L^2(\Gamma\backslash G)$ by lemma 1.38 and so $||\mathcal{D}_{\mathbf{v}}^{+}F_m||_2 \rightsquigarrow 0$ for $m \rightsquigarrow \infty$. That implies $F_\nu = 0$. □

So we obtain $\mathcal{D}_{\mathbf{v}}^{+}F_{qk-2} = \widetilde{f}$ and finally since $f \in \mathcal{O}(B)$

$$\left|\left|\widetilde{f}\right|\right|_2^2 = \left(\widetilde{f}, \mathcal{D}_{\mathbf{v}}^{+}F_{qk-2}\right) = -\left(\mathcal{D}_{\mathbf{v}}^{-}\widetilde{f}, F_{qk-2}\right) = 0 ,$$

which completes the proof of our main theorem. $\square$

Now let again Γ be arithmetic, and fix a k-admissible maximal loxodromic subgroup Γ_0 of Γ , $g \in G$ such that $\Gamma_0 \sqsubset gAMg^{-1}$ and $\chi \in (\mathbb{R}^q)^*$ such that for all $\mathbf{t} \in \mathbb{R}^q$ and $w \in M$ if $ga_{\mathbf{t}}wg^{-1} \in \Gamma_0$ then $j(w)^k = e^{2\pi i \chi \mathbf{t}}$. We will compute $\varphi_{\Gamma_0,\mathbf{l}} \in S_k(\Gamma)$, $\mathbf{l} \in \Lambda^* - \chi$, as a relative POINCARÉ series. Hereby $\equiv$ means equality up to a constant $\neq 0$ (not necessarily independent of Γ_0 and $\mathbf{l}$) .

Theorem 1.40 (computation of $\varphi_{\Gamma_0,\mathbf{l}}$) *If $k \geq \max(k_0, 2)$, where k_0 is given by* SATAKE*'s theorem, then for all $\mathbf{l} \in \Lambda^* - \chi$*

(i)

$$\varphi_{\Gamma_0,\mathbf{l}} \equiv \sum_{\gamma \in \Gamma_0 \backslash \Gamma} q|_\gamma$$

where

$$q := \int_{\mathbb{R}^q} e^{2\pi i \mathbf{l}\mathbf{t}} \Delta\left(\diamondsuit, ga_{\mathbf{t}}\mathbf{0}\right)^{-kP} \overline{j\left(ga_{\mathbf{t}}, \mathbf{0}\right)}^k d^q\mathbf{t} \in M\left(k, \Gamma_0\right) \cap L^1_k\left(\Gamma_0 \backslash B\right) .$$

(ii) In the case where $B = B_1 \times \cdots \times B_s$ and $B_1, \ldots, B_s$ are the unit balls of the full matrix spaces $\mathbb{C}^{p_1 \times q_1}, \ldots, \mathbb{C}^{p_s \times q_s}$ resp. , and $\mathbf{Z} \in B$ such that

$$g^{-1}\mathbf{Z} = \left(\begin{pmatrix} v_1 \\ \hline w_1 \end{pmatrix} \begin{matrix} \}\, q_1 \\ \}\, p_1 - q_1 \end{matrix}, \ldots, \begin{pmatrix} v_s \\ \hline w_s \end{pmatrix} \begin{matrix} \}\, q_s \\ \}\, p_s - q_s \end{matrix} \right)$$

with a triangular matrices $v_1, \ldots, v_s$ we can compute $q(\mathbf{Z})$ explicitly as

$$q\left(\mathbf{Z}\right) \equiv \left(\Delta\left(\mathbf{Z}, \mathbf{X}_{\varepsilon_0}\right) \Delta\left(\mathbf{Z}, \mathbf{X}_{-\varepsilon_0}\right)\right)^{-\frac{k}{2}P} \prod_{j=1}^{q} \left(\frac{1 + (v)_{jj}}{1 - (v)_{jj}}\right)^{\pi i l_j} ,$$

where $\varepsilon_0 \in \{\pm 1\}^q$ is arbitrary,

$$g\left(\left(\begin{array}{c} \begin{matrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_{q_1} \end{matrix} \\ \hline 0 \end{array} \right), \ldots, \left(\begin{array}{c} \begin{matrix} \varepsilon_{q-q_s+1} & & 0 \\ & \ddots & \\ 0 & & \varepsilon_q \end{matrix} \\ \hline 0 \end{array} \right) \right) ,$$

$\varepsilon \in \{\pm 1\}^q$, are the 2^q fixpoints of Γ_0 in the SHILOV *boundery of B , and*

$$v := \begin{pmatrix} v_1 & & & 0 \\ & v_2 & & \\ & & \ddots & \\ 0 & & & v_s \end{pmatrix} \in \mathbb{C}^{q \times q} .$$

Proof: (i) Let $\mathcal{F}$ be a fundamental domain of $\mathbb{R}^q/\Lambda$ and $f \in S_k(\Gamma)$, and define $h \in \mathcal{C}^\infty(\mathbb{R}^q \times M)^{\mathbb{C}}$ and $b_{\mathbf{l}}$ as in theorem 1.29 . Then by standard FOURIER expansion and theorem 1.17 we have

$$\begin{aligned}
b_{\mathbf{l}} &\equiv \int_{\mathcal{F}} e^{-2\pi i \mathbf{l}\mathbf{t}} h(\mathbf{t}, 1) d^q\mathbf{t} \\
&= \int_{\mathcal{F}} e^{-2\pi i \mathbf{l}\mathbf{t}} f(ga_{\mathbf{t}}\mathbf{0})\, j(ga_{\mathbf{t}}, \mathbf{0})^k\, d^q\mathbf{t} \\
&\equiv \int_{\mathcal{F}} e^{-2\pi i \mathbf{l}\mathbf{t}} \left(\Delta(\diamondsuit, ga_{\mathbf{t}}\mathbf{0})^{-kP}, f\right) j(ga_{\mathbf{t}}, \mathbf{0})^k\, d^q\mathbf{t} \\
&= \int_{\mathcal{F}} e^{-2\pi i \mathbf{l}\mathbf{t}} \int_G \overline{\left(\Delta(\diamondsuit, ga_{\mathbf{t}}\mathbf{0})^{-kP}\right)^\sim} \widetilde{f} j(ga_{\mathbf{t}}, \mathbf{0})^k\, d^q\mathbf{t}\,.
\end{aligned}$$

Since by SATAKE's theorem, $\widetilde{f} \in L^\infty(G)$, and

$$\begin{aligned}
&\int_{\mathcal{F}} \int_G \left| \left(\Delta(\diamondsuit, ga_{\mathbf{t}}\mathbf{0})^{-kP}\right)^\sim j(ga_{\mathbf{t}}, \mathbf{0})^k \right| d^q\mathbf{t} \\
&\quad \equiv \int_{\mathcal{F}} \int_B \left| \Delta(\diamondsuit, \mathbf{0})|_{(ga_{\mathbf{t}})^{-1}} \right| \Delta(\mathbf{Z}, \mathbf{Z})^{\left(\frac{k}{2}-1\right)P}\, dV_{\mathrm{Leb}} d^q\mathbf{t} \\
&\quad \equiv \int_B \Delta(\mathbf{Z}, \mathbf{Z})^{\left(\frac{k}{2}-1\right)P}\, dV_{\mathrm{Leb}} < \infty\,,
\end{aligned}$$

by TONELLI's and FUBINI's theorem we can interchange the order of integration:

$$\begin{aligned}
b_{\mathbf{l}} &\equiv \int_G \int_{\mathcal{F}} e^{-2\pi i \mathbf{l}\mathbf{t}} \overline{\left(\Delta(\diamondsuit, ga_{\mathbf{t}}\mathbf{0})^{-kP}\right)^\sim} j(ga_{\mathbf{t}}, \mathbf{0})^k\, d^q\mathbf{t}\, \widetilde{f} \\
&= \left(\int_{\mathcal{F}} e^{2\pi i \mathbf{l}\mathbf{t}} \Delta(\diamondsuit, ga_{\mathbf{t}}\mathbf{0})^{-kP}\, \overline{j(ga_{\mathbf{t}}, \mathbf{0})}^k d^q\mathbf{t}, f \right) \\
&= (q, f)_{\Gamma_0}\,,
\end{aligned}$$

by theorem 1.16 , where

$$\left(\int_{\mathcal{F}} e^{2\pi i \mathbf{l}\mathbf{t}} \Delta(\diamondsuit, ga_{\mathbf{t}}\mathbf{0})^{-kP}\, \overline{j(ga_{\mathbf{t}}, \mathbf{0})}^k d^q\mathbf{t} \right)^\sim \in L^1(G)\,,$$

$$\int_{\mathcal{F}} e^{2\pi i \mathbf{l}\mathbf{t}} \Delta(\diamondsuit, ga_{\mathbf{t}}\mathbf{0})^{-kP}\, \overline{j(ga_{\mathbf{t}}, \mathbf{0})}^k d^q\mathbf{t} \in \mathcal{O}(B)$$

since $\Delta(\diamondsuit, \mathbf{W}) \in \mathcal{O}(B)$ for all $\mathbf{W} \in B$ and the convergence of the integral is compact, and so

$$q' := \sum_{\gamma' \in \Gamma_0} \left(\int_{\mathcal{F}} e^{2\pi i \mathbf{l}\mathbf{t}} \Delta(\diamondsuit, ga_{\mathbf{t}}\mathbf{0})^{-kP}\, \overline{j(ga_{\mathbf{t}}, \mathbf{0})}^k d^q\mathbf{t} \right) \Bigg|_{\gamma'} \in M_k(\Gamma_0) \cap L^1_k(\Gamma_0 \backslash B)\,.$$

For all $\mathbf{Z} \in B$ we can compute $q'(\mathbf{Z})$ as

$$
\begin{aligned}
q'(\mathbf{Z}) &= \sum_{\gamma' \in \Gamma_0} \int_{\mathcal{F}} e^{2\pi i \mathbf{l t}} \Delta\left(\gamma' \mathbf{Z}, g a_{\mathbf{t}} \mathbf{0}\right)^{-kP} \overline{j\left(g a_{\mathbf{t}}, \mathbf{0}\right)}^{k} d^q \mathbf{t}\, j\left(\gamma', \mathbf{Z}\right)^k \\
&= \sum_{\gamma' \in \Gamma_0} \int_{\mathcal{F}} e^{2\pi i \mathbf{l t}} \Delta\left(\mathbf{Z}, {\gamma'}^{-1} g a_{\mathbf{t}} \mathbf{0}\right)^{-kP} \overline{j\left({\gamma'}^{-1} g a_{\mathbf{t}}, \mathbf{0}\right)}^{k} d^q \mathbf{t} \\
&\equiv \sum_{\mathbf{T} \in \Lambda} \int_{\mathcal{F}} e^{2\pi i \mathbf{l t}} \Delta\left(\mathbf{Z}, g a_{\mathbf{t}-\mathbf{T}} \mathbf{0}\right)^{-kP} \overline{j\left(g a_{\mathbf{t}-\mathbf{T}}, \mathbf{0}\right)}^{k} e^{2\pi i \chi \mathbf{T}} d^q \mathbf{t} \\
&= \sum_{\mathbf{T} \in \Lambda} \int_{\mathcal{F}} e^{2\pi i \mathbf{l}(\mathbf{t}-\mathbf{T})} \Delta\left(\mathbf{Z}, g a_{\mathbf{t}-\mathbf{T}} \mathbf{0}\right)^{-kP} \overline{j\left(g a_{\mathbf{t}-\mathbf{T}}, \mathbf{0}\right)}^{k} d^q \mathbf{t} \\
&= \int_{\mathbb{R}^q} e^{2\pi i \mathbf{l t}} \Delta\left(\mathbf{Z}, g a_{\mathbf{t}} \mathbf{0}\right)^{-kP} \overline{j\left(g a_{\mathbf{t}}, \mathbf{0}\right)}^{k} d^q \mathbf{t} =: q(\mathbf{Z}) \ .
\end{aligned}
$$

Again by theorem 1.16 we see that $\sum_{\gamma \in \Gamma_0 \backslash \Gamma} q'|_\gamma \in M_k(\Gamma) \cap L^1_k(\Gamma \backslash B)$, and so by SATAKE's theorem, even $\in S_k(\Gamma)$, such that

$$
b_{\mathbf{l}} \equiv \left(\sum_{\gamma \in \Gamma_0 \backslash \Gamma} q'|_\gamma, f \right)_\Gamma ,
$$

and so we conclude that $\varphi_{\Gamma_0, \mathbf{l}} \equiv \sum_{\gamma \in \Gamma_0 \backslash \Gamma} q'|_\gamma$. □

(ii) Let $\mathbf{Z} \in B$ such that

$$
g^{-1} \mathbf{Z} = \left(\begin{pmatrix} v_1 \\ \hline w_1 \end{pmatrix} \begin{matrix} \} \, q_1 \\ \} \, p_1 - q_1 \end{matrix} , \ldots , \begin{pmatrix} v_s \\ \hline w_s \end{pmatrix} \begin{matrix} \} \, q_s \\ \} \, p_s - q_s \end{matrix} \right)
$$

with triangular matrices $v_1, \ldots, v_s$. Then

$$
\begin{aligned}
q(\mathbf{Z}) &= \int_{\mathbb{R}^q} e^{2\pi i \mathbf{l}\mathbf{t}} \Delta\left(\mathbf{Z}, g a_{\mathbf{t}} \mathbf{0}\right)^{-kP} \overline{j\left(g a_{\mathbf{t}}, \mathbf{0}\right)}^{k} d^q \mathbf{t} \\
&= j\left(g^{-1}, \mathbf{Z}\right)^k \int_{\mathbb{R}^q} e^{2\pi i \mathbf{l}\mathbf{t}} \Delta\left(g^{-1}\mathbf{Z}, a_{\mathbf{t}} \mathbf{0}\right)^{-kP} \overline{j\left(a_{\mathbf{t}}, \mathbf{0}\right)}^{k} d^q \mathbf{t} \\
&= j\left(g^{-1}, \mathbf{Z}\right)^k \prod_{j=1}^{q} \int_{-\infty}^{\infty} e^{2\pi i l_j t} \left(1 - (v)_{jj} \tanh\, t\right)^{-kP} \frac{1}{(\cosh\, t)^{kP}} dt \\
&= j\left(g^{-1}, \mathbf{Z}\right)^k \prod_{j=1}^{q} \int_{-\infty}^{\infty} \frac{e^{2\pi i l_j t}}{\left(\cosh\, t - (v)_{jj} \sinh\, t\right)^{kP}} dt \\
&\equiv j\left(g^{-1}, \mathbf{Z}\right)^k \prod_{j=1}^{q} \frac{1}{\left(1 - (v)_{jj}^2\right)^{\frac{k}{2}P}} \left(\frac{1 + (v)_{jj}}{1 - (v)_{jj}}\right)^{\pi i l_j} \\
&= j\left(g^{-1}, \mathbf{Z}\right)^k \left(\prod_{j=1}^{q} \left(1 - (\varepsilon_0)_j (v)_{jj}\right) \left(1 + (\varepsilon_0)_j (v)_{jj}\right)\right)^{-\frac{k}{2}P} \times \\
&\quad \times \prod_{j=1}^{q} \left(\frac{1 + (v)_{jj}}{1 - (v)_{jj}}\right)^{\pi i l_j} \\
&\equiv \left(\Delta\left(\mathbf{Z}, \mathbf{X}_{\varepsilon_0}\right) \Delta\left(\mathbf{Z}, \mathbf{X}_{-\varepsilon_0}\right)\right)^{-\frac{k}{2}P} \prod_{j=1}^{q} \left(\frac{1 + (v)_{jj}}{1 - (v)_{jj}}\right)^{\pi i l_j} . \Box
\end{aligned}
$$

Chapter 2

Super manifolds and the concept of parametrization

2.1 Graded algebraic structures

Throughout this section let K be a field of characteristic $\neq 2$.

Definition 2.1 (graded vectorspace)

(i) A graded vectorspace over K is a K-vectorspace V together with a splitting $V = V_0 \oplus V_1$ of V into a direct sum of two K-vector spaces V_1 and V_2 . In this case V_0 is called the even and V_1 the odd part of V , $(V_1 \cup V_2) \setminus \{0\}$ is called the set of homogeneous elements of V , and for all homogeneous $v \in V$

$$\dot{v} := \begin{cases} 0 & \textit{if } v \in \mathcal{A}_0 \\ 1 & \textit{if } v \in \mathcal{A}_1 \end{cases}$$

is called the parity of v .

(ii) Let V and W be two graded vector spaces over K and $\varphi : V \to W$ be a linear map. Then φ is called graded if and only if $\varphi(V_0) \subset W_0$ and $\varphi(V_1) \subset W_1$.

(iii) Let $U \sqsubset V$ be a subspace of V . Then U is called a graded subspace of V if and only if $U = U_0 \oplus U_1$ where $U_0 := U \cap V_0$ and $U_1 := U \cap V_1$. In this case U itself is a graded vectorspace over K .

Definition 2.2 (graded algebra) *Let $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ be at the same time a graded vectorspace and an algebra over the field K .*

(i) We say $\mathcal{A}$ is a graded algebra if $\mathcal{A}_i\mathcal{A}_j \subset \mathcal{A}_{i+j}$ for all $i, j \in \mathbb{Z}_2$.

(ii) If $\mathcal{A}$ is a graded algebra then it is said to be graded commutative if for all homogeneous $a, b \in \mathcal{A}$

$$ab = (-1)^{\dot{a}\dot{b}} ba \,.$$

(v) Let $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ be a graded algebra over the field K , and let $\mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1$ be at the same time a graded vectorspace over K and a left- (right-)module over the algebra $\mathcal{A}$. Then $\mathcal{M}$ is called a left- (right-) graded module over the graded algebra $\mathcal{A}$ if and only if

$$\mathcal{A}_i \mathcal{M}_j \subset \mathcal{M}_{i+j}$$

resp.

$$\mathcal{M}_i \mathcal{A}_j \subset \mathcal{M}_{i+j}$$

for all $i, j \in \mathbb{Z}_2$.

$\mathcal{A}$ being an associative graded algebra clearly implies that $\mathcal{A}_0 \subset \mathcal{A}$ is a subalgebra and that $\mathcal{A}$ and $\mathcal{A}_1$ are bimoduls over $\mathcal{A}_0$. Especially $\mathcal{A}$ being a graded commutative algebra implies that $\mathcal{A}_0$ is commutative and that for all $\alpha, \beta \in \mathcal{A}_1$ we have $\alpha\beta = -\beta\alpha$ and so $\alpha^2 = 0$.

Every commutative algebra $\mathcal{A}$ is graded commutative as well if we split $\mathcal{A} = \mathcal{A} \oplus \{0\}$.

If $\mathcal{M}$ is a left- (right-) graded module over the graded commutative algebra $\mathcal{A}$ then $\mathcal{M}$ is at the same time a right- (left-) graded module over $\mathcal{A}$ by bilinear extension of

$$ma := (-1)^{\dot{m}\dot{a}} am$$

resp.

$$am := (-1)^{\dot{a}\dot{m}} ma$$

for all homogeneous $a \in \mathcal{A}$ and $m \in \mathcal{M}$. And so in this case we say that $\mathcal{M}$ is simply a graded module over $\mathcal{A}$.

Clearly if $\mathcal{A}$ is an associative graded commutative algebra , $\mathcal{B}$ is a sub graded algebra of $\mathcal{A}$ and $\mathcal{M}$ is a graded subspace of $\mathcal{A}$ invariant under left- and right-multiplication with elements of $\mathcal{B}$ then $\mathcal{M}$ is a graded module over $\mathcal{B}$.

The most important example of an associative graded commutative algebra over K is the exterior (Grassmann) algebra of K^n , $n \in \mathbb{N}$:

Let $n \in \mathbb{N}$, $\wp(n) := \wp(\{1, \dots, n\})$, $\wp_0(n) := \left\{ S \in \wp(n) \;\middle|\; 2 \mid \; |S| \right\}$, $\wp_1(n) := \left\{ S \in \wp(n) \;\middle|\; 2 \nmid \; |S| \right\}$, and let

$$\Lambda(K^n) = \left\{ \sum_{S \in \wp(n)} a_S e^S \;\middle|\; a_S \in K , \, S \in \wp(n) \right\}$$

be the exterior algebra of K^n with the abbreviation $e^S = e_{i_1} \wedge \dots \wedge e_{i_r}$ for all $S = \{i_1, \dots, i_r\} \in \wp(n)$, $i_1 < \dots < i_r$, where e_i denotes the i-th unit vector in K^n for $i = 1, \dots, n$. Then clearly

$$\Lambda(K^n) = \bigoplus_{r=0}^{n} \Lambda^{(r)}(K^n) ,$$

where $\Lambda^{(r)}(K^n) := \bigoplus_{S \in \wp(n) \, , \, |S| = r} K e^S$, and $\Lambda(K^n) = \Lambda_0(K^n) \oplus \Lambda_1(K^n)$, where

$$\Lambda_0(K^n) := \left\{ \sum_{S \in \wp_0(n)} a_S e^S \;\middle|\; a_S \in K , \, S \in \wp_0(n) \right\} = \bigoplus_{r \in \{0, \dots, n\} \, , \, 2 \mid r} \Lambda^{(r)}(K^n)$$

and

$$\Lambda_1(K^n) := \left\{ \sum_{S \in \wp_1(n)} a_S e^S \;\middle|\; a_S \in K , \, S \in \wp_1(n) \right\} = \bigoplus_{r \in \{0, \dots, n\} \, , \, 2 \nmid r} \Lambda^{(r)}(K^n)$$

is a unital associative graded commutative algebra. In $\Lambda(K^n)$ we have the multiplication rule

$$e^S e^T = \begin{cases} (-1)^{|T<S|} e^{S \cup T} & \text{if } S \cap T = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

for all $S, T \in \wp(n)$, where we use the abbreviation

$$|K < L| := \{(k, l) \in K \times L \,|\, k < l\}$$

for all $K, L \in \wp(n)$, and we have a so-called body map

$$^\# : \Lambda(K^n) \to K , \; \sum_{S \in \wp_0(n)} a_S e^S \mapsto a_\emptyset ,$$

which is a unital graded algebra epimorphism. Clearly $^\#|_K = \mathrm{id}$. The kernel of $^\#$ is precisely the set $\mathcal{N}$ of nilpotent elements in $\Lambda(K^n)$ and at the same time the ideal spanned by $\Lambda_1(K^n)$. We have $\mathcal{N}^n = Ke^{\{1,\ldots,n\}}$, $\mathcal{N}^{n+1} = 0$ and

$$\Lambda(K^n)/\mathcal{N} \simeq K$$

via $^\#$. Let $a \in \Lambda(K^n)$. Then a is invertible in $\Lambda(K^n)$ if and only if $a^\#$ is invertible in K , and in this case

$$a^{-1} = \frac{1}{a^\#} \sum_{r \in \mathbb{N}} \left(\frac{a^\# - a}{a^\#} \right)^r .$$

This is a consequence of a more general theorem:

Theorem 2.3 *Let $\mathcal{A}$ be a unital associative algebra over a field K , and let $a, b \in \mathcal{A}$ such that b is invertible and $(b-a)b^{-1}$ or equivalently $b^{-1}(b-a)$ is nilpotent. Then a is invertible and*

$$a^{-1} = b^{-1} \sum_{r=0}^{\infty} \left((b-a)b^{-1} \right)^r = \sum_{r=0}^{\infty} \left(b^{-1}(b-a) \right)^r b^{-1} .$$

Proof: Both sums are finite since $(b-a)b^{-1}$ and $b^{-1}(b-a)$ are nilpotent. Clearly

$$a = \left(1 + (a-b)b^{-1}\right) b = b \left(1 + b^{-1}(a-b)\right) ,$$

and so

$$\begin{aligned} a \left(b^{-1} \sum_{r=0}^{\infty} \left((b-a)\, b^{-1} \right)^r \right) &= \left(1 + (a-b)b^{-1}\right) \sum_{r=0}^{\infty} \left(-(a-b)b^{-1} \right)^r \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} \left(\sum_{r=0}^{\infty} \left(b^{-1}(b-a) \right)^r b^{-1} \right) a &= \sum_{r=0}^{\infty} \left(-b^{-1}(a-b) \right)^r \left(1 + b^{-1}(a-b) \right) \\ &= 1 . \end{aligned}$$

So a is invertible and the formula holds. $\square$

Let $\mathcal{A}$ be a unital associative graded algebra over the field K , $p, q, r, s \in \mathbb{N}$. Then we define the graded bimodule $\mathcal{A}^{(p|q)\times(r|s)} = \mathcal{A}_0^{(p|q)\times(r|s)} \oplus \mathcal{A}_1^{(p|q)\times(r|s)}$ of $(p|q) \times (r|s)$ - graded matrices over $\mathcal{A}$ by $\mathcal{A}^{(p|q)\times(r|s)} := \mathcal{A}^{(p+q)\times(r+s)}$ as a graded vectorspace with grading

$$\mathcal{A}_0^{(p|q)\times(r|s)} := \left\{ \left(\begin{array}{c|c} A & \beta \\ \hline \gamma & D \end{array} \right) \middle| \, A \in \mathcal{A}_0^{p\times r} \, , \, D \in \mathcal{A}_0^{q\times s} \, , \, \beta \in \mathcal{A}_1^{p\times s} \, , \, \gamma \in \mathcal{A}_1^{q\times r} \right\}$$

and

$$\mathcal{A}_1^{(p|q)\times(r|s)} = \left\{ \left(\begin{array}{c|c} \alpha & B \\ \hline C & \delta \end{array} \right) \middle| \, \alpha \in \mathcal{A}_1^{p\times r} \, , \, \delta \in \mathcal{A}_1^{q\times s} \, , \, B \in \mathcal{A}_0^{p\times s} \, , \, C \in \mathcal{A}_0^{q\times r} \right\} .$$

Clearly for all $i, j \in \mathbb{Z}_2$ if $g \in \mathcal{A}_i^{(p|q)\times(r|s)}$ and $h \in \mathcal{A}_j^{(r|s)\times(t|u)}$ then $gh \in \mathcal{A}_{i+j}^{(p|q)\times(t|u)}$. Especially $\mathcal{A}^{(p|q)\times(p|q)}$ is a unital associative graded algebra, and all units in $\mathcal{A}_0^{(p|q)\times(p|q)}$ form a group

$$\mathrm{GL}(p|q, \mathcal{A}) := \left(\mathcal{A}_0^{(p|q)\times(p|q)}\right)^\times = \left\{ \left(\begin{array}{c|c} A & \beta \\ \hline \gamma & D \end{array} \right) \in \mathcal{A}_0^{(p|q)\times(p|q)} \text{ invertible } \right\} .$$

In the special case where $\mathcal{A} = \Lambda(K^n)$ we have the body map

$$^\# : \Lambda(K^n)^{(p|q)\times(r|s)} \to K^{(p|q)\times(r|s)}$$

taken componentwise, which is a graded K-linear map. Especially $^\#$ is a unital graded algebra epimorphism in the case where $(p, q) = (r, s)$. Again all elements of $\ker {}^\#$ are nilpotent, and so we can apply theorem 2.3 , which gives here

Corollary 2.4 *Let*

$$g = \left(\begin{array}{c|c} A & \beta \\ \hline \gamma & D \end{array} \right) \in \Lambda(K^n)_0^{(p|q)\times(p|q)} .$$

Then $g \in \mathrm{GL}(p|q, \Lambda(K^n))$ *if and only if* $A^\# \in \mathrm{GL}(p, K)$ *and* $D^\# \in GL(q, K)$ *if and only if* $A \in \mathrm{GL}(p, \Lambda(K^n)_0)$ *and* $D \in \mathrm{GL}(q, \Lambda(K^n)_0)$ *. In this case*

$$\begin{aligned} g^{-1} &= \left(\begin{array}{c|c} A^{-1} & 0 \\ \hline 0 & D^{-1} \end{array} \right) \sum_{r=0}^n \left(\begin{array}{c|c} 0 & \beta D^{-1} \\ \hline \gamma A^{-1} & 0 \end{array} \right)^r \\ &= \sum_{r=0}^n \left(\begin{array}{c|c} 0 & A^{-1}\beta \\ \hline D^{-1}\gamma & 0 \end{array} \right)^r \left(\begin{array}{c|c} A^{-1} & 0 \\ \hline 0 & D^{-1} \end{array} \right) . \end{aligned}$$

We define the so-called Berezinian Ber on $\mathrm{GL}(p|q, \Lambda(K^n))$ as

$$\begin{aligned}\mathrm{Ber} : \mathrm{GL}(p|q, \Lambda(K^n)) &\to (\Lambda(K^n)_0)^\times , \\ \left(\begin{array}{c|c} A & \beta \\ \hline \gamma & D \end{array}\right) &\mapsto \det\left(A - \beta D^{-1}\gamma\right)(\det D)^{-1} .\end{aligned}$$

For checking well-definedness of Ber first observe that for all

$$\left(\begin{array}{c|c} A & \beta \\ \hline \gamma & D \end{array}\right) \in \mathrm{GL}(p|q, \Lambda(K^n))$$

$D \in \mathrm{GL}(q, \Lambda(K^n)_0)$ by corollary 2.4 , then $A - \beta D^{-1}\gamma$ and D are ordinary matrices of sizes $p \times p$ resp. $q \times q$ over the unital associative and commutative algebra $\Lambda(K^n)_0$, and so we can take the determinant of them. $A - \beta D^{-1}\gamma \in \mathrm{GL}(p, \Lambda(K^n))$ by theorem 2.3 , and so

$$\det\left(A - \beta D^{-1}\gamma\right)^\# = \det A^\# \in K^\times .$$

Therefore $\det A \in \Lambda(K^n)_0^\times$ and by the same reason $\det D \in \Lambda(K^n)_0^\times$.

Clearly for all

$$g = \left(\begin{array}{c|c} A & \beta \\ \hline \gamma & D \end{array}\right) \in \mathrm{GL}(p|q, \Lambda(K^n))$$

$(\mathrm{Ber}\ g)^\# = \det A^\# \det\left(D^\#\right)^{-1}$.

On $\Lambda(K^n)^{(p|q)\times(p|q)}$ we define the super trace as

$$\mathrm{str} : \Lambda(K^n)^{(p|q)\times(p|q)} \to \Lambda(K^n) , \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \mapsto \mathrm{tr}\, A - \mathrm{tr}\, D ,$$

which is clearly $\Lambda(K^n)$- linear and respects the grading.

Theorem 2.5

(i) Ber *is a group homomorphism.*

(ii) If $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{K} = \mathbb{C}$ *then the exponential series*

$$\exp : \Lambda(\mathbb{K}^n)^{(p|q)\times(p|q)} \to \Lambda(\mathbb{K}^n)^{(p|q)\times(p|q)}$$

converges absolutely and uniformly on compact sets, and for all $X \in \Lambda(\mathbb{K}^n)_0^{(p|q)\times(p|q)}$

$$\left(\Lambda(\mathbb{K}^n)_0, +\right) \to \left(\mathrm{GL}(p|q, \Lambda(\mathbb{K}^n)), \cdot\right),\ a \mapsto \exp(aX)$$

is a group homomorphism.

(iii) If $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{K} = \mathbb{C}$ *then*

$$\begin{array}{ccc} \mathrm{GL}(p|q, \Lambda(\mathbb{K}^n)) & \xrightarrow{\mathrm{Ber}} & \Lambda(\mathbb{K}^n)_0^\times \\ \exp \uparrow & \% & \uparrow \exp \\ \Lambda(\mathbb{K}^n)_0^{(p|q)\times(p|q)} & \underset{\mathrm{str}|_{\Lambda(\mathbb{K}^n)_0^{(p|q)\times(p|q)}}}{\longrightarrow} & \Lambda(\mathbb{K}^n)_0 \end{array} .$$

Proof: (i) This is theorem 2.27 of [4] . □
(ii) As a $\mathbb{K}$-vectorspace $\Lambda(\mathbb{K}^n)^{(p|q)\times(p|q)} \simeq \mathbb{K}^{2^n(p+q)^2}$ since each entry of $Y \in \Lambda(\mathbb{K}^n)^{(p|q)\times(p|q)}$ is an element of $\Lambda(\mathbb{K}^n) \simeq \mathbb{K}^{2^n}$ as a $\mathbb{K}$-vectorspace. By induction on $k \in \mathbb{N}$ we see that

$$\left|\left|Y^k\right|\right|_\infty \leq \left(2^n(p+q)\,||Y||_\infty\right)^k$$

for all $k \in \mathbb{N}$. So the exponential series exp converges absolutely and uniformly on compact subsets of $\Lambda(\mathbb{K}^n)^{(p|q)\times(p|q)}$. Now let $X \in \Lambda(\mathbb{K}^n)_0^{(p|q)\times(p|q)}$. Then since $\Lambda(\mathbb{K}^n)_0^{(p|q)\times(p|q)}$ is a closed subalgebra of $\Lambda(\mathbb{K}^n)^{(p|q)\times(p|q)}$ clearly $\exp X \in \Lambda(\mathbb{K}^n)_0^{(p|q)\times(p|q)}$. The rest goes as in classical analysis. □

(iii) This is an assertion in section 2.2 of [4] . □

Clearly since $^\#$ is a continuous unital graded algebra homomorphism we have $(\exp Y)^\# = \exp\left(Y^\#\right)$ for all $Y \in \Lambda(\mathbb{K}^n)^{(p|q)\times(p|q)}$.

Definition 2.6 (graded tensor product) *Let* $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ *and* $\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_1$ *be graded algebras over the field* K *. We define the graded tensor product*

$$\mathcal{A} \boxtimes \mathcal{B} = (\mathcal{A} \boxtimes \mathcal{B})_0 \oplus (\mathcal{A} \boxtimes \mathcal{B})_1$$

of $\mathcal{A}$ *and* $\mathcal{B}$ *as* $\mathcal{A} \boxtimes \mathcal{B} := \mathcal{A} \otimes \mathcal{B}$ *as a* K*-vectorspace given the grading*

$$(\mathcal{A} \boxtimes \mathcal{B})_0 := (\mathcal{A}_0 \otimes \mathcal{B}_0) \oplus (\mathcal{A}_1 \otimes \mathcal{B}_1)$$

and

$$(\mathcal{A} \boxtimes \mathcal{B})_1 = (\mathcal{A}_1 \otimes \mathcal{B}_0) \oplus (\mathcal{A}_0 \otimes \mathcal{B}_1)$$

and given the multiplication by linear extension of

$$(a \otimes b)\left(a' \otimes b'\right) := (-1)^{\dot{b}\dot{a}'}\left(aa'\right) \otimes \left(bb'\right)$$

for all homogeneous $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$. $\mathcal{A} \boxtimes \mathcal{B}$ is a graded algebra.

One easily verifies that $\mathcal{A} \boxtimes \mathcal{B}$ is associative resp. graded commutive, if $\mathcal{A}$ and $\mathcal{B}$ are associative resp. graded commutative.
The graded tensor product fulfills the following universal property: We have canonical graded embeddings $C_1 : \mathcal{A} \hookrightarrow \mathcal{A} \boxtimes \mathcal{B}\,,\ a \mapsto a \otimes 1$ and $C_2 : \mathcal{B} \hookrightarrow \mathcal{A} \boxtimes \mathcal{B}\,,\ b \mapsto 1 \otimes b$. If $\mathcal{C}$ is an associative graded commutative algebra over the field K and $\Phi_1 : \mathcal{A} \to \mathcal{C}$ and $\Phi_2 : \mathcal{B} \to \mathcal{C}$ are graded algebra homomorphisms then there exists a unique graded algebra homomorphism $\Psi : \mathcal{A} \boxtimes \mathcal{B} \to \mathcal{C}$ such that $\Phi_1 = \Psi \circ C_1$ and $\Phi_2 = \Psi \circ C_2$. Since $\Psi(a \otimes b) = \Phi_1(a)\Phi_2(b)$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$ we denote the induced graded algebra homomorphism Ψ by $\Phi_1 \otimes \Phi_2$.
The graded tensor product is commutative and associative in the sense that $\mathcal{A} \boxtimes \mathcal{B} \simeq \mathcal{B} \boxtimes \mathcal{A}$, where the isomorphism is given by linear extension of

$$a \otimes b \mapsto (-1)^{\dot{a}\dot{b}} b \otimes a$$

for all homogeneous $a \in \mathcal{A}$ and $b \in \mathcal{B}$, and if $\mathcal{C}$ is a third graded algebra then $(\mathcal{A} \boxtimes \mathcal{B}) \boxtimes \mathcal{C} \simeq \mathcal{A} \boxtimes (\mathcal{B} \boxtimes \mathcal{C})$. If $\mathcal{A} = \mathcal{A} \oplus \{0\}$ is an ordinary algebra regarded as a graded algebra then $\mathcal{A} \boxtimes \mathcal{B} = \mathcal{A} \otimes \mathcal{B}$ as algebras, and the grading of $\mathcal{A} \boxtimes \mathcal{B}$ is given by $(\mathcal{A} \boxtimes \mathcal{B})_0 = \mathcal{A} \otimes \mathcal{B}_0$ and $(\mathcal{A} \boxtimes \mathcal{B})_1 = \mathcal{A} \otimes \mathcal{B}_1$, and in this case we write $\mathcal{A} \otimes \mathcal{B}$ instead of $\mathcal{A} \boxtimes \mathcal{B}$.

Clearly $\Lambda\left(K^{m+n}\right) \simeq \Lambda\left(K^m\right) \boxtimes \Lambda\left(K^n\right)$ canonically for all $m, n \in \mathbb{N}$.

In the end of this section let us talk especially about graded algebras over $\mathbb{R}$ and $\mathbb{C}$ and the connection between them. Let $\mathcal{A}$ be a graded algebra over $\mathbb{R}$. Then the complexification $\mathcal{A}^{\mathbb{C}} := \mathbb{C} \otimes \mathcal{A}$ of $\mathcal{A}$ is a graded algebra over $\mathbb{C}$ with grading $\mathcal{A}^{\mathbb{C}} = \mathcal{A}_0^{\mathbb{C}} \oplus \mathcal{A}_1^{\mathbb{C}}$. If $\mathcal{A}$ is associative resp. graded commutative then so is $\mathcal{A}^{\mathbb{C}}$, and given two graded algebras $\mathcal{A}$, $\mathcal{B}$ over $\mathbb{R}$ we have

$$\mathcal{A}^{\mathbb{C}} \boxtimes_{\mathbb{C}} \mathcal{B}^{\mathbb{C}} = \left(\mathcal{A} \boxtimes_{\mathbb{R}} \mathcal{B}\right)^{\mathbb{C}} .$$

Clearly $\Lambda\left(\mathbb{R}^q\right)^{\mathbb{C}} = \Lambda\left(\mathbb{C}^q\right)$.

Definition 2.7 (graded involution) *Let $\mathcal{A}$ be a graded algebra over $\mathbb{C}$. A $\mathbb{C}$-antilinear graded map*

$$\bar{\ } : \mathcal{A} \to \mathcal{A}$$

is called a graded involution on $\mathcal{A}$ if and only if $\overline{ab} = \overline{b}\ \overline{a}$ and $\overline{\overline{a}} = a$ for all $a, b \in \mathcal{A}$.

Clearly if $\overline{}$ is a graded involution on the graded algebra $\mathcal{A}$ then it is an involution on $\mathcal{A}$ regarded as an ordinary algebra as well. By the way, if $\mathcal{A}$ is a unital algebra and $\overline{}$ is an involution on $\mathcal{A}$ then automatically

$$\overline{1} = 1\overline{1} = \overline{\overline{1\overline{1}}} = \overline{\overline{1}} = 1\,.$$

An easy calculation shows that given a graded commutative algebra $\mathcal{A}$ over $\mathbb{C}$ with involution $\overline{}$ then

$$^* : \mathcal{A}^{(p|q)\times(r|s)} \to \mathcal{A}^{(r|s)\times(p|q)}\,,\, g \mapsto \overline{g}^t = \overline{g^t}\,,$$

where $\diamondsuit^t$ denotes the usual transpose and $\overline{}$ is taken componentwise, is clearly $\mathbb{C}$-antilinear and respects the grading. $(g^*)^* = g$ for all $g \in \mathcal{A}^{(p|q)\times(r|s)}$, and if $g \in \mathcal{A}^{(p|q)\times(r|s)}$ and $h \in \mathcal{A}^{(r|s)\times(t|u)}$ then $(gh)^* = h^* g^*$. So especially $^* : \mathcal{A}^{(p|q)\times(p|q)} \to \mathcal{A}^{(p|q)\times(p|q)}$ is a graded involution.

Clearly $\exp(X^*) = (\exp X)^*$ for all $X \in \Lambda(K^n)_0^{(p|q)\times(p|q)}$, $\mathrm{str}(Y^*) = \overline{\mathrm{str}\, Y}$ for all $Y \in \Lambda(K^n)^{(p|q)\times(p|q)}$, and $\mathrm{Ber}\,(g^*) = \overline{\mathrm{Ber}\, g}$ for all $g \in \mathrm{GL}(p|q, \Lambda(K^n))$.

If $\mathcal{A}$ and $\mathcal{B}$ are two graded algebras over $\mathbb{C}$ with involution $\overline{}$ then the $\mathbb{C}$-antilinear extension of

$$\overline{a \otimes b}' := (-1)^{\dot{a}\dot{b}}\, \overline{a} \otimes \overline{b}$$

for all homogeneous $a \in \mathcal{A}$ and $b \in \mathcal{B}$ defines a graded involution $\overline{}'$ on $\mathcal{A} \boxtimes \mathcal{B}$. If $\mathcal{A}$ and $\mathcal{B}$ are unital then $\overline{}'$ restricted to $\mathcal{A}$ and $\mathcal{B}$ canonically embedded into $\mathcal{A} \boxtimes \mathcal{B}$ is just $\overline{}$, and so we denote the involution $\overline{}'$ on $\mathcal{A} \boxtimes \mathcal{B}$ again by the symbol $\overline{}$.

Theorem 2.8

(i) Let $\mathcal{A}$ be a graded commutative algebra over $\mathbb{R}$. Then the $\mathbb{C}$-antilinear map $\overline{} : \mathcal{A}^{\mathbb{C}} \to \mathcal{A}^{\mathbb{C}}$ given by $\mathbb{C}$-antilinear extension of

$$a \mapsto \begin{cases} a & \text{if } a \in \mathcal{A}_0 \\ ia & \text{if } a \in \mathcal{A}_1 \end{cases}$$

is a graded involution on $\mathcal{A}^{\mathbb{C}}$.

$$\mathcal{A}_0 = \left\{ a \in \mathcal{A}_0^{\mathbb{C}} \,\middle|\, \overline{a} = a \right\}$$

and

$$\mathcal{A}_1 = \left\{ \alpha \in \mathcal{A}_1^{\mathbb{C}} \;\middle|\; \overline{\alpha} = i\alpha \right\} .$$

(ii) Conversely let $\mathcal{A}$ be a graded commutative algebra over $\mathbb{C}$ with graded involution $\overline{}$. Then $\mathcal{A}_{\mathbf{R}} = (\mathcal{A}_{\mathbf{R}})_0 \oplus (\mathcal{A}_{\mathbf{R}})_1$ given by

$$(\mathcal{A}_{\mathbf{R}})_0 := \{ a \in \mathcal{A}_0 \mid \overline{a} = a \}$$

and

$$(\mathcal{A}_{\mathbf{R}})_1 := \{ \alpha \in \mathcal{A}_1 \mid \overline{\alpha} = i\alpha \}$$

is a graded commutative algebra over $\mathbb{R}$, $\mathcal{A}$ is the complexification of $\mathcal{A}_{\mathbf{R}}$, and the graded involution $\overline{}$ on $\mathcal{A}$ is given by $\mathbb{C}$-antilinear extension of

$$a \mapsto \begin{cases} a & \text{if } a \in (\mathcal{A}_{\mathbf{R}})_0 \\ ia & \text{if } a \in (\mathcal{A}_{\mathbf{R}})_1 \end{cases} .$$

(iii) Let $\mathcal{A}$ and $\mathcal{B}$ be graded commutative algebras over $\mathbb{R}$ and $\Phi : \mathcal{A} \to \mathcal{B}$ a graded algebra homomorphism. Then its unique $\mathbb{C}$-linear extension $\Phi^{\mathbb{C}} : \mathcal{A}^{\mathbb{C}} \to \mathcal{B}^{\mathbb{C}}$ is again a graded algebra homomorphism, and it respects $\overline{}$ given by (i) .

(iv) Conversely let $\mathcal{A}$ and $\mathcal{B}$ be graded commutative algebras over $\mathbb{C}$ with graded involutions $\overline{}$ and $\Phi : \mathcal{A} \to \mathcal{B}$ a graded algebra homomorphism respecting $\overline{}$. Then Φ restricts to a graded algebra homomorphism $\Phi_{\mathbf{R}} : \mathcal{A}_{\mathbf{R}} \to \mathcal{B}_{\mathbf{R}}$, and Φ is the unique $\mathbb{C}$-linear extension of $\Phi_{\mathbf{R}}$.

Proof: (i) For proving that $\overline{}$ is a graded involution it suffices to show that $\overline{ab} = \overline{b}\,\overline{a}$ and $\overline{\overline{a}} = a$ for all homogeneous $a, b \in \mathcal{A}$. Let $\alpha, \beta \in \mathcal{A}_1$. Then $\alpha\beta \in \mathcal{A}_0$. So

$$\overline{\alpha\beta} = \alpha\beta = -\beta\alpha = (i\beta)(i\alpha) = \overline{\beta}\,\overline{\alpha} ,$$

and

$$\overline{\overline{\alpha}} = \overline{i\alpha} = -i\overline{\alpha} = (-i)i\alpha = \alpha .$$

The other cases are similar. Now let $a = x + iy \in \mathcal{A}_0^{\mathbb{C}}$, $x, y \in \mathcal{A}_0$ such that $\overline{a} = a$. Then

$$x - iy = \overline{a} = a = x + iy ,$$

so $y = 0$ and $a \in \mathcal{A}_0$. Let $\alpha = \xi + i\eta \in \mathcal{A}_1^{\mathbb{C}}$, $\xi, \eta \in \mathcal{A}_1$ such that $\overline{\alpha} = i\alpha$. Then

$$i\,(\xi - i\eta) = \overline{\xi} - i\overline{\eta} = \overline{\alpha} = i\alpha = i\,(\xi + i\eta)\ ,$$

so $\eta = 0$ and $\alpha \in \mathcal{A}_1$. $\square$

(ii) For proving that $\mathcal{A}_{\mathbb{R}}$ is a graded commutative algebra over $\mathbb{R}$ it suffices to show that $ab \in \mathcal{A}_{\mathbb{R}}$ for all homogeneous $a, b \in \mathcal{A}_{\mathbb{R}}$. Let $\alpha, \beta \in (\mathcal{A}_{\mathbb{R}})_1$. Then $\alpha\beta \in \mathcal{A}_0$, and

$$\overline{\alpha\beta} = \overline{\beta}\,\overline{\alpha} = (i\beta)(i\alpha) = -\beta\alpha = \alpha\beta\,.$$

So even $\alpha\beta \in (\mathcal{A}_{\mathbb{R}})_0$. The other cases are done similar. For proving that $\mathcal{A}$ is the complexification of $\mathcal{A}_{\mathbb{R}}$ observe that for all $a \in \mathcal{A}_0$ and $\alpha \in \mathcal{A}_1$

$$a = \underbrace{\frac{a + \overline{a}}{2}}_{\in (\mathcal{A}_{\mathbb{R}})_0} + i\,\underbrace{(-i)\,\frac{a - \overline{a}}{2}}_{\in (\mathcal{A}_{\mathbb{R}})_0}$$

and

$$\alpha = \underbrace{\frac{\alpha - i\overline{\alpha}}{2}}_{\in (\mathcal{A}_{\mathbb{R}})_1} + i\,\underbrace{\frac{-i\alpha + \overline{\alpha}}{2}}_{\in (\mathcal{A}_{\mathbb{R}})_1}\ .$$

An easy calculation shows that $(\mathcal{A}_{\mathbb{R}})_0 \cap i\,(\mathcal{A}_{\mathbb{R}})_0 = (\mathcal{A}_{\mathbb{R}})_1 \cap i\,(\mathcal{A}_{\mathbb{R}})_1 = \{0\}$. The rest is trivial. $\square$

(iii) Clearly $\Phi^{\mathbb{C}}$ is a graded algebra homomorphism. For showing that it respects $\overline{}$ let $a = a_1 + ia_2 \in \mathcal{A}_0^{\mathbb{C}}$, $a_1, a_2 \in \mathcal{A}_0$ and $\alpha = \alpha_1 + i\alpha_2 \in \mathcal{A}_1^{\mathbb{C}}$, $\alpha_1, \alpha_2 \in \mathcal{A}_1$. Then

$$\Phi^{\mathbb{C}}\,(\overline{a}) = \Phi^{\mathbb{C}}\,(a_1 - ia_2) = \Phi\,(a_1) - i\Phi\,(a_2) = \overline{\Phi\,(a_1) + i\Phi\,(a_2)} = \overline{\Phi^{\mathbb{C}}(a)}$$

and

$$\Phi^{\mathbb{C}}\,(\overline{\alpha}) = \Phi^{\mathbb{C}}\,(i\alpha_1 + \alpha_2) = i\Phi\,(\alpha_1) + \Phi\,(\alpha_2) = \overline{\Phi\,(\alpha_1) + i\Phi\,(\alpha_2)} = \overline{\Phi^{\mathbb{C}}(\alpha)}\,.$$

$\square$

(iv) trivial.

We see that $\mathbb{C}$-linear extension induces a bijection between all graded algebra homomorphisms from $\mathcal{A}$ to $\mathcal{B}$ and all graded algebra homomorphisms from $\mathcal{A}^{\mathbb{C}}$ to $\mathcal{B}^{\mathbb{C}}$ respecting $\overline{}$.

Furthermore a straight forward calculation shows that given two graded commutative algebras $\mathcal{A}$ and $\mathcal{B}$ over $\mathbb{R}$ and graded involutions $\overline{}$ on $\mathcal{A}^{\mathbb{C}}$, $\mathcal{B}^{\mathbb{C}}$ and $(\mathcal{A} \boxtimes \mathcal{B})^{\mathbb{C}}$ according to theorem 2.8 (i) then again

$$\overline{a \boxtimes b} = (-1)^{\dot{a}\dot{b}}\, \overline{a} \boxtimes \overline{b}$$

for all homogeneous $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Definition 2.9 *Let $\mathcal{A}$ be a graded commutative algebra over $\mathbb{C}$ with graded involution* $\overline{}$ *.*

(i) The graded commutative $\mathbb{R}$-algebra $\mathcal{A}_{\mathbf{R}}$ given by

$$(\mathcal{A}_{\mathbf{R}})_0 := \{ a \in \mathcal{A}_0 \mid \overline{a} = a \}$$

and

$$(\mathcal{A}_{\mathbf{R}})_1 := \{ \alpha \in \mathcal{A}_1 \mid \overline{\alpha} = i\alpha \}$$

is called the real part of $\mathcal{A}$ with respect to $\overline{}$ *.*

(ii) Let $a \in \mathcal{A}$, and let $a = x + iy$ be the unique splitting of a such that $x, y \in \mathcal{A}_{\mathbf{R}}$. Then Re $a := x$ *is called the real part and* Im $a := y$ *is called the imaginary part of a with respect to* $\overline{}$ *.*

Clearly if $a \in \mathcal{A}$ then $a \in \mathcal{A}_0$ if and only if $\mathrm{Re}\, a, \mathrm{Im}\, a \in (\mathcal{A}_{\mathbf{R}})_0$ and $a \in \mathcal{A}_1$ if and only if $\mathrm{Re}\, a, \mathrm{Im}\, a \in (\mathcal{A}_{\mathbf{R}})_1$. For all $a \in \mathcal{A}_0$

$$\mathrm{Re}\, a = \frac{a + \overline{a}}{2}$$

and

$$\mathrm{Im}\, a = -i\, \frac{a - \overline{a}}{2}\,,$$

and for all $\alpha \in \mathcal{A}_1$

$$\mathrm{Re}\, \alpha = \frac{\alpha - i\overline{\alpha}}{2}$$

and

$$\mathrm{Im}\, \alpha = \frac{-i\alpha + \overline{\alpha}}{2}\,.$$

Lemma 2.10 *Let $n \in \mathbb{N}$.*

(i) There exists a unique unital graded algebra automorphism

$$\iota : \left(\Lambda\left(\mathbb{R}^n\right) \boxtimes_{\mathbb{R}} \Lambda\left(\mathbb{R}^n\right)\right)^{\mathbb{C}} \xrightarrow{\sim} \Lambda\left(\mathbb{C}^n\right) \boxtimes_{\mathbb{C}} \Lambda\left(\mathbb{C}^n\right)$$

such that

$$\iota\left(e_j \otimes 1\right) = \frac{e_j \otimes 1 - i\left(1 \otimes e_j\right)}{2}$$

and

$$\iota\left(1 \otimes e_j\right) = \frac{-i\left(e_j \otimes 1\right) + 1 \otimes e_j}{2}$$

for all $j = 1, \ldots, n$.

$$\iota^{-1}\left(e_j \otimes 1\right) = e_j \otimes 1 + i\left(1 \otimes e_j\right)$$

and

$$\iota^{-1}\left(1 \otimes e_j\right) = i\left(e_j \otimes 1\right) + 1 \otimes e_j$$

for all $j = 1, \ldots, n$.

(ii) There exists a unique graded involution $\overline{}$ on $\Lambda\left(\mathbb{C}^n\right) \boxtimes_{\mathbb{C}} \Lambda\left(\mathbb{C}^n\right)$ such that

$$\overline{e_j \otimes 1} = 1 \otimes e_j$$

for all $j = 1, \ldots, n$. $\overline{}$ is given by $\mathbb{C}$-antilinear extension of

$$\overline{e^S \otimes e^T} := (-1)^{\frac{|S|(|S|+1)}{2} + \frac{|T|(|T|+1)}{2}} e^T \otimes e^S$$

for all $S, T \in \wp(n)$. Let $\overline{}'$ be the graded involution on $\left(\Lambda\left(\mathbb{R}^n\right) \boxtimes_{\mathbb{R}} \Lambda\left(\mathbb{R}^n\right)\right)^{\mathbb{C}}$ given by theorem 2.8 (i) . Then $\overline{}$ is the unique graded involution on $\Lambda\left(\mathbb{C}^n\right) \boxtimes_{\mathbb{C}} \Lambda\left(\mathbb{C}^n\right)$ such that

$$\begin{array}{ccc} \left(\Lambda\left(\mathbb{R}^n\right) \boxtimes_{\mathbb{R}} \Lambda\left(\mathbb{R}^n\right)\right)^{\mathbb{C}} & \xrightarrow{\iota} & \Lambda\left(\mathbb{C}^n\right) \boxtimes_{\mathbb{C}} \Lambda\left(\mathbb{C}^n\right) \\ {}^{\overline{}'}\uparrow & \% & \uparrow^{\overline{}} \\ \left(\Lambda\left(\mathbb{R}^n\right) \boxtimes_{\mathbb{R}} \Lambda\left(\mathbb{R}^n\right)\right)^{\mathbb{C}} & \xrightarrow{\iota} & \Lambda\left(\mathbb{C}^n\right) \boxtimes_{\mathbb{C}} \Lambda\left(\mathbb{C}^n\right) \end{array} .$$

Proof: (i) Clearly there exist unique $\mathbb{C}$-vectorspace automorphisms Φ and Ψ on $\mathbb{C}^{2q}$ (with basis $\{e_1 \otimes 1, \ldots, e_n \otimes 1, 1 \otimes e_1, \ldots, 1 \otimes e_n\}$) , such that

$$\Phi\left(e_j \otimes 1\right) = \frac{e_j \otimes 1 - i\left(1 \otimes e_j\right)}{2} , \Phi\left(1 \otimes e_j\right) = \frac{-i\left(e_j \otimes 1\right) + 1 \otimes e_j}{2} ,$$

$$\Psi(e_j \otimes 1) = e_j \otimes 1 + i(1 \otimes e_j) \text{ and } \Psi(1 \otimes e_j) = i(e_j \otimes 1) + 1 \otimes e_j$$

for all $j = 1, \ldots, n$, clearly inverse to each other. So existence and uniqueness of ι follow by functoriality of the exterior algebra.

(ii) For proving uniqueness of $\overline{}$ observe that since $\overline{}$ is an involution $\overline{e_j \otimes 1} = 1 \otimes e_j$ implies also $\overline{1 \otimes e_j} = e_j \otimes 1$ and so

$$\overline{e^S \otimes e^T} := (-1)^{\frac{|S|(|S|+1)}{2} + \frac{|T|(|T|+1)}{2}} e^T \otimes e^S$$

for all $S, T \in \wp(n)$. For proving existence of $\overline{}$ and commutativity of the diagram let $\overline{}'$ be the unique graded involution on $\Lambda(\mathbb{C}^n) \boxtimes_{\mathbb{C}} \Lambda(\mathbb{C}^n)$ such that the diagram commutes. Then for all $j = 1, \ldots, n$

$$\begin{aligned} \overline{e_j \otimes 1} &= \overline{\iota(e_j \otimes 1 + i(1 \otimes e_j))} \\ &= \iota\left(\overline{e_j \otimes 1 + i(1 \otimes e_j)}'\right) \\ &= \iota(i(e_j \otimes 1) + 1 \otimes e_j) \\ &= 1 \otimes e_j . \Box \end{aligned}$$

In section 2.3 , when studying complex super open sets, we will see that ι is precisely the 'intertwiner' between odd holomorphic and odd real coordinate functions on complex super open sets. Observe that for all $j = 1, \ldots n$

$$\mathrm{Re}\ (e_j \otimes 1) = \iota(e_j \otimes 1)$$

and

$$\mathrm{Im}\ (e_j \otimes 1) = \iota(1 \otimes e_j)$$

with respect to $\overline{}$.

2.2 super manifolds - the real case

We will give a short description of what we mean by the category of super manifolds. Super manifolds have for example been studied in [4] , but a notion of parametrization concerning super manifolds seems to be new, never studied systematically. Let us start with a sub category, namely the category of super open sets, which is simpler to deal with. Let M be a manifold and $q \in \mathbb{N}$. Then we have a sheaf

$$\mathcal{D}\left(\diamondsuit^{|q}\right)_M := \mathcal{C}^\infty_M \otimes \Lambda(\mathbb{R}^q)$$

of unital associative graded commutative algebras on M and a unital graded sheaf epimorphism

$$^{\#}:\mathcal{D}\left(\diamondsuit^{|q}\right)_{M}\rightarrow\mathcal{C}_{M}^{\infty}\,,\ \sum_{S\in\wp(q)}f_{S}e^{S}\mapsto f_{\emptyset}$$

called the body map. Via the canonical unital graded sheaf embeddings $\mathcal{C}_{M}^{\infty}\hookrightarrow\left(\mathcal{D}\left(\diamondsuit^{|q}\right)_{M}\right)_{0}$ and $\Lambda\left(\mathbb{R}^{q}\right)\hookrightarrow\mathcal{D}\left(\diamondsuit^{|q}\right)_{M}$ we identify $\mathcal{C}_{M}^{\infty}$ and $\Lambda\left(\mathbb{R}^{q}\right)$ with graded subsheaves of $\mathcal{D}\left(\diamondsuit^{|q}\right)_{M}$. Then clearly $^{\#}$ is a projecton in the sense that $^{\#}|_{\mathcal{C}_{M}^{\infty}}=\mathrm{id}$.

As in $\Lambda\left(\mathbb{R}^{q}\right)$ itself the subsheaf $\mathcal{N}$ of $\mathcal{D}\left(\diamondsuit^{|q}\right)_{M}$ of all nilpotents elements is precisely the kernel of $^{\#}$ in $\mathcal{D}\left(\diamondsuit^{|q}\right)_{M}$ and at the same time the ideal in $\mathcal{D}\left(\diamondsuit^{|q}\right)_{M}$ spanned by $\left(\mathcal{D}\left(\diamondsuit^{|q}\right)_{M}\right)_{1}$. We have $\mathcal{N}^{q}=e^{\{1,\dots,q\}}\mathcal{C}_{M}^{\infty}\neq 0$, $\mathcal{N}^{q+1}=0$ and

$$\mathcal{D}\left(\diamondsuit^{|q}\right)_{M}\Big/\mathcal{N}\simeq\mathcal{C}_{M}^{\infty}$$

via $^{\#}$.

On $\mathcal{D}\left(\diamondsuit^{|q}\right)_{M}$ as a free 2^{q}-dimensional $\mathcal{C}_{M}^{\infty}$-module we will always use the uniformal structure of compact convergence in all derivatives. Then from classical analysis we know that given $U\subset\mathbb{R}^{p}$ open, $\mathcal{D}\left(U^{|q}\right)$ is complete, and given $V\subset U$ open the restriction map

$$|_{V}:\mathcal{D}\left(U^{|q}\right)\rightarrow\mathcal{D}\left(V^{|q}\right)$$

is a continuous unital graded algebra homomorphism, whose image is dense in $\mathcal{D}\left(V^{|q}\right)$.

Now let us consider the special case $M=\mathbb{R}^{p}$ for some $p\in\mathbb{N}$. Then in $\mathcal{D}\left(\mathbb{R}^{p|q}\right)$ we have the even coordinate functions

$$x_{1},\dots,x_{p}\in\mathcal{C}^{\infty}(\mathbb{R}^{p})\hookrightarrow\mathcal{D}\left(\mathbb{R}^{p|q}\right)_{0}$$

and the odd coordinate functions

$$\xi_{1}:=e_{1},\dots,\xi_{q}:=e_{q}\in\Lambda\left(\mathbb{R}^{q}\right)_{1}\hookrightarrow\mathcal{D}\left(\mathbb{R}^{p|q}\right)_{1}\,.$$

We define $\xi^{S}:=e^{S}$ for all $S\in\wp(q)$.

Definition 2.11 (super open sets)

(i) Let $(p,q)\in\mathbb{N}^{2}$ *and* $U\subset\mathbb{R}^{p}$ *be open. Then the pair* $U^{|q}:=(U,q)$ *is called a super open set of dimension* (p,q) *.* U *is called the body of* $U^{|q}$ *and* $^{\#}:\mathcal{D}\left(U^{|q}\right)\rightarrow\mathcal{C}^{\infty}(U)$ *the body map of* $\mathcal{D}\left(U^{|q}\right)$ *. If* $V\subset U$ *open,* $V^{|q}$ *is called a sub super open set of* $U^{|q}$ *. If* U *is even a domain then* $U^{|q}$ *is called a super domain.*

(ii) Let $U^{|q}$ and $V^{|s}$ be two super open sets, $\varphi : U \to V$ a $\mathcal{C}^\infty$-map and $\Phi : \mathcal{D}\left(V^{|s}\right) \to \mathcal{D}\left(U^{|q}\right)$ a unital graded algebra homomorphism. Then the pair (φ, Φ) is called a super morphism from $U^{|q}$ to $V^{|s}$ if and only if

$$(\Phi(f))^\# = f^\# \circ \varphi$$

for all $f \in \mathcal{D}\left(V^{|s}\right)$. In this case φ is called the body of (φ, Φ) .

(iii) Let (φ, Φ) be a super morphism from $U^{|q}$ to $V^{|s}$. Then (φ, Φ) is called a super embedding if and only if φ is an embedding and $\Phi\left(\mathcal{D}\left(V^{|s}\right)\right) \subset \mathcal{D}\left(U^{|q}\right)$ is dense.
(φ, Φ) is called a super projection if and only if φ is a projection and Φ is injective.

(iv) A super morphism (φ, Φ) from $U^{|q}$ to $V^{|s}$ is called a diffeomorphism if and only if φ is a diffeomorphism and Φ is an isomorphism. Then an easy calculation shows that $(\varphi, \Phi)^{-1} := \left(\varphi^{-1}, \Phi^{-1}\right)$ is a diffeomorphism from $V^{|s}$ to $U^{|q}$, which is called the inverse diffeomorphism to (φ, Φ) .

Let $U^{|q}$ be a (p, q)-dimensional and $V^{|s}$ be an (r, s)-dimensional super open set, and let (φ, Φ) be a super morphism from $U^{|q}$ to $V^{|s}$. Then comparing degrees of nilpotency in $\mathcal{D}\left(U^{|q}\right)$ and $\mathcal{D}\left(V^{|s}\right)$ one sees that if (φ, Φ) is a super embedding then $p \le r$ and $q \le s$, if (φ, Φ) is a super projection then $p \ge r$ and $q \ge s$, and finally if (φ, Φ) is a super diffeomorphism then $p = r$ and $q = s$. $\left(\mathrm{id}, {}^\#\right)$ is a super embedding from U into $U^{|q}$ for each super open set $U^{|q}$.

Clearly the set of all super open sets together with all super morphisms forms a category, where the composition of two super morphisms (φ, Φ) from $U^{|q}$ to $V^{|s}$ and (ψ, Ψ) from $V^{|s}$ to $W^{|u}$ is defined as

$$(\varphi, \Phi) \circ (\psi, \Psi) := (\varphi \circ \psi, \Psi \circ \Phi) ,$$

and $(\mathrm{id}, \mathrm{id})$ is the identity morphism from a super open set $U^{|q}$ to itself. The category of ordinary open subsets of all $\mathbb{R}^p$, $p \in \mathbb{N}$, together with $\mathcal{C}^\infty$-maps is a subcategory of the category of super open sets identifying each $U \subset \mathbb{R}^p$ open, $p \in \mathbb{N}$, with $(U, 0)$. The body map is precisely a functor from the category of super open sets to the category of ordinary open subsets, which restricted to the subcategory of open subsets is just the identity.

Given super open sets $U^{|q}, V^{|s}$ and $W^{|u}$, a super embedding (φ, Φ) from $U^{|q}$ to $V^{|s}$ and a super morphism (ψ, Ψ) from $V^{|s}$ to $W^{|u}$ we call the composition $(\psi, \Psi)|_{U^{|q}} := (\psi, \Psi) \circ (\varphi, \Phi)$ the restriction of (ψ, Ψ) to $U^{|q}$. In the trivial case where $U^{|q}$ is a super open set and $U' \subset U$ open we call the super embedding $(c, |_{U'})$ from $U'^{|q}$ to $U^{|q}$ (where $c : U' \hookrightarrow U$ denotes the inclusion map) the super inclusion from $U'^{|q}$ to $U^{|q}$. In this case given a super morphism (φ, Φ) from $U^{|q}$ to $V^{|s}$ the restriction to $U'^{|q}$ is just $(\varphi|_{U'}, |_{U'} \circ \Phi)$.

There is a nice characterization of super morphisms between super open sets, which shows in particular that in some sense given a super open set $U^{|q}$ the algebra $\mathcal{D}\left(U^{|q}\right)$ is 'spanned' by its coordinate functions:

Theorem 2.12 *Let $U^{|q}$ and $V^{|s}$ be two super open sets, and let $y_1, \ldots, y_r$ and $\eta_1, \ldots, \eta_s$ be the coordinate functions on $V^{|s}$.*

(i) Let $\Phi : \mathcal{D}\left(V^{|s}\right) \to \mathcal{D}\left(U^{|q}\right)$ be a unital graded algebra homomorphism. Then Φ is continuous, and there exists a unique $\mathcal{C}^\infty$-map $\varphi : U \to V$ such that (φ, Φ) is a super morphism from $U^{|q}$ to $V^{|s}$. Let $f_i := \Phi(y_i)$, $i = 1, \ldots, r$, $\lambda_j := \Phi(\eta_j)$, $j = 1, \ldots, s$. Then $\left(f_1^\#, \ldots, f_r^\#\right)(\mathbf{x}) \in V$ for all $\mathbf{x} \in U$, $\varphi = \left(f_1^\#, \ldots, f_r^\#\right)$, and for all $g = \sum_{S \in \wp(s)} g_S \eta^S \in \mathcal{D}\left(V^{|q}\right)$ we have

$$\Phi(g) = \sum_{S \in \wp(s)} \sum_{\mathbf{n} \in \mathbf{N}^r} \frac{1}{\mathbf{n}!} \left((\partial^{\mathbf{n}} g_S) \circ \varphi\right) \left(f_1 - f_1^\#, \ldots, f_r - f_r^\#\right)^{\mathbf{n}} \lambda^S \qquad (2.1)$$

in multi-index language, where we set $\lambda^S := \lambda_{t_1} \cdots \lambda_{t_m}$ for all $S = \{t_1, \ldots, t_m\} \in \wp(s)$, $1 \leq t_1 < \cdots < t_m \leq s$.

(ii) Conversely let $f_1, \ldots, f_r \in \mathcal{D}\left(U^{|q}\right)_0$ such that $\left(f_1^\#, \ldots, f_r^\#\right)(\mathbf{x}) \in V$ for all $\mathbf{x} \in U$, and let $\lambda_1, \ldots, \lambda_s \in \mathcal{D}\left(U^{|q}\right)_1$. Then there exists a unique unital graded algebra homomorphism $\Phi : \mathcal{D}\left(V^{|s}\right) \to \mathcal{D}\left(U^{|q}\right)$ given by formula 2.1 such that $\Phi(y_i) = f_i$, $i = 1, \ldots, r$, and $\Phi(\eta_j) = \lambda_j$, $j = 1, \ldots, s$.

Proof: (i) For proving uniqueness of φ let $\varphi = (\varphi_1, \ldots, \varphi_r) : U \to V$ be a $\mathcal{C}^\infty$-map, such that the pair (φ, Φ) is a super morphism from $U^{|q}$ to $V^{|s}$. Then for all $i = 1, \ldots, r$ one has

$$\varphi_i = y_i \circ \varphi = (\Phi(y_i))^\# = f_i^\# .$$

For proving existence we first show that $\left(f_1^\#, \ldots, f_r^\#\right)(\mathbf{x}) \in V$ for all $\mathbf{x} \in U$. So let $\mathbf{x}_0 \in U$. Then

$$\Psi : \mathcal{C}^\infty(V) \to \mathbb{R}, \, g \mapsto \Phi(g)^\# (\mathbf{x}_0)$$

is an algebra homomorphism hence lies in the spectrum of $\mathcal{C}^\infty(V)$. So from classical analysis we know that there exists $\mathbf{y}_0 \in V$ such that $\Psi(g) = g(\mathbf{y}_0)$ for all $g \in \mathcal{C}^\infty(V)$. Therefore for all $i = 1, \ldots, r$

$$f_i^{\#}(\mathbf{x}_0) = \Psi(y_i) = (\mathbf{y}_0)_i \ ,$$

so $\left(f_1^{\#}, \ldots, f_r^{\#}\right)(\mathbf{x}_0) = \mathbf{y}_0 \in V$. Define $\varphi := \left(f_1^{\#}, \ldots, f_r^{\#}\right) : U \to V$. For proving formula 2.1 let $g \in \mathcal{C}^\infty(V)$ and $\mathbf{x}_0 \in U$. It suffices to show that

$$\Phi(g)(\mathbf{x}_0) = \sum_{n \in \mathbb{N}^r} \frac{1}{n!} \partial^n g(\varphi(\mathbf{x}_0))\left((f_1, \ldots, f_r)(\mathbf{x}_0) - \varphi(\mathbf{x}_0)\right)^n$$

in $\Lambda(\mathbb{R}^q)$. Again from classical analysis we know that there exist $\Delta_{\mathbf{n}} \in \mathcal{C}^\infty(U)$, $\mathbf{n} \in \mathbb{N}^r, |\mathbf{n}| = s+1$, such that

$$g = \sum_{\mathbf{n} \in \mathbb{N}^r,\, |\mathbf{n}| \le s} \frac{1}{\mathbf{n}!} \partial^{\mathbf{n}} g(\varphi(\mathbf{x}_0))(\mathbf{y} - \varphi(\mathbf{x}_0))^{\mathbf{n}} + \sum_{\mathbf{n} \in \mathbb{N}^r,\, |\mathbf{n}| = s+1} (\mathbf{y} - \varphi(\mathbf{x}_0))^{\mathbf{n}} \Delta_{\mathbf{n}}$$

and therefore

$$\begin{aligned} \Phi(g) \quad &= \sum_{\mathbf{n} \in \mathbb{N}^r,\, |\mathbf{n}| \le s} \frac{1}{\mathbf{n}!} \partial^{\mathbf{n}} g(\varphi(\mathbf{x}_0))\ \left((f_1, \ldots, f_r) - \varphi(\mathbf{x}_0)\right)^{\mathbf{n}} \\ &+ \sum_{\mathbf{n} \in \mathbb{N}^r,\, |\mathbf{n}| = s+1} \left((f_1, \ldots, f_r) - \varphi(\mathbf{x}_0)\right)^{\mathbf{n}} \Phi(\Delta_{\mathbf{n}}) \ . \end{aligned}$$

Since all $\left(f_i - f_i^{\#}\right)(\mathbf{x}_0)$, $i = 1, \ldots, r$, are nilpotent elements in $\Lambda(\mathbb{R}^s)$ we have $\left((f_1, \ldots, f_r)(\mathbf{x}_0) - \varphi(\mathbf{x}_0)\right)^{\mathbf{n}} = 0$ for all $\mathbf{n} \in \mathbb{N}^r, |\mathbf{n}| = s+1$, and so the desired equation follows. Especially by equation 2.1 we see that Φ is continuous and that

$$\Phi(g)^{\#} = g_\emptyset \circ \varphi = g^{\#} \circ \varphi \,,$$

for all $g = \sum_{S \in \wp(s)} g_S \eta^S \in \mathcal{D}\left(V^{|q}\right)$, so (φ, Φ) is indeed a super morphism from $U^{|q}$ to $V^{|s}$. $\square$

(ii) For proving existence define

$$\begin{aligned} \Phi : \mathcal{D}\left(V^{|s}\right) \quad &\to \quad \mathcal{D}\left(U^{|q}\right) \,, \\ \sum_{S \in \wp(s)} g_S \eta^S \quad &\mapsto \quad \sum_{S \in \wp(s)} \sum_{\mathbf{n} \in \mathbb{N}^r} \frac{1}{\mathbf{n}!} \left((\partial^{\mathbf{n}} g_S) \circ \varphi\right) \left(f_1 - f_1^{\#}, \ldots, f_r - f_r^{\#}\right)^{\mathbf{n}} \lambda^S \,. \end{aligned}$$

Then clearly Φ is linear and respects the grading. For proving that Φ is also multiplicative one observes immediately that $\Phi\left(g\eta^S\right) = \Phi(g)\Phi\left(\eta^S\right) = \Phi(g)\lambda^S$ for all $g \in \mathcal{C}^\infty(V)$ and $S \in \wp(s)$, for all $S, T \in \wp(s)$ we have

$$\eta^S \eta^T = \begin{cases} (-1)^{|T<S|} \eta^{S \cup T} & \text{if } S \cap T = \emptyset \\ 0 & \text{otherwise} \end{cases} \,,$$

and since all $\lambda_j \in \mathcal{D}\left(U^{|q}\right)_1$, $j = 1, \ldots, s$,

$$\lambda^S \lambda^T = \begin{cases} (-1)^{|T<S|} \lambda^{S \cup T} & \text{if } S \cap T = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

as well.

For all $g, h \in \mathcal{C}^\infty(V)$

$$\begin{aligned} \Phi(g)\Phi(h) &= \left(\sum_{\mathbf{m} \in \mathbf{N}^r} \frac{1}{\mathbf{m}!} ((\partial^{\mathbf{m}} g) \circ \varphi) \left(f_1 - f_1^{\#}, \ldots, f_r - f_r^{\#} \right)^{\mathbf{m}} \right) \times \\ &\quad \times \left(\sum_{\mathbf{n} \in \mathbf{N}^r} \frac{1}{\mathbf{n}!} ((\partial^{\mathbf{n}} h) \circ \varphi) \left(f_1 - f_1^{\#}, \ldots, f_r - f_r^{\#} \right)^{\mathbf{n}} \right) . \end{aligned}$$

Since both sums are finite we can interchange the order of summation, so we obtain

$$\begin{aligned} \Phi(g)\Phi(h) &= \sum_{\mathbf{k} \in \mathbf{N}^r} \left(\sum_{\mathbf{m} \in \mathbf{N}^r, \mathbf{m} \leq \mathbf{k}} \frac{1}{\mathbf{m}!(\mathbf{k}-\mathbf{m})!} ((\partial^{\mathbf{m}} g) \circ \varphi) \left(\left(\partial^{\mathbf{k}-\mathbf{m}} h \right) \circ \varphi \right) \right) \times \\ &\quad \times \left(f_1 - f_1^{\#}, \ldots, f_r - f_r^{\#} \right)^{\mathbf{k}} \\ &= \sum_{\mathbf{k} \in \mathbf{N}^r} \frac{1}{\mathbf{k}!} \left(\left(\sum_{\mathbf{m} \in \mathbf{N}^r, \mathbf{m} \leq \mathbf{k}} \binom{\mathbf{k}}{\mathbf{m}} (\partial^{\mathbf{m}} g) \left(\partial^{\mathbf{k}-\mathbf{m}} h \right) \right) \circ \varphi \right) \times \\ &\quad \times \left(f_1 - f_1^{\#}, \ldots, f_r - f_r^{\#} \right)^{\mathbf{k}} \\ &= \sum_{\mathbf{k} \in \mathbf{N}^r} \frac{1}{\mathbf{k}!} \left(\left(\partial^{\mathbf{k}} (gh) \right) \circ \varphi \right) \left(f_1 - f_1^{\#}, \ldots, f_r - f_r^{\#} \right)^{\mathbf{k}} = \Phi(gh) , \end{aligned}$$

where we used LEIBNIZ' rule in multi-index language and that all
$f_i \in \mathcal{D}\left(U^{|q}\right)_0$, $i = 1, \ldots, p$.
Uniqueness follows directly from (i) . □

Theorem 2.12 shows that there is a bijection between the set of all super morphisms (φ, Φ) from $U^{|q}$ to $V^{|p}$, the set of all unital graded algebra morphisms $\Phi : \mathcal{D}\left(V^{|s}\right) \to \mathcal{D}\left(U^{|q}\right)$ and the set of all tuples $(f_1, \ldots, f_r, \lambda_1, \ldots, \lambda_s) \in \mathcal{D}\left(U^{|q}\right)_0^r \times \mathcal{D}\left(U^{|q}\right)_1^s$ such that the image of U under $\left(f_1^{\#}, \ldots, f_r^{\#}\right)$ lies in V . So in practice, since it is more convenient and analogous to classical analysis, we will identify a super morphism (φ, Φ) from $U^{|q}$ to $V^{|s}$ with its 'defining tuple'

$$(f_1, \ldots, f_r, \lambda_1, \ldots, \lambda_s) \in \mathcal{D}\left(U^{|q}\right)_0^r \times \mathcal{D}\left(U^{|q}\right)_1^s .$$

Then $\varphi = \left(f_1^\#, \dots, f_r^\#\right)$, and for all $g \in \mathcal{D}\left(V^{|s}\right)$ we can write

$$g\left(f_1, \dots, f_r, \lambda_1, \dots, \lambda_s\right) := \Phi(g) \in \mathcal{D}\left(U^{|q}\right) ,$$

regarding Φ as a 'plugging in' homomorphism, although this can be just formally.

Often the class of super morphisms between super open sets seems to be too restrictive, for example if $(f_1, \dots, f_r, \lambda) \in \mathcal{D}\left(U^{|q}\right)_0^r \times \mathcal{D}\left(U^{|q}\right)_1$ is a super morphism from a super open set $U^{|q}$ to $\mathbb{R}^{r|1}$ we have automatically $f_1, \dots, f_r \in \mathcal{C}^\infty(U)$, and the set of 'points' of $U^{|q}$ given as the set of all unital graded algebra homomorphisms from $\mathcal{D}\left(U^{|q}\right)$ to $\mathbb{R}$ is just U itself, so in particular it is not separating points on $\mathcal{D}\left(U^{|q}\right)$. Therefore it is useful to introduce a notion of parametrization where the 'parameters' are odd coordinate functions on some $\mathbb{R}^{0|n}$, $n \in \mathbb{N}$.

Before we do so we remark that in the category of super open sets there exists a cross product: If $U^{|q}$ and $V^{|s}$ are (p,q)- resp. (r,s)-dimensional super open sets the cross product of them is simply $U^{|q} \times V^{|s} := (U \times V, q+s)$, and from classical analysis we know that

$$\mathcal{D}\left(U^{|q} \times V^{|s}\right) = \mathcal{D}\left(U^{|q}\right) \hat{\boxtimes} \mathcal{D}\left(V^{|s}\right)$$

in the topology of $\mathcal{D}\left(U^{|q} \times V^{|s}\right)$. As a cross product it fulfills the following universal property, see paragraph 5.18 in [4] : There exist super projections (pr_1, C_1) and (pr_2, C_2) from $U^{|q} \times V^{|s}$ onto $U^{|q}$ resp. $V^{|s}$, where $\mathrm{pr}_1 : U \times V \to U$ and $\mathrm{pr}_2 : U \times V \to V$ denote the canonical projections and $C_1 : \mathcal{D}\left(U^{|q}\right) \hookrightarrow \mathcal{D}\left(U^{|q}\right) \hat{\boxtimes} \mathcal{D}\left(V^{|s}\right)$ and $C_2 : \mathcal{D}\left(V^{|s}\right) \hookrightarrow \mathcal{D}\left(U^{|q}\right) \hat{\boxtimes} \mathcal{D}\left(V^{|s}\right)$ the canonical embeddings, such that for any super open set $W^{|u}$ and super morphisms (φ_1, Φ_1) and (φ_2, Φ_2) from $W^{|u}$ to $U^{|q}$ resp. $V^{|s}$ there exists a unique super morphism (ψ, Ψ) from $W^{|u}$ to $U^{|q} \times V^{|s}$ such that

$$(\mathrm{pr}_1, C_1) \circ (\psi, \Psi) = (\varphi_1, \Phi_1)$$

and

$$(\mathrm{pr}_2, C_2) \circ (\psi, \Psi) = (\varphi_2, \Phi_2) .$$

Here we have $\psi = (\varphi_1, \varphi_2) : W \to U \times V$ and

$$\Psi = \Phi_1 \hat{\otimes} \Phi_2 : \mathcal{D}\left(U^{|q}\right) \hat{\boxtimes} \mathcal{D}\left(V^{|s}\right) \to \mathcal{D}\left(W^{|u}\right) , \ f \otimes g \mapsto \Phi_1(f) \Phi_2(g) .$$

If we write (φ_1, Φ_1) and (φ_2, Φ_2) in terms of their defining tuples

$$(f_1, \ldots, f_p, \lambda_1, \ldots, \lambda_q) \in \mathcal{D}\left(W^{|u}\right)_0^p \times \mathcal{D}\left(W^{|u}\right)_1^q$$

resp.

$$(g_1, \ldots, g_r, \mu_1, \ldots, \mu_s) \in \mathcal{D}\left(W^{|u}\right)_0^r \times \mathcal{D}\left(W^{|u}\right)_1^s$$

then simply

$$(\psi, \Psi) = (f_1, \ldots, f_p, g_1, \ldots, g_r, \lambda_1, \ldots, \lambda_q, \mu_1, \ldots, \mu_s) \ .$$

Definition 2.13 (parametrized super morphisms) *Let $n \in \mathbb{N}$ and $\mathcal{P} := \Lambda(\mathbb{R}^n) = \mathcal{D}\left(\mathbb{R}^{0|n}\right)$ with coordinate functions $\alpha_1, \ldots, \alpha_n$.*

(i) Let $U^{|q}$ and $V^{|s}$ be two super open sets and (φ, Φ) a super morphism from $U^{|q} \times \mathbb{R}^{0|n}$ to $V^{|s} \times \mathbb{R}^{0|n}$. Then (φ, Φ) is called an over $\mathcal{P}$ parametrized (or simply $\mathcal{P}$-) super morphism from $U^{|q}$ to $V^{|s}$ if and only if $\Phi|_{\mathcal{P}} = id$, in other words

$$\begin{array}{ccc} U^{|q} \times \mathbb{R}^{0|n} & \stackrel{(\varphi,\Phi)}{\longrightarrow} & V^{|s} \times \mathbb{R}^{0|n} \\ {\scriptstyle (\mathrm{pr}_2, C_2)} \searrow & \% & \swarrow {\scriptstyle (\mathrm{pr}_2, C_2)} \\ & \mathbb{R}^{0|n} & \end{array} \ .$$

In this case again $\varphi : U \to V$ is called the body of (φ, Φ) .

(ii) Let (φ, Φ) be a $\mathcal{P}$-super morphism from $U^{|q}$ to $V^{|s}$. Then it is called a $\mathcal{P}$-super projection resp. embedding resp. diffeomorphism if it is a super projection resp. embedding resp. diffeomorphism as an ordinary super morphism from $U^{|q} \times \mathbb{R}^{0|n}$ to $V^{|s} \times \mathbb{R}^{0|n}$.

From now on let $n \in \mathbb{N}$ and $\mathcal{P} = \Lambda(\mathbb{R}^n) = \mathcal{D}\left(\mathbb{R}^{|n}\right)$.

By theorem 2.12 there is clearly a bijection between the set of all $\mathcal{P}$-super morphisms (φ, Φ) from $U^{|q}$ to $V^{|p}$, the set of all unital graded algebra morphisms $\Phi : \mathcal{D}\left(V^{|s}\right) \to \mathcal{D}\left(U^{|q}\right) \boxtimes \mathcal{P}$ and the set of all tuples $(f_1, \ldots, f_r, \lambda_1, \ldots, \lambda_s) \in \left(\mathcal{D}\left(U^{|q}\right) \boxtimes \mathcal{P}\right)_0^r \times \left(\mathcal{D}\left(U^{|q}\right) \boxtimes \mathcal{P}\right)_1^s$ such that the image of U under $\left(f_1^\#, \ldots, f_r^\#\right)$ lies in V . So in practice we will again identify a $\mathcal{P}$-super morphism (φ, Φ) from $U^{|q}$ to $V^{|s}$ with its defining tuple $(f_1, \ldots, f_r, \lambda_1, \ldots, \lambda_s) \in \left(\mathcal{D}\left(U^{|q}\right) \boxtimes \mathcal{P}\right)_0^r \times \left(\mathcal{D}\left(U^{|q}\right) \boxtimes \mathcal{P}\right)_1^s$, again $\varphi = \left(f_1^\#, \ldots, f_r^\#\right)$.

Given $\mathcal{P}' := \Lambda\left(\mathbb{R}^{n'}\right)$ for some $n' \in \mathbb{N}$ every $\mathcal{P}$-super morphism (φ, Φ) from $U^{|q}$ to $V^{|s}$ can be also regarded as a $\mathcal{P} \boxtimes \mathcal{P}'$- super morphism from $U^{|q}$ to $V^{|s}$ using $\Phi \otimes \mathrm{id} : \mathcal{D}\left(V^{|s}\right) \boxtimes \mathcal{P} \boxtimes \mathcal{P}' \to \mathcal{D}\left(U^{|q}\right) \boxtimes \mathcal{P} \boxtimes \mathcal{P}'$ instead of Φ , in particular every usual super morphism from $U^{|q}$ to $V^{|s}$ can be regarded as

$\mathcal{P}$-super morphism from $U^{|q}$ to $V^{|s}$. Now if (φ, Φ) is a $\mathcal{P}$-super morphism from $U^{|q}$ to $V^{|s}$ and (ψ, Ψ) is a $\mathcal{P}$-super morphism from $V^{|s}$ to $W^{|u}$ then the composition $(\psi, \Psi) \circ (\varphi, \Phi)$ is a $\mathcal{P}$-super morphism from $U^{|q}$ to $W^{|u}$.
If (φ, Φ) is a super diffeomorphism from $U^{|q}$ to $V^{|s}$ then the inverse $\left(\varphi^{-1}, \Phi^{-1}\right)$ is a $\mathcal{P}$-super diffeomorphism from $V^{|s}$ to $U^{|q}$. The universal property of the cross product in the category of super open sets remains true even under $\mathcal{P}$-super morphisms:
Let $U^{|q}$, $V^{|s}$ and $W^{|u}$ be super open sets and (φ_1, Φ_1) and (φ_2, Φ_2) be $\mathcal{P}$-super morphisms from $W^{|u}$ to $U^{|q}$ resp. $V^{|s}$. Then there exists a unique $\mathcal{P}$-super morphism from $W^{|u}$ to $U^{|q} \times V^{|s}$ such that

$$(\mathrm{pr}_1, C_1 \otimes \mathrm{id}) \circ (\psi, \Psi) = (\varphi_1, \Phi_1)$$

and

$$(\mathrm{pr}_2, C_2 \otimes \mathrm{id}) \circ (\psi, \Psi) = (\varphi_2, \Phi_2) \ .$$

We have $\psi := (\varphi_1, \varphi_2) : W \to U \times V$ and

$$\Psi := \Phi_1|_{\mathcal{D}\left(U^{|q}\right)} \hat{\otimes} \Phi_2|_{\mathcal{D}\left(V^{|s}\right)} \otimes \mathrm{id} : \mathcal{D}\left(U^{|q}\right) \hat{\boxtimes} \mathcal{D}\left(V^{|s}\right) \boxtimes \mathcal{P} \to \mathcal{D}\left(W^{|u}\right) \boxtimes \mathcal{P} .$$

Recall that $\Phi_1 = \Phi_1|_{\mathcal{D}\left(U^{|q}\right)} \otimes \mathrm{id}$ and $\Phi_2 = \Phi_2|_{\mathcal{D}\left(U^{|q}\right)} \otimes \mathrm{id}$.

Let $U \subset \mathbb{R}^p$, $V \subset \mathbb{R}^r$ and $V' \subset V$ be open, and let $\varphi : U \to V$ be a $\mathcal{C}^\infty$-map such that $\varphi(U) \subset V'$. Then φ can also be regarded as a $\mathcal{C}^\infty$-map $U \to V'$. In super analysis there is an analogon to this fact:

Lemma 2.14 *Let (φ, Φ) be a $\mathcal{P}$-super morphism from $U^{|q}$ to $V^{|s}$ and $V' \subset V$ open such that $\varphi(U) \subset V'$. Then there exists a unique $\mathcal{P}$-super morphism (φ', Φ') from $U^{|q}$ to $V'^{|s}$ such that $(\varphi, \Phi) = (c, |_{V'}) \circ (\varphi', \Phi')$, where $(c, |_{V'})$ denotes the super inclusion from $V'^{|s}$ to $V^{|s}$.*

Proof: (φ', Φ') being a $\mathcal{P}$-super morphism from $U^{|q}$ to $V'^{|s}$,
$(\varphi, \Phi) = (c, |_{V'}) \circ (\varphi', \Phi')$ is equivalent to $\varphi = c \circ \varphi'$ and

$$\begin{tikzcd}
\mathcal{D}\left(V^{|s}\right) \boxtimes \mathcal{P} \arrow[rr, "|_{V'}"] \arrow[dr, "\Phi"'] & \% & \mathcal{D}\left(V'^{|s}\right) \boxtimes \mathcal{P} \arrow[dl, "\Phi'"] \\
& \mathcal{D}\left(U^{|q}\right) \boxtimes \mathcal{P} &
\end{tikzcd} \quad . \tag{2.2}$$

Clearly there exists a unique $\mathcal{C}^\infty$-map φ' such that $\varphi = c \circ \varphi'$. By formula 2.1 one sees that $\ker |_{V'} \sqsubset \ker \Phi$. Furthermore the image of $|_{V'}$ is dense in $\mathcal{D}\left(V'^{|s}\right) \boxtimes \mathcal{P}$, and Φ and Φ' are automatically continuous. Therefore there

exists a unique unital graded algebra homomorphism
$\Phi' : \mathcal{D}\left(V'^{|s}\right) \boxtimes \mathcal{P} \to \mathcal{D}\left(U^{|q}\right) \boxtimes \mathcal{P}$ such that 2.2 is true.
Now we have to prove that (φ', Φ') is a $\mathcal{P}$-super morphism from $U^{|q}$ to $V'^{|s}$. Showing $\Phi'(f)^{\#} = f^{\#} \circ \varphi'$ for all f in the image of $|_{V'}$ is an easy exercise, and since Φ' , $^{\#}$ and $\diamondsuit \circ \varphi'$ are continuous, it is true even for general $f \in \mathcal{D}\left(V'^{|s}\right) \boxtimes \mathcal{P}$. Since $\Phi\left(\alpha_j\right) = \alpha_j = \alpha_j|_{V'}$ for all $j = 1, \ldots, n$ trivially $\Phi'|_{\mathcal{P}} = \mathrm{id}$. □

We identify φ' and φ . Lemma 2.14 has an important consequence: it tells us that a $\mathcal{P}$-super morphism (φ, Φ) from $U^{|q}$ to $V^{|s}$, $U^{|q}$ and $V^{|s}$ being two super open sets, induces a whole morphism from $\left(U, \mathcal{D}\left(\diamondsuit^{|q}\right)_U\right)$ to $\left(V, \mathcal{D}\left(\diamondsuit^{|s}\right)_V\right)$ as ringed spaces, in exact:

Corollary 2.15 *Let $U^{|q}$ and $V^{|s}$ be two super open sets and* $(\varphi, \Phi) = (f_1, \ldots, f_r, \lambda_1, \ldots, \lambda_r)$ *a $\mathcal{P}$-super morphism from $U^{|q}$ to $V^{|s}$.*

(i) Then for each $W \subset V$ open there exists a unique $\mathcal{P}$-super morphism $\left(\varphi|_{\varphi^{-1}(W)}, \Phi_W\right)$ from $W^{|s}$ to $\varphi^{-1}(W)^{|q}$ such that

$$\begin{array}{ccc} U^{|q} & \stackrel{(\varphi,\Phi)}{\longrightarrow} & V^{|s} \\ \left(c, |_{\varphi^{-1}(W)}\right) \uparrow & \% & \uparrow \left(c', |_W\right) \\ \varphi^{-1}(W)^{|q} & \underset{\left(\varphi|_{\varphi^{-1}(W)}, \Phi_W\right)}{\longrightarrow} & W^{|s} \end{array} ,$$

where $\left(c, |_{\varphi^{-1}(W)}\right)$ and $\left(c', |_W\right)$ denote the canonical embeddings. We have $\Phi_V = \Phi$ and for all $W' \subset W \subset V$ open

$$\begin{array}{ccc} \mathcal{D}\left(W^{|s}\right) \boxtimes \mathcal{P} & \stackrel{\Phi_W}{\longrightarrow} & \mathcal{D}\left(\varphi^{-1}(W)^{|q}\right) \boxtimes \mathcal{P} \\ |_{W'} \downarrow & \% & \downarrow |_{\varphi^{-1}(W')} \\ \mathcal{D}\left(W'^{|s}\right) \boxtimes \mathcal{P} & \underset{\Phi_{W'}}{\longrightarrow} & \mathcal{D}\left(\varphi^{-1}(W')^{|q}\right) \boxtimes \mathcal{P} \end{array} ,$$

finally $\left(\varphi|_{\varphi^{-1}(W)}, \Phi_W\right) = (f_1, \ldots, f_r, \lambda_1, \ldots, \lambda_r)|_{\varphi^{-1}(W)}$ for all $W \subset V$ open.

(ii) Let $W \subset V$ be open. If (φ, Φ) is a super embedding then $\Phi_W\left(\mathcal{D}\left(W^{|s}\right)\right)$ is dense in $\mathcal{D}\left(\varphi^{-1}(W)^{|q}\right)$, if (φ, Φ) is a super projection then Φ_W is injective, and if (φ, Φ) is a super diffeomorphism then Φ_W is an isomorphism and $\Phi_W^{-1} = \left(\Phi^{-1}\right)_{\varphi^{-1}(W)}$.

Proof: (i) Let $W \subset V$ be open. Then one obtains the existence and uniqueness of $\left(\varphi|_{\varphi^{-1}(W)}, \Phi_W\right)$ by applying lemma 2.14 to the $\mathcal{P}$-super morphism $\left(\varphi|_{\varphi^{-1}(W)}, |_{\varphi^{-1}(W)} \circ \Phi\right)$ from $\varphi^{-1}(W)^{|q}$ to $V^{|s}$.
Now let $y_1, \ldots, y_r$ be the even and $\eta_1, \ldots, \eta_s$ be the odd coordinate functions on $V^{|s}$. Then for all $k = 1, \ldots, r$

$$\Phi_W\left(y_k|_W\right) = \Phi\left(y_k\right)|_{\varphi^{-1}(W)} = f_k|_{\varphi^{-1}(W)}$$

and for all $l = 1, \ldots, s$

$$\Phi_W\left(\eta_l\right) = \Phi\left(\eta_l\right)|_{\varphi^{-1}(W)} = \lambda_l|_{\varphi^{-1}(W)} \,.$$

Since $\Phi_V := \Phi$ fulfills the commuting diagram we have $\Phi_V = \Phi$. Now let $W' \subset W \subset V$ be open. Then for all $f \in \mathcal{D}\left(V^{|s}\right)$

$$\Phi_W\left(f|_W\right)|_{\varphi^{-1}(W')} = \Phi(f)|_{\varphi^{-1}(W')} = \Phi_{W'}\left(f|_{W'}\right) ,$$

and so since the image of $\mathcal{D}\left(V^{|s}\right)$ under $|_W$ is dense in $\mathcal{D}\left(W^{|s}\right)$ we get $|_{\varphi^{-1}(W')} \circ \Phi_W = \Phi_{W'} \circ |_{W'}$. $\square$

(ii) : Let (φ, Φ) be a super embedding. On one hand

$$\Phi_W\left(\mathcal{D}\left(V^{|s}\right)\Big|_W\right) \subset \Phi_W\left(\mathcal{D}\left(W^{|s}\right)\right) \subset \mathcal{D}\left(\varphi^{-1}(W)^{|q}\right) ,$$

and on the other hand

$$\begin{array}{rcl} \Phi_W\left(\mathcal{D}\left(V^{|s}\right)\Big|_W\right) & = & \Phi\left(\mathcal{D}\left(V^{|s}\right)\right)\Big|_{\varphi^{-1}(W)^{|q}} \\ & \underset{\text{dense}}{\subset} & \mathcal{D}\left(U^{|q}\right)\Big|_{\varphi^{-1}(W)^{|q}} \\ & \underset{\text{dense}}{\subset} & \mathcal{D}\left(\varphi^{-1}(W)^{|q}\right) \end{array}$$

since $|_{\varphi^{-1}(W)^{|q}}$ is continuous, so $\Phi_W\left(\mathcal{D}\left(W^{|s}\right)\right) \subset \mathcal{D}\left(\varphi^{-1}(W)^{|q}\right)$ is dense.

Now let (φ, Φ) be a super projection and $f \in \ker \Phi_W \sqsubset \mathcal{D}\left(W^{|s}\right)$. We have to show that $f = 0$. Let $\varepsilon \in \mathcal{C}^\infty(V)$ such that $\operatorname{supp} \varepsilon \in W$. Then $f\varepsilon \in \mathcal{D}\left(V^{|s}\right)$, and it suffices to show that $f\varepsilon = 0$. By formula 2.1 we have $\operatorname{supp} \Phi(f\varepsilon) \subset \varphi^{-1}(W)$, and on the other hand

$$\Phi(f\varepsilon)|_{\varphi^{-1}(W)} = \Phi_W\left(f\left(\varepsilon|_W\right)\right) = \Phi_W(f)\Phi_W\left(\varepsilon|_W\right) = 0\,.$$

Therefore $f\varepsilon = 0$ since Φ is injective.

Now let (φ, Φ) be a super diffeomorphism. Then for all $f \in \mathcal{D}\left(V^{|s}\right)$

$$\begin{array}{rcl} \left(\Phi^{-1}\right)_{\varphi^{-1}(W)}\left(\Phi_W\left(f|_W\right)\right) & = & \left(\Phi^{-1}\right)_{\varphi^{-1}(W)}\left(\Phi(f)|_{\varphi^{-1}(W)}\right) \\ & = & \Phi^{-1}\left(\Phi(f)\right)\big|_W = f|_W \,. \end{array}$$

So since the image of $\mathcal{D}\left(V^{|s}\right)$ under $|_W$ is dense in $\mathcal{D}\left(W^{|s}\right)$ and Φ and Φ^{-1} are continuous we have $\left(\Phi^{-1}\right)_{\varphi^{-1}(W)} \circ \Phi_W = \mathrm{id}$ and by the same calculation

$\Phi_W \circ \left(\Phi^{-1}\right)_{\varphi^{-1}(W)} = \mathrm{id}$ as well. $\square$

A simple calculation shows that if (φ, Φ) and (ψ, Ψ) are $\mathcal{P}$-super morphism from $U^{|q}$ to $V^{|s}$ resp. from $V^{|s}$ to $W^{|u}$ then for all $X \subset W$ open $(\Phi \circ \Psi)_X = \Phi_{\psi^{-1}(X)} \circ \Psi_X$.

Let $U^{|q}$ be a super open set, and define the continuous linear maps

$$\partial_{i|} : \mathcal{D}\left(U^{|q}\right) \to \mathcal{D}\left(U^{|q}\right) \, , \; f = \sum_{S \in \wp(q)} f_S \xi^S \mapsto \sum_{S \in \wp(q)} \left(\partial_i f_S\right) \xi^S \, ,$$

$i = 1, \ldots, p$, and

$$\partial_{|j} : \mathcal{D}\left(U^{|q}\right) \to \mathcal{D}\left(U^{|q}\right) \, , \; f = \sum_{S \in \wp(q)} f_S \xi^S \mapsto \sum_{S \in \wp(q)\, , \, j \notin S} (-1)^{|S<j|} f_{S \cup \{j\}} \xi^S \, ,$$

$j = 1, \ldots, q$. Then $\partial_{i|} \circ |_V = |_V \circ \partial_{i|}$ and $\partial_{|j} \circ |_V = |_V \circ \partial_{|j}$ for all $i = 1, \ldots, p$, $j = 1, \ldots, q$ and $V \subset U$ open. $\partial_{i|} x_k = \delta_{ik}$, $\partial_{i|} \xi_j = 0$, $\partial_{|j} x_i = 0$ and $\partial_{|j} \xi_l = \delta_{ik}$ for all $i, k = 1, \ldots, p$, $j, l = 1, \ldots, q$.
Clearly $\left(\partial_{i|} f\right)^{\#} = \partial_i \left(f^{\#}\right)$, $\partial_{i|} \mathcal{D}\left(U^{|q}\right)_0 , \partial_{|j} \mathcal{D}\left(U^{|q}\right)_1 \subset \mathcal{D}\left(U^{|q}\right)_0$, $\partial_{i|} \mathcal{D}\left(U^{|q}\right)_1 , \partial_{|j} \mathcal{D}\left(U^{|q}\right)_0 \subset \mathcal{D}\left(U^{|q}\right)_1$ for all $i = 1, \ldots, p$ and $j = 1, \ldots, q$, and we have a super product rule:

$$\partial_{i|}(fg) = \left(\partial_{i|} f\right) g + f \left(\partial_{i|} g\right)$$

and

$$\partial_{|j}(fg) = \left(\partial_{i|} f\right) g + (-1)^{\dot{f}} f \left(\partial_{i|} g\right)$$

for all $i = 1, \ldots, p$, $j = 1, \ldots, q$ and $f, g \in \mathcal{D}\left(U^{|q}\right)$, f homogeneous. So all $\partial_{i|}$ and $\partial_{|j}$ are super derivations on $\mathcal{D}\left(U^{|q}\right)$, and we call them the partial derivatives with respect to the coordinate functions x_j resp. ξ_j .

Definition 2.16 (super Jacobian) *Let $U^{|q}$ and $V^{|s}$ be (p, q)- resp. (r, s)-dimensional super open sets and $(\varphi, \Phi) = (f_1, \ldots, f_r, \lambda_1, \ldots, \lambda_s)$ a $\mathcal{P}$- super morphism from $U^{|q}$ to $V^{|s}$. Then the even $(r|s) \times (p|q)$ - graded matrix*

$$D(\varphi, \Phi) := \left(\begin{array}{c|c} \left(\partial_{i|} f_k\right)_{k \in \{1, \ldots, r\}\, , \, i \in \{1, \ldots, p\}} & -\left(\partial_{|j} f_k\right)_{k \in \{1, \ldots, r\}\, , \, j \in \{1, \ldots, q\}} \\ \hline \left(\partial_{i|} \lambda_l\right)_{l \in \{1, \ldots, s\}\, , \, i \in \{1, \ldots, p\}} & \left(\partial_{|j} \lambda_l\right)_{l \in \{1, \ldots, s\}\, , \, j \in \{1, \ldots, q\}} \end{array}\right)$$
$$\in \left(\mathcal{D}\left(U^{|q}\right) \otimes \mathcal{P}\right)_0^{(r|s) \times (p|q)}$$

is called the super Jacobian of the $\mathcal{P}$-super morphism (φ, Φ) .

Let $U^{|q}$, $V^{|s}$ and $W^{|u}$ be (p,q)- resp. (r,s)- resp. (t,u)-dimensional super open sets. If (φ, Φ) is a $\mathcal{P}$-super morphism from $U^{|q} \times V^{|s}$ to $W^{|u}$ then one can reorder the columns of $D(\varphi, \Phi)$ such that

$$D(\varphi, \Phi) = (D_1(\varphi, \Phi)\, D_2(\varphi, \Phi)) \ ,$$

where $D_1(\varphi, \Phi) \in \left(\mathcal{D}\left(U^{|q} \times V^{|s}\right) \otimes \mathcal{P}\right)_0^{(t|u)\times(p|q)}$ is the even graded matrix collecting all derivatives with resp. to the coordinate functions in $U^{|q}$ and $D_2(\varphi, \Phi) \in \left(\mathcal{D}\left(U^{|q} \times V^{|s}\right) \otimes \mathcal{P}\right)_0^{(t|u)\times(r|s)}$ is the even graded matrix collecting all derivatives with resp. to the coordinate functions in $V^{|s}$.
If (φ, Φ) is a $\mathcal{P}$-super morphism from $U^{|q}$ to $V^{|s} \times W^{|u}$ then one can reorder the rows of $D(\varphi, \Phi)$ such that

$$D(\varphi, \Phi) = \begin{pmatrix} D_1(\varphi, \Phi) \\ D_2(\varphi, \Phi) \end{pmatrix}$$

where $D_1(\varphi, \Phi) = D\left((\mathrm{pr}_1, C_1) \circ (\varphi, \Phi)\right) \in \left(\mathcal{D}\left(U^{|q}\right) \otimes \mathcal{P}\right)_0^{(r|s)\times(p|q)}$ and $D_2(\varphi, \Phi) = D\left((\mathrm{pr}_2, C_2) \circ (\varphi, \Phi)\right) \left(\mathcal{D}\left(U^{|q}\right) \otimes \mathcal{P}\right)_0^{(t|u)\times(p|q)}$.

Let us recall some properties of super Jacobians:

Lemma 2.17 *Let $U^{|q}$, $V^{|s}$ be (p,q)- resp. (r,s)-dimensional super open sets and $(\varphi, \Phi) = (f_1, \dots, f_r, \lambda_1, \dots, \lambda_s)$ a $\mathcal{P}$-super morphism from $U^{|q}$ to $V^{|s}$ and $D(\varphi, \Phi)$ its super Jacobian as a $\mathcal{P}$-super morphism.*

(i) The Jacobian of (φ, Φ) regarded as an ordinary super morphism from $U^{|q} \times \mathbb{R}^{0|n}$ to $V^{|s} \times \mathbb{R}^{0|n}$ has the form

$$D_{ordinary}(\varphi, \Phi) = \left(\begin{array}{c|c} D(\varphi, \Phi) & * \\ \hline 0 & 1 \end{array}\right) ,$$

where $*$ is an even $(r|s) \times (0|n)$ - graded matrix consisting of derivatives of $f_1, \dots, f_r, \lambda_1, \dots, \lambda_s$ with respect to the odd coordinate functions $\alpha_1, \dots, \alpha_n$.

(ii) The body of $D(\varphi, \Phi)$, taken componentwise, has the form

$$D(\varphi, \Phi)^\# := \left(\begin{array}{c|c} D\varphi & 0 \\ \hline 0 & * \end{array}\right) \in \mathcal{C}^\infty(U)_0^{(r,s)\times(p,q)} .$$

(iii) For all $W \subset V$ open $D\left(\varphi|_{\varphi^{-1}(W)}, \Phi_W\right) = D(\varphi, \Phi)|_{\varphi^{-1}(W)}$.

Proof: trivial.

Lemma 2.18

(i) Let $W^{|u}$ be a third super open set and (ψ, Ψ) a $\mathcal{P}$-super morphism from $V^{|s}$ to $W^{|u}$. Then the super Jacobian of $(\psi, \Psi) \circ (\varphi, \Phi)$ is precisely

$$\Phi\left(D(\psi, \Psi)\right) \cdot D(\varphi, \Phi)\,,$$

where $\Phi\left(D\left(\psi, \Psi\right)\right)$ is taken componentwise.

(ii) If (φ, Φ) is a super diffeomorphism then $D(\varphi, \Phi)$ is invertible as an even $(p,q) \times (p,q)$- graded matrix with entries in $\Lambda\left(\mathbb{R}^q\right) \boxtimes \mathcal{P}$, equivalently $D(\varphi, \Phi)^\# \in \mathcal{C}^\infty(U)_0^{(p,q)\times(p,q)}$ is invertible, and

$$D\left((\varphi, \Phi)^{-1}\right) = \Phi^{-1}\left((D(\varphi, \Phi))^{-1}\right),$$

where $\Phi^{-1}\left((D(\varphi, \Phi))^{-1}\right)$ is taken componentwise.

Proof: (i) In the ordinary case where $\mathcal{P} = \mathbb{R}$ this is corollary 5.5 of [4] [1]. The general case then follows easily from lemma 2.17 (i) . $\Box$
(ii) follows directly from (i) . $\Box$

The converse of (ii) is almost right, since in the super case we have an analogon to the local inversion theorem in classical analysis:

Theorem 2.19 (super analogon to the local inversion theorem)
Let $U^{|q}$ and $V^{|q}$ be two super open sets of dimension (p,q) , (φ, Φ) a $\mathcal{P}$-super morphism from $U^{|q}$ to $V^{|q}$ and $\mathbf{x}_0 \in U$.

(i) Let $D(\varphi, \Phi)(\mathbf{x}_0) \in \left(\Lambda\left(\mathbb{R}^q\right) \boxtimes \mathcal{P}\right)_0^{(p|q)\times(p|q)}$ be invertible, equivalently $D(\varphi, \Phi)^\#(\mathbf{x}_0) \in \mathbb{R}_0^{(p|q)\times(p|q)}$ be invertible. Then there exists an open neighbourhood $W \subset V$ of $\varphi(\mathbf{x}_0)$ such that $\left(\varphi|_{\varphi^{-1}(W)}, \Phi_W\right)$ is a super diffeomorphism from $\varphi^{-1}(W)^{|q}$ to $W^{|q}$.

(ii) Let φ be bijective and $D(\varphi, \Phi)(\mathbf{x}_0)$, equivalently $D(\varphi, \Phi)^\#(\mathbf{x}_0)$, be invertible for all $\mathbf{x}_0 \in U$. Then (φ, Φ) is a diffeomorphism.

Proof: If $\mathcal{P} = \mathbb{R}$ then (i) is precisely theorem 5.13 , and (ii) is precisely corollary 5.14 of [4] . The general case follows easily from the case $\mathcal{P} = \mathbb{R}$ via lemma 2.17 (i) . $\Box$

In super analysis there is an analogon for the implicit function theorem as well:

[1] Constantinescu and de Groote have the minus sign in the lower left corner of the super Jacobian, which of course does not change the result.

Theorem 2.20 (super analogon to the implicit function theorem) *Let $U^{|q}$ be a (p,q)-dimensional and $V^{|s}, W^{|s}$ be two (r,s)-dimensional super open sets. Let (φ,Φ) be a $\mathcal{P}$-super morphism from $U^{|q}\times V^{|s}$ to $W^{|s}$ and $(\mathbf{x}_0,\mathbf{y}_0)\in U\times V$ such that*

$$D_2(\varphi,\Phi)\,(\mathbf{x}_0,\mathbf{y}_0)\in\left(\Lambda\left(\mathbb{R}^q\right)\boxtimes\Lambda\left(\mathbb{R}^s\right)\boxtimes\mathcal{P}\right)_0^{(p|q)\times(p|q)}\ ,$$

equivalently $D_2(\varphi,\Phi)^\#\,(\mathbf{x}_0,\mathbf{y}_0)\in\mathbb{R}_0^{(p|q)\times(p|q)}$, is invertible, and define $\mathbf{z}_0:=\varphi\,(\mathbf{x}_0,\mathbf{y}_0)\in W$. Then there exists an open neighbourhood $A\subset U\times W$ of $(\mathbf{x}_0,\mathbf{z}_0)$ and a $\mathcal{P}$-super morphism (ψ,Ψ) from $A^{|q+s}$, which is actually a sub super open set of $U^{|q}\times V^{|s}$, to $V^{|s}$ such that $\psi\,(\mathbf{x}_0,\mathbf{z}_0)=\mathbf{y}_0$ and

$$(\varphi,\Phi)\circ\big((\mathrm{pr}_1|_A,\psi)\,,(|_A\circ C_1)\,\hat{\otimes}\Psi\big)\,=\,(\mathrm{pr}_2,C_2)|_A\,.$$

Proof: If $\mathcal{P}=\mathbb{R}$ this is precisely theorem 5.23 of [4] . For the general case apply theorem 5.23 of [4] to (φ,Φ) regarded as an ordinary super morphism from $U^{|q}\times\left(V^{|s}\times\mathbb{R}^{0|n}\right)$ to $W^{|s}\times\mathbb{R}^{0|n}$ since then the second part of the ordinary super Jacobian is

$$\begin{pmatrix} D_2(\varphi,\Phi) & * \\ 0 & 1\end{pmatrix}\,,$$

which so is invertible at $(\mathbf{x}_0,\mathbf{y}_0)$. Then it tells us that there exists an open neighbourhood $A\subset U\times W$ of $(\mathbf{x}_0,\mathbf{z}_0)$ and a super morphism (ψ,Ψ) from $A^{|q+s+n}$ as a super open subset of $U^{|q}\times\left(V^{|s}\times\mathbb{R}^{0|n}\right)$ to $V^{|s}\times\mathbb{R}^{0|n}$ such that $\psi\,(\mathbf{x}_0,\mathbf{z}_0)=\mathbf{y}_0$ and

$$(\varphi,\Phi)\circ\big((\mathrm{pr}_1|_A,\psi)\,,(|_A\circ C_1)\,\hat{\otimes}\Psi\big)=\left(\mathrm{pr}_2|_A\,,|_A\circ C_2\right)\,,$$

where (pr_2,C_2) denotes the canonical super projection from $U^{|q}\times\left(V^{|s}\times\mathbb{R}^{0|n}\right)$ to $V^{|s}\times\mathbb{R}^{0|n}$. It remains to prove that (ψ,Ψ) is a $\mathcal{P}$-super morphism from $A^{|q+s}$ to $V^{|s}$. For all $j=1,\ldots,n$

$$\begin{aligned}\Psi\,(\alpha_j) &= \left(\left(|_A\circ C_1\right)\Psi\right)(\alpha_j)=\left(\left(|_A\circ C_1\right)\Psi\right)\left(\Phi\,(\alpha_j)\right)\\ &= \left(|_A\circ C_2\right)(\alpha_j)=\alpha_j\,,\end{aligned}$$

and so $\Phi|_\mathcal{P}=\mathrm{id}$. $\square$

Definition 2.21 (parametrized super manifolds) *Let M be a p-dimensional real $\mathcal{C}^\infty$-manifold and $q\in\mathbb{N}$. Let $\mathcal{S}$ be a sheaf of unital graded $\mathbb{R}$-algebras over M and $^\#:\mathcal{S}\to\mathcal{C}_M^\infty$ a sheaf homomorphism.*

(i) The triple $\mathcal{M} := \left(M, \mathcal{S}, {}^{\#}\right)$ is called a (p,q)-dimensional over $\mathcal{P}$ parametrized (or simply $\mathcal{P}$-) super manifold if and only if there exists a sheaf embedding $\mathcal{P} \hookrightarrow \mathcal{S}$, for all $x_0 \in M$ an open neighbourhood $U \subset M$ of x_0 and a sheaf isomorphism $\Phi : \mathcal{S}|_U \xrightarrow{\sim} \mathcal{D}\left(\diamondsuit^{|q}\right)_U \boxtimes \mathcal{P}$ such that $\Phi|_{\mathcal{P}} = id$ and

$$\begin{array}{ccc} \mathcal{S}|_U & \stackrel{\Phi}{\longrightarrow} & \mathcal{D}\left(\diamondsuit^{|q}\right)_U \boxtimes \mathcal{P} \\ \# \searrow & \% & \swarrow \# \\ & \mathcal{C}^\infty_U & \end{array} .$$

In this case $\mathcal{M}^{\#} := M$ is called the body of the $\mathcal{P}$-super manifold $\mathcal{M}$, $\mathcal{S}$ the structural sheaf of $\mathcal{M}$ and ${}^{\#}$ the body map of $\mathcal{S}$. We write $\mathcal{D}(\mathcal{M}) := \mathcal{S}(M)$. In the case where $\mathcal{P} = \mathbb{R}$ we call $\mathcal{M}$ simply a super manifold.

(ii) If $U \subset M$ is open then the triple $\mathcal{U} := \left(U, \mathcal{S}|_U, {}^{\#}\big|_U\right)$ is called an open sub $\mathcal{P}$-super manifold of $\mathcal{M}$. It is a (p,q)-dimensional $\mathcal{P}$-super manifold itself.

Now it is important to see that $\mathcal{S}$ carries a well-defined uniformal structure of compact convergence in all derivatives via the local isomorphisms $\mathcal{S}|_U \simeq \mathcal{D}\left(\diamondsuit^{|q}\right)_U \boxtimes \mathcal{P}$, $U \subset M$ open.

For checkig well-definedness let $x_0 \in M$, $U \subset M$ be an open neighbourhood of x_0 and

$$\Phi : \mathcal{S}|_U \xrightarrow{\sim} \mathcal{D}\left(\diamondsuit^{|q}\right)_U \boxtimes \mathcal{P}$$

and

$$\Psi : \mathcal{S}|_U \xrightarrow{\sim} \mathcal{D}\left(\diamondsuit^{|q}\right)_U \boxtimes \mathcal{P}$$

be unital graded sheaf isomorphisms. Without loss of generality we may assume that $U \hookrightarrow \mathbb{R}^p$ via a $\mathcal{C}^\infty$-embedding. Therefore

$$\Phi_U \circ \Psi_U^{-1} : \mathcal{D}\left(U^{|q}\right) \boxtimes \mathcal{P} \to \mathcal{D}\left(U^{|q}\right) \boxtimes \mathcal{P}$$

is a unital graded algebra isomorphism, which is bicontinuous by theorem 2.12 .

For checking existence observe that compact convergence in all derivatives is already a local property.

From now on if $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{O}$ are $\mathcal{P}$-super manifolds then we always denote by M , N and O the bodies and by $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{R}$ the structural sheaves of

$\mathcal{M}$ resp. $\mathcal{N}$ resp. $\mathcal{O}$.

From classical analysis we know that given a $\mathcal{C}^\infty$-manifold M each open covering of M has a locally finite refinement, and to each locally finite covering there exists a $\mathcal{C}^\infty$-partition of unity, see for example theorem and proof of theorem 4.11 in [4]. There exists an analogon for super manifolds:

Lemma 2.22 *Let $\mathcal{M}$ be a (p,q)-dimensional super manifold and*

$$M = \bigcup_{\lambda \in \Lambda} U_\lambda$$

be a locally finite open covering. Then there exists a family $(\varepsilon_\lambda)_{\lambda\in\Lambda} \in \mathcal{D}(\mathcal{M})^\Lambda$ such that supp $\varepsilon_\lambda \subset U_\lambda$ *for all $\lambda \in \Lambda$ and*

$$1 = \sum_{\lambda\in\Lambda} \varepsilon_\lambda \,.$$

Proof: This is theorem 4.11 of [4] . $\square$

Using partitions of unity on the super manifold $\mathcal{M}$ one easily proves that for any $U \subset M$ open the algebra $\mathcal{S}(U)$ together with the uniformal structure of compact convergence in all derivatives is complete, given some $V \subset U$ open the restriction map $|_V : \mathcal{S}(U) \to \mathcal{S}(V)$ is a continuous unital graded algebra homomorphism, whose image is dense in $\mathcal{S}(V)$, and for all $U \subset M$ open the body map $^\# : \mathcal{S}(U) \to \mathcal{C}^\infty(U)$ is a continuous unital algebra epimorphism.

Later we will deal with quotient sheaves of the structural sheaf $\mathcal{S}$ of a super manifold $\mathcal{M}$, and so it is convenient to derive a lemma about such quotient sheaves from the existence of partitions of unity in $\mathcal{D}(\mathcal{M})$.

Lemma 2.23 *Let $\mathcal{M}$ be a super manifold and $\mathcal{I}$ an ideal sheaf of $\mathcal{S}$. Then for all $U \subset M$ open*

$$\mathcal{S}(U)/\mathcal{I}(U) = \left(\mathcal{S}/\mathcal{I}\right)(U) \,.$$

Proof: By definition of the quotient sheaf $\mathcal{S}/\mathcal{I}$, given $U \subset M$ open, we have $\mathcal{S}(U)/\mathcal{I}(U) \subset \left(\mathcal{S}/\mathcal{I}\right)(U)$, and for any $f \in \left(\mathcal{S}/\mathcal{I}\right)(U)$ there exists an open covering $U = \bigcup_{\lambda\in\Lambda} U_\lambda$ such that $f|_{U_\lambda} \in \mathcal{S}\left(U_\lambda\right)/\mathcal{I}\left(U_\lambda\right)$, see for example in [7] section 1.3 .

'$\subset$' : now trivial.
'$\supset$' : Let $f \in \left(\mathcal{S}/\mathcal{I}\right)(U)$. Then by definition of $\mathcal{S}/\mathcal{I}$ there exists an open covering

$$U = \bigcup_{\lambda\in\Lambda} U_\lambda$$

of U and for each $\lambda \in \Lambda$ a function $f_\lambda \in \mathcal{S}(U_\lambda)$ such that $f|_{U_\lambda} = f_\lambda + \mathcal{I}(U_\lambda)$ for all $\lambda \in \Lambda$. Without loss of generality we can assume that $(U_\lambda)_{\lambda\in\Lambda}$ is locally finite. So by lemma 2.22 there exists a family $(\varepsilon_\lambda)_{\lambda\in\Lambda} \in \mathcal{D}(\mathcal{M})^\Lambda$ such that $\operatorname{supp} \varepsilon_\lambda \subset U_\lambda$ for all $\lambda \in \Lambda$ and

$$1 = \sum_{\lambda\in\Lambda} \varepsilon_\lambda \,.$$

So clearly

$$g := \sum_{\lambda\in\Lambda} \varepsilon_\lambda f_\lambda \in \mathcal{S}(U)\,,$$

and so $g + \mathcal{I}(M) \in \mathcal{S}(U)/\mathcal{I}(U) \subset (\mathcal{S}/\mathcal{I})(U)$. Now let $\lambda \in \Lambda$. Then $\eta_\nu := f_\nu - f_\lambda \in \mathcal{I}(U_\nu \cap U_\lambda)$ for all $\nu \in \Lambda$, so

$$\begin{aligned}
(g + \mathcal{I}(U))\,|_{U_\lambda} &= \sum_{\nu\in\Lambda} (\varepsilon_\nu f_\nu)\,|_{U_\lambda} + \mathcal{I}(U_\lambda) \\
&= \left(\sum_{\nu\in\Lambda} \varepsilon_\nu|_{U_\lambda}\right) f_\lambda + \sum_{\nu\in\Lambda} (\varepsilon_\nu|_{U_\lambda})\,\eta_\nu + \mathcal{I}(U_\lambda) \\
&= f_\lambda + \mathcal{I}(U_\lambda) \\
&= f|_{U_\lambda}
\end{aligned}$$

as functions in $\mathcal{S}(U_\lambda)/\mathcal{I}(U_\lambda)$, where we used that $(\varepsilon_\nu|_{U_\lambda})\,\eta_\nu \in \mathcal{I}(U_\lambda)$ for all $\nu \in \Lambda$ since $\mathcal{I}$ is an ideal sheaf of $\mathcal{S}$. Since $\lambda \in \Lambda$ has been arbitrary and $\mathcal{S}/\mathcal{I}$ is a sheaf we get $f = g + \mathcal{I}(U)$. $\square$

Definition 2.24 (super morphisms) *Let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{P}$-super manifolds, $\varphi : M \to N$ a $\mathcal{C}^\infty$-map and $(\Phi_W)_{W\subset N \text{ open}}$ a family of unital graded algebra homomorphisms $\Phi_W : \mathcal{T}(W) \to \mathcal{S}\left(\varphi^{-1}(W)\right)$ such that for all $W' \subset W \subset N$ open*

$$\begin{array}{ccc}
\mathcal{T}(W) & \xrightarrow{\Phi_W} & \mathcal{S}\left(\varphi^{-1}(W)\right) \\
|_{W'} \downarrow & \% & \downarrow |_{\varphi^{-1}(W')} \\
\mathcal{T}(W') & \underset{\Phi_{W'}}{\longrightarrow} & \mathcal{S}\left(\varphi^{-1}(W')\right)
\end{array} ,$$

which means precisely that the pair $\Phi := \left(\varphi, (\Phi_W)_{W\subset N \text{ open}}\right)$ is a morphism from $(M, \mathcal{S})$ to $(N, \mathcal{T})$ as ringed spaces.

(i) The pair $\Phi := \left(\varphi, (\Phi_W)_{W \subset N \text{ open}}\right)$ *is called a* $\mathcal{P}$*-super morphism from* $\mathcal{M}$ *to* $\mathcal{N}$ *if and only if for all* $W \subset N$ *open* $\Phi_W|_{\mathcal{P}} = id$ *and*

$$\left(\Phi_W(f)\right)^{\#} = f^{\#} \circ \varphi|_{\varphi^{-1}(W)}$$

for all $f \in \mathcal{T}(W)$ *. In this case* $\Phi^{\#} := \varphi$ *is called the body of* Φ *.*

(ii) Let Φ *be a* $\mathcal{P}$*-super morphism from* $\mathcal{M}$ *to* $\mathcal{N}$ *. Then* Φ *is called a* $\mathcal{P}$*-super embedding if and only if* φ *is an embedding and all* $\Phi_W(\mathcal{T}(W)) \subset \mathcal{S}\left(\varphi^{-1}\right)$ *,* $W \subset N$ *open, are dense.*
Φ *is called a* $\mathcal{P}$*-super projection if and only if* φ *is a projection and all* Φ_W *,* $W \subset N$ *open, are injective.*

(iii) A $\mathcal{P}$*-super morphism* $\Phi = \left(\varphi, (\Phi_W)_{W \subset N \text{ open}}\right)$ *from* $\mathcal{M}$ *to* $\mathcal{N}$ *is called a* $\mathcal{P}$*-super diffeomorphism if and only if* φ *is a diffeomorphism and all* Φ_W *,* $W \subset N$ *open are isomorphisms. Then again an easy calculation shows that*

$$\Phi^{-1} := \left(\varphi^{-1}, \left(\Phi^{-1}_{\varphi(U)}\right)_{U \subset M \text{ open}}\right)$$

is a $\mathcal{P}$*-super diffeomorphism from* $\mathcal{N}$ *to* $\mathcal{M}$ *, which is called the inverse* $\mathcal{P}$*-super diffeomorphism to* (φ, Φ) *.*

Let $\mathcal{M}$ be a (p,q)-dimensional and $\mathcal{N}$ an (r,s)-dimensional $\mathcal{P}$-super manifold, and let $\left(\varphi, (\Phi_W)_{W \subset N \text{ open}}\right)$ be a $\mathcal{P}$-super morphism from $\mathcal{M}$ to $\mathcal{N}$. Since for all $x_0 \in M$ there exist open neighbourhoods $V \subset N$ of $\varphi(x_0)$ and $U \subset M$ of x_0 such that $\varphi(U) \subset V$, which can be assumed to be open subsets of $\mathbb{R}^p$ resp. $\mathbb{R}^r$ without loss of generality, such that $\mathcal{S}|_U \simeq \mathcal{D}\left(U^{|q}\right) \boxtimes \mathcal{P}$ and $\mathcal{T}|_V \simeq \mathcal{D}\left(V^{|s}\right) \boxtimes \mathcal{P}$ an easy calculation shows that all $\Phi_W : \mathcal{T}(W) \to \mathcal{S}\left(\varphi^{-1}(W)\right)$ are continuous. By the same argument one sees that if Φ is a $\mathcal{P}$-super embedding then $p \leq r$ and $q \leq s$, if Φ is a $\mathcal{P}$-super projection then $p \geq r$ and $q \geq s$, and finally if Φ is a $\mathcal{P}$-super diffeomorphism then $p = r$ and $q = s$.
For any super manifold $\mathcal{M}$ we have a canonical embedding $\left(\mathrm{id}, (\#)_{U \subset M \text{ open}}\right)$ from M into $\mathcal{M}$.

Clearly all $\mathcal{P}$-super manifolds together with all $\mathcal{P}$-super morphisms form a category, and the body map is precisely a functor from the category of $\mathcal{P}$-super manifolds to the category of ordinary $\mathcal{C}^\infty$-manifolds together with $\mathcal{C}^\infty$-maps between them. Given super manifolds $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{O}$ and super morphisms $\Phi := \left(\varphi, (\Phi_V)_{V \subset N \text{ open}}\right)$ from $\mathcal{M}$ to $\mathcal{N}$ and
$\Psi := \left(\psi, (\Psi_W)_{W \subset N \text{ open}}\right)$ from $\mathcal{N}$ to $\mathcal{O}$ the composition of $\Psi \circ \Phi$ is given by

$$\Psi \circ \Phi := \left(\psi \circ \varphi, \left(\Phi_{\psi^{-1}(W)} \circ \Psi_W\right)_{W \subset O \text{ open}}\right) ,$$

and it is a $\mathcal{P}$-super morphism from $\mathcal{M}$ to $\mathcal{O}$.

Given a $\mathcal{P}$-super morphism $\Phi = \left(\varphi, (\Phi_V)_{V \subset N \text{ open}}\right)$ from $\mathcal{M}$ to $\mathcal{N}$ for all $W \subset N$ open and $f \in \mathcal{T}(W)$ we write $f(\Phi) := \Phi_V(f) \in \mathcal{S}\left(\varphi^{-1}(V)\right)$ regarding Φ_W as a 'plugging in' homomorphism although again this can be meant just formally.

Every super open set $U^{|q}$ with $U \subset \mathbb{R}^p$ open can be regarded as a super manifold with structural sheaf $\mathcal{S} := \mathcal{D}\left(\diamondsuit^{|q}\right)_U$, and so the category of super open sets is a subcategory of the category of super manifolds.
Given $\mathcal{P}' := \Lambda\left(\mathbb{R}^{n'}\right)$ for some $n' \in \mathbb{N}$ every (p,q)-dimensional super manifold $\mathcal{M} := \left(M, \mathcal{S}, {}^\#\right)$ can be also regarded as a (p,q)-dimensional $\mathcal{P} \boxtimes \mathcal{P}'$- super manifold using $\mathcal{S} \boxtimes \mathcal{P}'$ as structural sheaf instead of $\mathcal{S}$, in particular every usual super manifold can be regarded as a $\mathcal{P}$-super manifold. Via this identification the super open sets together with $\mathcal{P}$-super morphisms form a subcategory of the category of $\mathcal{P}$-super manifolds. In general there are $\mathcal{P} \boxtimes \mathcal{P}'$- super morphisms between two $\mathcal{P}$-super manifolds that do not come from $\mathcal{P}$-super morphisms. The $\mathcal{P}$-super morphisms between super open sets defined in 2.13 for example are of this type. The category of $\mathcal{C}^\infty$-manifolds together with $\mathcal{C}^\infty$-maps is a subcategory of the category of super manifolds identifying a p-dimensional manifold M with the $(p,0)$-dimensional super manifold $(M, \mathcal{C}^\infty_M, \mathrm{id})$, and the body functor is the identity on this subcategory.

Clearly if $\mathcal{M}$ is a (p,q)-dimensional $\mathcal{P}$-super manifold then $\mathcal{M}$ is a usual $(p, q+n)$-dimensional super manifold as well, and there exists a canonical super projection $\left(\mathbf{0}, (C_W)_{W \subset M \text{ open}}\right)$ from $\mathcal{M}$ to $\mathbb{R}^{0|n}$ where $C_W : \mathcal{P} \hookrightarrow \mathcal{S}(W)$, $W \subset M$ open, denote the (canonical) unital graded algebra embeddings given by definition 2.21 .

Let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{P}$-super manifolds and Φ a $\mathcal{P}$-super embedding from $\mathcal{M}$ to $\mathcal{N}$. Then $\mathcal{M}$ is called a $\mathcal{P}$-sub super manifold of $\mathcal{N}$, and in this case M can be regarded as a usual $\mathcal{C}^\infty$-submanifold of N via the $\mathcal{C}^\infty$-embedding $\varphi := \Phi^\# : M \hookrightarrow N$. If $\mathcal{O}$ is a third $\mathcal{P}$-super manifold and Ψ a $\mathcal{P}$-super morphism from $\mathcal{N}$ to $\mathcal{O}$ then the composition $\Psi|_\mathcal{M} := \Psi \circ \Phi$ is called the restriction of Ψ to $\mathcal{M}$. In the special case where $\mathcal{M}$ is a $\mathcal{P}$-super manifold and $\mathcal{U}$ is an open sub $\mathcal{P}$-super manifold of $\mathcal{M}$ we have a canonical embedding $\mathcal{C} := \left(c, (|_{U \cap W})_{W \subset M \text{ open}}\right)$ from $\mathcal{U}$ into $\mathcal{M}$, which is called the super inclusion from $\mathcal{U}$ to $\mathcal{M}$, where $c : U \to M$ denotes the canonical inclusion, and in this case given a $\mathcal{P}$-super morphism Φ from $\mathcal{M}$ to another $\mathcal{P}$-super manifold $\mathcal{N}$ the restriction of Φ to $\mathcal{U}$ is just

$\left(\varphi|_U, \left(|_{U\cap\varphi^{-1}(W)} \circ \Phi_W\right)_{W\subset N \text{ open}}\right)$.

Clearly if $\mathcal{M}$ and $\mathcal{N}$ are two $\mathcal{P}$-super manifolds , $\Phi = \left(\varphi, (\Phi_W)_{W\subset M \text{ open}}\right)$ is a $\mathcal{P}$-super morphism from $\mathcal{M}$ to $\mathcal{N}$ and $\mathcal{U}$ is an open sub $\mathcal{P}$-super manifold of $\mathcal{N}$ such that $\varphi(M) \subset U := \mathcal{U}^\#$ then there exists a unique $\mathcal{P}$-super morphism Φ' from $\mathcal{M}$ to $\mathcal{U}$ such that $\Phi = \mathcal{C} \circ \Phi'$, where $\mathcal{C}$ denotes the canonical super inclusion from $\mathcal{U}$ into $\mathcal{N}$. Φ' is given by

$$\Phi' = \left(\varphi, (\Phi_W)_{W\subset U \text{ open}}\right) .$$

Again we identify Φ and Φ' .

Theorem 2.25 *Let $\mathcal{M}$ and $\mathcal{N}$ be super manifolds and $\eta : \mathcal{D}(\mathcal{N}) \to \mathcal{D}(\mathcal{M})$ be a unital graded algebra homomorphism. Then there exists a unique super morphism $\left(\varphi, (\Phi_V)_{V\subset N \text{ open}}\right)$ from $\mathcal{M}$ to $\mathcal{N}$ such that $\Phi_N = \eta$.*
In particular a super manifold $\mathcal{M}$ is uniquely defined up to super diffeomorphism by the graded algebra $\mathcal{D}(\mathcal{M})$.

Proof: This is precisely theorem 4.8 of [4] . □

From now on, given $\mathcal{P}$-super manifolds $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{O}$ and $\mathcal{P}$-super morphisms Φ from $\mathcal{M}$ to $\mathcal{N}$ and Ψ from $\mathcal{N}$ to $\mathcal{O}$ we write $\varphi := \Phi^\#$ and $\psi := \Psi^\#$, and we let $\Phi_W : \mathcal{T}(W) \to \mathcal{S}\left(\varphi^{-1}(W)\right)$ and $\Psi_X : \mathcal{R}(X) \to \mathcal{T}\left(\psi^{-1}(X)\right)$, $W \subset N$ resp, $X \subset O$ open, be the unital graded algebra homomorphisms building up Φ resp. Ψ . When dealing with super manifolds the notion of parametrized points is very usefull, since we can formally deal with them as with usual points of ordinary manifolds.

Definition 2.26 ($\mathcal{P}$-points) *Let $\mathcal{M}$ be a $\mathcal{P}$-super manifold. A $\mathcal{P}$-super morphism $\Xi = \left(x, (\Xi_U)_{U\subset M \text{ open}}\right)$ from $\mathbb{R}^{0|0}$ to $\mathcal{M}$, which then is automatically a $\mathcal{P}$-super embedding, is called an over $\mathcal{P}$ parametrized, or simply $\mathcal{P}$-, point of $\mathcal{M}$. We write $\Xi \in_\mathcal{P} \mathcal{M}$.*

Given a $\mathcal{P}$-point Ξ of the $\mathcal{P}$-super manifold $\mathcal{M}$, the body $x = \Xi^\#$ of Ξ is a usual point of $M = \mathcal{M}^\#$. By theorem 2.25 there is a bijection between $\mathcal{P}$-points of $\mathcal{M}$ and unital graded algebra homomorphisms $\eta : \mathcal{D}(\mathcal{M}) \to \Lambda(\mathbb{R}^n)$ with $\eta|_\mathcal{P} = \mathrm{id}$ such that $\eta = \Xi_M$ and $\eta(f)^\# = f^\#(x)$ for all $f \in \mathcal{D}(\mathcal{M})$. If $f \in \mathcal{D}(\mathcal{M})$ then we write $f(\Xi) := \Xi_M(f) \in \mathcal{P}$. If Φ is a super morphism from $\mathcal{M}$ to another super manifold $\mathcal{N}$ then $\Phi \circ \Xi$ is a $\mathcal{P}$ point of $\mathcal{N}$, and we write $\Phi(\Xi) := \Phi \circ \Xi$. Then clearly for all $V \subset N$ such that $x \in \varphi^{-1}(V)$ we have $\Phi_V\left(f|_V\right)(\Xi) = f\left(\Phi(\Xi)\right)$ for all $f \in \mathcal{S}(U)$, and if $\mathcal{O}$ is a third super manifold and Ψ is a super morphism from $\mathcal{N}$ to $\mathcal{O}$ then

$$(\Psi \circ \Phi)(\Xi) = \Psi(\Phi(\Xi)) .$$

In the special case where $\mathcal{M} = U^{|q}$ is a super open set by theorem 2.12 there is a bijection between all $\mathcal{P}$-points of $U^{|q}$ and the set of all $(a_1, \ldots, a_p, \pi_1, \ldots, \pi_q) \in \Lambda(\mathbb{R}^n)_0^p \times \Lambda(\mathbb{R}^n)_1^q$ such that $\left(a_1^{\#}, \ldots, a_p^{\#}\right) \in U$, which then is the body of the corresponding $\mathcal{P}$-point.

Lemma 2.27 *Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{P}$-super manifolds of dimensions (p,q) resp. (r,s) and $f \in \mathcal{D}(\mathcal{M})$, let Φ and Ψ be $\mathcal{P}$-super morphisms from $\mathcal{M}$ to $\mathcal{N}$, $n' \in \mathbb{N}$ and $\mathcal{P}' := \Lambda\left(\mathbb{R}^{n'}\right) = \mathcal{D}\left(\mathbb{R}^{0|n'}\right)$.*

(i) If $n' \geq q$ and $f(\Xi) = 0$ for all $\Xi \in_{\mathcal{P}\boxtimes\mathcal{P}'} \mathcal{M}$ then $f = 0$. So the $\mathcal{P}\boxtimes\mathcal{P}'$-points of $\mathcal{M}$ are seperating points in $\mathcal{D}(\mathcal{M})$.

(ii) If $n' \geq q$ and $\Phi(\Xi) = \Psi(\Xi)$ for all $\Xi \in_{\mathcal{P}\boxtimes\mathcal{P}'} \mathcal{M}$ then $\Phi = \Psi$.

(iii) If $n' \geq s$ and for all $\Theta \in_{\mathcal{P}\boxtimes\mathcal{P}'} \mathcal{N}$ there exists $\Xi \in_{\mathcal{P}\boxtimes\mathcal{P}'} \mathcal{M}$ such that $\Phi(\Xi) = \Theta$ then Φ is a $\mathcal{P}$-super projection.

(iv) If Φ is a $\mathcal{P}$-super embedding and $\Xi, \Theta \in_{\mathcal{P}} \mathcal{M}$ such that $\Phi(\Xi) = \Phi(\Theta)$ then $\Xi = \Theta$.

Proof: (i) Assume $n' \geq q$ and $f(\Xi) = 0$ for all $\Xi \in_{\mathcal{P}\boxtimes\mathcal{P}'} \mathcal{M}$ and let $U \subset M$ such that $\mathcal{S}|_U \simeq \mathcal{D}_U \boxtimes \Lambda(\mathbb{R}^n)$. Let $x \in U$. Then $\Lambda(\mathbb{R}^q) \hookrightarrow \mathcal{P}' = \Lambda\left(\mathbb{R}^{n'}\right)$ canonically since $n' \geq q$, and so $(x, \xi_1, \ldots, \xi_q) \in_{\mathcal{P}\boxtimes\mathcal{P}'} \mathcal{M}$. So

$$0 = f(x,\xi) = \sum_{S \in \wp(q)} \sum_{T \in \wp(n)} f_{ST}(x) \xi^S \alpha^T .$$

Therefore $f(x) = 0$ as an element of $\Lambda(\mathbb{R}^q) \boxtimes \Lambda(\mathbb{R}^n)$. Since $x \in U$ has been arbitrary we obtain $f|_U = 0$, and since U has been arbitrary even $f = 0$. $\Box$
(ii) Assume $n' \geq q$ and $\Phi(\Xi) = \Psi(\Xi)$ for all $\Xi \in_{\mathcal{P}\boxtimes\mathcal{P}'} \mathcal{M}$. By theorem 2.25 it suffices to show that $\Phi_N(f) = \Psi_N(f)$ for all $f \in \mathcal{D}(\mathcal{N})$. So let $f \in \mathcal{D}(\mathcal{N})$ and $\Xi \in_{\mathcal{P}\boxtimes\mathcal{P}'} \mathcal{M}$. Then

$$(\Phi_N(f) - \Psi_N(f))(\Xi) = f(\Phi(\Xi)) - f(\Psi(\Xi)) = 0 .$$

So we can apply (i) , which tells us that $\Phi_N(f) - \Psi_N(f) = 0$. $\Box$
(iii) Assume $n' \geq s$ and for all $\Theta \in_{\mathcal{P}\boxtimes\mathcal{P}'} \mathcal{N}$ there exists $\Xi \in_{\mathcal{P}\boxtimes\mathcal{P}'} \mathcal{M}$ such that $\Phi(\Xi) = \Theta$. Let $y \in N$. Then trivially $y \in_{\mathcal{P}\boxtimes\mathcal{P}'} \mathcal{N}$, and so there exists $\Xi \in_{\mathcal{P}\boxtimes\mathcal{P}'} \mathcal{M}$ such that $\Phi(\Xi) = y$. So $\varphi(x) = y$ if we define $x := \Xi^{\#} \in M$. Therefore φ is surjective.
Now let $W \subset N$ be open. Then $\mathcal{W} := \left(W, \mathcal{T}|_W, {}^{\#}\right)$ and $\mathcal{R} := \left(\varphi^{-1}(W), \mathcal{S}|_{\varphi^{-1}(W)}, {}^{\#}\right)$ are open sub $\mathcal{P}$-super manifolds of $\mathcal{N}$ resp. $\mathcal{M}$, and we have to show that $\Phi_W : \mathcal{D}(\mathcal{W}) = \mathcal{T}(W) \to \mathcal{S}\left(\varphi^{-1}(W)\right) = \mathcal{D}(\mathcal{R})$ is injective. So let $f \in \mathcal{D}(\mathcal{W})$ such that $\Phi_W(f) = 0$. Let $\Theta \in_{\mathcal{P}\boxtimes\mathcal{P}'} \mathcal{W}$ with

body $y := \Theta^{\#} \in W$. Then $\Theta \in_{\mathcal{P}\boxtimes\mathcal{P}'} \mathcal{N}$, and so there exists $\Xi \in_{\mathcal{P}\boxtimes\mathcal{P}'} \mathcal{M}$ with body $x := \Xi^{\#} \in M$ such that $\Phi(\Xi) = \Theta$. Since $\varphi(x) = y$ we even have $x \in \varphi^{-1}(W)$, and so $\Xi \in_{\mathcal{P}\boxtimes\mathcal{P}'} \mathcal{R}$. Clearly
$f(\Theta) = \Phi_W(f)(\Xi) = 0$. Since $\Theta \in_{\mathcal{P}\boxtimes\mathcal{P}'} \mathcal{W}$ has been arbitrary we obtain $f = 0$ by (i) . $\Box$
(iv) Assume Φ is a $\mathcal{P}$-super embedding and $\Phi(\Xi) = \Phi(\Theta)$ for some $\Xi, \Theta \in_{\mathcal{P}} \mathcal{M}$. By theorem 2.25 it suffices to show that $\Xi_M = \Theta_M$. Let $g \in \mathcal{D}(\mathcal{N})$. Then

$$\begin{aligned}\Xi_N\left(\Phi_N(g)\right) = \Phi_N(g)(\Xi) = g\left(\Phi(\Xi)\right) = g\left(\Phi(\Theta)\right) = \Phi_N(g)(\Theta)\\ = \Theta_N\left(\Phi_N(g)\right) .\end{aligned}$$

Therefore $\Xi_M|_{\Phi_N(\mathcal{D}(\mathcal{N}))} = \Theta_M|_{\Phi_N(\mathcal{D}(\mathcal{N}))}$. Since Φ is a $\mathcal{P}$-super embedding and so $\Phi_N\left(\mathcal{D}(\mathcal{N})\right) \subset \mathcal{D}(\mathcal{M})$ is dense and $\Xi_N : \mathcal{D}(\mathcal{M}) \to \mathcal{P}$ and $\Xi_N : \mathcal{D}(\mathcal{M}) \to \mathcal{P}$ are continuous we finally get $\Xi_M = \Theta_M$. $\Box$

A special case of a $\mathcal{P}$-super manifold is that of a discrete one.

Definition 2.28 *Let $\mathcal{M}$ be a $\mathcal{P}$-super manifold. Then it is called discrete if and only if it is of dimension $(0,0)$.*

Clearly, given a discrete $\mathcal{P}$-super manifold $\mathcal{M}$ its body M is a discrete topological space. So its structural sheaf is $\mathcal{P}$ itself, and therefore we see that each discrete $\mathcal{P}$-submanifold $\mathcal{M}$ is equal to its body M regarded as a $(0,0)$-dimensional $\mathcal{P}$-super manifold. Conversely any discrete topological space M is a 0-dimensional manifold. So the subcategory of discrete $\mathcal{P}$-super manifolds is equal to the category of sets (regarded as discrete topological spaces) together with arbitrary maps between them, and hereby the parametrization over $\mathcal{P}$ is meaningless.
Let $\mathcal{M}$ be a $\mathcal{P}$-super manifold. Then a discrete $\mathcal{P}$-submanifold of $\mathcal{M}$ is simply a set $\mathcal{N}$ of $\mathcal{P}$-points Ξ of $\mathcal{M}$ such that $N := \mathcal{N}^{\#}$ is a discrete subset of M and for all $x \in N$ there exists a unique $\Xi \in_{\mathcal{P}} \mathcal{N}$ such that $\Xi^{\#} = x$. So for the $\mathcal{P}$-embedding of a discrete topological space $N = \mathcal{N}$ into a $\mathcal{P}$-super manifold the parametrization over $\mathcal{P}$ is essential.

There are two possibilities of constructing non-trivial examples of $\mathcal{P}$-super manifolds. The first is a super analogon of defining a manifold M as a common zero set of functions on some $U \subset \mathbb{R}^p$ open, the second is a construction via a vectorbundle on a $\mathcal{C}^\infty$-manifold.

Theorem 2.29 *Let $U^{|q}$ be a (p,q)-dimensional super open set, $(\varphi, \Phi) = (f_1, \ldots, f_r, \lambda_1, \ldots, \lambda_s)$ be a $\mathcal{P}$-super morphism from $U^{|q}$ to $\mathbb{R}^{r|s}$ such that* rk $\mathcal{D}(\varphi,\Phi)^{\#}\left(\mathbf{x}_0\right) = r + s$ *for all $\mathbf{x}_0 \in U$. Let*

$$M := \varphi^{-1}(\mathbf{0}) \ ,$$

and let

$$\mathcal{S} := \left(\left(\mathcal{D}\left(\diamondsuit^{|q}\right)_U \boxtimes \mathcal{P}\right) \big/ \mathcal{I}\right)\Big|_M \ ,$$

where $\mathcal{I}$ denotes the graded ideal sheaf in $\mathcal{D}\left(\diamondsuit^{|q}\right)_U \boxtimes \mathcal{P}$ spanned by $f_1, \ldots, f_r$ and $\lambda_1, \ldots, \lambda_s$.

(i) $r \leq p$ and $s \leq q$, and $\mathcal{M} := \left(M, \mathcal{S}, {}^{\#}\right)$ is a $(p-r, q-s)$-dimensional $\mathcal{P}$-super manifold.

(ii) $\mathcal{M}$ is a $\mathcal{P}$-sub super manifold of $U^{|q}$, and

$$\mathcal{C} := \Big(c, \left(|_{M \cap V} \circ \rho_V\right)_{V \subset U \text{ open}}\Big)$$

is a $\mathcal{P}$-super embedding from $\mathcal{M}$ into $U^{|q}$, where $\rho : \mathcal{D}\left(\diamondsuit^{|q}\right)_U \boxtimes \mathcal{P} \to \left(\mathcal{D}\left(\diamondsuit^{|q}\right)_U \boxtimes \mathcal{P}\right) \big/ \mathcal{I}$ denotes the canonical unital graded sheaf projection.

Proof: (i) : Clearly $r \leq p$ and $s \leq q$, and M is a $(p-r)$-dimensional submanifold of U . The canonical unital graded sheaf embedding $\mathcal{P} \hookrightarrow \mathcal{D}\left(\diamondsuit^{|q}\right)_U$ clearly induces a unital graded sheaf morphism $\omega : \mathcal{P} \to \left(\mathcal{D}\left(\diamondsuit^{|q}\right) \big/ \mathcal{I}\right)\big|_M$, but right now it is not clear that ω is an embedding. From classical analysis we know that

$$\left(\mathcal{C}_U^\infty \Big/ \mathcal{I}^{\#}\right)\Big|_M \simeq \mathcal{C}_M^\infty \ ,$$

and therefore ${}^{\#} : \mathcal{D}\left(\diamondsuit^{|q}\right)_U \boxtimes \mathcal{P} \to \mathcal{C}_U^\infty$ induces a unital graded sheaf homomorphism ${}^{\#} : \mathcal{S} \to \mathcal{C}_M^\infty$. Now let $\mathbf{x}_0 \in M$. Then without loss of generality we can assume that $\mathbf{x}_0 = (\mathbf{y}_0, \mathbf{0})$ with $\mathbf{y}_0 \in \mathbb{R}^{p-r}$ and that there exist open neighbourhoods $V \subset \mathbb{R}^{p-r}$ of $\mathbf{x}_0$ and $W \subset \mathbb{R}^r$ of $\mathbf{0}$ such that $V \times W \subset U$, $f_i|_{V \times W} = z_i$, $i = 1, \ldots, r$, and $\lambda_j = \zeta_j$, $j = 1, \ldots, s$, where z_i denote the even and ζ_j the odd coordinate functions on $W^{|s}$. Let us check it:

> Since rk $D(\varphi, \Phi)^{\#} = s$ we may assume without loss of generality that $D_2(\varphi, \Phi)^{\#} \in \mathbb{R}_0^{(r,s)\times(r,s)}$ is invertible. Let $(\psi, \Psi) := \left(\left(\mathrm{pr}_1, \varphi\right), C_1 \hat{\otimes} \Phi\right)$. Then (ψ, Ψ) is a $\mathcal{P}$-super morphism from $U^{|q}$ to $\mathbb{R}^{p|q} = \mathbb{R}^{p-r|q-s} \times \mathbb{R}^{r|s}$ with defining tuple $(x_1, \ldots, x_{p-r}, f_1, \ldots, f_s, \xi_1, \ldots, \xi_{q-s}, \lambda_1, \ldots, \lambda_s)$. So after reordering the rows and columns in $D(\psi, \Psi)$ we have

$$D(\psi, \Psi) = \begin{pmatrix} 1 & 0 \\ D_1(\varphi, \Phi) & D_2(\varphi, \Phi) \end{pmatrix} ,$$

which is invertible at $\mathbf{x}_0$. Therefore by theorem 2.19 there exists an open neigbourhood $X \subset \mathbb{R}^{p|q}$ of $\psi(\mathbf{x}_0)$ such that $(\psi|_{\psi^{-1}(X)}, \Psi_X)$ is a $\mathcal{P}$-super diffeomorphism from $\psi^{-1}(X)^{|q}$ to $X^{|q}$. We have $\mathbf{x}_0' := \psi(\mathbf{x}_0) = (\mathbf{y}_0, \mathbf{0})$ if we define
$\mathbf{y}_0 := \mathrm{pr}_1(\mathbf{x}_0) \in \mathbb{R}^{p-r}$, and without loss of generality we can assume that $X = V \times W$ where $V \subset \mathbb{R}^{p-r}$ and $W \subset \mathbb{R}^r$ are open neighbourhoods of $\mathbf{y}_0$ resp. $\mathbf{0}$.
Let $M' := \psi(M) \cap X = V \times \{\mathbf{0}\} \subset V \times W$.

$$\begin{aligned}(\varphi', \Phi') &:= (\varphi, \Phi)|_{\psi^{-1}(X)} \circ \left(\psi|_{\psi^{-1}(X)}, \Psi_X\right)^{-1} \\ &= (\varphi, \Phi)|_{\psi^{-1}(X)} \circ \left(\psi^{-1}\big|_X, \left(\Psi^{-1}\right)_{\psi^{-1}(X)}\right)\end{aligned}$$

is a $\mathcal{P}$-super morphism from $V^{|q-s} \times W^{|s}$ to $\mathbb{R}^{r|s}$ with defining tuple $(z_1, \ldots, z_r, \zeta_1, \ldots, \zeta_s)$.

Let $\mathcal{I}'$ be the ideal sheaf on $V \times W$ generated by $(z_1, \ldots, z_r, \zeta_1, \ldots, \zeta_s)$. Again the canonical unital graded sheaf embedding $\mathcal{P} \hookrightarrow \mathcal{D}\left(\diamondsuit^{|q}\right)_U \boxtimes \mathcal{P}$ induces a unital graded sheaf morphism $\omega' : \mathcal{P} \to \left(\mathcal{D}\left(\diamondsuit^{|q}\right)/\mathcal{I}'\right)\big|_{M'}$. Let $Y \subset V \times W$ be open. Recall that $\Psi_Y : \mathcal{D}\left(Y^{|q}\right) \boxtimes \mathcal{P} \to \mathcal{D}\left(\psi^{-1}(Y)^{|q}\right) \boxtimes \mathcal{P}$ is a unital graded algebra isomorphism such that $\Psi_Y|_{\mathcal{P}} = \mathrm{id}$ and $\Psi_Y\left(f^{\#}\right) = \Psi_Y(f)^{\#}$ for all $f \in \mathcal{D}\left(Y^{|q}\right) \boxtimes \mathcal{P}$. Clearly $\Psi_Y(\mathcal{I}'(Y)) = \mathcal{I}\left(\psi^{-1}(Y)^{|q}\right)$ since Ψ_Y is an isomorphism. Let $f \in \mathcal{D}\left(Y^{|q}\right) \boxtimes \mathcal{P}$. Then by formula 2.1 f vanishes in a neighbourhood of $M' \cap Y$ if and only if $\Psi_Y(f) \in \mathcal{D}\left(\varphi^{-1}(Y)^{|q}\right) \boxtimes \mathcal{P}$ vanishes in a neighbourhood of $M \cap \varphi^{-1}(Y)$. So Ψ_Y induces a unital graded algebra isomorphism

$$\begin{aligned}\Psi_Y' : &\left(\left(\mathcal{D}\left(Y^{|q}\right) \boxtimes \mathcal{P}\right) \Big/ \mathcal{I}'(Y)\right)\Big|_{M' \cap Y} \\ &\xrightarrow{\sim} \left(\left(\mathcal{D}\left(\psi^{-1}(Y)^{|q}\right) \boxtimes \mathcal{P}\right) \Big/ \mathcal{I}\left(\psi^{-1}(Y)^{|q}\right)\right)\Big|_{M \cap \psi^{-1}(Y)^{|q}}\end{aligned}$$

such that $\Psi_Y'\left(f^{\#}\right) = \Psi_Y'(f)^{\#}$ for all
$f \in \left(\left(\mathcal{D}\left(Y^{|q}\right) \boxtimes \mathcal{P}\right)/\mathcal{I}'(Y)\right)\big|_{M' \cap Y}$ and $\Psi_Y' \circ \omega_Y' = \omega_{\psi^{-1}(Y)} \circ \Psi_Y'$.
Clearly for all $Y' \subset Y \subset U \cap V$ open
$\Psi_Y' \circ |_{M' \cap Y'} = |_{M \cap \psi^{-1}(Y')} \circ \Psi_Y$, and so $\left(\psi, (\Psi_Y')_{Y \subset U \times V \text{ open}}\right)$ is a whole isomorphism between the ringed spaces $(M', \mathcal{S}')$ and $(M \cap X, \mathcal{S}|_{X \cap M})$, where $\mathcal{S}' := \left(\left(\mathcal{D}\left(\diamondsuit^{|q}\right)_{U \times V} \boxtimes \mathcal{P}\right) \Big/ \mathcal{I}'\right)\Big|_{M'}$ on M' .

So we see that $M \cap (V \times W) = V \times \{\mathbf{0}\}$, and now it is obvious that $\omega|_{V\times\{\mathbf{0}\}}$ is an embedding and that there exists a canonical unital graded sheaf isomorphism $\Psi : \mathcal{S}|_{V\times\{\mathbf{0}\}} \xrightarrow{\sim} \mathcal{D}\left(\diamondsuit^{|q-s}\right)_{V\times\{\mathbf{0}\}} \boxtimes \mathcal{P}$ such that
$\Psi\left(\omega\left(\alpha_j\right)\right) = \alpha_j$ for all $j = 1, \dots, n$ and

$$\begin{array}{ccc} \mathcal{S}|_{V\times\{\mathbf{0}\}} & \xrightarrow{\Psi} & \mathcal{D}\left(\diamondsuit^{|q-s}\right)_{V\times\{\mathbf{0}\}} \boxtimes \mathcal{P} \\ {}_{\#}\searrow & \% & \swarrow_{\#} \\ & \mathcal{C}^\infty_{V\times\{\mathbf{0}\}} & \end{array} ,$$

and so since $\mathbf{x}_0 \subset M$ has been arbitrary $\mathcal{M}$ is indeed a $(p-r, q-s)$-dimensional $\mathcal{P}$-super manifold. □

(ii) trivial.

We say that the $\mathcal{P}$-super manifold $\mathcal{M}$ is defined by the equations $f_1 = 0\,, \dots, f_r = 0\,, \lambda_1 = 0\,, \dots, \lambda_s = 0$.

Let M be a p-dimensional manifold and E a q-dimensional $\mathcal{C}^\infty$-vectorbundle on M . Then the triple $\left(M, \Gamma^\infty_{\Lambda E}, {}^\#\right)$ is a (p,q)-dimensional super manifold, where $\Gamma^\infty_{\Lambda E}$ denotes the sheaf of $\mathcal{C}^\infty$-sections into the bundle ΛE and
${}^\# : \Gamma^\infty_{\Lambda E} \to \mathcal{C}^\infty$ denotes the sheaf projection onto the constant term.
BATCHELOR's theorem, theorem 4.29 of [4] , now says that all super manifolds can be obtained this way:

Theorem 2.30 *Let $\left(M, \mathcal{S}, {}^\#\right)$ be a super manifold of dimension (p,q) . Then there exists a q-dimensional $\mathcal{C}^\infty$-vectorbundle E on M such that*

$$\begin{array}{ccc} \mathcal{S} & \simeq & \Gamma^\infty_{\Lambda E} \\ {}_{\#}\searrow & \% & \swarrow_{\#} \\ & \mathcal{C}^\infty|_U & \end{array} .$$

E is uniquely defined by $\mathcal{S}$ up to isomorphism.

Clearly if E has dimension $q+n$ instead of q and there exists a $\mathcal{C}^\infty$-embedding $M \times \mathbb{R}^n \hookrightarrow E$ then this embedding induces a unital graded sheaf embedding $\mathcal{P} \hookrightarrow \Gamma^\infty_{\Lambda E}$, and so $\left(M, \Gamma^\infty_{\Lambda E}, {}^\#\right)$ is a (p,q)-dimensional $\mathcal{P}$-super manifold. It is not surprising that there is a formulation of BATCHELOR's theorem for $\mathcal{P}$-super manifolds as well:

Corollary 2.31 (BATCHELOR's theorem for $\mathcal{P}$-super manifolds)
Let $\left(M, \mathcal{S}, {}^\#\right)$ be a $\mathcal{P}$-super manifold of dimension (p,q) . Then there exists a $(q+n)$-dimensional $\mathcal{C}^\infty$- vectorbundle E on M , a $\mathcal{C}^\infty$-embedding

$M \times \mathbb{R}^n \hookrightarrow E$ and a unital graded sheaf isomorphism $\Phi : \mathcal{S} \xrightarrow{\sim} \Gamma^\infty_{\Lambda E}$ such that $\Phi(\alpha_i) = e_j$ for all $j = 1, \ldots, n$ and

$$\begin{array}{ccccc} \mathcal{S} & & \xrightarrow{\Phi} & & \Gamma^\infty_{\Lambda E} \\ & {}_\# \searrow & \% & \swarrow_\# & \\ & & \mathcal{C}^\infty_U & & \end{array} .$$

Again E is uniquely defined by $\mathcal{S}$ up to isomorphism.

Proof: Since $\left(M, \mathcal{S}, {}^\#\right)$ is an ordinary $(p, q+n)$-dimensional super manifold we can apply theorem 2.30 to it, which says that there exists a $(q+n)$-dimensional $\mathcal{C}^\infty$-vectorbundle E on M and a sheaf isomorphism $\Phi' : \mathcal{S} \xrightarrow{\sim} \Gamma^\infty_{\Lambda E}$ such that

$$\begin{array}{ccccc} \mathcal{S} & & \xrightarrow{\Phi'} & & \Gamma^\infty_{\Lambda E} \\ & {}_\# \searrow & \% & \swarrow_\# & \\ & & \mathcal{C}^\infty_U & & \end{array} .$$

One immediately sees that

$$\mathcal{N} / \mathcal{N}^2 \simeq \Gamma^\infty_E$$

via Φ' , where $\mathcal{N}$ denotes the subsheaf of all nilpotent elements in $\mathcal{S}$, and if $U \subset M$ such that $E|_U$ is trivial then for all $V \subset U$ open $\left(\mathcal{N} / \mathcal{N}^2\right)(V) = \mathcal{N}(V) / \mathcal{N}(V)^2$. Let $\mathcal{N}_\mathcal{P}$ be the set of all nilpotent elements in $\mathcal{P}$. Then

$$\mathbb{R}^n \simeq \mathcal{N}_\mathcal{P} / \mathcal{N}_\mathcal{P}^2 \hookrightarrow \mathcal{N} / \mathcal{N}^2 \simeq \Gamma^\infty_E$$

as a sheaf on M , and so $M \times \mathbb{R}^n \hookrightarrow E$ as a $\mathcal{C}^\infty$-vectorbundle on M ,

$$\Phi'(\alpha_j) \in \left(e_j + \mathcal{N}(\Gamma^\infty_{\Lambda E})^2\right) \cap (\Gamma^\infty_{\Lambda E})_1 ,$$

where $\mathcal{N}(\Gamma^\infty_{\Lambda E})$ denotes the subsheaf of $\Gamma^\infty_{\Lambda E}$ of nilpotent elements. Let $M = \bigcup_{\lambda \in \Lambda} U_\lambda$ be an open locally finite covering of M such that $E|_{U_\lambda}$ is trivial for all $\lambda \in \Lambda$, and let

$$1 = \sum_{\lambda \in \Lambda} \varepsilon_\lambda$$

be a $\mathcal{C}^\infty$-partition of unity on M such that supp $\varepsilon_\lambda \subset U_\lambda$ for all $\lambda \in \Lambda$. Then for all $\lambda \in \Lambda$ since $E|_{U_\lambda}$ is trivial we can find a $\mathcal{C}^\infty$-vectorbundle embedding

$$\rho_\lambda : E|_{U_\lambda} \hookrightarrow (\Lambda E)_1 |_{U_\lambda}$$

such that $\rho_\lambda(x, e_j) = \Phi'(\alpha_j)(x)$ for all $j = 1, \ldots, n$ and $x \in U_\lambda$, and

$$E|_{U_\lambda} \overset{\rho_\lambda}{\hookrightarrow} (\Lambda E)_1 |_{U_\lambda} \hookrightarrow \mathcal{N}(\Lambda E)|_{U_\lambda} \xrightarrow{\mathrm{pr}} \mathcal{N}(\Lambda E) \Big/ \mathcal{N}(\Lambda E)^2 |_{U_\lambda} \simeq E|_{U_\lambda}$$

gives the identity map. So we get a whole $\mathcal{C}^\infty$-vectorbundle embedding

$$\rho : E \hookrightarrow (\Lambda E)_1 \;,\; (x, \mathbf{v}) \mapsto \sum_{\lambda \in \Lambda \,,\, x \in U_\lambda} \varepsilon_\lambda \rho_\lambda (x, \mathbf{v}) \,,$$

where the sum is taken in the fibre. Again $\rho(x, e_j) = \Phi'(\alpha_j)(x)$ for all $j = 1, \ldots, n$ and $x \in M$, and

$$E \overset{\rho}{\hookrightarrow} (\Lambda E)_1 \hookrightarrow \mathcal{N}(\Lambda E) \xrightarrow{\mathrm{pr}} \mathcal{N}(\Lambda E) \Big/ \mathcal{N}(\Lambda E)^2 \simeq E$$

gives the identity map. By the universal property of the exterior algebra the $\mathcal{C}^\infty$-vectorbundle embedding ρ extends to a unital graded $\mathcal{C}^\infty$-algebra bundle automorphism χ on ΛE , and χ induces a unital graded sheaf automorphism Ψ of $\Gamma^\infty_{\Lambda E}$ such that $\Psi(e_j) = \Phi'(\alpha_j)$, $j = 1, \ldots, n$. So take $\Phi := \Psi^{-1} \circ \Phi'$. Since Ψ comes from the unital algebra bundle automorphism χ we fortunately have again

$$\begin{array}{ccc} \mathcal{S} & \stackrel{\Phi}{\longrightarrow} & \Gamma^\infty_{\Lambda E} \\ {}_\# \searrow & \% & \swarrow_\# \\ & \mathcal{C}^\infty_U & \end{array} \,,$$

and by theroem 2.30 E is uniquely defined by $\mathcal{S}$ up to isomorphism. □

Via the up to isomorphisms one-to-one-correspondence between super manifolds and finite-dimensional $\mathcal{C}^\infty$-vectorbundles over $\mathcal{C}^\infty$-manifolds one can show that there exists a cross product in the category of super manifolds, see theorem 5.21 of [4] : Let $\mathcal{M}$ be a (p, q)-dimensional and $\mathcal{N}$ be an (r, s)-dimensional super manifold. Without loss of generality we can assume that $\mathcal{S} = \Gamma^\infty(\Lambda E)$ and $\mathcal{T} = \Gamma^\infty(\Lambda F)$ where E is a q-dimensional $\mathcal{C}^\infty$-vectorbundle over M and F is an s-dimensional vectorbundle over N . Then

$$\mathcal{M} \times \mathcal{N} := \Big(M \times N, \Gamma^\infty \left(\Lambda \left(\mathrm{pr}_1^* E \oplus \mathrm{pr}_2^* F \right) \right), {}^\# \Big)$$

is a cross product of $\mathcal{M}$ and $\mathcal{N}$, where $\mathrm{pr}_1^* E$ and $\mathrm{pr}_2^* F$ denote the pullbacks of E resp. F under the canonical projections $\mathrm{pr}_1 : M \times N \to M$ resp. $\mathrm{pr}_2 : M \times N \to N$.
Since for all $U \subset M$ and $V \subset N$ open

$$\Gamma^\infty \left(\Lambda \left(\mathrm{pr}_1^* E \oplus \mathrm{pr}_2^* F \right) \right) (U \times V) = \mathcal{S}(U) \hat{\boxtimes} \mathcal{T}(V)$$

we denote the stucture sheaf $\Gamma^\infty\left(\Lambda\left(\mathrm{pr}_1^* E \oplus \mathrm{pr}_2^* F\right)\right)$ of $\mathcal{M} \times \mathcal{N}$ by $\mathrm{pr}_1^*\mathcal{S} \hat{\boxtimes} \mathrm{pr}_2^*\mathcal{T}$, where $\mathrm{pr}_1^*\mathcal{S}$ and $\mathrm{pr}_2^*\mathcal{T}$ denote the pullbacks of $\mathcal{S}$ resp. $\mathcal{T}$. $(f \otimes g)^\# := f^\# \otimes g^\# \in \mathcal{C}^\infty(U \times V)$ for all $U \subset M$ and $V \subset N$ open, $f \in \mathcal{S}(U)$ and $g \in \mathcal{T}(V)$.
$\mathcal{M} \times \mathcal{N}$ is a $(p+r, q+s)$-dimensional super manifold. We have the canonical super projections $\mathrm{Pr}_1 := \left(\mathrm{pr}_1, (\mathcal{C}_{1U})_{U \subset M \text{ open}}\right)$ and
$\mathrm{Pr}_2 := \left(\mathrm{pr}_2, (\mathcal{C}_{2,V})_{V \subset M \text{ open}}\right)$ from $\mathcal{M} \times \mathcal{N}$ onto $\mathcal{M}$ resp. $\mathcal{N}$, where $\mathrm{pr}_1 : M \times N \to M$ and $\mathrm{pr}_2 : M \times N \to N$ denote the canonical projections and

$$\mathcal{C}_{1,U} : \mathcal{S}(U) \hookrightarrow \mathcal{S}(U) \hat{\boxtimes} \mathcal{T}(N)$$

for all $U \subset M$ open and

$$\mathcal{C}_{1,V} : \mathcal{S}(V) \hookrightarrow \mathcal{S}(M) \hat{\boxtimes} \mathcal{T}(V)$$

for all $V \subset N$ open denote the canonical embeddings, and again we have the universal property: For any super manifold $\mathcal{O} = \left(O, \mathcal{R}, {}^\#\right)$ and super morphisms $\Phi_1 = \left(\varphi_1, (\Phi_{1,U})_{U \subset M \text{ open}}\right)$ and $\Phi_2 = \left(\varphi_2, (\Phi_{2,V})_{V \subset N \text{ open}}\right)$ from $\mathcal{O}$ to $\mathcal{M}$ resp. $\mathcal{N}$ there exists a unique super morphism
$\Psi = \left(\psi, (\Psi_W)_{W \subset M \times N \text{ open}}\right)$ from $\mathcal{O}$ to $\mathcal{M} \times \mathcal{N}$ such that

$$\mathrm{Pr}_1 \circ \Psi = \Phi_1$$

and

$$\mathrm{Pr}_2 \circ \Psi = \Phi_2 \,.$$

Hereby we have $\psi = (\varphi_1, \varphi_2) : O \to M \times N$ and

$$\Psi_{U \times V} = \left(|_{\varphi_1^{-1}(U) \cap \varphi_2^{-1}(V)} \circ \Phi_{1,U}\right) \left(|_{\varphi_1^{-1}(U) \cap \varphi_2^{-1}(V)} \circ \Phi_{2,V}\right) ,$$

more precisely

$$\begin{aligned} \Psi_{U \times V} : \quad & \mathcal{S}(U) \hat{\boxtimes} \mathcal{T}(V) \to \mathcal{R}\left(\varphi_1^{-1}(U) \cap \varphi^{-1}(V)\right) , \\ & f \otimes g \mapsto \left(\Phi_{1,U}(f)|_{\varphi_1^{-1}(U) \cap \varphi_2^{-1}(V)}\right) \left(\Phi_{2,V}(f)|_{\varphi_1^{-1}(U) \cap \varphi_2^{-1}(V)}\right) \end{aligned}$$

for all $U \subset M$ and $V \subset N$ open. As usual we identify each super morphism Ψ from $\mathcal{O}$ to $\mathcal{M} \times \mathcal{N}$ with its 'defining pair' $(\mathrm{Pr}_1 \circ \Psi, \mathrm{Pr}_2 \circ \Psi)$.

Now let $\mathcal{M} = \left(M, \mathcal{S}, {}^\#\right)$ be a (p, q)-dimensional and $\mathcal{N} = \left(N, \mathcal{T}, {}^\#\right)$ be an (r, s)-dimensional $\mathcal{P}$-super manifold. We will construct a $\mathcal{P}$-cross product $\mathcal{M} \times_{\mathcal{P}} \mathcal{N}$ of $\mathcal{M}$ and $\mathcal{N}$ in the category of $\mathcal{P}$-super manifolds, being a

$(p+r, q+s)$-dimensional $\mathcal{P}$-super manifold. The cross product $\mathcal{M} \times \mathcal{N}$ of $\mathcal{M}$ and $\mathcal{N}$ as usual super manifolds is a $(p+r, q+s)$-dimensional $\mathcal{P} \boxtimes \mathcal{P}$-super manifold since the embeddings $\mathcal{P} \hookrightarrow \mathcal{S}$ resp. $\mathcal{T}$ induce a unital graded sheaf embedding $\mathcal{P} \boxtimes \mathcal{P} \hookrightarrow \mathrm{pr}_1^* \mathcal{S} \hat{\boxtimes} \mathrm{pr}_2^* \mathcal{T}$. We have a diagonal embedding $\mathcal{C}_{\mathcal{P}} := (\mathbf{0}, m)$ from $\mathbb{R}^{|n}$ into $\mathbb{R}^{|n} \times \mathbb{R}^{|n}$, where the unital graded algebra projection $m : \mathcal{P} \boxtimes \mathcal{P} \to \mathcal{P}$ is defined by $R \otimes S \mapsto RS$, and m induces a unital graded algebra isomorphism

$$m' : (\mathcal{P} \boxtimes \mathcal{P}) / \mathcal{I}_{\mathcal{P}} \to \mathcal{P} ,$$

where $\mathcal{I}_{\mathcal{P}} := \ker m \sqsubset \mathcal{P} \boxtimes \mathcal{P}$, which is precisely the ideal in $\mathcal{P}$ spanned by $\alpha_j \otimes 1 - 1 \otimes \alpha_j$, $j = 1, \ldots, n$. m'^{-1} is given by
$R \mapsto (R \otimes 1) + \mathcal{I}_{\mathcal{P}} = (1 \otimes R) + \mathcal{I}_{\mathcal{P}}$. Let $\mathcal{I}$ be the ideal sheaf of $\mathrm{pr}_1^* \mathcal{S} \hat{\boxtimes} \mathrm{pr}_2^* \mathcal{T}$ spanned by $\alpha_j \otimes 1 - 1 \otimes \alpha_j$, $j = 1, \ldots, n$.

Theorem 2.32

(i)

$$\mathcal{M} \times_{\mathcal{P}} \mathcal{N} := \left(M \times N, \left(\mathrm{pr}_1^* \mathcal{S} \hat{\boxtimes} \mathrm{pr}_2^* \mathcal{T} \right) \Big/ \mathcal{I}, {}^{\#} \right)$$

is a $(p+r, q+s)$-dimensional $\mathcal{P}$-super manifold.

(ii) $\mathcal{C} := \left(id, (\rho_W)_{W \subset U \times V \text{ open}} \right)$ *is a super embedding from $\mathcal{M} \times_{\mathcal{P}} \mathcal{N}$ into $\mathcal{M} \times \mathcal{N}$, where $\rho : \mathrm{pr}_1^* \mathcal{S} \hat{\boxtimes} \mathrm{pr}_2^* \mathcal{T} \to \left(\mathrm{pr}_1^* \mathcal{S} \hat{\boxtimes} \mathrm{pr}_2^* \mathcal{T} \right) \Big/ \mathcal{I}$ denotes the canonical unital graded sheaf epimorphism, and*

$$\begin{array}{ccc} \mathcal{M} \times_{\mathcal{P}} \mathcal{N} & \overset{\mathcal{C}}{\hookrightarrow} & \mathcal{M} \times \mathcal{N} \\ \downarrow & \% & \downarrow \\ \mathbb{R}^{|n} & \underset{\mathcal{C}_{\mathcal{P}}}{\hookrightarrow} & \mathbb{R}^{|n} \times \mathbb{R}^{|n} \end{array} ,$$

where the arrows on the left and right side denote the canonical super projections.

Proof: (i) Clearly

$$m'^{-1} : \mathcal{P} \xrightarrow{\sim} (\mathcal{P} \boxtimes \mathcal{P}) / \mathcal{I}_{\mathcal{P}} \hookrightarrow \left(\mathrm{pr}_1^* \mathcal{S} \hat{\boxtimes} \mathrm{pr}_2^* \mathcal{T} \right) \Big/ \mathcal{I}$$

is a unital graded sheaf embedding. Let $U \subset M$ and $V \subset N$ be open such that $\mathcal{S}|_U = \mathcal{D}\left(\diamondsuit^{|q}\right)_U \boxtimes \mathcal{P}$ and $\mathcal{T}|_V = \mathcal{D}\left(\diamondsuit^{|s}\right)_V \boxtimes \mathcal{P}$. Then

$$\begin{aligned} \mathrm{id} \otimes m' : \left(\left(\mathrm{pr}_1^* \mathcal{S} \hat{\boxtimes} \mathrm{pr}_2^* \mathcal{T} \right) \Big/ \mathcal{I} \right) \Big|_{U \times V} &= \mathcal{D}\left(\diamondsuit^{|q+s}\right)_{U \times V} \boxtimes \left((\mathcal{P} \boxtimes \mathcal{P}) / \mathcal{I}_{\mathcal{P}} \right) \\ &\xrightarrow{\sim} \mathcal{D}\left(\diamondsuit^{|q+s}\right)_{U \times V} \boxtimes \mathcal{P} \end{aligned}$$

is an isomorphism, $(\mathrm{id} \otimes m') \circ m'^{-1} = \mathrm{id}$, and since $\mathcal{I} \sqsubset \ker {}^{\#}$,
${}^{\#} : \mathrm{pr}_1^* \mathcal{S} \,\hat{\boxtimes}\, \mathrm{pr}_2^* \mathcal{T} \to \mathcal{C}^{\infty}_{M \times N}$ induces a body map

$$
{}^{\#} : \left(\mathrm{pr}_1^* \mathcal{S} \,\hat{\boxtimes}\, \mathrm{pr}_2^* \mathcal{T} \right) \Big/ \mathcal{I} \to \mathcal{C}^{\infty}_{M \times N}
$$

such that $(1 \otimes m') \, (f)^{\#} = f^{\#}$ for all $W \subset U \times V$ and
$f \in \left(\mathrm{pr}_1^* \mathcal{S} \,\hat{\boxtimes}\, \mathrm{pr}_2^* \mathcal{T} \right) (W)$. □

(ii) trivial since $\rho \, (\alpha_j \otimes 1) = \alpha_j$ and $m \, (\alpha_j \otimes 1) = \alpha_j$ as well for all $j = 1, \ldots, n$. □

Let $\mathrm{Pr}_1' := \mathcal{C} \circ \mathrm{Pr}_1$ and $\mathrm{Pr}_2' := \mathcal{C} \circ \mathrm{Pr}_2$, where Pr_1 and Pr_2 denote the canonical projections from $\mathcal{M} \times \mathcal{N}$ onto $\mathcal{M}$ resp. $\mathcal{N}$. Then Pr_1' and Pr_2' are super morphisms from $\mathcal{M} \times_{\mathcal{P}} \mathcal{N}$ to $\mathcal{M}$ resp. $\mathcal{N}$. We can write $\mathrm{Pr}_1' = \left(\mathrm{pr}_1, \left(\mathcal{C}_{1,U}' \right)_{U \subset M \text{ open}} \right)$ and $\mathrm{Pr}_1' = \left(\mathrm{pr}_1, \left(\mathcal{C}_{1,U}' \right)_{U \subset M \text{ open}} \right)$ where $\mathcal{C}_{1,U}' = \mathcal{C}_{1,U} \circ \rho_{U \times N}$ and $\mathcal{C}_{2,V}' = \mathcal{C}_{1,V} \circ \rho_{M \times V}$ for all $U \subset M$ resp. $V \subset N$ open.

Theorem 2.33 Pr_1' *and* Pr_2' *are* $\mathcal{P}$*-super projections from* $\mathcal{M} \times_{\mathcal{P}} \mathcal{N}$ *onto* $\mathcal{M}$ *resp.* $\mathcal{N}$ *.*

Proof: Let $U \subset M$ be open. We would like to show that

$$
\mathcal{C}_{1,U}' : \mathcal{S}(U) \to \left(\mathcal{S}(U) \hat{\boxtimes} \mathcal{T}(N) \right) \Big/ \mathcal{I}(U \times N)
$$

is injective. So let $V \subset N$ be open, $V \neq \emptyset$. Without loss of generality we may assume that $\mathcal{S}|_U = \mathcal{D} \left(\diamondsuit^{|q} \right)_U \boxtimes \mathcal{P}$ and $\mathcal{T}|_V = \mathcal{D} \left(\diamondsuit^{|s} \right)_V \boxtimes \mathcal{P}$. Let

$$
f = \sum_{I \in \wp(n)} f_I \alpha^I \in \ker \mathcal{C}_{1,U}' \sqsubset \mathcal{S}(U) = \mathcal{D} \left(U^{|q} \right) \boxtimes \mathcal{P} .
$$

Then an easy calculation shows that

$$
0 = \mathcal{C}_{1,U}'(f) \big|_{U \times V} = \sum_{I \in \mathcal{P}(n)} f_I \otimes 1 \otimes m'^{-1} \, (\alpha)^I
$$

as an element of $\left(\mathcal{S}(U) \hat{\boxtimes} \mathcal{T}(V) \right) = \mathcal{D} \left(U^{|q} \right) \hat{\boxtimes} \mathcal{D} \left(V^{|s} \right) \boxtimes \left(\left(\mathcal{P} \boxtimes \mathcal{P} \right) / \mathcal{I}_{\mathcal{P}} \right)$, and since $m'^{-1} : \mathcal{P} \xrightarrow{\sim} \left(\mathcal{P} \boxtimes \mathcal{P} \right) / \mathcal{I}_{\mathcal{P}}$ is an isomorphism we have $f_I = 0$ for all $I \in \wp(n)$, and so $f = 0$.

Now let $j = 1, \ldots, n$ and $U \subset M$ be open. Then

$$
\mathcal{C}_{1,U}' \left(\alpha_j \right) = \rho_{U \times N} \left(\mathcal{C}_{1,\alpha} \left(\alpha_j \right) \right) = \varepsilon \left(\alpha_j \otimes 1 \right) = \alpha_j .
$$

So $\mathcal{C}_{1,U}|_{\mathcal{P}} = \mathrm{id}$. For proving that Pr_2' is a $\mathcal{P}$-super projection from $\mathcal{M} \times_{\mathcal{P}} \mathcal{N}$ onto $\mathcal{N}$ one has to do the same calculations again. $\square$

Again we have the universal property:

Theorem 2.34 *For any $\mathcal{P}$-super manifold $\mathcal{O} = \left(O, \mathcal{R}, {}^{\#}\right)$ and $\mathcal{P}$-super morphisms $\Phi_1 = \left(\varphi_1, \left(\Phi_{1,W}\right)_{W \subset O \text{ open}}\right)$ and $\Phi_2 = \left(\varphi_2, \left(\Phi_{2,W}\right)_{W \subset O \text{ open}}\right)$ there exists a unique $\mathcal{P}$-super morphism Ψ' from $\mathcal{O}$ to $\mathcal{M} \times_{\mathcal{P}} \mathcal{N}$ such that*

$$\mathrm{Pr}_1' \circ \Psi' = \Phi_1$$

and

$$\mathrm{Pr}_2' \circ \Psi' = \Phi_2 \,.$$

Let $\Psi := (\Phi_1, \Phi_2)$, going from $\mathcal{O}$ to $\mathcal{M} \times \mathcal{N}$. Then Ψ' is the unique $\mathcal{P}$-super morphism from $\mathcal{O}$ to $\mathcal{M} \times_{\mathcal{P}} \mathcal{N}$ such that $\Psi = \mathcal{C} \circ \Psi'$, and so in particular the body of Ψ' is again $\psi = (\varphi_1, \varphi_2) : O \to M \times N$.

Proof: Let us first show that there exists a unique $\mathcal{P}$-super morphism Ψ' from $\mathcal{O}$ to $\mathcal{M} \times_{\mathcal{P}} \mathcal{N}$ such that $\Psi = \mathcal{C} \circ \Psi'$. $\Psi' = \left(\psi', \left(\Psi_W'\right)_{W \subset M \times N \text{ open}}\right)$ being a $\mathcal{P}$-super morphism from $\mathcal{O}$ to $\mathcal{M} \times_{\mathcal{P}} \mathcal{N}$, $\Psi = \mathcal{C} \circ \Psi'$ is equivalent to $\psi' = \psi$ and

$$\Psi_W = \Psi_W' \circ \rho_W$$

for all $W \subset M \times N$ open. Let $W \subset M \times N$. Then $\Psi_W \left(\alpha_j \otimes 1 - 1 \otimes \alpha_j\right) = 0$ since $\Psi_{M \times N} \circ C_{1,M} = \Phi_{1,M}$ and $\Psi_{M \times N} \circ C_{2,N} = \Phi_{2,N}$, and so

$$\begin{aligned}
\Psi_{M \times N} \left(\alpha_j \otimes 1\right) &= \Psi_{M \times N} \left(C_{1,M} \left(\alpha_j\right)\right) = \Phi_{1,M} \left(\alpha_j\right) = \alpha_j = \Phi_{2,N} \left(\alpha_j\right) \\
&= \Psi_{M \times N} \left(C_{2,N} \left(\alpha_j\right)\right) = \Psi_{M \times N} \left(1 \otimes \alpha_j\right) \,.
\end{aligned}$$

It follows that $\ker \rho_W = \mathcal{I}(W) \sqsubset \ker \Psi_W$. Therefore since ρ_W is surjective there exists a unique unital graded algebra homomorphism

$$\Psi_W' : \left(\left(\mathrm{pr}_1^* \mathcal{S} \,\hat{\boxtimes}\, \mathrm{pr}_2^* \mathcal{T}\right) \big/ \mathcal{I}\right)(W) \to \mathcal{R}\left(\psi^{-1}(W)\right)$$

such that $\Psi_W = \Psi_W' \circ \rho_W$. Now we show that $\left(\psi, \left(\Psi_W\right)_{W \subset U \times V \text{ open}}\right)$ is indeed a $\mathcal{P}$-super morphism from $\mathcal{O}$ to $\mathcal{M} \times_{\mathcal{P}} \mathcal{N}$:

> Let $W' \subset W \subset M \times N$ be open. Since Ψ is already a super morphism from $\mathcal{O}$ to $\mathcal{M} \times \mathcal{N}$ we obtain

$$\begin{aligned}\Psi'_{W'} \circ |_{W'} \circ \rho_W &= \Psi'_{W'} \circ \rho_{W'} \circ |_{W'} \\ &= \Psi_{W'} \circ |_{W'} \\ &= |_{\psi^{-1}(W')} \circ \Psi_W \\ &= |_{\psi^{-1}(W')} \circ \Psi'_W \circ \rho_W \,.\end{aligned}$$

But ρ_W is surjective, and so we have $\Psi'_{W'} \circ |_{W'} = |_{\psi^{-1}(W')} \circ \Psi'_W$.

Let $W \subset M \times N$ be open and $f \in \left(\left(\mathrm{pr}_1^* \mathcal{S} \,\hat{\boxtimes}\, \mathrm{pr}_2^* \mathcal{T}\right) \big/ \mathcal{I}\right)(W)$. Then we have $f = g + \mathcal{I}(W)$ for some $g \in \left(\mathrm{pr}_1^* \mathcal{S} \,\hat{\boxtimes}\, \mathrm{pr}_2^* \mathcal{T}\right)(W)$ by lemma 2.23 . So we get

$$\Psi'_W(f)^{\#} = \Psi(g) = g^{\#} \circ \left(\psi|_{\psi^{-1}(W)}\right) = f^{\#} \circ \left(\psi|_{\psi^{-1}(W)}\right) \,.$$

Now let $j = 1, \ldots, n$. Then

$$\Psi'_{M\times N}\left(\alpha_j\right) = \Psi'_{M\times N}\left(\rho_{M\times N}\left(\alpha_j \otimes 1\right)\right) = \Psi_{M\times N}\left(\alpha \otimes 1\right) = \alpha_j \,,$$

and so we get $\Psi'_W|_{\mathcal{P}} = \mathrm{id}$ for all $W \subset M \times N$ open.

It remains to prove that, $\Psi' = \left(\psi', (\Psi'_W)_{W \subset M\times N \text{ open}}\right)$ being a $\mathcal{P}$-super morphism from $\mathcal{O}$ to $\mathcal{M} \times_{\mathcal{P}} \mathcal{N}$, $\Psi = \mathcal{C} \circ \Psi'$ is equivalent to $\mathrm{Pr}'_1 \circ \Psi' = \Phi_1$ and $\mathrm{Pr}'_2 \circ \Psi' = \Phi_2$.

'$\Rightarrow$' : Let $\Psi = \mathcal{C} \circ \Psi'$. Then

$$\mathrm{Pr}'_1 \circ \Psi' = \mathrm{Pr}_1 \circ \mathcal{C} \circ \Psi' = \mathrm{Pr}_1 \circ \Psi = \Phi_1$$

and by the same calculation $\mathrm{Pr}'_2 \circ \Psi' = \Phi_2$ as well.

'$\Leftarrow$' : Let $\mathrm{Pr}'_1 \circ \Psi' = \Phi_1$ and $\mathrm{Pr}'_2 \circ \Psi' = \Phi_2$, and define $\Pi := \mathcal{C} \circ \Psi'$. Then

$$\mathrm{Pr}_1 \circ \Pi = \mathrm{Pr}_1 \circ \mathcal{C} \circ \Psi' = \mathrm{Pr}'_1 \circ \Psi' = \Phi_1$$

and by the same calculation $\mathrm{Pr}_2 \circ \Pi = \Phi_1$ as well. So by the universal proverty of the cross product $\mathcal{M} \times \mathcal{N}$ we get $\Pi = \Psi$. $\Box$

Given $\mathcal{P}$-super manifolds $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{O}$, we denote the $\mathcal{P}$-projections Pr'_1 and Pr'_2 going from $\mathcal{M} \times_{\mathcal{P}} \mathcal{N}$ to $\mathcal{M}$ resp. $\mathcal{N}$ by Pr_1 resp. Pr_2 since there is no danger of confusion. Again given a $\mathcal{P}$-super morphism Ψ from $\mathcal{O}$ to $\mathcal{M} \times_{\mathcal{P}} \mathcal{N}$ we write $\Psi = (\mathrm{Pr}_1 \circ \Psi, \mathrm{Pr}_2 \circ \Psi)$. Finally by the universal property of the $\mathcal{P}$-cross product one sees that the $\mathcal{P}$-cross product is commutative and associative in the sense that given $\mathcal{P}$-super manifolds $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{O}$ there are canonical $\mathcal{P}$- super diffeomorphisms from $\mathcal{M} \times_{\mathcal{P}} \mathcal{N}$ to $\mathcal{N} \times_{\mathcal{P}} \mathcal{M}$ and from $(\mathcal{M} \times_{\mathcal{P}} \mathcal{N}) \times_{\mathcal{P}} \mathcal{O}$ to $\mathcal{M} \times_{\mathcal{P}} (\mathcal{N} \times_{\mathcal{P}} \mathcal{O})$, and so we simply write $\mathcal{M} \times_{\mathcal{P}} \mathcal{N} \times_{\mathcal{P}} \mathcal{O}$.

2.3 super manifolds - the complex case

Now we will treat holomorphic super manifolds. Let us again start with the subcategory of complex super open sets.

Let M be a holomorphic manifold. Then we define the sheaves

$$\mathcal{D}\left(\diamondsuit^{|q,\overline{q}}\right)_M := \left(\mathcal{C}^\infty_M\right)^{\mathbb{C}} \boxtimes \Lambda\left(\mathbb{C}^q\right) \boxtimes \Lambda\left(\mathbb{C}^q\right)$$

and

$$\mathcal{O}\left(\diamondsuit^{|q,\overline{q}}\right)_M := \mathcal{O}_M \otimes \Lambda\left(\mathbb{C}^q\right)$$

of unital associative graded commutative algebras on M and a graded involution $\overline{}$ on $\mathcal{D}\left(\diamondsuit^{|q,\overline{q}}\right)_M$ given by

$$\begin{aligned} \overline{} : \mathcal{D}\left(\diamondsuit^{|q,\overline{q}}\right)_M &\to \mathcal{D}\left(\diamondsuit^{|q,\overline{q}}\right)_M , \\ \sum_{S,T \in \wp(q)} f_{S,T}\, e^S \otimes e^T &\mapsto \sum_{S,T \in \wp(q)} (-1)^{\frac{|S|(|S|+1)}{2} + \frac{|T|(|T|+1)}{2}} \overline{f_{S,T}}\, e^T \otimes e^S , \end{aligned}$$

which is less complicated than it seems to be: This is precisely the unique involution $\overline{}$ on $\mathcal{D}\left(\diamondsuit^{|q,\overline{q}}\right)_M$ such that $\overline{}$ restricted to $\left(\mathcal{C}^\infty_M\right)^{\mathbb{C}}$ is just ordinary complex conjugation and $\overline{e_k \otimes 1} = 1 \otimes e_k$, $k = 1, \dots, q$.

We regard $\mathcal{O}\left(U^{|q,\overline{q}}\right)$ as a sub graded sheaf of $\mathcal{D}\left(\diamondsuit^{|q,\overline{q}}\right)_M$ via the embedding

$$\mathcal{O}\left(\diamondsuit^{|q,\overline{q}}\right)_M \hookrightarrow \mathcal{D}\left(\diamondsuit^{|q,\overline{q}}\right)_M , \; f \mapsto f \otimes 1 .$$

Again we have a body map

$$^\# : \mathcal{D}\left(\diamondsuit^{|q,\overline{q}}\right)_M \to \left(\mathcal{C}^\infty_M\right)^{\mathbb{C}} , \; \sum_{S,T \in \wp(q)} f_{S,T}\; e^S \otimes e^T \mapsto f_{\emptyset,\emptyset} ,$$

which respects $\overline{}$. Again on $\mathcal{D}\left(\diamondsuit^{|q,\overline{q}}\right)_M$ as a free 2^{2q}-dimensional $\mathcal{C}^\infty_M$-module we will always use the uniformal structure of compact convergence in all derivatives. So for all $U \subset M$ open $\mathcal{O}\left(U^{|q,\overline{q}}\right)$ is a unital closed sub graded algebra of $\mathcal{D}\left(U^{|q,\overline{q}}\right)$, and we call a function $f \in \mathcal{D}\left(U^{|q,\overline{q}}\right)$ holomorphic if and only if it belongs to $\mathcal{O}\left(U^{|q,\overline{q}}\right)$. The image of $\mathcal{O}\left(\diamondsuit^{|q,\overline{q}}\right)_M$ under the body map $^{\#}$ is precisely $\mathcal{O}_M$. We have

$$\mathcal{D}\left(\diamondsuit^{|q,\overline{q}}\right)_M \simeq \mathcal{D}\left(\diamondsuit^{|2q}\right)^{\mathbb{C}}$$

by lemma 2.10 .

Now let $M = \mathbb{C}^p$ for some $p \in \mathbb{N}$. Then on $\mathbb{C}^{p|q,\overline{q}}$ we have the even coordinate functions

$$z_1, \ldots, z_p \in \mathcal{O}\left(\mathbb{C}^p\right) \hookrightarrow \mathcal{O}\left(\mathbb{C}^{p|q,\overline{q}}\right)_0$$

and the odd coordinate functions

$$\zeta_1 := e_1, \ldots, \zeta_q := e_q \in \Lambda\left(\mathbb{C}^q\right)_1 \hookrightarrow \mathcal{O}\left(\mathbb{C}^{p|q,\overline{q}}\right)_1 .$$

We define $\zeta^S := e^S \in \Lambda\left(\mathbb{C}^q\right) \hookrightarrow \mathcal{O}\left(\mathbb{C}^{p|q,\overline{q}}\right)$ and $\overline{\zeta}^S := 1 \otimes e^S \in \Lambda\left(\mathbb{C}^q\right) \boxtimes \Lambda\left(\mathbb{C}^q\right)$ for all $S \in \wp(q)$.

Definition 2.35 (complex super open sets)

(i) Let $(p,q) \in \mathbb{N}^2$ and $U \subset \mathbb{C}^p$ be open. Then the triple $U^{|q,\overline{q}} := (U,q,q)$ is called a complex super open set of dimension (p,q) . U is called the body of $U^{p|q,\overline{q}}$ and $^{\#} : \mathcal{D}\left(U^{|q,\overline{q}}\right) \to \mathcal{C}^\infty(U)^{\mathbb{C}}$ the body map of $\mathcal{D}\left(U^{|q,\overline{q}}\right)$.

(ii) Let $U^{|q,\overline{q}}$ and $V^{r|s,\overline{s}}$ be two complex super open sets, $\varphi : U \to V$ a $\mathcal{C}^\infty$-map and $\Phi : \mathcal{D}\left(V^{|s,\overline{s}}\right) \to \mathcal{D}\left(U^{|q,\overline{q}}\right)$ a unital graded algebra homomorphism respecting $\overline{}$. Then the pair (φ, Φ) is called a super morphism from $U^{|q,\overline{q}}$ to $V^{|s,\overline{s}}$ if and only if

$$(\Phi(f))^{\#} = f^{\#} \circ \varphi$$

for all $f \in \mathcal{D}\left(V^{|s,\overline{s}}\right)$. In this case φ is called the body of (φ, Φ) . (φ, Φ) is called holomorphic if and only if φ is holomorphic and

$$\Phi\left(\mathcal{O}\left(V^{|s,\overline{s}}\right)\right) \subset \mathcal{O}\left(U^{|q,\overline{q}}\right) .$$

From now on let $n \in \mathbb{N}$ and $\mathcal{P} := \Lambda\left(\mathbb{R}^n\right) = \mathcal{D}\left(\mathbb{R}^{0|n}\right)$ with real odd coordinate functions $\alpha_1, \ldots, \alpha_n$. Then clearly $\mathcal{P}^{\mathbb{C}} = \Lambda\left(\mathbb{C}^n\right)$. As in the real case we have the wider class of $\mathcal{P}$-super morphisms between super open sets.

Definition 2.36 (complex $\mathcal{P}$-super morphisms) *Let $U^{|q,\overline{q}}$ and $V^{r|s,\overline{s}}$ be two complex super open sets, $\varphi : U \to V$ a $\mathcal{C}^\infty$-map and $\Phi : \mathcal{D}\left(V^{|s,\overline{s}}\right) \boxtimes \mathcal{P}^{\mathbb{C}} \to \mathcal{D}\left(U^{|q,\overline{q}}\right) \boxtimes \mathcal{P}^{\mathbb{C}}$ a unital graded algebra homomorphism respecting $\overline{}$ such that $\Phi|_{\mathcal{P}^{\mathbb{C}}} = id$. Then the pair (φ, Φ) is called a $\mathcal{P}$-super morphism from $U^{|q,\overline{q}}$ to $V^{|s,\overline{s}}$ if and only if*

$$(\Phi(f))^{\#} = f^{\#} \circ \varphi$$

for all $f \in \mathcal{D}\left(V^{|s,\overline{s}}\right)$. In this case again φ is called the body of (φ, Φ) . (φ, Φ) is called holomorphic if and only if φ is holomorphic and

$$\Phi\left(\mathcal{O}\left(V^{|s,\overline{s}}\right)\right) \subset \mathcal{O}\left(U^{|q,\overline{q}}\right) \boxtimes \mathcal{P}^{\mathbb{C}} .$$

Again the set of all complex super open sets together with $\mathcal{P}$-super morphisms forms a category, where the composition of two $\mathcal{P}$-super morphisms (φ, Φ) from $U^{|q,\overline{q}}$ to $V^{|s,\overline{s}}$ and (ψ, Ψ) from $V^{|s,\overline{s}}$ to $W^{|u,\overline{u}}$ is again defined as

$$(\varphi, \Phi) \circ (\psi, \Psi) := (\varphi \circ \psi, \Psi \circ \Phi) ,$$

and $(\mathrm{id}, \mathrm{id})$ is the identity morphism from a complex super open set $U^{|q,\overline{q}}$ to itself. Clearly if both (φ, Φ) and (ψ, Ψ) are holomorphic then so is $(\varphi, \Phi) \circ (\psi, \Psi)$.

Fortunately by lemma 2.10 each (p, q)-dimensional complex super open set can be regarded as a $(2p, 2q)$-dimensional real super open set. Given two complex super open sets $U^{|q,\overline{q}}$ and $V^{|s,\overline{s}}$ and a $\mathcal{P}$-super morphism (φ, Φ) from $U^{|q,\overline{q}}$ to $V^{|s,\overline{s}}$, then by theorem 2.8 $\left(\varphi, \Phi|_{\mathcal{D}\left(V^{|2s}\right)}\right)$ is a $\mathcal{P}$ - super morphism from $U^{|2q}$ to $V^{|2s}$. So we obtain a whole functor from the category of complex super open sets together with holomorphic $\mathcal{P}$-super morphisms to the category of real super open sets together with $\mathcal{P}$ - super morphisms forgetting about the 'complex structure' .

Therefore on a (p, q)-dimensional complex super open set regarded as a real $(2p, 2q)$-dimensional super open set we have the real even coordinate functions

$$x_k := \mathrm{Re}\ z_k = \frac{z_k + \overline{z_k}}{2} ,$$

$$y_k := \mathrm{Im}\ z_k = -i\, \frac{z_k - \overline{z_k}}{2} ,$$

$k = 1, \ldots, p$, and the real odd coordinate functions

$$\xi_l := \mathrm{Re}\ \zeta_l = \frac{\zeta_l - i\overline{\zeta_l}}{2} ,$$

and

$$\eta_l := \mathrm{Im}\ \zeta_l = \frac{-i\zeta_l + \overline{\zeta_l}}{2} \,,$$

$l = 1, \ldots, q$.

Theorem 2.37 *Let $U^{|q,\overline{q}}$ and $V^{|s,\overline{s}}$ be two complex super open sets. Let $w_1, \ldots, w_r$ and $\vartheta_1, \ldots, \vartheta_s$ be the coordinate functions on $V^{|s,\overline{s}}$.*

(i) Let $\Phi : \mathcal{D}\left(V^{|s,\overline{s}}\right) \boxtimes \mathcal{P}^{\mathbb{C}} \to \mathcal{D}\left(U^{|q,\overline{q}}\right) \boxtimes \mathcal{P}^{\mathbb{C}}$ be a unital graded algebra homomorphism respecting $\overline{}$. Then Φ is continuous, and there exists a unique $\mathcal{C}^\infty$-map $\varphi : U \to V$ such that (φ, Φ) is a $\mathcal{P}$-super morphism from $U^{|q,\overline{q}}$ to $V^{|s,\overline{s}}$. Let $f_k := \Phi(w_k) \in \left(\mathcal{D}\left(U^{|q,\overline{q}}\right) \boxtimes \mathcal{P}^{\mathbb{C}}\right)_0$, $k = 1, \ldots, r$, and $\lambda_l := \Phi(\vartheta_l) \in \left(\mathcal{D}\left(U^{|q,\overline{q}}\right) \boxtimes \mathcal{P}^{\mathbb{C}}\right)_1$, $l = 1, \ldots, s$. Then $\left(f_1^\#, \ldots, f_r^\#\right)(\mathbf{z}) \in V$ for all $\mathbf{z} \in U$, $\varphi = \left(f_1^\#, \ldots, f_r^\#\right)$, and for all

$$h = \sum_{S,T \in \wp(s)} h_{S,T}\ \vartheta^S \overline{\vartheta}^T \in \mathcal{D}\left(V^{|s,\overline{s}}\right) \hookrightarrow \mathcal{D}\left(V^{|s,\overline{s}}\right) \boxtimes \mathcal{P}^{\mathbb{C}}$$

we have

$$\begin{aligned} \Phi(h) \quad = \quad & \sum_{S,T \in \wp(s)} \sum_{\mathbf{m},\mathbf{n} \in \mathbf{N}^r} \frac{1}{\mathbf{m}!\mathbf{n}!} \left(\left(\partial^{\mathbf{m}} \overline{\partial}^{\mathbf{n}} h_{S,T}\right) \circ \varphi\right) \times \\ & \times \left(f_1 - f_1^\#, \ldots, f_r - f_r^\#\right)^{\mathbf{m}} \left(\overline{f_1} - \overline{f_1^\#}, \ldots, \overline{f_r} - \overline{f_r^\#}\right)^{\mathbf{n}} \lambda^S \overline{\lambda}^T \end{aligned} \tag{2.3}$$

in multi-index language, where we set $\lambda^S := \lambda_{t_1} \cdots \lambda_{t_m}$ and $\overline{\lambda}^S := \overline{\lambda_{t_1}} \cdots \overline{\lambda_{t_m}}$ for all $S = \{t_1, \ldots, t_m\} \in \wp(s)$, $1 \leq t_1 < \cdots < t_m \leq s$.

(ii) Conversely let $f_1, \ldots, f_r \in \left(\mathcal{D}\left(U^{|q,\overline{q}}\right) \boxtimes \mathcal{P}^{\mathbb{C}}\right)_0$ and $\lambda_1, \ldots, \lambda_s \in \left(\mathcal{D}\left(U^{|q,\overline{q}}\right) \boxtimes \mathcal{P}^{\mathbb{C}}\right)_1$ such that $\left(f_1^\#, \ldots, f_r^\#\right)(\mathbf{z}) \in V$ for all $\mathbf{z} \in U$. Then there exists a unique unital graded algebra homomorphism $\Phi : \mathcal{D}\left(V^{|s,\overline{s}}\right) \boxtimes \mathcal{P}^{\mathbb{C}} \to \mathcal{D}\left(U^{|q,\overline{q}}\right) \boxtimes \mathcal{P}^{\mathbb{C}}$ respecting $\overline{}$ given by $\mathcal{P}^{\mathbb{C}}$-linear extension of formula 2.3 such that $\Phi|_{\mathcal{P}^{\mathbb{C}}} = id$ and $\Phi(w_k) = f_k$, $k = 1, \ldots, r$, and $\Phi(\vartheta_l) = \lambda_l$, $l = 1, \ldots, s$.

(iii) Let $\Phi : \mathcal{D}\left(V^{|s,\overline{s}}\right) \boxtimes \mathcal{P}^{\mathbb{C}} \to \mathcal{D}\left(U^{|q,\overline{q}}\right) \boxtimes \mathcal{P}^{\mathbb{C}}$ be given by $\mathcal{P}^{\mathbb{C}}$-linear extension of formula 2.3 . Then $\Phi\left(\mathcal{O}\left(V^{|s,\overline{s}}\right)\right) \subset \mathcal{O}\left(U^{|q,\overline{q}}\right) \boxtimes \mathcal{P}$ if and only if $f_1, \ldots, f_r, \lambda_1, \ldots, \lambda_s \in \mathcal{O}\left(U^{|q,\overline{q}}\right) \boxtimes \mathcal{P}^{\mathbb{C}}$ if and only if (φ, Φ) is holomorphic.

Proof: (i) Since Φ respects $\overline{}$ and $\mathcal{D}\left(U^{|q,\overline{q}}\right) \boxtimes \mathcal{P}^{\mathbb{C}} \simeq \mathcal{D}\left(U^{|2q+n}\right)^{\mathbb{C}}$ the continuity of Φ and the existence and uniqueness of φ follow from theorem 2.8 (iv) and 2.12 (i) . The rest can be proven by the same calculations as in the proof of theorem 2.12 (i) using the fact that for all $\mathbf{w}_0 \in V$ there exist

$\Delta_{\mathbf{m},\mathbf{o}} \in \mathcal{C}^\infty(V)^\mathbb{C}$, $\mathbf{m},\mathbf{o} \in \mathbb{N}^r$, $|\mathbf{m}| = s+1$, $|\mathbf{o}| = s$, and $\Sigma_{\mathbf{m},\mathbf{o}} \in \mathcal{C}^\infty(V)^\mathbb{C}$, $\mathbf{m},\mathbf{o} \in \mathbb{N}^r$, $|\mathbf{m}| = s$, $|\mathbf{o}| = s+1$ such that

$$\begin{aligned} h &= \sum_{\mathbf{m},\mathbf{o}\in\mathbb{N}^r} \frac{1}{\mathbf{m}!\mathbf{o}!} \partial^\mathbf{m}\overline{\partial}^\mathbf{o} h\left(\mathbf{w}_0\right)\left(\mathbf{w}-\mathbf{w}_0\right)^\mathbf{m}\left(\overline{\mathbf{w}}-\overline{\mathbf{w}_0}\right)^\mathbf{o} \\ &\quad + \sum_{\mathbf{m},\mathbf{o}\in\mathbb{N}^r\,,\,|\mathbf{m}|=s+1\,,\,|\mathbf{o}|=s} \left(\mathbf{w}-\mathbf{w}_0\right)^\mathbf{m}\left(\overline{\mathbf{w}}-\overline{\mathbf{w}_0}\right)^\mathbf{o} \Delta_{\mathbf{m},\mathbf{o}} \\ &\quad + \sum_{\mathbf{m},\mathbf{o}\in\mathbb{N}^r\,,\,|\mathbf{m}|=s\,,\,|\mathbf{o}|=s+1} \left(\mathbf{w}-\mathbf{w}_0\right)^\mathbf{m}\left(\overline{\mathbf{w}}-\overline{\mathbf{w}_0}\right)^\mathbf{o} \Sigma_{\mathbf{m},\mathbf{o}} \,.\, \square \end{aligned}$$

(ii) Let x_k , y_k , $k = 1,\ldots,r$, and ξ_l , η_l , $l = 1,\ldots,s$, be the real coordinate functions on $V^{|s,\overline{s}}$. Since Φ is a unital graded algebra homomorphism respecting $^-$ by theorem 2.8 $\Phi\left(w_k\right) = f_k$, $k = 1,\ldots,r$, $\Phi\left(\vartheta_l\right) = \lambda_l$, $l = 1,\ldots,s$, and $\Phi|_{\mathcal{P}^\mathbb{C}} = \mathrm{id}$ is equivalent to

$$\begin{aligned} \Phi_\mathbb{R}\left(x_k\right) &= \mathrm{Re}\ f_k = \frac{f_k - \overline{f_k}}{2}\,, \\ \Phi_\mathbb{R}\left(y_k\right) &= \mathrm{Im}\ f_k = -i\,\frac{f_k - \overline{f_k}}{2} \in \left(\mathcal{D}\left(U^{|q,\overline{q}}\right)_\mathbb{R}\right)_0 \end{aligned}$$

for all $k = 1,\ldots,r$,

$$\Phi_\mathbb{R}\left(\alpha_t\right) = \alpha_t$$

for all $t = 1,\ldots,n$ and

$$\begin{aligned} \Phi_\mathbb{R}\left(\xi_l\right) &= \mathrm{Re}\ \lambda_l = \frac{\lambda_l - i\overline{\lambda_l}}{2}\,, \\ \Phi_\mathbb{R}\left(\eta_l\right) &= \mathrm{Im}\ \lambda_l = \frac{-i\lambda + \overline{\lambda_l}}{2} \in \left(\mathcal{D}\left(U^{|q,\overline{q}}\right)_\mathbb{R}\right)_1 \end{aligned}$$

for all $l = 1,\ldots,s$. Therefore uniqueness and existence of Φ follow by theorem 2.8 (iii) and 2.12 (ii) . $\square$

(iii) Assume $\Phi\left(\mathcal{O}\left(V^{|s,\overline{s}}\right)\right) \subset \mathcal{O}\left(U^{|q,\overline{q}}\right) \boxtimes \mathcal{P}^\mathbb{C}$. Then since $w_1,\ldots,w_r,\vartheta_1,\ldots,\vartheta_s \in \mathcal{O}\left(V^{|s,\overline{s}}\right)$ we have $f_k = \Phi\left(z_k\right) \in \mathcal{O}\left(U^{|q,\overline{q}}\right) \boxtimes \mathcal{P}^\mathbb{C}$ and $\lambda_l = \Phi\left(\vartheta_l\right) \in \mathcal{O}\left(U^{|q,\overline{q}}\right) \boxtimes \mathcal{P}^\mathbb{C}$, $k = 1,\ldots,r$, $l = 1,\ldots,s$.
Now assume all $f_k, \lambda_l \in \mathcal{O}\left(U^{|q,\overline{q}}\right) \boxtimes \mathcal{P}^\mathbb{C}$, $k = 1,\ldots,r$, $l = 1,\ldots,s$. Then we see that $\varphi = \left(f_1^\#,\ldots,f_r^\#\right)$ is holomorphic, and given some $h = \sum_{S\in\wp(s)} h_S\vartheta^S \in \mathcal{O}\left(V^{|s,\overline{s}}\right)$, all $h_S \in \mathcal{O}(V)$, $S \in \wp(s)$, by formula 2.3 we get

$$\begin{aligned} \Phi(h) &= \sum_{S\in\wp(s)} \sum_{\mathbf{m}\in\mathbb{N}^r} \frac{1}{\mathbf{m}!}\left(\left(\partial^\mathbf{m} h_S\right)\circ\varphi\right)\left(f_1 - f_1^\#,\ldots,f_r - f_r^\#\right)^\mathbf{m} \lambda^S \\ &\quad \in \mathcal{O}\left(U^{|q,\overline{q}}\right) \boxtimes \mathcal{P}^\mathbb{C}\,, \end{aligned}$$

and so (φ, Φ) is holomorphic.
Finally assume (φ, Φ) is holomorphic. Then by definition
$\Phi\left(\mathcal{O}\left(V^{|s,\overline{s}}\right)\right) \subset \mathcal{O}\left(U^{|q,\overline{q}}\right) \boxtimes \mathcal{P}^{\mathbb{C}}$. $\square$

By theorem 2.37 there is a bijection between the set of all $\mathcal{P}$-super morphisms (φ, Φ) from $U^{|q,\overline{q}}$ to $V^{|s,\overline{s}}$, the set of all unital graded algebra homomorphisms $\Phi : \mathcal{D}\left(V^{|s,\overline{s}}\right) \to \mathcal{D}\left(U^{|q,\overline{q}}\right) \boxtimes \mathcal{P}^{\mathbb{C}}$ and the set of all tuples

$$(f_1, \dots, f_r,\, \lambda_1, \dots, \lambda_s) \in \left(\mathcal{D}\left(U^{|q,\overline{q}}\right) \boxtimes \mathcal{P}^{\mathbb{C}}\right)_0^r \times \left(\mathcal{D}\left(U^{|q,\overline{q}}\right) \boxtimes \mathcal{P}^{\mathbb{C}}\right)_1^s$$

such that the image of U under $\left(f_1^{\#}, \dots, f_r^{\#}\right)$ lies in V . So again we will identify a $\mathcal{P}$-super morphism (φ, Φ) from $U^{|q,\overline{q}}$ to $V^{|s,\overline{s}}$ with its 'defining tuple'

$$(f_1, \dots, f_r,\, \lambda_1, \dots, \lambda_s) \in \left(\mathcal{D}\left(U^{|q,\overline{q}}\right) \boxtimes \mathcal{P}^{\mathbb{C}}\right)_0^r \times \left(\mathcal{D}\left(U^{|q,\overline{q}}\right) \boxtimes \mathcal{P}^{\mathbb{C}}\right)_1^s ,$$

$U^{|q,\overline{q}}$ and $V^{|s,\overline{s}}$ regarded as real super open sets of dimensions $(2p, 2q)$ resp. $(2r, 2s)$, (φ, Φ) has the defining tuple

$$\begin{aligned}
&(\mathrm{Re}\ f_1\,, \dots,\ \mathrm{Re}\ f_p\,,\ \mathrm{Im}\ f_1\,, \dots,\ \mathrm{Im}\ f_p\,,\ \mathrm{Re}\ \lambda_1\,, \dots,\ \mathrm{Re}\ \lambda_q\,,\\
&\ \mathrm{Im}\ \lambda_1\,, \dots,\ \mathrm{Im}\ \lambda_q)\\
&\quad \in \left(\mathcal{D}\left(U^{|2q}\right) \boxtimes \mathcal{P}\right)_0^{2r} \times \left(\mathcal{D}\left(U^{|2q}\right) \boxtimes \mathcal{P}\right)_1^{2s} .
\end{aligned}$$

For all $h \in \mathcal{D}\left(V^{|s,\overline{s}}\right) \boxtimes \mathcal{P}^{\mathbb{C}}$ we write

$$h\left(f_1, \dots, f_r,\, \lambda_1, \dots, \lambda_s\right) := \Phi(h) \in \mathcal{D}\left(U^{|q,\overline{q}}\right) \boxtimes \mathcal{P}^{\mathbb{C}} ,$$

regarding Φ as a 'plugging in' homomorphism.

Clearly in the category of complex super open sets together with holomorphic $\mathcal{P}$-super morphisms we have a cross product: If $U^{|q,\overline{q}}$ and $V^{|s,\overline{s}}$ are complex super open sets then the cross product of $U^{|q,\overline{q}}$ and $V^{|s,\overline{s}}$ is defined as

$$U^{|q,\overline{q}} \times V^{|s,\overline{s}} := (U \times V)^{|q+s,\overline{q+s}} ,$$

which is up to the additional complex structure equal to the real cross product. From classical analysis we know that

$$\mathcal{O}\left(U^{|q,\overline{q}} \times V^{|s,\overline{s}}\right) = \mathcal{O}\left(U^{|q,\overline{q}}\right) \hat{\boxtimes} \mathcal{O}\left(V^{|s,\overline{s}}\right) .$$

The canonical projections (pr_1, C_1) and (pr_2, C_2) from $U^{|q,\overline{q}} \times V^{|s,\overline{s}}$ to $U^{|q,\overline{q}}$ resp. $V^{|s,\overline{s}}$ turn out to be holomorphic. If $W^{|u,\overline{u}}$ is a third complex super open set and (φ_1, Φ_1) and (φ_2, Φ_2) are $\mathcal{P}$-super morphisms from $W^{|u,\overline{u}}$ to $U^{|q,\overline{q}}$ resp. $V^{|s,\overline{s}}$ with defining tuples

$$(f_1, \ldots, f_p, \lambda_1, \ldots, \lambda_q) \in \left(\mathcal{D}\left(W^{|u,\overline{u}}\right) \boxtimes \mathcal{P}^{\mathbb{C}}\right)_0^p \times \left(\mathcal{D}\left(U^{|q,\overline{q}}\right) \boxtimes \mathcal{P}^{\mathbb{C}}\right)_1^q ,$$

resp.

$$(g_1, \ldots, g_r, \mu_1, \ldots, \mu_s) \in \left(\mathcal{D}\left(W^{|u,\overline{u}}\right) \boxtimes \mathcal{P}^{\mathbb{C}}\right)_0^r \times \left(\mathcal{D}\left(U^{|q,\overline{q}}\right) \boxtimes \mathcal{P}^{\mathbb{C}}\right)_1^s ,$$

then the unique $\mathcal{P}$-super morphism Ψ from $W^{|u,\overline{u}}$ to $U^{|q,\overline{q}} \times V^{|s,\overline{s}}$ such that

$$(\mathrm{pr}_1, C_1 \otimes \mathrm{id}) \circ (\psi, \Psi) = (\varphi_1, \Phi_1)$$

and

$$(\mathrm{pr}_2, C_2 \otimes \mathrm{id}) \circ (\psi, \Psi) = (\varphi_2, \Phi_2)$$

has the defining tuple

$$\begin{aligned} &(f_1, \ldots, f_p, g_1, \ldots, g_r, \lambda_1, \ldots, \lambda_q, \mu_1, \ldots, \mu_s) \\ &\in \left(\mathcal{D}\left(W^{|u,\overline{u}}\right) \boxtimes \mathcal{P}^{\mathbb{C}}\right)_0^{p+r} \times \left(\mathcal{D}\left(U^{|q,\overline{q}}\right) \boxtimes \mathcal{P}^{\mathbb{C}}\right)_1^{q+s} , \end{aligned}$$

and so it is holomorphic if and only if (φ_1, Φ_1) and (φ_2, Φ_2) are holomorphic.

Clearly if (φ, Φ) is a holomorphic $\mathcal{P}$-super morphism from $U^{|q,\overline{q}}$ to $V^{|s,\overline{s}}$ and $V' \subset V$ open such that $\varphi(U) \subset V'$ then the unique $\mathcal{P}$-super morphism (φ', Φ') from $U^{|q,\overline{q}}$ to $V'^{|s,\overline{s}}$ such that $(\varphi, \Phi) = (c, |_{V'}) \circ (\varphi', \Phi')$, see theorem 2.14 in the real case, section 2.2 , is again holomorphic. We can deduce it from the fact that if we denote the defining tuple of (φ, Φ) by $(f_1, \ldots, f_r, \lambda_1, \ldots, \lambda_s)$ then $\varphi' = \varphi|_{\varphi^{-1}(V')}$ and the defining tuple of (φ', Φ') is just

$$\left(f_1|_{\varphi^{-1}(V')}, \ldots, f_r|_{\varphi^{-1}(V')}, \lambda_1|_{\varphi^{-1}(V')}, \ldots, \lambda_s|_{\varphi^{-1}(V')}\right) .$$

Let $U^{|q,\overline{q}}$ be a complex super open set, and define the continuous linear maps

$$\begin{aligned} \partial_{k|} : \mathcal{D}\left(U^{|q,\overline{q}}\right) &\to \mathcal{D}\left(U^{|q,\overline{q}}\right) , \\ f = \sum_{S,T \in \wp(q)} f_{S,T}\ \zeta^S \overline{\zeta}^T &\mapsto \sum_{S,T \in \wp(q)} (\partial_k f_{S,T})\, \zeta^S \overline{\zeta}^T , \end{aligned}$$

$$\begin{aligned}\overline{\partial}_{k|} : \mathcal{D}\left(U^{|q,\overline{q}}\right) &\to \mathcal{D}\left(U^{|q,\overline{q}}\right) , \\ f &\mapsto \sum_{S,T\in\wp(q)} \left(\overline{\partial}_k f_{S,T}\right) \zeta^S\overline{\zeta}^T ,\end{aligned}$$

$k = 1, \dots, p$,

$$\begin{aligned}\partial_{|l} : \mathcal{D}\left(U^{|q,\overline{q}}\right) &\to \mathcal{D}\left(U^{|q,\overline{q}}\right) , \\ f &\mapsto \sum_{S,T\in\wp(q),\, l\notin S} (-1)^{|S<l|} f_{S\cup\{l\},T}\, \zeta^S\overline{\zeta}^T\end{aligned}$$

and

$$\begin{aligned}\overline{\partial}_{|l} : \mathcal{D}\left(U^{|q,\overline{q}}\right) &\to \mathcal{D}\left(U^{|q,\overline{q}}\right) , \\ f &\mapsto \sum_{S,T\in\wp(q),\, l\notin T} (-1)^{|S|+|T<l|} f_{S,T\cup\{l\}}\, \zeta^S\overline{\zeta}^T ,\end{aligned}$$

$l = 1, \dots, q$. Clearly again $\left(\partial_{i|} f\right)^{\#} = \partial_i \left(f^{\#}\right)$ and $\left(\overline{\partial}_{i|} f\right)^{\#} = \overline{\partial}_i \left(f^{\#}\right)$,

$$\partial_{i|}\mathcal{D}\left(U^{|q}\right)_0 , \overline{\partial}_{i|}\mathcal{D}\left(U^{|q}\right)_0 , \partial_{|j}\mathcal{D}\left(U^{|q}\right)_1 , \overline{\partial}_{|j}\mathcal{D}\left(U^{|q}\right)_1 \subset \mathcal{D}\left(U^{|q}\right)_0 ,$$

$$\partial_{i|}\mathcal{D}\left(U^{|q}\right)_1 , \overline{\partial}_{i|}\mathcal{D}\left(U^{|q}\right)_1 , \partial_{|j}\mathcal{D}\left(U^{|q}\right)_0 , \overline{\partial}_{|j}\mathcal{D}\left(U^{|q}\right)_0 \subset \mathcal{D}\left(U^{|q}\right)_1$$

for all $i = 1, \dots, p$ and $j = 1, \dots, q$, and again we have a super product rule:

$$\partial_{i|}(fg) = \left(\partial_{i|} f\right) g + f \left(\partial_{i|} g\right) ,$$

$$\overline{\partial}_{i|}(fg) = \left(\overline{\partial}_{i|} f\right) g + f \left(\overline{\partial}_{i|} g\right) ,$$

$$\partial_{|j}(fg) = \left(\partial_{i|} f\right) g + (-1)^{\dot{f}} f \left(\partial_{i|} g\right)$$

and

$$\overline{\partial}_{|j}(fg) = \left(\overline{\partial}_{i|} f\right) g + (-1)^{\dot{f}} f \left(\overline{\partial}_{i|} g\right)$$

for all $i = 1, \dots, p$, $j = 1, \dots, q$ and $f, g \in \mathcal{D}\left(U^{|q}\right)$, f homogeneous. So all $\partial_{i|}$, $\overline{\partial}_{i|}$, $\partial_{|j}$ and $\overline{\partial}_{|j}$ are super derivations on $\mathcal{D}\left(U^{|q}\right)$, and we call them the partial derivatives with respect to the holomorphic coordinate functions z_j resp. ζ_j . We can extend them to continuous linear maps from $\mathcal{D}\left(U^{|q,\overline{q}}\right) \boxtimes \mathcal{P}^{\mathbb{C}}$ to $\mathcal{D}\left(U^{|q,\overline{q}}\right) \boxtimes \mathcal{P}^{\mathbb{C}}$ by right-$\mathcal{P}^{\mathbb{C}}$-linear extension. If we denote the unique $\mathbb{C}$-linear extensions of the real derivatives on $U^{|q,\overline{q}}$ regarded as a real $(2p, 2q)$-dimensional super open set by ∂_{x_k} , ∂_{y_k} , ∂_{ξ_l} resp. ∂_{η_l} , $k = 1, \ldots, p$, $l = 1, \ldots, q$, then we have the following basic properties, which can be proven by straight forward computation:

(i)

$$\begin{aligned}
\partial_{x_k} &= \partial_{k|} + \overline{\partial}_{k|} \, , \\
\partial_{y_k} &= i \left(\partial_{k|} - \overline{\partial}_{k|} \right) \, , \\
\partial_{\xi_l} &= \partial_{|l} + i \overline{\partial}_{|l} \, , \\
\partial_{\eta_l} &= i \partial_{|l} + \overline{\partial}_{|l} \, ,
\end{aligned}$$

(ii)

$$\begin{aligned}
\partial_{k|} &= \frac{\partial_{x_k} - i \partial_{y_k}}{2} \, , \\
\overline{\partial}_{k|} &= \frac{\partial_{x_k} + i \partial_{y_k}}{2} \, , \\
\partial_{|l} &= \frac{\partial_{\xi_l} - i \partial_{\eta_l}}{2} \, , \\
\overline{\partial}_{|l} &= \frac{-i \partial_{\xi_l} + \partial_{\eta_l}}{2} \, ,
\end{aligned}$$

(iii)

$$\begin{aligned}
\overline{\partial}_{k|} \overline{f} &= \overline{\partial_{k|} f} \, , \\
\overline{\partial}_{|l} \overline{f} &= (-1)^{\dot{f}+1} \, \overline{\partial_{|l} f} \, ,
\end{aligned}$$

(iv)

$$\begin{aligned}
\partial_{x_k} \overline{f} &= \overline{\partial_{x_k} f} \, , \\
\partial_{y_k} \overline{f} &= \overline{\partial_{y_k} f} \, , \\
\partial_{\xi_l} \overline{f} &= i \, (-1)^{\dot{f}+1} \, \overline{\partial_{\xi_l} f} \, , \\
\partial_{\eta_l} \overline{f} &= i \, (-1)^{\dot{f}+1} \, \overline{\partial_{\eta_l} f} \, ,
\end{aligned}$$

for all $k = 1, \ldots, p$, $l = 1, \ldots, q$ and homogeneous $f \in \mathcal{D}\left(U^{|q,\overline{q}}\right) \boxtimes \mathcal{P}^{\mathbb{C}}$.

Clearly if $f \in \mathcal{D}\left(U^{|q,\overline{q}}\right)$ then f is holomorphic if and only if $\overline{\partial}_{k|} f = \overline{\partial}_{|l} f = 0$ for all $k = 1, \ldots, p$ and $l = 1, \ldots, q$. $\mathcal{O}\left(U^{|q,\overline{q}}\right)$ is closed under all $\partial_{k|}$, $k = 1, \ldots, p$, and $\partial_{|l}$, $l = 1, \ldots, q$.

Definition 2.38 (complex and holomorphic super Jacobian) *Let $U^{|q,\overline{q}}$ and $V^{|s,\overline{s}}$ be two complex super open sets and (φ, Φ) a $\mathcal{P}$-super morphism from $U^{|q,\overline{q}}$ to $V^{|s,\overline{s}}$ with defining tuple $(f_1, \ldots, f_r, \lambda_1, \ldots, \lambda_s)$.*

(i) The even $(2r,2s)\times(2p,2q)$ - graded matrix

$$D_{\mathbb{C}}(\varphi,\Phi) \quad := \quad \left(\begin{array}{cc|cc} \left(\partial_{k|}f_t\right) & \left(\overline{\partial}_{k|}f_t\right) & -\left(\partial_{|l}f_t\right) & -\left(\overline{\partial}_{|l}f_t\right) \\ \hline \left(\partial_{k|}\overline{f_t}\right) & \left(\overline{\partial}_{k|}\overline{f_t}\right) & -\left(\partial_{|l}\overline{f_t}\right) & -\left(\overline{\partial}_{|l}\overline{f_t}\right) \\ \hline \left(\partial_{k|}\lambda_u\right) & \left(\overline{\partial}_{k|}\lambda_u\right) & \left(\partial_{|l}\lambda_u\right) & \left(\overline{\partial}_{|l}\lambda_u\right) \\ \hline \left(\partial_{k|}\overline{\lambda_u}\right) & \left(\overline{\partial}_{k|}\overline{\lambda_u}\right) & \left(\partial_{|l}\overline{\lambda_u}\right) & \left(\overline{\partial}_{|l}\overline{\lambda_u}\right) \end{array}\right)$$
$$\in \left(\mathcal{D}\left(U^{|q,\overline{q}}\right)\boxtimes\mathcal{P}^{\mathbb{C}}\right)_0^{(2r|2s)\times(2p|2q)} ,$$

where $k\in\{1,\ldots,p\}$, $l\in\{1,\ldots,q\}$, $t\in\{1,\ldots,r\}$ and $u\in\{1,\ldots,s\}$, is called the complex super Jacobian of (Φ,φ) .

(ii) If (φ,Φ) is holomorphic then the even $(r|s)\times(p|q)$ - graded matrix

$$D_{\mathrm{hol}}(\varphi,\Phi) \quad := \quad \left(\begin{array}{c|c} \left(\partial_{k|}f_t\right)_{t\in\{1,\ldots,r\},k\in\{1,\ldots,p\}} & -\left(\partial_{|l}f_t\right)_{t\in\{1,\ldots,r\},l\in\{1,\ldots,q\}} \\ \hline \left(\partial_{k|}\lambda_u\right)_{u\in\{1,\ldots,s\},k\in\{1,\ldots,p\}} & \left(\partial_{|l}\lambda_u\right)_{u\in\{1,\ldots,s\},l\in\{1,\ldots,q\}} \end{array}\right)$$
$$\in \left(\mathcal{O}\left(U^{|q,\overline{q}}\right)\boxtimes\mathcal{P}^{\mathbb{C}}\right)_0^{(r|s)\times(p|q)}$$

is called the holomorphic super Jacobian of (φ,Φ) .

Lemma 2.39 *Let $U^{|q,\overline{q}}$, $V^{|s,\overline{s}}$ and $W^{|u,\overline{u}}$ be complex super open sets, (φ,Φ) a $\mathcal{P}$-super morphism from $U^{|q,\overline{q}}$ to $V^{|s,\overline{s}}$ and (ψ,Ψ) a $\mathcal{P}$-super morphism from $V^{|s,\overline{s}}$ to $W^{|u,\overline{u}}$. For all $t,u\in\mathbb{N}$ define*

$$S_{t,u} := \left(\begin{array}{cc|cc} 1 & i\,1 & 0 & 0 \\ \hline 1 & -i\,1 & 0 & 0 \\ \hline 0 & 0 & 1 & i\,1 \\ \hline 0 & 0 & i\,1 & 1 \end{array}\right) \begin{array}{l} \}t \\ \}t \\ \}u \\ \}u \end{array} \in \mathbb{R}_0^{(2t,2u)\times(2t,2u)} .$$

(i)

$$D_{\mathbb{C}}(\varphi,\Phi) = S_{p,q}^{-1} D_{\mathbf{R}}(\varphi,\Phi) S_{r,s} ,$$

where $D_{\mathbf{R}}(\varphi,\Phi)$ denotes the real super Jacobian of (φ,Φ) .

(ii) The complex super Jacobian of $(\psi,\Psi)\circ(\varphi,\Phi)$ is precisely

$$\Phi\left(D_{\mathbb{C}}(\psi,\Psi)\right)\cdot D_{\mathbb{C}}(\varphi,\Phi) ,$$

where $\Phi\left(D_{\mathbb{C}}(\psi,\Psi)\right)$ is taken componentwise.

(iii) If $(p,q)=(r,s)$ and $\mathbf{z}_0\in U$ then $D_{\mathbb{C}}(\varphi,\Phi)(\mathbf{z}_0)$ is invertible if and only if $D_{\mathbf{R}}(\varphi,\Phi)(\mathbf{z}_0)$ is invertible.

(iv) If (φ, Φ) is a $\mathcal{P}$-super diffeomorphism then

$$D_{\mathbb{C}}\left((\varphi, \Phi)^{-1}\right) = \Phi^{-1}\left((D_{\mathbb{C}}(\varphi, \Phi))^{-1}\right),$$

where $\Phi^{-1}\left((D_{\mathbb{C}}(\varphi, \Phi))^{-1}\right)$ is taken componentwise.

(v) If (φ, Φ) is holomorphic then

$$D_{\mathbb{C}}(\varphi, \Phi) = \left(\begin{array}{c|c} \begin{array}{c|c} A & 0 \\ \hline 0 & \overline{A} \end{array} & \begin{array}{c|c} \beta & 0 \\ \hline 0 & -\overline{\beta} \end{array} \\ \hline \begin{array}{c|c} \gamma & 0 \\ \hline 0 & \overline{\gamma} \end{array} & \begin{array}{c|c} D & 0 \\ \hline 0 & \overline{D} \end{array} \end{array}\right), \tag{2.4}$$

where

$$D_{\mathrm{hol}}(\varphi, \Phi) = \left(\begin{array}{c|c} A & \beta \\ \hline \gamma & D \end{array}\right) \in \left(\mathcal{O}\left(U^{|q,\overline{q}}\right) \boxtimes \mathcal{P}^{\mathbb{C}}\right)_0^{(r|s)\times(p|q)}.$$

(vi) If (φ, Φ) and (ψ, Ψ) are holomorphic then the holomorphic super Jacobian of $(\psi, \Psi) \circ (\varphi, \Phi)$ is precisely

$$\Phi\left(D_{\mathrm{hol}}(\psi, \Psi)\right) \cdot D_{\mathrm{hol}}(\varphi, \Phi),$$

where again $\Phi\left(D_{\mathrm{hol}}(\psi, \Psi)\right)$ is taken componentwise.

(vii) If (φ, Φ) is holomorphic , $(p, q) = (r, s)$ and $\mathbf{z}_0 \in U$ then $D_{\mathrm{hol}}(\varphi, \Phi)(\mathbf{z}_0)$ is invertible if and only if $D_{\mathbb{C}}(\varphi, \Phi)(\mathbf{z}_0)$ is invertible.

(viii) If $(r, s) = (p, q)$ and (φ, Φ) is a holomorphic $\mathcal{P}$-super diffeomorphism then (φ, Φ) is biholomorphic, and

$$D_{\mathrm{hol}}\left((\varphi, \Phi)^{-1}\right) = \Phi^{-1}\left((D_{\mathrm{hol}}(\varphi, \Phi))^{-1}\right),$$

where $\Phi^{-1}\left((D_{\mathrm{hol}}(\varphi, \Phi))^{-1}\right)$ is taken componentwise.

Proof: (i) straight forward calculation.

(ii) combine (i) and lemma 2.18 (i) . □

(iii) trivial using (i) .

(iv) combine (i) and lemma 2.18 (ii) . □

(v) straight forward calculation.

(vi) combine (v) and lemma 2.18 (i) . $\square$

(vii) combine (v) and corollary 2.4 . $\square$

(viii) It suffices to show that $\Phi^{-1}\left(D_{\mathbb{C}}(\varphi,\Phi)^{-1}\right)$ is again of the form 2.4 , but this is an easy exercise using corollary 2.4 since Φ^{-1} respects $\overline{}$. $\square$

Corollary 2.40 (holomorphic super local inversion theorem) *Let $U^{|q,\overline{q}}$ and $V^{|q,\overline{q}}$ be two complex super open sets of dimension (p,q) , (φ,Φ) a holomorphic $\mathcal{P}$-super morphism from $U^{|q,\overline{q}}$ to $V^{|q,\overline{q}}$ and $\mathbf{z}_0 \in U$.*

(i) Let $D_{\text{hol}}(\varphi,\Phi)(\mathbf{z}_0) \in \left(\Lambda(\mathbb{C}^q) \boxtimes \mathcal{P}^{\mathbb{C}}\right)_0^{(p|q)\times(p|q)}$ be invertible, equivalently $D_{\text{hol}}(\varphi,\Phi)^{\#}(\mathbf{z}_0) \in \mathbb{C}_0^{(p|q)\times(p|q)}$ be invertible. Then there exists an open neighbourhood $W \subset V$ of $\varphi(\mathbf{z}_0)$ such that $\left(\varphi|_{\varphi^{-1}(W)}, \Phi_W\right)$ is a biholomorphic super morphism from $\varphi^{-1}(W)^{|q,\overline{q}}$ to $W^{|q,\overline{q}}$.

(ii) Let φ be bijective and $D(\varphi,\Phi)(\mathbf{z}_0)$, equivalently $D(\varphi,\Phi)^{\#}(\mathbf{z}_0)$, be invertible for all $\mathbf{z}_0 \in U$. Then (φ,Φ) is biholomorphic.

Proof: combine lemma 2.39 (iii) , (vii) and (viii) and theorem 2.19 . $\square$

Definition 2.41 (parametrized holomorphic super manifolds) *Let M be a p-dimensional holomorphic manifold and $q \in \mathbb{N}$. Let $\mathcal{S}$ be a sheaf of unital graded $\mathbb{C}$-algebras over M with involution $\overline{}$, $\mathcal{F}$ a sub graded sheaf of $\mathcal{S}$ and $^{\#} : \mathcal{S} \to (\mathcal{C}_M^\infty)^{\mathbb{C}}$ a sheaf homomorphism respecting $\overline{}$ such that the image of $\mathcal{F}$ under $^{\#}$ lies in $\mathcal{O}_M$.*

(i) The tuple $\mathcal{M} := \left(M, \mathcal{S}, \mathcal{F}, {}^{\#}\right)$ is called a (p,q)-dimensional holomorphic over $\mathcal{P}$ parametrized (or simply $\mathcal{P}$-) super manifold if and only if there exists a sheaf embedding $\mathcal{P}^{\mathbb{C}} \hookrightarrow \mathcal{F}$ respecting $\overline{}$, for all $x_0 \in M$ an open neighbourhood $U \subset M$ of x_0 and a sheaf isomorphism $\Phi : \mathcal{S}|_U \xrightarrow{\sim} \mathcal{D}\left(\diamondsuit^{|q,\overline{q}}\right)_U \boxtimes \mathcal{P}^{\mathbb{C}}$ respecting $\overline{}$ such that $\Phi|_{\mathcal{P}^{\mathbb{C}}} = id$,

$$\begin{array}{ccc} \mathcal{S}|_U & \xrightarrow{\Phi} & \mathcal{D}\left(\diamondsuit^{|q,\overline{q}}\right)_U \boxtimes \mathcal{P}^{\mathbb{C}} \\ \# \searrow & \% & \swarrow \# \\ & (\mathcal{C}_U^\infty)^{\mathbb{C}} & \end{array}$$

and the image of $\mathcal{F}|_U$ under Φ is precisely $\mathcal{O}\left(\diamondsuit^{|q,\overline{q}}\right)_U \boxtimes \mathcal{P}^{\mathbb{C}}$.
In this case $\mathcal{M}^{\#} := M$ is called the body of the $\mathcal{P}$-super manifold $\mathcal{M}$, $\mathcal{F}$ the structural sheaf of $\mathcal{M}$ and $^{\#}$ the body map of $\mathcal{S}$. We write $\mathcal{D}(\mathcal{M}) := \mathcal{S}(M)$ and $\mathcal{O}(\mathcal{M}) := \mathcal{F}(M)$. In the case where $n = 0$, equivalently $\mathcal{P} = \mathbb{R}$ we call $\mathcal{M}$ simply a holomorphic super manifold.

(ii) If $U \subset M$ is open then the tuple $\mathcal{U} := \left(U, \mathcal{S}|_U, \mathcal{F}_U, {}^\#\big|_U\right)$ is called an open sub $\mathcal{P}$-super manifold of $\mathcal{M}$. It is a (p,q)-dimensional holomorphic $\mathcal{P}$-super manifold itself.

(iii) Let $\mathcal{N} = \left(N, \mathcal{T}, \mathcal{G}, {}^\#\right)$ be another holomorphic $\mathcal{P}$-super manifold, $\varphi : M \to N$ a $\mathcal{C}^\infty$-map and $(\Phi_W)_{W \subset N \text{ open}}$ a family of unital graded algebra homomorphisms $\Phi_W : \mathcal{T}(W) \to \mathcal{S}\left(\varphi^{-1}(W)\right)$ respecting $^-$ such that for all $W' \subset W \subset N$ open

$$\begin{array}{ccc} \mathcal{T}(W) & \xrightarrow{\Phi_W} & \mathcal{S}\left(\varphi^{-1}(W)\right) \\ |_{W'} \downarrow & \% & \downarrow |_{\varphi^{-1}(W')} \\ \mathcal{T}(W') & \xrightarrow[\Phi_{W'}]{} & \mathcal{S}\left(\varphi^{-1}(W')\right) \end{array} .$$

Then the pair $\Phi := \left(\varphi, (\Phi_W)_{W \subset N \text{ open}}\right)$ is called a $\mathcal{P}$-super morphism from $\mathcal{M}$ to $\mathcal{N}$ if and only if for all $W \subset N$ open $\Phi_W|_{\mathcal{P}^\mathbb{C}} = id$ and

$$(\Phi_W(f))^\# = f^\# \circ \varphi|_{\varphi^{-1}(W)}$$

for all $f \in \mathcal{T}(W)$. In this case $\Phi^\# := \varphi$ is called the body of Φ . Φ is called holomorphic if and only if $\Phi_W\left(\mathcal{F}\left(\varphi^{-1}(W)\right)\right) \subset \mathcal{G}(W)$ for all $W \subset N$ open.

Again all holomorphic $\mathcal{P}$-super manifolds together with $\mathcal{P}$-super morphisms form a category. Given holomorphic $\mathcal{P}$-super manifolds $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{O}$ and $\mathcal{P}$-super morphisms Φ from $\mathcal{M}$ to $\mathcal{N}$ and Ψ from $\mathcal{N}$ to $\mathcal{O}$ the composition of Φ and Ψ is again given by

$$\Psi \circ \Phi := \left(\psi \circ \varphi, \left(\Phi_{\psi^{-1}(W)} \circ \Psi_W\right)_{W \subset O \text{ open}}\right) ,$$

and obviously if Φ and Ψ are holomorphic then so is $\Psi \circ \Phi$. Obviously by theorem 2.39 if Φ is a holomorphic $\mathcal{P}$-super diffeomorphism from $\mathcal{M}$ to $\mathcal{N}$ then it is biholomorphic.

By lemma 2.10 each holomorphic (p,q)-dimensional $\mathcal{P}$-super manifold can be regarded as a real $(2p,2q)$-dimensional $\mathcal{P}$-super manifold, and each $\mathcal{P}$-super morphism Ψ from $\mathcal{M}$ to $\mathcal{N}$ being two holomorphic $\mathcal{P}$-super manifolds can also be regarded as a real $\mathcal{P}$-super morphism. So we obtain a whole functor from the category of holomorphic $\mathcal{P}$-super manifolds together with holomorphic $\mathcal{P}$-super morphisms to the category of real $\mathcal{P}$-super manifolds together with $\mathcal{P}$-super morphisms fogetting about the 'holomorphic structure'.

Again in the category of holomorphic $\mathcal{P}$-super manifolds we have a $\mathcal{P}$-cross product: Given two holomorphic super manifolds $\mathcal{M} = \left(M, \mathcal{S}, \mathcal{F}, {}^\#\right)$ and $\mathcal{N} = \left(N, \mathcal{T}, \mathcal{G}, {}^\#\right)$ we have

$$\mathcal{M} \times_{\mathcal{P}} \mathcal{N} = \left(M \times N, \left(\mathrm{pr}_1^* \mathcal{S} \hat{\boxtimes} \mathrm{pr}_2^* \mathcal{T} \right) \Big/ \mathcal{I}, \left(\mathrm{pr}_1^* \mathcal{F} \hat{\boxtimes} \mathrm{pr}_2^* \mathcal{G} \right) \Big/ \mathcal{I}', ^{\#} \right) ,$$

where $\mathcal{I}$ denotes the ideal sheaf of $\mathrm{pr}_1^* \mathcal{S} \hat{\boxtimes} \mathrm{pr}_2^* \mathcal{T}$ spanned by $\alpha_t \otimes 1 - 1 \otimes \alpha_t$, $t = 1, \dots, n$, and $\mathcal{I}'$ denotes the ideal sheaf of $\mathrm{pr}_1^* \mathcal{F} \hat{\boxtimes} \mathrm{pr}_2^* \mathcal{G}$ spanned by $\alpha_t \otimes 1 - 1 \otimes \alpha_t$, $t = 1, \dots, n$. $\mathcal{P}^{\mathbb{C}} \hookrightarrow \left(\mathrm{pr}_1^* \mathcal{F} \hat{\boxtimes} \mathrm{pr}_2^* \mathcal{G} \right) \Big/ \mathcal{I}'$ via $R \mapsto R \otimes 1 + \mathcal{I}' = 1 \otimes R + \mathcal{I}'$.

$\mathcal{M} \times_{\mathcal{P}} \mathcal{N}$ regarded as a real $\mathcal{P}$-super manifold is precisely the $\mathcal{P}$-cross product $\mathcal{M} \times_{\mathcal{P}} \mathcal{N}$ of $\mathcal{M}$ and $\mathcal{N}$ regarded as real $\mathcal{P}$-super manifolds given the unique 'holomorphic structure' such that the canonical projections Pr_1' and Pr_2' from $\mathcal{M} \times_{\mathcal{P}} \mathcal{N}$ to $\mathcal{M}$ resp. $\mathcal{N}$ are holomorphic. Again given three holomorphic $\mathcal{P}$-super manifolds and $\mathcal{P}$-super morphism Φ , Ψ from $\mathcal{O}$ to $\mathcal{M}$ resp. $\mathcal{N}$ then the $\mathcal{P}$-super morphism (Φ, Ψ) from $\mathcal{O}$ to $\mathcal{M} \times_{\mathcal{P}} \mathcal{N}$ is holomorphic if and only if Φ and Ψ are holomorphic.

2.4 Super Lie groups and parametrized discrete subgroups

Again let $n \in \mathbb{N}$ and $\mathcal{P} := \Lambda(\mathbb{R}^n) = \mathcal{D}\left(\mathbb{R}^{0|n}\right)$.

Definition 2.42 ($\mathcal{P}$-super Lie groups)

(i) Let $\mathcal{G}$ be a real (p,q)-dimensional $\mathcal{P}$-super manifold and μ a $\mathcal{P}$-super diffeomorphism from $\mathcal{G} \times_{\mathcal{P}} \mathcal{G}$ to $\mathcal{G}$. The pair $(\mathcal{G}, \mu)$, or $\mathcal{G}$ for short, is called a (p,q)-dimesional $\mathcal{P}$-super Lie group if and only if there exist $e \in G := \mathcal{G}^{\#}$ and a $\mathcal{P}$-super morphism ι from $\mathcal{G}$ to $\mathcal{G}$ such that

$$\begin{array}{ccc} \mathcal{G} \times_{\mathcal{P}} \mathcal{G} \times_{\mathcal{P}} \mathcal{G} & \stackrel{(\mathrm{Pr}_1, \mu)}{\longrightarrow} & \mathcal{G} \times_{\mathcal{P}} \mathcal{G} \\ (\mu, \mathrm{Pr}_3) \downarrow & \% & \downarrow \mu \\ \mathcal{G} \times_{\mathcal{P}} \mathcal{G} & \underset{\mu}{\longrightarrow} & \mathcal{G} \end{array} \quad \textit{(associativity)} ,$$

$$\begin{array}{ccccc} & & \mathcal{G} \times_{\mathcal{P}} \mathcal{G} & & \\ & {}^{(e, \mathrm{Id})} \nearrow & \% & \searrow {}^{\mu} & \\ \mathcal{G} & & = & & \mathcal{G} \end{array} \quad \textit{(neutral property of } e \textit{)}$$

and

$$\begin{array}{ccc} \mathcal{G} & \stackrel{(\iota, \mathrm{Id})}{\hookrightarrow} & \mathcal{G} \times_{\mathcal{P}} \mathcal{G} \\ \downarrow & \% & \downarrow \mu \\ \{e\} & \hookrightarrow & \mathcal{G} \end{array} \quad \textit{(inversion property of } \iota \textit{)} .$$

$\mathcal{G}$ is called commutative if and only if in addition $\mu \circ (\mathrm{Pr}_2, \mathrm{Pr}_1) = \mu$ as a $\mathcal{P}$-super morphism from $\mathcal{G} \times_{\mathcal{P}} \mathcal{G}$ to $\mathcal{G}$. If $n = 0$ equivalently $\mathcal{P} = \mathbb{R}$ then $\mathcal{G}$ is simply called a super LIE *group. If $\mathcal{G}$ is a holomorphic (p, q)-dimensional $\mathcal{P}^{\mathbb{C}}$-super manifold then $\mathcal{G}$ is called a holomorphic $\mathcal{P}^{\mathbb{C}}$-super* LIE *group if and only if μ and ι are holomorphic.*

(ii) Let $(\mathcal{G}, \mu)$ and $(\mathcal{H}, \nu)$ be $\mathcal{P}$-super LIE *groups and Φ a $\mathcal{P}$-super morphism from $\mathcal{G}$ to $\mathcal{H}$. Then Φ is called a $\mathcal{P}$-super* LIE *group homomorphism if and only if $\varphi : G \to H$ is an ordinary group homomorphism, where $\varphi := \Phi^{\#}$ and $H := \mathcal{H}^{\#}$, and*

$$\begin{array}{ccc} \mathcal{G} \times_{\mathcal{P}} \mathcal{G} & \stackrel{(\Phi\circ\mathrm{Pr}_1, \Phi\circ\mathrm{Pr}_2)}{\longrightarrow} & \mathcal{H} \times_{\mathcal{P}} \mathcal{H} \\ \mu \downarrow & \% & \downarrow \nu \\ \mathcal{G} & \stackrel{\Phi}{\longrightarrow} & \mathcal{H} \end{array} .$$

If Φ is a $\mathcal{P}$-diffeomorphism and at the same time a $\mathcal{P}$-super LIE *group homomorphism then it is called a $\mathcal{P}$-super* LIE *group isomorphism. If Φ is an embedding then we call $\mathcal{G}$ a $\mathcal{P}$-sub super* LIE *group of $\mathcal{H}$.*

Clearly all $\mathcal{P}$-super LIE groups together with $\mathcal{P}$-super LIE group homomorphisms between them form a category. If $\mathcal{G}$ is a $\mathcal{P}$-super LIE group and $\mathcal{P}' := \Lambda\left(\mathbb{R}^{n'}\right) = \mathcal{D}\left(\mathbb{R}^{0|n'}\right)$ with $n' \in \mathbb{N}$ then $\mathcal{G}$ can also be regarded as a $\mathcal{P} \boxtimes \mathcal{P}'$ - super LIE group, more precisely the category of all $\mathcal{P}$- super LIE groups is a sub category of the category of all $\mathcal{P} \boxtimes \mathcal{P}'$- super LIE groups , and the category of usual $\mathcal{C}^\infty$-LIE groups together with $\mathcal{C}^\infty$- homomorphisms between them is a subcategory of the category of all super LIE groups.
If $\mathcal{G}$ is a $\mathcal{P}$-super LIE group then $G := \mathcal{G}$ is an ordinary $\mathcal{C}^\infty$-LIE group with multiplication $m := \mu^{\#} : G \times G \to G$, neutral element $e \in G$ and inversion map $i := \iota^{\#} : G \to G$, and if $\mathcal{G}$ is commutative then so is G . The body map is precisely a functor from the category of $\mathcal{P}$-super LIE groups to the category of usual $\mathcal{C}^\infty$-LIE groups, and restricted to the category of usual $\mathcal{C}^\infty$-LIE groups it is simply the identity functor.
Conversely if $n = 0$ equivalently $\mathcal{P} = \mathbb{R}$ and so $\mathcal{G}$ is simply a super LIE group then the canonical embedding $\left(\mathrm{id}, (^{\#})_{U \subset M \text{ open}}\right)$ from G into $\mathcal{G}$ is a super LIE group homomorphism.

If $\mathcal{G}$ is a $\mathcal{P}$-sub super LIE group of $\mathcal{H}$ then G can be regarded as an ordinary $\mathcal{C}^\infty$-sub LIE group of $H := \mathcal{H}^{\#}$ via the $\mathcal{C}^\infty$-LIE group embedding $\varphi := \Phi^{\#}$.

For any $g, h \in_{\mathcal{P}} \mathcal{G}$ let us write $gh := \mu(g, h) \in_{\mathcal{P}} \mathcal{G}$. Then clearly
$(gh)^{\#} = g^{\#} h^{\#}$. Let $\mathcal{G}$ be of dimension (p, q) , $n' \in \mathbb{N}$ such that $n' \geq 3q$ and $\mathcal{P}' := \Lambda\left(\mathbb{R}^{n'}\right)$. Then by theorem 2.27 (ii) the associativity is equivalent to

$(gh)j = g(hj)$ for all $g, h, j \in_{\mathcal{P} \boxtimes \mathcal{P}'} \mathcal{G}$, the neutral property of e is equivalent to $eg = g$ for all $g \in_{\mathcal{P} \boxtimes \mathcal{P}'} \mathcal{G}$, the inversion property of ι to $\iota(g)g = e$ for all $g \in_{\mathcal{P} \boxtimes \mathcal{P}'} \mathcal{G}$ and finally commutativity is equivalent to $gh = hg$ for all $g, h \in_{\mathcal{P} \boxtimes \mathcal{P}'} \mathcal{G}$.
Let $g \in_{\mathcal{P}} \mathcal{G}$. Define the left translation by g on $\mathcal{G}$ as $l_g := \mu \circ (g, \mathrm{id})$ and the right translation by g on $\mathcal{G}$ as $r_g := \mu \circ (\mathrm{id}, g)$, which both are $\mathcal{P}$-super morphisms from $\mathcal{G}$ to $\mathcal{G}$. Then one immediately sees that $l_e = \mathrm{Id}$, $l_g \circ l_h = l_{gh}$, $r_e = \mathrm{Id}$ and $r_g \circ r_h = r_{hg}$.

Theorem 2.43 *Let $\mathcal{G}$ be a $\mathcal{P}$-super* Lie *group, ι given as in definition 2.42 and $g, h, j \in_{\mathcal{P}} \mathcal{G}$.*

(i) $ge = g$ and $g\,\iota(g) = e$. So

$$\begin{array}{ccccc} & & \mathcal{G} \times_{\mathcal{P}} \mathcal{G} & & \\ & {\scriptstyle (\mathrm{Id},e)} \nearrow & \% & \searrow {\scriptstyle \mu} & \\ \mathcal{G} & & = & & \mathcal{G} \end{array}$$

and

$$\begin{array}{ccc} \mathcal{G} & \overset{(\mathrm{Id},\iota)}{\hookrightarrow} & \mathcal{G} \times_{\mathcal{P}} \mathcal{G} \\ \downarrow & \% & \downarrow \mu \\ \{e\} & \hookrightarrow & \mathcal{G} \end{array}$$

as well.

(ii) r_g and l_g are $\mathcal{P}$-super diffeomorphisms, and $l_g^{-1} = l_{\iota(g)}$ and $r_g^{-1} = r_{\iota(g)}$. In particular if $gh = gj$ then $h = j$ and if $gj = hj$ then $g = h$.

(iii) ι is uniquely determined by μ , it is a $\mathcal{P}$-super diffeomorphism, and $\iota^{-1} = \iota$.

(iv) $\iota\,(gh) = \iota(h)\,\iota(g)$, and so

$$\begin{array}{ccc} \mathcal{G} \times_{\mathcal{P}} \mathcal{G} & \xrightarrow{(\iota \circ \mathrm{Pr}_2, \iota \circ \mathrm{Pr}_1)} & \mathcal{G} \times_{\mathcal{P}} \mathcal{G} \\ \mu \downarrow & \% & \downarrow \mu \\ \mathcal{G} & \xrightarrow[\iota]{} & \mathcal{G} \end{array} .$$

Proof: Let (p, q) be the dimension of $\mathcal{G}$ as a $\mathcal{P}$-super manifold and $\mathcal{P}' := \Lambda\left(\mathbb{R}^{2q}\right)$.
(i) $ge = g$ and $g\,\iota(g) = e$ is proven as in classical algebra. Using $\mathcal{P} \boxtimes \mathcal{P}'$ instead of $\mathcal{P}$, theorem 2.27 (ii) gives the commutativity of the diagrams. $\square$
(ii) trivial using (i) .
(iii) Assume ι' is another $\mathcal{P}$-super morphism from $\mathcal{G}$ to $\mathcal{G}$ having the inversion property. Then $\iota(g)\,g = e = \iota'(g)\,g$ and so $\iota(g) = \iota'(g)$ by (ii) , furthermore

$g\,\iota^{-1}(g) = \iota\left(\iota^{-1}(g)\right)\,\iota^{-1}(g) = e = g\,\iota(g)$ by (i) , and so $\iota(g) = \iota^{-1}(g)$ by (ii) . Therefore using $\mathcal{P}\boxtimes\mathcal{P}'$ instead of $\mathcal{P}$, since $g \in_{\mathcal{P}} \mathcal{G}$ is arbitrary theorem 2.27 (ii) gives $\iota = \iota'$ and $\iota^{-1} = \iota$. □
(iv) $\iota\,(gh) = \iota(h)\,\iota(g)$ is proven as in classical algebra using (i) , the commutativity of the diagram then follows from theorem 2.27 (ii) using $\mathcal{P}\boxtimes\mathcal{P}'$ instead of $\mathcal{P}$. □

Since given a $\mathcal{P}$-super Lie group $\mathcal{G}$, ι is uniquely determined by μ and it is a $\mathcal{P}$-super diffeomorphism, we call it the inversion $\mathcal{P}$-super diffeomorphism of $\mathcal{G}$. For any $g \in_{\mathcal{P}} \mathcal{G}$ we write $g^{-1} := \iota(g)$.

Definition 2.44 ($\mathcal{P}$-super actions) *Let $\mathcal{G}$ be a $\mathcal{P}$-super Lie group , $\mathcal{M}$ a $\mathcal{P}$-super manifold and $\alpha : \mathcal{G} \times_{\mathcal{P}} \mathcal{M} \to \mathcal{M}$ a $\mathcal{P}$-super morphism.*

(i) α is called a $\mathcal{P}$-super action of $\mathcal{G}$ on $\mathcal{M}$ if and only if

$$\begin{array}{ccc} \mathcal{G} \times_{\mathcal{P}} \mathcal{G} \times_{\mathcal{P}} \mathcal{M} & \stackrel{(\mathrm{Pr}_1,\alpha)}{\longrightarrow} & \mathcal{G} \times_{\mathcal{P}} \mathcal{M} \\ (\mu, \mathrm{Pr}_3) \downarrow & \% & \downarrow \alpha \\ \mathcal{G} \times_{\mathcal{P}} \mathcal{M} & \underset{\alpha}{\longrightarrow} & \mathcal{M} \end{array} \quad \textit{(associativity)}$$

and

$$\begin{array}{ccc} & \mathcal{G} \times_{\mathcal{P}} \mathcal{M} & \\ {}^{(e,\mathrm{Id})}\nearrow & \% & \searrow^{\alpha} \\ \mathcal{M} & = & \mathcal{M} \end{array} \quad \textit{(neutral property of } e \textit{)} .$$

(ii) If α is a $\mathcal{P}$-super action and $\mathcal{N}$ is a $\mathcal{P}$-sub manifold of $\mathcal{M}$ then $\mathcal{N}$ is called α-invariant (or $\mathcal{G}$- invariant) if and only if there exists a $\mathcal{P}$-super morphism α' from $\mathcal{G} \times_{\mathcal{P}} \mathcal{N}$ to $\mathcal{N}$ such that

$$\begin{array}{ccc} \mathcal{G} \times_{\mathcal{P}} \mathcal{N} & \hookrightarrow & \mathcal{G} \times_{\mathcal{P}} \mathcal{M} \\ \alpha' \downarrow & \% & \downarrow \alpha \\ \mathcal{N} & \hookrightarrow & \mathcal{M} \end{array} .$$

(iii) If α is a $\mathcal{P}$-super action then α is called transitive if and only if there exists $x \in M := \mathcal{M}^{\#}$ such that $\alpha \circ (\mathrm{Id}, x)$ is a super projection from $\mathcal{G}$ onto $\mathcal{M}$.

If α is a $\mathcal{P}$-super action from $\mathcal{G} \times_{\mathcal{P}} \mathcal{M}$ to $\mathcal{M}$, $\mathcal{G}$ being a $\mathcal{P}$-super Lie group and $\mathcal{M}$ being a $\mathcal{P}$-super manifold, then $\alpha^{\#} : G \times M \to M$ is an ordinary action, if $\mathcal{N}$ is an α-invariant $\mathcal{P}$-sub super manifold of $\mathcal{M}$ then N is invariant under $\alpha^{\#}$, and finally if α is transitive then so is $\alpha^{\#}$.
Let $g \in_{\mathcal{P}} \mathcal{G}$ and $\Xi \in_{\mathcal{P}} \mathcal{M}$. Then we write $g\Xi := \alpha(g, \Xi)$. Let $\mathcal{G}$ be of dimension (p, q) and $\mathcal{M}$ of dimension (r, s) , let $n' \geq q + s$ and

$\mathcal{P}' := \Lambda\left(\mathbb{R}^{n'}\right)$. Then by theorem 2.27 the associativity in definition 2.44 is equivalent to $g(h\Xi) = (gh)\Xi$ for all $g, h \in_{\mathcal{P}\boxtimes\mathcal{P}'} \mathcal{G}$ and $\Xi \in_{\mathcal{P}\boxtimes\mathcal{P}'} \mathcal{M}$, so we write $gh\Xi := (gh)\Xi = g(h\Xi)$, and the neutral property of e is equivalent to $e\Xi = \Xi$ for all $\Xi \in_{\mathcal{P}\boxtimes\mathcal{P}'} \mathcal{M}$.

Let $g \in_{\mathcal{P}} \mathcal{G}$. Then $\alpha_g := \alpha \circ (g, \mathrm{Id})$ is a $\mathcal{P}$-super diffeomorphism from $\mathcal{M}$ to $\mathcal{M}$, and $\alpha_g^{-1} = \alpha_{g^{-1}}$.

Let $\mathcal{N}$ be an α-invariant $\mathcal{P}$-sub super manifold of $\mathcal{M}$ and α' be given by definition 2.44 (ii) . Then α' is uniquely determined by α and $\mathcal{N}$, and α' is again a $\mathcal{P}$-super action. This can easily be checked using $\mathcal{P}\boxtimes\mathcal{P}'$-points of $\mathcal{G}$ and $\mathcal{N}$. If $\mathcal{U}$ is an open sub $\mathcal{P}$-super manifold of $\mathcal{M}$ then clearly $\mathcal{U}$ is α-invariant if and only if $U := \mathcal{U}^{\#}$ is $\alpha^{\#}$ invariant, and in this case $\alpha' = \alpha|_{\mathcal{G}\times_{\mathcal{P}}\mathcal{U}}$, and so we write $\alpha|_{\mathcal{G}\times_{\mathcal{P}}\mathcal{N}} := \alpha'$ if $\mathcal{N}$ is an arbitrary α-invariant sub $\mathcal{P}$-super manifold of $\mathcal{M}$.

Let α be a transitive $\mathcal{P}$-super action of $\mathcal{G}$ on $\mathcal{M}$ and $x \in M$ such that $\alpha \circ (\mathrm{Id}, x)$ is a super projection from $\mathcal{G}$ onto $\mathcal{M}$. Then $\alpha \circ (\mathrm{Id}, y)$ is a super projection from $\mathcal{G}$ onto $\mathcal{M}$ for all $y \in M$. Let us check it:

> Let $\mathcal{G}$ be of dimension (p, q) . By definition $\alpha^{\#} \circ (\mathrm{id}, x) : G \to M$ is a projection, and so $\alpha^{\#}$ is transitive. Therefore there exists $g_0 \in G$ such that $y = g_0 x$. Let $\mathcal{P}' := \Lambda(\mathbb{R}^q) = \mathcal{D}\left(\mathbb{R}^{0|q}\right)$. Then we obtain $gy = gg_0x$ for all $g \in_{\mathcal{P}\boxtimes\mathcal{P}'} \mathcal{G}$. So
> $\alpha \circ (\mathrm{Id}, y) = \alpha \circ (\mathrm{Id}, x) \circ r_{g_0}$ by lemma 2.27 (ii) . Since r_{g_0} is a $\mathcal{P}$-super diffeomorphism from $\mathcal{G}$ to $\mathcal{G}$ we see that again $\alpha \circ (\mathrm{Id}, y)$ is a $\mathcal{P}$-super projection.

Clearly μ is a $\mathcal{P}$-super action of $\mathcal{G}$ on itself for each $\mathcal{P}$-super Lie group $\mathcal{G}$. Given two $\mathcal{P}$-super Lie groups $\mathcal{G}$ and $\mathcal{H}$, a $\mathcal{P}$-super Lie group homomorphism Φ from $\mathcal{G}$ to $\mathcal{H}$ and a $\mathcal{P}$-super action α of $\mathcal{H}$ on a $\mathcal{P}$-super manifold $\mathcal{M}$, $\alpha \circ (\Phi \circ \mathrm{Pr}_1, \mathrm{Pr}_2)$ is a $\mathcal{P}$-super action of $\mathcal{G}$ on $\mathcal{M}$, and in the case where Φ is an embedding we have $\alpha \circ (\Phi \circ \mathrm{Pr}_1, \mathrm{Pr}_2) = \alpha|_{\mathcal{G}\times_{\mathcal{P}}\mathcal{M}}$.

Let Γ be a group, M a $\mathcal{C}^\infty$-manifold and $\alpha : \Gamma \times M \to M$ be a discrete and fixpoint free $\mathcal{C}^\infty$-action of Γ on M . Then in classical analysis we can form the quotient $\Gamma\backslash M$, which then is again a $\mathcal{C}^\infty$-manifold locally diffeomorphic to M itself. In this case have a canonical sheaf isomorphism

$$\mathcal{C}^\infty_{\Gamma\backslash M} \simeq \left\{ f \in \mathcal{C}^\infty\left(\varphi^{-1}(\diamondsuit)\right) \,\middle|\, f\ \Gamma\text{-invariant} \right\} ,$$

where $\varphi : M \to \Gamma\backslash M$ denotes the canonical projection. In super analysis there is an analogon to this fact.

Recall that each group Γ can be regarded as a discrete $\mathcal{P}$-LIE group, and conversely each discrete $\mathcal{P}$-super LIE group is nothing but an ordinary group given the discrete topology. So given a group Γ and a $\mathcal{P}$-super manifold $\mathcal{M}$ we have a canonical bijection between all $\mathcal{P}$-super actions α of Γ on $\mathcal{M}$ and all mappings

$$\Gamma \to \{ \text{ P - super diffeomorphisms from } \mathcal{M} \text{ to } \mathcal{M} \} , \gamma \mapsto \alpha_\gamma$$

such that $\alpha_{\gamma\delta} = \alpha_\gamma \circ \alpha_\delta$ and $\alpha_e = \mathrm{Id}$.

Definition 2.45 *Let $\mathcal{G}$ be a $\mathcal{P}$-super* LIE *group, α a $\mathcal{P}$-super action of $\mathcal{G}$ on the $\mathcal{P}$-super manifold $\mathcal{M}$ and $f \in \mathcal{D}(\mathcal{M})$. f is called α-invariant (or $\mathcal{G}$-invariant) if and only if $\alpha_M(f) = \mathcal{C}_M(f)$, where α_M is the unital graded algebra homomorphism from $\mathcal{D}(\mathcal{M})$ to $\mathcal{D}(\mathcal{G} \times_{\mathcal{P}} \mathcal{M})$ coming from α and $\mathcal{C}_M$ is the canonical embedding from $\mathcal{D}(\mathcal{M})$ into $\mathcal{D}(\mathcal{G} \times_{\mathcal{P}} \mathcal{M})$.*

Let $\mathcal{G}$ be of dimension (p,q) , $\mathcal{M}$ be of dimension (r,s) , $n' \in \mathbb{N}$ such that $n' \geq q+s$ and $\mathcal{P}' := \Lambda\left(\mathbb{R}^{n'}\right)$. Let $f \in \mathcal{D}(\mathcal{M})$. Then by lemma 2.27 (i) f is α-invariant if and only if $f(g\Xi) = f(\Xi)$ for all $g \in_{\mathcal{P}\boxtimes\mathcal{P}'} \mathcal{G}$ and $\Xi \in_{\mathcal{P}\boxtimes\mathcal{P}'}$.

Theorem 2.46 (quotients of $\mathcal{P}$-super manifolds) *Let Γ be a group, $\mathcal{M}$ be a $\mathcal{P}$-super manifold of dimension (p,q) with structural sheaf $\mathcal{S}$, and let α be a $\mathcal{P}$-super action of Γ on $\mathcal{M}$ such that $\alpha^\#$ is a discrete and fixpoint free action of Γ on M . Let $\varphi : M \to \Gamma\backslash M$ denote the canonical projection, and let $\mathcal{Q}$ be the sheaf on $\Gamma\backslash M$ given by*

$$\mathcal{Q} := \left\{ f \in \mathcal{S}\left(\varphi^{-1}(\diamondsuit)\right) \middle|\, f \ \alpha\text{-invariant} \right\} .$$

For all $V \subset \Gamma\backslash M$ open let $\Phi_V : \mathcal{Q}(V) \hookrightarrow \mathcal{S}\left(\varphi^{-1}(V)\right)$ denote the canonical embedding.

(i) $\Gamma\backslash\mathcal{M} := \left(\Gamma\backslash M, \mathcal{Q}, ^\#\right)$ is a (p,q)-dimensional $\mathcal{P}$-super manifold, and $\Phi := \left(\varphi, (\Phi_V)_{V\subset\Gamma\backslash M \text{ open}}\right)$ is a $\mathcal{P}$-super projection from $\mathcal{M}$ onto $\Gamma\backslash\mathcal{M}$.

(ii) Φ is a local diffeomorphism, more precisely for each $x \in M$ there exists an open sub $\mathcal{P}$-super manifold $\mathcal{U}$ of $\mathcal{M}$ such that $x \in U := \mathcal{U}^\#$ and $\Phi|_{\mathcal{U}}$ is a diffeomorphism from $\mathcal{U}$ to $\mathcal{V} := \left(\phi(U), \mathcal{Q}|_{\phi(U)}, ^\#\right)$, which is actually an open sub $\mathcal{P}$-super manifold of $\Gamma\backslash\mathcal{M}$.

Proof: Since Γ acts discretely and without fixpoints on M via $\alpha^\#$, from classical analysis we know that $\Gamma\backslash M$ is a p-dimensional $\mathcal{C}^\infty$-manifold, $\varphi : M \to \Gamma\backslash M$ is an open $\mathcal{C}^\infty$-projection and induces a canonical sheaf isomorphism

$$\mathcal{C}^\infty_{\Gamma\backslash M} \simeq \left\{ f \in \mathcal{C}^\infty\left(\varphi^{-1}(\diamondsuit)\right) \middle|\, f \ \alpha^\#\text{-invariant} \right\} ,$$

and so we will identify these sheaves in what follows. Let $V \subset \Gamma\backslash M$ be open. Then clearly $\Phi(f)^{\#} = f^{\#} \circ \varphi$ for all $f \in \mathcal{Q}(V)$, and if $V' \subset V$ open as well

$$\begin{array}{ccc} \mathcal{Q}(V) & \stackrel{\Phi_V}{\longrightarrow} & \mathcal{S}\left(\varphi^{-1}(V)\right) \\ |_{V'} \downarrow & \% & \downarrow |_{\varphi^{-1}(V')} \\ \mathcal{Q}(V') & \underset{\Phi_{V'}}{\longrightarrow} & \mathcal{S}\left(\varphi^{-1}(V')\right) \end{array} .$$

Now let $x \in M$. Since $\alpha^{\#}$ is discrete and without fixpoints there exists an open neighbourhood $U \subset M$ of x such that $\gamma U \cap \gamma' U = \emptyset$ for all $\gamma, \gamma' \in \Gamma$, $\gamma \neq \gamma'$. Define $V := \varphi(U) \subset \Gamma\backslash M$. Then clearly $\varphi|_U : U \to V$ is a $\mathcal{C}^\infty$-diffeomorphism, and for all $W \subset V$ open

$$|_{\varphi^{-1}(W)\cap U} \circ \Phi_W : \mathcal{Q}(W) \to \mathcal{S}\left(\varphi^{-1}(W) \cap U\right)$$

is a unital graded algebra isomorphism.

(i) : Let $\varphi(x) = \Gamma x \in \Gamma\backslash M$, $x \in M$ and $U \subset M$ of x such that $\gamma U \cap \gamma' U = \emptyset$ for all $\gamma, \gamma' \in \Gamma$, $\gamma \neq \gamma'$. Without loss of generality we may assume that $\mathcal{S}|_U \simeq \mathcal{C}_U^\infty \otimes \Lambda(\mathbb{R}^q) \boxtimes \mathcal{P}$. Clearly $\mathcal{P} \hookrightarrow \mathcal{Q}$, and identifying V and U via φ , and so identifying W with $\varphi^{-1}(W) \cap U$ for all $W \subset V$ open, we see that $\Psi := \left(|_{\varphi^{-1}(W)\cap U} \circ \Phi_W\right)_{W \subset V \text{ open}}$ is a whole sheaf isomorphism from $\mathcal{Q}|_V$ to $\mathcal{S}|_V$ such that $\Psi|_{\mathcal{P}} = \mathrm{id}$ and

$$\begin{array}{ccc} \mathcal{Q}|_V & \stackrel{\Psi}{\longrightarrow} & \mathcal{S}|_V \simeq \mathcal{C}_V^\infty \otimes \Lambda(\mathbb{R}^q) \boxtimes \mathcal{P} \\ {}_{\#}\searrow & \% & \swarrow_{\#} \\ & \mathcal{C}_V^\infty & \end{array} .$$

So $\Gamma\backslash\mathcal{M}$ is a (p,q)-dimensional $\mathcal{P}$-super manifold. Since $\Phi_W|_{\mathcal{P}} = \mathrm{id}$ it is now obvious that Φ is a $\mathcal{P}$-super projection from $\mathcal{M}$ onto $\Gamma\backslash\mathcal{M}$. $\square$
(ii) : now trivial.

The most important example of a discrete and fixpoint free action α of a group Γ on a $\mathcal{P}$-super manifold $\mathcal{M}$ is the case where $\mathcal{M} = \mathcal{G}$ is a $\mathcal{P}$-super Lie group, $\Gamma = \Upsilon$ is a discrete $\mathcal{P}$-sub super Lie group of $\mathcal{G}$ and $\alpha = \mu|_{\Upsilon \times_{\mathcal{P}} \mathcal{M}}$. Recall that a discrete $\mathcal{P}$-sub super Lie group of $\mathcal{G}$ is nothing but a subset Υ of the set of all $\mathcal{P}$-points of $\mathcal{G}$ such that $\Gamma := \Upsilon^{\#}$ is discrete, $\gamma\delta \in_{\mathcal{P}} \Upsilon$ for all $\gamma, \delta \in_{\mathcal{P}} \Upsilon$ and for all $\gamma \in \Gamma := \Upsilon^{\#}$ there exists a unique $\gamma' \in_{\mathcal{P}} \Upsilon$ such that $\gamma = \gamma'^{\#}$.

Chapter 3

Super automorphic and super cusp forms

3.1 The general setting

Let $n, r \in \mathbb{N}$. Then $GL(n, \mathbb{C}) \times GL(r, \mathbb{C})$ is an open subset of $\mathbb{C}^{n^2+r^2}$, and so $sGL(n|r) := (GL(n, \mathbb{C}) \times GL(r, \mathbb{C}))^{|2nr, \overline{2nr}}$ is a complex super open set with even coordinate functions a_{ij} and $d_{kl} \in \mathcal{O}(sGL(n|r))_0$ and odd coordinate functions β_{il} and $\gamma_{kj} \in \mathcal{O}(sGL(n|r))_1$, $i, j = 1, \ldots, n$ and $k, l = 1, \ldots, r$. $sGL(n|r)$ is a holomorphic $\left(n^2 + r^2, 2nr\right)$-dimensional super Lie group with multiplication

$$\left(\begin{array}{c|c} A & \beta \\ \hline \gamma & D \end{array}\right) \otimes \left(\begin{array}{c|c} A & \beta \\ \hline \gamma & D \end{array}\right) = \left(\begin{array}{c|c} A \otimes A + \beta \otimes \gamma & A \otimes \beta + \beta \otimes D \\ \hline \gamma \otimes A + D \otimes \gamma & \gamma \otimes \beta + D \otimes D \end{array}\right),$$

where we use ordinary matrix multiplication, neutral element $1 \in GL(n, \mathbb{C}) \times GL(r, \mathbb{C})$ and inversion super diffeomorphism ι given by the ordinary matrix inversion

$$\left(\begin{array}{c|c} A & \beta \\ \hline \gamma & D \end{array}\right)^{-1} = \left(\begin{array}{c|c} A^{-1} & 0 \\ \hline 0 & D^{-1} \end{array}\right) \sum_{l=0}^{2nr} \left(\begin{array}{c|c} 0 & \beta D^{-1} \\ \hline \gamma A^{-1} & 0 \end{array}\right)^{l}$$

by corollary 2.4 . Clearly the body of $sGL(n|r)$ is $GL(n, \mathbb{C}) \times GL(r, \mathbb{C})$ together with ordinary matrix multiplication. Now let $p, q \in \mathbb{N} \setminus \{0\}$ such that $p + q = n$, and let us now sum up the coordinate functions into blocks according to

$$\left(\begin{array}{cc|c} A & B & \mu \\ C & D & \nu \\ \hline \rho & \sigma & E \end{array}\right) \begin{array}{l} \}p \\ \}q \\ \}r \end{array} .$$

Then using theorem 2.29 one can show that the equations

$$\left(\begin{array}{cc|c} A & B & \mu \\ C & D & \nu \\ \hline \rho & \sigma & E \end{array}\right)^* \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \left(\begin{array}{cc|c} A & B & \mu \\ C & D & \nu \\ \hline \rho & \sigma & E \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right),$$

$$\operatorname{Ber} \left(\begin{array}{cc|c} A & B & \mu \\ C & D & \nu \\ \hline \rho & \sigma & E \end{array}\right) = 1$$

or more explicitely

$$\begin{aligned} A^*A - C^*C + \rho^*\rho &= 1\,, \\ A^*B - C^*D + \rho^*\sigma &= 0\,, \\ B^*B - D^*D + \sigma^*\sigma &= -1\,, \\ \mu^*\mu - \nu^*\nu + E^*E &= 1\,, \\ \det \begin{pmatrix} A - \mu E^{-1}\rho & B - \mu E^{-1}\sigma \\ C - \nu E^{-1}\rho & D - \nu E^{-1}\sigma \end{pmatrix} &= \det E\,, \\ A^*\mu - C^*\nu + \rho^*E &= 0\,, \\ B^*\mu - D^*\nu + \sigma^*E &= 0 \end{aligned}$$

define a real $\left((p+q)^2 + r^2 - 1, 2(p+q)r\right)$-dimensional sub super manifold of $sGL\,(n|r)$, which we denote by $sSU(p,q|r)$. It turns out that $sSU(p,q|r)$ is even a real sub super LIE group of $sGL(n|r)$ on which the inversion map ι has a nice expression:

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \left(\begin{array}{cc|c} A & B & \mu \\ C & D & \nu \\ \hline \rho & \sigma & E \end{array}\right)^* \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} A^* & -C^* & \rho^* \\ -B^* & D^* & -\sigma^* \\ \hline \mu^* & -\nu^* & D^* \end{array}\right)$$

The body of $sSU(p,q|r)$ is

$$sS\left(U(p,q) \times U(r)\right) := \left\{ \left(\begin{array}{c|c} g & 0 \\ \hline 0 & E \end{array}\right) \in U(p,q) \times U(r) \,\middle|\, \det g = \det E \right\}$$

together with ordinary matrix multiplication. We call $sSU(p,q|r)$ the super special pseudo unitary group. Define the complex (pq, rq)-dimensional super domain $B^{p,q|r}$ as $B^{p,q|r} := \left(B^{p,q}\right)^{|qr,\overline{qr}}$, where

$$B^{p,q} := \left\{ \mathbf{Z} \in \mathbb{C}^{p\times q} \,\middle|\, \mathbf{Z}^*\mathbf{Z} \ll 1 \right\} \subset \mathbb{C}^{p\times q}$$

open, with holomorphic even coordinate functions $z_{ij} \in \mathcal{O}\left(B^{p,q|r}\right)_0$ and holomorphic odd coordinate functions $\zeta_{kj} \in \mathcal{O}\left(B^{p,q|r}\right)_1$, $i = 1,\dots,p$, $j = 1,\dots,q$, $k = 1,\dots,r$. Then we have a super action α of $sSU(p,q|r)$ on $B^{p,q|r}$ given by super fractional linear (Möbius) transformations

$$\left(\begin{array}{c} (A\mathbf{Z} + B + \mu\zeta)\,(C\mathbf{Z} + D + \nu\zeta)^{-1} \\ \hline (\rho\mathbf{Z} + \sigma + E\zeta)\,(C\mathbf{Z} + D + \nu\zeta)^{-1} \end{array}\right) .$$

α is holomorphic with respect to $\left(\begin{array}{c} \mathbf{Z} \\ \hline \zeta \end{array}\right)$ in the sense that if $f \in \mathcal{O}\left(B^{p,q|r}\right)$ then $f(\alpha) \in \mathcal{D}\left(sSU(p,q|r)\right)^{\mathbb{C}} \hat{\boxtimes} \mathcal{O}\left(B^{p,q|r}\right)$. Let us check that α is also transitive.

> We claim that $\alpha\left(\diamondsuit, \mathbf{0}\right)$ is a super projection from $sSU(p,q|r)$ onto $B^{p,q|r}$. For a proof define the super morphism Φ from $B^{p,q|r}$ to $sSU(p,q|r)$ by
>
> $$\begin{aligned} \left(\begin{array}{c|c} A & \mu \\ \hline \rho & E \end{array}\right) &:= \left(1 - \left(\begin{array}{c} \mathbf{Z} \\ \hline \zeta \end{array}\right)\left(\begin{array}{c|c} \mathbf{Z}^* & \zeta^* \end{array}\right)\right)^{-\frac{1}{2}} \\ &= \left(1 - \left(\begin{array}{c|c} \mathbf{Z}\mathbf{Z}^* & \mathbf{Z}\zeta^* \\ \hline \zeta\mathbf{Z}^* & \zeta\zeta^* \end{array}\right)\right)^{-\frac{1}{2}} , \end{aligned}$$
>
> $$\begin{aligned} \left(\begin{array}{c} B \\ \hline \sigma \end{array}\right) &:= \left(\begin{array}{c} \mathbf{Z} \\ \hline \zeta \end{array}\right)\left(1 - \left(\begin{array}{c|c} \mathbf{Z}^* & \zeta^* \end{array}\right)\left(\begin{array}{c} \mathbf{Z} \\ \hline \zeta \end{array}\right)\right)^{-\frac{1}{2}} \\ &= \left(\begin{array}{c} \mathbf{Z} \\ \hline \zeta \end{array}\right)\left(1 - \mathbf{Z}^*\mathbf{Z} - \zeta^*\zeta\right)^{-\frac{1}{2}} , \end{aligned}$$
>
> $$\begin{aligned} \left(\begin{array}{c|c} C & \nu \end{array}\right) &:= \left(\begin{array}{c|c} \mathbf{Z}^* & \zeta^* \end{array}\right)\left(1 - \left(\begin{array}{c} \mathbf{Z} \\ \hline \zeta \end{array}\right)\left(\begin{array}{c|c} \mathbf{Z}^* & \zeta^* \end{array}\right)\right)^{-\frac{1}{2}} \\ &= \left(\begin{array}{c|c} \mathbf{Z}^* & \zeta^* \end{array}\right)\left(1 - \left(\begin{array}{c|c} \mathbf{Z}\mathbf{Z}^* & \mathbf{Z}\zeta^* \\ \hline \zeta\mathbf{Z}^* & \zeta\zeta^* \end{array}\right)\right)^{-\frac{1}{2}} \end{aligned}$$
>
> and

$$D := \left(1 - \left(\begin{array}{c|c} \mathbf{Z}^* & \zeta^* \end{array} \right) \left(\begin{array}{c} \mathbf{Z} \\ \hline \zeta \end{array} \right) \right)^{-\frac{1}{2}} = \left(1 - \mathbf{Z}^*\mathbf{Z} - \zeta^*\zeta\right)^{-\frac{1}{2}} .$$

Then a simple calculation shows that Φ is even a cross section for $\alpha\left(\diamondsuit, \mathbf{0}\right)$ in the sense that the composition

$$B^{p,q|r} \xrightarrow{\Phi} sSU(p,q|r) \xrightarrow{\alpha(\diamondsuit,\mathbf{0})} B^{p,q|r}$$

gives the identity. So Φ is a super embedding from $B^{p,q|r}$ into $sSU(p,q|r)$ and $\alpha\left(\diamondsuit, \mathbf{0}\right)$ is a super projection from $sSU(p,q|r)$ onto $B^{p,q|r}$.

Now let $m \in \mathbb{N}$ and $\mathcal{P} := \Lambda\left(\mathbb{R}^m\right) = \mathcal{D}\left(\mathbb{R}^{0|m}\right)$ with the odd coordinate functions $\beta_1, \ldots, \beta_m \in \mathcal{D}\left(\mathbb{R}^{0|m}\right)_1$.

The stabilizer sub super Lie group of $\mathbf{0} \hookrightarrow B^{p,q|r}$ is $\mathcal{K} := sS\left(U(p|r) \times U(q)\right)$ which is a real $\left(p^2 + q^2 + r^2 - 1, 2pr\right)$-dimensional sub super Lie group of $sSU(p,q|r)$ given by the equations

$$\begin{aligned}
B &= 0\,, \\
C &= 0\,, \\
\nu &= 0\,, \\
\sigma &= 0\,, \\
\left(\begin{array}{c|c} A & \mu \\ \hline \rho & E \end{array} \right)^* \left(\begin{array}{c|c} A & \mu \\ \hline \rho & E \end{array} \right) &= 1\,, \\
D^*D &= 1\,, \\
\operatorname{Ber} \left(\begin{array}{c|c} A & \mu \\ \hline \rho & E \end{array} \right) \det D &= 1
\end{aligned}$$

or more explicitely

$$\begin{aligned}
B &= 0\,, \\
C &= 0\,, \\
\nu &= 0\,, \\
\sigma &= 0\,, \\
A^*A + \rho^*\rho &= 1\,, \\
D^*D &= 1\,, \\
\mu^*\mu + E^*E &= 1\,, \\
\det\left(A - \mu E^{-1}\rho\right)\det D &= \det E\,, \\
\\
A^*\mu + \rho^*E &= 0
\end{aligned}$$

in the sense that if $g \in_{\mathcal{P}} sSU(p,q|r)$ then $g\mathbf{0} = \mathbf{0}$ if and only if $g \in_{\mathcal{P}} \mathcal{K}$. Clearly the body of $\mathcal{K}$ is $K := sS\left((U(p) \times U(q)) \times U(r)\right)$, explicitely

$$K = \left\{ \left(\begin{array}{cc|c} \begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array} & & 0 \\ \hline & 0 & E \end{array} \right) \in U(p) \times U(q) \times U(r) \,|\, \det A \det D = \det E \right\} ,$$

which is also automatically the stabilizer of $\mathbf{0}$ in $s\left(SU(p,q) \times U(r)\right)$ acting on $B^{p,q}$ via $\alpha^{\#}$.

From now on let $\mathcal{G} := sSU(p,q|r)$, $\mathcal{B} := B^{p,q|r}$,

$G := \mathcal{G}^{\#} = s\left(SU(p,q) \times U(r)\right)$ and $B := \mathcal{B}^{\#} = B^{p,q}$. Then on $\mathcal{G} \times \mathcal{B}$ we define the function

$$j := \det\left(C\mathbf{Z} + D + \beta\zeta\right)^{-1} \in \left(\mathcal{D}\left(\mathcal{G}\right)^{\mathbb{C}} \hat{\boxtimes} \mathcal{O}(\mathcal{B})\right)_0 .$$

j fulfills the cocycle property

$$j\left(m \circ (\mathrm{Pr}_1, \mathrm{Pr}_2), \mathrm{Pr}_3\right) = j\left(\mathrm{Pr}_1, \alpha \circ (\mathrm{Pr}_2, \mathrm{Pr}_3)\right) j\left(\mathrm{Pr}_2, \mathrm{Pr}_3\right) ,$$

where we compare functions in $\mathcal{D}\left(\mathcal{G} \times \mathcal{G}\right)^{\mathbb{C}} \hat{\boxtimes} \mathcal{D}(\mathcal{B})$, or equivalently if $m \geq 2(p+q)r + qr$ then

$$j\left(gh, \Xi\right) = j\left(g, h\Xi\right) j\left(h, \Xi\right)$$

for all $g, h \in_{\mathcal{P}} \mathcal{G}$ and $\Xi \in_{\mathcal{P}} \mathcal{B}$.

From now on let $k \in \mathbb{Z}$ be fixed. Then we have $\mathcal{P}^{\mathbb{C}}$-linear continuous graded injections

$$| : \mathcal{D}(\mathcal{B}) \boxtimes \mathcal{P}^{\mathbb{C}} \to \mathcal{D}(\mathcal{G})^{\mathbb{C}} \hat{\boxtimes} \mathcal{D}(\mathcal{B}) \boxtimes \mathcal{P}^{\mathbb{C}} \, , \, f \mapsto f| := f(\alpha) j^k$$

and

$$\widetilde{\ } : \mathcal{D}(\mathcal{B}) \boxtimes \mathcal{P}^{\mathbb{C}} \to (\mathcal{D}(\mathcal{G}) \boxtimes \mathcal{P})^{\mathbb{C}} \, , \, f \mapsto \widetilde{f} := f\left(\alpha\left(\diamondsuit, \mathbf{0}\right)\right) j\left(\diamondsuit, \mathbf{0}\right)^k = f|_{\diamondsuit}\left(\mathbf{0}\right)$$

respecting $\bar{\ }$, in other words $f|_g(\Xi) = f(g\Xi)j(g,\Xi)^k$ and $\widetilde{f}(g) = f\left(g\mathbf{0}\right) j\left(g, \mathbf{0}\right)^k = f|_g\left(\mathbf{0}\right)$ for all $\Xi \in_{\mathcal{P}} \mathcal{B}$ and $g \in_{\mathcal{P}} \mathcal{G}$. Clearly $|$ is holomorphic in the sense that if $f \in \mathcal{O}(\mathcal{B}) \boxtimes \mathcal{P}^{\mathbb{C}}$ then $f| \in \mathcal{D}(\mathcal{G})^{\mathbb{C}} \hat{\boxtimes} \mathcal{O}(\mathcal{B}) \boxtimes \mathcal{P}^{\mathbb{C}}$.

From now on let Υ be a discrete $\mathcal{P}$-sub super Lie group of $\mathcal{G}$ with body $\Gamma \sqsubset G$.

Definition 3.1 (super automorphic forms) *Let $f \in \mathcal{O}(\mathcal{B}) \boxtimes \mathcal{P}^{\mathbb{C}}$. Then f is called a super automorphic form for Υ of weight k if and only if $\widetilde{f} \in (\mathcal{D}(\mathcal{G}) \boxtimes \mathcal{P})^{\mathbb{C}}$ is left-Υ-invariant or equivalently $f|_\gamma = f$ for all $\gamma \in_{\mathcal{P}} \Upsilon$. We denote the set of automorphic forms for Υ of weight k by $sM_k(\Upsilon)$. It is a graded sub $\mathcal{P}^{\mathbb{C}}$-module of $\mathcal{O}(\mathcal{B}) \boxtimes \mathcal{P}^{\mathbb{C}}$.*

Recall that $\Upsilon \backslash \mathcal{G}$ is a real $\left((p+q)^2 + r^2 - 1, 2(p+q)r\right)$-dimensional $\mathcal{P}$-super manifold with body $\Gamma \backslash G$ by theorem 2.46 , and $\widetilde{f}$ left-Υ-invariant means nothing but $\widetilde{f} \in \mathcal{D}\left(\Upsilon \backslash \mathcal{G}\right)^{\mathbb{C}}$.

Defining the space of cusp forms for a discrete $\mathcal{P}$-sub super Lie group Υ of $\mathcal{G}$ needs a notion of integrability in particular square integrability on $\mathcal{B}$ resp. $\mathcal{G}$ which seems to be very difficult to develop.

Therefore until the end of section 3.3 we restrict ourselves to the case where $m = 0$ equivalently $\mathcal{P} = \mathbb{R}$, and we call it the *non*-parametrized case.

Then $\Upsilon = \Gamma$ is nothing but a usual discrete subgroup of the $\mathcal{C}^\infty$-Lie group G . Clearly G is a sub super Lie group of $\mathcal{G}$, so G acts on $\mathcal{B}$ via $\alpha|_{G \times \mathcal{B}}$, and so we have a right action of G on $\mathcal{D}(\mathcal{B})$ given by

$$|_g : \mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{B}) \, , \, f \mapsto f|_g$$

for all $g \in G$. This action is clearly holomorphic in the sence that if $f \in \mathcal{O}(\mathcal{B})$ then again $f|_g \in \mathcal{O}(\mathcal{B})$ for all $g \in G$, and it respects the splittings

$$\mathcal{D}(\mathcal{B}) = \bigoplus_{(\mu,\nu) \in \{0,\dots,rq\}} \mathcal{D}^{(\mu,\nu)}(\mathcal{B}) \, ,$$

where

$$\mathcal{D}^{(\mu,\nu)}(\mathcal{B}) := \bigoplus_{I,J\in\wp(rq)\,,\,|I|=\mu\,,\,|J|=\nu} \mathcal{C}^\infty(B)\zeta^I\overline{\zeta}^J$$

and so also

$$\mathcal{O}(\mathcal{B}) = \bigoplus_{(\mu)\in\{0,\ldots,rq\}} \mathcal{O}^{(\mu)}(\mathcal{B})\,,$$

where

$$\mathcal{O}^{(\mu)}(\mathcal{B}) := \bigoplus_{I\in\wp(rq)\,,\,|I|=\mu} \mathcal{O}(B)\zeta^I = \mathcal{D}^{(\mu,0)}(\mathcal{B}) \cap \mathcal{O}(\mathcal{B})\,.$$

Now in the case of a usual discrete subgroup $\Gamma \sqsubset G$ another 'lift' seems to be more convenient:

$$\begin{aligned}
\widetilde{\ }' : \mathcal{D}(\mathcal{B}) &\rightarrow \mathcal{C}^\infty(G)^{\mathbb{C}} \otimes \Lambda\left(\mathbb{C}^{r\times q}\right) \boxtimes \Lambda\left(\mathbb{C}^{r\times q}\right) \\
&\qquad \simeq \mathcal{C}^\infty(G)^{\mathbb{C}} \otimes \mathcal{D}\left(\mathbb{C}^{0|rq,\overline{rq}}\right)\,, \\
f &\mapsto \widetilde{f}'\,,
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{f}'(g) &:= f|_g \left(\frac{\mathbf{0}}{\eta}\right) \\
&= f\left(g\left(\frac{\mathbf{0}}{\eta}\right)\right) j\left(g, \left(\frac{\mathbf{0}}{\eta}\right)\right)^k \\
&= f\left(g\left(\frac{\mathbf{0}}{\eta}\right)\right) j\left(g, \mathbf{0}\right)^k
\end{aligned}$$

for all $f \in \mathcal{D}(\mathcal{B})$ and $g \in G$, where we denote

$$j\left(g, \diamondsuit\right) = j\left(g, \left(\frac{\diamondsuit}{\zeta}\right)\right)\,,$$

since $j\left(g, \left(\frac{\diamondsuit}{\zeta}\right)\right) \in \mathcal{O}(B)$ and therefore 'independent' of ζ for all $g \in G$. Clearly again

$$
\begin{array}{ccc}
\mathcal{C}^\infty(G)^{\mathbb{C}} \otimes \mathcal{D}\left(\mathbb{C}^{0|rq,\overline{rq}}\right) & \stackrel{(g\diamondsuit)}{\longrightarrow} & \mathcal{C}^\infty(G)^{\mathbb{C}} \otimes \mathcal{D}\left(\mathbb{C}^{0|rq,\overline{rq}}\right) \\
\widetilde{\ }' \uparrow & \% & \uparrow \widetilde{\ }' \\
\mathcal{D}(\mathcal{B}) & \underset{|g}{\longrightarrow} & \mathcal{D}(\mathcal{B})
\end{array}
$$

for all $g \in G$, and clearly if $f \in \mathcal{O}(\mathcal{B})$ then

$$\widetilde{f}' \in \mathcal{C}^\infty(G)^{\mathbb{C}} \otimes \mathcal{O}\left(\mathbb{C}^{0|rq,\overline{rq}}\right) \simeq \mathcal{C}^\infty(G)^{\mathbb{C}} \otimes \Lambda\left(\mathbb{C}^{r\times q}\right) .$$

$f \in sM_k(\Gamma)$ if and only if $f \in \mathcal{O}(\mathcal{B})$ and $\widetilde{f}' \in \mathcal{C}^\infty(G)^{\mathbb{C}} \otimes \Lambda\left(\mathbb{C}^{r\times q}\right)$ is left-Γ-invariant.
Even $\widetilde{f}' \in \mathcal{C}^\infty(G)^{\mathbb{C}} \otimes \Lambda^{(\mu)}\left(\mathbb{C}^{r\times q}\right) \boxtimes \Lambda^{(\nu)}\left(\mathbb{C}^{r\times q}\right)$ if and only if $f \in \mathcal{D}^{(\mu,\nu)}(\mathcal{B})$ for all $f \in \mathcal{D}(\mathcal{B})$, and so $\widetilde{f}' \in \mathcal{C}^\infty(G)^{\mathbb{C}} \otimes \Lambda^{(\mu)}\left(\mathbb{C}^{r\times q}\right)$ if and only if $f \in \mathcal{O}^{(\mu)}(\mathcal{B})$ for all $f \in \mathcal{O}(\mathcal{B})$.

Since $\Lambda\left(\mathbb{C}^{r\times q}\right) \boxtimes \Lambda\left(\mathbb{C}^{r\times q}\right) \simeq \Lambda\left(\mathbb{C}^{2rq}\right)$ canonically we have a canonical scalar product $\langle\ ,\ \rangle$ (semilinear in the second entry) on $\Lambda\left(\mathbb{C}^{r\times q}\right) \boxtimes \Lambda\left(\mathbb{C}^{r\times q}\right)$ coming from the standard scalar product on $\mathbb{C}^{2rq}$. For all $a \in \Lambda\left(\mathbb{C}^{r\times q}\right) \boxtimes \Lambda\left(\mathbb{C}^{r\times q}\right)$ we write $|a| := \sqrt{\langle a, a\rangle}$.

We have a canonical embedding

$$G' := SU(p,q) \hookrightarrow G\,,\ g \mapsto \left(\begin{array}{c|c} g & 0 \\ \hline 0 & 1 \end{array}\right) ,$$

and the canonical projection

$$G \to U(r)\,,\ \left(\begin{array}{c|c} g & 0 \\ \hline 0 & E \end{array}\right) \mapsto E$$

induces a group isomorphism

$$G/G' \simeq U(r)\,.$$

Clearly $\alpha^{\#}$ extends the action of $G' = SU(p,q)$ on B by fractional linear (Möbius) transformations.

Since $G' = SU(p,q)$ is semisimple and $U(r)$ is compact both are unimodular. Furthermore the left- and right-invariant Haar measure on G' is even invariant under conjugation with elements of G . So a simple calculation shows that

$$\int_G \varphi := \int_{G/G'} \left(\int_{G'} \varphi(gn)dn\right) d\left(gG'\right)$$

is a left- and right-invariant HAAR measure on G , and therefore G is again unimodular. As in the higher rank case, chapter 1 , we have a 'scalar product'

$$(f,h)_\Gamma := \int_{\Gamma\backslash G} \left\langle \widetilde{h}', \widetilde{f}' \right\rangle$$

for all $f,g \in \mathcal{D}(\mathcal{B})$ such that $\left\langle \widetilde{h}', \widetilde{f}' \right\rangle \in L^1(\Gamma\backslash G)$ and for all $s \in \,]\,0,\infty\,]$

$$L^s_k(\Gamma\backslash\mathcal{B}) := \left\{ f \in \mathcal{D}(\mathcal{B}) \;\middle|\; \widetilde{f}' \text{ left-}\Gamma\text{-invariant and } \left|\widetilde{f}'\right| \in L^s(\Gamma\backslash G) \right\} .$$

Then clearly especially all $(\ ,\ ) := (\ ,\ )_{\{1\}}$ and all $L^s_k(\mathcal{B})$ are invariant under the action $|_g$, $g \in G$.

Definition 3.2 (super cusp forms in the *non*-parametrized case)
Let $f \in sM_k(\Gamma)$. f is called a super cusp form for Γ of weight k if and only if $f \in L^2_k(\Gamma\backslash\mathcal{B})$. The $\mathbb{C}$- vector space of all cups forms for Γ of weight k is denoted by $sS_k(\Gamma) := sM_k(\Gamma) \cap L^2_k(\Gamma\backslash\mathcal{B}) = \mathcal{O}(\mathcal{B}) \cap L^2_k(\Gamma\backslash\mathcal{B})$.

Let $\pi : G \to G/K \simeq B$ denote the canonical projection. Then clearly for all $L \subset B$ compact $\pi^{-1}(L) \subset G$ is again compact and there exists $C' \geq 0$ such that for all $h = \sum_{I,J\in\wp(r)} h_{IJ}\zeta^I\overline{\zeta}^J \in \mathcal{D}(\mathcal{B})$, all $h_{IJ} \in \mathcal{C}^\infty(B)^\mathbb{C}$, $I,J \in \wp(r)$, if we decompose $\widetilde{h}' = q_{IJ}\eta^I\overline{\eta}^J$, all $q_{IJ} \in \mathcal{C}^\infty(G)^\mathbb{C}$, $I,J \in \wp(r)$, then

$$||h_{IJ}||_{\infty,L} \leq C' \max_{I',J'\in\wp(r)} ||q_{I'J'}||_{\infty,\pi^{-1}(L)}$$

for all $I,J \in \wp(r)$. So we see that convergence with respect to $(\ ,\ )_\Gamma$ implies compact convergence, and so $sS_k(\Gamma)$ is a HILBERT space. Since all $\Lambda^{(\mu)}\left(\mathbb{C}^{r\times q}\right) \boxtimes \Lambda^{(\nu)}\left(\mathbb{C}^{r\times q}\right)$, $\mu,\nu = 0,\dots,rq$, are pairwise orthogonal with respect to $\langle\ ,\ \rangle$ we have an orthogonal splitting

$$sS_k(\Gamma) = \bigoplus_{\mu\in\{0,\dots,rq\}} sS^{(\mu)}_k(\Gamma)\,,$$

where

$$sS^{(\mu)}_k(\Gamma) := sS_k(\Gamma) \cap \mathcal{O}^{(\mu)}(\mathcal{B})$$

for all $\mu = 0,\dots,rq$.
As in the case of a usual bounded symmetric domain we would like to use relative POINCARÉ series

$$\sum_{\gamma\in\Gamma'\backslash\Gamma} f|_\gamma$$

for a subgroup $\Gamma' \subset \Gamma$.

Theorem 3.3 (convergence of relative POINCARÉ **series)** *Let* $\Gamma' \sqsubset \Gamma$ *be a subgroup and*

$$f \in sM_k\left(\Gamma'\right) \cap L_k^1\left(\Gamma' \backslash \mathcal{B}\right) .$$

Then

$$\Phi := \sum_{\gamma \in \Gamma' \backslash \Gamma} f|_\gamma \text{ and } \widetilde{\Phi}' := \sum_{\gamma \in \Gamma' \backslash \Gamma} \widetilde{f}'\left(\gamma \diamondsuit\right)$$

converge absolutely and uniformly on compact subsets of B *resp.* G *,*

$$\Phi \in sM_k\left(\Gamma\right) \cap L_k^1\left(\Gamma \backslash \mathcal{B}\right) ,$$

$\widetilde{\Phi}'$ *is the lift of* Φ *to* G *, and for all* $\varphi \in sM_k\left(\Gamma\right) \cap L_k^\infty\left(\Gamma \backslash \mathcal{B}\right)$ *we have*

$$\left(\Phi, \varphi\right)_\Gamma = \left(f, \varphi\right)_{\Gamma'} .$$

Proof: Let $g_0 \in G$ and $L \subset G$ be a compact neighbourhood of g_0 in G such that $\gamma L \cap L = \emptyset$ for all $\gamma \in \Gamma \setminus \{1\}$. Since the canonical projection $\pi : G \to G/K \simeq B$ is open, $\pi(L)$ is a compact neighbourhood of $g_0\mathbf{0}$ in B . Clearly since L is compact there exists $C' \geq 0$ such that for all $h = \sum_{I \in \wp(r)} h_I \zeta^I \in \mathcal{O}(\mathcal{B})$, all $h_I \in \mathcal{O}(B)$, $I \in \wp(r)$, if we decompose $\widetilde{h}' = q_I \eta^I$, all $q_I \in \mathcal{C}^\infty(G)^{\mathbb{C}}$, $I \in \wp(r)$, then

$$||h_I||_{\infty, \pi(L)} \leq C' \max_{J \in \wp(r)} ||q_J||_{\infty, L}$$

for all $I \in \wp(r)$ and

$$||q_J||_{\infty, L} \leq C' \max_{I \in \wp(r)} ||h_I||_{\infty, \pi(L)}$$

for all $J \in \wp(r)$. So by the mean value property of holomorphic functions applied to each $h_I \in \mathcal{O}(B)$, $I \in \wp(r)$, seperately there exists a neighbourhood $U \subset L$ of g_0 in G and $C \in \mathbb{R}$ such that for all $h \in \mathcal{O}(B)$ and $g \in U$

$$\left|\widetilde{h}'(g)\right| \leq C \int_L \left|\widetilde{h}'\right| .$$

So for all $g \in U$

$$\begin{aligned}\sum_{\gamma\in\Gamma'\backslash\Gamma}\left|\widetilde{f}'(\gamma g)\right| &= \sum_{\gamma\in\Gamma'\backslash\Gamma}\left|\widetilde{f|_\gamma}'(g)\right| \\ &\leq C\sum_{\gamma\in\Gamma'\backslash\Gamma}\int_L\left|\widetilde{f|_\gamma}'\right| \\ &= C\sum_{\gamma\in\Gamma'\backslash\Gamma}\int_L\left|\widetilde{f}'(\gamma\diamondsuit)\right| \\ &\leq C\int_{\Gamma'\backslash G}\left|\widetilde{f}'\right| < \infty\,.\end{aligned}$$

We see that $\widetilde{\Phi}'$ and, since for all $L\subset B$ again $\pi^{-1}(L)$ is compact, Φ as well converge absolutely and uniformly on compact subsets of B resp. G , and $\widetilde{\Phi}'$ is the lift of Φ to G . So clearly $\Phi\in sM_k(\Gamma)$.

$$\int_{\Gamma\backslash G}\left|\widetilde{\Phi}'\right| \leq \int_{\Gamma\backslash G}\sum_{\gamma\in\Gamma'\backslash\Gamma}\left|\widetilde{f}'(\gamma\diamondsuit)\right| = \int_{\Gamma'\backslash G}\left|\widetilde{f}'\right| < \infty\,,$$

and so $\Phi\in L^1_k(\Gamma\backslash B)$. Now let $\varphi\in L^\infty_k(\Gamma\backslash B)$. Then $\widetilde{f}'\widetilde{\varphi}'\in L^1(\Gamma'\backslash G)$, and so

$$(\Phi,\varphi)_\Gamma = \int_{\Gamma\backslash G}\sum_{\gamma\in\Gamma'\backslash\Gamma}\overline{\widetilde{f}'(\gamma\diamondsuit)}\widetilde{\varphi}' = \int_{\Gamma'\backslash G}\overline{\widetilde{f}'}\widetilde{\varphi}' = (f,\varphi)_{\Gamma'}\,.\square$$

From now on we assume $q=1$, and unlike in the higher rank case, where we had to do with an arbitrary bounded symmetric domain, in the special case of the unit ball $B=B^{p,1}\in\mathbb{C}^p$ now for our purposes it is more convenient to define classical automorphic resp. cusp forms on B with respect to the cocycle $j\in\mathcal{C}^\infty(G')^{\mathbb{C}}\hat{\otimes}\mathcal{O}(B)$ given by

$$j(g,\mathbf{z}) = (\mathbf{cz}+d)^{-1}$$

for all $g=\begin{pmatrix}A & \mathbf{b}\\ \mathbf{c} & d\end{pmatrix}\in G'$ and $\mathbf{z}\in B$, since it is the restriction of $j\in\left(\mathcal{D}(\mathcal{G})^{\mathbb{C}}\hat{\boxtimes}\mathcal{O}(\mathcal{B})\right)_0$ from above to $G'\times B$. Then $\det(\mathbf{z}\mapsto g\mathbf{z})' = j(g,\mathbf{z})^{p+1}$ for all $g\in G'$ and $\mathbf{z}\in B$. Again we can use the Jordan triple determinant $\Delta:\mathbb{C}^p\times\mathbb{C}^p\to\mathbb{C}$ which is given by

$$\Delta(\mathbf{z},\mathbf{w}) := 1-\mathbf{w}^*\mathbf{z}$$

for all $\mathbf{z},\mathbf{w}\in\mathbb{C}^p$. We recall the basic properties:

(i) $\Delta(\mathbf{0},\diamondsuit)=1$,

(ii) Δ is a sesqui polynomial, holomorphic in the first and antiholomorphic in the second variable,

(iii) $\Delta(\mathbf{z},\mathbf{w}) = \overline{\Delta(\mathbf{w},\mathbf{z})}$ for all $\mathbf{z},\mathbf{w} \in \mathbb{C}^p$ and $\Delta(\mathbf{z},\mathbf{z}) > 0$ for all $\mathbf{z} \in B$,

(iv) $|j(g,\mathbf{0})| = \Delta(g\mathbf{0}, g\mathbf{0})^{\frac{1}{2}}$ for all $g \in G$,

(v) $\Delta(g\mathbf{z}, g\mathbf{w}) = \Delta(\mathbf{z},\mathbf{w})\, j(g,\mathbf{z})\, \overline{j(g,\mathbf{w})}$ for all $g \in G$ and $\mathbf{z},\mathbf{w} \in B$, and

(vi) $\int_B \Delta(\mathbf{z},\mathbf{z})^\lambda \, dV_{\text{Leb}} < \infty$ if and only if $\lambda > -1$.

Because of (iv) and since $\left|\det(\mathbf{z} \mapsto g\mathbf{z})'\right| = |j(g,\mathbf{z})|^{p+1}$ for all $g \in G$ and $\mathbf{z} \in B$ on B we have the G-invariant volume element $\Delta(\mathbf{z},\mathbf{z})^{-(p+1)} dV_{\text{Leb}}$.

For all $I, J \in \wp(r)$, $h \in \mathcal{C}^\infty(B)^{\mathbb{C}}$, $\mathbf{z} \in B$ and $g = \left(\begin{array}{c|c} * & 0 \\ \hline 0 & E \end{array}\right) \in sSU(p,1|r)$ we have

$$\left. h\zeta^I \overline{\zeta}^J \right|_g (\mathbf{z}) = h(g\mathbf{z}) \,(E\eta)^I \left(\overline{E}\overline{\eta}\right)^J j(g,\mathbf{z})^{k+|I|+|J|} \, , \tag{3.1}$$

where $E \in U(r)$. So for all $s \in \left]0,\infty\right]$ and $f = \sum_{I,J \in \wp(r)} f_{IJ} \zeta^I \overline{\zeta}^J \in \mathcal{D}(\mathcal{B})$ such that $\widetilde{f}' \in \mathcal{C}^\infty(G)^{\mathbb{C}} \otimes \mathcal{D}\left(\mathbb{C}^{0|r,\overline{r}}\right)$ is left-Γ-invariant we have $f \in L^s_k(\Gamma\backslash\mathcal{B})$ if and only if

$$f_{IJ} \Delta(\mathbf{z},\mathbf{z})^{\frac{k+|I|+|J|}{2}} \in L^s(\Gamma\backslash B)$$

for all $I, J \in \wp(r)$ with respect to the G-invariant measure $\Delta(\mathbf{z},\mathbf{z})^{-(p+1)} dV_{\text{Leb}}$, and for all $f = \sum_{I,J \in \wp(r)} f_{IJ} \zeta^I \overline{\zeta}^J$,
$h = \sum_{I,J \in \wp(r)} h_{IJ} \zeta^I \overline{\zeta}^J \in \mathcal{D}(\Gamma\backslash B)$ such that $\left\langle \widetilde{h}', \widetilde{f}' \right\rangle \in L^1(\Gamma\backslash G)$

$$(f,h)_\Gamma \equiv \sum_{I,J \in \wp(r)} \int_{\Gamma\backslash B} \overline{f_{IJ}} h_{IJ} \Delta(\mathbf{z},\mathbf{z})^{\frac{k+|I|+|J|}{2} - (p+1)} \, dV_{\text{Leb}} \, .$$

In particular for all $s \in \left]0,\infty\right]$ and $I, J \in \wp(r)$ there is an embedding

$$L^s_{k+|I|+|J|}(B) \hookrightarrow L^s_k(\mathcal{B}) \, , \; f \mapsto f\zeta^I \overline{\zeta}^J$$

being unitary in the case $s = 2$ up to a constant $\neq 0$, and for all $s \in \left]0,\infty\right]$

$$L^s_k(\mathcal{B}) = \bigoplus_{I,J \in \wp(r)} L^s_{k+|I|+|J|}(B) \zeta^I \overline{\zeta}^J \, ,$$

where in the case $s = 2$ the sum is orthogonal.

Theorem 3.4 *Let $I \in \wp(r)$ and $k \geq 2p + 1 - |I|$. Then for all $\mathbf{w} \in B$*

$$\Delta(\diamondsuit, \mathbf{w})^{-k-|I|} \zeta^I \in L^1_k(\mathcal{B}) \, ,$$

and for all $f = \sum_{I \in \wp(r)} f_I \zeta^I \in \mathcal{O}(\mathcal{B}) \cap L^\infty_k(\mathcal{B})$ we have

$$\left(\Delta\left(\diamondsuit,\mathbf{w}\right)^{-k-|I|}\zeta^I,f\right)\equiv f_I\left(\mathbf{w}\right)\,,$$

where $\equiv$ *denotes equality up to a constant* $\neq 0$ *independent of* $\mathbf{w}$ *and* f *.*

Proof: Since for all $f\in\mathcal{O}(\mathcal{B})\cap L_k^\infty(\mathcal{B})$

$$\begin{aligned}\left(\Delta\left(\diamondsuit,\mathbf{w}\right)^{-k-|I|}\zeta^I,f\right) &= \left(\Delta\left(\diamondsuit,\mathbf{w}\right)^{-k-|I|}\zeta^I,f_I\zeta^I\right)\\ &\equiv \int_B\overline{\Delta\left(\diamondsuit,\mathbf{w}\right)^{-k-|I|}}f_I\Delta\left(\mathbf{z},\mathbf{z}\right)^{\frac{k+|I|}{2}-(p+1)}dV_{\mathrm{Leb}}\end{aligned}$$

it is the same calculation as in the proof of theorem 1.17 in the higher rank case. $\square$

Right now we see that there is a trivial special case, namely the case where $\Gamma\sqsubset G'=SU(p,1)\hookrightarrow G$.

Theorem 3.5 *Let* $\Gamma\sqsubset G'=SU(p,1)\hookrightarrow G$ *. Then for all* $I\in\wp(r)$ *the embedding*

$$S_{k+|I|}(\Gamma)\hookrightarrow sS_k(\Gamma)\,,\;f\mapsto f\zeta^I$$

is unitary up to a constant $\neq 0$ *, and*

$$sS_k(\Gamma)=\bigoplus_{I\in\wp(r)}S_{k+|I|}(\Gamma)\zeta^I$$

as an orthogonal sum.

Proof: obvious by formula (3.1) . $\square$

In the end let us compare the situation in the super case with that of the higher rank case in chapter 1 for again arbitrary $q\in\mathbb{N}\setminus\{0\}$. Obviously $K'=K\cap G'=S(U(p)\times U(q))$ is the stabilizer of $\mathbf{0}$ in G' . Let A denote the standard maximal split abelian subgroup of G' given by the image of the Lie group embedding

$\mathbb{R}^q \hookrightarrow G'$,

$$\mathbf{t} \mapsto a_{\mathbf{t}} := \left(\begin{array}{ccc|c|ccc}
\cosh t_1 & & 0 & & \sinh t_1 & & 0 \\
& \ddots & & 0 & & \ddots & \\
0 & & \cosh t_q & & 0 & & \sinh t_q \\
\hline
& 0 & & 1 & & 0 & \\
\hline
\sinh t_1 & & 0 & & \cosh t_1 & & 0 \\
& \ddots & & 0 & & \ddots & \\
0 & & \sinh t_q & & 0 & & \cosh t_q
\end{array}\right) .$$

Then A is at the same time a maximal abelian subgroup without compact factors, hence isomorphic to some $\mathbb{R}^\nu$, $\nu \in \mathbb{N}$, of G since $G/G' \simeq U(r)$ is compact. The centralizer M of A in K is the subgroup of K of all

$$\left(\begin{array}{ccc|c|ccc|c}
\varepsilon_1 & & 0 & & & & & \\
& \ddots & & 0 & & 0 & & \\
0 & & \varepsilon_q & & & & & \\
\hline
& 0 & & u & & & & 0 \\
\hline
& & & & \varepsilon_1 & & 0 & \\
& 0 & & & & \ddots & & \\
& & & & 0 & & \varepsilon_q & \\
\hline
& & & 0 & & & & E
\end{array}\right) ,$$

where $\varepsilon \in U(1)^q$, $u \in U(p-q)$ and $E \in U(r)$ such that $\varepsilon_1^2 \cdots \varepsilon_q^2 \det u = \det E$. Let $M' = K' \cap M = G' \cap M$ be the centralizer of A in K' .

Again on G we have an analytic multiflow $(\varphi_{\mathbf{t}})_{\mathbf{t}\in\mathbb{R}^q}$ given by the right translation by elements of A :

$$\varphi_{\mathbf{t}} : G \to G \, , \, g \mapsto g a_{\mathbf{t}} \, .$$

Th centralizer of G' in G is precisely

$$Z\left(G'\right) := \left\{ \left(\begin{array}{c|c} \varepsilon 1 & 0 \\ \hline 0 & E \end{array} \right) \middle| \; \varepsilon \in U(1) \, , \, E \in U(r) \, , \, \varepsilon^{p+q} = \det E \right\} \sqsubset M \, ,$$

and $G' \cap Z(G')$ is the centre of G' , which is finite and belongs to M' .

Lemma 3.6

$$G = G'Z\left(G'\right) \, .$$

Proof: Let $g = \left(\begin{array}{c|c} g' & 0 \\ \hline 0 & E \end{array}\right) \in G$. Then there exists $\varepsilon \in U(1)$ such that $\varepsilon^{p+q} = \det g' \in U(1)$, and so

$$g = \underbrace{\varepsilon^{-1}g'}_{\in G'} \underbrace{\left(\begin{array}{c|c} \varepsilon 1 & 0 \\ \hline 0 & E \end{array}\right)}_{\in Z(G')} . \square$$

So $K = K'Z\left(G'\right)$ and $M = M'Z\left(G'\right)$. Therefore if we decompose the adjoint representation of A as

$$\mathfrak{g} = \bigoplus_{\alpha \in \Phi} \mathfrak{g}^\alpha \, ,$$

where for all $\alpha \in (\mathbb{R}^q)^*$

$$\mathfrak{g}^\alpha := \left\{ \xi \in \mathfrak{g} \,\middle|\, \mathrm{Ad}_{a_\mathbf{t}}(\xi) = e^{\alpha \mathbf{t}} \right\}$$

and

$$\Phi := \{ \alpha \in (\mathbb{R}^q)^* \mid g^\alpha \neq 0 \}$$

then we see that theorem 1.7 remains true word by word, and we have $\mathfrak{g}^\alpha \sqsubset \mathfrak{g}'$ for all $\alpha \in \mathbb{R}^q \setminus \{\mathbf{0}\}$.

Lemma 3.7

$$N(A) = AN_K(A) = N(AM) \sqsubset N(M) \, .$$

Proof: We will use lemma 1.9 . For this purpose let $g = g'w \in G$, $g' \in G'$ and $w \in Z\left(G'\right)$.

'$N(A) \sqsubset AN_K(A)$' : Assume $g \in N(A)$. Then

$$g' = gw^{-1} \in N_{G'}(A) = AN_{K'}(A)$$

by lemma 1.9 , and so $g = g'w \in AN_K(A)$.

'$AN_K(A) \sqsubset N(AM)$' again trivial since M is the centralizer of A in K .

'$N(AM) \sqsubset N(M)$' : Assume $g \in N(AM)$. Then

$$g' \in N_{G'}\left(A\left(M \cap G'\right)\right) = N_{G'}\left(AM'\right) \sqsubset N_{G'}\left(M'\right)$$

by lemma 1.9 . So $g \in N(M)$.

'$N(AM) \sqsubset N(A)$' : Assume $g \in N(AM)$. Then

$$g' \in N_{G'}(AM) \sqsubset N_{G'}\left(AM'\right) = N_{G'}(A)$$

by lemma 1.9 . So $g \in N(A)$. $\square$

Again we define the WEYL group $W := N_K(A)/\,M$ acting on A via conjugation.

Lemma 3.8

$$W' \to W\,,\ gM' \mapsto gM\,,$$

where $W' := N_{K'}(A)/\,M'$ *denotes the* WEYL *group with respect to* G' *, is an isomorphism.*

Proof: Let $\varphi : N_{K'}(A) \to W\,,\ g \mapsto gM$, which is clearly a group homomorphism. Let $gM \in W$, $g \in N_K(A)$. Then we can write $g = g'w$ with $g' \in N_{K'}(A)$ and $w \in Z\left(G'\right) \sqsubset M$, so $\varphi(g') = gM$. Therefore φ is surjective. Now let $g \in \ker\varphi$. Then $g \in G' \cap M = M'$. Therefore $\ker\varphi = M'$. $\square$

From now on for simplicity we again assume $q = 1$. Then the root system Φ of G is simply $\Phi = \{-2, -1, 0, 1, 2\}$ if $p > 1$ and $\Phi = \{-2, 0, 2\}$ if $p = 1$, and the WEYL group degenerates to $W \simeq \{\pm 1\}$ changing sign on $\mathbb{R} \simeq A$.

Definition 3.9 *Let* $a \in G$ *.*

(i) a is called loxodromic if and only if there exists $g \in G$ *such that* $a \in gAMg^{-1}$ *.*

(ii) If a *is loxodromic, it is called regular if and only if* $a = ga_twg^{-1}$ *with* $t \in \mathbb{R} \setminus \{0\}$ *and* $w \in M$ *.*

(iii) If $a \in \Gamma$ *is regular loxodromic then it is called primitive in* Γ *if and only if* $a = a'^{\nu}$ *implies* $\nu \in \{\pm 1\}$ *for all loxodromic* $a' \in \Gamma$ *and* $\nu \in \mathbb{Z}$ *.*

Clearly for all $a \in \Gamma$ regular loxodromic there exists $a' \in \Gamma$ primitive regular loxodromic and $\nu \in \mathbb{N} \setminus \{0\}$ such that $a = a'^{\nu}$.

Theorem 3.10 *Let* $a \in G$ *be loxodromic,* $g \in G$ *,* $w \in M$ *and* $t \in \mathbb{R} \setminus \{\mathbf{0}\}$ *such that* $a = ga_twg^{-1}$ *. Then* g *is uniquely determined up to right translation by elements of* $AN_K(A)$ *, and* t *is uniquely determined up to sign.*

Proof: similar to the proof of theorem 1.11 (i) using lemma 3.7 instead of lemma 1.9 . □

3.2 Satake's theorem in the super case

Let $\Gamma \sqsubset G$ be a discrete subgroup. The main goal of this section is the following theorem:

Theorem 3.11 (Satake's theorem) *Assume* $\Gamma\backslash G$ *is compact* **or** $q = 1$ *,* $p \geq 2$ *and* $\Gamma \sqsubset G$ *is a lattice (discrete such that* vol $\Gamma\backslash G < \infty$ *,* $\Gamma\backslash G$ *not necessarily compact) . Then there exists* $k_0 \in \mathbb{N}$ *such that*

$$sS_k(\Gamma) = sM_k(\Gamma) \cap L_k^s(\Gamma\backslash\mathcal{B})$$

for all $s \in [1, \infty]$ *and* $k \geq k_0$ *.*

If $\Gamma\backslash G$ is compact then the assertion is trivial. In the case of $q = 1$, $p \geq 2$ and $\Gamma \sqsubset G$ being a lattice, not compact, we will give a proof in the end of this section using the so-called unbounded realization $\mathcal{H}$ of $\mathcal{B}$, which we will develop in the following. As in the higher rank case Satake's theorem and vol $\Gamma\backslash G < \infty$ imply that $sS_k(\Gamma)$ is finite dimensional for $k \geq k_0$ via lemma 12 of [1] section 10. 2 , see section 1.2 .

So let $q = 1$. As in the higher rank case define

$$\mathfrak{n} := \bigoplus_{\alpha\in\Phi_{>0}} \mathfrak{g}^\alpha ,$$

which is a sub Lie algebra of $\mathfrak{g}'$, and $N := \exp \mathfrak{n}$, which is a nilpotent sub Lie group of G' . As in the higher rank case we have an Iwasawa decomposition

$$G = KAN = NAK .$$

Clearly the group N is abelian in the case $p = 1$ and 2-step-nilpotent in the case $p > 1$. Let N' denote the centre of N . Then $N' = N$ if $p = 1$ and $N' = [N, N]$ if $p > 1$.

Now let $R \in G'^{\mathbb{C}} = SL(p+1, \mathbb{C})$ denote the partial Cayley transformation with respect to the tripotent $\mathbf{e}_1 = \begin{pmatrix} 1 \\ \hline \mathbf{0} \end{pmatrix} \begin{matrix} \leftarrow 1 \\ \}p-1 \end{matrix} \in \mathbb{C}^p$, see the end of section 1.1 . Via Möbius transformation it maps B biholomorphically onto the unbounded domain

$$H := \left\{ \mathbf{w} = \left(\begin{array}{c} w_1 \\ \hline \mathbf{w}_2 \end{array}\right) \begin{array}{l} \leftarrow 1 \\ \}p-1 \end{array} \in \mathbb{C}^p \,\middle|\, \mathrm{Re}\ w_1 > \frac{1}{2}\mathbf{w}_2^*\mathbf{w}_2 \right\} ,$$

$R\,\mathbf{0} = \mathbf{e}_1$, and we can compute R explicitely as

$$R = \left(\begin{array}{c|c|c} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \hline 0 & 1 & 0 \\ \hline -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{array}\right) \begin{array}{l} \leftarrow 1 \\ \}p-1 \\ \leftarrow p+1 \end{array} .$$

We see that $RG'R^{-1} \sqsubset G'^{\mathbb{C}} = SL(p+1,\mathbb{C}) \hookrightarrow GL(p+1,\mathbb{C}) \times GL(r,\mathbb{C})$ acts biholomorphically and transitively on H via Möbius transformations, and R commutes with all $g \in Z(G')$. Explicit calculations show that

$$a'_t := Ra_tR^{-1} = \left(\begin{array}{c|c|c} e^t & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & e^{-t} \end{array}\right) \begin{array}{l} \leftarrow 1 \\ \}p-1 \\ \leftarrow p+1 \end{array}$$

for all $t \in \mathbb{R}$, RNR^{-1} is the image of

$$\mathbb{R} \times \mathbb{C}^{p-1} \to RG'R^{-1} ,\ (\lambda, \mathbf{u}) \mapsto n'_{\lambda,\mathbf{u}} := \left(\begin{array}{c|c|c} 1 & \mathbf{u}^* & i\lambda + \frac{1}{2}\mathbf{u}^*\mathbf{u} \\ \hline 0 & 1 & \mathbf{u} \\ \hline 0 & 0 & 1 \end{array}\right) ,$$

which is a $\mathcal{C}^\infty$-diffeomorphism onto its image, with the multiplication rule

$$n'_{\lambda,\mathbf{u}} n'_{\mu,\mathbf{v}} = n'_{\lambda+\mu+\mathrm{Im}\ (\mathbf{u}^*\mathbf{v}),\mathbf{u}+\mathbf{v}}$$

for all $\lambda, \mu \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{C}^{p-1}$ and acting on H as pseudo translations

$$\mathbf{w} \mapsto \left(\begin{array}{c} w_1 + \mathbf{u}^*\mathbf{w}_2 + i\lambda + \frac{1}{2}\mathbf{u}^*\mathbf{u} \\ \hline \mathbf{w}_2 + \mathbf{u} \end{array}\right) ,$$

and $RN'R^{-1}$ is the image of the Lie group embedding

$$\mathbb{R} \to RG'R^{-1} ,\ \lambda \mapsto n'_{\lambda,\mathbf{0}}$$

acting on H as translations $\mathbf{w} \mapsto \mathbf{w} + i\lambda\mathbf{e}_1$.

Define $j(R, \diamondsuit) \in \mathcal{O}(B)$ as $j(R, \mathbf{z}) = \frac{\sqrt{2}}{1-z_1}$ for all $\mathbf{z} \in B$ and $j\left(R^{-1}, \diamondsuit\right) \in \mathcal{O}(H)$ as $j\left(R^{-1}, \mathbf{w}\right) := j\left(R, R^{-1}\mathbf{w}\right)^{-1} = \frac{\sqrt{2}}{1+w_1}$ for all $\mathbf{w} \in H$, and for all $g \in RGR^{-1}$ define $j(g, \diamondsuit) \in \mathcal{O}(H)$ as

$$j(g, \mathbf{w}) = j\left(R, R^{-1}g\mathbf{w}\right) j\left(R^{-1}gR, R^{-1}\mathbf{w}\right) j\left(R^{-1}, \mathbf{w}\right)$$

for all $\mathbf{w} \in H$. Then j remains a cocycle in the sense that for all $g, h \in RGR^{-1}$

$$j(gh, \mathbf{w}) = j(g, h\mathbf{w})\, j(h, \mathbf{w}) \ ,$$

and for all $g = \left(\begin{array}{cc|c} A & \mathbf{b} & \\ \hline \mathbf{c} & d & 0 \\ \hline \multicolumn{2}{c|}{0} & E \end{array}\right) \in RGR^{-1}$ an explicit computation of $j(g, \diamondsuit)$ gives

$$j(g, \mathbf{w}) = \frac{1}{\mathbf{c}\mathbf{w} + d}$$

for all $\mathbf{w} \in H$. Define the complex super domain $\mathcal{H}$ as $\mathcal{H} := H^{|r,\overline{r}}$. Then we have a right action of the group RGR^{-1} on $\mathcal{D}(\mathcal{H})$ given by

$$|_g : \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H})\, ,\ f \mapsto f\left(g\left(\frac{\diamondsuit}{\vartheta}\right)\right) j(g, \diamondsuit)^k$$

for all $g \in RGR^{-1}$, which is clearly holomorphic in the sense that if $f \in \mathcal{O}(\mathcal{H})$ then $f|_g \in \mathcal{O}(\mathcal{H})$ too. If we define

$$|_R : \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{B})\, ,\ f \mapsto f\left(R\left(\frac{\diamondsuit}{\vartheta}\right)\right) j(R, \diamondsuit)^k$$

and

$$|_{R^{-1}} : \mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{H})\, ,\ f \mapsto f\left(R^{-1}\left(\frac{\diamondsuit}{\vartheta}\right)\right) j\left(R^{-1}, \diamondsuit\right)^k \ ,$$

then we see that again if $f \in \mathcal{O}(\mathcal{H})$ then $f|_R \in \mathcal{O}(\mathcal{B})$, and if $f \in \mathcal{O}(\mathcal{B})$ then $f|_{R^{-1}} \in \mathcal{O}(\mathcal{H})$. $|_R$ and $|_{R^{-1}}$ are clearly invers to each other, and for all $g \in G$

$$\begin{array}{ccc} \mathcal{D}(\mathcal{H}) & \overset{|_{RgR^{-1}}}{\longrightarrow} & \mathcal{D}(\mathcal{H}) \\ |_R \ \downarrow & \% & \downarrow\ |_R \\ \mathcal{D}(\mathcal{B}) & \underset{|_g}{\longrightarrow} & \mathcal{D}(\mathcal{B}) \end{array} .$$

Now define the sesqui polynomial Δ' on $H \times H$, holomorphic in the first and antiholomorphic in the second variable, as

$$\Delta'(\mathbf{z}, \mathbf{w}) := \Delta\left(R^{-1}\mathbf{z}, R^{-1}\mathbf{w}\right) j\left(R^{-1}, \mathbf{z}\right)^{-1} \overline{j\left(R^{-1}, \mathbf{w}\right)}^{-1} = z_1 + \overline{w_1} - \mathbf{w}_2^* \mathbf{z}_2$$

for all $\mathbf{z}, \mathbf{w} \in H$. Clearly $\left|\det\left(\mathbf{z} \mapsto R\mathbf{z}\right)'\right| = |j(R, \mathbf{z})|^{p+1}$ for all $\mathbf{z} \in B$. So

$$\left|\det\left(\mathbf{w}\mapsto g\mathbf{w}\right)'\right| = \left|j\left(R,\mathbf{w}\right)\right|^{p+1} \ ,$$

$$\left|j\left(g,\mathbf{e}_1\right)\right| = \Delta'\left(g\mathbf{e}_1, g\mathbf{e}_1\right)^{\frac{1}{2}}$$

for all $g \in RGR^{-1}$ and $\Delta'\left(\mathbf{w},\mathbf{w}\right)^{-(p+1)} dV_{\text{Leb}}$ is the RGR^{-1} -invariant volume element on H . If $f = \sum_{I,J\in\wp(q)} f_{IJ}\zeta^I\overline{\zeta}^J \in \mathcal{D}(\mathcal{B})$, all $f_{IJ} \in \mathcal{C}^\infty(B)^{\mathbb{C}}$, $I,J \in \wp(q)$, then

$$f|_{R^{-1}} = \sum_{I,J\in\wp(q)} f_{IJ}\left(R^{-1}\diamondsuit\right) j\left(R^{-1},\diamondsuit\right)^{k+|I|+|J|} \vartheta^I\overline{\vartheta}^J \in \mathcal{D}(\mathcal{B}) \, ,$$

and if $f = \sum_{I,J\in\wp(q)} f_{IJ}\vartheta^I\overline{\vartheta}^J \in \mathcal{D}(\mathcal{H})$, all $f_{IJ} \in \mathcal{C}^\infty(H)^{\mathbb{C}}$, $I,J \in \wp(q)$, and $g = \left(\begin{array}{c|c} * & 0 \\ \hline 0 & E \end{array}\right) \in RGR^{-1}$, $E \in U(r)$, then

$$f|_g = \sum_{I,J\in\wp(q)} f_{IJ}\left(g\diamondsuit\right) j\left(g,\diamondsuit\right)^{k+|I|+|J|} \left(E\vartheta\right)^I \left(\overline{E\zeta}\right)^J \in \mathcal{D}(\mathcal{H}) \, .$$

Let $\partial H = \left\{\mathbf{w}\in\mathbb{C}^p \,\middle|\, \operatorname{Re} w_1 = \frac{1}{2}\mathbf{w}_2^*\mathbf{w}\right\}$ be the boundary of H in $\mathbb{C}^p$. Then Δ' and ∂H are RNR^{-1} -invariant, and RNR^{-1} acts transitively on ∂H and on each

$$\left\{\mathbf{w}\in H \,\middle|\, \Delta'\left(\mathbf{w},\mathbf{w}\right) = e^{2t}\right\} = RNa_t\mathbf{0} \, ,$$

$t \in \mathbb{R}$.

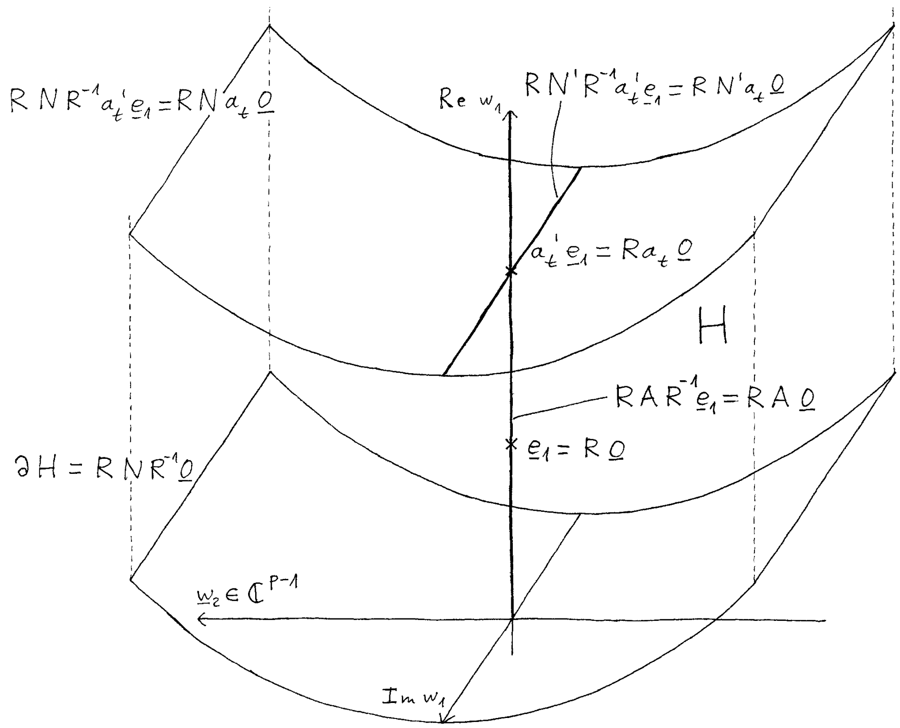

Figure 3.1: the geometry of H .

All geodesics in H can be written in the form

$$\mathbb{R} \to H \, , \, t \mapsto \mathbf{w}_t := Rga_t\mathbf{0} = RgR^{-1}a'_t\mathbf{e}_1$$

with some $g \in G$, and conversely all these curves are geodesics in H . We have to distinguish two cases: Either the goedesic connects ∞ with a point in ∂H , or it connects two points in ∂H . In the second case we have

$$\lim_{t \to \pm\infty} \Delta' \left(\mathbf{w}_t, \mathbf{w}_t\right) = 0 \, ,$$

so we may assume without loss of generality that $\Delta' \left(\mathbf{w}_t, \mathbf{w}_t\right)$ is maximal for $t = 0$, otherwise we have to reparametrize the geodesic using ga_T , $T \in \mathbb{R}$ appropiately chosen, instead of g .

Lemma 3.12

(i) Let

$$\mathbb{R} \to H \, , \, t \mapsto \mathbf{w}_t := Rga_t\mathbf{0} = RgR^{-1}a'_t\mathbf{e}_1$$

be a geodesic in H such that $\lim_{t\to\infty} \mathbf{w}_t = \infty$ and $\lim_{t\to-\infty} \in \partial H$ with respect to the euclidian metric on $\mathbb{C}^p$. Then for all $t \in \mathbb{R}$

$$\Delta' \left(\mathbf{w}_t, \mathbf{w}_t\right) = e^{2t} \Delta' \left(\mathbf{w}_0, \mathbf{w}_0\right) \, ,$$

and if instead $\lim_{t\to-\infty} \mathbf{w}_t = \infty$ *and* $\lim_{t\to\infty} \in \partial H$ *then*

$$\Delta'(\mathbf{w}_t, \mathbf{w}_t) = e^{-2t}\Delta'(\mathbf{w}_0, \mathbf{w}_0) \ .$$

(ii) Let

$$\mathbb{R} \to H \ , \ t \mapsto \mathbf{w}_t := Rga_t\mathbf{0} = RgR^{-1}a'_t\mathbf{e}_1$$

be a geodesic in H *connecting two points in* ∂H *such that* $\Delta'(\mathbf{w}_t, \mathbf{w}_t)$ *is maximal for* $t = 0$ *. Then*

$$\mathbb{R} \to \mathbb{R}_{>0} \ , \ t \mapsto \Delta'(\mathbf{w}_t, \mathbf{w}_t)$$

is strictly increasing on $\mathbb{R}_{\leq 0}$ *and strictly decreasing on* $\mathbb{R}_{\geq 0}$ *, and for all* $t \in \mathbb{R}$

$$\Delta'(\mathbf{w}_{-t}, \mathbf{w}_{-t}) = \Delta'(\mathbf{w}_t, \mathbf{w}_t)$$

and

$$e^{-2|t|}\Delta'(\mathbf{w}_0, \mathbf{w}_0) \leq \Delta'(\mathbf{w}_t, \mathbf{w}_t) \leq 4e^{-2|t|}\Delta'(\mathbf{w}_0, \mathbf{w}_0) \ .$$

Proof: (i) Since RNR^{-1} acts transitively on ∂H and Δ' is RNR^{-1} -invariant we can assume without loss of generality that the geodesic connects $\mathbf{0}$ and ∞ . But in H a geodesic is uniquely determined up to reparametrization by its endpoints. So we see that in the first case

$$w_t = a'_t x\mathbf{e}_1 = e^{2t}x\mathbf{e}_1$$

and in the second case

$$w_t = a'_{-t}x\mathbf{e}_1 = e^{-2t}x\mathbf{e}_1$$

both with an appropriately chosen $x \in \mathbb{R}_{>0}$. $\Box$

(ii) Let $u, y \in \mathbb{R}$ and $\mathbf{s} \in \mathbb{C}^{p-1}$ such that $y^2 + \mathbf{s}^*\mathbf{s} = 1$. Then

$$\mathbb{R} \to H \ , \ t \mapsto \mathbf{w}_t^{(u,y,\mathbf{s})} := \frac{e^u}{1 + y^2\tanh^2 t}\left(\frac{e^u\left(1 - y^2\tanh^2 t + 2iy\tanh t\right)}{\sqrt{2}\tanh t\,(1 + iy\tanh t)\,\mathbf{s}}\right)$$

is the geodesic through $e^{2u}\mathbf{e}_1$ in H since it is the image of the standard geodesic

$$\mathbb{R} \to B \ , \ t \mapsto a_t\mathbf{0} = \left(\frac{\tanh t}{\mathbf{0}}\right)$$

in B under the transformation

$$\underbrace{a'_u}_{\in RAR^{-1}\sqsubset RG'R^{-1}} \qquad \underbrace{R\left(\begin{array}{cc|c} iy & -\mathbf{s}^* & 0 \\ \mathbf{s} & -iy & 0 \\ \hline 0 & 0 & 1 \end{array}\right)}_{\in K'\sqsubset G'} .$$

So we see that $\partial_t \mathbf{w}_t^{(u,y,\mathbf{s})}\Big|_{t=0} = \begin{pmatrix} 2ie^{2u}y \\ \hline \sqrt{2}\,e^u\mathbf{s} \end{pmatrix} \in T_{e^{2u}\mathbf{e}_1}H$ is a unit vector with respect to RGR^{-1} -invariant metric on H .

Now since RNR^{-1} acts transitively on each

$$\left\{\mathbf{w} \in H \,\middle|\, \Delta'(\mathbf{w},\mathbf{w}) = e^{2t}\right\} = RNa_t\mathbf{0} ,$$

$t \in \mathbb{R}$, and Δ' is invariant under RNR^{-1} we may assume without loss of generality that $\mathbf{w}_0 = e^{2u}\mathbf{e}_1$ with an appropriate $u \in \mathbb{R}$. Since $\Delta'(\mathbf{w}_t,\mathbf{w}_t)$ is maximal for $t = 0$ we know that $\partial_t \mathbf{w}_t|_{t=0}$ is a unit vector in $i\mathbb{R} \oplus \mathbb{C}^{p-1} \sqsubset T_{\mathbf{e}_1}H$, and therefore there exist $y \in \mathbb{R}$ and $\mathbf{s} \in \mathbb{C}^{p-1}$ such that $y^2 + \mathbf{s}^*\mathbf{s} = 1$ and $\partial_t \mathbf{w}_t|_{t=0} = \begin{pmatrix} 2ie^{2u}y \\ \hline \sqrt{2}\,e^u\mathbf{s} \end{pmatrix}$. Since the geodesic is uniquely determined by $\mathbf{w}_0$ and $\partial_t \mathbf{w}_t|_{t=0}$ we see that $\mathbf{w}_t = \mathbf{w}_t^{(u,y,\mathbf{s})}$ for all $t \in \mathbb{R}$, and so a straight forward calculation shows that

$$\begin{aligned} \Delta'(\mathbf{w}_t,\mathbf{w}_t) &= 2e^{2u}\frac{1-\tanh^2 t}{1+y^2\tanh^2 t} \\ &= \frac{8e^{2u}}{(1+y^2)\left(e^{2t}+e^{-2t}\right)+2\mathbf{s}^*\mathbf{s}} . \end{aligned}$$

The rest is an easy exercise using $y^2 + \mathbf{s}^*\mathbf{s} = 1$. $\square$

For all $t \in \mathbb{R}$ and $\eta \subset N$ define $A_{<t} := \{a_\tau \mid \tau < t\} \subset A$ and $A_{>t} := \{a_\tau \mid \tau > t\} \subset A$.

Theorem 3.13 (a 'fundamental domain' for $\Gamma\backslash G$) *If $\Gamma\backslash G$ is not compact then there exist $\eta \subset N$ open and relatively compact , $t_0 \in \mathbb{R}$ and $\Xi \subset G'$ finite such that if we define*

$$\Omega := \bigcup_{g\in\Xi} g\eta A_{>t_0}K$$

then

(i) $g^{-1}\Gamma g \cap NZ(G') \sqsubset NZ(G')$ *and* $g^{-1}\Gamma g \cap N'Z(G') \sqsubset N'Z(G')$ *are lattices, and*

$$NZ(G') = \left(g^{-1}\Gamma g \cap NZ(G')\right)\eta Z(G')$$

for all $g \in \Xi$ *,*

(ii) $G = \Gamma\Omega$ *,*

(iii) the set $\{\gamma \in \Gamma \,|\, \gamma\Omega \cap \Omega \neq \emptyset\}$ *is finite.*

Proof: We use theorem 0.6 (i) - (iii) of [6] , which says the following:

> Let $\Gamma' \subset G'$ be an admissible discrete subgroup of G' . Then there exists $t_0' > 0$, an open, relatively compact subset $\eta_0 \subset N^+$, a finite set $\Xi \subset G'$, and an open, relatively compact subset Ω' of G' (Ξ being empty if G'/Γ' is compact, and Ω' being empty if G'/Γ' is non-compact) such that
>
> (i) For all $b \in \Xi$, $\Gamma \cap b^{-1}N^+b$ is a lattice in $b^{-1}N^+b$.
>
> (ii) For all $t > t_0'$ and for all open, relatively compact subsets η of N^+ such that $\eta \supset \eta_0$, if
>
> $$\Omega'_{t,\eta} = \Omega' \cup \left(\bigcup_{b\in\Xi} \sigma_{t,\eta} b\right) ,$$
>
> then $\Omega'_{t,\eta}\Gamma' = G'$, and
>
> (iii) the set $\left\{\gamma' \in \Gamma' \, , \, \Omega'_{t,\eta}\gamma' \cap \Omega'_{t,\eta} \neq \emptyset\right\}$ is finite.

Hereby G' is a connected semisimple LIE group of real rank 1 , $N^+ = N$ is the standard nilpotent sub LIE group of G' and $\sigma_{t,\eta} := K'A_{<t}\eta$ for all $t > 0$ and $\eta \subset N^+$ open and relatively compact, where A denotes the standard maximal non-compact abelian and K' the standard maximal compact sub LIE group of G' . Admissibility is a geometric property of the quotient $\Gamma'\backslash G'/K'$, roughly speaking Γ' is called admissible if and only if $\Gamma'\backslash G'/K'$ has only finitely many cusps.

Let us apply theorem 0.6 (i) - (iii) of [6] with $G' = SU(p,q) \hookrightarrow G$ and

$$\Gamma' := \left\{\gamma' \in G' \,\middle|\, \text{ there exists } w \in Z(G') \text{ such that } \gamma' w \in \Gamma\right\} \sqsubset G' ,$$

which of course is again a lattice such that $\Gamma'\backslash G'$ is not compact and so it is admissible in the sense of [6] by theorem 0.7 of [6] . By lemma 3.18 of [6] $g^{-1}\Gamma' g \cap N' \sqsubset N'$ is a lattice, and by lemma 3.16 of [6] applied with any

$\rho \in \Gamma' \cap N' \setminus \{1\}$ tells us that $\left(g^{-1}\Gamma' g \cap N\right) \setminus N$ is compact. So we see that there exist $t_0 \in \mathbb{R}$, $\eta \subset N$ open and relatively compact and $\Xi \subset G'$ finite such that for all $g \in \Xi$

$$\Gamma' \cap gNg^{-1} \sqsubset gNg^{-1}$$

is a lattice, $\Gamma'\Omega' = G'$ if we define $\Omega' = \bigcup_{b\in\Xi} b\eta A_{<t_0} K'$ and

$$\Delta := \left\{\gamma' \in \Gamma' \,\middle|\, \gamma'\Omega' \cap \Omega' \neq \emptyset\right\}$$

is finite.
(i) and (ii) : now trivial by definition of $\Gamma' \sqsubset G'$. $\Box$

(iii) : Let $\gamma = \gamma' w \in \Gamma$, $\gamma' \in \Gamma'$, $w \in Z(G')$, such that $\gamma\Omega \cap \Omega \neq \emptyset$. Then

$$\gamma'\Omega' Z\left(G'\right) \cap \Omega' Z\left(G'\right) \neq \emptyset .$$

Since $Z(G') \cap G' \sqsubset K'$ and Ω' is right-K'-invariant we have $\gamma'\Omega' \cap \Omega' \neq \emptyset$ as well and therefore $\gamma' \in \Delta$. Conversely $\gamma' Z(G')$ is compact and therefore $\Gamma \cap \gamma' Z(G')$ is finite for all $\gamma' \in \Gamma'$. $\Box$

Corollary 3.14 *Assume $\Gamma\backslash G$ is not compact, and let $t_0 \in \mathbb{R}$, $\eta \subset N$ and $\Xi \subset G$ be given by theorem 3.13 . Let $h \in \mathcal{C}(\Gamma\backslash G)^{\mathbb{C}}$ and $s \in]0, \infty]$. Then $h \in L^s(\Gamma\backslash G)$ if and only if $h(g\diamondsuit) \in L^s(\eta A_{>t_0} K)$ for all $g \in \Xi$.*

Proof: If $s = \infty$ then it is evident since $G = \Gamma\Omega$ by theorem 3.13 (ii) . Now assume $s \in \mathbb{R}_{>0}$, and assume $h \in L^s(\Gamma\backslash G)$.

$$S := \left|\{\gamma \in \Gamma \,|\, \gamma\Omega \cap \Omega \neq \emptyset\}\right| < \infty$$

by theorem 3.13 (iii) . Then for all $g \in \Xi$ we have

$$\int_{\eta A_{>t_0} K} |h(g\diamondsuit)|^s = \int_{g\eta A_{>t_0} K} |h|^s \leq \int_\Omega |h|^s \leq S \int_{\Gamma\backslash G} |h|^s < \infty .$$

Conversely assume $h(g\diamondsuit) \in L^s(\eta A_{>t_0} K)$ for all $g \in \Xi$. Then since $G = \Gamma\Omega$ by theorem 3.13 (ii) we obtain

$$\int_{\Gamma\backslash G} |h|^s \leq \int_\Omega |h|^s \leq \sum_{g\in\Xi} \int_{\eta A_{>t_0} K} |h(g\diamondsuit)|^s < \infty . \Box$$

Assume again $\Gamma\backslash G$ is not compact, and let $f \in sM_k(\Gamma)$ and $g \in \Xi$. Then we can decompose $f|_g|_{R^{-1}} = \sum_{I\in\wp(r)} q_I \vartheta^I \in \mathcal{O}(\mathcal{H})$, all $q_I \in \mathcal{O}(H)$, $I \in \wp(r)$, and by theorem 3.13 (i) we know that $g^{-1}\Gamma g \cap N' Z(G') \not\subset Z(G')$. So let $n \in g^{-1}\Gamma g \cap N' Z(G') \setminus Z(G')$,

$$RnR^{-1} = n'_{\lambda_0,\mathbf{0}} \left(\begin{array}{c|c} \varepsilon 1 & 0 \\ \hline 0 & E \end{array} \right) ,$$

$\lambda_0 \in \mathbb{R} \setminus \{0\}$, $\varepsilon \in U(1)$, $E \in U(r)$, $\varepsilon^{p+1} = \det E$.
$j\left(RnR^{-1}\right) := j\left(RnR^{-1}, \mathbf{w}\right) = \varepsilon^{-1} \in U(1)$ is independent of $\mathbf{w} \in H$. So there exists $\chi \in \mathbb{R}$ such that $j\left(RnR^{-1}\right) = e^{2\pi i\chi}$. Without loss of generality we can assume that E is diagonal, otherwise conjugate n with an appropriate element of $Z(G')$. So there exists $D \in \mathbb{R}^{r\times r}$ diagonal such that $E = \exp(2\pi i D)$. If $D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_r \end{pmatrix}$ and $I \in \wp(r)$ then we define $\mathrm{tr}_I D := \sum_{j\in I} d_j$.

Theorem 3.15 (Fourier expansion of $f|_g|_{R^{-1}}$)

(i) There exist unique $c_{I,m} \in \mathcal{O}\left(\mathbb{C}^{p-1}\right)$, $I \in \wp(r)$, $m \in \frac{1}{\lambda_0}(\mathbb{Z} - \mathrm{tr}_I D - (k+|I|)\chi)$, such that

$$q_I(\mathbf{w}) = \sum_{m \in \frac{1}{\lambda_0}(\mathbb{Z} - \mathrm{tr}_I D - (k+|I|)\chi)} c_{I,m}(\mathbf{w}_2) e^{2\pi m w_1}$$

for all $\mathbf{w} \in H$ and $I \in \wp(r)$, and so

$$f|_g|_{R^{-1}}(\mathbf{w}) = \sum_{I\in\wp(r)} \sum_{m \in \frac{1}{\lambda_0}(\mathbb{Z} - \mathrm{tr}_I D - (k+|I|)\chi)} c_{I,m}(\mathbf{w}_2) e^{2\pi m w_1} \vartheta^I$$

for all $\mathbf{w} = \left(\begin{array}{c} w_1 \\ \hline \mathbf{w}_2 \end{array} \right) \begin{array}{l} \leftarrow 1 \\ \}p-1 \end{array} \in H$, where the convergence is absolute and compact.

(ii) Assume $p \geq 2$. Then $c_{I,m} = 0$ for all $I \in \wp(r)$ and $m \in \frac{1}{\lambda_0}(\mathbb{Z} - \mathrm{tr}_I D - (k+|I|)\chi) \cap \mathbb{R}_{>0}$ (this is a super analogon for Koecher's principle, see for example in section 11.5 of [1]) , and if $\mathrm{tr}_I D + (k+|I|)\chi \in \mathbb{Z}$ then $c_{I,0}$ is a constant.

(iii) Assume again $p \geq 2$, and let $I \in \wp(r)$ and $s \in [1,\infty]$. If $\mathrm{tr}_I D + (k+|I|)\chi \notin \mathbb{Z}$ then

$$q_I \Delta'(\mathbf{w},\mathbf{w})^{\frac{k+|I|}{2}} \in L^s\left(R\eta A_{>t_0}\mathbf{0}\right)$$

with respect to the RGR^{-1} -invariant measure $\Delta'(\mathbf{w},\mathbf{w})^{-(p+1)} dV_{\mathrm{Leb}}$ on H . If $\mathrm{tr}_I D + (k+|I|)\chi \in \mathbb{Z}$ and $k \geq 2p - |I|$ then

$$q_I \Delta' (\mathbf{w}, \mathbf{w})^{\frac{k+|I|}{2}} \in L^s \left(R\eta A_{>t_0}\mathbf{0}\right)$$

with respect to the RGR^{-1} *-invariant measure on* H *if and only if* $c_{I,0} = 0$.

Proof: (i) $f|_g$ is $g^{-1}\Gamma g$ invariant, so we see that for all $\mathbf{w} \in H$

$$\begin{aligned}
\sum_{I \in \wp(r)} q_I (\mathbf{w}) \vartheta^I &= f|_g|_{R^{-1}} (\mathbf{w}) \\
&= f|_g|_n|_{R^{-1}} (\mathbf{w}) \\
&= \sum_{I \in \wp(r)} q_I (\mathbf{w} + i\lambda_0 \mathbf{e}_1) \left(E\vartheta j \left(RnR^{-1}\right)\right)^I j \left(RnR^{-1}\right)^k \\
&= \sum_{I \in \wp(r)} q_I (\mathbf{w} + i\lambda_0 \mathbf{e}_1) e^{2\pi i (\mathrm{tr}_I D + (k+|I|)\chi)} \vartheta^I .
\end{aligned}$$

Therefore for all $I \in \wp(r)$

$$q_I = q_I (\diamondsuit + i\lambda_0 \mathbf{e}_1) e^{2\pi i (\mathrm{tr}_I D + (k+|I|)\chi)} .$$

Let $I \in \wp(r)$. Then $h \in \mathcal{O}(H)$ given by

$$h(\mathbf{w}) := q_I (\mathbf{w}) e^{-2\pi i \frac{1}{\lambda_0} (\mathrm{tr}_I D + (k+|I|)\chi) w_1}$$

for all $\mathbf{w} \in H$ is $i\lambda_0 \mathbf{e}_1$ periodic, and therefore there exists $\widehat{h}$ holomorphic on

$$\widehat{H} := \left\{ \mathbf{z} = \left(\begin{array}{c} z_1 \\ \hline \mathbf{z}_2 \end{array} \right) \begin{array}{l} \leftarrow 1 \\ \}p-1 \end{array} \middle| \, |z_1| > e^{\frac{\pi}{\lambda_0} \mathbf{z}_2^* \mathbf{z}_2} \right\}$$

such that for all $\mathbf{w} \in H$

$$h(\mathbf{w}) = \widehat{h} \left(\begin{array}{c} e^{\frac{2\pi}{\lambda}} w_1 \\ \hline \mathbf{w}_2 \end{array} \right) .$$

Laurent expansion now tells us that there exist $a_{m,\mathbf{l}} \in \mathbb{C}$, $m' \in \mathbb{Z}$, $\mathbf{l} \in \mathbb{N}^{p-1}$, such that

$$\widehat{h}(\mathbf{z}) = \sum_{m' \in \mathbb{Z}} \sum_{\mathbf{l} \in \mathbb{N}^{p-1}} a_{m',\mathbf{l}} z_1^{m'} \mathbf{z}_2^{\mathbf{l}}$$

for all $\mathbf{z} = \left(\begin{array}{c} z_1 \\ \hline \mathbf{z}_2 \end{array} \right) \begin{array}{l} \leftarrow 1 \\ \}p-1 \end{array} \in \widehat{H}$, where the convergence is absolute and compact. Now let us define $d_{m'} \in \mathcal{O}\left(\mathbb{C}^{p-1}\right)$ as

$$d_{m'} (\mathbf{z}) := \sum_{\mathbf{l} \in \mathbb{N}^{p-1}} a_{m',\mathbf{l}} \mathbf{z}_2^{\mathbf{l}} ,$$

$m' \in \mathbb{Z}$. Then for all $\mathbf{w} \in H$

$$q_I(\mathbf{w})\, e^{-\frac{2\pi i}{\lambda_0}(\mathrm{tr}_I D + (k+|I|)\chi) w_1} = h(\mathbf{w}) = \sum_{m' \in \mathbb{Z}} d_{m'}(\mathbf{w}_2)\, e^{\frac{2\pi}{\lambda_0} m' w_1} .$$

So taking $c_m := d_{\lambda_0 m + \mathrm{tr}_I D + (k+|I|)\chi}$, $m \in \frac{1}{\lambda_0}(\mathbb{Z} - \mathrm{tr}_I D - (k+|I|)\chi)$, gives the desired result. Uniqueness follows from standard FOURIER theory. $\square$

(ii) *Step I* **Show that all q_I , $I \in \wp(r)$, are bounded on $RN\mathbf{0} = \{\mathbf{w} \in H \mid \Delta'(\mathbf{w}, \mathbf{w}) = 2\}$.**

Obviously all q_I , $I \in \wp(r)$, are bounded on $R\eta\mathbf{0}$ since $R\eta\mathbf{0}$ is relatively compact in H . Let $C \geq 0$ such that $|q_I| \leq C$ on $R\eta\mathbf{0}$ for all $I \in \wp(r)$. By theorem 3.13

$$RN\mathbf{0} = R\left(g^{-1}\Gamma g \cap NZ(G')\right)\eta\mathbf{0} .$$

So let $Rn'R^{-1} = n'_{\lambda',\mathbf{u}} \left(\begin{array}{c|c} \varepsilon' 1 & 0 \\ \hline 0 & E' \end{array}\right) \in g^{-1}\Gamma g \cap NZ(G')$, $\lambda' \in \mathbb{R}$, $\mathbf{u} \in \mathbb{C}^{p-1}$, $\varepsilon' \in U(1)$ and $E' \in (r)$. Then again

$$j\left(Rn'R^{-1}\right) := j\left(Rn'R^{-1}, \mathbf{w}\right) = \varepsilon'^{-1} \in U(1)$$

is independent of $\mathbf{w} \in H$.

$$\begin{aligned} \sum_{I \in \wp} q_I \vartheta^I &= f|_g|_{R^{-1}} \\ &= f|_g|_{n'}|_{R^{-1}} \\ &= \sum_{I \in \wp(r)} q_I\left(Rn'R^{-1}\diamondsuit\right)\left(E'\vartheta\right)^I \varepsilon'^{k+|I|} . \end{aligned}$$

$\Lambda(\mathbb{C}^r) \to \Lambda(\mathbb{C}^r)$, $\vartheta^I \mapsto (E'\vartheta)^I \varepsilon'^{k+|I|}$ is unitary, therefore

$$|q_I| \leq 2^r \left|q_I\left(Rn'R^{-1}\diamondsuit\right)\right| .$$

We see that $|q_I| \leq 2^r C$ on $RN\mathbf{0}$.

Step II **Show that**

$$\left|c_{I,m}(\mathbf{w}_2)\, e^{2\pi m w_1}\right| \leq ||q_I||_{\infty, RN\mathbf{0}}$$

on $RN\mathbf{0}$ for all $I \in \wp(r)$ and $m \in \frac{1}{\lambda_0}(\mathbb{Z} - \mathrm{tr}_I D - (k+|I|)\chi)$.

Let $I \in \wp(r)$ and $m \in \frac{1}{\lambda_0}(\mathbb{Z} - \mathrm{tr}_I D - (k+|I|)\chi)$. By classical FOURIER analysis

$$c_{I,m}\left(\mathbf{w}_2\right)e^{2\pi m w_1} = \frac{1}{\lambda_0}\int_0^{\lambda_0} q_I\left(\mathbf{w}+i\lambda\mathbf{e}_1\right)e^{-2\pi i m\lambda}$$

for all $\mathbf{w}\in H$, and since $\mathbf{w}+i\lambda\mathbf{e}_1 = n'_{\lambda,\mathbf{0}}\mathbf{w}\in RNR^{-1}\mathbf{w}$ the claim follows.

Step III **Conclusion.**

Let $I\in\wp(r)$ and $m\in\frac{1}{\lambda_0}\left(\mathbb{Z}-\mathrm{tr}_I D-(k+|I|)\chi\right)$. Let $\mathbf{u}\in\mathbb{C}^{p-1}$ be arbitrary. Then

$$\left(\frac{1+\frac{1}{2}\mathbf{u}^*\mathbf{u}}{\mathbf{u}}\right)\in RN\mathbf{0}\,,$$

and so

$$\left|c_{I,m}\left(\mathbf{u}\right)\right|\leq\left|\left|q_I\right|\right|_{\infty,RN\mathbf{0}}e^{-\pi m\mathbf{u}^*\mathbf{u}}\,.$$

Now the assertion follows by LIOUVILLE's theorem . $\square$

(iii) Let

$$\eta' := \left\{(iy,\mathbf{u})\in i\mathbb{R}\oplus\mathbb{C}^{p-1}\;\middle|\;\left(\frac{1+\frac{1}{2}\mathbf{u}^*\mathbf{u}+iy}{\mathbf{u}}\right)\in R\eta\mathbf{0}\right\}$$

be the projection of $R\eta\mathbf{0}$ onto $i\mathbb{R}\oplus\mathbb{C}^{p-1}$ in direction of $\mathrm{Re}\,w_1\in\mathbb{R}$. Then

$$\Psi:\mathbb{R}_{>e^{2t_0}}\times\eta'\to R\eta A_{>t_0}\mathbf{0}\,,\;(x,iy,\mathbf{u})\mapsto\left(\frac{x+\frac{1}{2}\mathbf{u}^*\mathbf{u}+iy}{\mathbf{u}}\right)$$

is a $\mathcal{C}^\infty$-diffeomorphism with determinant 1 , and

$$\Delta'\left(\Psi\left(x,iy,\mathbf{u}\right),\Psi\left(x,iy,\mathbf{u}\right)\right)=2x$$

for all $(x,iy,\mathbf{u})\in\mathbb{R}_{>e^{2t_0}}\times\eta'$. So

$$q_I\Delta'\left(\mathbf{w},\mathbf{w}\right)^{\frac{k+|I|}{2}}\in L^s\left(R\eta A_{>t_0}\mathbf{0}\right)$$

with respect to the measure $\Delta'\left(\mathbf{w},\mathbf{w}\right)^{-(p+1)}dV_{\mathrm{Leb}}$ if and only if

$$\left(q_I\circ\Psi\right)x^{\frac{k+|I|}{2}}\in L^s\left(\mathbb{R}_{>e^{2t_0}}\times\eta'\right)$$

with respect to the measure $x^{-(p+1)}dV_{\mathrm{Leb}}$.

Now assume either $\mathrm{tr}_I D+(k+|I|)\chi\notin\mathbb{Z}$ or $\mathrm{tr}_I D+(k+|I|)\chi\in\mathbb{Z}$ and $c_{I,0}\neq 0$. Then in both cases by (ii) we can write

$$q_I(\mathbf{w}) = \sum_{m \in \frac{1}{\lambda_0}(\mathbb{Z} - \mathrm{tr}_I D - (k+|I|)\chi) \cap \mathbb{R}_{<0}} c_{I,m}(\mathbf{w}_2)\, e^{2\pi m w_1}$$

for all $\mathbf{w} \in H$, where the sum converges absolutely and uniformly on compact subsets of H . Since $R\eta a_{t_0}\mathbf{0} \subset H$ is relatively compact we can define

$$C'' := e^{-2\pi M_0 e^{2t_0}} \sum_{m \in \frac{1}{\lambda_0}(\mathbb{Z} - \mathrm{tr}_I D - (k+|I|)\chi) \cap \mathbb{R}_{<0}} \left|\left| c_{I,m}(\mathbf{w}_2)\, e^{2\pi m w_1} \right|\right|_{\infty, R\eta a_{t_0}\mathbf{0}} < \infty \, .$$

If we define in addition

$$M_0 := \max \frac{1}{\lambda_0}(\mathbb{Z} - \mathrm{tr}_I D - (k+|I|)\chi) \cap \mathbb{R}_{<0} < 0$$

then we see that

$$|q_I(\mathbf{w})| \leq C'' e^{\pi M_0 \Delta'(\mathbf{w},\mathbf{w})}$$

for all $\mathbf{w} \in R\eta A_{>t_0}\mathbf{0}$, so

$$|q_I \circ \Psi| \leq C'' e^{2\pi M_0 x} \, ,$$

and so $x^{\frac{k+|I|}{2}}(q_I \circ \Psi) \in L^s(\mathbb{R}_{>e^{2t_0}} \times \eta')$ with respect to the measure $x^{-(p+1)} dV_{\mathrm{Leb}}$.

Conversely assume $\mathrm{tr}_I D + (k+|I|)\chi \in \mathbb{Z}$, $k \geq 2p - |I|$ and $c_{I,0} \neq 0$. Then as before we have the estimate

$$\left| \sum_{m \in \frac{1}{\lambda_0}\mathbb{N} \setminus \{0\}} c_{I,m}(\mathbf{w}_2)\, e^{2\pi m w_1} \right| \leq C'' e^{-\pi \Delta'(\mathbf{w},\mathbf{w})}$$

for all $\mathbf{w} \in R\eta A_{>t_0}\mathbf{0}$ if we define

$$C'' := e^{2\pi e^{2t_0}} \sum_{m \in \frac{1}{\lambda_0}(\mathbb{Z} - \mathrm{tr}_I D - (k+|I|)\chi) \cap \mathbb{R}_{<0}} \left|\left| c_{I,m}(\mathbf{w}_2)\, e^{2\pi m w_1} \right|\right|_{\infty, R\eta a_{t_0}\mathbf{0}} < \infty \, .$$

Therefore there exists $S \in \mathbb{R}_{\geq 0}$ such that

$$\left| \sum_{m \in \frac{1}{\lambda_0}\mathbb{N} \setminus \{0\}} c_{I,m}(\mathbf{w}_2)\, e^{2\pi m w_1} \right| \leq \frac{1}{2} |c_{I,0}|$$

and therefore $|q_I(\mathbf{w})| \geq \frac{1}{2}|c_{I,0}|$ for all $\mathbf{w} \in R\eta A_{>t_0}\mathbf{0}$ such that $\Delta'(\mathbf{w},\mathbf{w}) \geq S$. So $|(q_I \circ \Phi)(x, iy, \mathbf{u})| \geq \frac{1}{2}|c_{I0}|$ for all $(x, iy, \mathbf{u}) \in \mathbb{R}_{\geq S} \times \eta'$, and therefore definitely $x^{\frac{k+|I|}{2}} q_I \circ \Phi \notin L^s\left(\mathbb{R}_{>e^{2t_0}} \times \eta'\right)$ with respect to the measure $x^{-(p+1)} dV_{\mathrm{Leb}}$. $\square$

Now we prove theorem 3.11 in the case of $p \geq 2$ and $\Gamma \sqsubset G$ being a lattice, $\Gamma \backslash G$ not compact.

Let $k \geq k_0 := 2p \in \mathbb{N}$. Since $\mathrm{vol}\, \Gamma \backslash G < \infty$ it suffices to show that $f \in sM_k(\Gamma)$ and $\widetilde{f'} \in L^1(\Gamma \backslash G) \otimes \Lambda(\mathbb{C}^r)$ imply $\widetilde{f'} \in L^\infty(\Gamma \backslash G) \otimes \Lambda(\mathbb{C}^r)$. So let $f \in sM_k(\Gamma)$ such that $\widetilde{f'} \in L^1(\Gamma \backslash G) \otimes \Lambda(\mathbb{C}^r)$. Let $g \in \Xi$. By corollary 3.14 it is even enough to show that $\widetilde{f'}(g\diamondsuit) \in L^\infty(\eta A_{>t_0} K) \otimes \Lambda(\mathbb{C}^r)$.

Let $f|_g|_{R^{-1}} = \sum_{I \in \mathfrak{p}(r)} q_I \vartheta^I$, all $q_I \in \mathcal{O}(H)$, $I \in \wp(r)$. Then

$$f|_g = \sum_{I \in \wp(r)} q_I(R\diamondsuit) \zeta^I j(R, \diamondsuit)^{k+|I|} .$$

Since by corollary 3.14 $\widetilde{f'} \in L^1(\eta A_{>t_0} K)$ we conclude that

$$q_I(R\mathbf{z})\, j(R, \mathbf{z})^{k+|I|} \Delta(\mathbf{z}, \mathbf{z})^{\frac{k+|I|}{2}} \in L^1(\eta A_{>t_0} \mathbf{0})$$

with respect to the G-invariant measure on B or equivalently $q_I \Delta'(\mathbf{w}, \mathbf{w})^{\frac{k+|I|}{2}} \in L^1(R\eta A_{>t_0} \mathbf{0})$ for all $I \in \wp(r)$ with respect to the RGR^{-1} -invariant measure on H . So by theorem 3.15 (iii) we see that $q_I \Delta'(\mathbf{w}, \mathbf{w})^{\frac{k+|I|}{2}} \in L^\infty(R\eta A_{>t_0} \mathbf{0})$ as well or equivalently $q_I(R\mathbf{z})\, j(R, \mathbf{z})^{k+|I|} \Delta(\mathbf{z}, \mathbf{z})^{\frac{k+|I|}{2}} \in L^\infty(\eta A_{>t_0} \mathbf{0})$ for all $I \in \wp(r)$. Therefore

$$\widetilde{f'}(g\diamondsuit) \in L^\infty(\eta A_{>t_0} K) \otimes \Lambda(\mathbb{C}^r) . \square$$

3.3 A spanning set for the space of super cusp forms in the non-parametrized case

Again assume $q = 1$. Assume $\Gamma \backslash G$ compact **or** $p \geq 2$ and $\mathrm{vol}\, \Gamma \backslash G < \infty$ and $k \geq k_0$, where $k_0 \in \mathbb{N}$ is given by Satake's theorem, theorem 3.11 . Let $C > 0$. Let us first consider a regular loxodromic $\gamma_0 \in \Gamma$. Let $g \in G$, $w_0 \in M$ and $t_0 \in \mathbb{R} \setminus \{0\}$ such that $\gamma_0 = g a_{t_0} w_0 g^{-1}$.

There exists a torus $\mathbb{T} := \langle \gamma_0 \rangle \backslash gAM$ belonging to γ_0 . As in the higher rank case, chapter 1 , one can prove that $\mathbb{T}$ is independent of g up to right translation with an element of the Weyl group $W = N_K(A)/M$ using theorem 3.10 .

Let $f \in sS_k(\Gamma)$. Then $\widetilde{f'} \in \mathcal{C}^\infty(\Gamma \backslash G)^{\mathbb{C}} \otimes \Lambda(\mathbb{C}^r)$. Define $h \in \mathcal{C}^\infty(\mathbb{R} \times M)^{\mathbb{C}} \otimes \Lambda(\mathbb{C}^r)$ as

$$h(t,w) := \widetilde{f'}(g a_t w)$$

for all $(t,w) \in \mathbb{R} \times M$. Then clearly $h(t,w) = j(w)^k h(t,1,E\eta j(w))$, and so $h(t,w) = j(w)^{k+\mu} h(t,1,E\eta)$ if $f \in sS_k^{(\mu)}(\Gamma)$ for all $(t,w) \in \mathbb{R} \times M$, $w = \left(\begin{array}{c|c} * & 0 \\ \hline 0 & E \end{array}\right)$, $E \in U(r)$. Clearly $w_0 = \left(\begin{array}{c|c} * & 0 \\ \hline 0 & E_0 \end{array}\right)$ with some $E_0 \in U(r)$. So we can choose $g \in G$ such that E_0 is diagonal without changing $\mathbb{T}$. Choose $D \in \mathbb{R}^{r\times r}$ diagonal such that $\exp(2\pi i D) = E_0$ and $\chi \in \mathbb{R}$ such that $j(w_0) = e^{2\pi i \chi}$. D and χ are uniquely defined by w_0 up to $\mathbb{Z}$. If $D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_r \end{pmatrix}$ with $d_1 \ldots, d_r \in \mathbb{R}$ and $I \in \wp(r)$ then again we define $\mathrm{tr}_I D := \sum_{j\in I} d_j$.

Theorem 3.16 (Fourier **expansion of h)**

(i) $h(t+t_0,w) = h\left(t, w_0^{-1}w\right)$ *for all* $(t,w) \in \mathbb{R} \times M$ *, and there exist unique* $b_{I,m} \in \mathbb{C}$ *,* $I \in \wp(r)$ *,* $m \in \frac{1}{t_0}\left(\mathbb{Z} - (k+|I|)\chi - \mathrm{tr}_I D\right)$ *, such that*

$$h(t,w) = \sum_{I\in\wp(r)} j(w)^{k+|I|} \sum_{m \in \frac{1}{t_0}(\mathbb{Z}-(k+|I|)\chi - \mathrm{tr}_I D)} b_{I,m} e^{2\pi i m t} (E\eta)^I$$

for all $(t,w) \in \mathbb{R} \times M$ *,* $w = \left(\begin{array}{c|c} * & 0 \\ \hline 0 & E \end{array}\right)$ *,* $E \in U(r)$ *, where the sum converges uniformly in all derivatives.*

(ii) If $b_{I,m} = 0$ *for all* $I \in \wp(r)$ *and* $m \in \frac{1}{t_0}\left(\mathbb{Z} - (k+|I|)\chi - \mathrm{tr}_I D\right) \cap \left] -C, C\right[$ *then there exists* $H \in \mathcal{C}^\infty(\mathbb{R} \times M)^{\mathbb{C}} \otimes \Lambda(\mathbb{C}^r)$ *uniformly* Lipshitz *continuous with a* Lipshitz *constant* $C_2 \geq 0$ *independent of* γ_0 *such that*

$$h = \partial_t H ,$$

$$H(t,w) = j(w)^k H(t,1,E\eta j(w))$$

and

$$H(t+t_0,w) = H\left(t, w_0^{-1}w\right)$$

for all $(t,w) \in \mathbb{R} \times M$, $w = \left(\begin{array}{c|c} * & 0 \\ \hline 0 & E \end{array}\right)$, $E \in U(r)$ *.*

Proof: (i) Let $t \in \mathbb{R}$ and $w \in M$. Then

$$h\left(t+t_0,w\right) = \widetilde{f}'\left(a_{t_0}a_tw\right) = \widetilde{f}'\left(\gamma_0 w_0^{-1}a_tw\right) = \widetilde{f}'\left(a_tw_0^{-1}w\right) = h\left(t,w_0^{-1}w\right) ,$$

and so

$$\begin{aligned} h\left(t+t_0,1\right) &= h\left(t,w_0^{-1}\right) \\ &= j\left(w_0\right)^{-k} h\left(t,1,E_0^{-1}\eta j\left(w_0\right)^{-1}\right) \\ &= j\left(w_0\right)^{-k} \sum_{I\in\wp(r)} h\left(t,1\right) e^{-2\pi i \mathrm{tr}_I D}\eta^I j\left(w_0\right)^{-|I|} \\ &= \sum_{I\in\wp(r)} e^{-2\pi i((k+|I|)\chi+\mathrm{tr}_I D)} h_I(t,1)\eta^I . \end{aligned}$$

Therefore $h_I\left(t+t_0,1\right) = e^{-2\pi i((k+|I|)\chi+\mathrm{tr}_I D)} h_I(t,1)$ for all $I \in \wp(r)$, and the rest follows by standard FOURIER expansion. $\square$

(ii) Let $b_{I,m} = 0$ for all $I \in \wp(r)$ and
$m \in \frac{1}{t_0}\left(\mathbb{Z} - (k+|I|)\chi - \mathrm{tr}_I D\right) \cap \left]-C,C\right[$, and fix $I \in \wp(r)$. Then

$$h_I(\diamondsuit,1) = \sum_{m\in\frac{1}{t_0}(\mathbb{Z}-(k+|I|)\chi-\mathrm{tr}_I D)\cap\left]-C,C\right[} b_{I,m} e^{2\pi i m\diamondsuit} ,$$

and so we can apply the generalized reverse BERNSTEIN inequality, theorem 1.30 , to h_I with $q=1$, $\Lambda := t_0\mathbb{Z} \sqsubset \mathbb{R}$ lattice,
$\chi := (k+|I|)\chi + \mathrm{tr}_I D \in \mathbb{R}^* = \mathbb{R}$ and $\mathbf{v}' = 1$. Therefore we can define

$$H_I' := \widehat{h_I(\diamondsuit,1)} = \sum_{m\in\frac{1}{t_0}(\mathbb{Z}-(k+|I|)\chi-\mathrm{tr}_I D)\cap\left]-C,C\right[} \frac{b_{I,m}}{2\pi i m} e^{2\pi i m\diamondsuit} \in \mathcal{C}^\infty\left(\mathbb{R}\right)^{\mathbb{C}} .$$

By SATAKE's theorem , theorem 3.11 , since $f \in sS_k(\Gamma)$,
$\left|\widetilde{f}'\right| \in L^\infty(G)$, and so there exists a constant $C' > 0$ independent of γ_0 such that $||h_I||_\infty < C'$ for all $I \in \wp(r)$, and now theorem 1.30 tells us that

$$\left|\left|H_I'\right|\right|_\infty \leq \frac{6}{\pi C}\left|\left|h\left(\diamondsuit,1\right)\right|\right|_\infty \leq \frac{6C'}{\pi C} .$$

Clearly $h_I\left(\diamondsuit,1\right) = \partial_t H_I'$.

Since j is smooth on the compact set M , all $j^{k+|I|}(E\eta)^I$, $I \in \wp(r)$, are uniformly LIPSHITZ continuous on M with a common LIPSHITZ constant C'' independent of γ_0 . So we see that $H \in \mathcal{C}^\infty(\mathbb{R}, M)^{\mathbb{C}} \otimes \Lambda(\mathbb{C}^r)$ defined as

$$H(t,w) := \sum_{I \in \wp(r)} j(w)^{k+|I|} H'_I(t) (E\eta)^I$$

for all $(t,w) \in \mathbb{R} \times M$ is uniformly LIPSHITZ continuous with LIPSHITZ constant $C_2 := \left(\frac{6C''}{\pi C} + 1\right) C'$ independent of γ_0 , and the rest is trivial. □

Let $I \in \wp(r)$ and $m \in \frac{1}{t_0}(\mathbb{Z} - (k+|I|)\chi - \mathrm{tr}_I D)$. Since $sS_k(\Gamma)$ is a HILBERT space and $sS_k(\Gamma) \to \mathbb{C}$, $f \mapsto b_{I,m}$ is linear and continuous there exists exactly one $\varphi_{\gamma_0,I,m} \in sS_k(\Gamma)$ such that $b_{I,m} = (\varphi_{\gamma_0,I,m}, f)$ for all $f \in sS_k(\Gamma)$.
Clearly having g fixed, the family

$$\{\varphi_{\gamma_0,I,m}\}_{I \in \wp(r)\,,\, m \in \frac{1}{t_0}(\mathbb{Z}-(k+|I|)\chi - \mathrm{tr}_I D) \cap]-C,C[}$$

is independent of the choice of D and χ , but even independent of the choice of $g \in G$ itself up to multiplication with a unitary matrix with entries in $\mathbb{C}$ and invariant under conjugating γ_0 with elements of Γ . Let us check it.

Let $g' \in G$, $t'_0 \in \mathbb{R}$ and $w'_0 = \left(\begin{array}{c|c} * & 0 \\ \hline 0 & E'_0 \end{array}\right) \in M$, $E'_0 \in U(r)$ diagonal, such that also $\gamma_0 = g' a_{t'_0} w'_0 g'^{-1}$. Then by theorem 3.10 there exist $T \in \mathbb{R}$ and $n = \left(\begin{array}{c|c} * & 0 \\ \hline 0 & E_n \end{array}\right) \in N_K(A)$, $E_n \in U(r)$, such that $g' = g a_T n$. Then $a_{t'_0} = n^{-1} a_{t_0} n$, and so $t'_0 = t_0$ if $n \in M$ and $t'_0 = -t_0$ if $n \notin M$, $w'_0 = w_n^{-1} w_0 w_n$, and so $E'_0 = E_n^{-1} E_0 E_n$. $j(w'_0) = j(w_0) = e^{2\pi i \chi}$, and $E'_0 = \exp(2\pi i D')$ if we define $D' := E_n^{-1} D E_n$. Without loss of generality we may assume that two diagonal elements of D are equal if the corresponding diagonal elements of E_0 are equal. Then D' is again diagonal. Let $h' \in \mathcal{C}^\infty(\mathbb{R} \times M)^{\mathbb{C}} \otimes \Lambda(\mathbb{C}^r)$ be given by $h'(t,w) := \widetilde{f}'(g' a_t w)$ for all $t \in \mathbb{R}$ and $w \in M$. Then

$$\begin{aligned} h'(t,1) &= \widetilde{f}'(g a_T n a_t) \\ &= \widetilde{f}'(g a_T a_{t'} n) \\ &= j(n)^k \widetilde{f}'(g a_{\mathbf{t'+T}}, E_n \eta j(n)) \\ &= j(n)^k h\left(t' + T, 1, E_n \eta j(n)\right) , \end{aligned}$$

where we obtain t' by transforming t with the element $nM \in W$, so $t' = t$ if $n \in M$ and $t' = -t$ if $n \notin M$. If we decompose

$$h'(t,w) = \sum_{I \in \wp(r)} j(w)^{k+|I|} \sum_{m \in \frac{1}{t_0'}(\mathbb{Z}-(k+|I|)\chi - \mathrm{tr}_I D')} b'_{I,m} e^{2\pi i m t} (E\eta)^I$$

for all $(t,w) \in \mathbb{R} \times M$, all $b'_{l,m} \in \mathbb{C}$, $I \in \wp(r)$, $m \in \frac{1}{t_0'}(\mathbb{Z} - (k+|I|)\chi - \mathrm{tr}_I D')$, then we get

$$\begin{aligned}
&\sum_{I \in \wp(r)} \sum_{m \in \frac{1}{t_0'}(\mathbb{Z}-(k+|I|)\chi - \mathrm{tr}_I D')} b'_{I,m} e^{2\pi i m t} \eta^I \\
&= h'(t,1) \\
&= j(n)^k h\left(t'+T, 1, E_n \eta j(n)\right) \\
&= \sum_{I \in \wp(r)} j(n)^{k+|I|} \times \\
&\quad \times \sum_{m \in \frac{1}{t_0}(\mathbb{Z}-(k+|I|)\chi - \mathrm{tr}_I D)} b_{I,m} e^{2\pi i m (t'+T)} (E_n \eta)^I \ . \quad (3.2)
\end{aligned}$$

Let $\varphi'_{I,m} \in sS_k(\Gamma)$ such that $b'_{I,m} = \left(\varphi'_{I,m}, f\right)$ for all $f \in sS_k(\Gamma)$, $I \in \wp(r)$, $m \in \frac{1}{t_0'}(\mathbb{Z} - (k+|I|)\chi - \mathrm{tr}_I D')$.

Without loss of generality we can assume that either $n \in N_{K'}(A) \setminus M'$ or $n \in M$ itself and $T = 0$, since by lemma 3.8 a general $a_T n$ is a product of the two types.

In the first case $E_n = 1$, $t_0' = -t_0$ and $t' = -t$ for all $t \in \mathbb{R}$. So by equation 3.2 we see that $b'_{I,m'} = j(n)^{k+|I|} e^{2\pi i T} b_{I,m}$ for all $m \in \frac{1}{t_0}(\mathbb{Z} - (k+|I|)\chi - \mathrm{tr}_I D)$ and so $\varphi'_{I,m'} = j(n)^{k+|I|} e^{2\pi i T} \varphi_{\gamma_0,I,m}$, where we obtain m' by transforming m with $nM \in W$, so $m' = -m$.

Now let us treat the case $n \in M$ and $T = 0$. Then $t_0' = t_0$ and $t' = t$ for all $t \in \mathbb{R}$. Without loss of generality we can assume that either E_n is a permutation matrix or E_n stabilizes each eigenspace of E_0 , again a general E_n is a product of the two types since $E_n \in U(r)$ and $E_0' = E_n^{-1} E_0 E_n$ is again diagonal.

In the first case let $\sigma \in \mathfrak{S}(r)$ such that $E_n e_j = e_{\sigma(j)}$ for all $j = 1, \ldots, r$. Then we have $(E_n \eta j(n))^I = \varepsilon_I \eta^{\sigma^{-1}(I)} j(n)^{|I|}$ with some $\varepsilon_I \in \{\pm 1\}$ and $\mathrm{tr}_{\sigma^{-1}(I)} D' = \mathrm{tr}_I D$ for all $I \in \wp(r)$. So again by equation 3.2 we see that

$$b'_{\sigma^{-1}(I),m} = \varepsilon_I j(n)^{k+|I|} b_{I,m} \,,$$

and so $\varphi'_{\sigma^{-1}(I),m} = \varepsilon_I j(n)^{-k-|I|} \varphi_{\gamma_0,I,m}$.

In the second case, where E_n stabilizes each eigenspace of E_0 we have $E'_0 = E_0$ and therefore $D' = D$. Since $E_n \in U(r)$ there exists a unitary matrix $(\varepsilon_{IJ})_{I,J\in\wp(r)}$ with entries in $\mathbb{C}$ such that

$$(E_n\eta)^I = \sum_{J\in\wp(r)} \varepsilon_{IJ}\eta^J \,,$$

if $\varepsilon_{IJ} \neq 0$ then $|I| = |J|$, and since E_n stabilizes each eigenspace of D if $\varepsilon_{IJ} \neq 0$ then $\mathrm{tr}_I D = \mathrm{tr}_J D$. So by equation 3.2

$$\begin{aligned}
&\sum_{J\in\wp(r)} \sum_{m\in\frac{1}{t_0}(\mathbb{Z}-(k+|J|)\chi-\mathrm{tr}_J D)} b'_{J,m} e^{2\pi imt} \eta^J \\
&= \sum_{I\in\wp(r)} j(n)^{k+|I|} \sum_{m\in\frac{1}{t_0}(\mathbb{Z}-(k+|I|)\chi-\mathrm{tr}_I D)} b_{I,m} e^{2\pi im(t'+T)} (E_n\eta)^I \\
&= \sum_{J\in\wp(r)} j(n)^{k+|J|} \times \\
&\quad \times \sum_{m\in\frac{1}{t_0}(\mathbb{Z}-(k+|J|)\chi-\mathrm{tr}_J D)} \sum_{I\in\wp(r)} \varepsilon_{IJ} b_{I,m} e^{2\pi imT} \eta^J \,,
\end{aligned}$$

so $b'_{J,m} = j(n)^{k+|J|} \sum_{I\in\wp(r)} \varepsilon_{IJ} b_{I,m}$ and

$$\varphi'_{J,m} = j(n)^{-k-|J|} \sum_{I\in\wp(r)} \overline{\varepsilon_{IJ}} \varphi_{\gamma_0,I,m} \,.$$

Now let $\gamma \in \Gamma$ and $\gamma'_0 := \gamma\gamma_0\gamma^{-1}$. Then clearly $\gamma'_0 = \gamma g a_{t_0} w_0 (\gamma g)^{-1}$, and so, if we define $h' \in \mathcal{C}^\infty(\mathbb{R}\times M)^{\mathbb{C}} \otimes \Lambda(\mathbb{C}^r)$ by $h'(t,w) := \widetilde{f'}(\gamma g a_t w)$ for all $t \in \mathbb{R}$ and $w \in M$, then we get

$$h'(t,w) = \widetilde{f'}(\gamma g a_t w) = h(t,w)$$

by the left-Γ-invariance of $\widetilde{f'}$.

For the rest of the chapter for simplicity we write $m \in]-C, C[$ instead of $m \in \frac{1}{t_0}(\mathbb{Z} - (k+|I|)\chi - \mathrm{tr}_I D) \cap]-C, C[$. In the end we will compute $\varphi_{\gamma_0,I,m}$ as a relative Poincaré series.

Now we can state our main theorem: Let Ω be a fundamental set for all primitive regular loxodromic $\gamma_0 \in \Gamma$ modulo conjugation by elements of Γ .

Theorem 3.17 (spanning set for $sS_k(\Gamma)$) *Assume that the right translation of A on $\Gamma\backslash G$ is topologically transitive. Then*

$$\{\varphi_{\gamma_0,I,m} \mid \gamma_0 \in \Omega, I \in \wp(r), m \in]-C,C[\}$$

is a spanning set for $sS_k(\Gamma)$.

Proof: The Lie algebra embedding $\rho : \mathfrak{sl}(2,\mathbb{C}) \hookrightarrow \mathfrak{g}'^{\mathbb{C}}$ of theorem 1.32 in the higher rank case now has an explicit description:

$$\rho : \mathfrak{sl}(2,\mathbb{C}) \hookrightarrow \mathfrak{g}'^{\mathbb{C}} = \mathfrak{sl}(p+1,\mathbb{C}), \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mapsto \left(\begin{array}{c|c|c} a & 0 & b \\ \hline 0 & 0 & 0 \\ \hline c & 0 & -a \end{array}\right).$$

By theorem 1.32 or by explicit computation one can see that the preimage of $\mathfrak{g}'$ under ρ is $\mathfrak{su}(1,1)$, the preimage of $\mathfrak{k}'$ under ρ is $\mathfrak{s}(\mathfrak{u}(1) \oplus \mathfrak{u}(1)) \simeq \mathfrak{u}(1)$, and ρ lifts to a Lie group homomorphism

$$\widetilde{\rho} : SL(2,\mathbb{C}) \to G'^{\mathbb{C}} = SL(p+1,\mathbb{C}), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(\begin{array}{c|c|c} a & 0 & b \\ \hline 0 & 0 & 0 \\ \hline c & 0 & d \end{array}\right)$$

such that $\widetilde{\rho}(SU(1,1)) \sqsubset G'$.

Let us now identify the elements of $\mathfrak{g}$ with the corresponding left invariant differential operators, they are defined on a dense subset of $L^2(\Gamma\backslash G)$, and define

$$\begin{aligned} \mathcal{D} &:= \rho \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{a} \\ \mathcal{D}' &:= \rho \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \in \mathfrak{g}' \\ \phi &:= \rho \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathfrak{k}'. \end{aligned}$$

We see that $\mathfrak{a} = \mathbb{R}\mathcal{D}$, and so as left-invariant differential operator $\mathcal{D}$ generates the flow φ_t . Again the $\mathbb{R}$-linear span of $\mathcal{D}$, $\mathcal{D}'$ and ϕ is the 3-dimensional sub Lie algebra $\rho(\mathfrak{su}(1,1))$ of $\mathfrak{g}' \sqsubset \mathfrak{g}$, and we have the following commutation relations:

$$[\phi, \mathcal{D}] = 2\mathcal{D}' \, , \, \left[\phi, \mathcal{D}'\right] = -2\mathcal{D} \, , \, \left[\mathcal{D}, \mathcal{D}'\right] = -2\phi \, .$$

ϕ generates a subgroup of $K' \sqsubset K$, and

$$\mathbb{R}/2\pi\mathbb{Z} \to K \, , \, t \mapsto \exp(t\phi) = \rho \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}$$

is an isomorphism. Now define

$$\mathcal{D}^+ := \frac{1}{2}\left(\mathcal{D} - i\mathcal{D}'\right) \, , \, \mathcal{D}^- := \frac{1}{2}\left(\mathcal{D} + i\mathcal{D}'\right) \text{ and } \Psi := -i\phi$$

as left invariant differential operators on G . Then clearly

$$\left[\Psi, \mathcal{D}^+\right] = 2\mathcal{D}^+, \left[\Psi, \mathcal{D}^-\right] = -2\mathcal{D}^- \text{ and } \left[\mathcal{D}^+, \mathcal{D}^-\right] = \Psi \, .$$

As in the higher rank case we see that

$$\left(\mathcal{D}^+\right)^* = -\mathcal{D}^-, \left(\mathcal{D}^-\right)^* = -\mathcal{D}^+ \text{ and } \Psi^* = \Psi \, ,$$

and so by standard Fourier analysis

$$L^2\left(\Gamma\backslash G\right) = \widehat{\bigoplus_{\nu \in \mathbb{Z}}} H_\nu$$

as an orthogonal sum, where

$$H_\nu := \left\{ F \in L^2\left(\Gamma\backslash G\right) \cap \text{ domain } \Psi \,\middle|\, \Psi F = \nu F \right\}$$

for all $\nu \in \mathbb{Z}$. By a simple calculation we obtain

$$\mathcal{D}^+\left(H_\nu \cap \text{ domain } \mathcal{D}^+\right) \subset H_{\nu+2} \text{ and } \mathcal{D}^-\left(H_\nu \cap \text{ domain } \mathcal{D}^-\right) \subset H_{\nu-2}$$

for all $\nu \in \mathbb{Z}$.

Lemma 3.18 *For all $h \in \mathcal{O}(\mathcal{B}) \simeq \mathcal{O}(B) \otimes \Lambda\left(\mathbb{C}^r\right)$*

$$\mathcal{D}^- \widetilde{h}' = 0 \, .$$

Proof: Let $g \in G$. Then again $h|_g \in \mathcal{O}(\mathcal{B})$, and $\widetilde{h}'\left(g\diamondsuit\right) = \widetilde{h|_g}'$. So

$$\mathcal{D}^- \widetilde{h}'(g) = \mathcal{D}^-\left(\widetilde{h}'\left(g\diamondsuit\right)\right)(1) = \overline{\partial}_1 h|_g = 0 \, . \square$$

Lemma 3.19 *Let $f \in sS_k(\Gamma)$. Then $\widetilde{f}'$ is uniformly* Lipshitz *continuous.*

Proof: similar to the proof in the higher rank case, chapter 1 , using SATAKE's theorem, theorem 3.11 . □

Now let us return to the LIE group G . Choose a left invariant metric on G such that $\mathfrak{g}^\alpha$, $\alpha \in \Phi \setminus \{0\}$, $\mathfrak{a}$ and $\mathfrak{m}$ are pairwise orthogonal and the isomorphism $\mathbb{R} \simeq A \subset G$ is even isometric. Then since the flow $(\varphi_t)_{t\in\mathbb{R}}$ commutes with left translations it is partially hyperbolic with constant 1 , as one sees immediately in the root space decomposition of $\mathfrak{g}$. The corresponding splitting of the tangent bundle of G is the unique left invariant splitting given by

$$T_1G = \mathfrak{g} = \underbrace{\mathfrak{a} \oplus \mathfrak{m}}_{T_1^0:=} \oplus \underbrace{\bigoplus_{\alpha\in\Phi\,,\,\alpha>0} \mathfrak{g}^\alpha}_{T_1^-:=} \oplus \underbrace{\bigoplus_{\alpha\in\Phi\,,\,\alpha<0} \mathfrak{g}^\alpha}_{T_1^+:=} .$$

Indeed $T^0 \oplus T^+$, $T^0 \oplus T^-$, T^0 , T^+ and T^- are closed under the commutator since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ if $\alpha+\beta \in \Phi$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$ otherwise for all $\alpha, \beta \in \Phi$. So we can apply the partial ANOSOV closing lemma, theorem 1.21 , which here is again really convenient since G acts transitively and isometrically on itself by left translations.

As in the higher rank case for all $L \subset G$ compact, $T, \varepsilon > 0$ define

$$M_{L,T} := \left\{ g a_t g^{-1} \,\middle|\, g \in L, t \in [-T, T] \right\}$$

and

$$N_{L,T,\varepsilon} := \{ g \in G \mid \mathrm{dist}\,(g, M_{L,T}) \le \varepsilon \} .$$

Lemma 3.20 *For all $L \subset G$ compact there exist $T_0, \varepsilon_0 > 0$ such that $\Gamma \cap N_{L,T_0,\varepsilon_0} = \{1\}$.*

Proof: same as in the higher rank case, chapter 1 . □

Theorem 3.21

(i) For all $T_1 > 0$ there exist $C_1 \ge 1$ and $\varepsilon_1 > 0$ such that for all $x \in G$, $\gamma \in \Gamma$ and $T \ge T_1$ if

$$\varepsilon := d\,(\gamma x, x a_T) \le \varepsilon_1$$

then there exist $z \in G$, $w \in M$ and $t \in \mathbb{R} \setminus \{0\}$ such that $\gamma z = z a_t w$, $d\,((t, w), (T, 1)) \le C_1 \varepsilon$ and for all $\tau \in [0, T]$

$$d\,(x a_\tau, z a_\tau) \le C_1 \varepsilon \left(e^{-\tau} + e^{-(T-\tau)} \right) .$$

(ii) For all $L \subset G$ compact there exists $\varepsilon_2 > 0$ such that for all $x \in L$, $\gamma \in \Gamma$ and $T \in [0, T_0]$, $T_0 > 0$ given by lemma 3.20 , if

$$\varepsilon := d(\gamma x, x a_T) \leq \varepsilon_2$$

then $\gamma = 1$ and $T \leq \varepsilon$.

Proof: same as in the higher rank case. □

Now let $f \in sS_k(\Gamma)$ such that $(\varphi_{\gamma_0,I,m}, f) = 0$ for all $\varphi_{\gamma_0,I,m}$, $\gamma_0 \in \Gamma$ primitive loxodromic, $I \in \wp(r)$, $m \in]-C, C[$. We will show that $f = 0$ in several steps.

Lemma 3.22 *There exists $F \in \mathcal{C}(\Gamma\backslash G)^{\mathbb{C}}$ uniformly* Lipshitz *continuous on compact sets and differentiable along the flow φ_t such that*

$$f = \partial_\tau F(\Diamond a_\tau)|_{\tau=0} = \mathcal{D}F .$$

Proof: By assumption the right translation with A is topologically transitive on $\Gamma\backslash G$. So there exists $g_0 \in G$ such that $\Gamma g_0 A \subset G$ is dense. Define $s \in \mathcal{C}^\infty(\mathbb{R})^{\mathbb{C}} \otimes \Lambda(\mathbb{C}^r)$ by

$$s(t) := \int_0^t \widetilde{f}'(g_0 a_\tau)\, d\tau$$

for all $t \in \mathbb{R}$.

Step I **Show that for all $L \subset G$ compact there exist constants $C_3 \geq 0$ and $\varepsilon_3 > 0$ such that for all $t \in \mathbb{R}$, $T \geq 0$ and $\gamma \in \Gamma$ if $g_0 a_t \in L$ and**

$$\varepsilon := d(\gamma g_0 a_t, g_0 a_{t+T}) \leq \varepsilon_3$$

then $|s(t) - s(t+T)| \leq C_3 \varepsilon$.

Let $L \subset G$ be compact, $T_0 > 0$ be given by lemma 3.20 and $C_1 \geq 1$ and ε_1 be given by theorem 3.21 (i) with $T_1 := T_0$. Define $C_3 := \max\left(C_1(C_2 + 2c), \left|\left|\widetilde{f}'\right|\right|_\infty\right) \geq 0$, where $C_2 \geq 0$ is the Lipshitz constant from theorem 3.16 (ii) and $c \geq 0$ is the Lipshitz constant of $\widetilde{f}'$. Define $\varepsilon_3 := \min\left(\varepsilon_1, \varepsilon_2, \frac{T_0}{2C_1}\right) > 0$, where $\varepsilon_2 > 0$ is given by theorem 3.21 (ii) .

Let $t \in \mathbb{R}$, $T \geq 0$ and $\gamma \in \Gamma$ such that $g_0 a_t \in L$ and $\varepsilon := d\left(\gamma g_0 a_t, g_0 a'_{t+T}\right) \leq \varepsilon'$.

First assume $T \geq T_0$. Then by theorem 3.21 (i) since $\varepsilon \leq \varepsilon_1$ there exist $g \in G$, $w_0 \in M$ and $t_0 \in \mathbb{R} \setminus \{0\}$ such that $\gamma g = g a_{t_0} w_0$, $d\left((t_0, w_0), (T, 1)\right) \leq C_1 \varepsilon$, and for all $\tau \in [0, T]$

$$d\left(g_0 a_{t+\tau}, g a_\tau\right) \leq C_1 \varepsilon \left(e^{-\tau} + e^{-(T-\tau)}\right) .$$

We get

$$s(t+T) - s(t) = \underbrace{\int_0^T \widetilde{f}'\left(g a_\tau\right) d\tau}_{I_1 :=} + \underbrace{\int_0^T \left(\widetilde{f}'\left(g_0 a_{t+\tau}\right) - \widetilde{f}'\left(g a_\tau\right)\right) d\tau}_{I_2 :=}$$

and by the same calculation as in the proof of lemma 1.36 in the higher rank case $|I_2| \leq 2cC_1\varepsilon$.

Since $\gamma \in \Gamma$ is regular loxodromic there exists $\gamma_0 \in \Gamma$ primitive loxodromic and $\nu \in \mathbb{N} \setminus \{0\}$ such that $\gamma = \gamma_0^\nu$. $\gamma_0 \in gAWg^{-1}$ as well since theorem 3.10 tells us that $g \in G$ is already determined by γ up to right translation with elements of $AN_K(A)$. Choose $w' \in M$ such that $\gamma = g w' a_{t_0'} w_0' \left(g w'\right)^{-1}$ with $t_0' \in \mathbb{R}$ and $w_0' = \left(\begin{array}{c|c} * & 0 \\ \hline 0 & E_0' \end{array}\right) \in M$, $E_0' \in U(r)$ diagonal, and let $g' := g w'$.
We define $h \in \mathcal{C}^\infty\left(\mathbb{R} \times M\right)^{\mathbb{C}} \otimes \Lambda\left(\mathbb{C}^r\right)$ as

$$h(\tau, w) := \widetilde{f}'\left(g' a_\tau w\right) = \widetilde{f}'\left(g a_\tau w' w\right)$$

for all $\tau \in \mathbb{R}$ and $w \in M$. Then

$$I_1 = \int_0^T h\left(\tau, w'^{-1}\right) d\tau \, .$$

We can apply theorem 3.16 (i) and, since f is perpendicular to all $\varphi_{\gamma_0, I, m}$, $I \in \wp(r)$, $m \in]-C, C[$, 3.16 (ii) as well with $g' := g w'$ instead of g , and so

$$\begin{aligned} |I_1| &= \left|H\left(T, w'^{-1}\right) - H\left(0, w'^{-1}\right)\right| \\ &= \left|H\left(T, w'^{-1}\right) - H\left(t_0, w'^{-1} w_0\right)\right| \\ &\leq C_2 d\left((T, 1), (t_0, w_0)\right) \\ &\leq C_1 C_2 \varepsilon \, , \end{aligned}$$

where we used that $H\left(0, w'^{-1}\right) = H\left(t_0', w_0' w'^{-1}\right)$ and that we have chosen the left invariant metric on M , and the claim follows.
Now assume $T \leq T_0$. Then by theorem 3.21 (ii) since $\varepsilon \leq \varepsilon_0$ we get $T \leq \varepsilon$ and so

$$|s(t+T)-s(t)| = \left| \int_0^T \widetilde{f}'\left(g_0 a_{t+\tau}\right) d\tau \right| \leq \varepsilon \left\| \widetilde{f}' \right\|_\infty .$$

Step II **Show that there exists a unique** $F \in \mathcal{C}\left(\Gamma \backslash G\right)^{\mathbb{C}}$ **uniformly** LIPSHITZ **continuous on compact sets such that for all** $t \in \mathbb{R}$

$$s(t) = F\left(g_0 a_t\right) .$$

It is the same calculation as in the proof of theorem 1.36 in the higher rank case.

Step III **Show that** F **is differentiable along the diagonal flow and that for all** $g \in G$

$$\partial_\tau F\left(g a_\tau\right)|_{\tau=0} = \widetilde{f}'(g) .$$

again the same calculation as in the proof of theorem 1.36 in the higher rank case. □

Lemma 3.23

(i) For all $L \subset G$ compact there exists $\varepsilon_4 > 0$ such that for all $g, h \in L$ if g and h belong to the same T^--leaf and $d^-(g,h) \leq \varepsilon_4$ then

$$\lim_{t \to \infty} \left(F\left(g a_t\right) - F\left(h a_t\right)\right) = 0 ,$$

and if g and h belong to the same T^+-leaf and $d^+(g,h) \leq \varepsilon_4$ then

$$\lim_{t \to -\infty} \left(F\left(g a_t\right) - F\left(h a_t\right)\right) = 0 .$$

(ii) F is continuously differentiable along T^-- and T^+-leafs, more precisely if
$\rho : I \to G$ is a continuously differentiable curve in a T^--leaf then

$$\partial_t \left(F \circ \rho\right)(t) = -\int_0^\infty \partial_t \widetilde{f}'\left(\rho(t) a_\tau\right) d\tau ,$$

and if $\rho : I \to G$ is a continuously differentiable curve in a T^+-leaf then

$$\partial_t \left(F \circ \rho\right)(t) = \int_{-\infty}^0 \partial_t \widetilde{f}'\left(\rho(t) a_\tau\right) d\tau .$$

Proof: (i) Let $L \subset G$ be compact, and let $L' \subset G$ be a compact neighbourhood of L . Let $T_0 > 0$ be given by lemma 3.20 and $\varepsilon_2 > 0$ by theorem 3.21 (ii) both with respect to L' . Define

$$\varepsilon_4 := \frac{1}{3} \min \left(\varepsilon_1, \varepsilon_2, \frac{T_0}{2C_1} \right) > 0 \, ,$$

where $\varepsilon_1 > 0$ and $C_1 \geq 1$ are given by theorem 3.21 (i) with $T_1 := T_0$. Let $\delta_0 > 0$ such that $\overline{U_{\delta_0}(L)} \subset L'$ and let

$$\delta \in \,]\, 0, \min\left(\delta_0, \varepsilon_4\right) [\; .$$

Let $g, h \in L$ in the same T^--leaf such that $\varepsilon := d^-(g,h) \leq \varepsilon_4$. Fix some $T' > 0$. By assumption there exists $g_0 \in G$ such that $\Gamma g_0 A \underset{\text{dense}}{\subset} G$. So there exist $\gamma_g, \gamma_h \in \Gamma$ and $t_g, t_h \in \mathbb{R}$ such that

$$d\left(g a_t, \gamma_g g_0 a_{t_g + t}\right), d\left(h a_t, \gamma_h g_0 a_{t_h + t}\right) \leq \delta$$

for all $t \in [\,0, T'\,]$, and so especially $\gamma_g g_0 a_{t_g}, \gamma_h g_0 a_{t_h} \in L'$. We show that for all $t \in [\,0, T'\,]$

$$\left| F\left(\gamma_g g_0 a_{t_g+t}\right) - F\left(\gamma_h g_0 a_{t_h+t}\right) \right| \leq C_3' \left(\varepsilon e^{-t} + 2\delta \right)$$

with the same constant $C_3' \geq 0$ as in step I of the proof of lemma 3.22 with respect to L' .

Without loss of generality we may assume $T := t_h - t_g \geq 0$. Define $\gamma := \gamma_g \gamma_h^{-1}$. Then for all $t \in [\,0, T'\,]$

$$d\left(\gamma \gamma_g g_0 a_{t_g+t}, \gamma_g g_0 a_{t_g+t+T}\right) \leq \varepsilon e^{-t} + 2\delta$$

by the left invariance of the metric on G .

First assume $T \geq T_0$ and fix $t \in [\,0, T'\,]$. Then by theorem 3.21 (i) since $\varepsilon e^{-t} + 2\delta \leq \varepsilon + 2\delta \leq \min\left(\varepsilon_1, \frac{T_0}{2C_1}\right)$ there exist $z \in G$, $t_0 \in \mathbb{R}$ and $w \in M$ such that $\gamma z = z a_{t_0} w$,

$$d\left(\left(t_0, w\right), \left(T, 1\right)\right) \leq C_1 \left(2\delta + \varepsilon e^{-t}\right) \, ,$$

and for all $\tau \in [\,0, T\,]$

$$d\left(\gamma_g g_0 a_{t_g+t+\tau}, z a_\tau\right) \leq C_1 \left(\varepsilon e^{-t} + 2\delta\right) \left(e^{-\tau} + e^{-(T-\tau)}\right) .$$

And so by the same calculations as in the proof of lemma 1.36 we obtain the estimate

$$\left|F\left(\gamma_g g_0 a_{t_g+t}\right) - F\left(\gamma_g g_0 a_{t_h+t}\right)\right| \le C_3'\left(\varepsilon e^{-t} + 2\delta\right) .$$

Now assume $T \le T_0$. Then by theorem 3.21 (ii) since $\gamma_g g_0 a_{t_g} \in L'$ and $\varepsilon + 2\delta \le \varepsilon_2$ we obtain $\gamma = 1$ and so by the left invariance of the metric on G

$$d\left(1, a_T\right) \le \varepsilon e^{-T'} + 2\delta ,$$

therefore $T \le \varepsilon e^{-T'} + 2\delta$. So as in the proof of lemma 3.22

$$\begin{aligned}\left|F\left(\gamma_g g_0 a_{t_g+t}\right) - F\left(\gamma_g g_0 a_{t_h+t}\right)\right| &\le \left|\left|\widetilde{f}'\right|\right|_\infty \left(\varepsilon e^{-T'} + 2\delta\right) \\ &\le C_3'\left(\varepsilon e^{-t} + 2\delta\right) .\end{aligned}$$

Since F is left-Γ-invariant we have the desired estimate.

The rest goes exactly as in the higher rank case. □

(ii) same as in the higher rank case. □

Lemma 3.24

(i) $F \in L^2\left(\Gamma\backslash G\right) \otimes \Lambda\left(\mathbb{C}^r\right)$,

(ii) $\xi F \in L^2\left(\Gamma\backslash G\right) \otimes \Lambda\left(\mathbb{C}^r\right)$ *for all* $\xi \in \mathbb{R}\mathcal{D} \oplus \mathfrak{g} \cap \left(T^+ \oplus T^-\right)$.

Proof: (i) If $\Gamma\backslash G$ is compact then the assertion is trivial. So assume that $\Gamma\backslash G$ is not compact. Since vol $\left(\Gamma\backslash G\right) < \infty$ it suffices to prove that F is bounded, and by corollary 3.14 it is even enough to show that $F\left(g\diamondsuit\right)$ is bounded on $NA_{>t_0}K$ for all $g \in \Xi$. So let $g \in \Xi$.

Step I **Show that** $F\left(g\diamondsuit\right)$ **is bounded on** $Na_{t_0}K$ **.**

$F\left(g\diamondsuit\right)$ is bounded on $\eta a_{t_0}K$ since $\eta a_{t_0}K$ is relatively compact. On the other hand $F\left(g\diamondsuit\right)$ is left- $g^{-1}\Gamma g$ -invariant, so it is also bounded on

$$Na_{t_0}K = \left(g\Gamma g^{-1} \cap NZ\left(G'\right)\right)\eta a_{t_0}K$$

by theorem 3.13 (i) .

Step II **Show that there exists $C' \geq 0$ such that for all $g' \in NA_{>t_0}K$**

$$\left|\widetilde{f'}\left(gg'\right)\right| \leq \frac{C'}{\Delta'\left(Rg'\mathbf{0}, Rg'\mathbf{0}\right)} .$$

As in section 3.2 let $q_I \in \mathcal{O}(H)$ such that $f|_g|_{R^{-1}} = \sum_{I \in \wp(r)} q_I \vartheta^I$. Then since $\widetilde{f'}\left(g\diamondsuit\right) \in L^2\left(\eta A_{>t_0}K\right) \otimes \Lambda\left(\mathbb{C}^r\right)$ by theorem 3.15 we have FOURIER expansions

$$q_I\left(\mathbf{w}\right) = \sum_{m \in \frac{1}{\lambda_0}\left(\mathbb{Z} - \mathrm{tr}_I D - (k+|I|)\chi\right) \cap \mathbb{R}_{<0}} c_{I,m}\left(\mathbf{w}_2\right) e^{2\pi m w_1} \tag{3.3}$$

for all $I \in \wp(I)$ and $\mathbf{w} = \begin{pmatrix} w_1 \\ \hline \mathbf{w}_2 \end{pmatrix} \begin{array}{l} \leftarrow 1 \\ \}p-1 \end{array} \in H$, where $c_{I,m} \in \mathcal{O}\left(\mathbb{C}^{p-1}\right)$, $I \in \wp(r)$, $m \in \frac{1}{\lambda_0}\left(\mathbf{z} - \mathrm{tr}_I D - (k+|I|)\chi\right) \cap \mathbb{R}_{<0}$. Define

$$M_0 := \max \bigcup_{I \in \wp(r)} \frac{1}{\lambda_0}\left(\mathbb{Z} - \mathrm{tr}_I D - (k+|I|)\chi\right) \cap \mathbb{R}_{<0} < 0 .$$

$R\eta a_{t_0}\mathbf{0} \subset H$ is relatively compact, and so since the convergence of the FOURIER series 3.3 is absolute and compact we can define

$$\begin{aligned} C'' &:= e^{-2\pi M_0 e^{2t_0}} \times \\ &\times \max_{I \in \wp(r)} \sum_{m \in \frac{1}{\lambda_0}\left(\mathbb{Z} - \mathrm{tr}_I D - (k+|I|)\chi\right) \cap \mathbb{R}_{<0}} \left|\left|c_{I,m}\left(\mathbf{w}_2\right) e^{2\pi m w_1}\right|\right|_{\infty, R\eta a_{t_0}\mathbf{0}} < \infty . \end{aligned}$$

Then we have

$$\left|q_I\left(\mathbf{w}\right)\right| \leq C'' e^{\pi M_0 \Delta'(\mathbf{w},\mathbf{w})}$$

for all $I \in \wp(r)$ and $\mathbf{w} \in R\eta A_{>t_0}\mathbf{0}$. Now let $g' = \left(\begin{array}{c|c} * & 0 \\ \hline 0 & E' \end{array}\right) \in \eta A_{>0}K$, $E' \in U(r)$. Then

$$\begin{aligned} \widetilde{f'}\left(gg'\right) &= f|_g|_{R^{-1}}|_{RgR^{-1}}\left(\mathbf{e}_1\right) \\ &= f|_g|_{R^{-1}}\left(Rg'R^{-1}\begin{pmatrix} \mathbf{e}_1 \\ \hline \eta \end{pmatrix}\right) j\left(Rg'R^{-1}, \mathbf{e}_1\right)^k \\ &= f|_g|_{R^{-1}}\left(\frac{Rg'\mathbf{0}}{E\eta j\left(Rg'R^{-1}\right)}\right) j\left(Rg'R^{-1}, \mathbf{e}_1\right)^k \\ &= \sum_{I \in \wp(r)} q_I\left(Rg'\mathbf{0}\right)\left(E\eta\right)^I j\left(Rg'R^{-1}, \mathbf{e}_1\right)^{k+|I|} . \end{aligned}$$

Therefore since $\left|j\left(Rg'R^{-1},\mathbf{e}_1\right)\right| = \sqrt{\Delta'\left(Rg'\mathbf{0},Rg'\mathbf{0}\right)}$ we get

$$\begin{aligned}\left|\widetilde{f}'\left(gg'\right)\right| &\leq 2^r C'' e^{\pi M_0 \Delta'(Rg'\mathbf{0},Rg'\mathbf{0})} \times \\ &\quad\times\left(\Delta'\left(Rg'\mathbf{0},Rg'\mathbf{0}\right)^{\frac{k}{2}}+\Delta'\left(Rg'\mathbf{0},Rg'\mathbf{0}\right)^{\frac{k+r}{2}}\right).\end{aligned}$$

So we see that there exists $C'>0$ such that

$$\left|\widetilde{f}'\left(gg'\right)\right| \leq \frac{C'}{\Delta'\left(Rg'\mathbf{0},Rg'\mathbf{0}\right)}$$

for all $g' \in \eta A_{>t_0}K$, but on one hand $\widetilde{f}'\left(g\diamondsuit\right)$ is left- $g^{-1}\Gamma g$ -invariant, and on the other hand Δ' is $RNZ\left(G'\right)R^{-1}$ -invariant. Therefore the estimate is correct even for all

$$g' \in NA_{>t_0}K = \left(g\Gamma g^{-1}\cap NZ\left(G'\right)\right)\eta A_{>t_0}K$$

by theorem 3.13 (i) .

Step III **Conclusion: Prove that**

$$\left|F\left(g\diamondsuit\right)\right| \leq \left|\left|F\left(g\diamondsuit\right)\right|\right|_{\infty,Na_{t_0}K} + 2C'e^{-2t_0}$$

on $NA_{>t_0}K$ **.**

Let $g' \in G$ be arbitrary. We will show the estimate on $g'A \cap NA_{>t_0}K$.

$$\mathbb{R} \to H\,,\ t \mapsto \mathbf{w}_t := Rg'a_t\mathbf{0}$$

is a geodesic in H , and for all $t \in \mathbb{R}$ we have $g'a_t \in NA_{>t_0}K$ if and only if $\Delta'\left(\mathbf{w}_t,\mathbf{w}_t\right) > 2e^{2t_0}$. Now we have to distinguish two cases.

In the first case the geodesic connects ∞ with a point in ∂H . First assume that $\lim_{t\to\infty}\mathbf{w}_t = \infty$ and $\lim_{t\to-\infty}\mathbf{w}_t \in \partial H$. Then $\lim_{t\to\infty}\Delta'\left(\mathbf{w}_t,\mathbf{w}_t\right) = \infty$ and $\lim_{t\to-\infty}\Delta'\left(\mathbf{w}_t,\mathbf{w}_t\right) = 0$. So we may assume without loss of generality that $\Delta'\left(\mathbf{w}_0,\mathbf{w}_0\right) = 2e^{2t_0}$, and therefore $g' = g'a_0 \in Na_{t_0}K$ and $g'a_t \in NA_{>t_0}K$ if and only if $t>0$. So let $t>0$. Then

$$F\left(gg'a_t\right) = F\left(gg'\right) + \int_0^t \widetilde{f}'\left(gg'a_\tau\right)d\tau\,,$$

and so

$$\left|F\left(gg'a_t\right)\right| \leq \left|\left|F\left(g\diamondsuit\right)\right|\right|_{\infty,Na_{t_0}K} + \int_0^t\left|\widetilde{f}'\left(gg'a_\tau\right)\right|d\tau\,.$$

By step II and lemma 3.12 (i)

$$\begin{aligned}
\int_0^t \left|\widetilde{f}'\left(gg'a_\tau\right)\right| d\tau &\leq C'\int_0^t \frac{d\tau}{\Delta'\left(\mathbf{w}_\tau,\mathbf{w}_\tau\right)} \\
&= \frac{C'}{\Delta'\left(\mathbf{w}_0,\mathbf{w}_0\right)}\int_0^t e^{-2\tau}d\tau \\
&\leq C'e^{-2t_0}\,.
\end{aligned}$$

The case where $\lim_{t\to-\infty}=\infty$ and $\lim_{t\to\infty}\in\partial H$ is done similarly.

In the second case the geodesic connects two points in ∂H . Then without loss of generality we can assume that $\Delta'\left(R\mathbf{w}_t,R\mathbf{w}_t\right)$ is maximal for $t=0$. So if $\Delta'\left(\mathbf{w}_0,\mathbf{w}_0\right)<2e^{2t_0}$ we have $g'A\cap NA_{>t_0}K=\emptyset$. Otherwise by lemma 3.12 (ii) there exists $T\in\mathbb{R}_{\geq 0}$ such that $\Delta'\left(\mathbf{w}_T,\mathbf{w}_T\right)=\Delta'\left(\mathbf{w}_{-T},\mathbf{w}_{-T}\right)=2e^{2t_0}$, and since $\Delta'\left(\mathbf{w}_T,\mathbf{w}_T\right)\leq\frac{4}{e^{2|T|}}\Delta'\left(\mathbf{w}_0,\mathbf{w}_0\right)$ we see that

$$T\leq\frac{1}{2}\log\left(2\Delta'\left(\mathbf{w}_T,\mathbf{w}_T\right)\right)-t_0\,.$$

So $g'a_T,g'a_{-T}\in Na_{t_0}K$ and $g'a_t\in NA_{>t_0}K$ if and only if $t\in]-T,T[$. Let $t\in]-T,T[$ and assume $t\geq 0$ first. Then

$$F\left(gg'a_t\right)=F\left(gg'a_T\right)-\int_t^T\widetilde{f}'\left(gg'a_\tau\right)d\tau\,,$$

and so

$$\left|F\left(gg'a_t\right)\right|\leq||F\left(g\Diamond\right)||_{\infty,Na_{t_0}K}+\int_0^T\left|\widetilde{f}'\left(gg'a_\tau\right)\right|d\tau\,.$$

By step II and lemma 3.12 (ii) now

$$\begin{aligned}
\int_0^T \left|\widetilde{f}'\left(gg'a_\tau\right)\right| d\tau &\leq C'\int_0^T \frac{d\tau}{\Delta'\left(\mathbf{w}_\tau,\mathbf{w}_\tau\right)} \\
&\leq \frac{C'}{\Delta'\left(\mathbf{w}_0,\mathbf{w}_0\right)}\int_0^T e^{2\tau}d\tau \\
&\leq \frac{C'}{2\Delta'\left(\mathbf{w}_0,\mathbf{w}_0\right)}e^{2T} \\
&\leq 2C'e^{-2t_0}\,.
\end{aligned}$$

The case $t\leq 0$ is done similarly. $\Box$

(ii) Since on one hand $\partial_\tau F\left(\Diamond a_\tau\right)|_{\tau=0}=\widetilde{f}'\in L^2\left(\Gamma\backslash G\right)\otimes\Lambda\left(\mathbb{C}^r\right)$ and on the other hand vol $\left(\Gamma\backslash G\right)<\infty$ it suffices to show that ξF is bounded for all $\alpha\in\Phi\setminus\{0\}$ and $\xi\in\mathfrak{g}^\alpha$. So let $\alpha\in\Phi\setminus\{0\}$ and $\xi\in\mathfrak{g}^\alpha$. First assume $\alpha>0$, which clearly implies $\alpha\geq 1$ and $\xi\in T^-$. So there exists a continuously differential curve $\rho:I\to G$ contained in the T^--leaf containing 1 such that

$0 \in I$, $\rho(0) = 1$ and $\partial_t \rho(t)|_{t=0} = \xi$. Let $g \in G$. Then by theorem 3.23 (ii) we have

$$\begin{aligned}
(\xi F)(g) &= \partial_t F(g\rho(t))|_{t=0} \\
&= -\int_0^\infty \partial_t \widetilde{f'}(g\rho(t)a_\tau)\Big|_{t=0} d\tau \\
&= -\int_0^\infty \partial_t \widetilde{f'}(ga_\tau a_{-\tau}\rho(t)a_\tau)\Big|_{t=0} d\tau \\
&= -\int_0^\infty \left(\left(\mathrm{Ad}_{a_{-\tau}}(\xi)\right)\widetilde{f'}\right)(ga_\tau)\, d\tau \\
&= -\int_0^\infty e^{-\alpha\tau}\left(\xi\widetilde{f'}\right)(ga_\tau)\, d\tau\,,
\end{aligned}$$

so

$$|(\xi F)(g)| \leq c\,||\xi||_2 < \infty$$

where c is the LIPSHITZ constant of $\widetilde{f'}$. The case $\alpha < 0$ is done similarly. $\square$

Therefore by the FOURIER decomposition described above we have

$$F = \sum_{I \in \wp}\sum_{\nu \in \mathbb{Z}} F_{I\nu}\eta^I\,,$$

where $F_{I\nu} \in H_\nu$ for all $I \in \wp(r)$ and $\nu \in \mathbb{Z}$. $\mathcal{D} = \mathcal{D}^+ + \mathcal{D}^-$, and again a simple calculation shows that $\mathcal{D}^+$ and $\mathcal{D}^- \in \mathbb{R}\mathcal{D} \oplus \mathfrak{g} \cap (T^+ \oplus T^-)$, and so $\mathcal{D}^+F, \mathcal{D}^-F \in L^2(\Gamma\backslash G) \otimes \Lambda(\mathbb{C}^r)$ by lemma 3.24 . So we get the FOURIER decomposition of $\widetilde{f'}$ as

$$\widetilde{f'} = \mathcal{D}F = \sum_{I \in \wp(r)}\sum_{\nu \in \mathbb{Z}} \left(\mathcal{D}^+F_{I,\nu-2} + \mathcal{D}^-F_{I,\nu+2}\right)\eta^I$$

with $\mathcal{D}^+F_{I,\nu-2} + \mathcal{D}^-F_{I,\nu+2} \in H_\nu$ for all $\nu \in \mathbb{Z}$. But since $f \in sS_k(\Gamma)$ the FOURIER decomposition of $\widetilde{f'}$ is exactly

$$\widetilde{f'} = q_I\eta^I$$

with $q_I \in \mathcal{C}^\infty(G)^\mathbb{C} \cap H_{k+|I|}$, and so for all $I \in \wp(r)$ and $\nu \in \mathbb{Z}$

$$\mathcal{D}^+F_{I,\nu-2} + \mathcal{D}^-F_{I,\nu+2} = \begin{cases} q_I & \text{if } \nu = k+|I| \\ 0 & \text{otherwise} \end{cases}\,.$$

Lemma 3.25 *$F_{I,\nu} = 0$ for $I \in \wp(r)$ and $n \in \mathbb{N}_{\geq k+|I|}$.*

Proof: similar to the higher rank case. Apply the argument for each $I \in \wp(r)$ seperately. $\square$

So for all $I \in \wp(r)$ we obtain $\mathcal{D}^+ F_{I,k+|I|-2} = q_I$ and finally $\mathcal{D}^- q_I = 0$ by lemma 3.18 , since $f \in \mathcal{O}(\mathcal{B})$, so

$$||q_I||_2^2 = \left(q_I, \mathcal{D}^+ F_{I,\nu-2}\right) = -\left(\mathcal{D}^- q_I, F_{I,\nu-2}\right) = 0\,,$$

and so $\widetilde{f}' = 0$, which completes the proof of our main theorem. $\square$

Fix a regular loxodromic $\gamma_0 \in \Gamma$, $g \in G$, $g = \left(\begin{array}{c|c} * & 0 \\ \hline 0 & E \end{array}\right)$, $E \in U(r)$, such that $\gamma_0 = g a_{t_0} w_0 g^{-1} \in gAMg^{-1}$, $t_0 \in \mathbb{R} \setminus \{0\}$ and $w_0 = \left(\begin{array}{c|c} * & 0 \\ \hline 0 & E_0 \end{array}\right)$, $E_0 \in U(r)$ diagonal, $D \in \mathbb{R}^{r \times r}$ diagonal such that $\exp(2\pi i D) = E_0$ and $\chi \in \mathbb{R}$ such that $j(w_0) = e^{2\pi i \chi}$. Now we will compute $\varphi_{\gamma_0,I,m} \in sS_k(\Gamma)$, $I \in \wp(r)$, $m \in \frac{1}{t_0}\left(\mathbb{Z} - (k+|I|)\,\chi - \mathrm{tr}_I D\right)$, as a relative Poincaré series with respect to $\Gamma_0 := \langle \gamma_0 \rangle \sqsubset \Gamma$. Hereby again $\equiv$ means equality up to a constant $\neq 0$ (not necessarily independent of γ_0 , I and m) .

Theorem 3.26 (computation of $\varphi_{\gamma_0,I,m}$) *Let $I \in \wp(r)$ and $k \geq \max\left(k_0, 2p+1-|I|\right)$, where k_0 is given by* Satake*'s theorem, theorem 3.11 . Then for all $m \in \frac{1}{t_0}\left(\mathbb{Z} - (k+|I|)\,\chi - \mathrm{tr}_I D\right)$*

(i)

$$\varphi_{\gamma_0,I,m} \equiv \sum_{\gamma \in \Gamma_0 \backslash \Gamma} q|_\gamma$$

where

$$q := \int_{-\infty}^{\infty} e^{2\pi i m t} \Delta\left(\diamondsuit, g a_t \mathbf{0}\right)^{-k-|I|} \overline{j\left(g a_t, \mathbf{0}\right)}^{k+|I|} dt \left(E^{-1}\zeta\right)^I$$
$$\in sM_k\left(\Gamma_0\right) \cap L^1_k\left(\Gamma_0 \backslash \mathcal{B}\right) .$$

(ii) For all $\mathbf{z} \in B$ we have

$$q\left(\mathbf{z}\right) \equiv \left(\Delta\left(\mathbf{z}, \mathbf{X}^+\right) \Delta\left(\mathbf{z}, \mathbf{X}^-\right)\right)^{-\frac{k+|I|}{2}} \left(\frac{1+v_1}{1-v_1}\right)^{\pi i m} \left(E^{-1}\zeta\right)^I ,$$

where

$$\mathbf{X}^+ := g\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} \quad \textit{and } \mathbf{X}^- := g\begin{pmatrix}-1\\0\\\vdots\\0\end{pmatrix}$$

are the two fixpoints of γ_0 *in* ∂B *, and*

$$\mathbf{v} := g^{-1}\mathbf{z} \in B \subset \mathbb{C}^p .$$

Proof: (i) Let $f \in sS_k(\Gamma)$, and define $h = \sum_{J\in\wp(r)} h_J \eta^J \in \mathcal{C}^\infty(\mathbb{R}\times M)^{\mathbb{C}} \otimes \Lambda(\mathbb{C}^r)$, all $h_J \in \mathcal{C}^\infty(\mathbb{R}\times M)^{\mathbb{C}}$ and $b_{I,m} \in \mathbb{C}$, $I \in \wp(r)$, $m \in \frac{1}{t_0}(\mathbb{Z} - (k+|I|)\chi - \mathrm{tr}_I D)$ as in theorem 3.16 . Then by standard FOURIER expansion we have

$$\begin{aligned}
b_{I,m} &\equiv \int_0^{t_0} e^{-2\pi i m t} h_I(t,1)dt \\
&= \int_0^{t_0} e^{-2\pi i m t}\left\langle f\left(ga_t\left(\frac{\mathbf{0}}{\eta}\right)\right) j\left(ga_t,\mathbf{0}\right)^k, \eta^I\right\rangle dt \\
&\equiv \int_0^{t_0} e^{-2\pi i m t} \times \\
&\quad \times\left\langle \sum_{J\in\wp(r)} \left(\Delta\left(\diamondsuit, ga_t\mathbf{0}\right)^{-k-|J|}\zeta^J, f\right)(E\eta)^J j\left(ga_t,\mathbf{0}\right)^{k+|J|}, \eta^I\right\rangle dt \\
&\equiv \int_0^{t_0} e^{-2\pi i m t}\left(\Delta\left(\diamondsuit, ga_t\mathbf{0}\right)^{-k-|I|}\left(E^{-1}\zeta\right)^I, f\right) j\left(ga_t,\mathbf{0}\right)^{k+|I|} dt \\
&= \int_0^{t_0} e^{-2\pi i m t}\int_G\left\langle \widetilde{f}', \left(\Delta\left(\diamondsuit, ga_t\mathbf{0}\right)^{-k-|I|}\left(E^{-1}\zeta\right)^I\right)^{\sim\prime}\right\rangle \times \\
&\quad \times j\left(ga_t,\mathbf{0}\right)^{k+|I|} dt .
\end{aligned}$$

Since by SATAKE's theorem, theorem 3.11 , $\widetilde{f}' \in L^\infty(G)\otimes\Lambda(\mathbb{C}^r)$, and

$$\begin{aligned}
&\int_0^{t_0}\int_G\left|\left(\Delta\left(\diamondsuit, ga_t\mathbf{0}\right)^{-k-|I|}\left(E^{-1}\zeta\right)^I\right)^{\sim\prime} j\left(ga_t,\mathbf{0}\right)^{k+|I|}\right| dt \\
&\quad = \int_0^{t_0}\int_G\left|\left(\Delta\left(\diamondsuit,\mathbf{0}\right)^{-k-|I|}\zeta^I\right)^{\sim\prime}\left(\left(ga_t\right)^{-1}\diamondsuit\right)\right| dt \\
&\quad \equiv \int_G\left|\widetilde{\zeta^I}'\right| \\
&\quad = \int_G\left|j\left(\diamondsuit,\mathbf{0}\right)^{k+|I|}\right| \\
&\quad \equiv \int_B \Delta\left(\mathbf{Z},\mathbf{Z}\right)^{\frac{k+|I|}{2}-(p+1)} dV_{\mathrm{Leb}} < \infty ,
\end{aligned}$$

by TONELLI's and FUBINI's theorem we can interchange the order of integration:

$$\begin{aligned}
b_{I,m} &\equiv \int_G \int_0^{t_0} e^{-2\pi imt} \left\langle \widetilde{f}', \left(\Delta\left(\diamondsuit, ga_t\mathbf{0}\right)^{-k-|I|} \left(E^{-1}\zeta\right)^I\right)^{\sim\prime} \right\rangle \times \\
&\quad \times j\left(ga_t, \mathbf{0}\right)^{k+|I|} dt \\
&= \int_G \left\langle \widetilde{f}', \int_0^{t_0} e^{2\pi imt} \left(\Delta\left(\diamondsuit, ga_t\mathbf{0}\right)^{-k-|I|} \left(E^{-1}\zeta\right)^I\right)^{\sim\prime} \overline{j\left(ga_t, \mathbf{0}\right)}^{k+|I|} dt \right\rangle \\
&= \left(\int_0^{t_0} e^{2\pi imt} \Delta\left(\diamondsuit, ga_t\mathbf{0}\right)^{-k-|I|} \overline{j\left(ga_t, \mathbf{0}\right)}^{k+|I|} dt \left(E^{-1}\zeta\right)^I, f\right) \\
&= \left(q', f\right)_{\Gamma_0},
\end{aligned}$$

where

$$\left(\int_0^{t_0} e^{2\pi imt} \Delta\left(\diamondsuit, ga_t\mathbf{0}\right)^{-k-|I|} \overline{j\left(ga_t, \mathbf{0}\right)}^{k+|I|} dt \left(E^{-1}\zeta\right)^I\right)^{\sim\prime} \in L^1(G) \otimes \Lambda\left(\mathbb{C}^r\right),$$

$$\int_0^{t_0} e^{2\pi imt} \Delta\left(\diamondsuit, ga_t\mathbf{0}\right)^{-k-|I|} \overline{j\left(ga_t, \mathbf{0}\right)}^{k+|I|} dt \left(E^{-1}\zeta\right)^I \in \mathcal{O}(\mathcal{B})$$

since $\Delta\left(\diamondsuit, \mathbf{w}\right) \in \mathcal{O}(B)$ for all $\mathbf{w} \in B$ and the convergence of the integral is compact, and so by theorem 3.3

$$\begin{aligned}
q' &:= \sum_{\gamma' \in \Gamma_0} \int_0^{t_0} e^{2\pi imt} \Delta\left(\diamondsuit, ga_t\mathbf{0}\right)^{-k-|I|} \overline{j\left(ga_t, \mathbf{0}\right)}^{k+|I|} dt \left(E^{-1}\zeta\right)^I \Big|_{\gamma'} \\
&\quad \in sM_k\left(\Gamma_0\right) \cap L_k^1\left(\Gamma_0 \backslash \mathcal{B}\right).
\end{aligned}$$

Clearly

$$\begin{aligned}
&\Delta\left(\diamondsuit, ga_t\mathbf{0}\right)^{-k-|I|} \left(E^{-1}\zeta\right)^I \Big|_{\gamma_0} \\
&\quad = \Delta\left(\gamma_0\diamondsuit, ga_t\mathbf{0}\right)^{-k-|I|} \left(E_0E^{-1}\zeta\right)^I j\left(\gamma_0, \diamondsuit\right)^{k+|I|} \\
&\quad = \Delta\left(\diamondsuit, \gamma_0^{-1}ga_t\mathbf{0}\right)^{-k-|I|} \left(E_0E^{-1}\zeta\right)^I \overline{j\left(\gamma_0^{-1}, ga_t\mathbf{0}\right)}^{k+|I|},
\end{aligned}$$

so for all $\mathbf{z} \in B$ we can compute $q'\left(\mathbf{z}\right)$ as

$$\begin{aligned}
q'(\mathbf{z}) &= \sum_{\nu\in\mathbb{Z}} \int_0^{t_0} e^{2\pi imt} \Delta\left(\diamondsuit, ga_t\mathbf{0}\right)^{-k-|I|} \left(E^{-1}\zeta\right)^I \overline{j\left(ga_t,\mathbf{0}\right)}^{k+|I|} dt \Big|_{\gamma_0^\nu}(\mathbf{z}) \\
&= \sum_{\nu\in\mathbb{Z}} \int_0^{t_0} e^{2\pi imt} \Delta\left(\mathbf{z}, \gamma_0^{-\nu} ga_t\mathbf{0}\right)^{-k-|I|} \left(E_0^\nu E^{-1}\zeta\right)^I \times \\
&\quad \times \overline{j\left(\gamma_0^{-\nu} ga_t,\mathbf{0}\right)}^{k+|I|} dt \\
&= \sum_{\nu\in\mathbb{Z}} \int_0^{t_0} e^{2\pi imt} \Delta\left(\mathbf{z}, ga_{t-\nu t_0}\mathbf{0}\right)^{-k-|I|} \left(E^{-1}\zeta\right)^I e^{2\pi i\nu \operatorname{tr}_I D} \times \\
&\quad \times \overline{j\left(ga_{t-\nu t_0},\mathbf{0}\right)}^{k+|I|} e^{2\pi i\nu(k+|I|)\chi} dt \\
&= \sum_{\nu\in\mathbb{Z}} \int_0^{t_0} e^{2\pi im(t-\nu t_0)} \Delta\left(\mathbf{z}, ga_{t-\nu t_0}\mathbf{0}\right)^{-k-|I|} \overline{j\left(ga_{t-\nu t_0},\mathbf{0}\right)}^{k+|I|} dt \times \\
&\quad \times \left(E^{-1}\zeta\right)^I \\
&= \pm \int_{-\infty}^{\infty} e^{2\pi imt} \Delta\left(\mathbf{z}, ga_t\mathbf{0}\right)^{-k-|I|} \overline{j\left(ga_t,\mathbf{0}\right)}^{k+|I|} dt \left(E^{-1}\zeta\right)^I =: q(\mathbf{z}) \; .
\end{aligned}$$

Again by theorem 3.3 we see that $\sum_{\gamma\in\Gamma_0\backslash\Gamma} q|_\gamma \in sM_k(\Gamma) \cap L_k^1(\Gamma\backslash\mathcal{B})$, and so by Satake's theorem, theorem 3.11 , even $\in sS_k(\Gamma)$, such that

$$b_{I,m} \equiv \left(\sum_{\gamma\in\Gamma_0\backslash\Gamma} q|_\gamma, f \right)_\Gamma \; ,$$

and so we conclude that $\varphi_{\gamma_0,I,m} \equiv \sum_{\gamma\in\Gamma_0\backslash\Gamma} q|_\gamma$. $\square$

(ii)

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{2\pi imt} \Delta\left(\mathbf{z}, ga_t\mathbf{0}\right)^{-k-|I|} \overline{j\left(ga_t,\mathbf{0}\right)}^{k+|I|} dt \\
&\quad = j\left(g^{-1},\mathbf{z}\right)^{k+|I|} \int_{-\infty}^{\infty} e^{2\pi imt} \Delta\left(g^{-1}\mathbf{z}, a_t\mathbf{0}\right)^{-k-|I|} \overline{j\left(a_t,\mathbf{0}\right)}^{k+|I|} dt \\
&\quad = j\left(g^{-1},\mathbf{z}\right)^{k+|I|} \int_{-\infty}^{\infty} e^{2\pi imt} \left(1 - v_1 \tanh\, t\right)^{-k-|I|} \frac{1}{(\cosh\, t)^{k+|I|}} dt \\
&\quad = j\left(g^{-1},\mathbf{z}\right)^{k+|I|} \int_{-\infty}^{\infty} \frac{e^{2\pi imt}}{(\cosh\, t - v_1 \sinh\, t)^{k+|I|}} dt \\
&\quad \equiv j\left(g^{-1},\mathbf{z}\right)^{k+|I|} \frac{1}{\left(1 - v_1^2\right)^{\frac{k+|I|}{2}}} \left(\frac{1+v_1}{1-v_1}\right)^{\pi im} \\
&\quad = j\left(g^{-1},\mathbf{z}\right)^{k+|I|} \left(\left(1-v_1\right)\left(1+v_1\right)\right)^{-\frac{k+|I|}{2}} \left(\frac{1+v_1}{1-v_1}\right)^{\pi im} \\
&\quad \equiv \left(\Delta\left(\mathbf{z},\mathbf{X}^+\right)\Delta\left(\mathbf{z},\mathbf{X}^-\right)\right)^{-\frac{k+|I|}{2}} \left(\frac{1+v_1}{1-v_1}\right)^{\pi im} . \square
\end{aligned}$$

3.4 Super cusp forms in the parametrized case

Now we return to the general case where $\mathcal{G} := sSU(p,q|r)$, $\mathcal{P} := \Lambda(\mathbb{R}^m) = \mathcal{D}\left(\mathbb{R}^{0|m}\right)$, $m \in \mathbb{N}$, with the odd coordinate functions $\beta_1, \dots, \beta_m \in \mathcal{D}\left(\mathbb{R}^{0|m}\right)$ and Υ is a discrete $\mathcal{P}$-sub super LIE group of $\mathcal{G}$ with body $\Upsilon^\# = \Gamma$ being a discrete subgroup of $G = \mathcal{G}^\#$.

Let $\mathcal{M}$ be a real super manifold and $f \in (\mathcal{D}(\mathcal{M}) \boxtimes \mathcal{P})^{\mathbb{C}}$. Then f has a unique expansion

$$f = \sum_{I,J \in \wp(m)} f_I \beta^I$$

with $f_I \in \mathcal{D}(\mathcal{M})^{\mathbb{C}}$, $I \in \wp(m)$, and a notion of a (with respect to $\mathcal{P}$) relative body map and also of a relative degree seem to be useful: We define the relative body map $\#'$ as

$$\#' := \mathrm{id} \otimes \# : (\mathcal{D}(\mathcal{M}) \boxtimes \mathcal{P})^{\mathbb{C}} \to \mathcal{D}(\mathcal{M})^{\mathbb{C}} ,\, f = \sum_{I \in \wp(m)} f_I \beta^I \mapsto f^{\#'} := f_\emptyset ,$$

which is again a unital continuous graded algebra epimorphism, and the relative degree $\deg'$ as

$$\deg' f := \min\left\{ |I| \;\middle|\; I \in \wp(m) \text{ and } f_{IJ} \neq 0 \right\}$$

for all $f \in (\mathcal{D}(\mathcal{M}) \boxtimes \mathcal{P})^{\mathbb{C}}$ (in particular this implies $\deg' 0 = \infty$!) . Clearly $\deg'(f+h) \geq \min\left(\deg' f, \deg' h\right)$, $\deg'(fh) \geq \deg' f + \deg' h$ and $\deg' f - f^{\#'} \geq 1$ for all $f, g \in (\mathcal{D}(\mathcal{M}) \boxtimes \mathcal{P})^{\mathbb{C}}$. The kernel of $\#'$ is precisely

$$\mathcal{I}' := \left\{ f \in (\mathcal{D}(\mathcal{M}) \boxtimes \mathcal{P})^{\mathbb{C}} \;\middle|\; \deg' f \geq 1 \right\} ,$$

which is an ideal of $(\mathcal{D}(\mathcal{M}) \boxtimes \mathcal{P})^{\mathbb{C}}$ (only contained in the set of nilpotent elements of $(\mathcal{D}(\mathcal{M}) \boxtimes \mathcal{P})^{\mathbb{C}}$) . For all $\nu \in \mathbb{N}$ obviously

$$\mathcal{I}'^\nu = \left\{ f \in (\mathcal{D}(\mathcal{M}) \boxtimes \mathcal{P})^{\mathbb{C}} \;\middle|\; \deg' f \geq \nu \right\}$$

is the ideal spanned by the elements β^I , $I \in \wp(m)$, $|I| = \nu$. So $\mathcal{I}'^{m+1} = 0$. Clearly if $\mathcal{M}$ is a holomorphic super manifold then the image of $\mathcal{O}(\mathcal{M}) \boxtimes \mathcal{P}^{\mathbb{C}}$ under $\#'$ is $\mathcal{O}(\mathcal{M})$.

Proposition 3.27 *Let $g \in_{\mathcal{P}} \mathcal{G}$.*

(i) Let $f \in (\mathcal{D}(\mathcal{G}) \boxtimes \mathcal{P})^{\mathbb{C}}$ and $f(g\diamondsuit) = \sum_{I \in \wp(m)} q_I \beta^I$ with $q_I \in \mathcal{D}(\mathcal{G})^{\mathbb{C}}$ for all $I \in \wp(m)$. Then $f(g\diamondsuit)^{\#'} = f^{\#'}\left(g^\#\diamondsuit\right)$, $\deg' f(g\diamondsuit) = \deg' f$, and

$$q_I = f_I\left(g^\#\diamondsuit\right)$$

for all $I \in \wp(m)$ such that $|I| = \deg' f$.

(ii) Let $f \in \mathcal{D}(\mathcal{B}) \boxtimes \mathcal{P}^{\mathbb{C}}$ *and* $f|_g = \sum_{I\in\wp(m)} q_I \beta^I$ *with* $q_I \in \mathcal{D}(\mathcal{B})$ *for all* $I \in \wp(m)$ *. Then* $\deg' f|_g = \deg' f$ *, and*

$$q_I = f_I|_{g^\#}$$

for all $I \in \wp(m)$ *such that* $|I| = \deg' f$ *, in particular* $(f|_g)^{\#'} = \left(f^{\#'}\right)\Big|_{g^\#}$.

Proof: (i) Obviously it suffices to prove the assertion for $f \in \mathcal{D}(\mathcal{M}) \boxtimes \mathcal{P}$. The fact that $h(g\diamondsuit)^{\#'} = h^{\#'}\left(g^\#\diamondsuit\right)$ for all $h \in \mathcal{D}(\mathcal{G}) \boxtimes \mathcal{P}$ can be seen in the following commutative diagrams: Let

$$m' : \mathcal{P} \boxtimes \mathcal{D}(\mathcal{G}) \boxtimes \mathcal{P} \to \mathcal{D}(\mathcal{G}) \boxtimes \mathcal{P}\,,\; a \otimes h \otimes b \mapsto (-1)^{\dot{a}\dot{h}} h \otimes (ab)\,,$$

m be the multiplication on $\mathcal{G}$, which is a super morphism from $\mathcal{G}$ to $\mathcal{G} \times \mathcal{G}$, and let

$$(m) : \mathcal{D}(\mathcal{G}) \to \mathcal{D}(\mathcal{G} \times \mathcal{G}) = \mathcal{D}(\mathcal{G})\hat{\boxtimes}\mathcal{D}(\mathcal{G})\,,\; h \mapsto h(m)\,,$$

$$(g) : \mathcal{D}(\mathcal{G}) \to \mathcal{P}\,,\; h \mapsto h(g)$$

and

$$\left(g^\#\right) : \mathcal{C}^\infty(G) \to \mathbb{C}\,,\; h \mapsto h\left(g^\#\right)$$

be the 'plugging in' homomorphisms, which are graded algebra homomorphisms. Then

$$\begin{array}{ccc}
\mathcal{D}(\mathcal{G}) \boxtimes \mathcal{P} & & \\
(m) \otimes \mathrm{id} \downarrow & & \\
\mathcal{D}(\mathcal{G})\hat{\boxtimes}\mathcal{D}(\mathcal{G}) \boxtimes \mathcal{P} & \stackrel{\#\hat{\otimes}\mathrm{id}\otimes\mathrm{id}}{\longrightarrow} & \mathcal{C}^\infty(G)\hat{\otimes}\mathcal{D}(\mathcal{G})\hat{\boxtimes}\mathcal{P} \\
(g)\hat{\otimes}\mathrm{id} \otimes \mathrm{id} \downarrow & \% & \downarrow \left(g^\#\right)\hat{\otimes}\mathrm{id} \otimes \mathrm{id} \\
\mathcal{P} \boxtimes \mathcal{D}(\mathcal{G}) \boxtimes \mathcal{P} & \stackrel{\#\otimes\mathrm{id}\otimes\mathrm{id}}{\longrightarrow} & \mathcal{D}(\mathcal{G}) \boxtimes \mathcal{P} \\
m' \downarrow & \% & \downarrow \#' \\
\mathcal{D}(\mathcal{G}) \boxtimes \mathcal{P} & \stackrel{\#'}{\longrightarrow} & \mathcal{D}(\mathcal{G})
\end{array} \tag{3.4}$$

and

$$
\begin{array}{ccc}
\mathcal{D}(\mathcal{G}) \boxtimes \mathcal{P} & \xrightarrow{\#'} & \mathcal{D}(\mathcal{G}) \\
(m) \otimes \mathrm{id} \downarrow & \% & \downarrow | \\
\mathcal{D}(\mathcal{G}) \hat{\boxtimes} \mathcal{D}(\mathcal{G}) \boxtimes \mathcal{P} & \xrightarrow{\mathrm{id} \hat{\otimes} \mathrm{id} \otimes \#} & \mathcal{D}(\mathcal{G}) \hat{\boxtimes} \mathcal{D}(\mathcal{G}) \\
{}^{\#} \hat{\otimes} \mathrm{id} \otimes \mathrm{id} \downarrow & \% & \downarrow {}^{\#} \hat{\otimes} \mathrm{id} \\
\mathcal{C}^{\infty}(G) \hat{\otimes} \mathcal{D}(\mathcal{G}) \boxtimes \mathcal{P} & \xrightarrow{\mathrm{id} \hat{\otimes} \mathrm{id} \otimes \#} & \mathcal{C}^{\infty}(G) \hat{\otimes} \mathcal{D}(\mathcal{G}) \\
\left(g^{\#}\right) \hat{\otimes} \mathrm{id} \otimes \mathrm{id} \downarrow & \% & \downarrow \left(g^{\#}\right) \hat{\otimes} \mathrm{id} \\
\mathcal{D}(\mathcal{G}) \boxtimes \mathcal{P} & \xrightarrow{\#'} & \mathcal{D}(\mathcal{G})
\end{array} \quad . \qquad (3.5)
$$

The map in the lower left corner of 3.4 maps $h \in \mathcal{D}(\mathcal{G}) \boxtimes \mathcal{P}$ to $h(g\diamondsuit)^{\#'} \in \mathcal{D}(\mathcal{B})$, the map in the upper right corner of 3.5 maps $h \in \mathcal{D}(\mathcal{G}) \boxtimes \mathcal{P}$ to $h^{\#'}\left(g^{\#}\diamondsuit\right) \in \mathcal{D}(\mathcal{B})$, and finally the map in the upper right corner of 3.4 and the map in the lower left corner of 3.5 , both going from $\mathcal{D}(\mathcal{G}) \boxtimes \mathcal{P}$ to $\mathcal{D}(\mathcal{G})$, coincide.

Now let $\nu := \deg' f$. Since the map

$$
\mathcal{D}(\mathcal{G}) \boxtimes \mathcal{P} \to \mathcal{D}(\mathcal{G}) \boxtimes \mathcal{P} \,,\; h \mapsto h(g\diamondsuit)
$$

is $\mathcal{P}$-linear we have

$$
\begin{aligned}
\sum_{I \in \wp(m)} q_I &= f(g\diamondsuit) \\
&= \sum_{I \in \wp(m)\,,\,|I| \geq \nu} f_I(g\diamondsuit)\beta^I \\
&= \sum_{I \in \wp(m)\,,\,|I| = \nu} f_I(g\diamondsuit)^{\#'}\beta^I \\
&\quad + \sum_{I \in \wp(m)\,,\,|I| = \nu} \left(f_I(g\diamondsuit) - f_I(g\diamondsuit)^{\#'}\right)\beta^I \\
&\quad + \sum_{I \in \wp(m)\,,\,|I| \geq \nu+1} f_I(g\diamondsuit)\beta^I \,,
\end{aligned}
$$

but all $\left(f_I(g\diamondsuit) - f_I(g\diamondsuit)^{\#'}\right)\beta^I$, $I \in \wp(m)$, $|I| = \nu$, and $f_I(g\diamondsuit)\beta^I$, $I \in \wp(m)$, $|I| \geq \nu + 1$, belong to $\mathcal{I}'^{\nu+1}$. So $q_I = f_I(g\diamondsuit)^{\#'} = f_I\left(g^{\#}\diamondsuit\right)$. Since therefore $q_I = 0$ if and only if $f_I = 0$ we finally obtain $\deg' f|_g = \deg' f$. $\square$

(ii) follows from (i) since $\widetilde{h|_g} = \widetilde{h}(g\diamondsuit)$ for all $h \in \mathcal{D}(\mathcal{B}) \boxtimes \mathcal{P}^{\mathbb{C}}$, $\widetilde{f} = \sum_{I \in \wp(m)} \widetilde{f}_I \beta^I$ and $\widetilde{f}(g\diamondsuit) = \sum_{I \in \wp(m)} \widetilde{q}_I \beta^I$ with all $\widetilde{f}_I \,,\, \widetilde{q}_I \in \mathcal{D}(\mathcal{G})^{\mathbb{C}}$. $\square$

Corollary 3.28 *Let $f \in sM_k(\Upsilon)$. Then $f_I \in sM_k(\Gamma)$ for all $I \in \wp(m)$ such that $|I| = \deg' f$. In particular $f^{\#'} \in sM_k(\Gamma)$.*

Because of proposition 3.27 and corollary 3.28 the idea now is to define the space of super cusp forms $sS_k(\Upsilon)$ as a sub graded $\mathcal{P}^{\mathbb{C}}$-module of $sM_k(\Upsilon)$ having the following property:

> **If** $f = \sum_{I \in \wp(m)} f_I \beta^I \in sS_k(\Upsilon)$ **,** $f_I \in \mathcal{O}(\mathcal{B})$ **for all** $I \in \wp(m)$ **,**
> **and** $I_0 \in \wp(m)$ **such that** $|I_0| = \deg' f$ **then** $f_{I_0} \in sS_k(\Gamma)$ **.**

Assume that we have already successfully defined $sS_k(\Upsilon)$. Then in particular $f^{\#'} \in sS_k(\Gamma)$ for all $f \in sS_k(\Upsilon)$, and we have the following theorem:

Theorem 3.29 *Let $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ be a family in $sS_k(\Upsilon)$ with the following properties:*

{*i*} $\left\{\varphi_\lambda^{\#'}\right\}_{\lambda \in \Lambda}$ *is a spanning set for $sS_k(\Gamma)$,*

{*iii*} *all φ_λ , $\lambda \in \Lambda$ are homogeneous,*

{*ii*} *if $\{c_\lambda\}_{\lambda \in \Lambda}$ is a family in $\mathbb{C}$ such that $\sum_{\lambda \in \Lambda} c_\lambda \varphi_\lambda^{\#'}$ converges with respect to $(\ ,\ )_\Gamma$ then $\sum_{\lambda \in \Lambda} c_\lambda \varphi_\lambda$ converges uniformly on compact sets to a function in $sS_k(\Upsilon)$.*

Then $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ is a $\mathcal{P}^{\mathbb{C}}$-spanning set for $sS_k(\Upsilon)$, more precisely if $f \in sS_k(\Upsilon)$ then there exists a family $\{a_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{P}^{\mathbb{C}}$ such that

$$f = \sum_{\lambda \in \Lambda} \varphi_\lambda a_\lambda \,,$$

where the sum converges with respect to $(\ ,\ )_\Gamma$ in all components belonging to some $I \in \wp(m)$ with $|I| = \deg' f$ and uniformly in all derivatives on compact sets, and $\deg a_\lambda \geq \deg' f$ for all $\lambda \in \Lambda$.

Proof: If $f = 0$ then the assertion is obvious. Otherwise $\deg' f \leq m$, and so we prove the assertion for $f \in sS_k(\Upsilon) \setminus \{0\}$ by reverse induction on $\deg' f$. Let $f \in sS_k(\Upsilon)$ such that $\deg' f = m$. Then by corollary 3.28

$$f = f_{\{1,\ldots,m\}} \beta^{\{1,\ldots,m\}}$$

with $f_{\{1,\ldots,m\}} \in sS_k(\Gamma)$. So by property {i} there exists a family $\{c_\lambda\}_{\lambda \in \Lambda}$ in $\mathbb{C}$ such that

$$f_{\{1,\ldots,m\}} = \sum_{\lambda \in \Lambda} c_\lambda \varphi_\lambda^{\#'} \,,$$

where the convergence is with respect to $(\ ,\ )_\Gamma$ and so also uniformly on compact sets, even uniformly in all derivatives on compact sets. So if we define $a_\lambda := c_\lambda \beta^{\{1\dots,m\}} \in \mathcal{P}^{\mathbb{C}}$ for all $\lambda \in \Lambda$ then

$$\begin{aligned}\sum_{\lambda\in\Lambda} \varphi_\lambda a_\lambda &= \sum_{\lambda\in\Lambda} c_\lambda \varphi_\lambda^{\#'} \beta^{\{1\dots,m\}} \\ &\quad + \sum_{\lambda\in\Lambda} c_\lambda \left(\varphi_\lambda - \varphi_\lambda^{\#'}\right) \beta^{\{1\dots,m\}} \\ &= f + \sum_{\lambda\in\Lambda} c_\lambda \left(\varphi_\lambda - \varphi_\lambda^{\#'}\right) \beta^{\{1\dots,m\}} ,\end{aligned}$$

but $\left(\varphi_\lambda - \varphi_\lambda^{\#'}\right) \beta^{\{1\dots,m\}} \in \mathcal{I}^{m+1}$ and therefore it is $= 0$ for all $\lambda \in \Lambda$.

Now assume $\nu \in \{1, \dots, m-1\}$ and that for all $h \in sS_k(\Upsilon)$ with $\deg'(h) \geq \nu + 1$ there exists a family $\{b_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{P}^{\mathbb{C}}$ such that $f = \sum_{\lambda\in\Lambda} \varphi_\lambda b_\lambda$, where the sum converges with respect to $(\ ,\ )_\Gamma$ in all components belonging to some $I \in \wp(m)$ with $|I| = \deg' f$ and uniformly on compact sets. Let $f \in sS_k(\Upsilon)$ with $\deg'(f) = \nu$. Then if we decompose

$$f = \sum_{I\in\wp(m)\,,\,|I|\geq\nu} f_{IJ} \beta^I ,$$

$f_I \in \mathcal{O}(\mathcal{B})$ for all $I \in \wp(m)$, $|I| \geq \nu$, by corollary 3.28 we see that $f_I \in sS_k(\Gamma)$ for all $I \in \wp(m)$ such that $|I| = \nu$. Again by property {i} there exist families $\left\{c_\lambda^{(I)}\right\}_{\lambda\in\Lambda}$ in $\mathbb{C}$, $I \in \wp(m)$, $|I| = \nu$, such that

$$f_I = \sum_{\lambda\in\Lambda} c_\lambda^{(I)} \varphi_\lambda^{\#'}$$

for all $I \in \wp(m)$ such that $|I| = \nu$. Let $I \in \wp(m)$ such that $|I| = \nu$. By property {iii} $\sum_{\lambda\in\Lambda} c_\lambda^{(I)} \varphi_\lambda$ converges uniformly on compact sets to a function $F_I \in sS_k(\Upsilon)$, and $F_I^{\#'} = f_I$. So

$$\begin{aligned} h &:= f - \sum_{I\in\wp(m)\,,\,|I|=\nu} F_I \beta^I \\ &= -\sum_{I\in\wp(m)\,,\,|I|=\nu} \left(F_I - F_I^{\#'}\right) \beta^I \\ &\quad + \sum_{I\in\wp(m)\,,\,|I|\geq\nu+1} f_I \beta^I \\ &\in sS_k(\Upsilon)\end{aligned}$$

with $\deg' h \geq \nu+1$. So by induction hypothesis there exists a family $\{b_\lambda\}_{\lambda\in\Lambda}$ such that $h = \sum_{\lambda\in\Lambda} \varphi_\lambda b_\lambda$ in compact convergence and $\deg b_\lambda \geq \nu + 1$.

Therefore by property {ii} $f = \sum_{\lambda \in \Lambda} \varphi_\lambda a_\lambda$ in compact convergence, since all φ_λ are holomorphic with respect to $\left(\frac{\mathbf{Z}}{\zeta}\right)$ even in uniform convergence in all derivatives on compact sets, if we define

$$a_\lambda := \sum_{I \in \wp(m)\,,\,|I|=\nu} (-1)^{\nu+\dot{\varphi}_\lambda} c_\lambda^{(I)} \beta^I \ + b_\lambda \in \mathcal{P}^{\mathbb{C}}$$

for all $\lambda \in \Lambda$. Clearly $\deg a_\lambda \geq \nu$, and

$$\begin{aligned} f &= \sum_{\lambda \in \Lambda} \varphi_\lambda a_\lambda \\ &= \sum_{I \in \wp(m)\,,\,|I|=\nu} \ \sum_{\lambda \in \Lambda} c_\lambda^{(I)} \varphi_\lambda^{\#'} \beta^I \\ &\quad + \sum_{I \in \wp(m)\,,\,|I|=\nu} \ \sum_{\lambda \in \Lambda} c_\lambda^{(I)} \left(\varphi_\lambda - \varphi_\lambda^{\#'}\right) \beta^I + \sum_{\lambda \in \Lambda} b_\lambda \varphi_\lambda \,. \end{aligned}$$

All $\left(\varphi_\lambda - \varphi_\lambda^{\#'}\right) \beta^I$ and b_λ , $I \in \wp(n)$, $|I| = \nu$, $\lambda \in \Lambda$, belong to $\mathcal{I}^{\nu+1}$. Therefore we see that the convergence is with respect to $(\ ,\ )_\Gamma$ in all components belonging to some $I \in \wp(m)$ having $|I| = \nu$. $\square$

Now in the end let us consider three special cases where it is possible to define the space $sS_k(\Upsilon)$ having the desired property.

First case : $\Gamma \backslash G$ **compact.**

Then we define $sS_k(\Upsilon) := sM_k(\Upsilon)$.

Second case : **There exists a discrete subgroup** $\Gamma \sqsubset G$ **and a** $\mathcal{P}$**-element** $g \in_{\mathcal{P}} \mathcal{G}$ **such that** $\Upsilon = g\Gamma g^{-1}$ **.**

Without loss of generality we may assume that $g^\# = 1$. Then clearly $\Gamma = \Upsilon^\#$.

Proposition 3.30

$$\Phi : sM_k(\Gamma) \boxtimes \mathcal{P}^{\mathbb{C}} \to sM_k(\Upsilon)\,,\ f \mapsto f|_{g^{-1}}$$

is a $\mathcal{P}^{\mathbb{C}}$*-linear isomorphism, which respects the grading.*

Proof: simple calculation. $\Phi^{-1} : sM_k(\Upsilon) \to sM_k(\Gamma) \boxtimes \mathcal{P}^{\mathbb{C}}$ is given by $f \mapsto f|_g$. $\square$

Clearly Φ and Φ^{-1} are continuous with respect to compact convergence.

Definition 3.31 *The sub graded $\mathcal{P}^{\mathbb{C}}$-module*

$$sS_k(\Upsilon) := \Phi\left(sS_k(\Gamma) \boxtimes \mathcal{P}^{\mathbb{C}}\right)$$

of $sM_k(\Upsilon)$ is called the space of super cusp forms for Υ of weight k .

For proving that this definition is independent of the choice of $g \in_{\mathcal{P}} \mathcal{G}$ one has to consider a $\mathcal{P}$-point $g \in_{\mathcal{P}} \mathcal{G}$ commuting with all elements of Γ and to show that $|_g : sM_k(\Gamma) \boxtimes \mathcal{P}^{\mathbb{C}} \to sM_k(\Gamma) \boxtimes \mathcal{P}^{\mathbb{C}}$ maps $sS_k(\Gamma) \boxtimes \mathcal{P}^{\mathbb{C}}$ onto itself. Unfortunately this seems to be out of reach. However we have the following theorem:

Theorem 3.32

(i) Let $f = \sum_{I \in \wp(m)} f_I \beta^I \in sS_k(\Upsilon)$, $f_I \in \mathcal{O}(\mathcal{B})$ for all $I \in \wp(m)$, and $I_0 \in \wp(m)$ such that $|I| = \deg' f$. Then $f_{I_0} \in sS_k(\Gamma)$.

(ii) If $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ is a spanning set for $sS_k(\Gamma)$ then $\left\{\varphi_\lambda|_{g^{-1}}\right\}_{\lambda \in \Lambda}$ fulfills properties $\{i\}$, $\{ii\}$ and $\{iii\}$ of theorem 3.29 , and so it is a $\mathcal{P}^{\mathbb{C}}$-spanning set for $sS_k(\Upsilon)$ in the sense of theorem 3.29 .

Proof: (i) Let $\Phi^{-1}(f) = f|_g = \sum_{I \in \wp(m)} q_I \beta^I \in sS_k(\Gamma) \boxtimes \mathcal{P}^{\mathbb{C}}$, all $q_I \in sS_k(\Gamma)$. Then by proposition 3.27 (ii) $f_{I_0} = q_{I_0}$. $\square$
(ii) Properties {i} and {ii} are clearly fulfilled. For proving property {iii} let $\{c_\lambda\}_{\lambda \in \Lambda}$ be a family in $\mathbb{C}$ such that $\sum_{\lambda \in \Lambda} c_\lambda \varphi_\lambda$ converges with respect to $(\ ,\ )_\Gamma$. Then $f := \sum_{\lambda \in \Lambda} c_\lambda \varphi_\lambda \in sS_k(\Gamma)$, where the sum converges also uniformly on compact sets, and since all $\varphi_\lambda \in \mathcal{O}(\mathcal{B})$, $\lambda \in \Lambda$, even in compact convergence in all derivatives. $|_g : \mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{B})$ is continuous with respect to the uniform structure of compact convergence in all derivatives, so $f|_g := \sum_{\lambda \in \Lambda} \varphi_\lambda|_{g^{-1}}$ in compact convergence as well. $\square$

Third case : $p \geq 2$ **,** $q = 1$ **,** vol $\Gamma \backslash G < \infty$ **,** $\Gamma \backslash G$ **not compact, and** $k \geq k_0$ **, where** $k_0 \in \mathbb{N}$ **is given by** S**ATAKE's theorem, theorem 3.11 .**

We will use the Fourier expansion given by theorem 3.15 of section 3.2 . Before we do so we need some tools:

Lemma 3.33 *Let Φ be a $\mathcal{P}$-super* Lie *group homomorphism from $\mathbb{R}$ to $\mathcal{G}$ (this implies that $\Phi(t) \in_{\mathcal{P}} \mathcal{G}$ for all $t \in \mathbb{R}$) and $h \in (\mathcal{D}(\mathcal{G}) \boxtimes \mathcal{P})^{\mathbb{C}}$ left-$\Phi(1)$-invariant. Then there exists a unique splitting $h = \sum_{s \in \mathbb{Z}} b_s$,
$b_s \in (\mathcal{D}(\mathcal{G}) \boxtimes \mathcal{P})^{\mathbb{C}}$, $s \in \mathbb{Z}$, such that*

$$b_s(\Phi(t)\diamondsuit) = e^{2\pi i s t} b_s$$

for all $s \in \mathbb{Z}$ and $t \in \mathbb{R}$. $\deg' b_s \geq \deg h$ for all $s \in \mathbb{Z}$, and if h is homogeneous then all b_s are homorgeneous of the same parity.

Proof: Since $h \in (\mathcal{D}(\mathcal{G}) \boxtimes \mathcal{P})^{\mathbb{C}}$ is left-$\Phi(1)$-invariant we see that

$$h(\Phi \diamondsuit) \in (\mathcal{D}((\mathbb{R}/\mathbb{Z}) \times \mathcal{G}) \boxtimes \mathcal{P})^{\mathbb{C}} = \left(\mathcal{C}^\infty(\mathbb{R}/\mathbb{Z}) \hat{\otimes} \mathcal{D}(\mathcal{G}) \boxtimes \mathcal{P}\right)^{\mathbb{C}} .$$

If h is homogeneous then again $h(\Phi \diamondsuit)$ is homogeneous of the same parity. By proposition 3.27 (i) we have $\deg' h(\Phi \diamondsuit) = \deg' h$.

Let $\mathcal{S}$ denote the structural sheaf of $\mathcal{G}$ regarded as a real $\left((p+1)^2 + r^2 - 1, 2(p+1)r\right)$-dimensional super manifold (this means $\mathcal{D}(\mathcal{U}) = \mathcal{S}\left(\mathcal{U}^\#\right)$ for each open sub super manifold $\mathcal{U}$ of $\mathcal{G}$) . Then $h(\Phi \diamondsuit) \in \left(\mathcal{C}^\infty(\mathbb{R}/\mathbb{Z}) \hat{\otimes} \mathcal{S}(G) \boxtimes \mathcal{P}\right)^{\mathbb{C}}$, and for all $g \in G$ there exists an open neighbourhood $U \subset G$ of g such that $\mathcal{S}|_U \simeq \mathcal{C}_U^\infty \otimes \Lambda\left(\mathbb{R}^{2(p+1)r}\right) \simeq \mathcal{C}_U^\infty \otimes \mathcal{D}\left(\mathbb{R}^{0|2(p+1)r}\right)$. Let $\xi_1, \ldots, \xi_{2(p+1)r}$ be the odd coordinate functions on $\mathbb{R}^{0|2(p+1)r}$.

Let δ be the set of all $\left(U, (d_s)_{s\in\mathbb{Z}}\right)$ such that $U \subset G$ open, all $d_s \in (\mathcal{S}(U) \boxtimes \mathcal{P})^{\mathbb{C}}$, $s \in \mathbb{Z}$, and

$$h(\Phi \diamondsuit)|_{\mathbb{R}\times U} = \sum_{s\in\mathbb{Z}} d_s e^{2\pi i s\tau} ,$$

where we denote the coordinate function on $\mathbb{R}$ by τ .

Step I **Show that if** $\left(U, (d_s)_{s\in\mathbb{Z}}\right), \left(V, (h_s)_{s\in\mathbb{Z}}\right) \in \delta$ **then** $d_s \equiv h_s$ **on** $\mathbb{R} \times (U \cap V)$ **for all** $s \in \mathbb{Z}$ **.**

Let $g \in U \cap V$ be arbitrary. Then there exists an open neighbourhood $W \subset U \cap V$ of g such that $\mathcal{S}|_W \simeq \mathcal{C}_W^\infty \otimes \Lambda\left(\mathbb{R}^{2(p+1)r}\right)$, and without loss of generality we may assume equality. Then if we decompose

$$h(\Phi \diamondsuit)|_{\mathbb{R}\times W} = \sum_{S\in\wp(2(p+1)r)\,,\, I\in\wp(m)} q_{SI}\, \xi^S \beta^I ,$$

$$d_s|_{\mathbb{R}\times W} = \sum_{S\in\wp(2(p+1)r)\,,\, I\in\wp(m)} d_{s,SI}\, \xi^S \beta^I$$

and

$$h_s|_{\mathbb{R}\times W} = \sum_{S\in\wp(2(p+1)r)\,,\, I\in\wp(m)} h_{s,SI}\, \xi^S \beta^I ,$$

all $q_{SI} \in \mathcal{C}^\infty((\mathbb{R}/\mathbb{Z}) \times W)^{\mathbb{C}} = \left(\mathcal{C}^\infty(\mathbb{R}/\mathbb{Z}) \hat{\otimes} \mathcal{C}^\infty(W)\right)^{\mathbb{C}}$, $d_{s,SI}, h_{s,SI} \in \mathcal{C}^\infty(W)^{\mathbb{C}}$, then we see that

$$q_{SI} = \sum_{s\in\mathbb{Z}} d_{s,SI} e^{2\pi i s\tau} = \sum_{s\in\mathbb{Z}} h_{s,SI} e^{2\pi i s\tau} .$$

So $d_{s,SI} = h_{s,SI}$ for all $S \in \wp(2(p+1)r)$ and $I \in \wp(m)$ by classical FOURIER analysis.

Step II **Show that for all $g \in G$ there exists $\left(U, (d_s)_{s\in\mathbb{Z}}\right) \in \delta$ such that $g \in U$, $\deg' d_s \geq \deg' h$ for all $s \in \mathbb{Z}$ and if h is homogeneous then all d_s are homogeneous of the same parity.**

Let $g \in G$. Then again there exists an open neighbourhood $U \subset G$ of g such that $\mathcal{S}|_U \simeq \mathcal{C}_U^\infty \otimes \Lambda\left(\mathbb{R}^{2(p+1)r}\right)$, and again without loss of generality we assume equality. So we can decompose

$$h\left(\Phi \diamondsuit\right)|_{\mathbb{R}\times U} = \sum_{S\in\wp(2(p+1)r)\,,\,I\in\wp(m)} q_{SI}\,\xi^S\beta^I$$

with some $q_{SI} \in \mathcal{C}^\infty\left((\mathbb{R}/\mathbb{Z})\times U\right)^{\mathbb{C}}$, $S \in \wp(2(p+1)r)$, $I \in \wp(m)$. By classical FOURIER theory we see that there exist unique $d_{s,SI} \in \mathcal{C}^\infty(U)^{\mathbb{C}}$ such that

$$q_{SI} = \sum_{s\in\mathbb{Z}} d_{s,SI} e^{2\pi i s\tau}\,,$$

where the convergence is compact in all derivatives on $\mathbb{R} \times U$. If $I \in \wp(m)$ such that $|I| < \deg' h$ then all $q_{SI} = 0$ and so all $d_{s,SI} = 0$, $S \in \wp(2(p+1)r)$, $s \in \mathbb{Z}$. So clearly $\left(U, (d_s)_{s\in\mathbb{Z}}\right) \in \delta$ if we define $d_s := \sum_{S\in\wp(2(p+1)r)\,,\,I\in\wp(m)} d_{s,SI}\xi^S\beta^I \in \mathcal{S}(U)\boxtimes\mathcal{P}$, $s \in \mathbb{Z}$.

Step III **Conclusion.**

By step II

$$\bigcup_{\left(U,(d_s)_{s\in\mathbb{Z}}\right)\in\delta} U = G\,,$$

and so by step I since $(\mathcal{S}\boxtimes\mathcal{P})^{\mathbb{C}}$ is a sheaf on G we see that there exist unique $c_s \in (\mathcal{S}(G)\boxtimes\mathcal{P})^{\mathbb{C}} = (\mathcal{D}(\mathcal{G})\boxtimes\mathcal{P})^{\mathbb{C}}$, $s \in \mathbb{Z}$, such that

$$c_s|_U = d_s$$

for all $\left(U, (d_s)_{s\in\mathbb{Z}}\right) \in \delta$. Then $\deg' c_s \geq \deg' h$, if h is homogeneous then all c_s are homogeneous of the same parity, and clearly $c_s \in (\mathcal{D}(\mathcal{G})\boxtimes\mathcal{P})^{\mathbb{C}}$, $s \in \mathbb{Z}$ are unique such that

$$h\left(\Phi \diamondsuit\right) = \sum_{s\in\mathbb{Z}} c_s e^{2\pi i s\tau}\,.$$

Now for proving existence of $b_s \in (\mathcal{D}(\mathcal{G}) \boxtimes \mathcal{P})^{\mathbb{C}}$, $s \in \mathbb{Z}$, let $b_s := c_s$ for all $s \in \mathbb{Z}$. Then obviously

$$h = \sum_{s \in \mathbb{Z}} b_s ,$$

and for all $t \in \mathbb{R}$

$$\begin{aligned} \sum_{s \in \mathbb{Z}} b_s \left(\Phi(t) \diamondsuit \right) e^{2\pi i \tau} &= h \left(\Phi(\tau) \, \Phi(t) \diamondsuit \right) \\ &= h \left(\Phi(\tau + t) \right) \\ &= \sum_{s \in \mathbb{Z}} b_s e^{2\pi i s t} e^{2\pi i s \tau} . \end{aligned}$$

By uniqueness of $c_s \in (\mathcal{D}(\mathcal{G}) \boxtimes \mathcal{P})^{\mathbb{C}}$, $s \in \mathbb{Z}$, we see that $b_s \left(\Phi(t) \diamondsuit \right) = c_s \left(\Phi(t) \diamondsuit \right) = b_s e^{2\pi i s t}$ for all $s \in \mathbb{Z}$.

For proving uniqueness assume that $b_s \in (\mathcal{D}(\mathcal{G}) \boxtimes \mathcal{P})^{\mathbb{C}}$, $s \in \mathbb{Z}$, such that

$$b_s \left(\Phi(t) \diamondsuit \right) = e^{2\pi i s t} b_s$$

for all $s \in \mathbb{Z}$ and $t \in \mathbb{R}$ and $h = \sum_{s \in \mathbb{Z}} b_s$. Then

$$h \left(\Phi \, \diamondsuit \right) = \sum_{s \in \mathbb{Z}} b_s \left(\Phi \, \diamondsuit \right) = \sum_{s \in \mathbb{Z}} b_s e^{2\pi i s \tau} .$$

So by uniqueness of $c_s \in (\mathcal{D}(\mathcal{G}) \boxtimes \mathcal{P})^{\mathbb{C}}$, $s \in \mathbb{Z}$, we see that $b_s = c_s$ for all $s \in \mathbb{Z}$. $\square$

Now let $\eta \subset N$ open and relatively compact, $t_0 \in \mathbb{R}$ and $\Xi \subset G'$ finite be given by theorem 3.13 in section 3.2 with respect to Γ . For all $g \in \Xi$ let $n_g \in_{\mathcal{P}} g^{-1} \Upsilon g$ such that $n_g^{\#} \in g^{-1} \Gamma g \cap N' Z(G') \setminus Z(G')$. Then

$$R n_g^{\#} R^{-1} = n'_{\lambda_g, \mathbf{0}} \left(\begin{array}{c|c} \varepsilon_g 1 & 0 \\ \hline 0 & E_g \end{array} \right)$$

with appropriate $\lambda_g \in \mathbb{R} \setminus \{0\}$, $\varepsilon_g \in U(1)$ and $E_g \in U(r)$, $\varepsilon_g^{p+1} = \det E_g$. Let $g \in \Xi$ be arbitrary. As we have already seen in section 3.2 , $j \left(R n_g^{\#} R^{-1} \right) := j \left(R n_g^{\#} R^{-1}, \mathbf{w} \right) = \varepsilon_g^{-1} \in U(1)$ is independent of $\mathbf{w} \in H$. Let $\chi_g \in \mathbb{R}$ such that $j \left(R n_g^{\#} R^{-1}, \mathbf{w} \right) = e^{2\pi i \chi_g}$. Again without loss of generality we can assume that E_g is diagonal, and we may assume $\lambda_g > 0$. So choose

$$D_g = \begin{pmatrix} d_1^{(g)} & & 0 \\ & \ddots & \\ 0 & & d_r^{(g)} \end{pmatrix} \in \mathbb{R}^{r \times r}$$

diagonal such that $E_g = \exp(2\pi i D_g)$. Then clearly

$$\varphi_g : \mathbb{R} \hookrightarrow G \, , \, t \mapsto R^{-1} n'_{t\lambda_g, \mathbf{0}} \left(\begin{array}{c|c} e^{-2\pi i t \chi_g} 1 & 0 \\ \hline 0 & \exp(2\pi i t D_g) \end{array} \right) R$$

is a $\mathcal{C}^\infty$-LIE group embedding, and $\varphi(1) = n_g^\# \in g^{-1} \Gamma g$.

Let $f \in sM_k(\Gamma)$. Then $\widetilde{f}(g\diamondsuit) \in \mathcal{D}(\mathcal{G})^{\mathbb{C}}$ is left-$\varphi_g(1)$-invariant, and by lemma 3.33 applied with $\mathcal{P} = \mathbb{R}$ we see that there exists a unique splitting

$$\widetilde{f}(g\diamondsuit) = \sum_{s \in \mathbb{Z}} b_{g,s} \, ,$$

$b_{g,s} \in \mathcal{D}(\mathcal{G})^{\mathbb{C}}$, $s \in \mathbb{Z}$, such that

$$b_{g,s}(\varphi_g(t)\diamondsuit) = e^{2\pi i s t} b_{g,s}$$

for all $s \in \mathbb{Z}$ and $t \in \mathbb{R}$, and a straight forward calculation shows that for all $s \in \mathbb{Z}$

$$b_{g,s} = \left(\left(\sum_{I \in \wp(r)} c_{I, m_{I,s}}(\mathbf{w}_2) e^{2\pi m_{I,s} w_1} \vartheta^I \right) \Big|_R \right)^\sim ,$$

where

$$m_{I,s} := \frac{1}{\lambda_g} \left(s - \mathrm{tr}_I \, D_g - (k + |I|) \chi_g \right) \in \frac{1}{\lambda_g} \left(\mathbb{Z} - \mathrm{tr}_I \, D_g - (k + |I|) \chi_g \right)$$

and $c_{I, m_{I,s}} \in \mathcal{O}\left(\mathbb{C}^{p-1}\right)$ is given by theorem 3.15 (i) for all $I \in \wp(r)$.

From now on we have to make the additional assumption that for all $g \in \Xi$ we can choose n_g , χ_g and D_g such that

$$\mathrm{tr}_I \, D_g + (k + |I|) \chi_g \in]-1, 0]$$

and all $d_k^{(g)}, d_k^{(g)} - d_l^{(g)} \notin \mathbb{Z} \setminus \{0\}$, $k, l \in \{1, \dots, r\}$.

Proposition 3.34 *$f \in sS_k(\Gamma)$ if and only if $b_{g,s} = 0$ for all $g \in \Xi$ and $s \geq 0$.*

Proof: Let $g \in \Xi$ and $I \in \wp(m)$. Since $s \in \mathbb{Z}$ and we assume $\mathrm{tr}_I D_g + (k + |I|)\,\chi_g \in]-1, 0]$ we have $s \geq 0$ if and only if $m_{s,I} \geq 0$.

'$\Rightarrow$' : Let $f \in sM_k(\Gamma)$ and $g \in \Xi$. Let $f|_g|_{R^{-1}} = \sum_{I \in \wp(r)} q_I \vartheta^I$, all $q_I \in \mathcal{O}(H)$. Then

$$f|_g = \sum_{I \in \wp(r)} q_I(R\diamondsuit)\zeta^I j(R, \diamondsuit)^{k+|I|} .$$

Since $\widetilde{f'} \in L^2(\Gamma\backslash G) \otimes \Lambda(\mathbb{C}^r)$, and so $\widetilde{f'}(g\diamondsuit) \in L^2(\eta A_{>t_0} K) \otimes \Lambda(\mathbb{C}^r)$ by corollary 3.14 of section 3.2 , we conclude that

$$q_I(R\mathbf{z})\, j(R, \mathbf{z})^{k+|I|}\, \Delta(\mathbf{z}, \mathbf{z})^{\frac{k+|I|}{2}} \in L^2(\eta A_{>t_0}\mathbf{0})$$

with respect to the G-invariant measure $\Delta(\mathbf{z}, \mathbf{z})^{-(p+1)}\, dV_{\mathrm{Leb}}$ on B or equivalently $q_I \Delta'(\mathbf{w}, \mathbf{w})^{\frac{k+|I|}{2}} \in L^2(R\eta A_{>t_0}\mathbf{0})$ for all $I \in \wp(r)$ with respect to the RGR^{-1}-invariant measure $\Delta'(\mathbf{w}, \mathbf{w})^{-(p+1)}\, dV_{\mathrm{Leb}}$ on H . So by theorem 3.15 (ii) and (iii) of section 3.2 we see that $c_{I,m} = 0$ for all $m \geq 0$ and $I \in \wp(m)$, and so $b_{g,s} = 0$ for all $s \geq 0$.

'$\Leftarrow$' : Conversely assume that $b_{g,s} = 0$ for all $g \in \Xi$ and $s \geq 0$. Let $g \in \Xi$. Then we have $c_{I,m} = 0$ for all $m \geq 0$ and $I \in \wp(m)$, and by corollary 3.14 it suffices to show that

$$\widetilde{f'}(g\diamondsuit) \in L^2(\eta A_{>t_0} K) \otimes \Lambda(\mathbb{C}^r) .$$

If we decompose $f|_g|_{R^{-1}} = \sum_{I \in \wp(r)} q_I \vartheta^I$, all $q_I \in \mathcal{O}(H)$, we obtain $q_I \Delta'(\mathbf{w}, \mathbf{w})^{\frac{k+|I|}{2}} \in L^2(R\eta A_{>t_0}\mathbf{0})$ for all $I \in \wp(r)$ with respect to the RGR^{-1}-invariant measure $\Delta'(\mathbf{w}, \mathbf{w})^{-(p+1)}\, dV_{\mathrm{Leb}}$ on H by theorem 3.15 (iii) of section 3.2 , and so

$$q_I(R\mathbf{z})\, j(R, \mathbf{z})^{k+|I|}\, \Delta(\mathbf{z}, \mathbf{z})^{\frac{k+|I|}{2}} \in L^2(\eta A_{>t_0}\mathbf{0})$$

with respect to the G-invariant measure $\Delta(\mathbf{z}, \mathbf{z})^{-(p+1)}\, dV_{\mathrm{Leb}}$ on B . So we see that $\widetilde{f'}(g\diamondsuit) \in L^2(\eta A_{>t_0} K) \otimes \Lambda(\mathbb{C}^r)$. $\Box$

Again let $g \in \Xi$.

Theorem 3.35 *There exists a unique $\mathcal{P}$-super* Lie *group embedding Φ_g from $\mathbb{R}$ to $\mathcal{G}$ such that $\Phi_g(1) = n_g$ and $\Phi_g^{\#} = \varphi_g$.*

Proof: Since we did not introduce the concept of super Lie algebras of super Lie groups and the concept of a super chart we are only able to give a sketch.

We use the exponential mapping exp which is a holomorphic super morphism from $\mathbb{C}^{n^2+r^2|2nr,\overline{2nr}}$ with even coordinate functions a'_{ij} and $d'_{kl} \in \mathcal{O}(sGL(n|r))_0$ and odd coordinate functions β'_{il} and $\gamma'_{kj} \in \mathcal{O}(sGL(n|r))_1$, $i,j = 1,\ldots,n$ and $k,l = 1,\ldots,r$, to $sGL(n,r)$, $n = p+1$, given by

$$\exp\left(\begin{array}{c|c} A' & \beta' \\ \hline \gamma' & D' \end{array}\right)$$

defined via the exponential power series, see theorem 2.5 of section 2.1 . Since the body $(D\exp)^{\#}(\mathbf{0})$ of the super Jacobian of exp at $\mathbf{0}$ is the identity matrix one sees that exp is locally biholomorphic at $\mathbf{0} \in \mathbb{C}^{n^2+r^2}$ by theorem 2.40 (i) . Let us again sum up the coordinate functions of $\mathbb{C}^{n^2+r^2|2nr,\overline{2nr}}$ in blocks according to

$$\left(\begin{array}{cc|c} A' & B' & \mu' \\ C' & D' & \nu' \\ \hline \rho' & \sigma' & E' \end{array}\right) \begin{array}{l} \}p \\ \leftarrow p+1 \\ \}r \end{array} ,$$

and let $\mathcal{V}$ be the real $\left(n^2+r^2-1, 2nr\right)$-dimensional sub super manifold of $\mathbb{C}^{n^2+r^2|2nr,\overline{2nr}}$ given by the equations

$$\left(\begin{array}{cc|c} A' & B' & \mu' \\ C' & D' & \nu' \\ \hline \rho' & \sigma' & E' \end{array}\right)^* \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) = -\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)\left(\begin{array}{cc|c} A' & B' & \mu' \\ C' & D' & \nu' \\ \hline \rho' & \sigma' & E' \end{array}\right),$$

$$s\mathrm{tr}\left(\begin{array}{cc|c} A' & B' & \mu' \\ C' & D' & \nu' \\ \hline \rho' & \sigma' & E' \end{array}\right) = 0$$

or more explicitely

$$\begin{aligned} A'^* &= -A' , \\ C'^* &= B' , \\ D'^* &= -D' , \\ E'^* &= -E' , \\ \mathrm{tr}A' + \mathrm{tr}D' &= \mathrm{tr}E' , \\ \rho'^* &= -\mu' \\ \sigma'^* &= \nu' . \end{aligned}$$

Then clearly $\mathcal{V}$ is diffeomorphic to $\mathbb{R}^{n^2+r^2-1|2nr}$, and exp restricts to a super morphism $\exp_{\mathcal{V}}$ from $\mathcal{V}$ to $\mathcal{G} = sSU(p,1|r)$. $\mathcal{V}^\# = \mathfrak{g}$ is the LIE algebra of $G = \mathcal{G}^\#$, and $\exp_{\mathcal{V}}^\# = \exp_G : \mathfrak{g} \to G$ is the usual exponential mapping. Again $\exp_{\mathcal{V}}$ is a local diffeomorphism at $\mathbf{0} \in \mathfrak{g} = \mathcal{V}^\#$, and so we can regard $\mathcal{V}$ as the 'super LIE algebra' of the super LIE group $\mathcal{G}$.
Since $\exp_{\mathcal{V}}$ is a local super diffeomorphism and its body $\exp_G$ maps $\mathbf{0}$ to 1 we will use $\exp_{\mathcal{V}}$ as a local chart of $\mathcal{G}$ at 1 when talking about super Jacobians, in other words given an open sub super manifold $\mathcal{U}$ of $\mathcal{G}$ such that $\mathcal{U}^\#$ is a neighbourhood of $1 \in G$ which is small enough we identify $\mathcal{U}$ with the unique super open subset $\mathcal{W}$ of $\mathcal{V}$ such that $\mathbf{0} \in \mathcal{W}^\#$ and $\exp_{\mathcal{V}}|_{\mathcal{W}}$ is a super diffeomorphism from $\mathcal{W}$ to $\mathcal{U}$.

Now let

$$X := R^{-1} \left(\begin{array}{cc|c} \begin{array}{c|c} 0 & \lambda_g \\ \hline 0 & 0 \end{array} & & 0 \\ \hline & 0 & 2\pi i D_g \end{array} \right) \begin{array}{l} \leftarrow 1 \\ \}p \\ \}r \end{array} \quad R \in \mathfrak{g} \, .$$

Then $\varphi_g = \exp_G(\Diamond X)$, and so especially $n_g^\# = \exp_G(X)$. It suffices to show that $\exp_{\mathcal{V}}(X + \Diamond)$ is a local diffeomorphism at $\mathbf{0}$ since then there exists a unique $Y \in_{\mathcal{P}} \mathcal{V}$ with body $Y^\# = X$ and $\exp_{\mathcal{V}}(Y) = n_g$, and so $\Phi_g := \exp_{\mathcal{V}}(\Diamond Y)$ is the unique $\mathcal{P}$-super LIE group embedding from $\mathbb{R}$ to $\mathcal{G}$ such that $\Phi_g^\# = \varphi_g$ and $\Phi_g(1) = n_g$.

Since the left translation $l_{\left(n_g^\#\right)^{-1}}$ is a super diffeomorphism from $\mathcal{G}$ to $\mathcal{G}$ it is even enough to show that $l_{\left(n_g^\#\right)^{-1}} \circ \exp_{\mathcal{V}}(\Diamond + X)$ is a local super diffeomorphism at $\mathbf{0} \in \mathfrak{g}$. $l_{\left(n_g^\#\right)^{-1}} \circ \exp_{\mathcal{V}}(X + \Diamond)$ has the body

$$\left(l_{\left(n_g^\#\right)^{-1}} \circ \exp_{\mathcal{V}}(X + \Diamond) \right)^\# = l_{\left(n_g^\#\right)^{-1}} \circ \exp_{\mathfrak{g}}(X + \Diamond) : \mathfrak{g} \to G$$

mapping $\mathbf{0}$ to 1 . So by theorem 2.19 of section 2.2 it suffices to show that $D\left(l_{\left(n_g^\#\right)^{-1}} \circ \exp_{\mathcal{V}}(X + \Diamond) \right)^\# (\mathbf{0})$ is invertible.

Since $D\left(l_{\left(n_g^\#\right)^{-1}} \circ \exp_{\mathcal{V}}(X + \Diamond) \right)^\# (\mathbf{0})$ only involves terms which are constant or linear with respect to the odd coordinate functions we may replace these odd coordinate functions by even ones.

So let $\exp'$ denote the exponential map from $\mathbb{C}^{(n+r)\times(n+r)}$, which is the LIE algebra of $GL(n+r,\mathbb{C})$, to $GL(n+r,\mathbb{C})$. Then $\exp_G$ is the restriction of

$\exp'$ to $\mathfrak{g} \hookrightarrow \mathbb{C}^{(n+r)\times(n+r)}$ going from $\mathfrak{g}$ to $G \hookrightarrow GL(n+r,\mathbb{C})$. Let V' denote the $\big((n+r)^2-1\big)$-dimensional $\mathbb{R}$-subspace of $\mathbb{C}^{(n+r)\times(n+r)}$ containing all matrices

$$\left(\begin{array}{cc|c} A' & B' & M' \\ C' & D' & N' \\ \hline P' & Q' & E' \end{array}\right) \begin{array}{l} \}p \\ \leftarrow p+1 \\ \}r \end{array}$$

such that

$$\begin{aligned} A'^* &= -A'\,, \\ C'^* &= B'\,, \\ D'^* &= -D'\,, \\ E'^* &= -E'\,, \\ \mathrm{tr}A' + \mathrm{tr}D' &= \mathrm{tr}E'\,, \\ P'^* &= -M'\,, \\ Q'^* &= N'\,. \end{aligned}$$

Then of course V' is not a sub Lie algebra of $\mathbb{C}^{(n+r)\times(n+r)}$. But still the image of V' under the differential of $l_{\left(n_g^\#\right)^{-1}} \circ \exp'(\diamondsuit + X)$ taken at $\mathbf{0}$ again lies in V' , and $D\left(l_{\left(n_g^\#\right)^{-1}} \circ \exp_{\mathcal{V}}(\diamondsuit + X)\right)(\mathbf{0})^\#$ is equal to the differential of $l_{\left(n_g^\#\right)^{-1}} \circ \exp'(\diamondsuit + X)$ taken at $\mathbf{0}$ and restricted to V' . So it suffices to show that the differential of $l_{\left(n_g^\#\right)^{-1}} \circ \exp'(\diamondsuit + X)$ is an automorphism of $\mathbb{C}^{(n+r)\times(n+r)}$.

We use theorem 1.7 of chapter II section 1.4 in [9] , which says the following:

> Let G be a Lie group with Lie algebra $\mathfrak{g}$. The exponential mapping of the manifold $\mathfrak{g}$ into G has the differential
>
> $$D\exp_X = D\left(l_{\exp X}\right)_e \circ \frac{1-e^{-\mathrm{ad}_X}}{\mathrm{ad}_X} \qquad (X \in \mathfrak{g})\,.$$
>
> As usual, $\mathfrak{g}$ is here identified with the tangent space $\mathfrak{g}_X$.

Hereby e denotes the unit element of the Lie group G .

So we see that $D\left(l_{\left(n_g^\#\right)^{-1}} \circ \exp'(\diamondsuit + X)\right)(\mathbf{0}) = \frac{1-e^{-\mathrm{ad}_X}}{\mathrm{ad}_X}$. We can split $X = X_d + X_n$, where $X_d \in \mathbb{C}^{(n+r)\times(n+r)}$ is a diagonalizable and

$X_n \in \mathbb{C}^{(n+r)\times(n+r)}$ is a nilpotent matrix. Clearly X_d has the eigenvalues $0, 2\pi i d_1^{(g)}, \ldots, 2\pi i d_r^{(g)}$. So $\mathrm{ad}_X = \mathrm{ad}_{X_d} + \mathrm{ad}_{X_n}$, ad_{X_d} is diagonalizable and ad_{X_n} is nilpotent. Let S denote the set of eigenvalues of ad_{X_d} . Then

$$S \subset \left\{0, 2\pi i d_1^{(g)}, \ldots, 2\pi i d_r^{(g)}\right\} \cup \left\{ 2\pi i \left(d_i^{(g)} - d_j^{(g)}\right) \,\middle|\, i, j \in 1, \ldots, r \right\} .$$

So again $\frac{1-e^{-\mathrm{ad}_X}}{\mathrm{ad}_X}$ as a linear operator from $\mathfrak{g}$ to $\mathfrak{g}$ splits into a sum of a diagonalizable operator and a nilpotent operator. The diagonalizable summand of $\frac{1-e^{-\mathrm{ad}_X}}{\mathrm{ad}_X}$ has precisely the eigenvalues $\frac{1-e^{-s}}{s}$, $s \in S$, which all are different from 0 since by assumtion $S \cap (2\pi i\mathbb{Z} \setminus \{0\}) = \emptyset$. So

$$D\left(l_{\left(n_g^{\#}\right)^{-1}} \circ \exp'(\diamondsuit + X)\right)(\mathbf{0}) = \frac{1 - e^{-\mathrm{ad}_X}}{\mathrm{ad}_X}$$

is indeed an automorphism. $\square$

Let $f \in sM_k(\Upsilon)$. Then $\widetilde{f}\,(g\diamondsuit) \in (\mathcal{D}(\mathcal{G}) \boxtimes \mathcal{P})^{\mathbb{C}}$ is left-$\Phi_g(1)$-invariant since $\Phi_g(1) \in g^{-1}\Upsilon g$ and $\widetilde{f}\,(g\diamondsuit)$ is left-$g^{-1}\Upsilon g$-invariant. So by lemma 3.33 there exists a unique splitting

$$\widetilde{f}(g\diamondsuit) = \sum_{s\in\mathbb{Z}} h_{g,s} \,, \tag{3.6}$$

$h_{g,s} \in (\mathcal{D}(\mathcal{G}) \boxtimes \mathcal{P})^{\mathbb{C}}$, $s \in \mathbb{Z}$, such that for all $s \in \mathbb{Z}$ and $t \in \mathbb{R}$

$$h_{g,s}\left(\Phi(t)\diamondsuit\right) = e^{2\pi i t} h_{g,s} \,.$$

$\deg' h_{g,s} \geq \deg' f\,(g\diamondsuit) = \deg' f$ by lemma 3.33 .

Definition 3.36

$$sS_k(\Upsilon) := \{\, f \in sM_k(\Upsilon) \,|\, h_{g,s} = 0 \text{ in 3.6 for all } g \in \Xi \text{ and } s \geq 0\}$$

is called the space of super cusp forms for Υ *of weight* k *.*

From lemma 3.33 we deduce that if f is homogeneous then in the splitting 3.6 all $h_{g,s}$, $g \in \Xi$ and $s \in \mathbb{Z}$, are homogeneous of the same parity. So we see that again $sS_k(\Upsilon)$ is a sub graded $\mathcal{P}^{\mathbb{C}}$-module of $sM_k(\Upsilon)$.

Theorem 3.37 *Let* $f = \sum_{I\in\wp(m)} f_I \beta^I \in sS_k(\Upsilon)$ *, all* $f_I \in \mathcal{O}(\mathcal{B})$ *, and* $I_0 \in \wp(m)$ *such that* $|I_0| = \deg' f$ *. Then indeed* $f_{I_0} \in sS_k(\Gamma)$ *.*

Proof: Let $g \in \Xi$. Clearly $f_{I_0} \in sM_k(\Gamma)$ by corollary 3.28 . So we have a unique splitting

$$\widetilde{f_{I_0}}(g\diamondsuit) = \sum_{s\in\mathbb{Z}} b_{g,s} \, ,$$

$b_{g,s} \in \mathcal{D}(\mathcal{G})^{\mathbb{C}}$, $g \in \Xi$, $s \in \mathbb{Z}$, such that

$$b_{g,s}\left(\varphi_g(t)\diamondsuit\right) = e^{2\pi i s t} b_{g,s}$$

for all $g \in \Xi$, $s \in \mathbb{Z}$ and $t \in \mathbb{R}$. On the other hand we have the splitting 3.6 , and by proposition 3.34 it suffices to show that $b_{g,s} = h_{g,s}^{(I_0)}$ for all $s \in \mathbb{Z}$ if we decompose

$$h_{g,s} = \sum_{I\in\wp(m)} h_{g,s}^{(I)} \beta^I \, ,$$

$h_{g,s}^{(I)} \in \mathcal{D}(\mathcal{G})^{\mathbb{C}}$, $I \in \wp(m)$, for all $s \in \mathbb{Z}$.

$$\begin{aligned}
\sum_{I\in\wp(m)} \widetilde{f}_I\left(g\diamondsuit\right)\beta^I &= \widetilde{f}\left(g\diamondsuit\right) \\
&= \sum_{s\in\mathbb{Z}} h_{g,s} \\
&= \sum_{I\in\wp(m)} \left(\sum_{s\in\mathbb{Z}} h_{g,s}^{(I)}\right)\beta^I \, .
\end{aligned}$$

Therefore since all $\widetilde{f}_I\left(g\diamondsuit\right), h_{g,s}^{(I)} \in \mathcal{D}(\mathcal{G})^{\mathbb{C}}$ we see that $\widetilde{f}_I\left(g\diamondsuit\right) = \sum_{s\in\mathbb{Z}} h_{g,s}^{(I)}$ for all $I \in \wp(m)$. On the other hand for all $t \in \mathbb{R}$ we have

$$h_{g,s}\left(\Phi_g(t)\diamondsuit\right) = \sum_{I\in\wp(m)} e^{2\pi i t} h_{g,s}^{(I)} \beta^I$$

with $e^{2\pi i t} h_{g,s}^{(I)} \in \mathcal{D}(\mathcal{G})^{\mathbb{C}}$. Therefore by proposition 3.27 (i) since $\Phi_g(t)^{\#} = \varphi_g(t)$ we obtain

$$e^{2\pi i t} h_{g,s}^{(I_0)} = h_{g,s}^{(I_0)}\left(\varphi_g(t)\diamondsuit\right) \, .$$

So by uniqueness of $b_{g,s} \in \mathcal{D}(\mathcal{G})^{\mathbb{C}}$, $s \in \mathbb{Z}$, we have $b_{g,s} = h_{g,s}^{(I_0)}$ for all $s \in \mathbb{Z}$. $\square$

Further research has to show if in the *first* and the *third* case it is possible to find $\psi_{\gamma_0,I,m} \in sS_k(\Upsilon)$, $\gamma_0 \in \Omega$, $I \in \wp(r)$ and $m \in]-C, C[$, satisfying $\psi_{\gamma_0,I,m}^{\#'} = \varphi_{\gamma_0,I,m}$, $\gamma_0 \in \Omega$, $I \in \wp(r)$ and $m \in]-C, C[$, and conditions {ii} and {iii} of theorem 3.29 , where Ω and $\varphi_{\gamma_0,I,m}$, $\gamma_0 \in \Omega$, $I \in \wp(r)$ and $m \in]-C, C[$, are given by theorem 3.17 .

Chapter 4

Super numbers and super functions

In chapter 2 we introduced (p,q)-dimensional super open sets $U^{|q}$ as ringed spaces, which means in terms of sheaves over their bodies U being ordinary open subsets of $\mathbb{R}^p$ resp. $\mathbb{C}^p$, which are interpreted as the sheaves of $\mathcal{C}^\infty$-functions on $U^{|q}$ itself. The goal of this chapter now is to show that there also exists a description of super open sets in terms of points using super numbers, which have been considered for example in section 1.1 of [17] , which is equivalent to that by sheaves.

We set $\wp := \{I \subset \mathbb{Z} \text{ finite}\}$, $\wp_0 := \left\{ I \in \wp \,\middle|\, 2 \mid |I| \right\}$, and $\wp_1 := \left\{ I \in \wp \,\middle|\, 2 \nmid |I| \right\}$. Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. $\Lambda^{\mathbb{K}} := \mathbb{K}^\wp$,

$$\Lambda_0^{\mathbb{K}} := \left\{ (a_I)_{I\in\wp} \in \Lambda^{\mathbb{K}} \,\middle|\, a_I = 0 \text{ for all } I \in \wp_1 \right\} \simeq \mathbb{K}^{\wp_0}$$

and

$$\Lambda_1^{\mathbb{K}} := \left\{ (a_I)_{I\in\wp} \in \Lambda^{\mathbb{K}} \,\middle|\, a_I = 0 \text{ for all } I \in \wp_0 \right\} \simeq \mathbb{K}^{\wp_1} .$$

Then clearly $\Lambda^{\mathbb{K}} = \Lambda_0^{\mathbb{K}} \oplus \Lambda_1^{\mathbb{K}}$ as $\mathbb{K}$-vector spaces. On $\Lambda^{\mathbb{K}}$ we use the uniformal structure of pointwise convergence. So $\Lambda^{\mathbb{K}}$ is complete, $\Lambda_0^{\mathbb{K}}$ and $\Lambda_1^{\mathbb{K}}$ form closed subspaces of $\Lambda^{\mathbb{K}}$, and all

$$]-\varepsilon, \varepsilon[^{\wp'} \times \mathbb{K}^{\wp \setminus \wp'} \subset \Omega$$

with $\varepsilon > 0$ and $\wp' \subset \wp$ finite form a basis of neighbourhoods of $0 \in \Lambda^{\mathbb{K}}$.
If for all $I \in \wp$ we define $E_I := 1_{\{I\}}$ we can write the elements of $\Lambda^{\mathbb{K}}$ as

$$(a_I)_{I\in\wp} = \sum_{I\in\wp} a_I E_I .$$

On $\Lambda^{\mathbb{K}}$ we define a multiplication by

$$(a_I)_{I\in\wp}(b_I)_{I\in\wp} := \left(\sum_{J\subset I}(-1)^{|J<I\setminus J|}\,a_J b_{I\setminus J}\right)_{I\in\wp}$$

with the abbreviation $|K<L| := |\{(r,s)\in K\times L \,|\, r<s\}|$ for all $K,L\in\wp$. Especially for $I,J\in\wp$ this means

$$E_I E_J = \begin{cases} (-1)^{|J<I|}\,E_{I\cup J} & \text{if } I\cap J=\emptyset \\ 0 & \text{otherwise} \end{cases} .$$

One immediately verifies that this multiplication is continuous and that $\Lambda^{\mathbf{K}}$ together with this multiplication is a unital associative algebra over $\mathbb{K}$ with unit element $E_\emptyset$. In addition one can easily verify

Theorem 4.1 $\Lambda^{\mathbf{K}} = \Lambda_0^{\mathbf{K}} \oplus \Lambda_1^{\mathbf{K}}$ *is a unital associative graded commutative algebra, which is called the graded algebra of super numbers.*

We can regard $\mathbb{K}$ as a subalgebra of $\Lambda_0^{\mathbf{K}}$ via the embedding

$$\mathbb{K} \hookrightarrow \Lambda_0^{\mathbf{K}}\,,\; x\mapsto x\,E_\emptyset\,,$$

which is clearly a homeomorphism onto its image and respects scalar multiplication, and therefore we will identify $1 = E_\emptyset$ in what follows.
We also have a so-called body map

$$^{\#} : \Lambda^{\mathbf{K}} \to \mathbb{K}\,,\; z=\sum_{I\in\wp} a_I E_I \mapsto z^{\#} := a_\emptyset\,,$$

which is a continuous, open and surjective algebra projection, and it is clear that $^{\#}\big|_{\mathbb{K}} = id$ and $^{\#}\big|_{\Lambda_1^{\mathbf{K}}} = 0$.
For every $z := \sum_{I\in\wp} a_I E_I \in \Lambda^{\mathbf{K}}$ we define the degree of z as

$$\deg z := \min\left\{ |I| \;\middle|\; I\in\wp \text{ and } a_I\neq 0\right\} .$$

Then we see immediately $\deg(z+w)\geq \min(\deg z, \deg w)$ and $\deg(zw)\geq \deg z+\deg w$ for all $z,w\in\Lambda^{\mathbf{K}}$, $\deg\left(z-z^{\#}\right)\geq 2$ for all $z\in\Lambda_0^{\mathbf{K}}$ and $\deg z\geq 1$ for all $z\in\Lambda_1^{\mathbf{K}}$.

Lemma 4.2

(i) Let $(z_n)_{n\in\mathbf{N}}$ be a sequence in $\Lambda^{\mathbf{K}}$ with the property $\deg z_n \rightsquigarrow \infty$ *. Then* $z_n \rightsquigarrow 0$ *, and*

$$\sum_{n\in\mathbf{N}} z_n$$

is convergent.

(ii) Let $(a_{\mathbf{n}})_{\mathbf{n}\in\mathbb{N}^p} \in \left(\Lambda^{\mathbb{K}}\right)^{\mathbb{N}^p}$, $\mathbf{z} \in \left(\Lambda^{\mathbb{K}}\right)^p$ *with* $\deg z_i \geq 1$ *for all* $i = 1, \dots, p$ *. Then*

$$\sum_{\mathbf{n}\in\mathbb{N}^p} a_{\mathbf{n}} \mathbf{z}^{\mathbf{n}}$$

converges.

Proof: (i) Let $z_n = \sum_{I\in\wp} c_I^{(n)}$ for all $n \in \mathbb{N}$. Fix an $I \in \wp$. Then $\deg z_n > |I|$ and so $c_I^{(n)} = 0$ for almost all $n \in \mathbb{N}$. So clearly in the sum $\sum_{n\in\mathbb{N}} c_I^{(n)}$ only a finite number of terms are $\neq 0$, and therefore it converges. $\square$

(ii) For all $\mathbf{n} \in \mathbb{N}^p$ we have $\deg(a_{\mathbf{n}}\mathbf{z}^{\mathbf{n}}) \geq \deg a_{\mathbf{n}} + \sum_{i=1}^p n_i \deg z_i \geq |\mathbf{n}|$, and we can apply (i) . $\square$

Corollary 4.3 $z \in \Lambda^{\mathbb{K}}$ *is invertible if and only if* $z^\# \neq 0$ *. In this case*

$$z^{-1} = \frac{1}{z^\#} \sum_{n\in\mathbb{N}} \left(\frac{z^\# - z}{z^\#}\right)^n .$$

If $z \in \Lambda_0^{\mathbb{K}}$ *then again* $z^{-1} \in \Lambda_0^{\mathbb{K}}$ *.*

Proof: Let $z \in \Lambda^{\mathbb{K}}$ such that $z^\# \neq 0$. Then convergence of the sum is clear by lemma 4.2 , and since multiplication in $\Lambda^{\mathbb{K}}$ is continuous we have

$$\begin{aligned} z\frac{1}{z^\#} \sum_{n\in\mathbb{N}} \left(\frac{z^\# - z}{z^\#}\right)^n &= \frac{z}{z^\#} \sum_{n\in\mathbb{N}} \left(1 - \frac{z}{z^\#}\right)^n \\ &= \sum_{n\in\mathbb{N}} \left(\frac{z}{z^\#}\right)^n - \sum_{n\in\mathbb{N}} \left(\frac{z}{z^\#}\right)^{n+1} \\ &= 1 , \end{aligned}$$

and by the same calculation $\frac{1}{z^\#} \sum_{n\in\mathbb{N}} \left(\frac{z^\# - z}{z^\#}\right)^n z = 1$. $\square$

Now we have to treat the cases $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$ seperately.

4.1 The real case

Let $\mathbb{K} := \mathbb{R}$ and $\Lambda := \Lambda^{\mathbb{R}}$. We define $\mathbb{R}^{p|q} := \Lambda_0^p \times \Lambda_1^q$ for all $p, q \in \mathbb{N}$. Then again we have a body map

$$\begin{aligned} \# : \quad & \mathbb{R}^{p|q} \to \mathbb{R}^p , \\ & (\mathbf{z}, \zeta) = (z_1, \dots, z_p, \zeta_1, \dots, \zeta_q) \mapsto (\mathbf{z}, \zeta)^\# := \left(z_1^\#, \dots, z_p^\#\right) = \mathbf{z}^\# . \end{aligned}$$

Let $U \underset{\text{open}}{\subset} \mathbb{R}^p$. Then we define $U^{|q} := \left\{ (\mathbf{z}, \zeta) \in \mathbb{R}^{p|q} \,\middle|\, (\mathbf{z}, \zeta)^\# \in U \right\}$. We have $U^{|q} \underset{\text{open}}{\subset} \mathbb{R}^{p|q}$ and $\left(U^{|q}\right)^\# = U$. Conversely if $\Omega \underset{\text{open}}{\subset} \mathbb{R}^{p|q}$ then clearly $\Omega^\# \underset{\text{open}}{\subset} \mathbb{K}^p$ and $\Omega \subset \left(\Omega^\#\right)^{|q}$, see figure 4.1 below.

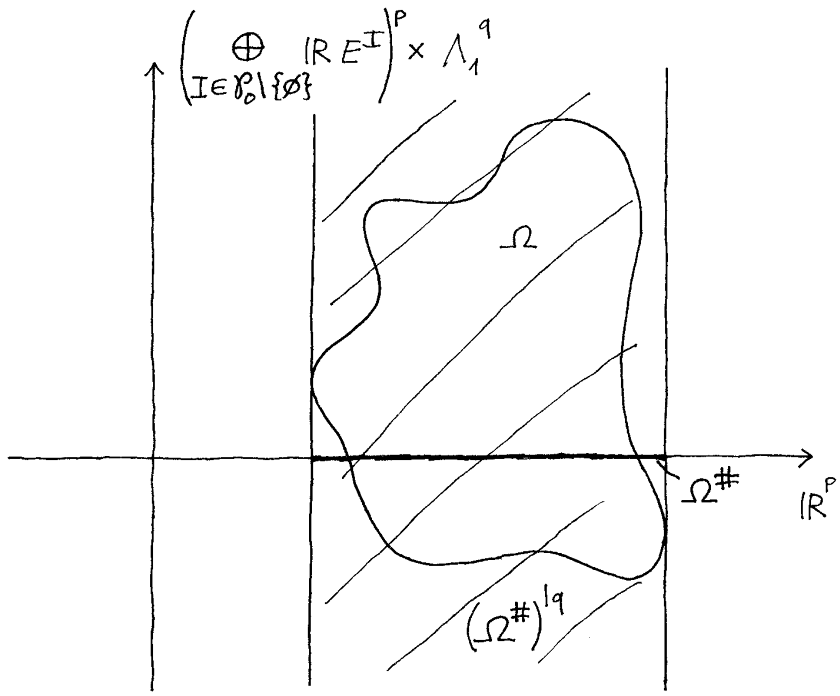

Figure 4.1: Ω , $\Omega^\#$ and $\left(\Omega^\#\right)^{|q}$.

If $M \subset \mathbb{R}^{p|q}$, the set $\Lambda^M = \Lambda_0^M \oplus \Lambda_1^M$ of all functions $f : M \to \Lambda$ forms a unital associative graded commutative algebra by pointwise addition and multiplication, and we consider Λ as the sub graded algebra of Λ^M containing precisely the constant functions. Then clearly $\mathcal{C}(M, \Lambda)$ is a sub graded algebra of Λ^M containing Λ .

Theorem 4.4 *For each $U \underset{\text{open}}{\subset} \mathbb{R}^p$ we have an algebra embedding*

$$\widehat{\ } : \mathcal{C}^\infty(U) \hookrightarrow \mathcal{C}\left(U^{|0}, \Lambda_0\right) , \; f \mapsto \widehat{f}$$

where for all $\mathbf{z} \in U^{|0}$

$$\widehat{f}(\mathbf{z}) := \sum_{\mathbf{n}\in\mathbf{N}^p} \frac{1}{\mathbf{n}!} \partial^{\mathbf{n}} f\left(\mathbf{z}^\#\right) \left(\mathbf{z} - \mathbf{z}^\#\right)^{\mathbf{n}}$$

in multi-index language. Clearly $\left.\widehat{f}\right|_U = f$ *, and if* $\partial^{\mathbf{n}} f_k \rightsquigarrow \partial^{\mathbf{n}} f$ *pointwise for all* $\mathbf{n} \in \mathbb{N}^p$ *then again* $\widehat{f_k} \rightsquigarrow \widehat{f}$ *pointwise.*

Proof: The sum is convergent by lemma 4.2 (ii) . Injectivity is clear by the property $f = \left.\widehat{f}\right|_U$. So let us prove the conservation of multiplication. For all $f, g \in \mathcal{C}^\infty(U)$ and $\mathbf{z} \in U^{|0}$ we have

$$\begin{aligned}\widehat{f}(\mathbf{z})\widehat{g}(\mathbf{z}) &= \left(\sum_{\mathbf{m}\in\mathbb{N}^p} \frac{1}{\mathbf{m}!}\partial^{\mathbf{m}} f\left(\mathbf{z}^\#\right)\left(\mathbf{z}-\mathbf{z}^\#\right)^{\mathbf{m}}\right) \times \\ &\qquad \times \left(\sum_{\mathbf{n}\in\mathbb{N}^p} \frac{1}{\mathbf{n}!}\partial^{\mathbf{n}} g\left(\mathbf{z}^\#\right)\left(\mathbf{z}-\mathbf{z}^\#\right)^{\mathbf{n}}\right) .\end{aligned}$$

Since restricted to any fixed component in both sums only a finite number of terms is $\neq 0$ we can interchange the order of summation. So

$$\begin{aligned}\widehat{f}(\mathbf{z})\widehat{g}(\mathbf{z}) &= \sum_{\mathbf{k}\in\mathbb{N}^p}\left(\sum_{\mathbf{m}\in\mathbb{N}^p,\mathbf{m}\le\mathbf{k}} \frac{1}{\mathbf{m}!(\mathbf{k}-\mathbf{m})!}\partial^{\mathbf{m}} f\left(\mathbf{z}^\#\right)\partial^{\mathbf{k}-\mathbf{m}} g\left(\mathbf{z}^\#\right)\right) \times \\ &\qquad \times \left(\mathbf{z}-\mathbf{z}^\#\right)^{\mathbf{k}} \\ &= \sum_{\mathbf{k}\in\mathbb{N}^p}\frac{1}{\mathbf{k}!}\left(\sum_{\mathbf{m}\in\mathbb{N}^p,\mathbf{m}\le\mathbf{k}} \binom{\mathbf{k}}{\mathbf{m}}\partial^{\mathbf{m}} f\left(\mathbf{z}^\#\right)\partial^{\mathbf{k}-\mathbf{m}} g\left(\mathbf{z}^\#\right)\right)\left(\mathbf{z}-\mathbf{z}^\#\right)^{\mathbf{k}} \\ &= \sum_{\mathbf{k}\in\mathbb{N}^p}\frac{1}{\mathbf{k}!}\partial^{\mathbf{k}}(fg)\left(\mathbf{z}^\#\right)\left(\mathbf{z}-\mathbf{z}^\#\right)^{\mathbf{k}} = \widehat{(fg)}(\mathbf{z}) ,\end{aligned}$$

where we used LEIBNIZ' rule in multi-index language.
To see that $\widehat{f}$ is continuous let $\mathbf{z} \in U^{|0}$, $I \in \wp$, and let F_I be the I-th component function of $\widehat{f}$. Then since $\deg\left(z_i - z_i^\#\right) \geq 2$, $i = 1, \ldots, p$, we have

$$F_I(\mathbf{z}) = \sum_{\mathbf{n}\in\mathbb{N}^p,\mathbf{n}\le\left(\frac{|I|}{2}\right)^p} \frac{1}{\mathbf{n}!}\partial^{\mathbf{n}} f\left(\mathbf{z}^\#\right)\left(\left(\mathbf{z}-\mathbf{z}^\#\right)^{\mathbf{n}}\right)_I . \square$$

More precisely we have a sheef embedding $\mathcal{C}^\infty_{\mathbb{R}^p} \hookrightarrow \mathcal{C}\left(\Diamond^{|0}, \Lambda_0\right)_{\mathbb{R}^p}$ since if V is an open subset of U then $V^{|0} \subset U^{|0}$ and $\widehat{f|_V} = \left.\widehat{f}\right|_{V^{|0}}$.

Lemma 4.5 *Let* $U \underset{open}{\subset} \mathbb{R}^p$ *,* $f \in \mathcal{C}^\infty(U)$ *and* $\mathbf{b} \in U^{|0}$ *. Then for all* $\mathbf{z} \in \mathbb{R}^{p|0}$ *with* $\mathbf{z}^\# = \mathbf{0}$

$$\widehat{f}(\mathbf{b}+\mathbf{z}) = \sum_{\mathbf{n}\in\mathbb{N}^p} \frac{1}{\mathbf{n}!}\widehat{\partial^{\mathbf{n}} f}(\mathbf{b})\mathbf{z}^{\mathbf{n}} .$$

Proof: Let $\mathbf{z} \in \mathbb{R}^{p|0}$ with $\mathbf{z}^{\#} = \mathbf{0}$. Then

$$\begin{aligned}
\widehat{f}(\mathbf{b}+\mathbf{z}) &= \sum_{\mathbf{n}\in\mathbb{N}^p} \frac{1}{\mathbf{n}!}\partial^{\mathbf{n}} f\left(\mathbf{b}^{\#}\right)\left(\mathbf{b}+\mathbf{z}-\mathbf{b}^{\#}\right)^{\mathbf{n}} \\
&= \sum_{\mathbf{n}\in\mathbb{N}^p} \frac{1}{\mathbf{n}!}\partial^{\mathbf{n}} f\left(\mathbf{b}^{\#}\right) \sum_{\mathbf{m}\in\mathbb{N}^p,\mathbf{m}\leq\mathbf{n}} \binom{\mathbf{n}}{\mathbf{m}} \left(\mathbf{b}-\mathbf{b}^{\#}\right)^{\mathbf{n}-\mathbf{m}} \mathbf{z}^{\mathbf{m}} .
\end{aligned}$$

Since restricted to any fixed component in the summation over $\mathbf{n}$ only a finite number of terms is $\neq 0$ we can interchange the order of summation. So

$$\begin{aligned}
\widehat{f}(\mathbf{b}+\mathbf{z}) &= \sum_{\mathbf{m}\in\mathbb{N}^p} \left(\sum_{\mathbf{k}\in\mathbb{N}^p} \frac{1}{(\mathbf{m}+\mathbf{k})!}\partial^{\mathbf{m}+\mathbf{k}} f\left(\mathbf{b}^{\#}\right) \binom{\mathbf{m}+\mathbf{k}}{\mathbf{m}} \left(\mathbf{b}-\mathbf{b}^{\#}\right)^{\mathbf{k}} \right) \mathbf{z}^{\mathbf{m}} \\
&= \sum_{\mathbf{m}\in\mathbb{N}^p} \frac{1}{\mathbf{m}!} \left(\sum_{\mathbf{k}\in\mathbb{N}^p} \frac{1}{\mathbf{k}!}\partial^{\mathbf{k}}\partial^{\mathbf{m}} f\left(\mathbf{b}^{\#}\right) \left(\mathbf{b}-\mathbf{b}^{\#}\right)^{\mathbf{k}} \right) \mathbf{z}^{\mathbf{m}} \\
&= \sum_{\mathbf{m}\in\mathbb{N}^p} \frac{1}{\mathbf{m}!} \widehat{\partial^{\mathbf{m}} f}(\mathbf{b})\mathbf{z}^{\mathbf{m}} . \square
\end{aligned}$$

Lemma 4.6 *Let* $f(\mathbf{x}) := \sum_{\mathbf{n}\in\mathbb{N}^p} \frac{a_{\mathbf{n}}}{\mathbf{n}!} (\mathbf{x}-\mathbf{c})^{\mathbf{n}}$ *be a power series convergent in* $U \subset \mathbb{R}^p$ *with* $\mathbf{c} \in \mathbb{R}^p$ *and all* $a_{\mathbf{n}} \in \mathbb{R}$ *. Then for all* $\mathbf{z} \in U^{|0}$

$$\widehat{f}(\mathbf{z}) = \sum_{\mathbf{n}\in\mathbb{N}^p} \frac{a_{\mathbf{n}}}{\mathbf{n}!} (\mathbf{z}-\mathbf{c})^{\mathbf{n}} .$$

Proof: Let $\mathbf{z} \in U^{|0}$. Then

$$\begin{aligned}
\widehat{f}(\mathbf{z}) &= \sum_{\mathbf{n}\in\mathbb{N}^p} \frac{1}{\mathbf{n}!}\partial^{\mathbf{n}} f\left(\mathbf{z}^{\#}\right)\left(\mathbf{z}-\mathbf{z}^{\#}\right)^{\mathbf{n}} \\
&= \sum_{\mathbf{n}\in\mathbb{N}^p} \frac{1}{\mathbf{n}!} \left(\sum_{\mathbf{k}\in\mathbb{N}^p} \frac{a_{\mathbf{n}+\mathbf{k}}}{\mathbf{k}!} \left(\mathbf{z}^{\#}-\mathbf{c}\right)^{\mathbf{k}} \right) \left(\mathbf{z}-\mathbf{z}^{\#}\right)^{\mathbf{n}} .
\end{aligned}$$

Since restricted to any fixed component in the summation over $\mathbf{n}$ only a finite number of terms is $\neq 0$ we can interchange the order of summation again. So we have

$$\begin{aligned}
\widehat{f}(\mathbf{z}) &= \sum_{\mathbf{m}\in\mathbb{N}^p} a_{\mathbf{m}} \sum_{\mathbf{n}\in\mathbb{N}^p,\mathbf{n}\leq\mathbf{m}} \frac{1}{\mathbf{n}!(\mathbf{m}-\mathbf{n})!} \left(\mathbf{z}^{\#}-\mathbf{c}\right)^{\mathbf{m}-\mathbf{n}} \left(\mathbf{z}-\mathbf{z}^{\#}\right)^{\mathbf{n}} \\
&= \sum_{\mathbf{m}\in\mathbb{N}^p} \frac{a_{\mathbf{m}}}{\mathbf{m}!} \sum_{\mathbf{n}\in\mathbb{N}^p,\mathbf{n}\leq\mathbf{m}} \binom{\mathbf{m}}{\mathbf{n}} \left(\mathbf{z}^{\#}-\mathbf{c}\right)^{\mathbf{m}-\mathbf{n}} \left(\mathbf{z}-\mathbf{z}^{\#}\right)^{\mathbf{n}} \\
&= \sum_{\mathbf{m}\in\mathbb{N}} \frac{a_{\mathbf{m}}}{\mathbf{m}!} (\mathbf{z}-\mathbf{c})^{\mathbf{m}} . \square
\end{aligned}$$

In the following we will discuss more in detail mappings from $\Omega \underset{\text{open}}{\subset} \mathbb{R}^{p|q}$ to Λ , and we will always use the following notations:
We set $(\mathbf{z},\zeta) = (z_1,\dots,z_p,\zeta_1,\dots\zeta_q) \in \Omega$ with for all $i = 1,\dots,p$ respectively $j = 1,\dots,q$: $z_i = \sum_{J\in\wp_0} x_{iJ}E_J \in \mathbb{R}^{1|0}$, $\zeta_j = \sum_{J\in\wp_1} y_{jJ}E_J \in \mathbb{R}^{0|1}$, all $x_{iJ}, y_{jJ} \in \mathbb{R}$. We define

$$\zeta^S := \zeta_{s_1}\cdots\zeta_{s_r}$$

for all $S = \{s_1,\dots,s_r\} \subset \{1,\dots,q\}$ and $s_1 < \cdots < s_r$. And if there is no danger of confusion we denote by the same symbols the projections

$$z_i : \mathbb{R}^{p|q} \to \mathbb{R}^{1|0}\,,\ (\mathbf{z},\zeta) \mapsto z_i$$

for $i = 1,\dots,p$,

$$\zeta_j : \mathbb{R}^{p|q} \to \mathbb{R}^{0|1}\,,\ (\mathbf{z},\zeta) \mapsto \zeta_j$$

for $j = 1,\dots,q$ and

$$\zeta^S : \mathbb{R}^{p|q} \to \Lambda\,,\ (\mathbf{z},\zeta) \mapsto \zeta^S$$

for all $S \subset \{1,\dots,q\}$.
If $f \in \Lambda^\Omega$ then we write $f = \sum_{I\in\wp} F_I E_I$ with uniquely determined component functions $F_I : \Omega \to \mathbb{R}$. Let $(\mathbf{b},\beta) \in \Omega$ and $I \in \wp$. If $i \in \{1,\dots,p\}$ and $J \in \wp_0$, and $F_I(\mathbf{z},\zeta)$ differentiable with respect to x_{iJ} at $(\mathbf{b},\beta)$ then we define

$$\partial_{i|J} F_I(\mathbf{b},\beta) := \left.\frac{\partial F_I(\mathbf{z},\zeta)}{\partial x_{iJ}}\right|_{(\mathbf{z},\zeta)=(\mathbf{b},\beta)} .$$

And also if $j \in \{1,\dots,q\}$, $J \in \wp_1$ and $F_I(\mathbf{z},\zeta)$ is partially differentiable with respect to y_{jJ} at $(\mathbf{b},\beta)$ we define

$$\partial_{|jJ} F_I(\mathbf{b},\beta) := \left.\frac{\partial F_I(\mathbf{z},\zeta)}{\partial y_{jJ}}\right|_{(\mathbf{z},\zeta)=(\mathbf{b},\beta)} .$$

Definition 4.7 *Let $\Omega \underset{open}{\subset} \mathbb{R}^{p|q}$, $f : \Omega \to \Lambda$ and $(\mathbf{b},\beta) \in \Omega$. f is called differentiable at $(\mathbf{b},\beta)$ if and only if there exist $\Omega' \underset{open}{\subset} \Omega$ such that $(\mathbf{b},\beta) \in \Omega'$ and $\Delta_i, \Sigma_j : \Omega' \to \Lambda$, $i = 1,\dots,p$, $j = 1,\dots,q$, continuous at $(\mathbf{b},\beta)$ such that for all $(\mathbf{z},\zeta) \in \Omega'$*

$$f(\mathbf{z},\zeta) = f(\mathbf{b},\beta) + \sum_{i=1}^{p}(z_i - b_i)\Delta_i(\mathbf{z},\zeta) + \sum_{j=1}^{q}(\zeta_i - \beta_i)\,\Sigma_j(\mathbf{z},\zeta)\,.$$

If f is differentiable at $(\mathbf{b},\beta)$ then we call $\partial_{i|} f(\mathbf{b},\beta) := \Delta_i(\mathbf{b},\beta)$,
$i = 1,\dots,p$, and $\partial_{|j} f(\mathbf{b},\beta) := \Sigma_j(\mathbf{b},\beta)$, $j = 1,\dots,q$, the partial derivatives of f at $(\mathbf{b},\beta)$. If f is differentiable at each $(\mathbf{b},\beta) \in \Omega$ then f is said

to be differentiable, and $\partial_{i|} f : \Omega \to \Lambda$, $(\mathbf{z}, \zeta) \mapsto \partial_{i|} f(\mathbf{z}, \zeta)$, $i = 1, \ldots, p$, *and* $\partial_{|j} f : \Omega \to \Lambda$, $(\mathbf{z}, \zeta) \mapsto \partial_{|j} f(\mathbf{z}, \zeta)$, $j = 1, \ldots, q$, *are called the partial derivatives of* f .

In general the functions Δ_i , $i \in \{1, \ldots, p\}$, Σ_j , $j \in \{1, \ldots, q\}$, are not uniquely determined by f and $(\mathbf{b}, \beta)$. Let us check that however $\partial_{i|} f(\mathbf{b}, \beta)$, $i \in \{1, \ldots, p\}$, $\partial_{|j} f(\mathbf{b}, \beta)$, $j \in \{1, \ldots, q\}$, are well-defined:

> Let $i \in \{1, \ldots, p\}$. Then $\Delta_i(\mathbf{b}, \beta) = \partial_t f\left(\mathbf{b} + t\mathbf{e}_i, \beta\right)|_{t=0}$, $t \in \mathbb{R}$, is independent of the choice of $\Delta_1, \ldots, \Delta_p$, $\Sigma_1, \ldots, \Sigma_q$.
>
> Now let $j \in \{1, \ldots, q\}$ and $I \in \wp$, and let $\Sigma_{jI} : \Omega \to \mathbb{R}$ be the I-th component function of Σ_j . Choose $J \in \wp_1$ such that $J < I$. Then $\Sigma_{jI}(\mathbf{b}, \beta) = \partial_{|jJ} F_{I \cup J} = \partial_t F_{I \cup J}\left(\mathbf{b}, \beta + tE_J\mathbf{e}_j\right)|_{t=0}$, $t \in \mathbb{R}$, is independent of the choice of $\Delta_1, \ldots, \Delta_p$, $\Sigma_1, \ldots, \Sigma_q$ as well.

If $\Omega \underset{\text{open}}{\subset} \mathbb{R}^{p|q}$, then clearly the set of all differentiable mappings $\Omega \to \Lambda$ forms a sub graded algebra of $\mathcal{C}\left(\Omega, \Lambda\right)$ containing Λ , all $\partial_{i|}$ and $\partial_{|j}$ are 0 on Λ , and the super product rule holds:

$$\left(\partial_{i|} f\right)^{\cdot} = \dot{f} \,,\; \left(\partial_{|j} f\right)^{\cdot} = \dot{f} + 1 \,,$$

$$\partial_{i|}\left(fg\right) = \left(\partial_{i|} f\right) g + f\left(\partial_{i|} g\right)$$

and

$$\partial_{|j}\left(fg\right) = \left(\partial_{|j} f\right) g + (-1)^{\dot{f}} f\left(\partial_{|j} g\right)$$

for all differentiable $f, g : \Omega \to \Lambda$, f homogeneous.

If $\Omega \underset{\text{open}}{\subset} \mathbb{R}^{p|q}$ then we define $\mathcal{D}(\Omega, \Lambda)$ to be the set of all $f \in \Lambda^{\Omega}$ that are continuous with respect to $\mathbf{z}$ and partially differentiable with respect to all z_i ,
$i = 1, \ldots, p$, and ζ_j , $j = 1, \ldots, q$.
Then $\mathcal{D}(\Omega, \Lambda)$ is a sub graded algebra of Λ^{Ω} containing Λ and, as we will see, of $\mathcal{C}(\Omega, \Lambda)$ as well.

One goal of this chapter is the following theorem, which we will prove later.

Theorem 4.8 *Let* $\Omega \underset{\textit{open}}{\subset} \mathbb{R}^{p|q}$ *such that* $\Omega_{\mathbf{x}} := \left\{(\mathbf{z}, \zeta) \in \Omega \;\middle|\; (\mathbf{z}, \zeta)^{\#} = \mathbf{x}\right\}$ *is connected for all* $\mathbf{x} \in \Omega^{\#}$ *, see figure 4.2 below. Then we have isomorphisms*

$$\left(\Lambda\left(\mathbb{R}^q\right)\otimes\mathcal{C}^\infty\left(\Omega^\#\right)\right)\hat{\boxtimes}\Lambda\simeq\mathcal{C}^\infty\left(\left(\Omega^\#\right)^{|q},\Lambda\right)\simeq\mathcal{D}(\Omega,\Lambda)$$

as unital graded algebras, where on $\mathcal{C}^\infty\left(\Omega^\#\right)$ *we use the uniformal structure of compact convergence in all derivatives and on* $\mathcal{C}^\infty\left(\left(\Omega^\#\right)^{|q},\Lambda\right)$ *that of pointwise convergence.*
The first isomorphism is the unique $\mathbb{R}$*-linear and continuous map given by* $e^S\otimes f\otimes E_I\mapsto\zeta^S\widehat{f}E_I$ *for all* $f\in\mathcal{C}^\infty\left(\Omega^\#\right)$, $S\in\wp(q)$, $I\in\wp$, *and the second is given by the restriction map.*

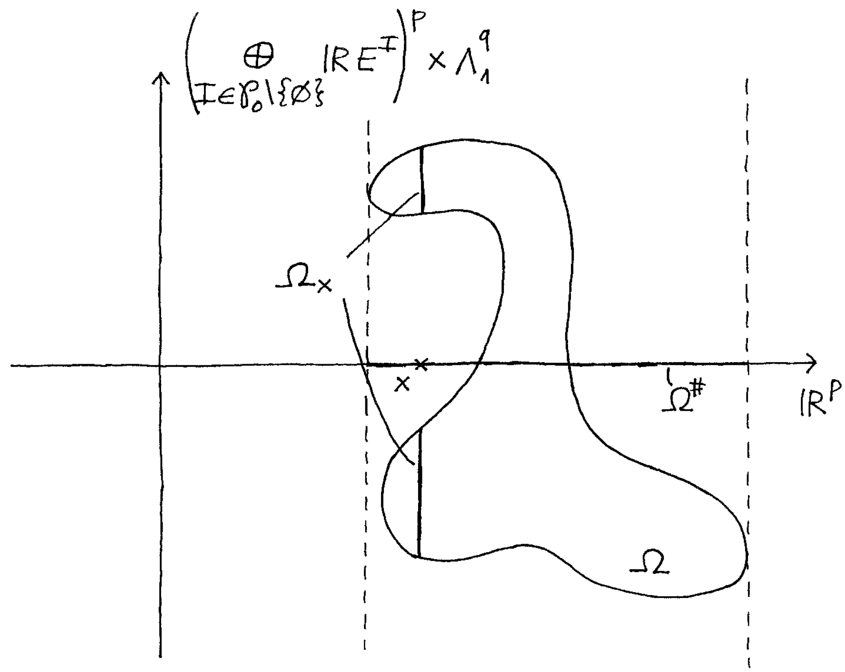

Figure 4.2: $\Omega_{\mathbf{x}}$ in theorem 4.8 .

Let $\Omega\underset{\text{open}}{\subset}\mathbb{R}^{p|q}$. We say Ω is of cube type if and only if there exist $\wp'\subset\wp\setminus\{\emptyset\}$ finite and $\varepsilon>0$ such that for all $(\mathbf{z},\zeta)\in\left(\Omega^\#\right)^{|q}$:
$(\mathbf{z},\zeta)\in\Omega$ if and only if $x_{iI},y_{jJ}\in\,]-\varepsilon,\varepsilon[$ for all $i=1,\dots,p$, $j=1,\dots,q$, $I\in\wp'\cap\wp_0$, $J\in\wp'\cap\wp_1$.

In this case clearly $\Omega_{\mathbf{x}}\underset{\text{open , convex}}{\subset}\mathbb{R}^{p|q}_{\mathbf{x}}$ for all $\mathbf{x}\in\Omega^\#$ and $\Omega^\#\subset\Omega$. Clearly $U^{|q}$ is of cube type for all $U\underset{\text{open}}{\subset}\mathbb{R}^p$.

Theorem 4.9 *Let* $\Omega \underset{open}{\subset} \mathbb{R}^{p|0}$ *be of cube type and* $f \in \Lambda^{\Omega}$ *. Then there are equivalent*

(i) f *is differentiable,*

(ii) $f \in \mathcal{D}(\Omega, \Lambda)$,

(iii) all F_I , $I \in \wp$, *are continuous and fulfill the following system of differential equations*

$$\partial_{i|J} F_I = \begin{cases} (-1)^{|I \setminus J < J|} \partial_{i|\emptyset} F_{I \setminus J} & \text{if } J \subset I \\ 0 & \text{otherwise} \end{cases}$$

for all $i = 1, \ldots, p$, $I \in \wp$ *and* $J \in \wp_1$,

(iv) there exists a family $(f_I)_{I \in \wp} \in \left(\mathcal{C}^\infty\left(\Omega^\#\right)\right)^\wp$ *such that*

$$f(\mathbf{z}) = \sum_{I \in \wp} \widehat{f}_I(\mathbf{z}) E_I$$

for all $\mathbf{z} \in \Omega$.

In this case $f^* := \sum_{I \in \wp} \widehat{f}_I E_I : \left(\Omega^\#\right)^{|0} \to \Lambda$ *is the unique extension in* $\mathcal{D}\left(\left(\Omega^\#\right)^{|0}\right)$ *of* f , f^* *is differentiable,* $\partial_{i|} f^* \in \mathcal{D}\left(\left(\Omega^\#\right)^{|0}\right)$, *and*

$$\partial_{i|} f^* = \sum_{I \in \wp} \widehat{\partial_i f_I} E_I$$

for all $i = 1, \ldots, p$.

Notice that $f_I = F_I|_{\Omega^\#}$, $I \in \wp$, but in general $\widehat{f}_I \neq F_I$ since $\widehat{f}_I(\mathbf{z}) \notin \mathbb{R}$ for $\mathbf{z} \in \Omega \setminus \Omega^\#$. Theorem 4.8 will show that the family $(f_I)_{I \in \wp}$ in (iv) is uniquely determined by f .

Proof: (iv) $\Rightarrow$ (i) : Let $U \underset{\text{open}}{\subset} \mathbb{R}^p$, $(f_I)_{I \in \wp} \in (\mathcal{C}^\infty(U))^\wp$, $f = \sum_{I \in \wp} \widehat{f}_I E_I$ and $\mathbf{b} \in U^{|0}$. Let $g := f\left(\diamondsuit + \mathbf{b} - \mathbf{b}^\#\right) : U^{|0} \to \Lambda$.

First we prove that g is again of the form $g = \sum_{K \in \wp} \widehat{g_K} E_K$ with $(g_K)_{K \in \wp} \in (\mathcal{C}^\infty(U))^\wp$.

So let $\mathbf{z} \in U^{|0}$ and $a_J^{(\mathbf{n})} \in \mathbb{R}$, $J \in \wp_0$, $\mathbf{n} \in \mathbb{N}^p$, such that for all $\mathbf{n} \in \mathbb{N}^p$

$$\left(\mathbf{b} - \mathbf{b}^\#\right)^{\mathbf{n}} = \sum_{J \in \wp_0} a_J^{(\mathbf{n})} E_J .$$

So by lemma 4.5

$$\begin{aligned} g(\mathbf{z}) &= \sum_{I\in\wp} \widehat{f}_I\left(\mathbf{z}+\mathbf{b}-\mathbf{b}^{\#}\right) E_I \\ &= \sum_{I\in\wp}\sum_{\mathbf{n}\in\mathbb{N}^p} \frac{1}{\mathbf{n}!}\widehat{\partial^{\mathbf{n}} f_I}(\mathbf{z})\left(\mathbf{b}-\mathbf{b}^{\#}\right)^{\mathbf{n}} E_I \\ &= \sum_{I\in\wp}\sum_{\mathbf{n}\in\mathbb{N}^p} \frac{1}{\mathbf{n}!}\widehat{\partial^{\mathbf{n}} f_I}(\mathbf{z}) \sum_{J\in\wp_0} a_J^{(\mathbf{n})} E_J E_I\,. \end{aligned}$$

Since $\deg\left(\mathbf{b}-\mathbf{b}^{\#}\right) \geq 2$ we have $a_J^{(\mathbf{n})} = 0$ for all $\mathbf{n}\in\mathbb{N}^p$, $J\in\wp$ with $|\mathbf{n}| > \frac{|J|}{2}$. Therefore for each fixed component the summation is finite, and so we can interchange the order of summation. So

$$\begin{aligned} & g(\mathbf{z}) \\ &= \sum_{K\in\wp_0}\left(\sum_{J\in\wp_0, J\subset K} (-1)^{|K\setminus J<J|} \sum_{\mathbf{n}\in\mathbb{N}^p, |\mathbf{n}|\leq\frac{|K|}{2}} \frac{1}{\mathbf{n}!} a_J^{(\mathbf{n})} \widehat{\partial^{\mathbf{n}} f_{K\setminus J}}(\mathbf{z})\right) E_K \\ &= \sum_{K\in\wp} \widehat{g_K}(\mathbf{z}) E_K\,, \end{aligned}$$

if we define for all $K\in\wp$

$$g_K := \sum_{J\in\wp_0, J\subset K} (-1)^{|K\setminus J<J|} \sum_{\mathbf{n}\in\mathbb{N}^p, |\mathbf{n}|\leq\frac{|K|}{2}} \frac{1}{\mathbf{n}!} a_J^{(\mathbf{n})} \partial^{\mathbf{n}} f_{K\setminus J} \in \mathcal{C}^\infty(U)\,.\,\square$$

Now we would like to show that f is differentiable at $\mathbf{b}$. Since all $g_K \in \mathcal{C}^\infty(U)$, $K\in\wp$, there exist $\Delta_{iK}\in\mathcal{C}^\infty(U)$, $i=1,\dots,p$, $K\in\wp$, such that for all $\mathbf{x}\in U$ and $K\in\wp$

$$g_K(\mathbf{x}) = g_K\left(\mathbf{b}^{\#}\right) + \sum_{i=1}^p \left(\mathbf{x}_i - \mathbf{b}_i^{\#}\right)\Delta_{iK}(\mathbf{x})\,.$$

If we apply $\widehat{\ }$ to these equations we obtain for all $\mathbf{z}\in U^{|0}$

$$\begin{aligned} f(\mathbf{z}) &= g\left(\mathbf{z}-\mathbf{b}+\mathbf{b}^{\#}\right) = \sum_{K\in\wp} \widehat{g_K}\left(\mathbf{z}-\mathbf{b}+\mathbf{b}^{\#}\right) E_K \\ &= \sum_{K\in\wp} g_K\left(\mathbf{b}^{\#}\right) E_K + \sum_{i=1}^p \sum_{K\in\wp} (z_i - b_i)\,\widehat{\Delta_{iK}}\left(\mathbf{z}-\mathbf{b}+\mathbf{b}^{\#}\right) E_K \\ &= f(\mathbf{b}) + \sum_{i=1}^p (z_i - b_i)\,\Delta_i(\mathbf{z})\,, \end{aligned}$$

where

$$\Delta_i : U^{|0} \to \Lambda\,,\ \mathbf{z} \mapsto \sum_{K \in \wp} \widehat{\Delta_{iK}} \left(\mathbf{z} - \mathbf{b} + \mathbf{b}^{\#}\right) E_K$$

is a continuous function because all $\widehat{\Delta_{iK}}$ are continuous, and restricted to any fixed component in the sum only a finite number of terms is $\neq 0$. So f is differentiable at $\mathbf{b}$, and for all $i = 1, \ldots, p$

$$\begin{aligned}
\partial_{i|} f(\mathbf{b}) &= \Delta_i(\mathbf{b}) = \sum_{K \in \wp} \Delta_{iK} \left(\mathbf{b}^{\#}\right) E_K = \sum_{K \in \wp} \partial_i g_K \left(\mathbf{b}^{\#}\right) E_K \\
&= \sum_{K \in \wp} \sum_{J \in \wp_0\,,\, J \subset K} (-1)^{|K \setminus J < J|} \sum_{\mathbf{n} \in \mathbb{N}^p, |\mathbf{n}| \leq \frac{|K|}{2}} \frac{1}{\mathbf{n}!} \partial^{\mathbf{n}} \partial_i f_{K \setminus J} \left(\mathbf{b}^{\#}\right) a_J^{(\mathbf{n})} E_K \\
&= \sum_{I \in \wp} \sum_{\mathbf{n} \in \mathbb{N}^p} \frac{1}{\mathbf{n}!} \partial^{\mathbf{n}} \partial_i f_I \left(\mathbf{b}^{\#}\right) \sum_{J \in \wp_0} a_J^{(\mathbf{n})} E_J E_I \\
&= \sum_{I \in \wp} \sum_{\mathbf{n} \in \mathbb{N}^p} \frac{1}{\mathbf{n}!} \partial^{\mathbf{n}} \partial_i f_I \left(\mathbf{b}^{\#}\right) \sum_{J \in \wp_0} \left(\mathbf{b} - \mathbf{b}^{\#}\right)^{\mathbf{n}} E_I \\
&= \sum_{I \in \wp} \widehat{\partial_i f_I}(\mathbf{b}) E_I\,. \square
\end{aligned}$$

(i) $\Rightarrow$ (ii) : trivial.
(ii) $\Rightarrow$ (iii) : Let $f \in \mathcal{D}(\Omega)$, which means differentiable with respect to all z_i , $i = 1, \ldots, p$, and continuous. Then clearly all F_I are continuous. For proving the system of differential equations let $\mathbf{b} = (b_1, \ldots, b_p) \in \Omega$ and $i \in \{1, \ldots, p\}$ be arbitrary , $b_i = \sum_{I \in \wp_0} b_{iI} E_I$, all $b_{iI} \in \mathbb{R}$. $\Omega \underset{\text{open}}{\subset} \mathbb{R}^{p|0}$ and f is differentiable at $\mathbf{b}$ with respect to z_i . Therefore there exist $\Omega' \underset{\text{open}}{\subset} \mathbb{R}^{1|0}$ such that $b_i \in \Omega'$ and $(b_1, \ldots, b_{i-1}, z_i, b_{i+1}, \ldots, b_p) \in \Omega$ for all $z_i \in \Omega'$ and $\Delta : \Omega' \to \Lambda$ continuous at b_i such that

$$f(b_1, \ldots, b_{i-1}, z_i, b_{i+1}, \ldots, b_p) = f(\mathbf{b}) + \Delta(z_i)(z_i - b_i)$$

for all $z_i \in \Omega'$. Let Δ_J be the J-th component function of Δ for all $J \in \wp$. Then for all $z_i \in \Omega'$

$$\begin{aligned}
&\sum_{I \in \wp} F_I(b_1, \ldots, b_{i-1}, z_i, b_{i+1}, \ldots, b_p) E_I \\
&= \sum_{I \in \wp} F_I(\mathbf{b}) E_I + \sum_{K \in \wp_0} (x_{iK} - b_{iK})\, E_K \sum_{J \in \wp} \Delta_J\,(z_i)\, E_J \\
&= \sum_{I \in \wp} \left(F_I(\mathbf{b}) + \sum_{K \in \wp_0, K \subset I} (-1)^{|I \setminus K < K|} (x_{iK} - b_{iK})\, \Delta_{I \setminus K}\,(z_i) \right) E_I\,.
\end{aligned}$$

So for all $I \in \wp$ and $z_i \in \Omega'$

$$\begin{aligned}&F_I\left(b_1,\ldots,b_{i-1},z_i,b_{i+1},\ldots,b_p\right)\\ &= F_I(\mathbf{b}) + \sum_{K\in\wp_0, K\subset I} (-1)^{|I\setminus K<K|}\left(x_{iK}-b_{iK}\right)\Delta_{I\setminus K}\left(z_i\right)\,.\end{aligned}$$

We see that F_I is partially differentiable at $\mathbf{b}$ with respect to x_{iJ} for all $I \in \wp$, $J \in \wp_0$, and

$$\partial_{i|J}F_I(\mathbf{b}) = \begin{cases} (-1)^{|I\setminus J<J|}\Delta_{i,I\setminus J}(\mathbf{b}) & \text{if } J\subset I\\ 0 & \text{otherwise}\end{cases}\,.$$

Especially, if we set $J=\emptyset$, we obtain $\Delta_{iR}(\mathbf{b}) = \partial_{i|\emptyset}F_R(\mathbf{b})$ for all $R\in\wp$. $\square$

(iii) $\Rightarrow$ (iv) Assume that all F_I are continuous and fulfill the system of differential equations.

Step I **Show that all F_I are affine linear with respect to x_{iJ} , $J\subset I, J\neq\emptyset$, and $\mathcal{C}^\infty$ with respect to $\mathbf{x}_\emptyset := \left(x_{1\emptyset},\ldots x_{p\emptyset}\right)\in\Omega^\#$.**

Let $i\in\{1,\ldots,p\}$, $I\in\wp$, $J\in\wp_0\setminus\{\emptyset\}$ with $J\subset I$. Then $J\not\subset I\setminus J$, and therefore $\partial_{i|J}F_I = (-1)^{|I\setminus J<J|}\partial_{i|\emptyset}F_{I\setminus J}$ is independent of x_{iJ} . So F_I is affine linear in x_{iJ} .

Now we prove by induction on n that all F_I are $\mathcal{C}^n$ with repect to $\mathbf{x}_\emptyset$ for arbitrary $n\in\mathbb{N}$.

> Since f is continuous, all F_I are continuous with respect to $\mathbf{x}_\emptyset$. Now let $i\in\{1,\ldots,p\}$, $n\in\mathbb{N}$ such that all F_R are $\mathcal{C}^n$ with repect to $\mathbf{x}_\emptyset$, and let $I\in\wp$, $\mathbf{b}\in\Omega$. Choose $S\in\wp_0\setminus\{\emptyset\}$ such that $S<I$.
>
> Since $\Omega \underset{\text{open}}{\subset} \mathbb{R}^{p|0}$, there exist $\varepsilon>0$ and $\Omega' \underset{\text{open}}{\subset} \Omega$ such that $\mathbf{b}\in\Omega'$ and $\Omega' + \left[\,0,\varepsilon\,\right]E_S\mathbf{e}_j\subset\Omega$. So for all $\mathbf{z}\in\Omega'$
>
> $$\begin{aligned}\partial_{i|\emptyset}F_I(\mathbf{z}) &= \partial_{i|S}F_{I\cup S}(\mathbf{z})\\ &= \frac{F_{I\cup S}\left(\mathbf{z}+\varepsilon E_S\mathbf{e}_i\right)-F_{I\cup S}(\mathbf{z})}{\varepsilon}\,,\end{aligned}$$
>
> because $F_{I\cup S}$ is affine linear with respect to x_{iS} .
>
> By assumption the right hand side is $\mathcal{C}^n$ with respect to $\mathbf{x}_\emptyset$, and so is the left hand side. This means $F_I|_{\Omega'}$ is $\mathcal{C}^{n+1}$ with respect

to $\mathbf{x}_\emptyset$. Since $\mathbf{b}$ was arbitrary, the F_I itself is $\mathcal{C}^{n+1}$ with respect to $\mathbf{x}_\emptyset$. $\square$

Step II **Show that all F_I and so f are determined by $f|_{\Omega^\#}$.**

Since the system is linear, we only have to prove that $f|_{\Omega^\#} = 0$ implies $f = 0$. So let $f|_{\Omega^\#} = 0$.

We will show that $F_I = 0$ for all $I \in \wp$ by induction on $|I|$. This of course will imply $f = 0$.

Assume $n \in \mathbb{N}$ with the property that $F_K = 0$ for all $K \in \wp$ with $|K| < n$, and let $I \in \wp$ with $|I| = n$. For any $i \in \{1, \ldots, p\}$, $J \in \wp_0 \setminus \{\emptyset\}$ with $J \subset I$ we have by (ii)

$$\partial_{i|J} F_I = (-1)^{|I \setminus J < J|} \partial_{i|\emptyset} F_{I \setminus J} \,,$$

which is 0 by assumption, since $|I \setminus J| \leq |I| - 2 < n$. So F_I is independ of all x_{iJ} , $i = 1, \ldots, p$, $J \in \wp_0 \setminus \{\emptyset\}$, and so for all $\mathbf{z} \in \Omega$

$$F_I(\mathbf{z}) = F_I\left(\mathbf{z}^\#\right) = 0 \,,$$

because $\mathbf{z}^\# \in \Omega^\#$. $\square$

This automatically proves the uniqueness of f^* as well, because $U^{|q}$ is of cube type for all $U \underset{\text{open}}{\subset} \mathbb{R}^{p|0}$.

Step III **Conclusion .**

Define $f_I := F_I|_{\Omega^\#} \in \mathcal{C}^\infty\left(\Omega^\#\right)$ by step II for all $I \in \wp$, and $g := \sum_{I \in \wp} \widehat{f}_I E_I : \left(\Omega^\#\right)^{|0} \to \Lambda$, which then fulfills (iv) . So, since we proved already (iv) $\Rightarrow$ (i) $\Rightarrow$ (iii) , we know that g is differentiable and fulfills (iii) as well. Finally we have

$$g|_{\Omega^\#} = \sum_{I \in \wp} \widehat{f}_I\Big|_{\Omega^\#} E_I = \sum_{I \in \wp} f_I E_I = \sum_{I \in \wp} F_I|_{\Omega^\#} E_I = f|_{\Omega^\#} \,.$$

So we can apply step II , which tells us $f = g|_\Omega$. $\square$

A similar result holds for $\Omega \subset \mathbb{R}^{0|1}$:

Theorem 4.10 *Let* $\Omega \underset{open}{\subset} \mathbb{R}^{0|1}$ *be of cube type and* $f \in \Lambda^\Omega$ *. Then there are equivalent*

(i) f *is differentiable,*

(ii) F_I , $I \in \wp$ *, fulfill the following system of differential equations*

$$\partial_J F_I = \begin{cases} (-1)^{|I\setminus J<J|+|I\setminus J<K|}\partial_K F_{(I\setminus J)\cup K} & \textit{if } J \subset I \\ 0 & \textit{otherwise} \end{cases}$$

for all $I \in \wp$ *,* $J \in \wp_1$ *and* $K \in \wp_1$ *with* $K \cap (I \setminus J) = \emptyset$ *,*

(iii) there exist $a, d \in \Lambda$ *such that* $f(\zeta) = a + \zeta d$ *for all* $\zeta \in \Omega$ *.*

In this case $f^* : \mathbb{R}^{0|1} \to \Lambda, \zeta \mapsto a + \zeta d$ *is again the unique differentiable extension of* f *, and* $\partial_{|1} f^* = d$ *.*

Proof: (iii) $\Rightarrow$ (i) : It is obvious that every $f : \mathbb{R}^{0|1} \to \Lambda, \zeta \mapsto a + \zeta d$ where $a, d \in \Lambda$ is differentiable with $\partial_{|1} f = d$. Since $f^*(0) = a = f(0)$ and $\partial_{|1} f^*(0) = d = \partial_{|1} f(0)$ clearly f^* is uniquely determined by f . $\square$
(i) $\Rightarrow$ (ii) : Assume f differentiable and let $\beta \in \Omega$ be arbitrary. As in the proof of theorem 4.9 we get: F_I is partially differentiable at β with respect to all y_J for all $I \in \wp, J \in \wp_1$, and

$$\partial_J F_I(\beta) = \begin{cases} (-1)^{|I\setminus J<J|}\Sigma_{I\setminus J}(\beta) & \text{if } J \subset I \\ 0 & \text{otherwise} \end{cases} .$$

So if $K \in \wp_1$ with $K \cap (I \setminus J)$, we get

$$\Sigma_{I\setminus J}(\beta) = (-1)^{|I\setminus J<K|}\partial_J F_{(I\setminus J)\cup K}(\beta) . \square$$

(ii) $\Rightarrow$ (iii) : Let F_I , $I \in \wp$ fulfill the system of differential equations.

Step I **Prove that all** $\partial_J F_I$ **are constant .**

We show $\partial_S \partial_T F_I = 0$ for all $I \in \wp$, $S, T \in \wp_1$, $S, T \subset I$.

Let $I \in \wp$, $S, T \in \wp_1$ with $S, T \subset I$.

First assume $S \cap T \neq \emptyset$. Choose $K \in \wp_1$ such that $K < I$. Then $K < I \setminus T$ and $S \not\subset (I \setminus T) \cup K$. Therefore by (ii)

$$\partial_T F_I = (-1)^{|I\setminus T<T|}\partial_K F_{(I\setminus T)\cup K}$$

is independent of y_S , and so $\partial_S \partial_T F_I = 0$. Especially if we set $S := T := J$ we see that F_I is affine linear with respect to y_J for all $I \in \wp$, $J \in \wp_1$, $J \subset I$.

Now let $I \in \wp$, $S, T \in \wp_1$ with $S, T \subset I$ and $S \cap T = \emptyset$. Choose $L, M \in \wp_1$ such that $M < L < I$. Then we have $L < I \setminus S$, and so by (ii)

$$\partial_S F_I = (-1)^{|I \setminus S < S|} \partial_L F_{(I \setminus S) \cup L} .$$

Since all F_K , $K \in \wp$ are affine linear with respect to all y_J , $J \in \wp_1$, $J \subset K$, we can interchange partial derivatives. Clearly $M < (I \setminus (S \cup T)) \cup L$, and so again by (ii) , since $|L < T| = |L|\,|T|$ is an odd number,

$$\begin{aligned} \partial_S \partial_T F_I &= (-1)^{|I \setminus S < S|} \partial_L \partial_T F_{(I \setminus S) \cup L} \\ &= (-1)^{|I \setminus S < S| + |(I \setminus (S \dot{\cup} T)) \dot{\cup} L < T|} \partial_L \partial_M F_{(I \setminus (S \cup T)) \cup L \cup M} \\ &= (-1)^{|I \setminus S < S| + |I \setminus T < T| + |S < T| + 1} \partial_L \partial_M F_{(I \setminus (S \cup T)) \cup L \cup M} . \end{aligned}$$

But S and T are arbitrary, so this holds again when S and T are interchanged. So we get

$$\partial_S \partial_T F_I = \partial_T \partial_S F_I = (-1)^{|S < T| + |T < S|} \partial_S \partial_T F_I = -\partial_S \partial_T F_I ,$$

because $|S < T| + |T < S| = |S|\,|T|$ is again an odd number. And this implies $\partial_S \partial_T F_I = 0$. $\square$

Step II **Conclusion .**

For all $I \in \wp$ we have $\{\min(I \cup \{0\}) - 1\} \in \wp_1$, so we can define $d_I := \partial_{\{\min(I \cup \{0\}) - 1\}} F_{I \cup \{\min(I \cup \{0\}) - 1\}} \in \mathbb{R}$, $d := \sum_{I \in \wp} d_I E_I \in \Lambda$.
Let $I \in \wp$, $J \in \wp_1$ and $K := \{\min((I \setminus J) \cup \{0\}) - 1\} < I \setminus J$. Then by (ii)

$$\partial_J F_I = (-1)^{|I \setminus J < J|} \partial_K F_{(I \setminus J) \cup K} = (-1)^{|I \setminus J < J|} d_{I \setminus J} ,$$

so for all $I \in \wp$ and $\zeta \in \Omega$

$$F_I(\zeta) = F_I(0) + \sum_{J \in \wp_1, J \subset I} (-1)^{|I \setminus J < J|} y_J d_{I \setminus J} ,$$

and finally, if we define $a := f(0) \in \Lambda$,

$$\begin{aligned} f(\zeta) &= f(0) + \sum_{I \in \wp} \sum_{J \in \wp_1, J \subset I} (-1)^{|I \setminus J < J|} y_J d_{I \setminus J} E_I \\ &= a + \sum_{J \in \wp_1} y_J E_J \sum_{K \in \wp} d_K E_K \\ &= a + \zeta d \, . \square \end{aligned}$$

Corollary 4.11 *Let $\Omega \subset \mathbb{R}^{p|q}$ be of cube type and $f : \Omega \to \Lambda$. Then there are equivalent*

(i) f is arbitrarily often differentiable,

(ii) $f \in \mathcal{D}(\Omega, \Lambda)$,

(iii) there exists a family $(f_{IS})_{I \in \wp, S \in \wp(q)} \in \left(\mathcal{C}^\infty\left(\Omega^\#\right)\right)^{\wp \times \wp(q)}$ such that for all $(\mathbf{z}, \zeta) \in \left(\Omega^\#\right)^{|q}$

$$f(\mathbf{z}, \zeta) = \sum_{I \in \wp} \sum_{S \in \wp(q)} \zeta^S \widehat{f_{IS}}(\mathbf{z}) E_I \, .$$

Again the function

$$f^* := \sum_{I \in \wp} \sum_{S \in \wp(q)} \left(\zeta^S \widehat{f_{IS}}\right) E_I : \left(\Omega^\#\right)^{|q} \to \Lambda$$

is the unique extension in $\mathcal{D}\left(\left(\Omega^\#\right)^{|q}\right)$ of f . f^* is arbitrarily often differentiable, for all $i = 1, \ldots, p$ we have

$$\partial_{i|} f^* = \sum_{I \in \wp} \sum_{S \in \wp(q)} \zeta^S \widehat{\partial_i f_{IS}} E_I$$

and for all $j = 1, \ldots, q$

$$\partial_{|j} f^* = \sum_{I \in \wp} \sum_{S \in \wp(q), j \in S} (-1)^{|S \setminus \{j\} < j|} \zeta^{S \setminus \{j\}} \widehat{f_{IS}} E_I \, .$$

Proof: (i) $\Rightarrow$ (ii) : trivial.
(ii) $\Rightarrow$ (iii) and uniqueness: by induction on q . If $q = 0$ then it follows immediately from theorem 4.9 .
So let us assume $\Omega \subset \mathbb{R}^{p|q+1}$ of cube type and $f \in \mathcal{D}(\Omega, \Lambda)$. Then there exist $\Omega' \subset \mathbb{R}^{p|q}$ and $\Omega'' \subset \mathbb{R}^{0|1}$ both of cube type such that $\Omega = \Omega' \times \Omega''$. So for all $(\mathbf{z}, \zeta') \in \Omega'$ we have a differentiable function $f(\mathbf{z}, \zeta', \diamondsuit) : \Omega'' \to \Lambda$,

and so by theorem 4.10 there exist unique $g, h : \Omega' \to \Lambda$ such that for all $(\mathbf{z}, \zeta) = (\mathbf{z}, \zeta', \zeta_{q+1}) \in \Omega$

$$f(\mathbf{z}, \zeta) = g\left(\mathbf{z}, \zeta'\right) + \zeta_{q+1} h\left(\mathbf{z}, \zeta'\right) .$$

$g = f|_{\Omega' \times \{0\}} \in \mathcal{D}\left(\Omega', \Lambda\right)$. Now we prove that $h \in \mathcal{D}\left(\Omega', \Lambda\right)$.

For all $(\mathbf{z}, \zeta) = (\mathbf{z}, \zeta', \zeta_{q+1}) \in \Omega$ we have

$$\begin{aligned}
&\sum_{I \in \wp} F_I(\mathbf{z}, \zeta) E_I \\
&= \sum_{I \in \wp} G_I\left(\mathbf{z}, \zeta'\right) E_I + \sum_{J \in \wp_1} y_{q+1,J} E_J \sum_{K \in \wp} H_K\left(\mathbf{z}, \zeta'\right) E_K \\
&= \sum_{I \in \wp} \left(G_I(\mathbf{z}, \zeta') + \sum_{J \in \wp_1, J \subset I} (-1)^{|I \setminus J < J|} H_{I \setminus J}\left(\mathbf{z}, \zeta'\right) y_{q+1,J} \right) E_I ,
\end{aligned}$$

and so

$$F_I(\mathbf{z}, \zeta) = G_I\left(\mathbf{z}, \zeta'\right) + \sum_{J \in \wp_1, J \subset I} (-1)^{|I \setminus J < J|} H_{I \setminus J}\left(\mathbf{z}, \zeta'\right) y_{q+1,J} .$$

Finally we get

$$H_K = \partial_{|q+1,S} F_{K \cup S}$$

for all $K \in \wp$ and $S \in \wp_1$ with $S < K$, and therefore one can easily verify that H_I , $I \in \wp$, fulfill (iii) of theorem 4.9 with respect to z and (ii) of theorem 4.10 with respect to all $\zeta_1, \ldots, \zeta_q$, and so $h \in \mathcal{D}(\Omega', \Lambda)$ by theorems 4.9 and 4.10 .

By induction hypothesis there exist families $\left(g_{IR}\right)_{I \in \wp, R \in \wp(q)}$ and $\left(h_{IR}\right)_{I \in \wp, R \in \wp(q)} \in \left(\mathcal{C}^\infty\left(\Omega^\#\right)\right)^{\wp \times \wp(q)}$ such that for all $(\mathbf{z}, \zeta') \in \Omega'$

$$g\left(\mathbf{z}, \zeta'\right) = \sum_{I \in \wp} \sum_{R \in \wp(q)} \zeta'^R \widehat{g_{IR}}(\mathbf{z}) E_I$$

and

$$h\left(\mathbf{z}, \zeta'\right) = \sum_{I \in \wp} \sum_{R \in \wp(q)} \zeta'^R \widehat{h_{IR}}(\mathbf{z}) E_I .$$

So since $\zeta'^R = \zeta^R$ and $\zeta_{q+1} \zeta^R = (-1)^{|R|} \zeta^{R \cup \{q+1\}}$ for all $R \in \wp(q)$, we have for all $(\mathbf{z}, \zeta) \in \Omega$

$$\begin{aligned} f(\mathbf{z},\zeta) &= g\left(\mathbf{z},\zeta'\right)+\zeta_{q+1}h\left(\mathbf{z},\zeta'\right) \\ &= \sum_{I\in\wp}\sum_{S\in\wp(q+1)}\zeta^S\widehat{f_{IS}}(\mathbf{z})E_I\,, \end{aligned}$$

if we define

$$f_{IS}:=\begin{cases} (-1)^{|S\backslash\{q+1\}|}h_{I,S\backslash\{q+1\}} & \text{if } q+1\in S \\ g_{IS} & \text{otherwise} \end{cases}.$$

By the same argument it ca be seen that if $f^*\in\mathcal{D}\left(\left(\Omega^{\#}\right)^{|q+1},\Lambda\right)$ is an extension of f then $f^*=g^*+\zeta_{q+1}h^*$ where g^* and h^* are the unique extensions of g and h , and so f^* is uniquely determined by f . $\square$

(iii) $\Rightarrow$ (i) and partial derivatives: by induction on q . If $q=0$ then it follows directly from theorem 4.9 .

Now let us assume $U\underset{\text{open}}{\subset}\mathbb{R}^p$ and $(f_{IS})_{I\in\wp,S\in\wp(q+1)}\in\left(\mathcal{C}^\infty(U)\right)^{\wp\times\wp(q+1)}$ such that for all $(\mathbf{z},\zeta)\in U^{|q+1}$

$$f(\mathbf{z},\zeta)=\sum_{I\in\wp}\sum_{S\in\wp(q+1)}\zeta^S\widehat{f_{IS}}(\mathbf{z})E_I\,.$$

Then $f(\mathbf{z},\zeta)=g\left(\mathbf{z},\zeta'\right)+\zeta_{q+1}h\left(\mathbf{z},\zeta'\right)$ for all $(\mathbf{z},\zeta',\zeta_{q+1})\in U^{|q+1}$ with

$$\begin{aligned} g &= \sum_{I\in\wp}\sum_{R\in\wp(q)}\zeta^R\widehat{f_{IR}}E_I, \\ h &= \sum_{I\in\wp}\sum_{R\in\wp(q)}(-1)^{|R|}\zeta^R\widehat{f_{R\cup\{q+1\}}}E_I:U^{|q}\to\Lambda\,. \end{aligned}$$

Let $(\mathbf{b},\beta)=(\mathbf{b},\beta',\beta_{q+1})\in U^{|q+1}$. By induction hypothesis g and h are differentiable. So there exist $\Delta_1,\ldots,\Delta_p$, $\Sigma_1,\ldots,\Sigma_q$, $\Delta'_1,\ldots,\Delta'_i$, $\Sigma'_1,\ldots,\Sigma'_j:U^{|q}\to\Lambda$ continuous at $(\mathbf{b},\beta)$ such that

$$g\left(\mathbf{z},\zeta'\right)=g\left(\mathbf{b},\beta'\right)+\sum_{i=1}^{p}\left(z_i-b_i\right)\Delta_i\left(\mathbf{z},\zeta'\right)+\sum_{j=1}^{q}\left(\zeta'_i-\beta_i\right)\Sigma_i\left(\mathbf{z},\zeta'\right)$$

and

$$h\left(\mathbf{z},\zeta'\right)=h\left(\mathbf{b},\beta'\right)+\sum_{i=1}^{p}\left(z_i-b_i\right)\Delta'_i\left(\mathbf{z},\zeta'\right)+\sum_{j=1}^{q}\left(\zeta'_i-\beta_i\right)\Sigma'_i\left(\mathbf{z},\zeta'\right)$$

for all $(\mathbf{z},\zeta')\in U^{|q}$. So for all $(\mathbf{z},\zeta)=(\mathbf{z},\zeta',\zeta_{q+1})\in U^{p|q+1}$

$$
\begin{aligned}
f(\mathbf{z},\zeta) &= g\left(\mathbf{z},\zeta'\right)+\zeta_{q+1}h\left(\mathbf{z},\zeta'\right)\\
&= g(\mathbf{b},\beta)+\beta_{q+1}h(\mathbf{b},\beta)+\sum_{i=1}^{p}\left(z_i-b_i\right)\left(\Delta_i\left(\mathbf{z},\zeta'\right)+\zeta_{q+1}\Delta_i'\left(\mathbf{z},\zeta'\right)\right)\\
&\quad +\sum_{j=1}^{q}\left(\zeta_i-\beta_i\right)\left(\Sigma_j\left(\mathbf{z},\zeta'\right)-\zeta_{q+1}\Sigma_j'\left(\mathbf{z},\zeta'\right)\right)\\
&\quad +\left(\zeta_{q+1}-\beta_{q+1}\right)h(\mathbf{b},\beta)\,.
\end{aligned}
$$

So we see that f is differentiable at $(\mathbf{b},\beta)$,
$\partial_{i|}f(\mathbf{b},\beta)=\partial_{i|}g\left(\mathbf{b},\beta'\right)+\beta_{q+1}\partial_{i|}h(\mathbf{b},\beta)$ for all $i=1,\ldots,p$,
$\partial_{|j}f(\mathbf{b},\beta)=\partial_{|j}g\left(\mathbf{b},\beta'\right)-\beta_{q+1}\partial_{|j}h(\mathbf{b},\beta)$ for all $j=1,\ldots,q$ and
$\partial_{|q+1}f(\mathbf{b},\beta)=h(\mathbf{b},\beta)$.
By induction hypothesis

$$
\begin{aligned}
\partial_{i|}g\left(\mathbf{b},\beta'\right) &= \sum_{I\in\wp}\sum_{R\in\wp(q)}\beta'^{R}\widehat{\partial_i f_{IR}}(\mathbf{b})E_I,\\
\partial_{i|}h\left(\mathbf{b},\beta'\right) &= \sum_{I\in\wp}\sum_{R\in\wp(q)}(-1)^{|R|}\beta'^{R}\partial_i\widehat{f_{R\cup\{q+1\}}}(\mathbf{b})E_I
\end{aligned}
$$

for all $i=1,\ldots,p$,

$$
\begin{aligned}
\partial_{|j}g\left(\mathbf{b},\beta'\right) &= \sum_{I\in\wp}\sum_{R\in\wp(q),j\in R}(-1)^{|R\setminus\{j\}<j|}\beta'^{R\setminus\{j\}}\widehat{f_{IR}}(\mathbf{b})E_I,\\
\partial_{|j}h\left(\mathbf{b},\beta'\right) &= \sum_{I\in\wp}\sum_{R\in\wp(q),j\in R}(-1)^{|R\setminus\{j\}<j|+|R|}\beta'^{R\setminus\{j\}}\widehat{f_{R\cup\{q+1\}}}(\mathbf{b})E_I
\end{aligned}
$$

for all $j=1,\ldots,q$. So if $j\in\{1,\ldots,q\}$ one has

$$
\begin{aligned}
\partial_{|j}f(\mathbf{b},\beta) &= \sum_{I\in\wp}\sum_{R\in\wp(q),j\in R}(-1)^{|R\setminus\{j\}<j|}\beta'^{R\setminus\{j\}}\widehat{f_{IR}}(\mathbf{b})E_I\\
&\quad -\beta_{q+1}\sum_{I\in\wp}\sum_{R\in\wp(q),j\in R}(-1)^{|R\setminus\{j\}<j|+|R|}\beta'^{R\setminus\{j\}}\widehat{f_{R\cup\{q+1\}}}(\mathbf{b})E_I\\
&= \sum_{I\in\wp}\sum_{S\in\wp(q+1),j\in S}(-1)^{|S\setminus\{j\}<j|}\beta^{S\setminus\{j\}}\widehat{f_{IS}}(\mathbf{b})E_I\,.
\end{aligned}
$$

With similar calculations in the cases $i=1,\ldots,p$ and $j=q+1$ the rest follows as well. □

Corollary 4.12 *Let* $\Omega\underset{open}{\subset}\mathbb{R}^{p|q}$ *such that* $\Omega_{\mathbf{x}}=\left\{(\mathbf{z},\zeta)\in\Omega\,\middle|\,(\mathbf{z},\zeta)^{\#}=\mathbf{x}\right\}$ *is connected for all* $\mathbf{x}\in\Omega^{\#}$ *. Then the same holds as in corollary 4.11 .*

Proof: (i) $\Rightarrow$ (ii) : again trivial.
(ii) $\Rightarrow$ (iii) and uniqueness: Let α be the set of all (U, g) where $U \underset{\text{open}}{\subset} \Omega^{\#}$ and $g \in \mathcal{D}\left(U^{|q}, \Lambda\right)$ such that $g \equiv f$ on $U^{|q} \cap \Omega$.

Step I **Show that if** (U, g) **and** $(V, h) \in \alpha$ **then** $g \equiv h$ **on** $(U \cap V)^{|q}$ **.**

Let $(\mathbf{z}, \zeta) \in (U \cap V)^{|q} \subset \left(\Omega^{\#}\right)^{|q}$. Then there exists $(\mathbf{z}', \zeta') \in \Omega \cap (U \cap V)^{|q}$ such that $\mathbf{z}'^{\#} = \mathbf{z}^{\#}$. Since $\Omega \cap (U \cap V)^{|q} \underset{\text{open}}{\subset} \mathbb{R}^{p|q}$ there exists $\Omega' \underset{\text{open}}{\subset} \mathbb{R}^{p|q}$ of cube type such that $\mathbf{0} \in \Omega'$ and

$$\left(\mathbf{z}', \zeta'\right) + \Omega' \subset \Omega \cap (U \cap V)^{|q} .$$

Therefore $g\left(\diamondsuit + (\mathbf{z}', \zeta')\right)|_{\left(\Omega'^{\#}\right)^{|q}}$ and $h\left(\diamondsuit + (\mathbf{z}', \zeta')\right)|_{\left(\Omega'^{\#}\right)^{|q}}$ both are differentiable extensions of $f\left(\diamondsuit + (\mathbf{z}', \zeta')\right)|_{\Omega'}$ to $\left(\Omega'^{\#}\right)^{|q}$, and so $g\left(\diamondsuit + (\mathbf{z}', \zeta')\right) = h((\diamondsuit + (\mathbf{z}', \zeta'))$ on $\left(\Omega'^{\#}\right)^{|q}$ by the uniqueness in corollary 4.11 . Since $(\mathbf{z}, \zeta) - (\mathbf{z}', \zeta') \in \left(\Omega'^{\#}\right)^{|q}$, we have $g(\mathbf{z}, \zeta) = h(\mathbf{z}, \zeta)$.

Step II **Show that for all** $(\mathbf{b}, \beta) \in \Omega$ **there exists** $(U, g) \in \alpha$ **such that** $\mathbf{b}^{\#} \in U$ **.**

Let $(\mathbf{b}, \beta) \in \Omega$. Since $\Omega \underset{\text{open}}{\subset} \mathbb{R}^{p|q}$, there exists $\Omega' \underset{\text{open}}{\subset} \mathbb{R}^{p|q}$ of cube type such that $\mathbf{0} \in \Omega'$ and $(\mathbf{b}, \beta) + \Omega' \subset \Omega$. Let $U := ((\mathbf{b}, \beta) + \Omega')^{\#} \underset{\text{open}}{\subset} \Omega^{\#}$. Then clearly $\mathbf{b}^{\#} \in U$. By corollary 4.11 there exists $h \in \mathcal{D}\left(\left(\Omega'^{\#}\right)^{|q}, \Lambda\right)$ such that $h = f(\diamondsuit + (\mathbf{b}, \beta))$ on Ω' , because $f(\diamondsuit + (\mathbf{b}, \beta))|_{\Omega'} \in \mathcal{D}\left(\Omega', \Lambda\right)$. So if we define $g := h(\diamondsuit - (\mathbf{b}, \beta)) \in \mathcal{D}\left(U^{|q}, \Lambda\right)$ we have $g \equiv f$ on $(\mathbf{b}, \beta) + \Omega'$.

Now let us prove $g \equiv f$ on $U^{|q} \cap \Omega$.

> So let $(\mathbf{z}, \zeta) \in U^{|q} \cap \Omega$ and $\mathbf{x} := \mathbf{z}^{\#} \in U$. Since $U = ((\mathbf{b}, \beta) + \Omega')^{\#}$ there exists $(\mathbf{z}', \zeta') \in (\mathbf{b}, \beta) + \Omega'$ such that $\mathbf{x} = \mathbf{z}'^{\#}$. Let X be the set of all $(\mathbf{z}'', \zeta'') \in \Omega_{\mathbf{x}}$ such that
>
> $$\partial_{i_1|J_1} \ldots \partial_{i_r|J_r} \partial_{|j_1 K_1} \ldots \partial_{|j_s K_s} \left(F_I - G_I\right)\left(\mathbf{z}'', \zeta''\right) = 0$$
>
> for all $i_1, \ldots, i_r \in \{1, \ldots, p\}$, $J_1, \ldots, J_r \in \wp_0 \setminus \{\emptyset\}$, $j_1, \ldots, j_s \in \{1, \ldots, q\}$, $K_1, \ldots, K_s \in \wp_1$, $I \in \wp$. Then clearly $X \underset{\text{closed}}{\subset} \Omega_x$, and $(\mathbf{z}', \zeta') \in X$ since $(\mathbf{b}, \beta) + \Omega' \underset{\text{open}}{\subset} \Omega$ is a neighbourhood of $(\mathbf{z}', \zeta')$. But since f and g both fulfill the systems of differential equations given in theorem 4.9 (iii) and theorem

4.10 (ii) for each $\zeta_1, \ldots, \zeta_q$, all F_I , G_I , $I \in \wp$, are locally affine linear with respect to all x_{iJ} , $i = 1, \ldots, p$, $J \subset \wp_0 \setminus \{\emptyset\}$ and y_{jK} , $j = 1, \ldots, q$, $K \subset \wp_1$. So $X \underset{\text{open}}{\subset} \Omega_{\mathbf{x}}$.

And since $\Omega_{\mathbf{x}}$ is connected we have $X = \Omega_{\mathbf{x}}$, so $(\mathbf{z}, \zeta) \in X$ and so $f(\mathbf{z}, \zeta) = g(\mathbf{z}, \zeta)$. $\square$

Step III **Conclusion.**

Since by step II

$$\bigcup_{(U,g)\in\alpha} U^{|q} = \left(\Omega^{\#}\right)^{|q} ,$$

$g = h$ on $(U \cap V)^{|q}$ for all (U, g) and $(V, h) \in \alpha$ by step I there exists a unique function $f^* \in \mathcal{D}\left(\left(\Omega^{\#}\right)^{|q}\right)$ such that $f^*|_{U^{|q}} = g$ for all $(g, U) \in \alpha$. So $f = f^*|_{\Omega}$, and f^* is the only function in $\mathcal{D}\left(\left(\Omega^{\#}\right)^{|q}\right)$ which has this property. Since $\left(\Omega^{\#}\right)^{|q}$ is of cube type, we see that f^* has the required form by corollary 4.11 . $\square$

(iii) $\Rightarrow$ (i) : same as in in the proof of corollary 4.11 . $\square$

Now we prove theorem 4.8 .

Let $U := \Omega^{\#}$ and

$$\Phi : \left(\Lambda\left(\mathbb{R}^q\right) \otimes \mathcal{C}^\infty(U)\right) \hat{\boxtimes} \Lambda \to \mathcal{C}^\infty\left(U^{|q}, \Lambda\right)$$

be the unique $\mathbb{R}$-linear and continuous map given by
$e^S \otimes f \otimes E_I \mapsto \left(\zeta^S \widehat{f}\right) E_I$ for all $f \in \mathcal{C}^\infty(U)$, $S \in \wp(q)$, $I \in \wp$. Then Φ is surjective by lemma 4.11 since $U^{|q}$ is of cube type. To prove injectivity let

$$\mathfrak{f} := \sum_{I\in\wp} \sum_{S\in\wp(q)} e^S \otimes f_{IS} \otimes E_I \in \left(\Lambda\left(\mathbb{R}^q\right) \otimes \mathcal{C}^\infty(U)\right) \hat{\boxtimes} \Lambda$$

such that

$$0 = \Phi(\mathfrak{f}) = \sum_{I\in\wp} \sum_{S\in\wp(q)} \zeta^S \widehat{f_{IS}} E_I ,$$

and let $\mathbf{x} \in U$. We prove by induction on $|S|$ that all $f_{IS}(\mathbf{x}) = 0$, $I \in \wp$, $S \in \wp(q)$.

Let $n \in \mathbb{N}$ such that $f_{KR}(\mathbf{x}) = 0$ for all $K \in \wp$, $R \in \wp(q)$ with $|R| < n$, and let $I \in \wp$ and $S \in \wp(q)$ with $|S| = n$. Define $\zeta_j := E_{\{j+\max I\}}$ for all $j = 1, \ldots, q$. Then $(\mathbf{x}, \zeta) \in U^{|q}$, and

$$
\begin{aligned}
0 &= \Phi(f)(\mathbf{x},\zeta) = \sum_{K\in\wp}\sum_{R\in\wp(q)} E_{R+\max I} f_{KR}(\mathbf{x}) E_K \\
&= \sum_{L\in\wp}\ \sum_{R\in\wp(q), R+\max I\subset L} (-1)^{|L\setminus(R+\max I)<R+\max I|} \times \\
&\qquad \times f_{L\setminus(R+\max I),R}(\mathbf{x}) E_L .
\end{aligned}
$$

Since all $f_{KR}(\mathbf{x}) \in \mathbb{R}$, $K \in \wp$, $R \in \wp(q)$ this implies

$$
\sum_{R\in\wp(q), R+\max I\subset L} (-1)^{|L\setminus(R+\max I)<(R+\max I)|} f_{L\setminus(R+\max I),R}(\mathbf{x}) = 0
$$

for all $L \in \wp$. If we set $L := I\dot{\cup}(S + \max I)$ we see that for all $R \in \wp(q)$: $R + \max I \subset L$ implies $R \subset S$ and so either $R = S$ or $|R| < n$. By induction hypothesis we obtain $f_{IS}(\mathbf{x}) = 0$. $\Box$

For proving the conservation of multiplication and grading let $e^R \otimes f \otimes E_I$ and $e^S \otimes g \otimes E_K \in (\Lambda(\mathbb{R}^q) \otimes \mathcal{C}^\infty(U)) \boxtimes \Lambda$ and $(\mathbf{z}, \zeta) \in \mathbb{R}^{p|q}$. Then we have

$$
e^R e^S = \begin{cases} (-1)^{|S<R|} e^{R\cup S} & \text{if } R\cap S = \emptyset \\ 0 & \text{otherwise} \end{cases} ,
$$

and since all $\zeta_j \in \Lambda_1$

$$
\zeta^R \zeta^S = \begin{cases} (-1)^{|S<R|} \zeta^{R\cup S} & \text{if } R\cap S = \emptyset \\ 0 & \text{otherwise} \end{cases}
$$

as well.

Of course since Λ is a graded algebra $\dot{\zeta}^S \equiv |S|$ and $\dot{E}_I \equiv |I| \mod 2$. So $E_I \zeta^S = (-1)^{|I||S|} \zeta^S E_I$. This all implies

$$
\begin{aligned}
&\Phi\left(\left(e^R \otimes f \otimes E_I\right)\left(e^S \otimes g \otimes E_K\right)\right)(\mathbf{z},\zeta) \\
&= \Phi\left((-1)^{|I||S|} e^R e^S \otimes fg \otimes E_I E_K\right)(\mathbf{z},\zeta) \\
&= (-1)^{|I||S|} \zeta^R \zeta^S f(\mathbf{z}) g(\mathbf{z}) E_I E_K \\
&= \zeta^R f(\mathbf{z}) E_I g(\mathbf{z}) \zeta^S E_K = \left(\Phi\left(e^S \otimes f \otimes E_I\right)\Phi\left(e^R \otimes g \otimes E_K\right)\right)(\mathbf{z},\zeta) .
\end{aligned}
$$

$e^R \otimes f \otimes E_I$ is homogeneous of parity $\equiv |S| + |I| \mod 2$ by definition of the graded tensor product, and

$$\Phi(e^S \otimes f \otimes E_I)(\mathbf{z}, \zeta) = \zeta^S \widehat{f}(\mathbf{z}) E_I \in \Lambda$$

is clearly homogeneous with parity $\equiv |S| + |I| \mod 2$ as well.

Now let $|_\Omega : \mathcal{C}^\infty\left(U^{|q}\right) \to \mathcal{D}(\Omega)$ be the restriction map. Then it is clearly a graded algebra homomorphism, and by corollary 4.12 we have $^* : \mathcal{D}(\Omega) \to \mathcal{C}^\infty\left(U^{|q}\right)$, $f \mapsto f^*$ as an inverse homomorphism. $\square$

Now we will deduce a lemma which will be useful in what follows.

Lemma 4.13 *Let $U \underset{open}{\subset} \mathbb{R}^p$ and $f \in \mathcal{D}\left(U^{|q}, \Lambda\right)$.*

(i) For all $(\mathbf{b}, \beta) \in U^{|q}$ there exist $\Delta_1, \ldots, \Delta_p$, $\Sigma_1, \ldots, \Sigma_q \in \mathcal{D}\left(U^{|q}, \Lambda\right)$ such that for all $(\mathbf{z}, \zeta) \in U^{|q}$

$$f(\mathbf{z}, \zeta) = f(\mathbf{b}, \beta) + \sum_{i=1}^{p} (z_i - b_i)\, \Delta_i(\mathbf{z}, \zeta) + \sum_{j=1}^{q} (\zeta_j - \beta_j)\, \Sigma_j(\mathbf{z}, \zeta)\,.$$

(ii) f is determined by the functions $\left(\partial_{|j_1} \ldots \partial_{|j_r} f\right)\big|_U \in \mathcal{C}^\infty(U) \otimes \Lambda$, $\{j_1, \ldots, j_r\} \subset \{1, \ldots, q\}$ with $j_1 < \cdots < j_r$.

Proof: By corollary 4.11 and theorem 4.8 we can write

$$f = \sum_{I \in \wp} \sum_{S \in \wp(q)} \zeta^S \widehat{f_{IS}} E_I$$

with uniquely determined $f_{IS} \in \mathcal{C}^\infty(U)$, $I \in \wp$, $S \in \wp(q)$ because $U^{|q} \subset \mathbb{R}^{p|q}$ is of cube type.
(i) Since $f\left(\diamondsuit + \left(\mathbf{b} - \mathbf{b}^\#, \beta\right)\right) \in \mathcal{D}\left(U^{|q}, \Lambda\right)$ we can say without loss of generality $\mathbf{b} = \mathbf{b}^\# \in U$ and $\beta = \mathbf{0}$. Since all $f_{I\emptyset} \in \mathcal{C}^\infty(U)$ there exist $\Delta_{Ii} \in \mathcal{C}^\infty(U)$, $I \in \wp$, $i = 1, \ldots, p$, such that

$$f_{I\emptyset}(\mathbf{x}) = f_{I\emptyset}(\mathbf{b}) + \sum_{i=1}^{p} (x_i - b_i) \Delta_{Ii}(\mathbf{x})$$

for all $I \in \wp$ and $\mathbf{x} \in U$. If we apply $\widehat{\ }$ to these equations we obtain

$$\begin{aligned} f(\mathbf{z}, \zeta) &= \sum_{I \in \wp} f_{I\emptyset}(\mathbf{b}) E_I + \sum_{i=1}^{p} (z_i - b_i) \sum_{I \in \wp} \widehat{\Delta_{Ii}}(\mathbf{z}) E_I \\ &\quad + \sum_{j=1}^{q} \zeta_j \sum_{I \in \wp} \sum_{S \in \wp(q), j = \min S} \zeta^{S \setminus \{j\}} \widehat{f_{IS}}(\mathbf{z}) E_I \\ &= f(\mathbf{b}, \mathbf{0}) + \sum_{i=1}^{p} (z_i - b_i)\, \Delta_i(\mathbf{z}, \zeta) + \sum_{j=1}^{q} \zeta_j \Sigma_j(\mathbf{z}, \zeta) \end{aligned}$$

for all $(\mathbf{z},\zeta) \in U^{|q}$ with $\Delta_i := \sum_{I\in\wp} \widehat{\Delta_{Ii}} E_I$, $\Sigma_j := \sum_{I\in\wp} \sum_{S\in\wp(q), j=\min S} \zeta^{S\setminus\{j\}} \widehat{f_{IS}} \in \mathcal{D}\left(U^{|q}, \Lambda\right)$, $i = 1, \ldots, p$, $j = 1, \ldots, q$. $\Box$

(ii) Since all $\partial_{j|}$ are linear it suffices to show that if $\left(\partial_{|j_1} \ldots \partial_{|j_r} f\right)\big|_U = 0$ for all $r \in \mathbb{N}$, $(j_1, \ldots, j_r) \in \wp(q)^r$ then $f = 0$. So assume all $\left(\partial_{|j_1} \ldots \partial_{|j_r} f\right)\big|_U = 0$. Let $R = \{j_1, \ldots, j_r\} \in \wp(q)$ with $j_1 < \cdots < j_r$. Then for all $\mathbf{x} \in U$ by the derivation rule given in corollary 4.11

$$0 = \partial_{|j_1} \ldots \partial_{|j_r} f(\mathbf{x}, \mathbf{0}) = \sum_{I\in\wp} f_{IR}(\mathbf{x}) E_I \,.$$

But all $f_{IR}(\mathbf{x}) \in \mathbb{R}$, so since $\mathbf{x} \in U$ was arbitrary all $f_{IR} = 0$, and this implies $f = 0$. $\Box$

If $f \in \mathcal{D}\left(U^{|q}\right)$ then especially $\mathbf{z}^\# = \mathbf{b}^\#$ implies $f(\mathbf{z},\zeta)^\# = f(\mathbf{b},\beta)^\#$ for all $(\mathbf{z},\zeta), (\mathbf{b},\beta) \in U^{|q}$.

From now on let $\Omega \underset{\text{open}}{\subset} \mathbb{R}^{p|q}$ and $\Omega' \underset{\text{open}}{\subset} \mathbb{R}^{r|s}$ such that $\Omega_{\mathbf{x}}$ and $\Omega'_{\mathbf{x}'}$ are connected for all $\mathbf{x} \in \Omega^\#$ resp. $\mathbf{x}' \in \Omega'^\#$. We define

$$\mathcal{D}(\Omega) := \Psi\left(\left(\Lambda\left(\mathbb{R}^q\right) \otimes \mathcal{C}^\infty\left(\Omega^\#\right)\right) \otimes 1\right) \subset \mathcal{D}(\Omega, \Lambda) \,,$$

where $\Psi : \left(\Lambda\left(\mathbb{R}^q\right) \otimes \mathcal{C}^\infty\left(\Omega^\#\right)\right) \hat{\boxtimes} \Lambda \to \mathcal{D}(\Omega, \Lambda)$ is the isomorphism given by theorem 4.8 . Then $\mathcal{D}(\Omega)$ is a sub graded algebra of $\mathcal{D}(\Omega, \Lambda)$ **not** containing Λ , more precisely $\mathcal{D}(\Omega) \cap \Lambda = \mathbb{R}$. We can characterize $\mathcal{D}(\Omega)$ as follows:

Theorem 4.14

a) $\mathcal{D}(\Omega)$ is closed under derivation.

b) Let $f \in \mathcal{D}(\Omega, \Lambda)$. Then there are equivalent

(i) $f \in \mathcal{D}(\Omega)$,

(ii) there exists a family $\left(f_S\right)_{S\in\wp(q)} \in \mathcal{C}^\infty\left(\Omega^\#\right)^{\wp(q)}$ such that

$$f^* = \sum_{S\in\wp(q)} \zeta^S \widehat{f_S} \,,$$

(iii) $f^*(\mathbf{z},\zeta) \in \overline{\langle \mathbf{z}, \zeta\rangle}$ for all $(\mathbf{z},\zeta) \in \left(\Omega^\#\right)^{|q}$, where $\overline{\langle \mathbf{z}, \zeta\rangle}$ is defined to be the smallest closed sub graded algebra of Λ containing all $z_i, \ldots, z_p$, $\zeta_i, \ldots, \zeta_q$,

(iv) $\partial_{|j_1} \ldots \partial_{|j_r} f^*(\mathbf{x}, \mathbf{0}) \in \mathbb{R}$ for all $\{j_1, \ldots, j_r\} \in \{1, \ldots, q\}^r$ and $\mathbf{x} \in \Omega^\#$.

c) $\Lambda(\mathbb{R}^q) \otimes \mathcal{C}^\infty\left(\Omega^\#\right) \simeq \mathcal{D}(\Omega)$ *as graded algebras where the isomorphism is the unique* $\mathbb{R}$*-linear map given by* $e^S \otimes f \mapsto \zeta^S \widehat{f}$ *for all* $f \in \mathcal{C}^\infty\left(\Omega^\#\right)$ *and* $S \in \wp(q)$ *.*

d) $\mathcal{D}(\Omega, \Lambda) = \mathcal{D}(\Omega) \hat{\boxtimes} \Lambda$ *, where on* $\mathcal{D}(\Omega, \Lambda)$ *we use the topology coming from* $\left(\Lambda(\mathbb{R}^q) \otimes \mathcal{C}^\infty\left(\Omega^\#\right)\right) \hat{\boxtimes} \Lambda$ *via the isomorphism* Ψ *.*

Proof: b) (i) $\Leftrightarrow$ (ii) : trivial.

a) Let $f \in \mathcal{D}(\Omega)$. Then by b) (i) $\Leftrightarrow$ (ii) there exists a family $(f_S)_{S \in \wp(q)} \in \mathcal{C}^\infty\left(\Omega^\#\right)^{\wp(q)}$ such that $f^* = \sum_{S \in \wp(q)} \zeta^S \widehat{f_S}$.
So

$$\left(\partial_{i|} f\right)^* = \partial_{i|} f^* = \sum_{S \in \wp(q)} \zeta^S \widehat{\partial_i f_S} \in \mathcal{D}(\Omega)$$

for all $i = 1, \ldots, p$ and

$$\left(\partial_{|j} f\right)^* = \partial_{|j} f^* = \sum_{S \in \wp(q), j \in S} (-1)^{|S<j|} \zeta^{S \setminus \{j\}} \widehat{f_S} \in \mathcal{D}(\Omega)$$

for all $j = 1, \ldots, q$ by corollary 4.11 . $\square$

b) (ii) $\Rightarrow$ (iii) : trivial.
(iii) $\Rightarrow$ (iv) : By (iii) we have $g^*(\mathbf{x}, \mathbf{0}) \in \overline{\langle \mathbf{x}, \mathbf{0} \rangle} = \mathbb{R}$ for all $g \in \mathcal{D}(\Omega)$ and $\mathbf{x} \in \Omega^\#$. Since $\mathcal{D}(\Omega)$ closed under derivation by a) we get (iv) . $\square$

(iv) $\Rightarrow$ (ii) : By corollary 4.12 we can write

$$f^* = \sum_{I \in \wp} \sum_{S \in \wp(q)} \zeta^S \widehat{f_{IS}} E_I ,$$

where $(f_{IS})_{I \in \wp, S \in \wp(q)} \in \left(\mathcal{C}^\infty\left(\Omega^\#\right)\right)^{\wp \times \wp(q)}$. So let $R = \{j_1, \ldots, j_r\} \in \wp(q)$ with $j_1 < \cdots < j_r$ and $\mathbf{x} \in \Omega^\#$. Then

$$\partial_{|j_1} \cdots \partial_{|j_r} f^*(\mathbf{x}, \mathbf{0}) = \sum_{I \in \wp} f_{IR}(\mathbf{x}) E_I \in \mathbb{R}$$

by (iv) . Since all $f_{IR}(\mathbf{x}, \mathbf{0}) \in \mathbb{R}$ this means $f_{IR}(\mathbf{x}) = 0$ for all $I \in \wp \setminus \{\emptyset\}$. Because R and $\mathbf{x}$ are arbitrary we have

$$f^* = \sum_{S \in \wp(q)} \zeta^S \widehat{f_{\emptyset S}} . \square$$

c) trivial since it is the restriction of the isomorphism Ψ .
d) trivial since $\Psi(1 \otimes 1 \otimes E_I) = E_I$. $\square$

The next two theorems show that $\left(\Omega^{\#}\right)^{|q}$ is essentially determined by the algebra $\mathcal{D}(\Omega)$.

Let $U \underset{\text{open}}{\subset} \mathbb{R}^n$ and $\mathcal{S}\left(\mathcal{C}^\infty(U)\right)$ be the spectrum of $\mathcal{C}^\infty(U)$, which is the set of all unital algebra homomorphisms $\eta : \mathcal{C}^\infty(U) \to \mathbb{R}$. Then from analysis we know that there is a canonical bijection

$$U \to \mathcal{S}\left(\mathcal{C}^\infty(U)\right) \ , \ \mathbf{a} \mapsto \eta_{\mathbf{a}}$$

with $\eta_{\mathbf{a}}(f) := f(\mathbf{a})$ for all $f \in \mathcal{C}^\infty(U)$ and $\mathbf{a} \in U$. There is an analogous result for $\mathcal{D}(\Omega)$:

Let $\mathcal{S}\left(\mathcal{D}(\Omega, \Lambda)\right)$ be the set of all graded algebra homomorphisms $\psi : \mathcal{D}(\Omega, \Lambda) \to \Lambda$ being the identity on Λ , and let $\mathcal{S}\left(\mathcal{D}(\Omega)\right)$ be the set of all unital graded algebra homomorphisms $\eta : \mathcal{D}(\Omega) \to \Lambda$.

Theorem 4.15 *We have bijections*

$$\left(\Omega^{\#}\right)^{|q} \to \mathcal{S}\left(\mathcal{D}(\Omega, \Lambda)\right) \ , \ (\mathbf{b}, \beta) \mapsto \psi_{(\mathbf{b},\beta)} \ ,$$

where $\psi_{(\mathbf{b},\beta)}(f) := f^*(\mathbf{b}, \beta)$ *for all* $f \in \mathcal{D}(\Omega, \Lambda)$ *, and*

$$\left(\Omega^{\#}\right)^{|q} \to \mathcal{S}\left(\mathcal{D}(\Omega)\right) \ , \ (\mathbf{b}, \beta) \mapsto \eta_{(\mathbf{b},\beta)} \ ,$$

where $\eta_{(\mathbf{b},\beta)}(g) := g^*(\mathbf{b}, \beta)$ *for all* $g \in \mathcal{D}(\Omega)$ *and* $(\mathbf{b}, \beta) \in \left(\Omega^{\#}\right)^{|q}$ *.*

Proof: For proving surjectivity let first $\psi \in \mathcal{S}\left(\mathcal{D}(\Omega, \Lambda)\right)$, $b_i := \psi(z_i) \in \Lambda_0$, $i = 1, \ldots, p$, $\mathbf{b} := (b_1, \ldots, b_p)$, $\beta_j := \psi(\zeta_i) \in \Lambda_1$, $j = 1, \ldots, q$, and $\beta := (\beta_1, \ldots, \beta_q)$. Since $\mathcal{C}^\infty\left(\Omega^{\#}\right) \to \mathbb{R}$, $h \mapsto \psi\left(\widehat{h}\right)^{\#}$ is again a unital algebra homomorphism, there exists $\mathbf{a} \in \Omega^{\#}$ such that $\psi\left(\widehat{h}\right)^{\#} = h(\mathbf{a})$ for all $h \in \mathcal{C}^\infty(U)$. So $\mathbf{b}^{\#} = \mathbf{a}$, and so $(\mathbf{b}, \beta) \in \left(\Omega^{\#}\right)^{|q}$. Now let $f \in \mathcal{D}(\Omega, \Lambda)$. Then by lemma 4.13 (i) there exist $\Delta_1, \ldots, \Delta_p$, $\Sigma_1, \ldots, \Sigma_q \in \mathcal{D}\left(\left(\Omega^{\#}\right)^{|q}, \Lambda\right)$ such that for all $(\mathbf{z}, \zeta) \in \left(\Omega^{\#}\right)^{|q}$

$$f^*(\mathbf{z}, \zeta) = f^*(\mathbf{b}, \beta) + \sum_{i=1}^{p} (z_i - b_i)\, \Delta_i(\mathbf{z}, \zeta) + \sum_{j=1}^{q} (\zeta_j - \beta_j)\, \Sigma_j(\mathbf{z}, \zeta) \,.$$

We see that $\psi(f) = f^*(\mathbf{b}, \beta)$. This proves $\psi = \psi_{(\mathbf{b},\beta)}$.

Now let $\eta \in \mathcal{S}\left(\mathcal{D}(\Omega)\right)$. Then by Λ-linear extension there exists $\psi \in \mathcal{S}\left(\mathcal{D}(\Omega, \Lambda)\right)$ such that $\eta = \psi|_{\mathcal{D}(\Omega)}$. So there exists $(\mathbf{b}, \beta) \in \left(\Omega^{\#}\right)^{|q}$ such that $\psi = \psi_{(\mathbf{b},\beta)}$, and so $\eta = \eta_{(\mathbf{b},\beta)}$.

Injectivity is clear because all coordinate functions $z_1, \ldots, z_p$, $\zeta_1, \ldots, \zeta_q$ belong to $\mathcal{D}\left(\left(\Omega^{\#}\right)^{|q}\right)$. $\square$

Definition 4.16 *Let $\varphi : \Omega \to \Omega'$.*

(i) φ is called a $\mathcal{D}$-map if and only if all component functions $\varphi_{i|} : \Omega \to \mathbb{R}^{1|0}$, $i = 1, \ldots, r$, and $\varphi_{|j} : \Omega \to \mathbb{R}^{0|1}$, $j = 1, \ldots, s$, belong to $\mathcal{D}(\Omega)$.

(ii) φ is called a diffeomorphism if and only if φ is bijective and φ and φ^{-1} are $\mathcal{D}$-maps.

Clearly if $\varphi : \Omega \to \Omega'$ is a $\mathcal{D}$-map then there exists a unique extension $\varphi^* : \left(\Omega^{\#}\right)^{|q} \to \left(\Omega'^{\#}\right)^{|s}$ of φ which is as well a $\mathcal{D}$-map. For all $i = 1, \ldots, r$ and $j = 1, \ldots, s$ we have $(\varphi^*)_{i|} = \varphi^*_{i|}$ and $(\varphi^*)_{|j} = \varphi^*_{|j}$.
In the super case a chain rule holds as well:

Proposition 4.17 *Let $f \in \mathcal{D}(\Omega')$ and*

$$\varphi = (\varphi_{1|}, \ldots, \varphi_{r|}, \varphi_{|1}, \ldots, \varphi_{|s}) : \Omega \to \Omega'$$

be a $\mathcal{D}$-map. Then $f \circ \varphi \in \mathcal{D}(\Omega)$, and for all $i = 1, \ldots, p$ and $j = 1, \ldots, q$ we have

$$\partial_{i|} (f \circ \varphi) = \sum_{k=1}^{r} \left(\partial_{i|}\varphi_{k|}\right) \left((\partial_{k|} f) \circ \varphi\right) + \sum_{l=1}^{s} \left(\partial_{i|}\varphi_{|l}\right) \left((\partial_{|l} f) \circ \varphi\right)$$

and

$$\partial_{|j} (f \circ \varphi) = \sum_{k=1}^{r} \left(\partial_{|j}\varphi_{k|}\right) \left((\partial_{k|} f) \circ \varphi\right) + \sum_{l=1}^{s} \left(\partial_{|j}\varphi_{|l}\right) \left((\partial_{|l} f) \circ \varphi\right) .$$

Proof: Let $(\mathbf{b}, \beta) \in \Omega$. Then $\varphi(\mathbf{b}, \beta) \in \Omega'$, and so by lemma 4.13 (i) there exist $\Delta_1, \ldots, \Delta_r$, $\Sigma_1, \ldots, \Sigma_s \in \mathcal{D}(\Omega', \Lambda)$ such that for all $(\mathbf{z}', \zeta') \in \Omega'$

$$\begin{aligned} f\left(\mathbf{z}', \zeta'\right) &= f\left(\varphi(\mathbf{b}, \beta)\right) + \sum_{k=1}^{r} \left(z'_k - \varphi_{k|}(\mathbf{b}, \beta)\right) \Delta_k\left(\mathbf{z}', \zeta'\right) \\ &\quad + \sum_{l=1}^{s} \left(\zeta'_l - \varphi_{|l}(\mathbf{b}, \beta)\right) \Sigma_l\left(\mathbf{z}', \zeta'\right) , \end{aligned}$$

and there exist R_{ki} , S_{kj} , T_{li} and $U_{lj} \in \mathcal{D}(\Omega, \Lambda)$, $i = 1, \ldots, p$, $j = 1, \ldots, q$, $k = 1, \ldots, r$, $l = 1, \ldots, s$, such that for all $k = 1, \ldots, r$, $l = 1, \ldots, s$ and $(\mathbf{z}, \zeta) \in \Omega$

$$\varphi_{k|}(\mathbf{z}, \zeta) = \varphi_{k|}(\mathbf{b}, \beta) + \sum_{i=1}^{p} (z_i - b_i) R_{ki}(\mathbf{z}, \zeta) + \sum_{j=1}^{q} (\zeta_j - \beta_j) S_{kj}(\mathbf{z}, \zeta) ,$$

and

$$\varphi_{|l}(\mathbf{z},\zeta) = \varphi_{|l}(\mathbf{b},\beta) + \sum_{i=1}^{p} (z_i - b_i)\, T_{li}(\mathbf{z},\zeta) + \sum_{j=1}^{q} (\zeta_j - \beta_j)\, U_{lj}(\mathbf{z},\zeta)\,.$$

So for all $(\mathbf{z},\zeta) \in \Omega$ we get

$$\begin{aligned}
&(f\circ\varphi)(\mathbf{z},\zeta) = (f\circ\varphi)(\mathbf{b},\beta) + \sum_{k=1}^{r} \left(\varphi_{k|}(\mathbf{z},\zeta) - \varphi_{k|}(\mathbf{b},\beta)\right) \Delta_k\left(\varphi(\mathbf{z},\zeta)\right) \\
&\quad + \sum_{l=1}^{s} \left(\varphi_{|l}(\mathbf{z},\zeta) - \varphi_{|l}(\mathbf{b},\beta)\right) \Sigma_l\left(\varphi(\mathbf{z},\zeta)\right) \\
&= (f\circ\varphi)(\mathbf{b},\beta) \\
&+ \sum_{i=1}^{p} (z_i - b_i) \left(\sum_{k=1}^{r} R_{ki}(\mathbf{z},\zeta)\Delta_k\left(\varphi(\mathbf{z},\zeta)\right) + \sum_{l=1}^{s} T_{li}(\mathbf{z},\zeta)\Sigma_l\left(\varphi(\mathbf{z},\zeta)\right)\right) \\
&+ \sum_{j=1}^{q} (\zeta_j - \beta_j) \left(\sum_{k=1}^{r} S_{kj}(\mathbf{z},\zeta)\Delta_k\left(\varphi(\mathbf{z},\zeta)\right) + \sum_{l=1}^{s} U_{lj}(\mathbf{z},\zeta)\Sigma_l\left(\varphi(\mathbf{z},\zeta)\right)\right),
\end{aligned}$$

so $f\circ\varphi$ is differentiable and so $f\circ\varphi \in \mathcal{D}(\Omega,\Lambda)$. For all $i = 1,\ldots,p$ and $j = 1,\ldots,q$

$$\begin{aligned}
\partial_{i|}\left(f\circ\varphi\right)(\mathbf{b},\beta) &= \sum_{k=1}^{r} R_{ki}(\mathbf{b},\beta)\Delta_k\left(\varphi(\mathbf{b},\beta)\right) + \sum_{l=1}^{s} T_{li}(\mathbf{b},\beta)\Sigma_l\left(\varphi(\mathbf{b},\beta)\right) \\
&= \sum_{k=1}^{r} \partial_{i|}\varphi_{k|}(\mathbf{b},\beta)\partial_{k|}f\left(\varphi(\mathbf{b},\beta)\right) \\
&\quad + \sum_{l=1}^{s} \partial_{i|}\varphi_{|l}(\mathbf{b},\beta)\partial_{|l}f\left(\varphi(\mathbf{b},\beta)\right)
\end{aligned}$$

and

$$\begin{aligned}
\partial_{|j}\left(f\circ\varphi\right)(\mathbf{b},\beta) &= \sum_{k=1}^{r} S_{kj}(\mathbf{b},\beta)\Delta_k\left(\varphi(\mathbf{b},\beta)\right) + \sum_{l=1}^{s} U_{lj}(\mathbf{b},\beta)\Sigma_l\left(\varphi(\mathbf{b},\beta)\right) \\
&= \sum_{k=1}^{r} \partial_{|j}\varphi_{k|}(\mathbf{b},\beta)\partial_{k|}f\left(\varphi(\mathbf{b},\beta)\right) \\
&\quad + \sum_{l=1}^{s} \partial_{|j}\varphi_{|l}(\mathbf{b},\beta)\partial_{|l}f\left(\varphi(\mathbf{b},\beta)\right)\,.
\end{aligned}$$

For proving $f\circ\varphi \in \mathcal{D}(\Omega)$ let $(\mathbf{z},\zeta) \in \Omega$. Then since all component functions of φ belong to $\mathcal{D}(\Omega)$ and $f \in \mathcal{D}(\Omega')$ we have

$$(f \circ \varphi)(\mathbf{z}, \zeta) \in \overline{\left\langle \varphi_{1|}(\mathbf{z}, \zeta), \ldots, \varphi_{r|}(\mathbf{z}, \zeta), \varphi_{|1}(\mathbf{z}, \zeta), \ldots, \varphi_{|s}(\mathbf{z}, \zeta) \right\rangle} \subset \overline{\langle \mathbf{z}, \zeta \rangle} \,. \square$$

So every $\mathcal{D}$-map $\varphi : \Omega \to \Omega'$ induces a unital graded algebra homomorphism $\Phi : \mathcal{D}(\Omega') \to \mathcal{D}(\Omega)$, $f \mapsto f \circ \varphi$. The converse is almost true.

Theorem 4.18 *Let $\Phi : \mathcal{D}(\Omega') \to \mathcal{D}(\Omega)$ be a unital graded algebra homomorphism. Then*

(i) there exists a unique map $\varphi : \Omega \to \left(\Omega'^{\#}\right)^{|s}$ such that $\Phi(f) = f^* \circ \varphi$ for all $f \in \mathcal{D}(\Omega')$,

(ii) φ is a $\mathcal{D}$-map, and $\Phi(f)^* = f^* \circ \varphi^*$ for all $f \in \mathcal{D}(\Omega')$,

(iii) Φ is an isomorphism if and only if φ^* is a diffeomorphism.

Proof: (i) For proving uniqueness let

$$\varphi = (\varphi_{1|}, \ldots, \varphi_{r|}, \varphi_{|1}, \ldots, \varphi_{|s}) : \Omega \to \left(\Omega'^{\#}\right)^{|s}$$

with component functions $\varphi_{1|}, \ldots, \varphi_{r|} \in \Lambda_0^{\Omega}$ and $\varphi_{|1}, \ldots, \varphi_{|s} \in \Lambda_1^{\Omega}$ such that $\Phi(f)^* = f^* \circ \varphi$ for all $f \in \mathcal{D}(\Omega')$. Let z'_i , $i = 1, \ldots, r$, and $\zeta'_j \in \mathcal{D}(\Omega')$, $j = 1, \ldots, s$, be the coordinate functions on Ω' . Then $\varphi_{i|} = {z'_i}^* \circ \varphi = \Phi(z'_i)$ and $\varphi_{|j} = {\zeta'_j}^* \circ \varphi = \Phi(\zeta'_j)$ for all $i = 1, \ldots, r$ and $j = 1, \ldots, s$.

For proving existence notice that since Φ is a unital graded algebra homomorphism it induces a map

$$\mathcal{S}(\mathcal{D}(\Omega)) \to \mathcal{S}\left(\mathcal{D}(\Omega')\right) \,,\, \eta \mapsto \eta \circ \Phi$$

and so by theorem 4.15 a map $\widetilde{\varphi} : \left(\Omega^{\#}\right)^{|q} \to \left(\Omega'^{\#}\right)^{|q}$ such that $\eta_{(\mathbf{b},\beta)} \circ \Phi = \eta_{\widetilde{\varphi}(\mathbf{b},\beta)}$ for all $(\mathbf{b}, \beta) \in \left(\Omega^{\#}\right)^{|q}$. So for all $f \in \mathcal{D}(\Omega')$ and $(\mathbf{b}, \beta) \in \left(\Omega^{\#}\right)^{|q}$

$$\Phi(f)^*(\mathbf{b}, \beta) = \left(\eta_{(\mathbf{b},\beta)} \circ \Phi\right)(f) = \eta_{\widetilde{\varphi}(\mathbf{b},\beta)}(f) = f^*\left(\widetilde{\varphi}(\mathbf{b}, \beta)\right) ,$$

and so $\Phi(f)^* = f^* \circ \widetilde{\varphi}$. If we define $\varphi := \widetilde{\varphi}|_{\Omega}$ we get $\Phi(f) = f^* \circ \varphi$ for all $f \in \mathcal{D}(\Omega')$. $\square$

(ii) Let

$$\widetilde{\varphi} = (\widetilde{\varphi}_{1|}, \ldots, \widetilde{\varphi}_{r|}, \widetilde{\varphi}_{|1}, \ldots, \widetilde{\varphi}_{|s}) : \left(\Omega^{\#}\right)^{|q} \to \left(\Omega'^{\#}\right)^{|s}$$

with component functions $\widetilde{\varphi}_{1|}, \ldots, \widetilde{\varphi}_{r|} \in \Lambda_0^{\Omega}$ and $\widetilde{\varphi}_{|1}, \ldots, \widetilde{\varphi}_{|s} \in \Lambda_1^{\Omega}$. Let z'_i , $i = 1, \ldots, r$, and $\zeta'_j \in \mathcal{D}\left(\left(\Omega^{\#}\right)^{|q}\right)$, $j = 1, \ldots, s$, be the coordinate functions on $\left(\Omega'^{\#}\right)^{|s}$. Then $\widetilde{\varphi}_{i|} = z'_i \circ \widetilde{\varphi} = \Phi(z'_i)^* \in \mathcal{D}\left(\left(\Omega^{\#}\right)^{|q}\right)$ and $\widetilde{\varphi}'_{|j} = \zeta_j \circ \widetilde{\varphi} = \Phi(\zeta'_j)^* \in \mathcal{D}\left(\left(\Omega^{\#}\right)^{|q}\right)$ for all $i = 1, \ldots, r$ and $j = 1, \ldots, s$.

So $\widetilde{\varphi}$ is a $\mathcal{D}$-map, and since $\varphi = \widetilde{\varphi}|_\Omega$, φ is again a $\mathcal{D}$-map and $\widetilde{\varphi} = \varphi^*$. $\square$

(iii) By theorem 4.8 we have $\mathcal{D}(\Omega) \simeq \mathcal{D}\left(\left(\Omega^{\#}\right)^{|q}\right)$ and
$\mathcal{D}(\Omega') \simeq \mathcal{D}\left(\left(\Omega'^{\#}\right)^{|s}\right)$ via $\;^*$, which are unital graded algebra isomorphisms. So Φ induces a unital graded algebra homomorphism

$$\Psi : \mathcal{D}\left(\left(\Omega'^{\#}\right)^{|s}\right) \to \mathcal{D}\left(\left(\Omega^{\#}\right)^{|q}\right) .$$

$\Psi(g) = g \circ \varphi^*$ for all $g \in \mathcal{D}\left(\left(\Omega'^{\#}\right)^{|s}\right)$ as we saw in the proof of (i) .
'$\Rightarrow$' : Let Φ be an isomorphism. Then so is Ψ .
By (i) and (ii) there exists a $\mathcal{D}$-map $\rho : \left(\Omega'^{\#}\right)^{|s} \to \left(\Omega^{\#}\right)^{|q}$ such that $\Psi^{-1}(h) = h \circ \rho$ for all $h \in \mathcal{D}\left(\left(\Omega^{\#}\right)^{|q}\right)$. So for $g \in \mathcal{D}\left(\left(\Omega'^{\#}\right)^{|s}\right)$ we have

$$g = \Psi\left(\Phi(g)\right) = g \circ \varphi^* \circ \rho .$$

So by the uniqueness in (i) we obtain $\varphi^* \circ \rho = \mathrm{id}$ and by the same calculation $\rho \circ \varphi^* = \mathrm{id}$.
'$\Leftarrow$' : Let φ^* be a diffeomorphism. Then φ^{-1} induces a unital graded algebra isomorphism

$$\Xi : \mathcal{D}\left(\left(\Omega^{\#}\right)^{|s}\right) \to \mathcal{D}\left(\left(\Omega'^{\#}\right)^{|q}\right) , \, h \mapsto h \circ \varphi^{-1} .$$

For $g \in \mathcal{D}\left(\left(\Omega'^{\#}\right)^{|s}\right)$ we have

$$\Xi\left(\Psi(g)\right) = g \circ \varphi \circ \varphi^{-1} = g .$$

So $\Xi \circ \Psi = \mathrm{id}$ and by the same calculation $\Psi \circ \Xi = \mathrm{id}$. Therefore Ψ and so Φ are isomorphisms. $\square$

To conclude this section we make the following definition:

Definition 4.19 *Let $\Omega \subset \mathbb{R}^{p|q}$. Ω is called super open in $\mathbb{R}^{p|q}$ if and only if $\Omega^{\#} \underset{open}{\subset} \mathbb{R}^p$ and $\Omega = \left(\Omega^{\#}\right)^{|q}$.*

In this case we regard $\mathcal{D}\left(\Omega\right) \simeq \Lambda\left(\mathbb{R}^q\right) \otimes \mathcal{C}^\infty\left(\Omega^{\#}\right)$ as the natural analogon of $\mathcal{C}^\infty$-functions for the super open set Ω . There are two reasons for doing so. The first is: if $U \underset{\text{open}}{\subset} \mathbb{R}^n$ then

$$\mathcal{C}^\infty(U, \Lambda) = \mathcal{C}^\infty(U)\hat{\otimes}\Lambda .$$

The second is: in the case $\Omega \subset \mathbb{R}^{p|0}$ we have $\mathcal{D}(\Omega) \simeq \mathcal{C}^\infty\left(\Omega^\#\right)$.
For all $f \in \mathcal{C}^\infty\left(\Omega^\#\right)$ we identify $\widehat{f}$ and f , and if $f \in \mathcal{D}(\Omega)$ we write

$$f = \sum_{I \in \wp(q)} \zeta^I f_I$$

with uniquely determined $f_I \in \mathcal{C}^\infty\left(\Omega^\#\right)$, $I \in \wp(q)$. On $\mathcal{D}(\Omega)$ the body map now simply occurs as the restriction map

$$^\# : \mathcal{D}(\Omega) \to \mathcal{C}^\infty\left(\Omega^\#\right) , \; f \mapsto f|_{\Omega^\#} ,$$

which is a continuous unital graded algebra epimorphism.

4.2 The complex case

Now let $\mathbb{K} := \mathbb{C}$. Then $\Lambda^\mathbb{C}$ is the complexification of Λ . There is a graded involution on $\Lambda^\mathbb{C}$:

$$\overline{} : \Lambda^\mathbb{C} \to \Lambda^\mathbb{C} , \; w = \sum_{I \in \wp} a_I E_I \mapsto \overline{w} = \sum_{I \in \wp} \overline{a_{-I}} E_I .$$

Clearly $\overline{}$ is a homeomorphism. $\deg \overline{w} = \deg w$ for all $w \in \Lambda^\mathbb{C}$, $\overline{}$ commutes with the body map $^\#$, and restricted to $\mathbb{C}$ it is just the complex conjugation.

By the way: Let $\overline{}'$ be the involution on $\Lambda^\mathbb{C}$ given by theorem 2.8 (i) , in particular $\overline{w}' = w$ if $w \in \Lambda_0$ and $\overline{w}' = iw$ if $w \in \Lambda_1$. Then an easy calculation shows that there exists a unique unital bicontinuous graded algebra automorphism $\rho : \Lambda^\mathbb{C} \to \Lambda^\mathbb{C}$ such that

$$\rho\left(E_{\{n\}}\right) = \frac{E_{\{n\}} - iE_{\{-n\}}}{2}$$

for all $n \in \mathbb{Z}$.

$$\rho^{-1}\left(E_{\{n\}}\right) = E_{\{n\}} + iE_{\{-n\}}$$

for all $n \in \mathbb{Z}$, and

$$\begin{array}{ccc} \Lambda^\mathbb{C} & \xrightarrow{\rho} & \Lambda^\mathbb{C} \\ {}^{\overline{}'}\uparrow & \% & \uparrow^{\overline{}} \\ \Lambda^\mathbb{C} & \xrightarrow{\rho} & \Lambda^\mathbb{C} \end{array} .$$

Let $\Lambda_r := \left(\Lambda^\mathbb{C}\right)_\mathbb{R}$ be the real part of $\Lambda^\mathbb{C}$ with respect to the graded involution $\overline{}$. Then $\rho|_\Lambda : \Lambda \to \Lambda_r$ is a bicontinuous unital graded isomorphism.

Now we define $\mathbb{C}^{p|q,\overline{q}} := \left(\Lambda_0^{\mathbb{C}}\right)^p \times \left(\Lambda_1^{\mathbb{C}}\right)^q$ for all $p, q \in \mathbb{N}$. Then again we have a body map

$$\begin{aligned} \# : \quad & \mathbb{C}^{p|q,\overline{q}} \to \mathbb{C}^p , \\ & (\mathbf{w}, \vartheta) = (w_1, \ldots, w_p, \vartheta_1, \ldots, \vartheta_q) \mapsto (\mathbf{w}, \vartheta)^\# := \left(w_1^\#, \ldots, w_p^\#\right) = \mathbf{w}^\# . \end{aligned}$$

Let $U \underset{\text{open}}{\subset} \mathbb{C}^p$. Then we define $U^{|q,\overline{q}} := \left\{ (\mathbf{w}, \vartheta) \in \mathbb{C}^{p|q,\overline{q}} \,\middle|\, (\mathbf{w}, \vartheta)^\# \in U \right\}$. We have $U^{|q,\overline{q}} \underset{\text{open}}{\subset} \mathbb{C}^{p|q,\overline{q}}$ and $\left(U^{|q,\overline{q}}\right)^\# = U$. Conversely if $\Omega \underset{\text{open}}{\subset} \mathbb{C}^{p|q,\overline{q}}$ then clearly $\Omega^\# \underset{\text{open}}{\subset} \mathbb{C}^p$ and $\Omega \subset \left(\Omega^\#\right)^{|q,\overline{q}}$.

If $M \subset \mathbb{C}^{p|q,\overline{q}}$, the set $\left(\Lambda^{\mathbb{C}}\right)^M = \left(\Lambda_0^{\mathbb{C}}\right)^M \oplus \left(\Lambda_1^{\mathbb{C}}\right)^M$ of all functions $f : M \to \Lambda^{\mathbb{C}}$ forms a unital associative graded commutative algebra by pointwise addition and multiplication, and we consider $\Lambda^{\mathbb{C}}$ as the sub graded algebra of $\left(\Lambda^{\mathbb{C}}\right)^M$ containing precisely the constant functions. Then clearly $\mathcal{C}\left(M, \Lambda^{\mathbb{C}}\right)$ is a sub graded algebra of $\left(\Lambda^{\mathbb{C}}\right)^M$ containing $\Lambda^{\mathbb{C}}$.

Theorem 4.20 *For each $U \underset{open}{\subset} \mathbb{C}^p$ we have an algebra embedding*

$$\widehat{\ } : \mathcal{C}^\infty(U)^{\mathbb{C}} \hookrightarrow \mathcal{C}\left(U^{|0,\overline{0}}, \Lambda_0^{\mathbb{C}}\right) , \; f \mapsto \widehat{f}$$

where for all $\mathbf{w} \in U^{|0,\overline{0}}$

$$\widehat{f}(\mathbf{w}) := \sum_{\mathbf{m},\mathbf{n} \in \mathbb{N}^p} \frac{1}{\mathbf{m}!\mathbf{n}!} \partial^{\mathbf{m}} \overline{\partial}^{\mathbf{n}} f\left(\mathbf{w}^\#\right) \left(\mathbf{w} - \mathbf{w}^\#\right)^{\mathbf{m}} \overline{\left(\mathbf{w} - \mathbf{w}^\#\right)}^{\mathbf{n}} .$$

Clearly again $\widehat{f}|_U = f$, and if $\partial^{\mathbf{n}} f_k \rightsquigarrow \partial^{\mathbf{n}} f$ and $\overline{\partial}^{\mathbf{n}} f_k \rightsquigarrow \overline{\partial}^{\mathbf{n}} f$ pointwise for all $\mathbf{n} \in \mathbb{N}^p$ then again $\widehat{f_k} \rightsquigarrow \widehat{f}$ pointwise.

Proof: same as in the real case since $\deg \overline{\mathbf{w}} = \deg \mathbf{w}$ for all $\mathbf{w} \in \Lambda^{\mathbb{C}}$.

Again we have a sheef embedding $\left(\mathcal{C}^\infty_{\mathbb{C}^p}\right)^{\mathbb{C}} \hookrightarrow \mathcal{C}\left(\diamondsuit^{|0,\overline{0}}, \Lambda_0^{\mathbb{C}}\right)$.

Lemma 4.21 *Let $U \underset{open}{\subset} \mathbb{C}^p$, $f \in \mathcal{C}^\infty(U)^{\mathbb{C}}$ and $\mathbf{b} \in U^{|0,\overline{0}}$. Then for all $\mathbf{w} \in \mathbb{C}^{p|0,\overline{0}}$ with $\mathbf{w}^\# = 0$*

$$\widehat{f}(\mathbf{b} + \mathbf{w}) = \sum_{\mathbf{m},\mathbf{n} \in \mathbb{N}^p} \frac{1}{\mathbf{m}!\mathbf{n}!} \widehat{\partial^{\mathbf{m}} \overline{\partial}^{\mathbf{n}} f}\left(\mathbf{b}\right) \mathbf{w}^{\mathbf{m}} \overline{\mathbf{w}}^{\mathbf{n}} .$$

Proof: same as in the real case.

Lemma 4.22 *Let*

$$f(\mathbf{z}) := \sum_{\mathbf{m},\mathbf{n} \in \mathbb{N}^p} \frac{a_{\mathbf{mn}}}{\mathbf{m}!\mathbf{n}!} \left(\mathbf{z} - \mathbf{c}\right)^{\mathbf{m}} \overline{\left(\mathbf{z} - \mathbf{c}\right)}^{\mathbf{n}}$$

be a power series convergent in $U \subset \mathbb{C}^p$ with $\mathbf{c} \in \mathbb{C}^p$ and all $a_{\mathbf{mn}} \in \mathbb{C}$. Then for all $\mathbf{w} \in U^{|0,\bar{0}}$

$$\widehat{f}(\mathbf{w}) = \sum_{\mathbf{m},\mathbf{n} \in \mathbb{N}^p} \frac{a_{\mathbf{mn}}}{\mathbf{m}!\mathbf{n}!} (\mathbf{w} - \mathbf{c})^{\mathbf{m}} \overline{(\mathbf{w} - \mathbf{c})}^{\mathbf{n}} .$$

Proof: same as in the real case.

There is a canonical restriction of this construction to $(\Lambda_r)_0$:

Lemma 4.23 *Let $U \underset{open}{\subset} \mathbb{C}^p$ and $f \in \mathcal{C}^\infty(U)^{\mathbb{C}}$. Then*

(i) if $U \cap \mathbb{R}^p \neq \emptyset$ and $g := f|_{U \cap \mathbb{R}^p}$ we have

$$\widehat{f}(\mathbf{w}) = \sum_{\mathbf{n} \in \mathbb{N}^p} \frac{1}{\mathbf{n}!} \partial^{\mathbf{n}} g \left(\mathbf{w}^\# \right) \left(\mathbf{w} - \mathbf{w}^\# \right)^{\mathbf{n}}$$

for all $\mathbf{w} \in U^{|0,\bar{0}} \cap (\Lambda_r)_0^p$.

(ii) $\widehat{\overline{f}} = \overline{\widehat{f}}$, and so if $f(U) \subset \mathbb{R}$ then $f \left(U^{|0,\bar{0}} \right) \subset (\Lambda_r)_0$.

Proof: (i) Let $\mathbf{w} \in U^{|0,\bar{0}} \cap (\Lambda_r)_0^p$. Then

$$\widehat{f}(\mathbf{w}) = \sum_{\mathbf{m},\mathbf{n} \in \mathbb{N}^p} \frac{1}{\mathbf{m}!\mathbf{n}!} \partial^{\mathbf{m}} \overline{\partial}^{\mathbf{n}} f \left(\mathbf{w}^\# \right) \left(\mathbf{w} - \mathbf{w}^\# \right)^{\mathbf{m}+\mathbf{n}} .$$

Since for each fixed component the sum over $\mathbf{m}$ and $\mathbf{n}$ is finite we can interchange the order of summation. So

$$\begin{aligned}
\widehat{f}(\mathbf{w}) &= \sum_{\mathbf{r} \in \mathbb{N}^p} \sum_{\mathbf{n} \in \mathbb{N}^p, \mathbf{n} \leq \mathbf{r}} \frac{1}{(\mathbf{r} - \mathbf{n})!\mathbf{n}!} \partial^{\mathbf{r}-\mathbf{n}} \overline{\partial}^{\mathbf{n}} f \left(\mathbf{w}^\# \right) \left(\mathbf{w} - \mathbf{w}^\# \right)^{\mathbf{r}} \\
&= \sum_{\mathbf{r} \in \mathbb{N}^p} \frac{1}{\mathbf{r}!} \sum_{\mathbf{n} \in \mathbb{N}^p, \mathbf{n} \leq \mathbf{r}} \binom{\mathbf{r}}{\mathbf{n}} \partial^{\mathbf{r}-\mathbf{n}} \overline{\partial}^{\mathbf{n}} f \left(\mathbf{w}^\# \right) \left(\mathbf{w} - \mathbf{w}^\# \right)^{\mathbf{r}} \\
&= \sum_{\mathbf{r} \in \mathbb{N}^p} \frac{1}{\mathbf{r}!} \left(\partial + \overline{\partial} \right)^{\mathbf{r}} f \left(\mathbf{w}^\# \right) \left(\mathbf{w} - \mathbf{w}^\# \right)^{\mathbf{r}} \\
&= \sum_{\mathbf{r} \in \mathbb{N}^p} \frac{1}{\mathbf{r}!} \partial^{\mathbf{r}} g \left(\mathbf{w}^\# \right) \left(\mathbf{w} - \mathbf{w}^\# \right)^{\mathbf{r}} . \square
\end{aligned}$$

(ii) Let $\mathbf{w} \in U^{|0,\bar{0}}$. Then

$$\begin{aligned}
\widehat{\overline{f}}(\mathbf{w}) &= \sum_{\mathbf{m},\mathbf{n} \in \mathbb{N}^p} \frac{1}{\mathbf{m}!\mathbf{n}!} \partial^{\mathbf{m}} \overline{\partial}^{\mathbf{n}} \overline{f} \left(\mathbf{w}^\# \right) \left(\mathbf{w} - \mathbf{w}^\# \right)^{\mathbf{m}} \overline{\left(\mathbf{w} - \mathbf{w}^\# \right)}^{\mathbf{n}} \\
&= \overline{\sum_{\mathbf{m},\mathbf{n} \in \mathbb{N}^p} \frac{1}{\mathbf{m}!\mathbf{n}!} \overline{\partial}^{\mathbf{m}} \partial^{\mathbf{n}} f \left(\mathbf{w}^\# \right) \left(\mathbf{w} - \mathbf{w}^\# \right)^{\mathbf{n}} \overline{\left(\mathbf{w} - \mathbf{w}^\# \right)}^{\mathbf{m}}} \\
&= \overline{\widehat{f}(\mathbf{w})} . \square
\end{aligned}$$

As in the real case for the following we need some notations:
We set $(\mathbf{w}, \vartheta) = (w_1, \ldots, w_p, \vartheta_1, \ldots \vartheta_q) \in \Omega$ with for all $i = 1, \ldots, p$ respectively $j = 1, \ldots, q$: $w_i = \sum_{J \in \wp_0} z_{iJ} E_J \in \mathbb{C}^{1|0,\overline{0}}$,
$\vartheta_j = \sum_{J \in \wp_1} r_{jJ} E_J \in \mathbb{C}^{0|1,\overline{1}}$, all $z_{iJ}, r_{jJ} \in \mathbb{C}$. We define

$$\vartheta^S := \vartheta_{s_1} \cdots \vartheta_{s_r}$$

for all $S = \{s_1, \ldots, s_r\} \subset \{1, \ldots, q\}$ and $s_1 < \cdots < s_r$.

If $f \in \left(\Lambda^{\mathbb{C}}\right)^{\Omega}$ then we write $f = \sum_{I \in \wp} F_I E_I$ with uniquely determined component functions $F_I : \Omega \to \mathbb{C}$. Let $(\mathbf{b}, \beta) \in \Omega$ and $I \in \wp$. If $i \in \{1, \cdots, p\}$ and $J \in \wp_0$, and $F_I(\mathbf{w}, \vartheta)$ is partially differentiable with respect to z_{iJ} at $(\mathbf{b}, \beta)$ then we define

$$\partial_{i|J} F_I(\mathbf{b}, \beta) := \left.\frac{\partial F_I(\mathbf{w}, \vartheta)}{\partial r_{iJ}}\right|_{(\mathbf{w},\vartheta)=(\mathbf{b},\beta)}$$

and

$$\overline{\partial}_{i|J} F_I(\mathbf{b}, \beta) := \left.\frac{\partial F_I(\mathbf{w}, \vartheta)}{\partial \overline{z_{iJ}}}\right|_{(\mathbf{w},\vartheta)=(\mathbf{b},\beta)} .$$

And also if $j \in \{1, \cdots, q\}$, $J \in \wp_1$ and $F_I(\mathbf{w}, \vartheta)$ differentiable with respect to r_{jJ} at $(\mathbf{b}, \beta)$ we define

$$\partial_{|jJ} F_I(\mathbf{b}, \beta) := \left.\frac{\partial F_I(\mathbf{w}, \vartheta)}{\partial r_{jJ}}\right|_{(\mathbf{w},\vartheta)=(\mathbf{b},\beta)}$$

and

$$\overline{\partial}_{|jJ} F_I(\mathbf{b}, \beta) := \left.\frac{\partial F_I(\mathbf{w}, \vartheta)}{\partial \overline{r_{jJ}}}\right|_{(\mathbf{w},\vartheta)=(\mathbf{b},\beta)} .$$

Definition 4.24 *Let* $\Omega \underset{open}{\subset} \mathbb{C}^{p|q,\overline{q}}$, $f : \Omega \to \Lambda^{\mathbb{C}}$ *and* $(\mathbf{b}, \beta) \in \Omega$. f *is called differentiable at* $(\mathbf{b}, \beta)$ *if and only if there exist* $\Omega' \underset{open}{\subset} \Omega$ *such that* $(\mathbf{b}, \beta) \in \Omega'$ *and* $\Delta_i, \Delta_i', \Sigma_j, \Sigma_j' : \Omega' \to \Lambda^{\mathbb{C}}$, $i = 1, \ldots, p$, $j = 1, \ldots, q$, *continuous at* $(\mathbf{b}, \beta)$ *such that for all* $(\mathbf{w}, \vartheta) \in \Omega'$

$$\begin{aligned} f(\mathbf{w}, \vartheta) &= f(\mathbf{b}, \beta) + \sum_{i=1}^{p} (w_i - b_i)\, \Delta_i(\mathbf{w}, \vartheta) + \sum_{i=1}^{p} \overline{(w_i - b_i)} \Delta_i'(\mathbf{w}, \vartheta) \\ &\quad + \sum_{j=1}^{q} (\vartheta_i - \beta_i)\, \Sigma_j(\mathbf{w}, \vartheta) + \sum_{j=1}^{q} \overline{(\vartheta_i - \beta_i)} \Sigma_j'(\mathbf{w}, \vartheta) . \end{aligned}$$

If f *is differentiable at* $(\mathbf{b}, \beta)$ *then we call* $\partial_{i|} f(\mathbf{b}, \beta) := \Delta_i(\mathbf{b}, \beta)$, $\overline{\partial}_{i|} f(\mathbf{b}, \beta) := \Delta_i'(\mathbf{b}, \beta)$, $i = 1, \ldots, p$, $\partial_{|j} f(\mathbf{b}, \beta) := \Sigma_j(\mathbf{b}, \beta)$, $\overline{\partial}_{|j} f(\mathbf{b}, \beta) := \Sigma_j'(\mathbf{b}, \beta)$, $j = 1, \ldots, q$, *the partial derivatives of* f *at*

$(\mathbf{b}, \beta)$. *If* $\overline{\partial}_{i|} f(\mathbf{b}, \beta) = \overline{\partial}_{|j} f(\mathbf{b}, \beta) = 0$ *for all* $i = 1, \ldots, p$, $j = 1, \ldots, q$ *then* f *is called complex differentiable at* $(\mathbf{b}, \beta)$.

If f *is differentiable at each* $(\mathbf{b}, \beta) \in \Omega$ *then* f *is said to be differentiable, and* $\partial_{i|} f, \overline{\partial}_{i|} f : \Omega \to \Lambda^{\mathbb{C}}$, $i = 1, \ldots, p$, *and* $\partial_{|j} f, \overline{\partial}_{|j} f : \Omega \to \Lambda^{\mathbb{C}}$, $j = 1, \ldots, q$, *are called the partial derivatives of* f . *If* f *is complex differentiable at each* $(\mathbf{b}, \beta) \in \Omega$ *then* f *is said to be holomorphic.*

Notice that again $\partial_{i|} f(\mathbf{b}, \beta)$, $\overline{\partial}_{i|} f(\mathbf{b}, \beta)$, $i \in \{1, \ldots, p\}$, and $\partial_{|j} f(\mathbf{b}, \beta)$, $\overline{\partial}_{|j} f(\mathbf{b}, \beta)$, $j \in \{1, \ldots, q\}$, are well-defined. This can be proven by calculations similar to that in the real case.

If $\Omega \underset{\text{open}}{\subset} \mathbb{C}^{p|q, \overline{q}}$, then clearly the set of all differentiable mappings $\Omega \to \Lambda^{\mathbb{C}}$ forms a sub graded algebra of $\mathcal{C}\left(\Omega, \Lambda^{\mathbb{C}}\right)$ invariant under $\overline{}$, which contains the holomorphic mappings $\Omega \to \Lambda^{\mathbb{C}}$ as a graded sub algebra, both containing $\Lambda^{\mathbb{C}}$.

All $\partial_{i|}$, $\overline{\partial}_{i|}$, $\partial_{|j}$ and $\overline{\partial}_{|j}$ are 0 on $\Lambda^{\mathbb{C}}$, and again a super product rule holds:

$$\left(\partial_{i|} f\right)^{\cdot} = \left(\overline{\partial}_{i|} f\right)^{\cdot} = \dot{f} \; , \; \left(\partial_{|j} f\right)^{\cdot} = \left(\overline{\partial}_{|j} f\right)^{\cdot} = \dot{f} + 1 \, ,$$

$$\partial_{i|}\left(fg\right) = \left(\partial_{i|} f\right) g + f\left(\partial_{i|} g\right) \; , \; \overline{\partial}_{i|}\left(fg\right) = \left(\overline{\partial}_{i|} f\right) g + f\left(\overline{\partial}_{i|} g\right) \, ,$$

$$\partial_{|j}\left(fg\right) = \left(\partial_{|j} f\right) g + (-1)^{\dot{f}} f\left(\partial_{|j} g\right)$$

and

$$\overline{\partial}_{|j}\left(fg\right) = \left(\overline{\partial}_{|j} f\right) g + (-1)^{\dot{f}} f\left(\overline{\partial}_{|j} g\right)$$

for all differentiable $f, g : \Omega \to \Lambda$, f homogeneous.

If $\Omega \underset{\text{open}}{\subset} \mathbb{C}^{p|q, \overline{q}}$ then we define as in the real case $\mathcal{D}\left(\Omega, \Lambda^{\mathbb{C}}\right)$ to be the set of all $f \in \left(\Lambda^{\mathbb{C}}\right)^{\Omega}$ that are continuous with respect to $\mathbf{w}$ and partially differentiable with respect to all w_i , $i = 1, \ldots, p$, and ϑ_j , $j = 1, \ldots, q$. We define $\mathcal{O}\left(\Omega, \Lambda^{\mathbb{C}}\right)$ to be the set containing all $f \in \mathcal{D}\left(\Omega, \Lambda^{\mathbb{C}}\right)$ that are partially holomorphic with respect to each w_i and ϑ_j .
Then $\mathcal{D}\left(\Omega, \Lambda^{\mathbb{C}}\right)$ is a sub graded algebra of $\left(\Lambda^{\mathbb{C}}\right)^{\Omega}$ invariant under $\overline{}$ and containing $\Lambda^{\mathbb{C}}$, and $\mathcal{O}\left(\Omega, \Lambda^{\mathbb{C}}\right)$ is a sub graded algebra of $\mathcal{D}\left(\Omega, \Lambda^{\mathbb{C}}\right)$ containing $\Lambda^{\mathbb{C}}$ but of course **not** invariant under $\overline{}$.
The main goal in the complex case is:

Theorem 4.25 *Let* $\Omega \underset{open}{\subset} \mathbb{C}^{p|q,\overline{q}}$ *such that* $\Omega_{\mathbf{z}} := \left\{ (\mathbf{w}, \vartheta) \in \Omega \;\middle|\; (\mathbf{w}, \vartheta)^{\#} = \mathbf{z} \right\}$ *for all* $\mathbf{z} \in \Omega^{\#}$ *is connected.*

(i) We have isomorphisms

$$\left(\Lambda \left(\mathbb{C}^q \right) \boxtimes \Lambda \left(\mathbb{C}^q \right) \otimes \mathcal{C}^\infty \left(\Omega^{\#} \right)^{\mathbb{C}} \right) \hat{\boxtimes} \Lambda^{\mathbb{C}} \simeq \mathcal{C}^\infty \left(\left(\Omega^{\#} \right)^{|q,\overline{q}}, \Lambda^{\mathbb{C}} \right) \simeq \mathcal{D} \left(\Omega, \Lambda^{\mathbb{C}} \right)$$

as unital graded algebras, where on $\mathcal{C}^\infty \left(\Omega^{\#} \right)^{\mathbb{C}}$ *we use the uniformal structure given by* $f_n \rightsquigarrow 0$ *if and only if* $\partial^{\mathbf{k}} \overline{\partial}^{\mathbf{l}} f_n \rightsquigarrow 0$ *compact for all* $\mathbf{k}, \mathbf{l} \in \mathbb{N}^p$ *and pointwise convergence on* $\mathcal{C}^\infty \left(\left(\Omega^{\#} \right)^{|q,\overline{q}}, \Lambda^{\mathbb{C}} \right)$ *.*

The first isomorphism is the unique $\mathbb{C}$*-linear and continuous map given by* $e^S \otimes e^T \otimes f \otimes E_I \mapsto \vartheta^S \overline{\vartheta}^T \widehat{f} E_I$ *for all* $f \in \mathcal{C}^\infty \left(\Omega^{\#} \right)^{\mathbb{C}}$ *,* $S, T \in \wp(q)$ *,* $I \in \wp$ *, and the second is given by the restriction map.*

(ii) The above isomorphism restricts to an isomorphism

$$\left(\Lambda \left(\mathbb{C}^q \right) \otimes \mathcal{O} \left(\Omega^{\#} \right) \right) \hat{\boxtimes} \Lambda^{\mathbb{C}} \simeq \mathcal{O} \left(\Omega, \Lambda^{\mathbb{C}} \right) .$$

Let $\Omega \underset{\text{open}}{\subset} \mathbb{C}^{p|q,\overline{q}}$. Now we say Ω is of cube type if and only if there exist $\wp' \subset \wp \setminus \{\emptyset\}$ finite and $\varepsilon > 0$ such that for all $(\mathbf{w}, \vartheta) \in \left(\Omega^{\#} \right)^{|q,\overline{q}}$: $(\mathbf{w}, \vartheta) \in \Omega$ if and only if $|z_{iJ}|, |r_{jJ}| < \varepsilon$ for all $i = 1, \ldots, p$, $j = 1, \ldots, q$, $I \in \wp' \cap \wp_0$, $J \in \wp' \cap \wp_1$.

In this case clearly $\Omega_{\mathbf{z}} \underset{\text{open , convex}}{\subset} \mathbb{C}_{\mathbf{z}}^{p|q,\overline{q}}$ for all $\mathbf{z} \in \Omega^{\#}$, and $\Omega^{\#} \subset \Omega$. Clearly $U^{|q,\overline{q}}$ is of cube type for all $U \underset{\text{open}}{\subset} \mathbb{C}^p$.

Theorem 4.26 *Let* $\Omega \underset{open}{\subset} \mathbb{C}^{p|0,\overline{0}}$ *be of cube type and* $f \in \left(\Lambda^{\mathbb{C}} \right)^{\Omega}$ *. Then there are equivalent*

(i) *f is differentiable,*

(ii) $f \in \mathcal{D} \left(\Omega, \Lambda^{\mathbb{C}} \right)$ *,*

(iii) *all* F_I *,* $I \in \wp$ *, are differentiable with respect to all* z_{iJ} *,* $i = 1, \ldots, p$ *,* $J \in \wp_0$ *, continuous, and they fulfill the following system of differential equations*

$$\partial_{i|J} F_I = \begin{cases} (-1)^{|I \setminus J < J|} \partial_{i|\emptyset} F_{I \setminus J} & \text{if } J \subset I \\ 0 & \text{otherwise} \end{cases}$$

and

$$\overline{\partial}_{i|J} F_I = \begin{cases} (-1)^{|I\setminus(-J)<-J|}\overline{\partial}_{i|\emptyset} F_{I\setminus(-J)} & \text{if } J \subset -I \\ 0 & \text{otherwise} \end{cases}$$

for all $i = 1, \ldots, p$, $I \in \wp$ *and* $J \in \wp_1$,

(iv) there exists a family $(f_I)_{I\in\wp} \in \left(\mathcal{C}^\infty\left(\Omega^\#\right)^{\mathbb{C}}\right)^{\wp}$ *such that*

$$f(\mathbf{w}) = \sum_{I\in\wp} \widehat{f}_I(\mathbf{w}) E_I$$

for all $\mathbf{w} \in \Omega$.

In this case $f^* := \sum_{I\in\wp} \widehat{f}_I E_I : \left(\Omega^\#\right)^{|0,\overline{0}} \to \Lambda^{\mathbb{C}}$ *is the unique extension in* $\mathcal{D}\left(\left(\Omega^\#\right)^{|0,\overline{0}}\right)$ *of* f , f^* *is differentiable,* $\partial_{i|} f^*$ *and* $\overline{\partial}_{i|} f^*$ *are again in* $\mathcal{D}\left(\left(\Omega^\#\right)^{|0,\overline{0}}\right)$ *, and*

$$\partial_{i|} f^* = \sum_{I\in\wp} \widehat{\partial_i f_I} E_I$$

and

$$\overline{\partial}_{i|} f^* = \sum_{I\in\wp} \widehat{\overline{\partial}_i f_I} E_I$$

for all $i = 1, \ldots, p$.

Proof: almost the same as the proof of theorem 4.9 in the real case. *Step I* of (iii) $\Rightarrow$ (iv) is different:

Show that all F_I are affine linear with respect to z_{iJ} , $J \subset I$, $J \neq \emptyset$, if $-J \not\subset I$, affine antilinear with respect to z_{iJ} , $-J \subset I$, $J \neq \emptyset$, if $J \not\subset I$ and $\mathcal{C}^\infty$ with respect to $\mathbf{z}_\emptyset := (z_{1\emptyset}, \ldots, z_{p\emptyset}) \in \Omega^\#$.

Let $i \in \{1, \ldots, p\}$ and $I \in \wp_0$. Let $J \in \wp_0 \setminus \{\emptyset\}$ such that $J \subset I$ and $-J \not\subset I$. Then $J, -J \not\subset I \setminus J$, and so $\partial_{k|J} F_I = (-1)^{|I\setminus J<J|}\partial_{k|\emptyset} F_{I\setminus J}$ is independent of z_{iJ} . So F_I is affine linear with respect to z_{iJ} .
Now let $J \in \wp_0 \setminus \{\emptyset\}$ such that $-J \subset I$ and $J \not\subset I$. Then $J, -J \not\subset I \setminus (-J)$, and so again $\overline{\partial}_{k|J} F_I = (-1)^{|I\setminus(-J)<-J|}\overline{\partial}_{k|\emptyset} F_{I\setminus(-J)}$ is independent of z_{iJ} . So here F_I is affine antilinear with respect to z_{iJ} .

Now we prove by induction on n that all F_I are $\mathcal{C}^n$ with repect to $\mathbf{z}_\emptyset$ for arbitrary $n \in \mathbb{N}$.

Since f is continuous, all F_I are continuous with respect to $\mathbf{z}_\emptyset$.
Now let $n \in \mathbb{N}$ such that all F_R are $\mathcal{C}^n$ with respect to $\mathbf{z}_\emptyset$, and let $i \in \{1, \dots, p\}$, $I \in \wp$, $b \in \Omega$.

Choose $S \in \wp_0 \setminus \{\emptyset\}$ such that $-S < I < S$.

Since $\Omega \underset{\text{open}}{\subset} \mathbb{C}^{p|0,\overline{0}}$, there exist $\varepsilon > 0$ and $\Omega' \underset{\text{open}}{\subset} \Omega$ such that $b \in \Omega'$ and $\Omega' + [0, \varepsilon] E_S \mathbf{e}_j \subset \Omega$. So for all $\mathbf{w} \in \Omega'$

$$\begin{aligned} \partial_{i|\emptyset} F_I(\mathbf{w}) &= \partial_{i|-S} F_{I \cup (-S)}(\mathbf{w}) \\ &= \frac{F_{I\cup(-S)}\left(\mathbf{w} + \varepsilon E_{-S} \mathbf{e}_i\right) - F_{I\cup S}(\mathbf{w})}{\varepsilon}, \end{aligned}$$

because $F_{I\cup(-S)}$ is affine linear with respect to $w_{i,-S}$, and

$$\begin{aligned} \overline{\partial}_{i|\emptyset} F_I(\mathbf{w}) &= \overline{\partial}_{i|S} F_{I \cup S}(\mathbf{w}) \\ &= \frac{F_{I\cup S}\left(\mathbf{w} + \varepsilon E_{S} \mathbf{e}_i\right) - F_{I\cup S}(\mathbf{w})}{\varepsilon}, \end{aligned}$$

because $F_{I\cup S}$ is affine antilinear with respect to z_{iS} .

By assumption in both cases the right hand side is $\mathcal{C}^n$ with respect to $\mathbf{z}_\emptyset$, and so is the left hand side. That means $F_I|_{\Omega'}$ is $\mathcal{C}^{n+1}$ with respect to $\mathbf{z}_\emptyset$. Since $\mathbf{b}$ was arbitrary, F_I itself is $\mathcal{C}^{n+1}$ with respect to $\mathbf{z}_\emptyset$. $\square$

A similar result holds again for $\Omega \subset \mathbb{C}^{0|1,\overline{1}}$:

Theorem 4.27 *Let $\Omega \underset{open}{\subset} \mathbb{C}^{0|1,\overline{1}}$ be of cube type and $f \in \left(\Lambda^{\mathbb{C}}\right)^{\Omega}$. Then there are equivalent*

(i) f is differentiable,

(ii) F_I , $I \in \wp$, are differentiable with respect to all r_J and fulfill the following system of differential equations

$$\partial_J F_I = \begin{cases} (-1)^{|I\setminus J<J|+|I\setminus J<L|} \partial_L F_{(I\setminus J)\cup L} & \text{if } J \subset I \\ 0 & \text{otherwise} \end{cases}$$

for all $I \in \wp$, $J \in \wp_1$ and $L \in \wp_1$ with $L \cap (I \setminus J) = \emptyset$, and

$$\overline{\partial}_K F_I = \begin{cases} (-1)^{|I\setminus(-K)<-K|+|I\setminus(-K)<-L|} \overline{\partial}_L F_{(I\setminus(-K))\cup(-L)} & \text{if } K \subset -I \\ 0 & \text{otherwise} \end{cases}$$

for all $I \in \wp$, $K \in \wp_1$ and $L \in \wp_1$ with $(-L) \cap (I \setminus (-K)) = \emptyset$, *(iii) there exist* $a, b, c, d \in \Lambda^{\mathbb{C}}$ *such that* $f(\vartheta) = a + \vartheta b + \overline{\vartheta} c + \vartheta\overline{\vartheta} d$ *for all* $\vartheta \in \Omega$ *.*

In this case $f^* : \mathbb{C}^{0|1,\overline{1}} \to \Lambda^{\mathbb{C}}$, $\vartheta \mapsto a + \vartheta b + \overline{\vartheta} c + \vartheta\overline{\vartheta} d$ *is again the unique differentiable extension of* f , $\partial_{|1} f^*(\vartheta) = b + \overline{\vartheta} d$ *and* $\overline{\partial}_{|1} f^*(\vartheta) = c - \vartheta d$ *for all* $\vartheta \in \mathbb{C}^{0|1,\overline{1}}$ *.*

Proof: almost the same as the proof of theorem 4.10 in the real case. Now (ii) $\Rightarrow$ (iii) is different:

Let F_I , $I \in \wp$ fulfill the system of differential equations.

Step I **Show that all** $\partial_J F_I$ **,** $J \subset I$ **, are independent of** r_K **if** $K \not\subset -(I \setminus J)$ **and antiholomorphic in** r_K **if** $K \subset -(I \setminus J)$ **, that all** $\overline{\partial}_K F_I$ **,** $K \subset -I$ **, are independent of** r_J **if** $J \not\subset I \setminus (-K)$ **and holomorphic in** r_J **if** $J \subset I \setminus (-K)$ **and that all** $\partial_J \overline{\partial}_K F_I = \overline{\partial}_K \partial_J F_I$ **,** $J \subset I$ **,** $K \subset -I$ **,** $J \cap (-K) = \emptyset$ **, are constant.**

First we show $\partial_S \partial_T F_I = 0$ for all $I \in \wp$, $S, T \in \wp_1$, $S, T \subset I$ and $\overline{\partial}_S \overline{\partial}_T F_I = 0$ for all $I \in \wp$, $S, T \in \wp_1$, $S, T \subset -I$.

> First assume $S \cap T \neq \emptyset$. Then $\partial_S \partial_T F_I = 0$ and $\overline{\partial}_S \overline{\partial}_T F_I = 0$ follow as in the real case. Since $J \neq \emptyset$ for all $J \in \wp_1$ we see that F_I is polynomial in r_J and $\overline{r_J}$ of partial degrees ≤ 1 for all $I \in \wp$, $J \in \wp_1$. And so as in the real case we can interchange partial derivatives.
>
> Now assume $S \cap T = \emptyset$. Then $\partial_S \partial_T F_I = 0$ and $\overline{\partial}_S \overline{\partial}_T F_I = 0$ follow again as in the real case since we can interchange partial derivatives. $\square$

So clearly all $\partial_J \overline{\partial}_K F_I$, $J \subset I$, $K \subset -I$ are constant.

It remains to show that all $\overline{\partial}_K \partial_J F_I = 0$, $J \subset I$, $K \subset -I$, $J \cap (-K) \neq \emptyset$. So let $I \in \wp$, $J, K \in \wp_1$ such that $J \subset I$, $K \subset -I$ and $J \cap (-K) \neq \emptyset$. Choose $M \in \wp_1$ such that $M < I$. Then $K \not\subset -((I \setminus J) \cup M)$ and so

$$\overline{\partial}_K \partial_J F_I = (-1)^{|I \setminus J < J|} \partial_M \overline{\partial}_K F_{(I \setminus J) \cup M} = 0 \, .$$

Step II **Conclusion .**

For all $I \in \wp$ we have $\{\min (I \cup \{0\}) - 1\}, \{\min (I \cup \{0\}) - 2\} \in \wp_1$, so we can define

$b_I := \partial_{\{\min(I\cup\{0\})-1\}} F_{I\cup\{\min(I\cup\{0\})-1\}}|_{r_K=0, K\subset -I} \in \mathbb{C}$,
$b := \sum_{I\in\wp} b_I E_I \in \Lambda^{\mathbb{C}}$,

$c_I := \overline{\partial}_{\{-\min(I\cup\{0\})+1\}} F_{I\cup\{\min(I\cup\{0\})-1\}}|_{r_K=0, K\subset I} \in \mathbb{C}$,
$c := \sum_{I\in\wp} c_I E_I \in \Lambda^{\mathbb{C}}$,

$d_I := \partial_{\{\min(I\cup\{0\})-2\}} \overline{\partial}_{\{-\min(I\cup\{0\})+1\}} F_{I\cup\{\min(I\cup\{0\})-2\}\cup\{\min(I\cup\{0\})-1\}} \in \mathbb{C}$,
$d := \sum_{I\in\wp} d_I E_I \in \Lambda^{\mathbb{C}}$.

Let $I \in \wp$.

Let $J \in \wp_1$ with $J \subset I$ and $S := \{\min ((I \setminus J) \cup \{0\}) - 1\} < I \setminus J$. Then

$$\begin{aligned} \partial_J F_I|_{r_K=0, K\subset -(I\setminus J)} &= (-1)^{|I\setminus J < J|} \partial_S F_{(I\setminus J)\cup S}|_{r_K=0, K\subset -(I\setminus J)} \\ &= (-1)^{|I\setminus J < J|} b_{I\setminus J} . \end{aligned}$$

Let $K \in \wp_1$ with $K \subset -I$ and $T := \{-\min ((I \setminus (-K)) \cup \{0\}) - 1\}$. Then $-T < I \setminus (-K)$ and so

$$\begin{aligned} \overline{\partial}_K F_I|_{r_J=0, J\subset I\setminus(-K)} &= (-1)^{|I\setminus(-K) < -K|} \partial_T F_{(I\setminus(-K))\cup(-T)}|_{r_J=0, J\subset I\setminus(-K)} \\ &= (-1)^{|I\setminus(-K) < -K|} b_{I\setminus(-K)} . \end{aligned}$$

Now let $J, K \in \wp_1$ such that $J \subset I$, $K \subset -I$ and $J \cap (-K) = \emptyset$. We prove that

$$\partial_J \overline{\partial}_K F_I = (-1)^{|-K<J| + |I\setminus(J\cup(-K)) < J\cup(-K)|} d_{I\setminus(J\cup(-K))} .$$

We define $S := \{-\min(I \setminus ((-K) \cup J)) + 1\}$ and
$T := \{\min(I \setminus ((-K) \cup J)) - 2\} < (I \setminus ((-K) \cup J)) \cup (-S)$.
Choose $M, N \in \wp_1$ such that $M > (-I) \cup T$ and $N < I \cup (-M)$.
Then

$$
\begin{aligned}
\partial_J \overline{\partial}_K F_I &= (-1)^{|I\setminus(-K)<-K|} \partial_J \overline{\partial}_M F_{(I\setminus(-K))\cup(-M)} \\
&= (-1)^{|I\setminus(-K)<-K|+|(I\setminus((-K)\cup J))\cup(-M)<J|} \times \\
&\quad \times \partial_N \overline{\partial}_M F_{(I\setminus(J\cup(-K)))\cup(-M)\cup N} \\
&= (-1)^{|I\setminus(-K)<-K|+|I\setminus((-K)\cup J)<J|+1} \times \\
&\quad \times \partial_N \overline{\partial}_M F_{(I\setminus(J\cup(-K)))\cup(-M)\cup N} \,,
\end{aligned}
$$

since $-M < I \setminus (-K)$, $N < (I \setminus (-K)) \cup (-M)$, and
$|-M < J| = |M|\,|J|$ is an odd number. Further

$$
\begin{aligned}
\partial_N \overline{\partial}_M F_{(I\setminus(J\cup(-K)))\cup(-M)\cup N} &= (-1)^{|N<-M|+|N<-T|} \times \\
&\quad \times \partial_N \overline{\partial}_T F_{(I\setminus(J\cup(-K)))\cup N\cup(-T)} \\
&= \partial_N \overline{\partial}_T F_{(I\setminus(J\cup(-K)))\cup N\cup(-T)} \\
&= \partial_S \overline{\partial}_T F_{(I\setminus(J\cup(-K)))\cup S\cup(-T)} \\
&= d_{I\setminus(J\cup(-K))} \,,
\end{aligned}
$$

since $-M, -T < I \setminus (J \cup (-K))$,
$|N < -M| + |N < -T| = |M|\,|N| + |N|$ is an even number, and
$N, S < (-T) \cup (I \setminus (J \cup /(-K)))$. Finally we have

$$
\begin{aligned}
& |I \setminus (-K) < -K| + |I \setminus (J \cup (-K)) < J| + 1 \\
&\equiv |I \setminus (J \cup (-K)) < -K| + |J < -K| \\
&\quad + |I \setminus ((-K) \cup J) < J| + |J|\,|K| \\
&\equiv |I \setminus (J \cup (-K)) < J \cup (-K)| + |-K < J| \quad \mod 2 \,. \square
\end{aligned}
$$

So for all $I \in \wp$ and $\vartheta \in \Omega$

$$
\begin{aligned}
F_I(\vartheta) &= F_I(0) + \sum_{J\in\wp_1, J\subset I} (-1)^{|I\setminus J<J|} r_J b_{I\setminus J} \\
&\quad + \sum_{K\in\wp_1, K\subset -I} (-1)^{|I\setminus(-K)<-K|} \overline{r_K} b_{I\setminus(-K)} \\
&\quad + \sum_{J,K\in\wp_1, J\subset I, K\subset -I, J\cap(-K)=\emptyset} (-1)^{|-K<J|+|I\setminus((-K)\cup J)<J\cup(-K)|} \times \\
&\quad \times r_J \overline{r_K} d_{(I\setminus(J\cup(-K)))} \,,
\end{aligned}
$$

and finally, if we define $a := f(0) \in \Lambda^{\mathbb{C}}$,

$$\begin{aligned} f(\vartheta) &= f(0)+\sum_{I\in\wp}\sum_{J\in\wp_1, J\subset I}(-1)^{|I\setminus J<J|}r_J b_{I\setminus J}E_I \\ &\quad +\sum_{I\in\wp}\sum_{K\in\wp_1, K\subset -I}(-1)^{|I\setminus(-K)<-K|}\overline{r_K}b_{I\setminus(-K)}E_I \\ &\quad +\sum_{I\in\wp}\sum_{J,K\in\wp_1, J\subset I, K\subset -I, J\cap(-K)=\emptyset}(-1)^{|-K<J|+|I\setminus((-K)\cup J)<J\cup(-K)|}\times \\ &\quad \times r_J\overline{r_K}d_{(I\setminus(J\cup(-K)))}E_I \\ &= a+\sum_{J\in\wp_1}r_J E_J\sum_{L\in\wp}b_L E_L+\sum_{K\in\wp_1}\overline{r_K}E_{-K}\sum_{M\in\wp}b_M E_M \\ &\quad +\sum_{J\in\wp_1}r_J E_J\sum_{K\in\wp_1}\overline{r_K}E_{-K}\sum_{T\in\wp}d_T E_T \\ &= a+\vartheta b+\overline{\vartheta}c+\vartheta\overline{\vartheta}d\,.\square \end{aligned}$$

Corollary 4.28 *Let $\Omega\subset\mathbb{C}^{p|q,\overline{q}}$ be of cube type and $f:\Omega\to\Lambda^{\mathbb{C}}$. Then there are equivalent*

(i) f is arbitrarily often differentiable,

(ii) $f\in\mathcal{D}(\Omega,\Lambda^{\mathbb{C}})$,

(iii) there exists a family $(f_{IST})_{I\in\wp,S,T\in\wp(q)}\in\left(\mathcal{C}^\infty\left(\Omega^{\#}\right)^{\mathbb{C}}\right)^{\wp\times\wp(q)^2}$ such that for all $(\mathbf{w},\vartheta)\in\left(\Omega^{\#}\right)^{|q,\overline{q}}$

$$f(\mathbf{w},\vartheta)=\sum_{I\in\wp}\sum_{S,T\in\wp(q)}\vartheta^S\overline{\vartheta}^T\widehat{f_{IST}}(\mathbf{w})E_I\,.$$

Again the function

$$f^*:=\sum_{I\in\wp}\sum_{S,T\in\wp(q)}\vartheta^S\overline{\vartheta}^T\widehat{f_{IST}}E_I:\left(\Omega^{\#}\right)^{|q,\overline{q}}\to\Lambda^{\mathbb{C}}$$

is the unique extension in $\mathcal{D}\left(\left(\Omega^{\#}\right)^{|q,\overline{q}}\right)$ of f . For all $i=1,\dots,p$ we have

$$\partial_{i|}f^*=\sum_{I\in\wp}\sum_{S,T\in\wp(q)}\vartheta^S\overline{\vartheta}^T\widehat{\partial_i f_{IST}}E_I$$

and

$$\overline{\partial}_{i|}f^*=\sum_{I\in\wp}\sum_{S,T\in\wp(q)}\vartheta^S\overline{\vartheta}^T\widehat{\overline{\partial}_i f_{IST}}E_I\,,$$

and for all $j=1,\dots,q$

$$\partial_{|j} f^* = \sum_{I \in \wp} \sum_{S \in \wp(q), j \in S} \sum_{T \in \wp(q)} (-1)^{|S \setminus \{j\} < j|} \vartheta^{S \setminus \{j\}} \overline{\vartheta}^T \widehat{f_{IST}} E_I$$

and

$$\overline{\partial}_{|j} f^* = \sum_{I \in \wp} \sum_{S \in \wp(q)} \sum_{T \in \wp(q), j \in T} (-1)^{|S| + |T \setminus \{j\} < j|} \vartheta^S \overline{\vartheta}^{T \setminus \{j\}} \widehat{f_{IST}} E_I \,.$$

Proof: similar to the proof of theorem 4.11 in the real case.

Corollary 4.29 *Let* $\Omega \underset{open}{\subset} \mathbb{C}^{p|q,\overline{q}}$ *such that* $\Omega_{\mathbf{z}} = \left\{ (\mathbf{w}, \vartheta) \in \Omega \,\middle|\, (\mathbf{w}, \vartheta)^\# = \mathbf{z} \right\}$ *is connected for all* $\mathbf{z} \in \Omega^\#$ *. Then the same holds as in corollary 4.28 .*

Proof: similar to the proof of theorem 4.12 in the real case.

Now we prove theorem 4.25 .

(i) similar to the proof of theorem 4.8 in the real case.

(ii) We prove that $\mathcal{O}\left(\Omega, \Lambda^{\mathbb{C}}\right) = \Phi\left(\left(\Lambda\left(\mathbb{C}^q\right) \otimes 1 \otimes \mathcal{O}\left(\Omega^\#\right)\right) \hat{\boxtimes} \Lambda^{\mathbb{C}}\right)$.
So let first

$$\mathfrak{f} = \sum_{I \in \wp} \sum_{S, T \in \wp(q)} e^S \otimes e^T \otimes f_{IST} \otimes E_I \in \left(\Lambda\left(\mathbb{C}^q\right) \boxtimes \Lambda\left(\mathbb{C}^q\right) \otimes \mathcal{C}^\infty\left(\Omega^\#\right)^{\mathbb{C}}\right) \hat{\boxtimes} \Lambda^{\mathbb{C}}$$

such that $\Phi(\mathfrak{f}) = \sum_{I \in \wp} \sum_{S, T \in \wp(q)} \vartheta^S \overline{\vartheta}^T \widehat{f_{IST}} E_I \in \mathcal{O}\left(\Omega, \Lambda^{\mathbb{C}}\right)$, let $R \in \wp(q) \setminus \{\emptyset\}$ and $j := \min R$. Then since $\Phi(\mathfrak{f}) \in \mathcal{O}\left(\Omega, \Lambda^{\mathbb{C}}\right)$

$$0 = \overline{\partial}_{|j} \Phi(\mathfrak{f}) = \sum_{I \in \wp} \sum_{S \in \wp(q)} \sum_{T \in \wp(q), i \in T} (-1)^{|S| + |T \setminus \{j\} < j|} \vartheta^S \overline{\vartheta}^{T \setminus \{j\}} \widehat{f_{IST}} E_I \,,$$

which is the image of

$$\sum_{I \in \wp} \sum_{S \in \wp(q)} \sum_{T \in \wp(\mathbb{C}^q), j \in T} (-1)^{|S| + |T \setminus \{j\} < j|} e^S \otimes e^{T \setminus \{j\}} \otimes f_{IST} \otimes E_I$$

under Φ . Since Φ is an isomorphism and $j \in R$ we obtain $f_{ISR} = 0$ for all $S \in \wp(q)$ and $I \in \wp$. But $R \in \wp(q) \setminus \{\emptyset\}$ was arbitrary, so since $e^\emptyset = 1$ we get

$$\mathfrak{f} = \sum_{I \in \wp} \sum_{S \in \wp(\mathbb{C}^q)} e^S \otimes 1 \otimes f_{IS\emptyset} \otimes E_I \,.$$

Now let $i \in \{1, \ldots, p\}$. Then since $\Phi(\mathfrak{f}) \in \mathcal{O}\left(\Omega, \Lambda^{\mathbb{C}}\right)$

$$0 = \overline{\partial}_{i|}\Phi(\mathfrak{f}) = \sum_{I \in \wp} \sum_{S \in \wp(q)} \vartheta^S \widehat{\overline{\partial}_i f_{IS\emptyset}} E_I \,.$$

And so by the same reason all $\overline{\partial}_i f_{IS\emptyset} = 0$, $I \in \wp$, $S \in \wp(q)$. That means all $f_{IS\emptyset}$, $I \in \wp$, $S \in \wp(q)$, are holomorphic.

Now let

$$\mathfrak{f} = \sum_{I \in \wp} \sum_{S \in \wp(q)} e^S \otimes f_{IS} \otimes E_I \in \left(\Lambda\left(\mathbb{C}^q\right) \otimes 1 \otimes \mathcal{O}\left(\Omega^{\#}\right)\right) \hat{\boxtimes} \Lambda^{\mathbb{C}} \,.$$

Then it is easy to check that $\overline{\partial}_{i|}\Phi(\mathfrak{f}) = \overline{\partial}_{|j}\Phi(\mathfrak{f}) = 0$ for all $i = 1, \ldots, p$, $j = 1, \ldots, q$. $\square$

Especially we see that if $\Omega \underset{\text{open}}{\subset} \mathbb{C}^{p|q,\overline{q}}$ then $\mathcal{O}\left(\Omega, \Lambda^{\mathbb{C}}\right)$ is closed under derivation, and if in addition $\Omega_{\mathbf{z}}$ is connected for all $\mathbf{z} \in \Omega^{\#}$ then $f \in \mathcal{O}(\Omega)$ implies $f^* \in \mathcal{O}\left(\left(\Omega^{\#}\right)^{|q,\overline{q}}\right)$.

Here again we have a useful lemma:

Lemma 4.30 *Let* $U \underset{open}{\subset} \mathbb{C}^p$ *and* $f \in \mathcal{D}\left(U^{|q,\overline{q}}, \Lambda^{\mathbb{C}}\right)$.

(i) For all $(\mathbf{b}, \beta) \in U^{|q,\overline{q}}$ *there exist* $\Delta_1, \ldots, \Delta_p$, $\Delta'_1, \ldots, \Delta'_p$, $\Sigma_1, \ldots, \Sigma_q$, $\Sigma'_1, \ldots, \Sigma'_q \in \mathcal{D}\left(U^{|q,\overline{q}}, \Lambda^{\mathbb{C}}\right)$ *such that for all* $(\mathbf{w}, \vartheta) \in U^{|q,\overline{q}}$

$$\begin{aligned} f(\mathbf{w}, \vartheta) &= f(\mathbf{b}, \beta) + \sum_{i=1}^{p} \left(w_i - b_i\right) \Delta_i(\mathbf{w}, \vartheta) + \sum_{i=1}^{p} \left(\overline{w_i} - \overline{b_i}\right) \Delta'_i(\mathbf{w}, \vartheta) \\ &\quad + \sum_{j=1}^{q} \left(\vartheta_j - \beta_j\right) \Sigma_j(\mathbf{w}, \vartheta) + \sum_{j=1}^{q} \left(\overline{\vartheta_j} - \overline{\beta_j}\right) \Sigma'_j(\mathbf{w}, \vartheta) \,. \end{aligned}$$

(ii) If $f \in \mathcal{O}\left(U^{|q,\overline{q}}, \Lambda^{\mathbb{C}}\right)$ *then for all* $(\mathbf{b}, \beta) \in U^{|q,\overline{q}}$ *there exist* $\Delta_1, \ldots, \Delta_p$, $\Sigma_1, \ldots, \Sigma_q \in \mathcal{O}\left(U^{|q,\overline{q}}, \Lambda^{\mathbb{C}}\right)$ *such that for all* $(\mathbf{w}, \vartheta) \in U^{|q,\overline{q}}$

$$f(\mathbf{w}, \vartheta) = f(\mathbf{b}, \beta) + \sum_{i=1}^{p} \left(w_i - b_i\right) \Delta_i(\mathbf{w}, \vartheta) + \sum_{j=1}^{q} \left(\vartheta_j - \beta_j\right) \Sigma_j(\mathbf{w}, \vartheta) \,.$$

(iii) f *is determined by the functions*

$$\left(\partial_{|j_1} \ldots \partial_{|j_r} \overline{\partial}_{|k_1} \ldots \overline{\partial}_{|k_s} f\right)|_U \in \mathcal{C}^\infty(U)^{\mathbb{C}} \otimes \Lambda^{\mathbb{C}} \,,$$

$\{j_1, \ldots, j_r\}$, $\{k_1, \ldots, k_s\} \subset \{1, \ldots, q\}$ *with* $j_1 < \cdots < j_r$, $k_1 < \cdots < k_s$.

Proof: similar to the proof of theorem 4.13 in the real case.

From now on let $\Omega \underset{\text{open}}{\subset} \mathbb{C}^{p|q,\overline{q}}$ and $\Omega' \underset{\text{open}}{\subset} \mathbb{C}^{r|s,\overline{s}}$ such that $\Omega_{\mathbf{z}}$ and $\Omega'_{\mathbf{z}'}$ are connected for all $\mathbf{z} \in \Omega^{\#}$ resp. $\mathbf{z}' \in \Omega'^{\#}$. We define

$$\mathcal{D}(\Omega) := \Psi\left(\left(\Lambda\left(\mathbb{C}^q\right) \boxtimes \Lambda\left(\mathbb{C}^q\right) \otimes \mathcal{C}^\infty\left(\Omega^{\#}\right)\right)^{\mathbb{C}} \otimes 1\right) \subset \mathcal{D}\left(\Omega, \Lambda^{\mathbb{C}}\right),$$

where $\Psi : \left(\Lambda\left(\mathbb{C}^q\right) \boxtimes \Lambda\left(\mathbb{C}^q\right) \otimes \mathcal{C}^\infty\left(\Omega^{\#}\right)\right)^{\mathbb{C}} \hat{\boxtimes} \Lambda^{\mathbb{C}} \to \mathcal{D}\left(\Omega, \Lambda^{\mathbb{C}}\right)$ is the isomorphism given by theorem 4.25 (i) , and $\mathcal{O}(\Omega) := \mathcal{D}(\Omega) \cap \mathcal{O}\left(\Omega, \Lambda^{\mathbb{C}}\right)$. Then $\mathcal{D}(\Omega)$ is a sub graded algebra of $\mathcal{D}\left(\Omega, \Lambda^{\mathbb{C}}\right)$ invariant under $\overline{}$, and $\mathcal{O}(\Omega)$ is a sub graded algebra of $\mathcal{O}\left(\Omega, \Lambda^{\mathbb{C}}\right)$, both **not** containing $\Lambda^{\mathbb{C}}$, more precisely

$$\mathcal{D}(\Omega) \cap \Lambda^{\mathbb{C}} = \mathcal{O}(\Omega) \cap \Lambda^{\mathbb{C}} = \mathbb{C}\,.$$

We can characterize $\mathcal{D}(\Omega)$ and $\mathcal{O}(\Omega)$ as follows:

Theorem 4.31

a) $\mathcal{D}(\Omega)$ *and* $\mathcal{O}(\Omega)$ *are closed under derivation.*

b) Let $f \in \mathcal{D}\left(\Omega, \Lambda^{\mathbb{C}}\right)$ *. Then there are equivalent*

(i) $f \in \mathcal{D}(\Omega)$ *,*

(ii) there exists a family $\left(f_{ST}\right)_{S,T \in \wp(q)} \in \left(\mathcal{C}^\infty\left(\Omega^{\#}\right)^{\mathbb{C}}\right)^{\wp(q)^2}$ *such that*

$$f^* = \sum_{S,T \in \wp(q)} \vartheta^S \overline{\vartheta}^T \widehat{f_{ST}}\,,$$

(iii) $f^*(\mathbf{w}, \vartheta) \in \overline{\left\langle \mathbf{w}, \overline{\mathbf{w}}, \vartheta, \overline{\vartheta} \right\rangle}$ *for all* $(\mathbf{w}, \vartheta) \in \left(\Omega^{\#}\right)^{|q,\overline{q}}$ *, where* $\overline{\left\langle \mathbf{w}, \overline{\mathbf{w}}, \vartheta, \overline{\vartheta} \right\rangle}$ *is the smallest closed graded subalgebra of* $\Lambda^{\mathbb{C}}$ *invariant under* $\overline{}$ *containing all* $w_i, \ldots, w_p$ *and* $\vartheta_i, \ldots, \vartheta_q$ *,*

(iv) $\partial_{|j_1} \ldots \partial_{|j_r} \overline{\partial}_{|k_1} \ldots \overline{\partial}_{|k_s} f^*(\mathbf{z}, \mathbf{0}) \in \mathbb{C}$ *for all* $(j_1, \ldots, j_r) \in \{1, \ldots, q\}^r$ *,* $(k_1, \ldots, k_s) \in \{1, \ldots, q\}^s$ *and* $\mathbf{z} \in \Omega^{\#}$ *.*

c) Let $f \in \mathcal{O}\left(\Omega, \Lambda^{\mathbb{C}}\right)$ *. Then there are equivalent*

(i) $f \in \mathcal{O}(\Omega)$ *,*

(ii) there exists a family $\left(f_S\right)_{S \in \wp(q)} \in \mathcal{O}\left(\Omega^{\#}\right)^{\wp(q)}$ *such that*

$$f^* = \sum_{S \in \wp(q)} \vartheta^S \widehat{f_S}\,,$$

(iii) $f^*(\mathbf{w},\vartheta) \in \overline{\langle \mathbf{w},\vartheta\rangle}$ *for all* $(\mathbf{w},\vartheta) \in \left(\Omega^\#\right)^{|q,\overline{q}}$ *, where* $\overline{\langle \mathbf{w},\vartheta\rangle}$ *is the smallest closed graded subalgebra of* $\Lambda^{\mathbb{C}}$ *containing all* $w_i,\dots,w_p$ *and* $\vartheta_i,\dots,\vartheta_q$ *,*

(iv) $\partial_{|j_1}\dots\partial_{|j_r} f^*(\mathbf{z},\mathbf{0}) \in \mathbb{C}$ *for all* $(j_1,\dots,j_r) \in \{1,\dots,q\}^r$ *and* $\mathbf{z} \in \Omega^\#$ *.*

d) $\Lambda(\mathbb{C}^q) \boxtimes \Lambda(\mathbb{C}^q) \otimes \mathcal{C}^\infty\left(\Omega^\#\right)^{\mathbb{C}} \simeq \mathcal{D}(\Omega)$ *and* $\Lambda(\mathbb{C}^q) \otimes \mathcal{O}\left(\Omega^\#\right) \simeq \mathcal{O}(\Omega)$ *as graded algebras where the isomorphisms are the unique* $\mathbb{C}$*-linear maps given by* $e^S \otimes e^T \otimes f \mapsto \vartheta^S\overline{\vartheta}^S\widehat{f}$ *for all* $f \in \mathcal{C}^\infty\left(\Omega^\#\right)^{\mathbb{C}}$ *,* S *and* $T \in \wp(q)$ *resp.* $e^S \otimes f \mapsto \vartheta^S\widehat{f}$ *for all* $f \in \mathcal{O}\left(\Omega^\#\right)$ *and* $S \in \wp(q)$ *.*

e) $\mathcal{D}\left(\Omega,\Lambda^{\mathbb{C}}\right) = \mathcal{D}(\Omega)\hat{\boxtimes}\Lambda^{\mathbb{C}}$ *and* $\mathcal{O}\left(\Omega,\Lambda^{\mathbb{C}}\right) = \mathcal{O}(\Omega)\hat{\boxtimes}\Lambda^{\mathbb{C}}$ *, where on* $\mathcal{D}\left(\Omega,\Lambda^{\mathbb{C}}\right)$ *we use the topology coming from* $\left(\Lambda\left(\mathbb{C}^q\right) \otimes \mathcal{C}^\infty\left(\Omega^\#\right)^{\mathbb{C}}\right)\hat{\boxtimes}\Lambda^{\mathbb{C}}$ *via the isomorphism* Ψ *.*

Proof: similar to the proof of theorem 4.14 in the real case.

The last two theorems show that again in the complex case $\left(\Omega^\#\right)^{|q,\overline{q}}$ is essentially determined by the algebra $\mathcal{D}(\Omega)$.

Let $U \underset{\text{open}}{\subset} \mathbb{C}^n$, $\mathcal{S}\left(\mathcal{C}^\infty(U)^{\mathbb{C}}\right)$ be the spectrum of $\mathcal{C}^\infty(U)^{\mathbb{C}}$, more precisely the set of all unital algebra homomorphisms $\eta : \mathcal{C}^\infty(U)^{\mathbb{C}} \to \mathbb{C}$ which respect $\overline{}$. Then from analysis we know that there is a canonical bijection

$$U \to \mathcal{S}\left(\mathcal{C}^\infty(U)^{\mathbb{C}}\right) , \mathbf{a} \mapsto \eta_{\mathbf{a}} ,$$

where $\eta_{\mathbf{a}}(f) := f(\mathbf{a})$ for all $f \in \mathcal{C}^\infty(U)^{\mathbb{C}}$ and $\mathbf{a} \in U$. There is an analogous result for $\mathcal{D}(\Omega)$:

Let $\mathcal{S}\left(\mathcal{D}\left(\Omega,\Lambda^{\mathbb{C}}\right)\right)$ be the set of all graded algebra homomorphisms $\psi : \mathcal{D}\left(\Omega,\Lambda^{\mathbb{C}}\right) \to \Lambda^{\mathbb{C}}$ respecting $\overline{}$ and being the identity on $\Lambda^{\mathbb{C}}$, and let $\mathcal{S}\left(\mathcal{D}(\Omega)\right)$ be the set of all unital graded algebra homomorphisms $\eta : \mathcal{D}(\Omega) \to \Lambda^{\mathbb{C}}$ respecting $\overline{}$.

Theorem 4.32 *We have bijections*

$$\left(\Omega^\#\right)^{|q,\overline{q}} \to \mathcal{S}\left(\mathcal{D}\left(\Omega,\Lambda^{\mathbb{C}}\right)\right) , (\mathbf{b},\beta) \mapsto \psi_{(\mathbf{b},\beta)} ,$$

where $\psi_{(\mathbf{b},\beta)}(f) := f^*(\mathbf{b},\beta)$ *for all* $f \in \mathcal{D}\left(\Omega,\Lambda^{\mathbb{C}}\right)$ *, and*

$$\left(\Omega^\#\right)^{|q,\overline{q}} \to \mathcal{S}\left(\mathcal{D}(\Omega)\right) , (\mathbf{b},\beta) \mapsto \eta_{(\mathbf{b},\beta)} ,$$

where $\eta_{(\mathbf{b},\beta)}(g) := g^*(\mathbf{b},\beta)$ *for all* $g \in \mathcal{D}(\Omega)$ *and* $(\mathbf{b},\beta) \in \left(\Omega^\#\right)^{|q,\overline{q}}$ *.*

Proof: similar to the proof of theorem 4.15 in the real case.

Definition 4.33 *Let $\varphi : \Omega \to \Omega'$.*

(i) φ is called a $\mathcal{D}$-map if and only if all component functions $\varphi_{i|} : \Omega \to \mathbb{C}^{1|0,\overline{0}}$, $i = 1, \ldots, r$, and $\varphi_{|j} : \Omega \to \mathbb{C}^{0|1,\overline{1}}$, $j = 1, \ldots, s$, belong to $\mathcal{D}(\Omega)$.

(ii) φ is called a diffeomorphism if and only if φ is bijective and φ and φ^{-1} are $\mathcal{D}$-maps.

(iii) φ is called holomorphic if and only if all component functions of φ belong to $\mathcal{O}(\Omega)$.

(iv) φ is called biholomorphic if and only if φ is bijective and φ and φ^{-1} are holomorphic.

Clearly again if $\varphi : \Omega \to \Omega'$ is a $\mathcal{D}$-map then there exists a unique extension $\varphi^* : \left(\Omega^{\#}\right)^{|q,\overline{q}} \to \left(\Omega'^{\#}\right)^{|s,\overline{s}}$ of φ which is again a $\mathcal{D}$-map. If φ is holomorphic then again φ^* is holomorphic. For all $i = 1, \ldots, r$ and $j = 1, \ldots, s$ we have $(\varphi^*)_{i|} = \varphi^*_{i|}$ and $(\varphi^*)_{|j} = \varphi^*_{|j}$.
Here again we have a chain rule:

Lemma 4.34 *Let $f \in \mathcal{D}(\Omega')$ and $\varphi = \left(\varphi_{1|}, \ldots, \varphi_{r|}, \varphi_{|1}, \ldots, \varphi_{|s}\right) : \Omega \to \Omega'$ be a $\mathcal{D}$-map.*

(i) $f \circ \varphi \in \mathcal{D}(\Omega)$, and for all $i = 1, \ldots, p$ and $j = 1, \ldots, q$ we have

$$\begin{aligned}
\partial_{i|}\left(f \circ \varphi\right) &= \sum_{k=1}^{r} \left(\partial_{i|}\varphi_{k|}\right)\left(\left(\partial_{k|}f\right) \circ \varphi\right) + \sum_{k=1}^{r} \overline{\left(\overline{\partial}_{i|}\varphi_{k|}\right)}\left(\left(\overline{\partial}_{k|}f\right) \circ \varphi\right) \\
&\quad + \sum_{l=1}^{s} \left(\partial_{i|}\varphi_{|l}\right)\left(\left(\partial_{|l}f\right) \circ \varphi\right) + \sum_{l=1}^{s} \overline{\left(\overline{\partial}_{i|}\varphi_{|l}\right)}\left(\left(\overline{\partial}_{|l}f\right) \circ \varphi\right) ,
\end{aligned}$$

$$\begin{aligned}
\overline{\partial}_{i|}\left(f \circ \varphi\right) &= \sum_{k=1}^{r} \left(\overline{\partial}_{i|}\varphi_{k|}\right)\left(\left(\partial_{k|}f\right) \circ \varphi\right) + \sum_{k=1}^{r} \overline{\left(\partial_{i|}\varphi_{k|}\right)}\left(\left(\overline{\partial}_{k|}f\right) \circ \varphi\right) \\
&\quad + \sum_{l=1}^{s} \left(\overline{\partial}_{i|}\varphi_{|l}\right)\left(\left(\partial_{|l}f\right) \circ \varphi\right) + \sum_{l=1}^{s} \overline{\left(\partial_{i|}\varphi_{|l}\right)}\left(\left(\overline{\partial}_{|l}f\right) \circ \varphi\right) ,
\end{aligned}$$

$$\begin{aligned}
\partial_{|j}\left(f \circ \varphi\right) &= \sum_{k=1}^{r} \left(\partial_{|j}\varphi_{k|}\right)\left(\left(\partial_{k|}f\right) \circ \varphi\right) - \sum_{k=1}^{r} \overline{\left(\overline{\partial}_{|j}\varphi_{k|}\right)}\left(\left(\overline{\partial}_{k|}f\right) \circ \varphi\right) \\
&\quad + \sum_{l=1}^{s} \left(\partial_{|j}\varphi_{|l}\right)\left(\left(\partial_{|l}f\right) \circ \varphi\right) + \sum_{l=1}^{s} \overline{\left(\overline{\partial}_{|j}\varphi_{|l}\right)}\left(\left(\overline{\partial}_{|l}f\right) \circ \varphi\right)
\end{aligned}$$

and

$$\begin{aligned}\overline{\partial}_{|j}\left(f\circ\varphi\right) &= \sum_{k=1}^{r}\left(\overline{\partial}_{|j}\varphi_{k|}\right)\left(\left(\partial_{k|}f\right)\circ\varphi\right)-\sum_{k=1}^{r}\overline{\left(\partial_{|j}\varphi_{k|}\right)}\left(\left(\overline{\partial}_{k|}f\right)\circ\varphi\right)\\ &+\sum_{l=1}^{s}\left(\overline{\partial}_{|j}\varphi_{|l}\right)\left(\left(\partial_{|l}f\right)\circ\varphi\right)+\sum_{l=1}^{s}\overline{\left(\partial_{|j}\varphi_{|l}\right)}\left(\left(\overline{\partial}_{|l}f\right)\circ\varphi\right)\,.\end{aligned}$$

(ii) If $f\in\mathcal{O}\left(\Omega'\right)$ and φ holomorphic then $f\circ\varphi\in\mathcal{O}\left(\Omega\right)$.

Proof: similar to the real case.

So every $\mathcal{D}$-map $\varphi:\Omega\to\Omega'$ induces a unital graded algebra homomorphism $\Phi:\mathcal{D}(\Omega')\to\mathcal{D}(\Omega)\,,\ f\mapsto f\circ\varphi$, which respects $\overline{}$, and if φ is holomorphic then $\Phi\left(\mathcal{O}\left(\Omega'\right)\right)\subset\mathcal{O}\left(\Omega\right)$. The converse is almost true.

Theorem 4.35 *Let $\Phi:\mathcal{D}(\Omega')\to\mathcal{D}(\Omega)$ be a unital graded algebra homomorphism which respects $\overline{}$. Then*

(i) there exists a unique map $\varphi:\Omega\to\left(\Omega'^{\#}\right)^{|s,\overline{s}}$ such that $\Phi(f)=f^*\circ\varphi$ for all $f\in\mathcal{D}(\Omega')$,

(ii) φ is a $\mathcal{D}$-map, and $\Phi(f)^*=f^*\circ\varphi^*$ for all $f\in\mathcal{D}(\Omega')$,

(iii) Φ is an isomorphism if and only if φ^* is a diffeomorphism,

(iv) $\Phi\left(\mathcal{O}\left(\Omega'\right)\right)\subset\mathcal{O}(\Omega)$ if and only if φ is holomorphic,

(v) Φ is an isomorphism and $\Phi\left(\mathcal{O}\left(\Omega'\right)\right)=\mathcal{O}\left(\Omega'\right)$ if and only if φ^* is biholomorphic.

Proof: (i) , (ii) and (iii) : similar to the proof of theorem 4.18 in the real case.
(iv) '$\Rightarrow$' : Let $\Phi\left(\mathcal{O}\left(\Omega'\right)\right)\subset\mathcal{O}(\Omega)$ and let $w'_1,\ldots,w'_r$, $\vartheta'_1,\ldots,\vartheta'_s\in\mathcal{O}\left(\Omega'\right)$ be the coordinate functions on Ω' . Then we have $\varphi_{i|}=w'^*_i\circ\varphi=\Phi(w'_i)\in\mathcal{O}(\Omega)$ and $\varphi_{|j}=\vartheta'^*_j\circ\varphi=\Phi\left(\vartheta'_j\right)\in\mathcal{O}(\Omega)$ for all $i=1,\ldots,r$, $j=1,\ldots,s$. So φ is holomorphic.
'$\Leftarrow$' : trivial since if $f\in\mathcal{O}\left(\Omega'\right)$ then $f^*\in\mathcal{O}\left(\left(\Omega^{\#}\right)^{|s,\overline{s}}\right)$. $\square$
(v) apply (iv) to φ^* and $\left(\varphi^*\right)^{-1}$. $\square$

Definition 4.36 *Let $\Omega\subset\mathbb{C}^{p|q,\overline{q}}$. Ω is called super open in $\mathbb{C}^{p|q,\overline{q}}$ if and only if $\Omega^{\#}\underset{open}{\subset}\mathbb{C}^p$ and $\Omega=\left(\Omega^{\#}\right)^{|q,\overline{q}}$.*

In this case for the same reasons as in the real case we regard $\mathcal{D}\left(\Omega\right)=\mathcal{C}^\infty(\Omega)\simeq\Lambda\left(\mathbb{C}^q\right)\boxtimes\Lambda\left(\mathbb{C}^q\right)\otimes\mathcal{C}^\infty\left(\Omega^{\#}\right)^{\mathbb{C}}$ as the natural analogon of $\mathcal{C}^\infty$-functions resp. $\mathcal{O}\left(\Omega\right)\simeq\Lambda\left(\mathbb{C}^q\right)\otimes\mathcal{O}\left(\Omega^{\#}\right)$ as the natural analogon of holomorphic functions for the super open set Ω .

For all $f \in \mathcal{C}^{\infty}\left(\Omega^{\#}\right)^{\mathbb{C}}$ we identify $\widehat{f}$ and f , and so if $f \in \mathcal{D}(\Omega)$ we write

$$f = \sum_{I,J \in \wp(q)} \vartheta^I \overline{\vartheta}^J f_{IJ}$$

with uniquely determined $f_{IJ} \in \mathcal{C}^{\infty}\left(\Omega^{\#}\right)^{\mathbb{C}}$, $I, J \in \wp(q)$. Again on $\mathcal{D}(\Omega)$ the body map simply occurs as the restriction map

$$^{\#} : \mathcal{D}(\Omega) \to \mathcal{C}^{\infty}\left(\Omega^{\#}\right)^{\mathbb{C}} , \; f \mapsto f|_{\Omega^{\#}} ,$$

which is a continuous unital graded algebra epimorphism.
In terms of $f = \sum_{I,J \in \wp(q)} \vartheta^I \overline{\vartheta}^J f_{IJ} \in \mathcal{D}(\Omega)$ we have

$$\overline{f} = \sum_{I,J \in \wp(q)} (-1)^{\frac{(|I|+|J|)(|I|+|J|-1)}{2}} \vartheta^I \overline{\vartheta}^J f_{JI} .$$

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Zusammenfassung der Hauptresultate

Ausgangspunkt der in der vorliegenden Dissertation vorgestellten Forschungsarbeit bilden zwei Artikel von Svetlana KATOK und Tatyana FOTH:

- FOTH, Tatyana and KATOK, Svetlana: Spanning sets for automorphic forms and dynamics of the frame flow on complex hyperbolic spaces, [5] ,
- KATOK, Svetlana: Livshitz theorem for the unitary frame flow, [11] .

In diesen Artikeln werden Erzeugendensysteme für den Raum der Spitzenformen (cusp forms) konstruiert für den Fall eines beschränkten symmetrischen Gebietes B in $\mathbb{C}^n$ vom Rang 1 , welches nach Klassifikation biholomorph zur gewöhnlichen Einheitskugel in $\mathbb{C}^n$ ist, und eines Gitters $\Gamma \sqsubset G = \mathrm{Aut}_1(B)$. Dabei ist nach einem berühmten Satz von H. CARTAN die 1-Zusammenhangskomponente $\mathrm{Aut}_1(B)$ der Automorphismengruppe eines komplexen beschränkten symmetrischen Gebietes B stets eine halbeinfache LIE-Gruppe vom Hermiteschen Typ, und wir nennen eine diskrete Untergruppe Γ von $G = \mathrm{Aut}_1(B)$ ein Gitter genau dann wenn $\mathrm{vol}\, \Gamma\backslash G < \infty$.

Definition 0.1 (Automorphe Formen und Spitzenformen) *Sei G eine* LIE*-Gruppe, welche transitiv und holomorph auf B operiert, in der Regel $G = \mathrm{Aut}_1(B)$, und sei $\Gamma \sqsubset G$ eine diskrete Untergruppe. Seien $k \in \mathbb{N}$ und $j \in \mathcal{C}^\infty(G \times B)^{\mathbb{C}}$ ein Kozykel, holomorph in der zweiten Variablen. In der Regel ist $j(g\diamondsuit) = \det g'$ für alle $g \in G$.*

(i) Eine Funktion $f \in \mathcal{O}(B)$ heißt eine automorphe Form vom Gewicht k bzgl. Γ genau dann wenn gilt

$$f|_\gamma = f$$

für alle $\gamma \in \Gamma$ oder äquivalent $\widetilde{f} \in \mathcal{C}^\infty(G)^{\mathbb{C}}$ ist links-Γ-invariant, wobei $f|_g(\mathbf{Z}) := f(g\mathbf{Z})\, j(g, \mathbf{Z})^k$ und $\widetilde{f}(g) := f|_g(\mathbf{0})$ für alle $g \in G$ und $\mathbf{Z} \in B$. Der $\mathbb{C}$-Vektorraum der automorphen Formen vom Gewicht k bzgl. Γ wird mit $M_k(\Gamma)$ bezeichnet.

(ii) Eine automorphe Form $f \in M_k(\Gamma)$ heißt Spitzenform vom Gewicht k bzgl. Γ genau dann wenn gilt $\widetilde{f} \in L^2(\Gamma\backslash G)$. Der $\mathbb{C}$-Vektorraum aller Spitzenformen wird mit $S_k(\Gamma)$ bezeichnet. $S_k(\Gamma)$ ist ein Hilbert*-Raum mit dem Skalarprodukt*

$$(f, h)_\Gamma := \int_{\Gamma\backslash G} \overline{\widetilde{f}}\,\widetilde{h}$$

für alle $f, h \in S_k(\Gamma)$.

Foth und Katok benutzen einen neuen geometrischen Ansatz basierend auf dem Konzept hyperbolischer Flüsse auf Mannigfaltigkeiten. Dieses Konzept stammt ursprünglich aus der Theorie der dynamischen Systeme, siehe z. B. [10] . Grob gesagt, nennt man einen Fluß $(\varphi_t)_{t\in\mathbb{R}}$ auf einer Riemannschen Mannigfaltigkeit M hyperbolisch, wenn eine orthogonale und $(\varphi_t)_{t\in\mathbb{R}}$-stabile Zerlegung $TM = T^+ \oplus T^- \oplus T^0$ des Tangentialbündels TM existiert, sodass $(\varphi_t)_{t\in\mathbb{R}}$ gleichmäßig expandiert auf T^+ , gleichmäßig kontrahiert auf T^- , das Differential von $(\varphi_t)_{t\in\mathbb{R}}$ auf T^0 isometrisch wirkt und schließlich T^0 von $\partial_t\varphi_t$ erzeugt wird. Das berühmte Anosov-Schließungslemma (closing lemma) besagt, dass im Falle eines hyperbolischen Flusses auf einer Riemannschen Mannigfalitigkeit sich „neben" einer „fast" geschlossenen Bahn eines Punktes von M stets eine komplett geschlossene Bahn befindet. In der Tat folgt aus der Theorie der halbeinfachen Lie-Gruppen von Hermiteschem Typ, insbesondere aus der Wurzelzerlegung der Lie Algebra $\mathfrak{g}$ von $G = \mathrm{Aut}_1(B)$ bzgl. einer Cartan-Unteralgebra, dass der geodätische Fluss auf dem Einheitstangentenbündel $S(B)$ eines komplexen beschränkten symmetrischen Gebietes B vom Rang 1 hyperbolisch ist.

Ziel der vorliegenden Arbeit ist es nun, diese Artikel auf komplexe beschränkte symmetrische Gebiete B vom Rang > 1 sowie auf superautomorphe Formen und Super-Spitzenformen zu verallgemeinern.

In Kapitel 1 beschäftigen wir uns mit beschränkten symmetrischen Gebieten höheren Ranges. Dabei ist nach Klassifikation jedes beschränkte symmetrische Gebiet $B \subset \mathbb{C}^n$ biholomorph zur Einheitskugel in $\mathbb{C}^n$, $\mathbb{C}^n$ betrachtet als ein Hermitesches Jordan-Tripelsystem. Es gelingt mit Hilfe des Foth/Katokschen Ansatzes ein Erzeugensystem für $S_k(\Gamma)$ zu konstruieren für den Fall eines Produktes $B = B_1 \times \cdots \times B_q$ komplexer beschränkter symmetrischer Gebiete $B_1, \ldots, B_q$ vom Rang 1 (B besitzt damit Rang q) und kokompakter diskreter Untergruppe

$$\Gamma \sqsubset G = \mathrm{Aut}_1(B) = \mathrm{Aut}_1(B_1) \times \cdots \times \mathrm{Aut}_1(B_q)$$

unter Zusatzbedingungen (Satz 1.31 in Abschnitt 1.4). Hierfür müssen zunächst Hilfsmittel bereitgestellt bzw. auf den höheren Rang verallgemeinert werden. Zu diesen zählen insbesondere:

- Eine Klärung des Zusammenhangs zwischen maximal flachen total geodätischen (MFTG) Untermannigfaltigkeiten von B und maximal Abelschen Untergruppen ohne kompakten Anteil (maximal split Abelian subgroups) von G . Maximal Abelsche Untergruppen ohne kompakten Anteil von G stehen in Einszueinsbeziehung zu CARTAN-Unteralgebren der LIE-Algebra $\mathfrak{g}$ von G vermöge $\exp_G$.

  Hierfür benötigen wir die volle Theorie halbeinfacher LIE-Gruppen vom Hermiteschen Typ sowie die Hermitescher JORDAN-Tripelsysteme.

- Eine Verallgemeinerung der Theorie hyperbolischer Flüsse auf *partiell* hyperbolische Flüsse. Hierbei nennen wir einen Fluss $(\varphi_t)_{t\in\mathbf{R}}$ auf einer Riemannschen Mannigfaltigkeit M partiell hyperbolisch, wenn eine orthogonale und $(\varphi_t)_{t\in\mathbf{R}}$-stabile Zerlegung $TM = T^+ \oplus T^- \oplus T^0$ des Tangentialbündels TM existiert, sodass $(\varphi_t)_{t\in\mathbf{R}}$ gleichmäßig expandiert auf T^+ , gleichmäßig kontrahiert auf T^- , das Differential von $(\varphi_t)_{t\in\mathbf{R}}$ auf T^0 isometrisch wirkt und lediglich $\partial_t\varphi_t$ in T^0 enthalten ist. Wir entwickeln ein partielles ANOSOV-Schließungslemma für partiell hyperbolische Flüsse, welches besagt, dass sich „neben" einer Bahn, die sich modulo der T_0-Blätterung „fast" schließt, stets eine Bahn befindet, die sich modulo der T_0-Blätterung komplett schließt (Abschnitt 1.3).

Im zweiten Teil der Arbeit nun beschäftigen wir uns mit super-automorphen Formen und Super-Spitzenformen. Zu diesem Zweck befassen wir uns in Kapitel 2 zunächst mit ($\mathbb{Z}_2$-) graduierten algebraischen Strukturen, insbesondere graduierten Algebren, sowie mit der allgemeinen Theorie der Supermannigfaltigkeiten. Dabei heißt eine Algebra $\mathcal{A}$ über einem Körper K der Charakteristik $\neq 2$ eine graduierte Algebra genau dann wenn sie als K-Vektorraum in eine direkte Summe $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ zerfällt mit der Eigenschaft

$$\mathcal{A}_i\mathcal{A}_j \subset \mathcal{A}_{i+j}$$

für alle $i, j \in \mathbb{Z}_2$. Das Hauptbeispiel einer unitalen assoziativen graduierten (sogar graduiert-kommutativen) Algebra ist die GRASSMANN-Algebra $\Lambda(V)$ über einem K-Vektorraum V . In diesem Fall haben wir eine sogenannte Rumpfabbildung (body map) $^\# : \Lambda(V) \to K$, welche jedem Element aus $\Lambda(V)$ seinen konstanten Term zuordnet, und welche ein Algebraepimorphismus ist.

Eine Supermannigfaltigkeit ist kurz gesagt ein Objekt mit einem Paar $(p, q) \in \mathbb{N}^2$ als Dimension. Charakteristisch für eine Supermannigfaltigkeit $\mathcal{M}$ der Dimension (p, q) sind:

(i) $\mathcal{M}$ besitzt einen sogenannten Rumpf (body) $M = \mathcal{M}^\#$, welcher eine gewöhnliche $\mathcal{C}^\infty$-Mannigfaltigkeit der Dimension p ist.

(ii) $\mathcal{M}$ zugeordnet ist die graduierte Algebra $\mathcal{D}(\mathcal{M})$ der Superfunktionen auf $\mathcal{M}$. Ihre Elemente sind die globalen Schnitte einer Garbe $\mathcal{S}$, der sogenannten Strukturgarbe von $\mathcal{M}$, über M , welche lokal isomorph zu $\mathcal{C}^\infty_M \otimes \Lambda(\mathbb{R}^q)$ ist.

(iii) Es gibt eine Rumpfabbildung $^\# : \mathcal{S} \to \mathcal{C}^\infty_M$, und diese ist ein Garbenepimorphismus.

Besonders einfache Supermannigfaltigkeiten sind die super-offenen Mengen. Eine super-offene Menge $U^{|q}$ der Dimension (p, q) ist eine Supermannigfaltigkeit, deren Rumpf U eine offene Teilmenge des $\mathbb{R}^p$, und deren Strukturgarbe gerade $\mathcal{S} = \mathcal{C}^\infty_U \otimes \Lambda(\mathbb{R}^q)$ ist. Auf einer super-offenen Menge $U^{|q}$ der Dimension (p, q) haben wir also die geraden Koordinatenfunktionen $x_1, \ldots, x_p \in \mathcal{C}^\infty(U) \hookrightarrow \mathcal{D}\left(U^{|q}\right)_0$ und die ungeraden Koordinatenfunktionen $\zeta_1 := e_1, \ldots, \zeta_q := e_q \in \Lambda(\mathbb{R}^q) \hookrightarrow \mathcal{D}\left(U^{|q}\right)_1$.
In dieser Arbeit wird nun das Konzept einer Parametrisierung eingeführt, wobei die „Parameter" die (ungeraden) Koordinatenfunktionen $\alpha_1, \ldots, \alpha_n$ aus einer GRASSMANN-Algebra $\mathcal{P} := \Lambda(\mathbb{R}^n) = \mathcal{D}\left(\mathbb{R}^{|n}\right)$, $n \in \mathbb{N}$, sind. Dieses Konzept ist offenbar neu. Hier die Definition einer $\mathcal{P}$-Supermannigfaltigkeit bzw. eines $\mathcal{P}$-Supermorphismus:

Definition 0.2

(i) Seien M eine p-dimensionale $\mathcal{C}^\infty$-Mannigfaltigkeit und $q \in \mathbb{N}$. Sei $\mathcal{S}$ eine Garbe unitaler graduierter Algebren über M mit einer Garbeneinbettung $\mathcal{P} \hookrightarrow \mathcal{S}$ und einem Garbenhomomorphismus $^\# : \mathcal{S} \to \mathcal{C}^\infty_M$. Dann heißt das Tripel $\mathcal{M} := \left(M, \mathcal{S}, ^\#\right)$ eine (p, q)-dimensionale über $\mathcal{P}$ parametrisierte (oder kurz $\mathcal{P}$-) Supermannigfaltigkeit genau dann wenn für alle $x_0 \in M$ eine offene Umgebung $U \subset M$ sowie ein Garbenisomorphismus $\Phi : \mathcal{S}|_U \xrightarrow{\sim} \mathcal{C}^\infty_U \otimes \Lambda(\mathbb{R}^q) \boxtimes \mathcal{P}$ existieren, sodass $\Phi|_\mathcal{P} = id$ und

$$\begin{array}{ccc} \mathcal{S}|_U & \xrightarrow{\Phi} & \mathcal{C}^\infty_U \otimes \Lambda(\mathbb{R}^q) \boxtimes \mathcal{P} \\ \# \searrow & \% & \swarrow \# \\ & \mathcal{C}^\infty_U & \end{array} .$$

$M := \mathcal{M}^{\#}$ heißt dabei der Rumpf (body) und $\mathcal{S}$ die Strukturgarbe der Supermannigfaltigkeit $\mathcal{M}$. Wir schreiben $\mathcal{D}(\mathcal{M}) := \mathcal{S}(M)$.

(ii) Seien $\mathcal{M} = \left(M, \mathcal{S}, {}^{\#}\right)$ und $\mathcal{N} = \left(N, \mathcal{T}, {}^{\#}\right)$ zwei $\mathcal{P}$-Supermannigfaltigkeiten, $\varphi : M \to N$ eine $\mathcal{C}^\infty$-Abbildung und $(\Phi_W)_{W \subset N \text{ offen}}$ eine Familie von unitalen graduierten Algebrahomomorphismen $\Phi_W : \mathcal{T}(W) \to \mathcal{S}\left(\varphi^{-1}(W)\right)$ mit der Eigenschaft, dass für alle $W' \subset W \subset N$ offen

$$\begin{array}{ccc} \mathcal{T}(W) & \xrightarrow{\Phi_W} & \mathcal{S}\left(\varphi^{-1}(W)\right) \\ |_{W'} \downarrow & \% & \downarrow |_{\varphi^{-1}(W')} \\ \mathcal{T}(W') & \underset{\Phi_{W'}}{\longrightarrow} & \mathcal{S}\left(\varphi^{-1}(W')\right) \end{array}$$

(das heißt gerade, dass das Paar $\Phi := \left(\varphi, (\Phi_W)_{W \subset N \text{ offen}}\right)$ ein Morphismus der geringten Räume $(M, \mathcal{S})$ und $(N, \mathcal{T})$ ist). Dann heißt das Paar $\Phi := \left(\varphi, (\Phi_W)_{W \subset N \text{ offen}}\right)$ ein $\mathcal{P}$-Supermorphismus von $\mathcal{M}$ nach $\mathcal{N}$ genau dann wenn für alle $W \subset N$ offen $\Phi_W|_{\mathcal{P}} = id$ und für alle $f \in \mathcal{T}(W)$

$$(\Phi_W(f))^{\#} = f^{\#} \circ \varphi|_{\varphi^{-1}(W)}$$

gilt.

Setzen wir $n = 0$ (äquivalent $\mathcal{P} = \mathbb{R}$) , so erhalten wir die Definition einer gewöhnlichen Supermannigfaltigkeit bzw. eines gewöhnlichen Supermorphismus zurück. Es zeigt sich, dass die $\mathcal{P}$-Supermannigfaltigkeiten zusammen mit $\mathcal{P}$-Supermorphismen eine Kategorie bilden, welche eine echte Erweiterung der Kategorie der Supermannigfaltigkeiten mit Supermorphismen darstellt. Als Hauptresultat der Untersuchungen über $\mathcal{P}$-Supermannigfaltigkeiten lässt sich sicherlich das Ergebnis bezeichnen, dass auch in der Kategorie der $\mathcal{P}$-Supermannigfaltigkeiten ein Kreuzprodukt existiert (siehe die Sätze am Schluss von Abschnitt 2.2): Sind nämlich $\mathcal{M} = \left(M, \mathcal{S}, {}^{\#}\right)$ und $\mathcal{N} = \left(N, \mathcal{T}, {}^{\#}\right)$ zwei $\mathcal{P}$-Supermannigfaltigkeiten der Dimension (p, q) bzw. (r, s) , so ist eine Realisierung ihres Kreuzproduktes gegeben durch

$$\mathcal{M} \times_{\mathcal{P}} \mathcal{N} = \left(M \times N, \left(\mathrm{pr}_1^* \mathcal{S} \hat{\boxtimes} \mathrm{pr}_2^* \mathcal{T}\right) \Big/ \mathcal{I}, {}^{\#}\right) ,$$

wobei $\mathcal{I}$ die Idealgarbe von $\mathrm{pr}_1^* \mathcal{S} \hat{\boxtimes} \mathrm{pr}_2^* \mathcal{T}$ bezeichnet, welche von allen $\alpha_j \otimes 1 - 1 \otimes \alpha_j$, $j = 1, \ldots, n$, erzeugt wird, und $\boxtimes$ das graduierte Tensorprodukt bezeichnet. $\mathcal{P}$ wird eingebettet in die Garbe $\left(\mathrm{pr}_1^* \mathcal{S} \hat{\boxtimes} \mathrm{pr}_2^* \mathcal{T}\right) \Big/ \mathcal{I}$ gemäß

$$\mathcal{P} \hookrightarrow \left(\mathrm{pr}_1^* \mathcal{S} \hat{\boxtimes} \mathrm{pr}_2^* \mathcal{T}\right) \Big/ \mathcal{I} \, , \; R \mapsto R \otimes 1 + \mathcal{I} = 1 \otimes R + \mathcal{I} \, .$$

$\mathcal{M} \times_{\mathcal{P}} \mathcal{N}$ ist also eine $(p+r, q+s)$-dimensionale $\mathcal{P}$-Supermannigfaltigkeit mit Rumpf $M \times N$.

Der Grund, warum wir das Konzept der Parametrisierung einführen ist - abgesehen von der Eleganz der Theorie selbst - der folgende: Für die Definition von super-automorphen Formen bzw. Super-Spitzenformen benötigen wir analog zum klassischen Fall eine diskrete Untergruppe einer Super-LIE-Gruppe, welche auf einem beschränkten symmetrischen Supergebiet operiert. Eine Super-LIE-Gruppe $\mathcal{G}$ besitzt als Rumpf eine gewöhnliche $\mathcal{C}^\infty$-LIE-Gruppe $G := \mathcal{G}^\#$, und es zeigt sich, dass eine diskrete Untergruppe Γ von $\mathcal{G}$ nichts anderes ist als eine diskrete Untergruppe ihres Rumpfes G . Betrachten wir stattdessen *parametrisierte* diskrete Untergruppen von $\mathcal{G}$, so erhalten wir eine echt größere Klasse von Untergruppen, nicht notwendigerweise im Rumpf G enthalten.

In Kapitel 3 untersuchen wir nun super-automorphe Formen und Super-Spitzenformen. Wir beschränken uns dabei auf den Fall eines beschränkten symmetrischen Supergebietes $\mathcal{B} := B^{p,q|r}$, $p, q, r \in \mathbb{N}$, $p, q \geq 1$, welches die eindeutig bestimmte komplexe super-offene Menge von komplexer Dimension (pq, rq) ist mit der Matrizeneinheitskugel $B := B^{p,q} \subset \mathbb{C}^{p \times q}$ als Rumpf, und der super-speziellen pseudounitaren Gruppe $\mathcal{G} := sSU(p, q|r)$. $sSU(p, q|r)$ ist eine reelle $\big((p+q)^2 + r^2 - 1, 2(p+q)r\big)$-dimensionale Super-LIE-Gruppe mit Rumpf

$$sS\left(U(p,q) \times U(r)\right) := \left\{ \left(\begin{array}{c|c} g & 0 \\ \hline 0 & E \end{array} \right) \in U(p,q) \times U(r) \;\middle|\; \det g = \det E \right\} ,$$

und sie operiert transitiv auf $\mathcal{B}$ durch gebrochen rationale (MÖBIUS-) Transformation. Da zur Zeit noch unklar ist, wie das Kozept eines Fundamentalbereiches einer diskreten Untergruppe Γ einer gewöhnlichen LIE-Gruppe auf parametrisierte diskrete Untergruppen einer Super-LIE-Gruppe zu verallgemeinern ist, können wir bislang nur für den Fall einer nicht-parametrisierten diskreten Untergruppe eine Definition des Raums der Spitzenformen als HILBERT-Raum geben. Im Falle einer nicht-parametrisierten diskreten Untergruppe Γ gelingt es uns, mit Hilfe des FOTH/KATOKschen Ansatzes ein Erzeugendensystem für den Raum $sS_k(\Gamma)$ der Super-Spitzenformen zum Gewicht k zu konstruieren für $q = 1$ und $\Gamma\backslash G$ kokompakt **oder** $q = 1$, $p \geq 2$ und vol $\Gamma\backslash G < \infty$ unter der Zusatzvoraussetzung, dass die Rechtstranslation der maximal Abelschen Untergruppe A von G ohne kompakten Anteil (da $q = 1$ also $A \simeq \mathbb{R}$) auf $\Gamma\backslash G$ topologisch transitiv ist (Satz 3.17 in Abschnitt 3.3). Zu diesem Zweck wird zunächst ein Analogon des Satzes von

SATAKE für super-automorphe Formen gewonnen (Satz 3.11 in Abschnitt 3.2):

> Sei $\Gamma\backslash G$ kompakt **oder** $q = 1$, $p \geq 2$ und $\Gamma \sqsubset G$ ein Gitter. Dann existiert ein $k_0 \in \mathbb{N}$ mit der Eigenschaft, dass
>
> $$sS_k(\Gamma) = sM_k(\Gamma) \cap L^s\left(\Gamma\backslash B\right)$$
>
> bzgl. eines geeigneten Maßes auf $\Gamma\backslash B$ für alle $s \in [1, \infty]$ und $k \geq k_0$, wobei $sM_k(\Gamma)$ den Raum der super-automorphen Formen zum Gewicht k bzgl. Γ bezeichnet.

Im ersten Fall ist die Aussage trivial, und dort $sS_k(\Gamma) = sM_k(\Gamma)$ für alle $k \in \mathbb{Z}$. Im zweiten Fall läuft der Beweis analog zum klassischen Fall über die FOURIER-Entwicklung einer super-automorphen Form an den Spitzen des Quotienten $\Gamma\backslash B$, und wir erhalten $k_0 = 2p$.

Für parametrisierte diskrete Untergruppen (also diskrete $\mathcal{P}$-Untergruppen mit einem geeigneten $\mathcal{P} = \Lambda(\mathbb{R}^n)$, $n \in \mathbb{N}$) Υ von $\mathcal{G}$ erzielen wir partielle Resultate indem wir Υ als eine Störung ihres Rumpfes $\Gamma := \Upsilon^\#$ betrachten. In drei Spezialfällen gelingt es uns, den Raum $sS_k(\Upsilon)$ der Spitzenformen vom Gewicht k bzgl. Υ als $\mathcal{P}^\mathbb{C}$-Untermodul des $\mathcal{P}^\mathbb{C}$-Moduls $sM_k(\Upsilon)$ der super-automorphen Funktionen vom Gewicht k bzgl. Υ zu definieren. Es sind die folgenden Fälle:

(i) $\Gamma\backslash G$ kompakt. In diesem Falle definieren wir $sS_k(\Upsilon) := sM_k(\Upsilon)$,

(ii) Es existiert ein parametrisiertes Element $g \in_\mathcal{P} \mathcal{G}$ mit der Eigenschaft, dass $\Upsilon = g\Gamma g^{-1}$. In diesem Falle definieren wir $sS_k(\Upsilon)$ als das Bild von $sS_k(\Gamma) \boxtimes \mathcal{P}^\mathbb{C}$ unter dem Isomorphismus

$$\Phi : sM_k(\Gamma) \boxtimes \mathcal{P}^\mathbb{C} \to sM_k(\Upsilon)\,,\ f \mapsto f|_{g^{-1}}\,.$$

(iii) $q = 1$, $p \geq 2$, $\mathrm{vol}\left(\Gamma\backslash G\right) < \infty$, jedoch $\Gamma\backslash G$ **nicht** kompakt unter einer Zusatzbedingung. Dieses ist der deutlich schwierigste Fall, und hier benutzen wir wieder die FOURIER-Entwicklung einer Spitzenform $f \in sS_k(\Gamma)$ an den Spitzen des Quotienten $\Gamma\backslash B$.

Ziel ist es nun ausgehend von einem Erzeugendensystem $(\varphi_\lambda)_{\lambda\in\Lambda}$ von $sS_k(\Gamma)$, die Spitzenformen $\varphi_\lambda \in sS_k(\Gamma)$ so zu Elementen $\psi_\lambda \in sS_k(\Upsilon)$ zu deformieren, dass $(\psi_\lambda)_{\lambda\in\Lambda}$ ein Erzeugendensystem des $\mathcal{P}^\mathbb{C}$-Moduls darstellt. Dies gelingt im Fall (ii) : Wir definieren hier $\psi_\lambda := \varphi_\lambda|_{g^{-1}}$ und weisen nach, dass tatsächlich in einem gewissen Sinne $(\psi_\lambda)_{\lambda\in\Lambda}$ ein Erzeugendensystem von $sS_k(\Upsilon)$ ist (Satz 3.32 (ii) in Abschnitt 3.4). In den Fällen (i) und (iii) besteht hier weiterer Klärungsbedarf.

In Kapitel 4 schließlich wird eine punktweise Realisierung von super-offenen Mengen gegeben unter Benutzung von Superzahlen im Unterschied zu Kapitel 2 , wo wir super-offene Mengen als geringte Räume beschreiben. Auch dieser Aspekt der Theorie der Supermannigfaltigkeiten ist augenscheinlich neu, obwohl Superzahlen z. B. von B. de Witt in [17] betrachtet werden. Wir zeigen, dass für eine super-offene Menge $U^{|q}$ die graduierte Algebra $\mathcal{D}\left(U^{|q}\right) = \mathcal{C}^\infty(U) \otimes \Lambda\left(\mathbb{R}^q\right)$ der Superfunktionen auf $U^{|q}$ nichts anderes ist als die in einem gewissen Sinne reduzierte graduierte Algebra der stetigen und partiell differenzierbaren Funktionen auf dem topologischen Raum $U^{|q}$ (Satz 4.8 in Abschnitt 4.1), und dass die Algebra $\mathcal{D}\left(U^{|q}\right)$ die Menge $U^{|q}$ bis auf Diffeomorphismen eindeutig bestimmt (Sätze 4.15 und 4.18 in Abschnitt 4.1). Bemerkenswert in diesem Zusammenhang ist, dass $\mathcal{D}\left(U^{|q}\right)$ gleichzeitig die (reduzierte) graduierte Algebra aller unendlich oft differenzierbaren Funktionen auf $U^{|q}$ ist.
Mit Hilfe der punktweisen Realisierung von super-offenen Mengen wird es möglicherweise in der Zukunft gelingen, Fundamentalbereiche für *parametrisierte* diskrete Untergruppen beschreiben zu können.

Ich möchte an dieser Stelle noch einmal allen Mitgliedern des Fachbereichs, die mich wohlwollend während meiner Zeit in Marburg begleitet haben, für ihre Unterstützung danken, dazu zähle ich insbesondere meinen Betreuer Prof. Dr. H. Upmeier sowie den Zweitgutachter der Dissertation Prof. Dr. F. W. Knöller .

Erklärung

Ich versichere, dass ich die vorliegende Dissertation - einschließlich beigefügter Zeichnungen - selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe. Die Dissertation habe ich in der vorliegenden oder ähnlichen Form noch nicht zu Prüfungszwecken eingereicht.

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Marburg, den *30. März 2007*