MAGKS

Joint Discussion Paper
Series in Economics

by the Universities of

Aachen · Gießen · Göttingen
Kassel · Marburg · Siegen

ISSN 1867-3678

No. 46-2022

Andreas Marcus Gohs

The Choice of GARCH Models to Forecast Value-at-Risk for Currencies (Euro Exchange Rates), Crypto Assets (Bitcoin and Ethereum), Gold, Silver and Crude Oil: Automated Processes, Statistical Distribution Models and the Specification of the Mean Equation

This paper can be downloaded from:

https://www.uni-marburg.de/en/fb02/research-groups/economics/macroeconomics/research/magks-joint-discussion-papers-in-economics

**Coordination: Bernd Hayo • Philipps-University Marburg
School of Business and Economics • Universitätsstraße 24, D-35032 Marburg
Tel: +49-6421-2823091, Fax: +49-6421-2823088, e-mail: hayo@wiwi.uni-marburg.de**

The Choice of GARCH Models to Forecast Value-at-Risk for Currencies (Euro Exchange Rates), Crypto Assets (Bitcoin and Ethereum), Gold, Silver and Crude Oil: Automated Processes, Statistical Distribution Models and the Specification of the Mean Equation

Andreas Marcus Gohs
University of Kassel
Institute of Economics

JEL Classification: G17 F31 G01 G11

Keywords: Conditional volatility, Skew Student T, Markov Switching MS-GARCH, Multivariate GARCH, Mean Excess Loss, Default Correlation, Software R

Abstract

Regular or automated processes require reliable software applications that provide accurate volatility and Value-at-Risk forecasts. The univariate and multivariate GARCH models proposed in the literature are reviewed and the suitability of selected R functions for automated forecasting systems is discussed. With the Markov-switching GARCH function constructed for modelling regime changes, parameter estimates are reliably obtained in studies with moving time windows. In contrast, in the case of structural breaks or outliers, the algorithm of the ordinary GARCH function often does not return valid parameter estimates and fails.

VaR prognoses are produced for extreme quantiles (up to 99.9%) and three alternative distribution assumptions (Skew Student-T, Student-T and Gaussian). Accurate one-day-ahead VaR predictions up to the 99% quantile are generally obtained for the time series when Skew Student-T distributed innovations are assumed. The VaR exceedance rates and their percentage deviations from the target alpha as well as the mean and median excess loss are reported.

The accompanying mean equation is often omitted when fitting GARCH models to heteroskedastic time series. The impact of this on the accuracy of VaR forecasts is investigated.

Coefficients of the ordinary (Pearson) and the default correlation are calculated for moving time windows. Since the calculated default correlation depends on the VaR forecasts, analyses are performed for different quantiles, the ordinary and the MS-GARCH function and specifications of mean equations.

Introduction

Preliminary considerations

Market participants have an interest in protecting themselves against the consequences of excessive losses in the event of fluctuating market prices of their exposures (e.g. securities investments, foreign currency risks from international trading, etc.). For risk management, the use of the Value-at-Risk (VaR) measure in particular has become established. The VaR indicates the loss from an exposure that will not be exceeded with a certain (high) probability over a predetermined period of time. In order to avoid liquidity bottlenecks, coverage capital in the amount of the VaR can be held for a risky position or an entire portfolio. GARCH models have become established for volatility forecasts and the VaR forecasts generated from them. The present work is intended to contribute to achieving accurate VaR forecasts.

Various extensions for GARCH models are presented in the literature. These can basically be justified by the fact that time series can be based on various stochastic processes of data generation or that random individual events can also be represented in the time series. Since time series can differ in their properties (e.g. volatility structures, structural breaks, regimes, outliers, etc.), different suitable models are also conceivable. However, the accuracy of volatility or VaR forecasts is essentially influenced not only by the structure of the fitted model equation but also by the distribution model used for the innovations (i.e. residuals): the normal distribution, which is traditionally used for modelling scientific phenomena, is less suitable for modelling return series. The Gaussian bell curve does not do justice to the frequency of extreme outcomes, i.e. masses at the edges of the return distributions, which are of particular interest for the VaR measure. The frequency of extreme returns (or losses) is clearly underestimated with the normal distribution. In recent years, distribution models have been proposed that provide a much more accurate fit to return series. However, no new standard model has yet been found or established that adequately represents the fat tails of the distributions. Obviously, the distribution models that have been focused on recently are also not able to model the extreme tail of a return distribution quite accurately.

Taking into account the descriptions of ARMA GARCH models in scientific studies, it can be assumed here that a textbook adaptation of these methods to financial market time series is dispensed with in the application. It is assumed here that in applications of forecasting models, e.g. in the financial industry, a simple lag structure is usually chosen for GARCH models. An implementation of complex GARCH models in daily applications may be dispensed with. However, such a procedure may be justified due to the sufficient approximation of complex GARCH models by simple ones. Furthermore, the scientific studies support the assumption that modelling the dynamics of the expected value or the mean return is dispensed with. I.e. the textbook establishment of a mean value equation within the framework of GARCH models is (entirely) dispensed with. This approach is also justified if the abandonment of the expected value or the mean value of a time series can be justified by the fact that it tends towards zero for high-frequency time series.

Structure of the paper

The next section provides an overview of recent literature examining the VaR forecasting performance of different variants of *GARCH* models under different distributional assumptions. The research topic is then described in more detail. Afterwards, the time series used for the study are presented. With regard to the different asset classes or risky exposures represented by the time series used, an explanation of their economic relevance and risk management is provided.

This is followed by a chronological description of the crises or major events that have historically caused stronger price fluctuations and have caused temporary or permanent changes on the world markets. These crises and changes are also reflected in the time series.

In a further part, the requirements of automated forecasting systems for VaR forecasts are discussed. In this context, preliminary considerations are also made about flexible variants of *GARCH* models and their R-functions, which guarantee stable forecasts. In addition, variance-covariance matrices are presented, which are used to represent the relationships between different risky exposures.

A section on methods provides an overview of the univariate and *Multivariate GARCH* models proposed in the literature. ARIMA models, which are used to model the mean equations supplementing the volatility equations, are discussed first. With regard to own VaR forecasts, descriptions of the study design are provided. The Value-at-Risk concept and other risk management concepts are presented.

The results of the study are discussed in detail in a further section. This is followed by a summary and hints or suggestions for further research.

Literature Review

Heracleous (2003) inserts the Student-T-distribution in univariate and multivariate volatility models.

Yoon & Kang (2007) fit Fractionally Integrated GARCH (FIGARCH)-models to Japanese financial time series. The authors find more accurate VaR estimations with the Skewed Student-T compared to the Student-T or Normal Distributions. FIGARCH is an extension of Baillie, Bollerslev & Mikkelsen (1996) to model long memory volatility processes.[1]

Osterrieder & Lorenz (2016) recognise: *„Bitcoin returns are much more volatile (albeit with volatility levels decreasing over the course of the last few years), much riskier and exhibit heavier tail behaviour than the traditional fiat currencies.*[…] *Using the traditional tail risk measures Value-at-Risk and expected shortfall, we could quantify that extreme events lead to losses in Bitcoin which are about eight times higher than what we can expect from the G10 currencies."*

The authors consider the Generalized Pareto distribution (GPD) and the Generalized Extreme Value distribution because of their important role in Extreme Value Theory. They also compute VaR (95%-quantile) for the Bitcoin assuming Gaussian distributed innovations.

Colucci (2016) suggests the use of Gaussian innovations for the 95% VaR forecasts, but prefers Student-T innovations for the extreme tail of the Bitcoin return distribution. However, he obtains the best results for the extreme tail and in particular the regulatory relevant 99% quantile with the Historical Filtered Bootstrap (HFBVaR-N) model.

Colucci (2018) considers Gaussian and Student-T distributed innovations in One day-ahead VaR prognosis for the Bitcoin. He also applies more sophisticated methods such as the Historical Simulation and the Extreme Value Theory Historical Filtered Bootstrap. For the mean equation, he fits AR(1) models.

[1] Cf. Tayefi & Ramanathan (2012) concerning FIGARCH and related time series models

Ardia et al. (2019b) find clear evidence of regime changes in the Bitcoin return series for a period from 13 June 2014 to 2 March 2018. The authors conclude that the MS-GARCH model with two regimes considerably outperforms the one day ahead VaR forecast of the ordinary GARCH model (with one regime). The Skewed Student-T -distribution provides a particularly good fit. However, the study by Ardia et al. (2019b) is limited to Bitcoin time series with their particular characteristics (including high volatility). However, the question arises whether the MS-GARCH model with employed distribution Skew Student-T also provides a better fit than the ordinary GARCH model for other time series. This paper aims to clarify this question.

Research Topic

The paper reviews and discusses extensions of the GARCH model and simplified implementations in practical applications.

The ordinary GARCH and the Markov Switching GARCH models are used to produce one-day-ahead Value-at-Risk forecasts. The VaR forecasts are made on the basis of variable time series segments, i.e. rolling and growing time windows. Different distribution assumptions (Gaussian, Student-T, and Skew Student-T) are run and accuracies of VaR forecasts are calculated for a range of quantiles of the distributions.

Correlations and default correlations are calculated for different asset pairs. The development of the default correlations is shown by means of successively moving time windows. From this, it is possible to see whether historical events have changed these relationships.

It is examined how simplifying (i.e. imprecise) procedures in modelling the dynamics of the expected value (i.e. the mean equation) using ARIMA affect the VaR forecast accuracy and the calculated default correlations.

Time Series Data

For the study, exchange rates for currencies whose time series are provided by the *European Central Bank (ECB)* were used, among others.[2] The euro exchange rates are given in indirect quotation: US dollar (USD/EUR), Japanese yen (JPY/EUR), Chinese yuan renminbi (CNY/EUR), British pound sterling (GBP/EUR), Swiss franc (CHF/EUR). The time series of the euro exchange rates are provided in trading-day periodicity (i.e. for five days per week, excluding public holidays) for the period from 4 January 1999 to 3 January 2020. The time series for the Chinese yuan renminbi starts on 13 January 2000.

In addition, the price of Brent crude oil in US dollars per barrel, the gold price in US dollars per troy ounce and the silver price in US dollars per troy ounce were available. The time series were used in trading-day periodicity for the period from 3 January 2006 to 13 February 2020. They were retrieved from the internet platform *finanzen.net*.[3]

Furthermore, time series of the US dollar prices of two crypto-assets were available. These are reported in daily periodicity (i.e. seven days per week, excluding holidays). The Bitcoin (USD/BTC) time series is available for the period from 1 October 2013 to 3 January 2020, the Ethereum

[2] The exchange rates ("Collection: Average of observations through period (A)") were retrieved from the ECB's website on 4 January 2020: http://sdw.ecb.europa.eu/browse.do?node=9691296

[3] Closing prices were downloaded from https://www.finanzen.net/rohstoffe/

(USD/ETH) time series starts on 9 August 2015. The data source for the crypto assets is the internet platform *CoinDesk*.[4]

At irregular intervals, no quotations were recorded for some days. Due to the different public holidays in the countries, these days only partially coincide in the time series.

Figures of the time series used in the study are presented in the appendix.

Risky Exposures or Asset Classes

Besides shares and bonds as classic asset classes, precious metals were also considered. In the low-interest phase of the past years, however, bonds were hardly attractive for investors. Investors were looking for alternative investments that promised higher yields. As private wealth grew, there was also a need to diversify capital to reduce or avoid cluster risks. As alternative investments, real estate became more and more in the public's awareness due to rising real estate prices. The emerging crypto-assets (so-called "*cryptocurrencies*") were also perceived by some investors as an interesting investment alternative. Crypto assets are fungible, which is supported by the possibility to buy shares. This new form of investment has brought investors enormous speculative profits at times since its emergence. As the "leading cryptocurrency", bitcoin initially recorded enormous price increases in its comparatively short history, but also high losses in the meantime. The hope that it could achieve the status of an alternative means of payment (independent of governments and central banks) has not yet been fulfilled, mainly because of its strong price fluctuations.

From the perspective of an investor in a country with a stable currency (i.e. not subject to significant inflation), investments in foreign currencies tend to play a subordinate role. However, foreign currencies can be held for intended trading transactions, which may be acquired at a supposedly favourable exchange rate. Investments in foreign currencies can also be made speculatively and indirectly, e.g. through foreign stocks. From an economic perspective, exchange rate developments play a role depending on the size of the country and the extent of its international trade relations (i.e. its external balance).

The oil price can affect the economic development of both oil-exporting and oil-importing countries. Due to existing dependencies for energy supply, the oil price has a strong impact on industrial companies.

Market participants' need for instruments to hedge against (loss) risks in the event of high volatility of an underlying security (or exchange rate) is met with standardised (or fungible) financial products such as futures and warrants. However, derivative financial products can be used not only to hedge actual positions of their associated underlying securities in an investor's portfolio, but also (in the absence of regulation) for speculative purposes with sometimes considerable risks and opportunities. In this context, the volatility of the price of the underlying security has a significant influence on the derivative price. Duan (1995) therefore proposed a GARCH option pricing model that reflects changes in the conditional volatility of the underlying asset. However, the historical trading of derivative financial instruments has amplified the market price volatilities (of the underlying securities).

For the unhedged exposures, risk management should preserve coverage capital in the amount of the VaR to ensure solvency. GARCH models are likewise used for VaR forecasts.

[4] Closing prices were obtained for the crypto assets at www.coindesk.com

Historical Events with an Impact on International Financial Markets and Exchange Rates

Especially in phases of upheaval, uncertainty increases. In crisis scenarios, the information situation can change from day to day, so that investors become more insecure or the uncertainty decreases again. This leads to drastic price changes in short periods of time, accompanied by higher fluctuation amplitudes or changes in the time-related volatilities. The events stand out in the time series as structural breaks or phases of increased volatilities.

Historically, the following significant events, among others, led to permanently changed market conditions:

In 1944, 44 countries signed an agreement in Bretton Woods (USA) that included the establishment of the World Bank and the International Monetary Fund (IMF) as well as a system of fixed exchange rates. The system of fixed exchange rates came into force in 1946 and linked major currencies such as the Japanese yen, the British pound, the Swiss franc and the German mark to the US dollar as the lead currency. There was a gold standard for the US dollar. However, as early as the 1960s, the US dollar could no longer be covered by the gold reserves of the Federal Reserve Bank. In August 1971, the gold dollar standard collapsed when the Fed refused to meet its obligation to exchange the US dollar for gold at a fixed price. The fixed exchange rate system agreed in the Bretton Woods Agreement was also temporarily suspended in 1971 and finally ended in 1973. The dissolution of the system led to fluctuations in the then flexible exchange rates.[5]

The oil price shocks in 1973 and 1979/1980 led to recessions in industrialised countries and to a rise in inflation and interest rates.

On *Black Monday* (19 October 1987), stock prices plummeted on the *Hong Kong Stock Exchange* and around the world during the day. It was the first stock market crash after the Second World War.

Japan experienced an economic crisis when price bubbles burst on the stock and real estate markets in 1990. In the following decade, the ongoing recession could not be overcome even by keeping key interest rates low.

Iraq's capture of Kuwait in August 1990 triggered the First Iraq War (Second Gulf War). Oil prices temporarily rose sharply due to fears of an oil price crisis. In early 1991, allied troops led by the United States carried out a ground offensive to free Kuwait. US President Bush declared the fighting over at the end of February 1991.

The Asian crisis of 1997/1998 was a financial, monetary and economic crisis that started in Thailand and spread to the Tiger and Panther states in March 1997. The People's Republic of China, however, was hardly affected.

On 2 May 1998, heads of state and government of several present-day EU countries decided to introduce the common currency, the euro. On 31 December 1998, the exchange rates of the currencies of participating countries were fixed and on 1 January 1999 the euro was initially introduced as book money.[6] On 1 January 2002, the euro also replaced the old currencies as cash.

[5] Due to the then flexible exchange rates, a need for hedging instruments such as futures contracts and warrants against the new currency uncertainties was also allowed to arise.

[6] In 1999, the following countries launched the euro: Austria, Belgium, Finland, France, Germany, Ireland, Italy, Luxembourg, Monaco, Netherlands, Portugal, San Marino, Spain and the Vatican City. 2001 Greece, 2002

In March 2000, the dot-com bubble burst, which had been built up by exaggerated profit expectations for technology companies of the new economy.

The terrorist attacks of 11 September 2001 on the *World Trade Center* in New York, among others, provided a rationale for US warfare against Afghanistan and Iraq. In October 2001, NATO troops under US leadership invaded Afghanistan, and in March 2003, US troops invaded Iraq (Second Iraq War). This led to destabilisation in the Middle East.

The subprime crisis was triggered by a bursting of the price bubble in the US real estate market after real estate prices peaked in 2006. Debtors with poor credit ratings ("subprime") benefited from years of rising real estate prices that allowed them to take out new loans. When the price bubble burst, the debts were not matched by sufficiently high real estate assets. However, US mortgage banks had made their claims tradable through securitisation and tranching and partially refinanced themselves through global trading. The contents of the traded packages of pro-rata loan receivables could no longer be transparently traced. However, they were given ratings that did not do justice to the actual volume of bad loans. The lack of transparency about shares of bad loans in bank portfolios worldwide led to a crisis of trust, so that banks no longer lent money to each other because of the insolvency risks. On 15 September 2008, the US investment bank *Lehman Brothers* went bankrupt, which is considered the climax of the financial crisis.

After the experience of the insolvency of the US investment bank *Lehman Brothers*, other countries tried to protect their (system-relevant) banks from insolvency. The government intervention schemes led to a sharp increase in public debt. The US central bank *Fed* and the *European Central Bank ECB* pursued a policy of low key interest rates and bought up bonds on a massive scale to counteract an impending recession.[7]

The *"euro crisis"* began in October 2009 when the new government of Greece declared earlier declarations of net new debt in 2009 to be false.[8] To avoid sovereign defaults due to excessive public debt, the *European Financial Stabilisation Facility (EFSF)* was established in 2010 and the *European Stability Mechanism (ESM)* in 2012. Greece was particularly affected by the threat of insolvency in 2015 and 2016 and repeatedly demanded loans from the other euro member states.

On 23 June 2016, an EU referendum was held on whether the United Kingdom should remain in the EU (the so-called *"Brexit referendum"*). In the referendum, 51.89% of voters voted in favour of leaving the EU. The EU referendum on Brexit led to a strong devaluation of the British pound against other leading currencies. The British government repeatedly agreed with the European Council to postpone the exit date. There were renegotiations on the withdrawal agreement in particular because of domestic political resistance in connection with the *"backstop" clause*, which was supposed to prevent a hard border between Ireland and the United Kingdom. The United Kingdom exited the EU on 31 January 2020.

In March 2020, a pandemic emerged due to the uncontainable spread of Covid 19 infections. Global stock markets plummeted as a recession was feared. Although the pandemic was still ongoing in 2020/2021, stock markets recovered and indices climbed to new highs. In 2020, the Corona crisis has made it difficult to turn away from the zero interest rate policy and a reduction in public debt.

Kosovo and Montenegro, 2007 Slovenia, 2008 Malta and Cyprus, 2009 Slovakia, 2011 Estonia, 2014 Latvia and Andorra, 2015 Lithuania.

[7] The *ECB*'s key interest rate was at an interim high of 4.25% on 9 July 2008 and fell to 0% by March 2016.

[8] After Greece's joining the euro area, it was questioned whether Greece actually fulfilled the entry criteria.

On the Requirements of Automated Value-at-Risk Forecasting Systems and Data Sets Suitable for Them

Established models for time series analysis or volatility forecasting

In order to find a suitable forecasting model, the properties of the time series must be taken into account. Time series of financial markets usually exhibit time-varying volatility: Graphical representations of heteroskedastic return series show that clusters of high and low volatility alternate. These reflect turbulent or tranquil market phases. The associated price level time series (e.g. of Bitcoin prices, exchange rates, gold prices, etc.) are usually also characterised by a lack of stationarity, i.e. a time-varying mean value (i.e. deterministic trend) or a volatility that increases with the forecast horizon across all boundaries (i.e. stochastic trend).[9] The price changes (or log returns) calculated by difference taking (if necessary after logrithmisation) are usually stationary, but still autocorrelated; i.e. there are dependencies between successive time series values or between their squares or time-related volatilities.[10]

Generalised Autoregressive Conditional Heteroscedasticity (GARCH) models have become established for modelling and forecasting time-dependent volatility. The (ordinary) GARCH model contains a volatility equation that is supplemented by a mean equation adjusted to the time series. *Autoregressive Integrated Moving Average (ARIMA)* models are usually used as time series models for forecasting the time-dependent expected value - i.e. forecast (mean) value - of a variable. However, if the mean or expected value (e.g. of the percentage change in the exchange rate) per period is approximately zero for high-frequency time series, the fitting of an (exact) mean equation is often neglected. The study examines, among other things, on the basis of the available time series, what effect the neglect of the mean equation can have on the accuracy of VaR forecasts.

The importance of flexible models and stable software applications

For regular applications such as daily VaR forecasts, it would be desirable to have a stable estimation algorithm that delivers a valid (and plausible) value even in unfavourable (data) constellations. Finally, in case of an interruption of the programme, a case-specific investigation of the cause or modification of the programme has to be carried out and, if necessary, even a substitute solution has to be justified. This requires a costly deployment of professional staff. If an interruption was triggered at the point of a structural break, a shortening of the time series by the preceding time period is also not necessarily possible if the following time period is too short for a time series analysis. The modelling of the structural break is then hardly to avoid. In addition, supervisory regulations may require the inclusion of a longer history of time series data. Especially when processing large amounts of data, such as when running through all elements of a variance-covariance matrix with repeated application of an algorithm for estimating (GARCH) parameters of the (univariate) models and calculating forecast values, however, if the stability is fundamentally insufficient, the estimation algorithm must be expected to abort and thus interrupt further programme runs. Complete calculations for all elements of a variance-covariance matrix are necessary, however, if this matrix is included in further calculation operations to determine the total risk of a portfolio (i.e. the portfolio variance).

[9] Cf. Hamilton (1994) or Hassler (2003) on the properties of economic time series, especially instationarity.
[10] The time-dependent volatilities can be approximated by squared returns.

Arguments for preferring univariate methods in automated applications

Processes (including analysis and forecasting tools) should be largely automated and trouble-free. For the exclusive purpose of a prediction, it makes sense to first consider the use of univariate procedures. Data mining of one's own historical values of a time series is advantageous, as there is then no dependence on the availability of explanatory variables. This availability would also have to be ensured for future applications, as in the case of regular Value-at-Risk forecasts. Univariate forecast models are also more easily transferable to other applications or other time series. This argues for the application of univariate methods in automated analysis and forecasting tools. When multivariate methods are used, matching explanatory variables would have to be found or the model structure modified if they were to be transferred to other time series to be explained or forecast. However, these are not necessarily available in the appropriate periodicity and for the same time period. Even with panel data sets, there may be a need for adjustment when selecting multivariate time series. In addition, working with multivariate methods or panel data sets increases the risk that one of the time series included has particular characteristics that the estimation algorithm cannot handle. In the study, a lack of return of valid estimated values occurred especially in the case of outliers at the current edge of a time series. Such circumstances may require an adjustment of the application by an econometrician or data scientist. However, if adjustments to the programme are required on a regularly basis, this would run counter to a high degree of automation and thus to an application that is as accessible as possible. Multivariate procedures enable the explanation of correlations between variables or may be necessary for this purpose. However, they should only be considered as a secondary option for automated forecasting for the reasons mentioned above. Moreover, univariate ARMA-GARCH models usually already provide a high forecasting accuracy for time-ordered data, which is hardly outperformed by multivariate procedures.

Alternative distribution models to the normal distribution

The accuracy of the predicted conditional volatilities and the Value-at-Risk forecasts calculated from them depends in particular on the distribution selected for parametric models, as the results of this study also confirm. However, the normal distribution discovered by C.F. Gauss (1777 - 1855) and established as the standard model in statistics does not do justice to the relative frequencies of extreme values in the two "fat tails" of the actual distributions of financial market data. Extreme values of returns actually occur much more frequently than the Gaussian distribution would suggest. Nevertheless, the normal distribution is also frequently used in forecasts with time series models (such as ARMA-GARCH): Read (1998), for example, claims that parametric models based on the assumption of normal distribution are comparatively easy to use and very popular. However, Bollerslev (1987) had already proposed to use the Student-T distribution for modelling. He had called this modified model "GARCH-T".[11] The Student-T distribution usually provides a better fit to actual return distributions because of its higher kurtosis compared to the Gaussian distribution, but it is also symmetric. However, actual return distributions are usually asymmetric: negative returns are often higher in amount than positive returns. This is also the reason why speculative bubbles burst faster than they have built up. Asymmetric distributions such as the Skew Student-T distribution do justice to this circumstance. However, extensions to the GARCH model have also been proposed that

[11] Heracleous (2007) investigates the ability of the GARCH-T model to estimate the correct number of degrees of freedom.

capture asymmetry through their model structure. That is, the effects of returns (or residuals) with different signs or ranges of values are modelled differently.

Alternative distribution models and modified GARCH models offered by statistical software R

In past years, studies often still used VaR calculations with the normal distribution. For standard (time series analytical) applications, however, Statistical Analysis Software now offers alternative distribution models to choose from. The software R, for example, offers the normal distribution and the Skew Student-T distribution, [12] among others, via the function *garchFit* of the package *"fGarch"* for estimating GARCH models. The same applies to the *MS-GARCH* function provided via the R package *"MSGARCH"*.[13] It is more flexible than the ordinary *GARCH* function due to the possibility of formulating multiple variance equations and was created for modelling time series with changing regimes. Thus, the *MS-GARCH* model should also be able to capture structural breaks and possibly also outlier values. Time series of the value development of internationally traded assets, exchange rates, etc., some of which encompass decades, are likely to show structural breaks due to macroeconomic and global economic relationships that have changed in the meantime. If these structural breaks are not modelled, the fitted time series models do not reflect the actual dynamics on the markets and biased parameters are estimated. In the study, the R-function *garchFit* estimation algorithm for the ordinary *GARCH* model also frequently failed when the processed time series section contained outliers. In particular, the attempt to fit the *GARCH* model to the time series of the CHF/EUR exchange rate did not provide a valid estimate for a long sequence of successively changed time windows that contained the clear outlier values. In contrast, the fit with the R function *MS-GARCH* (R package *"MSGARCH"*) was successful. It should be assumed here that the flexible *MS-GARCH* model also captures or at least approximates other proposed variants of *GARCH* models. Asymmetric GARCH models that treat e.g. positive and negative returns differently (e.g. by case distinction in parameter estimation) can possibly also be approximated by the switching regime model, provided that some market phases contain predominantly positive returns and others predominantly negative returns.[14]

Multivariate time series and Multivariate GARCH Models

The time series available for the study differ with respect to the daily data of missing quotes due to deviations on trading days and holidays (see section Time Series Data). Multivariate procedures (e.g. *Multivariate GARCH* models), however, require time series that are compatible with each other, so that they should match in their daily data of missing values and other properties. In multivariate methods, the optimised model order must fit all time series included. Since *GARCH(g,a)* and *ARMA(p,q)* models with low lag orders g, a, p and q (at most two each) generally approximate higher model orders well, the use of a uniform lag order for all included time series is often uncomplicated and appropriate. Under certain circumstances, however, the estimation algorithm of a more complex (e.g. *Multivariate GARCH* model) may fail because one of the time series may not allow a fit (with the specific lag order). Accordingly, this will happen in frequent applications such as VaR forecasts

[12] The forecasts carried out for the study are predominantly based on these applications.

[13] See Ardia (2019a) on probability density function (PDF) of the R package *"MSGARCH"* and in particular Trottier and Ardia (2016) on the density function of the Skew Student-T-distribution as well as Hansen (1994) on the Student-T-density function and a skewed generalisation of it.

[14] Cf. Schoffer (2003) on asymmetric GARCH models

produced on a daily basis or extensive variance-covariance matrices. In contrast, when fitting *GARCH* models to univariate time series, an individual optimisation of the lag structure is possible.

Alternative methods for the calculation of Value-at-Risk forecasts

Since time immemorial, non-parametric methods have also been used for VaR forecasting, such as *"Historical Simulation"*. In addition, forecasting systems that use machine learning algorithms or artificial intelligence have been developed more recently. With the help of these methods, multivariate data sets could possibly be used to better explain the exceeding of VaR values in certain periods or to make more differentiated probability statements. However, the VaR forecast quality of the (extended) parametric GARCH models with regard to the coverage of the entire time series range is already high and can thus hardly be improved upon using alternative methods. Moreover, the machine learning or AI methods were not (originally) created for the purpose of explorative studies.[15] In case of doubt, the effects of data mining, such as empirical significance levels that are too high, should also be taken into account.

Variance-covariance matrices

Risk management requires an assessment or forecast of the development of all risky exposures. If necessary, only exposures with potentially substantial effects, such as a threat to the company's survival, are to be taken into account. Since a company usually pursues various risky activities or is exposed to risks, their covariances (or correlations) must also be taken into account: These are of interest in determining the overall risk of a company. Moreover, in the case of strong dependencies, several risky investments could fail at the same time. For companies in the financial industry (banks, investment companies, etc.) it also makes sense to set up separate variance-covariance matrices for the management of investment portfolios of individual clients (groups), even if the company itself is not (directly) exposed to these risks. This poses particular challenges for investment companies that invest in many different assets. In a static view, the variance-covariance matrix **Σ** can be represented as follows:

$$\sum = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1k} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{k1} & \sigma_{k2} & \cdots & \sigma_k^2 \end{pmatrix}$$

The variance-covariance matrix **Σ** contains the variances on the main diagonal and the covariances of the different exposures on the outside. It is symmetric, since $\sigma_{ij} = \sigma_{ji}$. The elements of the variance-covariance matrix to be considered are thus already completely contained in the upper or lower triangular matrix. Their number grows disproportionately with the number *k* of exposures:

$$\frac{k\ (k+1)}{2}$$

In a static perspective, the (unconditional) variance of a portfolio σ_P^2 (representing the total risk) of a number k of exposures, where n_i represents the weighting of exposure (e.g. number of units of security) *i* in the portfolio, is as follows:

[15] However, a research strand could also evolve here that is concerned with extracting explanations from the results of AI applications.

$$\sigma_P^2 = \boldsymbol{n}' \, \boldsymbol{\Sigma} \, \boldsymbol{n} = (n_1 \, n_2 \, \cdots \, n_k) \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1k} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{k1} & \sigma_{k2} & \cdots & \sigma_k^2 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_k \end{pmatrix}$$

Sophisticated (Multivariate) GARCH models and considerations concerning stable software applications

When the first *Multivariate GARCH* models were presented at the end of the 1980s, the computing power of personal computers available at the time hardly allowed (regular) estimates of univariate GARCH models or predictions for all elements of a time-conditional variance-covariance matrix.[16] Restrictive *Multivariate GARCH* models, which make certain assumptions regarding the elements of a variance-covariance matrix or their relationships, should allow a joint modelling of the dynamics of all elements.[17] On the one hand, restrictions are necessary that allow an estimation of the *Multivariate GARCH* models at all and fulfil the requirements of convergence and positive definiteness of generated variance-covariances.[18] On the other hand, the restrictions should allow relationships between the assets and their dynamic developments to be represented realistically. For example, Constant Conditional Correlation (CCC) models assume a constant correlation between asset returns over time. Accordingly, only the dynamics of the volatilities of the assets would have to be modelled, i.e. the dynamics of the elements on the main diagonal of a variance-covariance matrix. If necessary, it would have to be checked in a preliminary study whether such an assumption of constant (default) correlations between different asset (classes) of a portfolio is justified.[19]

Since today's computing power allows a mass of forecasts of conditional variances and covariances in a short time, *Multivariate GARCH* models have probably lost importance for forecasting purposes.[20] However, *Multivariate GARCH* models offer the possibility of modelling volatility spillovers as dynamic relationships between time series.

For mass-used software applications, a high degree of stability and reliability is desirable: In the case of permanently repeated applications, an estimation algorithm is too often interrupted even if it fails only in the case of rarely occurring data constellations. However, the personnel effort for necessary model adjustments would be inefficient and possibly also not compatible with time limits. While the flexibility of a more sophisticated model such as *MS-GARCH* allows for a more stable application than

[16] In addition to the low computing capacity, suitable software applications were not available and certain risk management concepts such as the Value-at-Risk ratio were not yet established.

[17] Realistically, however, it should be assumed that only selected, substantial exposures of a company are actually included in the application.

[18] The requirement of positive definiteness for variance-covariance matrices corresponds to that of positivity for the scalar quantity variance. It ensures the calculation of a positive total variance (or portfolio variance) from the variance-covariance matrix with arbitrary weights *n*. The requirement of positive definiteness tends to complicate the operationalisability of a multivariate GARCH model as the variance-covariance matrix grows. This is particularly true in the absence of restrictions on the structure of the GARCH model that guarantee positive definiteness from the outset.

[19] This is contradicted by the fact that in times of crisis there is often hardly any investment that is not affected by losses in value. Correlations increase. This can also be observed for exposures whose correlations are quite stable for a long period in normal times. The protective diversification effect of a portfolio is thus lost especially in times of crisis, so that the aim of value preservation is impaired.

[20] This also means that ongoing controlling of the restrictions is no longer necessary (i.e. checking compliance with positive definiteness, rearranging assembled time series (i.e. the multivariate context) and, if necessary, making adjustments to the variance-covariance matrix or models to comply with it).

the standard *GARCH* model, in other cases (unnecessary) complexity causes instability. The use of simplified models can be justified if the performance (here forecast quality) is not significantly reduced. In this context, the effect of neglecting the mean equation, which according to the textbook complements the volatility equation of a *GARCH* model, is also examined here.

Forecasting Methodology and Study Design

ARIMA models

For the application of the *Autoregressive Integrated Moving Average ARIMA(p,d,q)* method introduced by Box and Jenkins (1970), it is necessary to determine the integration order *d* of a time series to be analysed.[21] Usually prices X_t as well as their logarithms are integrated of first order, in short $X_t \sim I(1)$.[22] Their first differences $y_t = X_t - X_{t-1}$, i.e. price changes as well as logarithmised returns $y_t = \log(X_t) - \log(X_{t-1})$ (i.e. percentage price changes) are then integrated of order zero, $y_t \sim I(0)$, or stationary.

ARMA(p,q) models can then be fitted to the stationary time series obtained by differentiations. In *ARMA(p,q)* models, the current values are regressed on lagged values and innovations (i.e. residuals) of a stationary time series:

$$y_t = \varphi_0 + \varphi_1 y_{t-1} + \cdots + \varphi_p y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}$$

A more compact representation of the equation is:

$$y_t = \varphi_0 + \sum_{i=1}^{p} \varphi_i y_{t-i} + \varepsilon_t + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j}$$

A distribution assumption is made for the innovative variable ε_t (which results from a random process), e.g. the normal distribution is assumed, i.e. $\varepsilon_t \sim N(\mu_\varepsilon = 0, \sigma_\varepsilon^2)$.

The values of the parameters φ_i and θ_j (and the residuals ε_t) are estimated in the regression analysis that way that the residuals are zero on average.

The values of the parameters φ_i and θ_j represent the impact of the *i*-th lagged AR- (y_{t-i}) and the *j*-th lagged MA-term (ε_{t-j}), respectively, on the current value y_t of the time series.

The *ARMA(p,q)* model order states that the influences of the time series value lagged by $i = p$ periods (here trading days) and the residual lagged by $j = q$ periods on the current value are necessarily non-zero. The influences of the intermediate lags, on the contrary, can also be non-existent (i.e. equal to zero).

In the study, the lag structure chosen for a particular time series was optimised using the *Bayes Information Criterion (BIC)*. The lags *p* and *q* were each constrained to up to five in order to limit the number of model orders (or lag combinations) played through. Usually, models with *p* and $q \leq 2$ already provide a sufficient fit to time series. The possibility of approximating higher ones by simple

---

[21] In order to determine the orders of integration d of the available time series, Augemented *Dickey Fuller (ADF)* unit root tests were performed with the R package "urca".
If a time series X_t is integrated of order *d*, a stationary time series y_t is obtained for the first time after *d* times differentiation, e.g. for $d = 2$ it follows: $y_t = \Delta^2 X_t = \Delta(X_t - X_{t-1}) = X_t - 2X_{t-1} + X_{t-2}$. An ARMA(p,q) model can then be fitted to the stationary time series y_t. Cf. Hamilton (1994) on time series analysis.

[22] Logarithmising changes the distribution of a variable. If, for example, prices are log-normally distributed, logarithmising produces a normal distribution that is symmetrical.

ARMA model orders results from the invertibility of AR(p) or MA(q) processes. Cf. Hamilton (1994) on the AR(∞) representation of an MA(1) process.

Study design to analyse the impact of inaccurately modelling the dynamics of the expected value or even omitting the mean equation

For heteroscedastic time series and VaR forecasting, the focus is on volatility. However, the paper also examines how less accurate modelling of the dynamics of the expected value affects the VaR forecast. It should be noted, however, that the results depend on the frequency of a time series and the forecast horizon, each of which corresponds to one day in the study. Indications that an inaccurate adjustment of the mean equation is common in scientific studies can be seen in the presentations in the papers themselves. For example, Koether (2005) uses the "mean equation" $X_t = \sigma_t \epsilon_t$, i.e. he even omits the estimation of the constant before fitting the variance equation. In its formal representation, the time series value X_t (i.e. not the residual) is incorporated directly into the variance equation: $\sigma_t^2 = \omega + \alpha X_{t-1}^2 + \beta \sigma_{t-1}^2$.[23] Colucci (2018) uses an AR(1) model for the mean equation in the context of estimating a Student-T-GARCH(1,1) model as well as a *Historical Filtered Bootstrap VaR* model.

Here it is assumed that in practice (i.e. in the financial industry) the mean equation is also neglected in GARCH modelling. This is probably due to practical and economic reasons, among others:

On the one hand, automation processes may require a sacrifice of (exact) model adjustments (of the mean value equation), since more complex models may also bear a higher probability of a failure of the estimation algorithm, i.e. a break down of the forecasting system. In particular, frequent use of the procedure could possibly result in additional personnel costs. On the other hand, the expected value for high-frequency return series is usually close to zero. In relation to the assets of investment banks, this may nevertheless result in high capital buffers to be held in order to comply with regulatory requirements or to actually secure liquidity.

The question arises whether a (significant) change in the VaR forecast accuracy results from an inaccurate adjustment of the mean equation. This is examined here as an example on the basis of the available time series and in connection with the MS GARCH function, which proved to be stable in contrast to the ordinary GARCH function of the R software. The following four alternatives for modelling the mean are played out:

1) The fitting of a mean equation is dispensed with, i.e. the variance equation is fitted directly to the return time series.
2) Only the mean return (here via the estimated constant) is extracted from the return time series before fitting the variance equation.
3) An *AR(1)* model is fitted to the return time series and the variance equation is fitted to its residuals.
4) As described above, an *ARMA(p,q)* model with optimised model order is fitted according to the Bayes information criterion and then a variance equation is fitted to the residuals obtained.

This implies or follows:

[23] The notation from Koether (2005) is quoted unchanged. He also denotes the conditional variance with σ_t^2. In contrast, in the further course of this paper, the conditional variance is again denoted by h_t^2 and the unconditional variance by σ^2 for a clear distinction.

Ad 1) The assumed forecast value for the mean return in the next period is zero, $\hat{y}_{T+1} = 0$. This can be justified, if necessary, by the fact that the mean return or change in high-frequency financial market time series relates to a very small time span and is thus approximately zero.

Ad 2) The mean return can be obtained by regressing the return series on a constant: $y_t = \varphi_0 + \varepsilon_t$. The expected return of a future period $T + \tau$ then corresponds to the mean value: $\hat{y}_{T+\tau} = \hat{\varphi}_0$. This approach certainly makes sense if the (trading) daily time division is not yet fine enough so that the mean return is significantly different from zero. However, the autoregressively explainable portion may be negligible.

Ad 3) The autoregressive process AR(1) can be written as follows:

$$y_t = \varphi_0 + \varphi_1 y_{t-1} + \varepsilon_t$$

According to the equation of the AR(1) model, the current return value y_t is explained by its preceding value y_{t-1}, which is incorporated with a share φ_1, as well as by an innovation ε_t in period *t*. Here, a dynamic of the expected value is assumed, which can essentially be captured by an AR(1) term. The constant φ_0 is also taken into account in this paper, but may not be in studies by other authors. If observations (of an exchange rate) are available up to trading day *T*, the one-step ahead (point estimator) forecast $\hat{y}_{T+1}$ for the next trading day *T+1* is received by application of

$$\hat{y}_{T+1} = \hat{\varphi}_0 + \hat{\varphi}_1 y_T$$

taking $E(\varepsilon_{T+1}) = 0$. The parameter estimate $\hat{\varphi}_1$ (as well as estimates for the historical residuals up to period *T*) can be received with the Maximum Likelihood method.[24]

Ad 4) The *ARMA(p,q)* process can be represented as follows:

$$y_t = \varphi_0 + \varphi_1 y_{t-1} + \varphi_2 y_{t-2} + \cdots + \varphi_p y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q}$$

From the *ARMA(p,q)* model presented above, the following one-step forecast function is obtained:[25]

$$\hat{y}_{T+1} = \hat{\varphi}_0 + \hat{\varphi}_1 y_T + \hat{\varphi}_2 y_{T-1} + \ldots + \hat{\varphi}_p y_{T-p+1} + \hat{\theta}_1 \hat{\varepsilon}_T + \hat{\theta}_2 \hat{\varepsilon}_{T-1} + \cdots + \hat{\theta}_q \hat{\varepsilon}_{T-q+1}$$

In addition to the parameter values, the residuals are also estimated up to the current period.[26]

GARCH models

To model the time-conditional variance h_t of a heteroscedastic time series y_t, from which the residuals u_t are obtained after estimating a mean equation, Engle (1982) proposed the *Autoregressive Conditional Heteroscedasticity (ARCH)* model.

The conditional variance h_t over time in Engle's *ARCH*(a) model, when taking into account the influences of the residuals (i.e. innovations) from up to a previous periods, results from

$$h_t = \alpha_0 + \sum_{i=1}^{a} \alpha_i \, u_{t-i}^2$$

Bollerslev (1986) then formulated the *Generalised Autoregressive Conditional Heteroscedasticity (GARCH)* model. The variance equation of the *GARCH*(g, a) model is

[24] For this study the parameter estimates are done with the R-function "ARIMA", choosing the Maximum Likelihood method "ML".
[25] Forecasts based on estimated ARMA(p,q) models were made using the R function "predict".
[26] In the case of multi-level forecasts for h periods in advance, their expected value of zero is used for the future residuals, cf. Hamilton (1994).

$$h_t = \alpha_0 + \sum_{i=1}^{a} \alpha_i \, u_{t-i}^2 + \sum_{j=1}^{g} \gamma_j \, h_{t-j}$$

with consideration of the influences of the lags $a \geq 1$ and $g \geq 1$, whereby analogous to the ARMA model intermediate lags can also be without influence.

Variances are positive. The positivity is ensured by the restrictions:

$$\alpha_0 > 0, \ \alpha_i \geq 0, \gamma_j \geq 0$$

Moreover, the time-conditional variance converges when the parameter sum is less than one:

$$\sum_{i=1}^{a} \alpha_i + \sum_{j=1}^{g} \gamma_j < 1$$

In the case of heteroscedastic time series with conditional variance h_t, a mean value equation is also formulated in conjunction with the volatility equation of the GARCH model when proceeding according to the textbook, which can be represented as follows for the AR(1) process, for example:

$$y_t = \varphi_0 + \varphi_1 \, y_{t-1} + \sqrt{h_t} \, \epsilon_t$$

The decomposition $u_t = \sqrt{h_t} \epsilon_t$ with independently and identically distributed $\epsilon_t \sim i.i.d$, for example standard normally distributed $\epsilon_t \sim N(0,1)$, is performed.

Extended univariate GARCH models

Various extensions to *GARCH* models have been proposed in the literature to fit time series with special characteristics.[27]

Nelson (1991) proposed the *Exponential GARCH* (*EGARCH* for short) model, for which various variations are again presented in the literature.

$$\ln(h_t) = \alpha_0 + \sum_{i=1}^{a} \alpha_i \, u_{t-i}^2 + \sum_{j=1}^{g} \gamma_j \ln(h_{t-j})$$

Here, the positivity of the conditional variance is ensured by modelling its logarithmic size. Nelson and Cao (1992) claim that the non-negativity requirements of the ordinary *GARCH* model are too restrictive. In the *EGARCH* model, these restrictions are not imposed. In the literature and on the internet, modifications of models are also proposed under the name *EGARCH*, which, in addition to the logarithmisation of the heteoskedastic variance, also take asymmetric effects into account.[28] Asymmetry often refers to the assumption that bad news (associated with negative returns) increases time-varying volatility to a greater extent than positive news. Such effects can be captured by the models presented below. However, generalisations of the models are also conceivable that operate at a value other than the zero value or even make case distinctions for more than two value ranges.

In the *Threshold GARCH* (*TGARCH* for short) model, the different treatment of negative and positive shocks is carried out by taking into account not only the influence of the innovations but also that of their absolute values. Since the variance equation no longer contains squared innovations (as in the ordinary *GARCH* model) to account for the changing sign, it is also appropriate to no longer include the time-lagged variance but the standard deviation as the explanatory variable:

[27] Various special GARCH models are presented in Ali (2013) and their goodness of fit is compared on an application basis, taking into account various distribution models (normal distribution, Student-T).

[28] Cf. Dhamija & Bhalla (2010)

$$h_t = \alpha_0 + \sum_{i=1}^{a} \alpha_i(|u_{t-i}| - \delta_i u_{t-i}) + \sum_{j=1}^{g} \gamma_j \sqrt{h_{t-j}}$$

The *GJR-GARCH* model is represented by the volatility equation

$$h_t = \alpha_0 + \sum_{i=1}^{a} (\beta_i I_{t-i} + \alpha_i) \cdot u_{t-i}^2 + \sum_{j=1}^{g} \gamma_j h_{t-j}$$

with indicator variable

$$I_{t-i} = \begin{cases} 1 & if\ u_{t-i} < 0 \\ 0 & if\ u_{t-i} \geq 0 \end{cases}$$

The term GJR is due to the initials of the authors Glosten, Jagannathan and Runkle (1993).

With a positive parameter β_i, a negative innovation (or return) thus leads to higher conditional volatility in the following period than a positive one of the same amount. It is obvious that generalisations of the model are possible, with the indicator variable set at a threshold other than zero.

It must be taken into account with the special GARCH models that special requirements (i.e. deviating from the ordinary GARCH model) may have to be fulfilled in order to generate positive time-dependent variances and for the process to be stationary, i.e. the variance does not grow with increasing time distance across all boundaries.

In the *GJR* model, the positivity of the conditional variance is ensured by the restrictions:

$$\alpha_0 > 0, \alpha_i \geq 0, \alpha_i + \beta_i \geq 0, \gamma_j \geq 0$$

Stationarity (i.e. convergence) is ensured if

$$\sum_{i=1}^{a} \left(\frac{\beta_i}{2} + \alpha_i\right) + \sum_{j=1}^{g} \gamma_j < 1$$

is fulfilled.

Under the names and abbreviations, not necessarily standard specific models are found in the literature and on the internet, but partly generalised, more restrictive, modified and even different models. The *EGARCH*, *T-GARCH*, *GJR-GARCH* and other specific univariate *GARCH* models are also described by Ali (2013). Ali (2013) deals with environmental issues and presents empirical results for the special univariate models, for which he also uses different distributions. For some models, including *E-GARCH* and *T-GARCH*, he reports *"not radically different"* results. Koether (2005) describes with *Treshold GARCH* a model equation which, due to its generalised representation, encompasses a class of special models. In contrast, his description of the *Exponential GARCH* model only considers a sum of ARCH terms, not the GARCH component.

Markov-switching GARCH

Markow-switching GARCH models allow the modelling of regime changes in the volatility structure of time series.[29] Gray (1996) was a pioneer in proposing to model conditional distributions by means of regime-switching processes.

[29] For the study, *Markov-switching GARCH* models were estimated using the *R* package "MSGARCH", which is described in Ardia et al. (2019a). This *R* package also allows the prediction of the conditional variance as well as the Value-at-Risk and the expected shortfall.

Variance equations result for the regimes $k = 1, \dots, K$ here presented analogously to Ardia et al. (2019a) for an *MS(K)-GARCH(1,1)* process:

$$h_{k,t} = \alpha_{0,k} + \alpha_{1,k}\ y_{t-1}^2 + \beta_k\ h_{k,t-1}$$

Ardia et al. (2019a) explain that covariance-stationarity in each regime requires $\alpha_{1,k} + \beta_k < 1$ and positivity is ensured, if $\alpha_{0,k} > 0$, $\alpha_{1,k} > 0$, and $\beta_k \geq 0$.

A state variable s_t determines the regime of the process in time t.[30]

The probability of remaining in or switching to another of the K regimes at time $T + 1$ depends on the regime k prevailing at the current time T. The transitions between the K regimes are formulated by a $(K \times K)$ probability matrix.

According to Ardia et al. (2019), the conditional variance $h_{k',t}$ of the *MS(K)-GARCH(1,1)* process y_t in period t is thus a function of the observation y_{t-1} and the conditional variance $h_{k,t-1}$ of the previous period as well as the regime-dependent parameter vector $\boldsymbol{\theta_k}$:

$$h_{k',t} = f(y_{t-1}, h_{k,t-1}, \boldsymbol{\theta_k})$$

For the present study, the optimal number of up to five regimes of *GARCH(1,1)* processes was obtained for each time series segment using the Bayes information criterion.

Multivariate GARCH models

In order to (fully) capture the risk of a securities portfolio, all risky exposures (contracts, foreign exchange reserves, financial derivatives, etc.) of a company, etc., variance-covariance matrices for the rates of change in value of the exposures must be set up.

As an example, a variance-covariance matrix for only $K = 2$ exposures is given:

$$\sum = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$$

The variance-covariance matrix Σ contains the variances on the main diagonal and the covariances outside the main diagonal. It is symmetrical, since $\sigma_{ij} = \sigma_{ji}$. The number of elements of the variance-covariance matrix to be considered grows disproportionately with the number *K* of exposures:

$$\frac{K\ (K+1)}{2}$$

In a static view, the (unconditional) return variance for a portfolio σ_P^2 results from a number *K* of securities or exposures, where n_i represents the number of securities (or, more generally, the weighting of the exposure) *i* in the portfolio:

$$\sigma_P^2 = n'\ \Sigma\ n = (n_1\ n_2\ \cdots\ n_k) \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1k} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{k1} & \sigma_{k2} & \cdots & \sigma_k^2 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_k \end{pmatrix}$$

For example, with only two securities, the portfolio variance is calculated as follows:

[30] Ardia et al. (2019) explain that the state variable dynamics can alternatively be formulated according to Haas, Mittnik & Paolella (2004a) or Haas et al. (2004b). The specification of the *MS-GARCH* model from Haas et al. (2004a) was selected via the *R* package “MSGARCH“.

$$\sigma_P^2 = n' \, \Sigma \, n = (n_1 \;\; n_2) \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

However, in the case of heteroskedasticity over time and a possible change in the portfolio composition in each period, these dynamics must be taken into account.

The time-dependend portfolio variance $h_{P,t}$ is calculated as

$$h_{P,t} = n'_t \, H_t \, n_t = (n_{1,t} \; n_{2,t} \; \cdots \; n_{k,t}) \begin{pmatrix} h_{11,t} & h_{12,t} & \cdots & h_{1k,t} \\ h_{21,t} & h_{22,t} & \cdots & h_{2k,t} \\ \vdots & \vdots & \ddots & \vdots \\ h_{k1,t} & h_{k2,t} & \cdots & h_{kk,t} \end{pmatrix} \begin{pmatrix} n_{1,t} \\ n_{2,t} \\ \vdots \\ n_{k,t} \end{pmatrix}$$

With only two exposures, a (2×2)-matrix of time-conditional variances results, with symmetry $h_{12,t} = h_{21,t}$:

$$H_t = \begin{pmatrix} h_{11,t} & h_{12,t} \\ h_{12,t} & h_{22,t} \end{pmatrix}$$

For the prediction of the time-dependent variances and covariances, in general $(k^2 + k)/2$ or, in the concrete example, three individual equations can be set up. Fitting *GARCH(1,1)* models results in the following system of equations:[31]

$$h_{11,t} = c_{11} + \alpha_{11} u_{1,t-1}^2 + \gamma_{11} h_{11,t-1}$$
$$h_{12,t} = c_{12} + \alpha_{22} u_{1,t-1} u_{2,t-1} + \gamma_{22} h_{12,t-1}$$
$$h_{22,t} = c_{22} + \alpha_{33} u_{2,t-1}^2 + \gamma_{33} h_{22,t-1}$$

An advantage of single equations is the possibility of optimising the model order of each single equation. Nevertheless, the presented system of equations for *GARCH(1,1)* models can also be presented in matrix notation:

The result is the *Multivariate GARCH(1,1)* process in *Diagonal VECH* representation:

$$\begin{pmatrix} h_{11,t} \\ h_{12,t} \\ h_{22,t} \end{pmatrix} = \begin{pmatrix} c_{11} \\ c_{12} \\ c_{22} \end{pmatrix} + \begin{pmatrix} \alpha_{11} & 0 & 0 \\ 0 & \alpha_{22} & 0 \\ 0 & 0 & \alpha_{33} \end{pmatrix} \cdot \begin{pmatrix} u_{1,t-1}^2 \\ u_{1,t-1} u_{2,t-1} \\ u_{2,t-1}^2 \end{pmatrix} + \begin{pmatrix} \gamma_{11} & 0 & 0 \\ 0 & \gamma_{22} & 0 \\ 0 & 0 & \gamma_{33} \end{pmatrix} \cdot \begin{pmatrix} h_{11,t-1} \\ h_{12,t-1} \\ h_{22,t-1} \end{pmatrix}$$

The parameter vector $\boldsymbol{c}$ can alternatively also be represented as a diagonal parameter matrix C, where the main diagonal contains the parameters c_{ij}. For variance-covariance matrices of arbitrary dimension and arbitrary lag structure g and a of a *GARCH(g,a)* process, the following short representation results:

$$vech(H_t) = C + \sum_{i=1}^{a} A_i \cdot vech(\boldsymbol{u}_{\boldsymbol{t-1}} \cdot \boldsymbol{u'}_{\boldsymbol{t-1}}) + \sum_{j=1}^{g} \Gamma_j \cdot vech(H_{t-i})$$

Here, the parameter matrices A_i and Γ_j for the periods delayed up to a and g, respectively, and C for the constants are diagonal matrices.[32] Diagonal matrices only contain non-zero elements on the main diagonal.

Here, the vector of time-conditional (co-)variances to be explained or forecast is obtained by applying the vector operator $vech(.)$, which stacks the elements of the upper triangular part of an

[31] Analogously, mean value equations are to be included if necessary, which are omitted in the following presentation (and, if applicable, in the actual application).

[32] The constants *C* are partially omitted.

$(n \times n)$-matrix on top of each other in such a way that a vector of length $(K^2 + K)/2$ results. For the example of a portfolio of two securities, it follows:

$$vech(H_t) = \begin{pmatrix} h_{11,t} \\ h_{12,t} \\ h_{22,t} \end{pmatrix}$$

Multivariate GARCH models in *Diagonal VEC* representation were proposed by Bollerslev et al (1988).[33] In the *Diagonal VECH* model, each element of the (co-)variance matrix is explained only by its own past values or innovations.

In place of ensuring the positivity of a (predicted) variance in the univariate case (i.e. in the single equation model), in the multivariate context there is the condition of positive definiteness to be fulfilled. According to Attanasio (1991), the variance-covariance matrices predicted with the *Diagonal VEC* representation are positive definite if the parameter matrices C, A_i und Γ_i as well as the initial time-conditional variance-covariance matrices corresponding to the lag structure are positive definite. Engle and Kroner (1995) explain that the *Diagonal VEC* representation is covariance stationary exactly when all eigenvalues of the matrix M formed as the sum of the parameter matrices A_i und Γ_j are absolutely less than one.

$$M = \sum_{i=1}^{a} A_i + \sum_{j=1}^{g} \Gamma_j$$

However, generalisations are also conceivable, so that a conditional (co-)variance is not only explained from its own past values and innovations, but corresponding influences of the other (co-)variances are also captured. This enables the modelling of volatility jumps between time series. This makes it possible to show that, if necessary (and to what extent and with what time lag), a change in the conditional volatility on one market spills over to another market.

The *VECH* model was introduced by Bollerslev, Engle and Wooldridge (1988):

$$\begin{pmatrix} h_{11,t} \\ h_{12,t} \\ h_{22,t} \end{pmatrix} = \begin{pmatrix} c_{11} \\ c_{12} \\ c_{22} \end{pmatrix} + \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \cdot \begin{pmatrix} u_{1,t-1}^2 \\ u_{1,t-1}u_{2,t-1} \\ u_{2,t-1}^2 \end{pmatrix} + \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix} \cdot \begin{pmatrix} h_{11,t-1} \\ h_{12,t-1} \\ h_{22,t-1} \end{pmatrix}$$

For variance-covariance matrices of arbitrary size as well as arbitrary lag structures g und a of a *GARCH(g,a)* approach and, if necessary, taking into account further explanatory variables $\boldsymbol{x_t}$, the following concise representation results:

$$vech(H_t) = C_0 + C_1 \cdot vech(\boldsymbol{x_t}) + \sum_{i=1}^{a} A_i \cdot vech(\boldsymbol{u_{t-1}} \cdot \boldsymbol{u'_{t-1}}) + \sum_{j=1}^{g} \Gamma_j \cdot vech(H_{t-i})$$

Since the number of parameters to be estimated is a multiple of the number of elements of the variance-covariance matrix and also the positive definiteness of the predicted variance-covariance matrices has to be guaranteed, there are limited application possibilities.

If time series with regime changes are concerned, *Multivariate GARCH* models could also be set up to represent the volatilities or covariances of possible regime combinations of one or more time series.

[33] In general, the vec(.) operator is used to write the elements of a matrix as a vector by stacking them column by column. Due to the symmetry of (co-)variance matrices, only the upper triangular part of a matrix is taken over when using the vech(.) operator, thus avoiding a double entry of elements.

In practice, portfolios are restructured when, for example, newly issued securities are included, warrants expire, new transactions with foreign currency risks are entered into, etc. These constant changes in variance-covariance matrices would also require the necessary checks or assurances of the positive definiteness of the covariance matrices. This is difficult to do with large covariance matrices, or most likely with sophisticated additional analysis software. Since adjustments might be necessary in every change period if the requirements are not met, it makes sense from the outset to set up an efficient strategy for risk measurement or volatility forecasting.

Due to their ("understanding") complexity, multivariate GARCH models are certainly less frequently used in practice. For controlling the requirements regarding positive definiteness and convergence, it is recommended to work with clear models. I.e. the covariance matrices should be as small in size and number of parameters as possible. This applies all the more to relatively short time series, which require a consideration of the degrees of freedom.

When modelling or selecting a *Multivariate GARCH* model proposed in the literature, the user should consider which volatility spillover effects exist between the time series used. The models should take these relationships into account suitably, succinctly but completely (not only in the sense of fully representing an economic theory, but also in the sense of avoiding biased parameter estimates). Insights into possible volatility-spillover relationships can be derived theoretically, if necessary. In addition, (time-conditional or unconditional) correlations can be calculated in advance to find out which time series pairs show a (significant or not only weak) correlation. In order to enable controlling of the requirements regarding positive definiteness and convergence in the case of daily updated time series and, if necessary, reallocations of large investment portfolios, this should also be carried out with (partially) automated support. But the applications developed for this purpose should (initially) also contain manageable problems in the sense of the developer or data scientist (who must correct initial errors). If necessary, it makes sense to automatically decompose large covariance matrices into $(2x2)$-covariance matrices and to run through them for a search for volatility spillover effects.[34]

Contrary to the time of the emergence of (multivariate) GARCH models in the late 1980s to mid-1990s, it should also be possible to estimate univariate GARCH models for all combinations of a variance-covariance matrix of a certain size in single equations due to today's computing power. If necessary, individual equations for covariances or correlations can also be used.

However, in order to make the (possibly daily) run-through process of all elements of a covariance matrix possible, a stable application function is required. If the run-through process is aborted, individual problem treatments are often required, which are time-consuming. In the present study, it was shown that the R function *"MSGARCH"* provided stably estimated parameters, while the estimation algorithm of the ordinary *GARCH* function frequently aborted.

Especially for investment companies that hold many different securities, a time-dependent forecast of a variance-covariance matrix that takes into account covariances for all pairs of securities is hardly feasible. However, individual securities investments with a small share of assets are negligible. In

[34] However, the influence of a third time series variable remains unconsidered in the individual sub-models with this approach. Analogous to the context of a multivariate regression analysis, if a (significant) third variable is omitted, the relationships between the two variables taken into account may be distorted. However, it is questionable whether such a systematic error arises in the given context and it may be negligible. In order to significantly reduce the possibility of such an effect, it may also be possible to work with $(3x3)$-matrices.

addition, other investments are due to several risk factors, for example, shares carry a country risk or foreign currency risk in addition to their sector risk, among others.

If a company is exposed to a large number of risky exposures, the question of complexity reduction arises. The variants of multivariate GARCH models proposed in the literature may already offer solutions for this:

The Baba, Engle, Kraft and Kroner *(BEKK)* representation of a *Multivariate GARCH* process is presented in Engle & Kroner (1995). For a bivariate *GARCH(1,1)* process it can be represented in matrix notation as follows: [35]

$$H_t = \begin{pmatrix} h_{11,t} & h_{12,t} \\ h_{12,t} & h_{22,t} \end{pmatrix} = \begin{pmatrix} c_{11} & 0 \\ c_{21} & c_{22} \end{pmatrix} \cdot \begin{pmatrix} c_{11} & c_{21} \\ 0 & c_{22} \end{pmatrix}$$
$$+ \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \cdot \begin{pmatrix} u_{1,t-1}^2 & u_{1,t-1}u_{2,t-1} \\ u_{1,t-1}u_{2,t-1} & u_{2,t-1}^2 \end{pmatrix} \cdot \begin{pmatrix} \alpha_{11} & \alpha_{21} \\ \alpha_{12} & \alpha_{22} \end{pmatrix}$$
$$+ \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} \cdot \begin{pmatrix} h_{11,t-1} & h_{12,t-1} \\ h_{12,t-1} & h_{22,t-1} \end{pmatrix} \cdot \begin{pmatrix} \gamma_{11} & \gamma_{21} \\ \gamma_{12} & \gamma_{22} \end{pmatrix}$$

The *BEKK* representation allows a more parsimonious parameterisation. The representations for the bivariate case show that instead of a (3x3) parameter matrix, a (2x2) matrix is pre-multiplied and post-multiplied respectively. The flexibility of the *BEKK* model with regard to parameters to be estimated is therefore more restricted compared to the "Full" representation shown above: The parameterisation of the resulting three (co-)variance equations is no longer done independently, but restrictions are created within and between the equations via the combinations of the estimated parameters. However, this also has the advantage that the generated variance-covariance matrices are positive definite under weak assumptions.

Alexander & Chibumba (1997) proposed the *Orthogonal Multivariate GARCH* model (*O-GARCH* for short). This is designed to identify orthogonal factors that are uncorrelated using principal component analysis. This provides a dimensionality reduction in which the few significant risk drivers are identified as principal components behind many risky exposures.

The use of factor analysis procedures with the aim of identifying factors that are as uncorrelated as possible (but possibly also correlated) should certainly be considered for huge investment portfolios consisting of many different assets. [36] Risks of individual equity investments can then be attributed, for example, to their main underlying risks, such as sector and country risks. The residual individual risk of a share remaining after deduction of the country and sector portion can possibly be neglected for a small number of stock certificates. However, the respective portions of different factor characteristics (e.g. country and industry risks) may have to be determined for the individual stocks. Due to the dimension reduction, conditional variance-covariance matrices can then be generated whose elements represent individual factors (countries or industries). The principle of using variance-covariance matrices for the risk factors underlying the exposures is basically applicable to all specifications of *Multivariate GARCH* models. Only uncorrelated factors are not always found. Industries and countries can be similar and have relationships.

---

[35] It should be noted that in the following system of equations residuals u_t are listed, while in the original literature returns r_t are sometimes used directly. In this way, however, the authors indirectly show that they do not model the dynamics of the expected values of the time series, i.e. the mean equations.

[36] Cf. Hair et al. (2010) on factor analysis

Instead of directly forecasting time-conditional variance-covariance matrices, it has also been suggested to treat time-conditional variances and correlations separately. For example, time-conditional variances could be predicted using univariate *GARCH* models that are specifically fitted. The possibilities for forecasting time-conditional correlations can be distinguished between *Dynamic Conditional Correlation (DCC)* and *Constant Conditional Correlation (CCC)* models.

CCC models were proposed by Bollerslev (1990). Here, the time-conditional variance-covariance matrix H_t can be constructed using a diagonal matrix D_t, which contains the time-conditional standard deviations on the main diagonal, and a matrix *R* of time-constant correlation coefficients:

$$H_t = D_t \cdot R \cdot D_t$$

$$H_t = \begin{pmatrix} h_{11,t} & h_{12,t} \\ h_{12,t} & h_{22,t} \end{pmatrix} = \begin{pmatrix} \sqrt{h_{11,t}} & 0 \\ 0 & \sqrt{h_{22,t}} \end{pmatrix} \cdot \begin{pmatrix} 1 & \rho_{12} \\ \rho_{12} & 1 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{h_{11,t}} & 0 \\ 0 & \sqrt{h_{22,t}} \end{pmatrix}$$

DCC models were proposed by Engle & Sheppard (2001), where the conditional variance-covariance matrix H_t is calculated as follows:

$$H_t = D_t \cdot R_t \cdot D_t$$

The use of time-conditional correlations is reflected in the notation in such a way that, deviating from the representation of the *CCC* model, the correlation coefficients in the matrix R_t are indexed with time index *t* in the *DCC* model:

$$R_t = \begin{pmatrix} 1 & \rho_{12,t} \\ \rho_{12,t} & 1 \end{pmatrix}$$

For a more flexible modelling, mixed forms could also be considered. For this purpose, it would make sense to identify which exposures show significant and medium-strong to strong correlations at all. If a correlation coefficient is not subject to any significant dynamics, it can be approximated by a time-constant correlation coefficient. Then only the time-conditional volatilities would have to be forecast. The time-conditional covariances then result in a further step as products of the time-conditional standard deviations and the time-constant correlation coefficients.

The Value-at-Risk

Capital buffers are retained to hedge against insolvency in the event of significant losses from exposures. To determine these cover amounts the Value-at-Risk concept has become established in risk management and has been included in regulatory requirements.[37] The VaR method became popular through the *RiskMetrics* risk management tool published in 1994 by the US investment bank *J.P. Morgan*.

The Value-at-Risk (VaR) is the anticipated loss of an exposure, which will not be exceeded in a specified time period from an adverse market movement with a predetermined probability.[38]

For the purpose of evaluating the accuracy of a VaR forecast methodology, such forecasts are considered for a range of time periods. The prediction function is optimal if the VaR forecasts are not exceeded at exactly (1-α)·100% of the forecasts.

---

[37] Cf. for example Basel Committee on Banking Supervision (2019), Federal Register (1996), Greenspan (1996)

[38] Basel Committee on Banking Supervision (2019) defines: *"Value at risk (VaR): a measure of the worst expected loss on a portfolio of instruments resulting from market movements over a given time horizon and a pre-defined confidence level."*

From the mean equation estimated (usually with ARIMA model), a point forecast is generated, i.e. the expected value (of return) of an exposure is calculated. The VaR forecast is then derived as a deviation from the point forecast using the volatility forecast from the GARCH model.[39] This deviation is calculated via the α-quantile of the assumed distribution.

The calculation of VaR as an absolute value results from the fact that it is usually expressed as a positive value like a loss.

Assuming normally distributed returns with $r_t \sim N(y_t, \sqrt{h_t})$, the quantile for the VaR forecast is calculated by subtracting a number of z_α conditional standard deviation units from the expected return:

$$\hat{q}_\alpha = \hat{y}_{T+1} - z_\alpha \cdot \sqrt{\hat{h}_{T+1}}.$$

The VaR forecast for the subsequent period $T+1$ with selected probability $1-\alpha$ is therefore calculated as follows:[40]

$$\widehat{VaR}_{T+1}(1-\alpha) = V_T \cdot \left| \hat{y}_{T+1} - z_\alpha \cdot \sqrt{\hat{h}_{T+1}} \right|$$

and

- z_α is the α-quantile of the standard normal distribution $N(0,1)$;[41]
- $\hat{y}_{T+1}$ is the point forecast of the return for the next period $T+1$, which results from the mean value equation;
- $\hat{h}_{T+1}$ is the predicted conditional variance in the next period;
- V_T is the value or price of the asset at the current time *T* .

The probability that the (negative) change in value in the subsequent period is greater than the VaR in absolute terms is then α:

$$Pr\left(V_T \cdot r_{T+1} \leq -\widehat{VaR}_{T+1}(1-\alpha)\right) = \alpha$$

The Mean Excess Loss

The *Mean Excess Loss (EL)* (also termed *Conditional VaR* or *Expected Shortfall*) is the average loss beyond the Value-at-Risk prognosis in the excess cases.[42]

It is calculated as

$$E(EL_{T+1}(1-\alpha)) = E(V_T \cdot r_{T+1} | V_T \cdot r_{T+1} \leq -\widehat{VaR}_{T+1}(1-\alpha))$$

The Median Excess Loss

Additionally, in the study the median loss beyond the Value-at-Risk prognosis is calculated.

$$Median\left(EL_{T+1}(1-\alpha)\right) =$$

[39] The GARCH model is used to forecast the conditional variance h, from which the conditional standard deviation $\sqrt{h}$ for the VaR forecast is calculated as the root.
[40] Cf. also Artzner et al. (1999) for the calculation of Value-at-Risk and Expected Shortfall.
[41] Cf. Sukono et al. (2019) on the quantile function or VaR calculation for the (Skewed) Student-T-distribution.
[42] Basel Committee on Banking Supervision (2019) defines: *"Expected shortfall (ES): a measure of the average of all potential losses exceeding the VaR at a given confidence level."*

$$\begin{cases} (V_T \cdot r_{T+1})_{\left\lceil \frac{n+1}{2} \right\rceil} | V_T \cdot r_{T+1} \leq -\widehat{VaR}_{T+1}(1-\alpha) \; for \; odd \; n \\ \frac{1}{2}\left((V_T \cdot r_{T+1})_{\left\lceil \frac{n}{2} \right\rceil} + (V_T \cdot r_{T+1})_{\left\lceil \frac{n}{2}+1 \right\rceil} \right) | V_T \cdot r_{T+1} \leq -\widehat{VaR}_{T+1}(1-\alpha) for \; even \; n \end{cases}$$

The usual case distinction for the median calculation is made for an even or odd number *n* of exceedances of the VaR forecasts, ordered by size.

The Not Exhausted VaR

In addition to the Mean Excess Loss and the Median Excess Loss, corresponding calculations are also carried out for these trading days on which the VaR was not exceeded. The results of these calculations are also presented in the appendix in tables entitled "Not Exhausted VaR".

Default Correlations (Correlations of the exceedances of Value-at-Risk forecasts)

For investment decisions and hedging strategies, it is also interesting whether correlations exist between the exceedances of Value-at-Risk forecasts for exposures. The term "default correlation" was adopted from credit risk management, referring to the correlations of defaults (or insolvencies) of borrowers.

In the world of set theory, $D_{i,t}$ describes the event that borrower *i* defaults in period *t* or that the Value-at-Risk forecast for exposure *i* was exceeded on (possibly the end of) trading day *t*. The associated default probability or exceedance probability is described by

$$\pi_{i,t} = Pr(D_{i,t})$$

The joint occurrence of the exceedances of two VaR forecasts for risk positions *i* and *j* on trading day *t* can then be determined by the intersection of $(D_{i,t} \cap D_{j,t})$, and the corresponding probability of event is

$$\pi_{ij,t} = Pr(D_{i,t} \cap D_{j,t})$$

The correlation of two exceedance events or default correlation $\rho_{ij,t}$ is calculated taking into account the joint probabilities:[43]

$$\rho_{ij,t} = \frac{\pi_{ij,t} - \pi_{i,t} \cdot \pi_{j,t}}{\sqrt{\pi_{i,t} \cdot (1-\pi_{i,t}) \cdot \pi_{j,t} \cdot (1-\pi_{j,t})}}$$

If the exceedances of the VaR forecasts of both exposures *i* and *j* are independent of each other, the probability (or relative frequency) of both VaR forecasts being exceeded on one day is equal to the product of their individual default probabilities or exceedance probabilities. The numerator and the total expression $\rho_{ij,t}$ are then equal to zero.

For an operationalisation (i.e. programming), dichotomous variables $X_{i,t}$ can be produced for all exposures *i*, *j* with

$$x_{i,t} = \begin{cases} 1 & VaR - prediction \; for \; i \; is \; exceeded \; in \; t \\ 0 & VaR - prediction \; for \; i \; is \; NOT \; exceeded \; in \; t \end{cases}$$

and probability functions

[43] Cf. Liu et al. (2015)

$$f(X_{i,t}) = \begin{cases} \pi_{i,t} & for\ x_{i,t} = 1 \\ 1 - \pi_{i,t} & for\ x_{i,t} = 0 \\ 0 & else \end{cases}$$

Relationships between asset correlations and VaR-implied as well as expected shortfall-implied correlations are shown in Liu (2016).

In an empirical investigation Servigny & Renault (2002) find that "*there does not seem to be an obvious link between equity correlation and asset correlation and this casts some doubt about the value of equities as precise indicators of default correlations*".

The results of the present study show that for some pairs of assets, the ordinary correlations and default correlations have similar magnitudes or temporal dynamics.

Design of the Ex-Post forecasts

For the generation of the forecasts, the time series models (incl. determinations of the model order and parameter estimates) were fitted again in the study for each of the successively changing time series sections (growing or rolling time windows).

The tables "Percentage deviation of exceedance rate from alpha" also contain mean values of the percentage deviations (i.e. the forecast errors) over all time series or quantiles (alpha 1% to 10%) as well as the corresponding Mean Absolute Percentage (Error) deviations, the mean values of positive deviations and the mean values of negative deviations.

A positive default correlation would mean that, for example, an exceedance of the VaR forecast for the US dollar/euro exchange rate would most likely be accompanied by an exceedance of the VaR forecast for the Chinese renminbi/euro exchange rate.

Put simply, high losses on one currency go hand in hand with high losses on the other. Provided that the corresponding other currency is held, an offset can be achieved if necessary, so that, for example, a European company can offset high exchange-rate-driven losses on liabilities in Chinese renminbi with high profits on receivables in US dollars.

Discussion of the Results

Key findings

Numerous univariate and multivariate extensions for GARCH models are presented in the literature, which can generally be used for forecasting conditional volatilities and Value-at-Risk calculated from them. In addition, alternative methods for VaR forecasting have been proposed, including non-parametric methods such as historical simulation.

In the theoretical part of the paper, different variants of GARCH models are presented. For own Value-at-Risk forecasts, the focus is on the *MS-GARCH* model, which was constructed for modelling time series with regime changes.

Since the structure of the *MS-GARCH* model permits flexibility, it is assumed here that it also captures structural breaks and outliers in time series.[44] If this is true, the *MS-GARCH* model would also encompass, replace or approximate other variants of GARCH models: For example, the

[44] Structural breaks divide a time series into different sections, which could be interpreted as single-occurrence regimes and are thus also captured by the *MS-GARCH* model.

Threshold GARCH model proposed earlier in the literature treats positive and negative returns differently. Price losses may be accompanied by greater fluctuations than price gains. However, this is also due to different temporal market phases in which speculative bubbles build up or burst, crises cause recessions or upswings occur. Since some market phases are characterised by frequently positive returns, but in others predominantly negative ones occur, these can possibly be interpreted as regimes.

For the study, Value-at-Risk forecasts were produced using the *MS-GARCH* and the ordinary *GARCH* model. For this purpose, time series of Euro exchange rates as well as of rates in US Dollars of the crypto assets Bitcoin and Ethereum as well as of gold, silver and crude oil were available. Quite accurate VaR forecasts were achieved with both the ordinary and the *MS GARCH* model (also for high quantiles up to 98% and partly beyond), provided a suitable distribution assumption (Skew Student-T) was made for the residuals or time series values.

Distribution models and stability

VaR forecast accuracies were evaluated for three alternative distribution models (Gaussian, Student-T, Skew Student-T). These distribution models are offered for selection via the R packages, among others, for both the ordinary *GARCH* and the *MS-GARCH* functions.

With the normal distribution, fairly accurate 95% VaR forecasts are obtained. However, for higher VaR quantiles (VaR > 95% or alpha < 5%) it provides an inadequate fit. The Skewed Student-T-distribution provides a fairly accurate fit to the time series used up to the 98% or 99% quantile and in some cases for even higher quantiles. Nevertheless, even the Skewed Student-T-distribution cannot accurately fit the fat tails of the actual return distributions.

In ex-post predictions (one day-ahead) of VaR due to shifting time series sections (rolling and growing time windows), the estimation algorithm of the ordinary *GARCH* model interrupted repeatedly. In contrast, the R function *MS-GARCH* reliably delivered parameter estimates for all time series sections, which serve as the basis for the VaR calculation.

The time series of the CHF/EUR exchange rate contains two clear outliers. The estimation algorithm of the ordinary *GARCH* function generally returned an error message instead of valid parameter estimates if the moving time window contained one or even both outliers.

For other time series, the estimation algorithm was also unable to output valid parameter values for some time windows. Instability occurred more frequently in the study with rolling time windows, especially with outliers of current values.[45]

This concerns several time series or study designs of miving windows (see appendix).

In this respect, the *MS GARCH* function proved to be better suited for modelling the present time series than the ordinary *GARCH* function.

Moreover, it should be taken into account that when fitting ordinary GARCH models to time series with structural breaks, biased parameter values may also be estimated.

The results (see appendix) show, however, that the *MS GARCH* function does not necessarily (constantly) provide more accurate forecasts than the ordinary *GARCH* function, evaluated on the basis of the results for the Skewed Student-T distribution.

[45] The same should apply to structural breaks, which are, however, more difficult to detect. Extensive tests for structural breaks in the time series were not carried out in order to limit the scope of the study.

Correlation and default correlation

Correlations and default correlations were calculated between the euro exchange rates of major currencies and between the US dollar prices of Bitcoin, Ethereum, gold, silver and crude oil.

Since the calculated default correlation coefficients depend on the VaR forecasts, the effects of different forecasting procedures on these are analysed (forecast with ordinary *GARCH* versus *MS-GARCH* function, variants in the mean equation used).

For the management of a portfolio of assets, not only the volatilities of the return series may be of interest, but also the covariances or correlations of these time series with each other. After all, the maintenance of a company's solvency would be endangered especially in the case of significant (total) losses on several exposures. Basically, the correlation matrix (pairwise correlations) of all substantial exposures should be calculated. This provides information about further meaningful analyses. In the case of medium or even strong or perfect positive or negative correlation of two exposures, it may be useful to determine the dynamics (i.e. development over time) of the correlation. This can then be taken into account when forecasting conditional variances and covariances. In the study, selected dynamic developments of the coefficients of the ordinary and the default correlation are presented (see appendix).

Correlations and default correlations were calculated between the euro exchange rates of major currencies and between the US dollar exchange rates of Bitcoin, Ethereum, gold, silver and crude oil.

Time developments of (default) correlations of the USD/EUR exchange rate with other time series were calculated via successively changing time series sections (growing time windows, for the ordinary correlation also rolling ones). They are shown graphically in the appendix. The default correlations of the exchange rates are or are developing quite steadily, apart from temporary changes during the financial crisis. For gold, silver and oil, there are converging trends. For the relatively short time series of Bitcoin and Ethereum, stronger changes in the correlations are evident, but they remain around the coefficient value of zero. For the entire period, it was examined how the calculated default correlations are influenced by the fitted model variants or distribution assumptions.

Time-conditional correlation and selection of Multivariate GARCH models

The paper argues that the *Multivariate GARCH* models proposed since the 1980s may have lost their importance (if they ever had any importance for forecasting) due to the computing power of modern personal computers.[46] Today, it is possible to forecast all elements of variance-covariance matrices of a certain size and within a reasonable time using single equations or univariate GARCH models. Thus, if necessary, specific and exact adjustments of time series models to individual time series or volatility equations can be made. Error handling (especially missing outputs of the estimation algorithm) is also less problematic, since only the forecast for a single exposure or a single variance or covariance is affected. For forecasts of the time-conditional variance-covariance matrix by means of *Multivariate GARCH* models, on the other hand, special requirements for the invertibility of the matrices, stationarity and positive definiteness have to be fulfilled. This is necessary in order to forecast conditional covariances or correlations for the composed assets with the *Multivariate GARCH* model. In daily applications for forecasting purposes, however, difficulties are likely to arise

[46] An important application of multivariate GARCH models is the identification of volatility spillovers between assets.

regularly, since even a current outlier value can prevent the estimation of the *Multivariate GARCH* model. Changes to model structures or collections of included time series would then be necessary and may have to be documented.

If *Multivariate GARCH* models are nevertheless considered for forecasting purposes or explorative studies, the question of the model variant to be selected arises for economical modelling or feasibility. In particular, a distinction can be made between *Dynamic Conditional Correlation (DCC)* and *Constant Conditional Correlation (CCC)* models. In the latter, the correlations are assumed to be constant over time, so that only the volatilities as elements on the main diagonal of a variance-covariance matrix are to be modelled or forecast as time-dependent.

Furthermore, the choice of a multivariate GARCH model requires weighing up and, if necessary, trying out different models: On the one hand, it should be ensured that volatility spillovers between time series are modelled realistically. Therefore, restrictions in the parameters and the model structure that suppress a realistic representation of the correlations should be avoided. Otherwise, it cannot be ruled out that biased parameter values are estimated. On the other hand, restrictions on the model must be made in such a way that requirements (including positive definiteness) are fulfilled that allow an estimate for the given multivariate time series at all and lead to plausible results. A control of the model structure that ensures an application to multivariate time series is rather possible for manageable models with a small number of parameters to be estimated.

The results of the present study can be used for a decision on model selection. It is recommended to conduct similar preliminary studies for alternative time series. For example, the results (see appendix) show that the (default) correlations of Bitcoin, Ethereum, gold, silver and crude oil with the USD/EUR exchange rate are rather weak. The determined (default) correlations fluctuate over time, partly in the positive and negative range around zero. It can therefore be considered to use the conditional correlation in *CCC* models as an (estimated) constant or even to set it equal to zero for the assets concerned, even if they are subject to directionless variation or directional dynamics. In the case of the (default) correlations between different exchange rates, modelling of the dynamics can also be dispensed with even in the case of (medium) strength if the (default) correlations are just constant or show hardly any variation over a long period of time. This is particularly true if the (default) correlation is perfect due to exchange rate pegging and thus stable. On the other hand, if the (default) correlation is medium to strong and there are recognisable dynamics, this would have to be depicted using *Dynamic Conditional Correlation (DCC)* models. Alternatively, if multivariate GARCH models are not used, the time-conditional covariance or correlation can be predicted using single equation models.

VaR exceedances and default correlation

To estimate a reliable (Pearson's) correlation coefficient, it would be advantageous to have a minimum number of approximately ten pairwise observations of both variables. Due to the rarity of VaR overruns, long time series are required for the calculation of default correlations (i.e. correlations of the exceedance events, i.e. when considering exposures in pairs, both VaR forecasts are exceeded in one period). Particularly with the extreme quantiles, a mass of observations is required to obtain robust coefficients. For the 1% VaR, for example, one exceedance is expected in one hundred trading days or ten exceedances in 1000 trading days. If two time series of this length are available, this might be sufficient for the calculation of a coefficient. However, with such a long observation period it is possible that the market conditions (or correlations) have changed in the meantime. However, these changes are not reflected by a single coefficient. If time series of this length are not yet available for new assets (e.g. crypto assets), the determination of the default correlation is also impaired. The question therefore arises whether the coefficients of the default correlations can be replaced by the ordinary (Pearson's) correlation coefficients as proxies or at least explained (e.g. via regression models).[47] If so, changes in the ordinary correlation coefficients (calculated for short time periods, such as 30 trading days) also indicate changes in the risk of pairwise defaults or VaR exceedances. Alternatively, the current change in the default correlation coefficient should be tracked, which is calculated on the basis of moving time windows. Due to the high weight of historical values, even slight changes in the default correlation coefficient can indicate strong current (daily) changes. In the appendix, developments of default correlations of selected exposures are presented over growing time windows. The figures show that the default correlations become more stable with growing time windows. Due to the proportion of outdated values, however, effects due to current market changes are less and less clearly recognisable. Nevertheless, trends are recognisable, i.e. it becomes apparent whether the default correlations tend to remain stable in the long term or tend to increase or decrease. This is informative for long-term or strategic decisions.

However, unforeseeable events and fluctuations in market value can greatly increase the insolvency risk of a company in the short term. For an appropriate reaction in a short period of time, changes in the value of the exposures and their volatilities and correlations should therefore be monitored continuously. It is therefore recommended to calculate Pearson's correlation coefficients for rolling time windows as in the study. These tend to vary more the shorter the time windows are.

If the rolling time windows show a strong dynamic of correlations, it makes sense to model the conditional correlations between the exchange rates. In principle, various univariate or multivariate GARCH models can be considered for this purpose. Multivariate GARCH models can also be used to model spillover effects between the time series. Volatility spillover means that a change in the fluctuation margins on one market can influence those on another market (with a time lag).

The default correlations between the USD/EUR exchange rate returns and the asset returns (BTC, ETH, gold, silver, oil) were temporarily medium-sized and subject to dynamics. However, as long as the (default) correlations are weak or fluctuate closely around zero in the positive and negative range, the modelling of their dynamics can also be dispensed with or even a missing (default) correlation can be assumed for forecast models.

A strong depreciation of the euro against the US dollar is also accompanied by a strong depreciation of the euro against other major currencies, especially the CNY. This relationship diminished for the Japanese yen and the British pound during the financial crisis. For the Chinese Remnimbi, a slight dip in the already high default correlation can also be seen at the beginning of 2008. For the Swiss franc, an increase between 2008 and 2010 is more noticeable.

The results illustrated in the appendix indicate that default correlations can be replaced or explained by correlations or that this is true for their changes. This should be investigated in more detail in further studies.

[47] Alternatively, coefficients of default correlations can also be derived theoretically or empirical values can be used.

Neglecting the mean equation

The effects of neglecting the mean equation on the accuracy of the VaR forecasts were also investigated. For the time series at hand, it was found that even less accurate VaR forecasts are obtained for quantiles above 95 % (or alpha < 5 %) when ARIMA models are fitted according to the textbook manner. That is, forecast accuracy increases in the absence of the mean (i.e. assumed mean zero) or inaccurate modelling of the dynamics of the expected value. One explanation for this phenomenon is that a deviation of the predicted from the actual expected value of the conditional distribution causes a shift of the density function on the abscissa. In the case of a positive mean return in the history, its neglect leads to a higher VaR quantile (lower alpha) being used than reported, since the predicted expected value is zero. Thus, more regulatory capital would be deposited for a risk position than required. The percentage VaR exceedance frequency would be below its corresponding target alpha. Neglecting a negative mean, on the other hand, results in correspondingly opposite effects. If the normal distribution is used for the VaR calculation, it is probably often the case in practice that the volatility equation is fitted directly to the return values. In a textbook way, however, the volatility equation would have to be fitted to the residuals resulting from the fitted mean equation. The ignorance can also be justified by the fact that the mean return is usually close to zero for time series with daily periodicity or even higher frequency. However, this does not apply to the Bitcoin time series and other crypto assets, which at times recorded enormous daily returns. In addition, it must be taken into account that for large funds managed by investment banks, even returns in the proportionate per mille range represent large amounts or a high coverage capital.

When using a distribution other than the Gaussian distribution, such as Student-T or Skew Student-T for the VaR forecast, other parameters have to be estimated. These also determine the skewness or kurtosis of the fitted distribution. When using the R-functions for these special distributions, the various parameters are output, including the mean value. If these R-functions also allow a forecast of time-dependent density functions or quantiles (i.e. Value-at-Risk), the mean value is taken into account. However, an independent modelling of the dynamics of the mean (and, by the way, also skewness) via individual equations may not be sufficiently taken into account.

In any case, there is the problem that the "fat tails" (extreme quantiles) of the actual return distributions are poorly represented by the fitted distributions.

Precise modelling of the mean has the effect that the fitted density function accurately represents the actual distribution in its entirety, i.e. in particular the quantiles of the central mass. However, the extreme quantiles of the actual distributions, which are of interest for VaR, are usually inaccurately covered by the fitted distributions. When the mean is inaccurately estimated, the mass shifts so that the fitted density function then hits certain quantiles of the actual distribution - possibly in the extreme tail - more accurately (but other quantiles even more inaccurately). [48]
However, with increasing distance from the mean, a certain (constant) mass under the density function (of the normal distribution) is represented by an increasing range on the abscissa. The change in the mean has a stronger effect on more extreme quantiles.[49] It can be seen from the tables

[48] The corresponding quantiles in the other tail of the distribution are hit even worse because of the shift in the density function.
[49] A more detailed explanation: If the mean (of the normal distribution) is shifted, the range (i.e. the difference) between the actually applied and the assumed (i.e. reported) quantile value is the same for all quantile values. However, as the distance from the mean increases, the area above a constant range (i.e. the probability mass

in the appendix that in concrete application cases a lack of consideration of the expected value has led to particularly accurate VaR forecasts at various extreme quantiles. In this case, less precise modelling of the mean equation was accompanied by a gain in forecast accuracy.

Altogether, it can be stated that the omission of modelling (the dynamics) of the expected value is usually justified, as the impact on forecast accuracy is only minor. However, this is not necessarily true in the case of the formation (or bursting) of speculative bubbles with enormous mean price changes. In any case, based on the sign of the mean of historical returns, an assessment should be made of the impact of this neglect on VaR quantiles of interest and the coverage capital to be held. A (complex) mean equation should rather be omitted if the estimation algorithm frequently interrupts during regular forecast generation.

By manipulating or (possibly permissible under supervisory law) subsequently changing the estimated (ARMA GARCH) parameter values representing the mass of a distribution, a better adjustment to the mass in the marginal areas of the distributions may be possible. When using the normal distribution, there is thus scope for design in that the density function is shifted by changing the mean value or the variance is compressed or stretched by changing the variance.

By manipulating or subsequently changing the estimated parameter values that represent the mass of a distribution, a better fit to the mass in the tails of the distribution can be achieved. There is scope when using the normal distribution in that the density function can be shifted by changing the mean or compressed or stretched by changing the variance.

Conclusion and Suggestions

The digital transformation promotes the increasing introduction of automated or partially automated applications in all possible areas of activity. In order to increase efficiency in companies, process flows should overlap as automatically as possible. This can be achieved by barrier-free interfaces which, for example, enable the linking of a wide variety of applications.[50] Since individual error handling in the event of system failures can hardly be managed by personnel, these complex processing sequences require stable functions in the individual applications.

Risk management has its origins in the financial industry, but is becoming increasingly important for companies in all sectors and legal forms. International institutions and standards as well as national legislation, which demand the introduction of risk management in companies, are also contributing to this. This development was justified in the past by insolvencies of large companies with considerable disadvantages for society (i.e. employees of the affected companies themselves, but also taxpayers, etc.). Many companies are currently in the process of introducing risk management systems. At the same time, there is a shortage of skilled personnel to set up and maintain the systems. This makes it all the more important to strive for a largely automated and barrier-free flow of the processes that are new for the companies.[51]

---

associated with the error) decreases. However, this area increases in relation to the remaining area (or alpha) in the tail of the distribution. Thus, with more extreme quantiles, the deviation of the actual VaR exceedance probability from the given alpha also increases.

[50] This concerns the compatibility of different software products. But also within a programme, intermediate results of functions are no longer read out and manually entered into other functions or programmes as input. Instead, a function that forecasts conditional volatilities, for example, submits this forecast value at runtime directly to the next function as input, so that the latter calculates and spits out the VaR, for example.

[51] These processes involve or require the processing of information or data.

The present study is essentially concerned with the key figure Value-at-Risk, which has gained particular importance in risk management. In addition to an evaluation of the accuracy of Value-at-Risk forecasts, it was of interest to find out which functions of the software R are used to obtain reliable (in the sense of stable programme processes) results.[52]

To investigate this, analyses and forecasts were carried out for the present work based on successively changed time windows. These are carried out using various functions of the *R* software.

The paper describes the ordinary GARCH model and extensions, especially for modelling and analysis of univariate and multivariate time series. Among other things, an overview of multivariate GARCH models is given, which could be used for forecasts of time-conditional variance-covariance matrices. VaR forecasts for portfolio compositions of assets can then be calculated from the forecast variance-covariance matrices. In this context, it is discussed whether or which variants of multivariate GARCH models are most likely to be used for forecasts or automated forecasting systems. However, there are arguments in favour of forecasting the individual elements (variances and covariances) using specifically fitted univariate GARCH models.

Software routines that are as stable as possible are required for frequent and automated forecasting. The study with rolling and growing time windows revealed that the *Markov switching GARCH* model of the *R* package *"MSGARCH"* ensures a flexible adaptation to time series and reliably delivers parameter estimation results. In contrast, the estimation algorithm of the *R* function of the ordinary *GARCH* model broke down more frequently in the case of outliers and structural breaks.

However, the accuracy of a VaR forecast was essentially determined by the adjusted distribution model.

Since it has been shown in the past that the normal distribution is not suitable for modelling financial market data, other distributions for the residuals (or returns) have also been considered. For example, the software R also offers the Student t distribution and the Skewed Student t distribution as alternatives in various applications (including for estimating GARCH and MS GARCH models).

Both, but most likely the Skewed Student t-distribution, provide a better approximation to the actual distributions. Calculations were also performed for high VaR quantiles. The actual distributions typically have more mass (so-called "fat tails") at the edges (especially above the 95% or below the 5% quantile) compared to the normal distribution. Accordingly, extreme return values occur more frequently than would be assumed by the normal distribution.

The effects of neglecting or designing the mean equation on the Value-at-Risk forecast using GARCH models were also investigated. It was shown that in some cases even more accurate forecasts are achieved for certain extreme quantiles.

In addition, calculations were made in the study on the development of the (ordinary) correlation and default correlation. These illustrations should contribute to a better decision regarding the modelling (and, if necessary, model selection) for the dynamics of the coefficients.

As far as the default correlation is concerned, the result for some pairwise exposures is that it hardly decreases even with increasing extreme quantiles. In this case, high losses would be expected for both exposures if VaR were exceeded.

[52] *"R is a free software environment for statistical computing and graphics."* , see https://www.r-project.org/

Since the calculated default correlations depend on their underlying VaR forecasts, it was also examined how a consideration of the expected value (or modelling of the mean equation) affects them. It turns out, however, that the coefficients of the default correlations are hardly changed.

Since calculated default correlations require long time series, possibly based on periods with intervening changes in market conditions, further research should address how to obtain short-term dynamics of this coefficient.

An evaluation of the accuracy of VaR forecasts refers to the mass of trading days for which the forecasts are made.

If key figures are actually relevant for insolvency prevention and are not only calculated because of regulatory requirements, further analyses and forecasts should be carried out for individual trading days in addition to the VaR calculation:

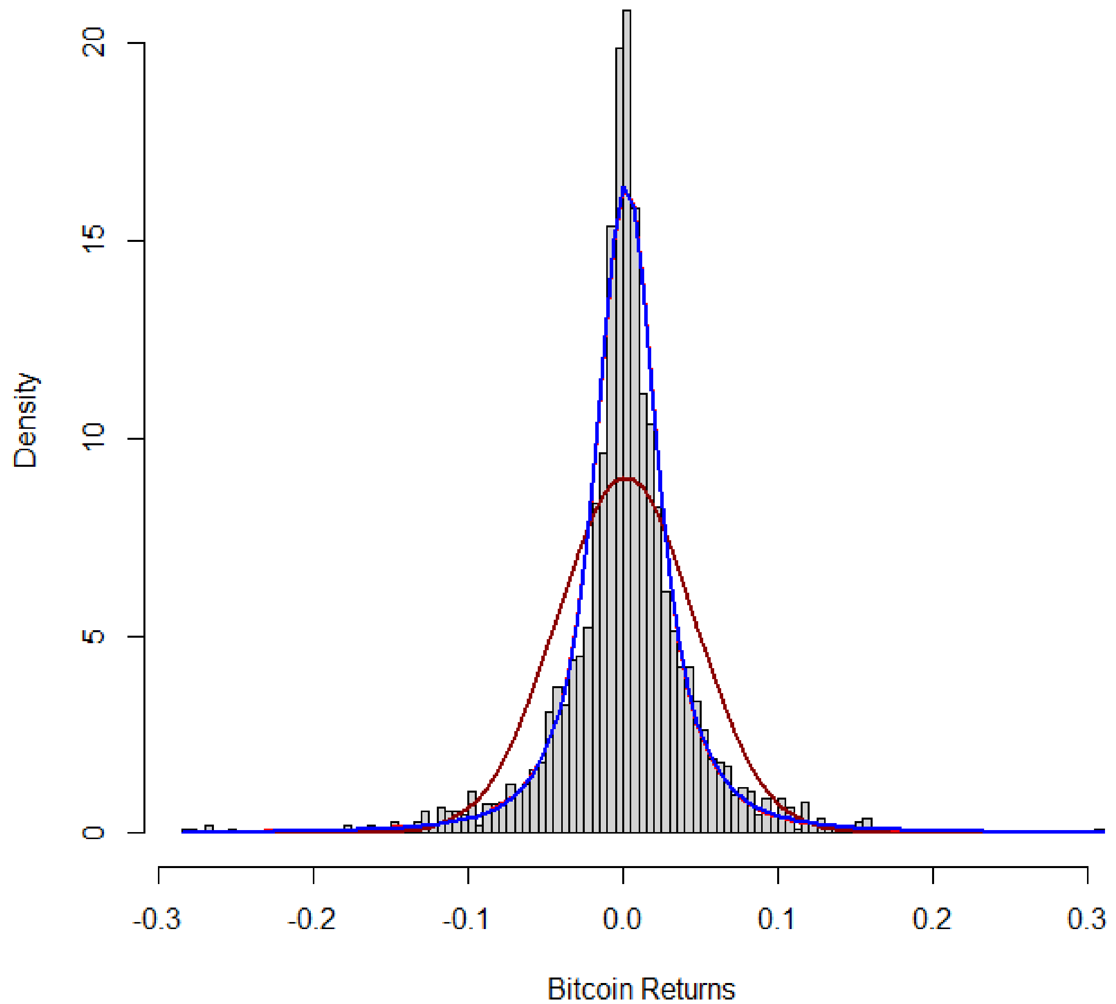
Kwon (2021) for example extends the CAViaR model of Engle and Manganelli (2004) to study potential drivers of the Bitcoin's 5% and 1% VaR. The author finds for the 1% VaR variables related to the macroeconomy as key drivers. For the 5% VaR the author identifies for example positive relationships to the Bitcoin trading volume and the Internet search index, and negative responses to volatility on the Chinese stock market.

APPENDIX

Histograms of bitcoin returns and USD/EUR exchange rate returns and fitted density functions

Histograms of Bitcoin returns and USD/EUR exchange rate returns are presented below. Density functions of the normal distribution (dark red), the Student t-distribution (light red) and the Skew Student t-distribution (blue) are fitted to these distributions.

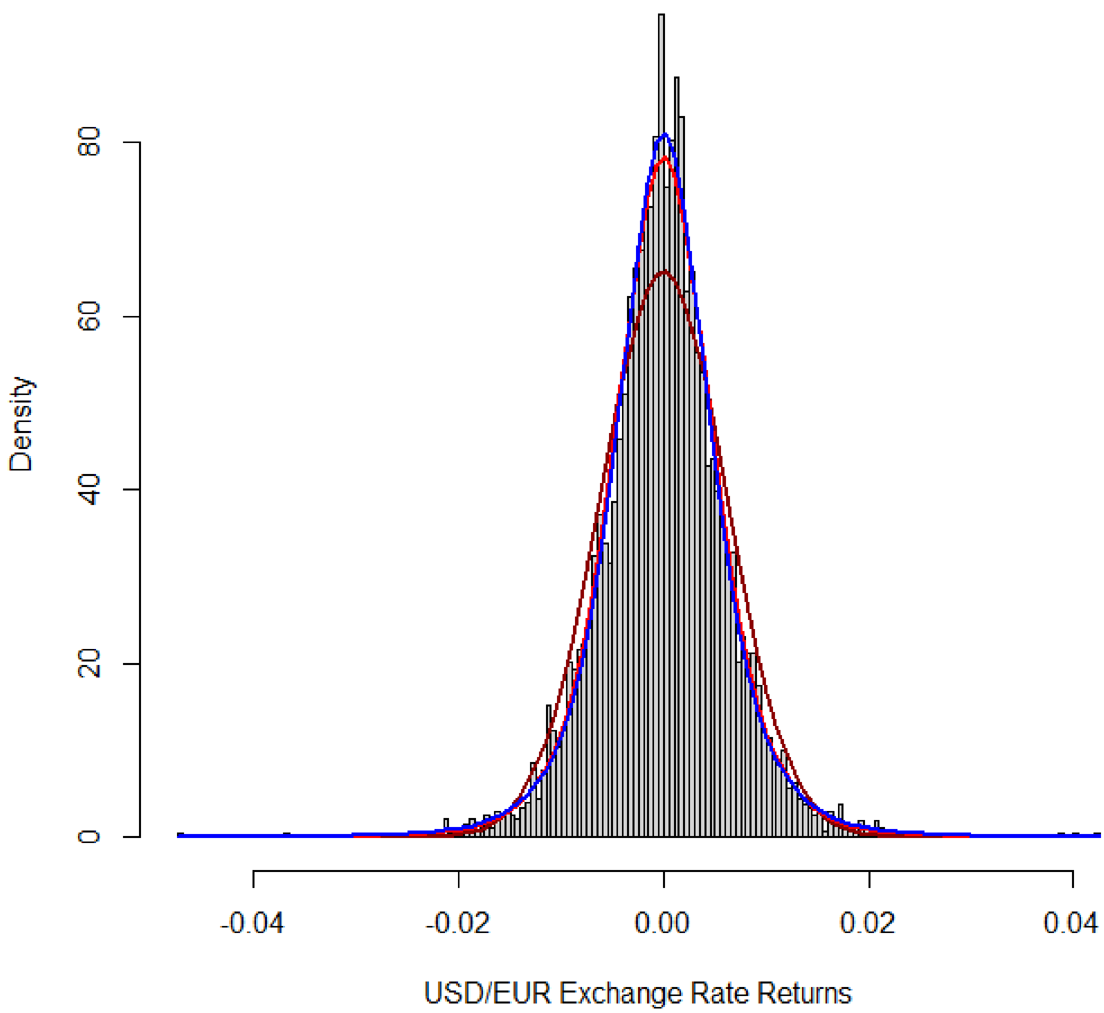
Figure 1 Histogram of daily Bitcoin returns and fitted density functions



Especially for the extreme bitcoin returns, it is apparent that the normal distribution does not provide a suitable fit. The density function of the Student t-distribution is overlaid by that of the Skew Student t-distribution in both figures. The Skew Student t-distribution provides an even better fit.

Parameters of the Student t and skew Student t distributions were estimated with the functions *stdFit* and *sstdFit* of the R package *fGarch*.

Figure 2 Histogram of daily US Dollar/Euro exchange rate returns and fitted density functions



Exemplary calculations for the Value-at-Risk forecast error when neglecting the mean value

The density functions were fitted to the (unconditional) distributions. For the Gaussian distribution, the two parameters mean and standard deviation were estimated with the R functions *mean* and *sd*. The estimated parameter values were transferred to the *dnorm* functions to calculate the density of the normal distribution. The graphs of these density functions were created with the R function *curve*.

Parameters of the Student t and Skew Student t distributions were estimated with the functions *stdFit* and *sstdFit* of the R package *fGarch*. These also include estimates of the mean values alongside the other parameters of these distributions.

Quantiles were obtained via the R-functions *quantile* (actual distribution), *qnorm* (Gaussian distribution), *qstd* (Student t) and *qsstd* (Skew Student t).

The following two tables show that the three distribution functions provide different precision fits to the quantiles.

As an example, it is also shown how neglecting the mean can affect these results. For this purpose, the unrealistic scenario was assumed that the returns are actually normally distributed. However, the established distribution functions do not provide a suitable fit, especially for the extreme returns at the tails of the distributions.

Table 1 VaR forecast error due to a neglected negative mean if USD/EUR exchange rate returns were normally distributed

alpha	10	5	2,5	2	1	0,5	0,25	0,1
Quantiles of USD/EUR returns and fitted distributions								
actual	-0,735	-0,986	-1,208	-1,294	-1,595	-1,861	-2,135	-2,495
Gaussian	-0,786	-1,009	-1,202	-1,259	-1,426	-1,579	-1,721	-1,894
Student t	-0,711	-0,967	-1,229	-1,316	-1,599	-1,906	-2,245	-2,752
Skew Student t	-0,715	-0,974	-1,239	-1,328	-1,614	-1,926	-2,270	-2,784
Overestimated quantile values when neglecting the negative mean value								
actual	-0,734	-0,985	-1,207	-1,293	-1,594	-1,860	-2,134	-2,494
Gaussian	-0,785	-1,008	-1,201	-1,258	-1,425	-1,578	-1,720	-1,893
Student t	-0,710	-0,966	-1,228	-1,315	-1,598	-1,905	-2,244	-2,751
Skew Student t	-0,714	-0,973	-1,238	-1,327	-1,613	-1,925	-2,269	-2,783
Forecast error (%) of exceedance probability if returns were Gaussian distributed								
Prob.-Gaussian	10,030	5,018	2,510	2,008	1,005	0,502	0,251	0,101
Percentage Error	0,299	0,351	0,398	0,412	0,454	0,493	0,529	0,574

If the USD/EUR daily returns were actually Gaussian distributed, neglecting the negative mean return would mean that, for example, the 99% VaR (i.e. alpha = 1%) would actually be set at the 98.995% VaR (i.e. alpha = 1.005%). The mass of the fat tail up to this quantile is thus overestimated by 0,454%. Thus, a smaller capital buffer would be held than actually required, but the VaR forecast would be exceeded more frequently.

Table 2 VaR forecast error due to a neglected mean if Bitcoin returns were normally distributed

alpha	10	5	2,5	2	1	0,5	0,25	0,1
Quantiles of Bitcoin returns and fitted distributions								
actual	-4,33	-6,80	-9,89	-10,94	-13,03	-16,41	-18,94	-26,78
Gaussian	-5,50	-7,11	-8,51	-8,92	-10,13	-11,24	-12,26	-13,52
Student t	-3,84	-6,07	-9,04	-10,22	-14,77	-21,13	-30,07	-47,76
Skew Student t	-3,79	-5,97	-8,89	-10,04	-14,49	-20,71	-29,44	-46,70
Underestimated quantile values when neglecting the positive mean value								
actual	-4,51	-6,98	-10,07	-11,12	-13,21	-16,59	-19,12	-26,96
Gaussian	-5,68	-7,29	-8,69	-9,10	-10,31	-11,42	-12,44	-13,70
Student t	-4,02	-6,25	-9,22	-10,40	-14,95	-21,31	-30,26	-47,96
Skew Student t	-3,97	-6,15	-9,06	-10,21	-14,66	-20,87	-29,60	-46,83
Forecast error (%) of exceedance probability if returns were Gaussian distributed								
Prob.-Gaussian	9,30	4,59	2,27	1,81	0,90	0,44	0,22	0,09
Percentage Error	-6,98	-8,15	-9,18	-9,48	-10,38	-11,21	-11,98	-12,92

If bitcoin daily returns were actually normally distributed, neglecting the high mean return would result in, for example, the 99% VaR (i.e. alpha = 1%) actually being set at the 99.1% VaR (i.e. alpha = 0.9%). Thus, a higher capital buffer would be held than actually required. However, the VaR forecast would be exceeded less frequently. The number of trading days with exceedances would decrease by 10.38%.

Value-at-Risk forecasts based on growing time windows - ordinary GARCH(g,a) model

ARMA(p,q)- GARCH(g,a) model -- Skew Student-T distribution

One day-ahead Value-at-risk forecasts with maximum lags of two for g and a in GARCH(g,a) volatility equations and five for p and q in ARMA(p,q) mean equations:

For the GBP/EUR and CHF/EUR exchange rates, the ordinary GARCH function did not produce a result for some time series segments. Due to the repeatedly interrupted estimation algorithm, the VaR forecasts were also not produced for the entire period of the time series in consideration.

Table 3 Exceedance rates for selected Value-at-Risk quantiles

VaR	90	95	97,5	98	99	99,5	99,75	99,9
alpha	10	5	2,5	2	1	0,5	0,25	0,1
BTC	9,885	5,195	2,667	2,115	0,690	0,230	0,046	0,046
ETH	9,291	6,083	3,543	2,741	1,136	0,535	0,267	0,134
USD	9,914	4,919	2,666	2,009	1,108	0,563	0,338	0,131
GBP	9,409	4,961	2,126	1,535	0,748	0,354	0,197	0,118
CHF	11,805	6,607	3,813	3,212	1,645	0,914	0,522	0,366
JPY	9,951	5,332	2,647	2,140	1,051	0,544	0,225	0,094
CNY	9,725	4,467	2,332	1,819	1,067	0,573	0,356	0,099
Oil	10,515	5,453	2,936	2,265	1,314	0,699	0,364	0,168
Gold	10,069	4,876	2,530	2,082	0,843	0,474	0,237	0,158
Silver	10,341	5,223	2,362	1,916	1,024	0,420	0,184	0,079

Table 4 Percentage deviation of exceedance rate from alpha

VaR	90	95	97,5	98	99	99,5	99,75	99,9	Mean	M Abs	M Pos	M Neg
alpha	10	5	2,5	2	1	0,5	0,25	0,1	alpha 10% to 1%			
BTC	-1,1	3,9	6,7	5,7	-31,0	-54,0	-81,6	-54,0	-3,2	9,7	5,4	-16,1
ETH	-7,1	21,7	41,7	37,0	13,6	7,0	7,0	33,7	21,4	24,2	28,5	-7,1
USD	-0,9	-1,6	6,6	0,5	10,8	12,7	35,2	31,4	3,1	4,1	6,0	-1,2
GBP	-5,9	-0,8	-15,0	-23,2	-25,2	-29,1	-21,3	18,1	-14,0	14,0		-14,0
CHF	18,0	32,1	52,5	60,6	64,5	82,8	108,9	265,6	45,6	45,6	45,6	
JPY	-0,5	6,6	5,9	7,0	5,1	8,9	-9,9	-6,1	4,8	5,0	6,2	-0,5
CNY	-2,7	-10,7	-6,7	-9,1	6,7	14,6	42,3	-1,2	-4,5	7,2	6,7	-7,3
Oil	5,1	9,1	17,4	13,3	31,4	39,8	45,4	67,8	15,3	15,3	15,3	
Gold	0,7	-2,5	1,2	4,1	-15,7	-5,1	-5,1	58,1	-2,4	4,8	2,0	-9,1
Silver	3,4	4,5	-5,5	-4,2	2,4	-16,0	-26,5	-21,3	0,1	4,0	3,4	-4,9
Mean	0,9	6,2	10,5	9,2	6,3	6,2	9,4	39,2	6,6			
M Abs	4,6	9,3	15,9	16,5	20,7	27,0	38,3	55,7		13,4		
M Pos	6,8	13,0	18,9	18,3	19,2	27,6	47,8	79,1			15,2	
M Neg	-3,0	-3,9	-9,1	-12,2	-24,0	-26,1	-28,9	-20,6				-10,4

Table 5 Mean Excess Loss for selected VaR quantiles

VaR	90	95	97,5	98	99	99,5	99,75	99,9
alpha	10	5	2,5	2	1	0,5	0,25	0,1
BTC	-3,31	-3,41	-2,82	-2,55	-3,00	-3,74	-5,89	-2,71
ETH	-4,41	-3,99	-3,62	-3,73	-4,70	-5,21	-5,00	-4,71
USD	-0,32	-0,32	-0,32	-0,35	-0,36	-0,42	-0,41	-0,50
GBP	-0,21	-0,18	-0,17	-0,18	-0,18	-0,19	-0,18	-0,10
CHF	-0,24	-0,26	-0,27	-0,27	-0,31	-0,34	-0,38	-0,27
JPY	-0,44	-0,42	-0,43	-0,43	-0,47	-0,48	-0,65	-0,95
CNY	-0,31	-0,33	-0,34	-0,36	-0,34	-0,37	-0,33	-0,56
Oil	-1,16	-1,08	-1,04	-1,10	-1,02	-1,02	-1,03	-0,98
Gold	-0,78	-0,85	-0,88	-0,88	-1,23	-1,45	-2,00	-1,74
Silver	-1,49	-1,61	-2,00	-2,07	-2,12	-2,71	-3,73	-6,25

Table 6 Median Excess Loss for selected VaR quantiles

VaR	90	95	97,5	98	99	99,5	99,75	99,9
alpha	10	5	2,5	2	1	0,5	0,25	0,1
BTC	-1,95	-2,85	-1,87	-1,58	-1,49	-2,19	-5,89	-2,71
ETH	-2,95	-2,62	-1,79	-1,98	-2,96	-2,53	-2,47	-4,71
USD	-0,22	-0,20	-0,21	-0,21	-0,21	-0,28	-0,21	-0,20
GBP	-0,16	-0,12	-0,09	-0,14	-0,10	-0,14	-0,16	-0,03
CHF	-0,14	-0,15	-0,14	-0,14	-0,18	-0,19	-0,36	-0,18
JPY	-0,31	-0,27	-0,24	-0,25	-0,31	-0,26	-0,27	-0,38
CNY	-0,20	-0,22	-0,22	-0,23	-0,23	-0,24	-0,18	-0,28
Oil	-0,80	-0,74	-0,72	-0,77	-0,72	-0,74	-0,60	-0,75
Gold	-0,49	-0,56	-0,49	-0,44	-0,55	-0,64	-1,77	-1,29
Silver	-0,88	-0,84	-1,23	-1,26	-1,01	-1,26	-1,04	-4,57

Table 7 Mean Not Exhausted VaR for selected quantiles

VaR	90	95	97,5	98	99	99,5	99,75	99,9
alpha	10	5	2,5	2	1	0,5	0,25	0,1
BTC	5,45	7,43	9,99	10,98	14,64	19,52	25,94	37,70
ETH	8,32	10,61	13,18	14,08	17,31	21,33	26,25	34,51
USD	0,84	1,02	1,21	1,27	1,47	1,67	1,89	2,19
GBP	0,65	0,78	0,90	0,94	1,08	1,22	1,36	1,56
CHF	0,38	0,48	0,60	0,64	0,78	0,95	1,15	1,47
JPY	1,00	1,25	1,50	1,58	1,85	2,14	2,44	2,87
CNY	0,83	1,01	1,20	1,26	1,46	1,67	1,88	2,17
Oil	2,78	3,43	4,11	4,33	5,05	5,80	6,57	7,64
Gold	1,58	2,00	2,47	2,63	3,16	3,75	4,41	5,38
Silver	2,61	3,39	4,29	4,61	5,72	6,98	8,45	10,78

Table 8 Median Not Exhausted VaR for selected quantiles

VaR	90	95	97,5	98	99	99,5	99,75	99,9
alpha	10	5	2,5	2	1	0,5	0,25	0,1
BTC	4,32	6,15	8,50	9,39	12,64	17,02	22,59	32,68
ETH	6,42	8,52	10,94	11,82	14,67	18,21	22,52	29,83
USD	0,76	0,94	1,13	1,19	1,39	1,59	1,80	2,09
GBP	0,55	0,67	0,80	0,83	0,96	1,09	1,23	1,40
CHF	0,29	0,37	0,47	0,51	0,62	0,75	0,91	1,16
JPY	0,87	1,11	1,34	1,42	1,68	1,94	2,23	2,62
CNY	0,73	0,92	1,11	1,17	1,37	1,57	1,78	2,07
Oil	2,37	3,02	3,68	3,90	4,58	5,29	6,01	6,99
Gold	1,34	1,77	2,22	2,37	2,88	3,42	4,01	4,90
Silver	2,24	3,01	3,85	4,14	5,16	6,31	7,64	9,68

ARMA(p,q)- GARCH(g,a) model -- Student-T distribution

For the CHF/EUR exchange rates, the ordinary GARCH function did not produce a result for some time series segments.

Table 9 Exceedance rates for selected Value-at-Risk quantiles

VaR	90	95	97,5	98	99	99,5	99,75	99,9
alpha	**10**	**5**	**2,5**	**2**	**1**	**0,5**	**0,25**	**0,1**
BTC	9,931	5,333	3,034	2,483	0,920	0,322	0,138	0,092
ETH	8,757	5,414	2,741	2,072	0,735	0,468	0,267	0,000
USD	9,557	4,281	2,197	1,727	0,976	0,488	0,244	0,131
GBP	9,144	4,675	2,028	1,633	0,845	0,469	0,207	0,113
CHF	13,508	7,697	4,563	3,925	2,190	1,126	0,639	0,426
JPY	10,383	5,652	2,967	2,347	1,145	0,676	0,376	0,113
CNY	9,705	4,467	2,293	1,858	1,028	0,593	0,277	0,099
Oil	11,074	5,928	3,244	2,657	1,706	0,867	0,447	0,196
Gold	10,965	5,825	3,084	2,399	1,186	0,474	0,290	0,185
Silver	11,759	6,404	3,228	2,520	1,522	0,761	0,289	0,105

Table 10 Percentage deviation of exceedance rate from alpha

VaR	90	95	97,5	98	99	99,5	99,75	99,9	Mean	M Abs	M Pos	M Neg
alpha	**10**	**5**	**2,5**	**2**	**1**	**0,5**	**0,25**	**0,1**	**alpha 10% to 1%**			
BTC	-0,7	6,7	21,4	24,1	-8,0	-35,6	-44,8	-8,0	8,7	12,2	17,4	-4,4
ETH	-12,4	8,3	9,6	3,6	-26,5	-6,4	7,0	-100,0	-3,5	12,1	7,2	-19,5
USD	-4,4	-14,4	-12,1	-13,6	-2,4	-2,4	-2,4	31,4	-9,4	9,4		-9,4
GBP	-8,6	-6,5	-18,9	-18,3	-15,5	-6,1	-17,4	12,7	-13,6	13,6		-13,6
CHF	35,1	53,9	82,5	96,2	119,0	125,1	155,6	325,9	77,4	77,4	77,4	
JPY	3,8	13,0	18,7	17,3	14,5	35,2	50,2	12,7	13,5	13,5	13,5	
CNY	-2,9	-10,7	-8,3	-7,1	2,8	18,6	10,7	-1,2	-5,2	6,4	2,8	-7,2
Oil	10,7	18,6	29,8	32,8	70,6	73,4	79,0	95,7	32,5	32,5	32,5	
Gold	9,6	16,5	23,4	19,9	18,6	-5,1	16,0	84,5	17,6	17,6	17,6	
Silver	17,6	28,1	29,1	26,0	52,2	52,2	15,5	5,0	30,6	30,6	30,6	

Mean	4,8	11,4	17,5	18,1	22,5	24,9	26,9	45,9	14,9			
M Abs	10,6	17,7	25,4	25,9	33,0	36,0	39,8	67,7		22,5		
M Pos	15,4	20,7	30,6	31,4	46,3	60,9	47,7	81,1			28,9	
M Neg	-5,8	-10,5	-13,1	-13,0	-13,1	-11,1	-21,5	-36,4				-11,1

ARMA(p,q)- GARCH(g,a) model -- Gaussian distribution

For the CHF/EUR exchange rates, the ordinary GARCH function did not produce a result for some time series segments.

Table 11 Exceedance rates for selected Value-at-Risk quantiles

VaR	90	95	97,5	98	99	99,5	99,75	99,9
alpha	**10**	**5**	**2,5**	**2**	**1**	**0,5**	**0,25**	**0,1**
BTC	7,494	4,828	3,816	3,494	2,437	1,793	1,195	0,966
ETH	7,086	4,612	2,941	2,406	1,738	1,070	0,869	0,602
USD	8,825	4,281	2,516	2,103	1,427	0,826	0,544	0,338
GBP	8,374	4,431	2,197	1,840	1,202	0,732	0,507	0,282
CHF	9,819	5,272	3,783	3,219	2,093	1,529	1,127	0,684
JPY	9,294	5,407	3,136	2,629	1,690	1,070	0,770	0,488
CNY	8,757	4,250	2,550	1,996	1,364	0,850	0,613	0,316
Oil	10,487	6,096	3,663	3,216	2,013	1,482	0,895	0,587
Gold	8,408	4,902	3,057	2,609	1,950	1,054	0,791	0,474
Silver	8,556	4,987	3,123	2,677	1,995	1,654	1,181	0,919

Table 12 Percentage deviation of exceedance rate from alpha

VaR	90	95	97,5	98	99	99,5	99,75	99,9	Mean	M Abs	M Pos	M Neg
alpha	**10**	**5**	**2,5**	**2**	**1**	**0,5**	**0,25**	**0,1**	**alpha 10% to 1%**			
BTC	-25,1	-3,4	52,6	74,7	143,7	258,6	378,2	865,5	48,5	59,9	90,3	-14,3
ETH	-29,1	-7,8	17,6	20,3	73,8	113,9	247,6	501,6	15,0	29,7	37,3	-18,4
USD	-11,8	-14,4	0,6	5,1	42,7	65,2	117,8	238,0	4,5	14,9	16,2	-13,1
GBP	-16,3	-11,4	-12,1	-8,0	20,2	46,5	102,8	181,6	-5,5	13,6	20,2	-11,9
CHF	-1,8	5,4	51,3	61,0	109,3	205,8	350,7	584,1	45,0	45,8	56,7	-1,8
JPY	-7,1	8,1	25,4	31,4	69,0	114,0	207,9	388,2	25,4	28,2	33,5	-7,1
CNY	-12,4	-15,0	2,0	-0,2	36,4	70,0	145,1	216,3	2,2	13,2	19,2	-9,2
Oil	4,9	21,9	46,5	60,8	101,3	196,4	257,9	487,2	47,1	47,1	47,1	
Gold	-15,9	-2,0	22,3	30,5	95,0	110,9	216,3	374,4	26,0	33,1	49,3	-8,9
Silver	-14,4	-0,3	24,9	33,9	99,5	230,7	372,4	818,6	28,7	34,6	52,8	-7,3
Mean	-12,9	-1,9	23,1	31,0	79,1	141,2	239,7	465,6	23,7			
M Abs	13,9	9,0	25,6	32,6	79,1	141,2	239,7	465,6		32,0		
M Pos	4,9	11,8	27,0	39,7	79,1	141,2	239,7	465,6			32,5	
M Neg	-14,9	-7,7	-12,1	-4,1								-9,7

Table 13 Mean Excess Loss for selected VaR quantiles

VaR	90	95	97,5	98	99	99,5	99,75	99,9
alpha	**10**	**5**	**2,5**	**2**	**1**	**0,5**	**0,25**	**0,1**
BTC	-3,38	-3,53	-3,08	-2,97	-2,97	-2,96	-3,49	-3,40
ETH	-4,25	-4,02	-4,04	-4,35	-4,26	-5,15	-4,86	-5,22
USD	-0,32	-0,34	-0,33	-0,34	-0,32	-0,37	-0,40	-0,45
GBP	-0,26	-0,24	-0,27	-0,27	-0,26	-0,29	-0,30	-0,38
CHF	-0,16	-0,18	-0,17	-0,18	-0,20	-0,20	-0,21	-0,24
JPY	-0,43	-0,41	-0,42	-0,43	-0,43	-0,46	-0,46	-0,47
CNY	-0,32	-0,34	-0,33	-0,36	-0,34	-0,37	-0,35	-0,44
Oil	-1,14	-1,06	-1,05	-1,02	-1,06	-0,95	-1,10	-1,08
Gold	-0,81	-0,83	-0,86	-0,89	-0,82	-1,11	-1,18	-1,57
Silver	-1,54	-1,65	-1,84	-1,94	-1,96	-1,81	-1,96	-1,83

Table 14 Median Excess Loss for selected VaR quantiles

VaR	90	95	97,5	98	99	99,5	99,75	99,9
alpha	**10**	**5**	**2,5**	**2**	**1**	**0,5**	**0,25**	**0,1**
BTC	-2,30	-2,64	-2,16	-2,00	-1,83	-1,94	-2,03	-1,64
ETH	-2,72	-2,59	-2,12	-2,35	-2,26	-3,64	-3,86	-3,60
USD	-0,20	-0,22	-0,23	-0,23	-0,15	-0,20	-0,25	-0,24
GBP	-0,18	-0,14	-0,18	-0,17	-0,15	-0,15	-0,14	-0,14
CHF	-0,10	-0,11	-0,10	-0,09	-0,11	-0,09	-0,10	-0,19
JPY	-0,29	-0,26	-0,24	-0,24	-0,26	-0,31	-0,24	-0,20
CNY	-0,19	-0,22	-0,21	-0,22	-0,20	-0,26	-0,21	-0,29
Oil	-0,79	-0,74	-0,70	-0,71	-0,76	-0,58	-0,80	-0,80
Gold	-0,54	-0,53	-0,52	-0,47	-0,31	-0,46	-0,54	-0,74
Silver	-0,80	-0,89	-1,11	-1,23	-1,18	-1,00	-0,99	-0,92

Table 15 Mean Not Exhausted VaR for selected quantiles

VaR	90	95	97,5	98	99	99,5	99,75	99,9
alpha	**10**	**5**	**2,5**	**2**	**1**	**0,5**	**0,25**	**0,1**
BTC	6,00	7,31	8,52	8,87	9,89	10,85	11,73	12,87
ETH	9,57	11,67	13,51	14,04	15,72	17,23	18,69	20,48
USD	0,88	1,05	1,22	1,27	1,43	1,57	1,70	1,87
GBP	0,71	0,86	0,99	1,03	1,15	1,27	1,38	1,51
CHF	0,35	0,42	0,48	0,50	0,56	0,62	0,67	0,73
JPY	1,04	1,25	1,44	1,50	1,68	1,85	2,01	2,20
CNY	0,85	1,02	1,19	1,24	1,39	1,53	1,66	1,82
Oil	2,79	3,33	3,84	4,00	4,46	4,91	5,32	5,84
Gold	1,66	2,01	2,34	2,43	2,73	2,99	3,25	3,57
Silver	2,81	3,42	3,97	4,14	4,65	5,13	5,57	6,11

Value-at-Risk forecasts based on growing time windows - MS(k)-GARCH(1,1) model

ARMA(p,q) - MS(k)-GARCH(1,1) model -- Skew Student-T distribution

One day-ahead Value-at-Risk forecasts with up to five regimes (i.e. the k optimised according to the BIC criterion is between 1 and 5).

Results from the estimation algorithm were obtained for all time series sections.

Table 16 Exceedance rates for selected Value-at-Risk quantiles

VaR	90	95	97,5	98	99	99,5	99,75	99,9
alpha	**10**	**5**	**2,5**	**2**	**1**	**0,5**	**0,25**	**0,1**
BTC	10,437	5,517	3,356	2,805	1,517	0,736	0,276	0,138
ETH	10,361	6,684	4,011	3,409	1,471	0,936	0,468	0,134
USD	9,820	4,957	2,478	1,990	1,070	0,582	0,376	0,131
GBP	9,763	5,238	2,666	2,084	1,239	0,620	0,300	0,169
CHF	9,895	4,694	2,347	1,971	1,014	0,582	0,319	0,207
JPY	9,444	4,863	2,328	1,915	0,920	0,563	0,207	0,094
CNY	9,231	4,151	2,234	1,819	1,008	0,593	0,395	0,158
Oil	10,431	5,509	3,104	2,573	1,510	0,923	0,531	0,252
Gold	9,673	4,692	2,451	2,056	0,817	0,422	0,211	0,158
Silver	10,131	5,066	2,283	1,942	1,102	0,420	0,184	0,079

Table 17 Percentage deviation of exceedance rate from alpha

VaR	90	95	97,5	98	99	99,5	99,75	99,9	Mean	M Abs	M Pos	M Neg
alpha	**10**	**5**	**2,5**	**2**	**1**	**0,5**	**0,25**	**0,1**	**alpha 10% to 1%**			
BTC	4,4	10,3	34,3	40,2	51,7	47,1	10,3	37,9	28,2	28,2	28,2	
ETH	3,6	33,7	60,4	70,5	47,1	87,2	87,2	33,7	43,0	43,0	43,0	
USD	-1,8	-0,9	-0,9	-0,5	7,0	16,4	50,2	31,4	0,6	2,2	7,0	-1,0
GBP	-2,4	4,8	6,6	4,2	23,9	23,9	20,2	69,0	7,4	8,4	9,9	-2,4
CHF	-1,1	-6,1	-6,1	-1,4	1,4	16,4	27,7	106,5	-2,7	3,2	1,4	-3,7
JPY	-5,6	-2,7	-6,9	-4,2	-8,0	12,7	-17,4	-6,1	-5,5	5,5		-5,5
CNY	-7,7	-17,0	-10,7	-9,1	0,8	18,6	58,1	58,1	-8,7	9,0	0,8	-11,1
Oil	4,3	10,2	24,2	28,6	51,0	84,6	112,5	151,7	23,7	23,7	23,7	
Gold	-3,3	-6,2	-2,0	2,8	-18,3	-15,7	-15,7	58,1	-5,4	6,5	2,8	-7,4
Silver	1,3	1,3	-8,7	-2,9	10,2	-16,0	-26,5	-21,3	0,3	4,9	4,3	-5,8

Mean	-0,8	2,7	9,0	12,8	16,7	27,5	30,7	51,9	8,09			
M Abs	3,5	9,3	16,1	16,4	21,9	33,9	42,6	57,4		13,46		
M Pos	3,4	12,1	31,4	29,3	24,1	38,4	52,3	68,3			20,05	
M Neg	-3,6	-6,6	-5,9	-3,6	-13,1	-15,8	-19,9	-13,7				-6,56

Table 18 Mean Excess Loss for selected VaR quantiles

VaR	90	95	97,5	98	99	99,5	99,75	99,9
alpha	10	5	2,5	2	1	0,5	0,25	0,1
BTC	-3,19	-3,46	-3,17	-3,07	-2,97	-3,17	-4,91	-5,11
ETH	-4,28	-4,06	-4,01	-3,91	-4,97	-4,44	-4,47	-7,04
USD	-0,32	-0,32	-0,34	-0,35	-0,37	-0,43	-0,43	-0,68
GBP	-0,25	-0,24	-0,25	-0,27	-0,26	-0,30	-0,41	-0,50
CHF	-0,22	-0,26	-0,35	-0,37	-0,54	-0,75	-1,19	-1,62
JPY	-0,44	-0,42	-0,45	-0,44	-0,51	-0,47	-0,71	-1,00
CNY	-0,32	-0,35	-0,35	-0,36	-0,37	-0,39	-0,37	-0,53
Oil	-1,18	-1,14	-1,08	-1,08	-1,04	-1,01	-1,03	-1,13
Gold	-0,80	-0,86	-0,88	-0,86	-1,22	-1,53	-2,19	-1,93
Silver	-1,49	-1,60	-1,98	-1,95	-1,86	-2,58	-3,74	-6,81

Table 19 Median Excess Loss for selected VaR quantiles

VaR	90	95	97,5	98	99	99,5	99,75	99,9
alpha	10	5	2,5	2	1	0,5	0,25	0,1
BTC	-1,83	-2,40	-2,08	-2,05	-1,57	-1,19	-3,40	-3,29
ETH	-2,80	-2,71	-2,39	-2,25	-3,61	-3,10	-2,12	-7,04
USD	-0,22	-0,20	-0,21	-0,20	-0,22	-0,27	-0,22	-0,73
GBP	-0,18	-0,15	-0,17	-0,17	-0,15	-0,12	-0,19	-0,44
CHF	-0,11	-0,11	-0,13	-0,12	-0,17	-0,21	-0,25	-0,24
JPY	-0,30	-0,26	-0,25	-0,28	-0,32	-0,21	-0,42	-0,46
CNY	-0,19	-0,24	-0,21	-0,22	-0,26	-0,28	-0,14	-0,27
Oil	-0,82	-0,80	-0,71	-0,72	-0,64	-0,60	-0,68	-0,80
Gold	-0,52	-0,58	-0,44	-0,43	-0,60	-0,67	-1,95	-1,42
Silver	-0,87	-0,84	-1,21	-1,17	-0,80	-1,12	-0,79	-4,73

Table 20 Mean Not Exhausted VaR for selected quantiles

VaR	90	95	97,5	98	99	99,5	99,75	99,9
alpha	10	5	2,5	2	1	0,5	0,25	0,1
BTC	5,22	6,79	8,58	9,19	11,19	13,38	15,76	19,15
ETH	7,90	9,80	11,84	12,55	14,87	17,62	20,65	25,06
USD	0,85	1,02	1,20	1,26	1,45	1,64	1,84	2,10
GBP	0,68	0,82	0,95	0,99	1,14	1,28	1,42	1,61
CHF	0,42	0,52	0,63	0,67	0,80	0,94	1,09	1,30
JPY	1,02	1,27	1,52	1,61	1,89	2,18	2,48	2,89
CNY	0,84	1,02	1,20	1,27	1,46	1,65	1,84	2,10
Oil	2,77	3,38	3,99	4,18	4,78	5,37	5,94	6,68
Gold	1,58	2,01	2,49	2,66	3,19	3,78	4,42	5,32
Silver	2,63	3,43	4,36	4,70	5,84	7,12	8,56	10,70

ARMA(p,q) - MS(k)-GARCH(1,1) model -- Student-T distribution

One day-ahead Value-at-Risk forecasts with up to five regimes (i.e. the k optimised according to the BIC criterion is between 1 and 5).

Results from the estimation algorithm were obtained for all time series sections.

Table 21 Exceedance rates for selected Value-at-Risk quantiles

VaR	90	95	97,5	98	99	99,5	99,75	99,9
alpha	**10**	**5**	**2,5**	**2**	**1**	**0,5**	**0,25**	**0,1**
BTC	10,575	5,563	3,494	2,943	1,609	0,736	0,276	0,138
ETH	9,893	5,682	3,275	2,406	1,070	0,602	0,267	0,134
USD	9,538	4,375	2,347	1,765	1,033	0,526	0,319	0,150
GBP	9,388	4,600	2,065	1,784	0,976	0,451	0,207	0,131
CHF	10,308	5,163	2,572	2,159	1,258	0,601	0,394	0,225
JPY	9,707	5,276	2,722	2,234	1,051	0,620	0,357	0,131
CNY	9,211	4,032	2,194	1,779	0,988	0,553	0,376	0,178
Oil	10,654	5,845	3,691	3,104	1,874	1,091	0,727	0,364
Gold	10,253	5,324	3,005	2,293	1,081	0,501	0,264	0,158
Silver	10,630	5,643	2,808	2,231	1,312	0,656	0,315	0,105

Table 22 Percentage deviation of exceedance rate from alpha

VaR	90	95	97,5	98	99	99,5	99,75	99,9	Mean	M Abs	M Pos	M Neg
alpha	**10**	**5**	**2,5**	**2**	**1**	**0,5**	**0,25**	**0,1**	**alpha 10% to 1%**			
BTC	5,7	11,3	39,8	47,1	60,9	47,1	10,3	37,9	33,0	33,0	33,0	
ETH	-1,1	13,6	31,0	20,3	7,0	20,3	7,0	33,7	14,2	14,6	18,0	-1,1
USD	-4,6	-12,5	-6,1	-11,8	3,3	5,1	27,7	50,2	-6,3	7,7	3,3	-8,7
GBP	-6,1	-8,0	-17,4	-10,8	-2,4	-9,9	-17,4	31,4	-8,9	8,9		-8,9
CHF	3,1	3,3	2,9	8,0	25,8	20,2	57,7	125,3	8,6	8,6	8,6	
JPY	-2,9	5,5	8,9	11,7	5,1	23,9	42,7	31,4	5,7	6,8	7,8	-2,9
CNY	-7,9	-19,4	-12,2	-11,0	-1,2	10,7	50,2	77,9	-10,3	10,3		-10,3
Oil	6,5	16,9	47,7	55,2	87,4	118,1	190,8	263,5	42,7	42,7	42,7	
Gold	2,5	6,5	20,2	14,7	8,1	0,2	5,4	58,1	10,4	10,4	10,4	
Silver	6,3	12,9	12,3	11,5	31,2	31,2	26,0	5,0	14,9	14,9	14,9	
Mean	0,2	3,0	12,7	13,5	22,5	26,7	40,0	71,5	10,38			
M Abs	4,7	11,0	19,8	20,2	23,2	28,7	43,5	71,5		15,79		
M Pos	4,8	10,0	23,3	24,1	28,6	30,8	46,4	71,5			18,15	
M Neg	-4,5	-13,3	-11,9	-11,2	-1,8	-9,9	-17,4					-8,54

ARMA(p,q) - MS(k)-GARCH(1,1) model -- Gaussian distribution

One day-ahead Value-at-Risk forecasts with up to five regimes (i.e. the k optimised according to the BIC criterion is between 1 and 5).

Results from the estimation algorithm were obtained for all time series sections.

Table 23 Exceedance rates for selected Value-at-Risk quantiles

VaR	90	95	97,5	98	99	99,5	99,75	99,9
alpha	**10**	**5**	**2,5**	**2**	**1**	**0,5**	**0,25**	**0,1**
BTC	10,345	5,241	3,080	2,805	1,747	1,057	0,552	0,276
ETH	8,890	4,612	2,406	2,005	1,203	0,869	0,468	0,267
USD	8,562	4,149	2,385	2,028	1,220	0,657	0,469	0,300
GBP	8,731	4,525	2,234	1,859	1,127	0,620	0,319	0,169
CHF	8,111	4,074	2,572	2,065	1,296	0,789	0,526	0,338
JPY	9,181	5,407	3,023	2,478	1,202	0,601	0,451	0,225
CNY	8,401	3,914	2,332	1,917	1,127	0,672	0,573	0,297
Oil	10,347	5,817	3,719	3,132	1,902	1,202	0,755	0,475
Gold	9,568	4,955	2,768	2,214	1,292	0,685	0,527	0,395
Silver	9,554	5,171	2,808	2,257	1,391	0,735	0,420	0,367

Table 24 Percentage deviation of exceedance rate from alpha

VaR	90	95	97,5	98	99	99,5	99,75	99,9	Mean	M Abs	M Pos	M Neg
alpha	**10**	**5**	**2,5**	**2**	**1**	**0,5**	**0,25**	**0,1**	**alpha 10% to 1%**			
BTC	3,4	4,8	23,2	40,2	74,7	111,5	120,7	175,9	29,3	29,3	29,3	
ETH	-11,1	-7,8	-3,7	0,3	20,3	73,8	87,2	167,4	-0,4	8,6	10,3	-7,5
USD	-14,4	-17,0	-4,6	1,4	22,0	31,4	87,8	200,4	-2,5	11,9	11,7	-12,0
GBP	-12,7	-9,5	-10,6	-7,1	12,7	23,9	27,7	69,0	-5,4	10,5	12,7	-10,0
CHF	-18,9	-18,5	2,9	3,3	29,6	57,7	110,3	238,0	-0,3	14,6	11,9	-18,7
JPY	-8,2	8,1	20,9	23,9	20,2	20,2	80,2	125,3	13,0	16,3	18,3	-8,2
CNY	-16,0	-21,7	-6,7	-4,1	12,7	34,4	129,3	196,5	-7,2	12,2	12,7	-12,1
Oil	3,5	16,3	48,8	56,6	90,2	140,5	202,0	375,4	43,1	43,1	43,1	
Gold	-4,3	-0,9	10,7	10,7	29,2	37,1	110,9	295,4	9,1	11,2	16,9	-2,6
Silver	-4,5	3,4	12,3	12,9	39,1	47,0	68,0	267,5	12,7	14,4	16,9	-4,5
Mean	-8,3	-4,3	9,3	13,8	35,1	57,7	102,4	211,1	9,12			
M Abs	9,7	10,8	14,5	16,0	35,1	57,7	102,4	211,1		17,21		
M Pos	3,5	8,2	19,8	18,7	35,1	57,7	102,4	211,1			17,03	
M Neg	-11,3	-12,6	-6,4	-5,6								-8,96

Value-at-Risk forecasts based on growing time windows - MS(k)-GARCH(1,1) model

AR(1)- MS(k)-GARCH(1,1) model -- Skew Student-T distribution

One day-ahead Value-at-Risk forecasts with up to five regimes (i.e. the *k* optimised according to the BIC criterion is between 1 and 5).

Results from the estimation algorithm were obtained for all time series sections.

Table 25 Exceedance rates for selected Value-at-Risk quantiles

VaR	90	95	97,5	98	99	99,5	99,75	99,9
alpha	10	5	2,5	2	1	0,5	0,25	0,1
BTC	10,161	5,609	3,356	2,851	1,471	0,828	0,276	0,092
ETH	10,561	6,551	4,078	3,342	1,537	0,869	0,535	0,134
USD	9,839	4,919	2,610	2,009	1,051	0,601	0,376	0,131
GBP	9,857	5,238	2,572	2,122	1,202	0,620	0,300	0,188
CHF	9,970	4,694	2,328	1,971	1,033	0,544	0,338	0,207
JPY	9,407	4,900	2,385	1,934	0,920	0,563	0,207	0,094
CNY	9,132	4,230	2,313	1,858	1,048	0,613	0,395	0,178
Oil	10,487	5,425	3,104	2,489	1,454	0,867	0,531	0,280
Gold	9,647	4,718	2,557	2,030	0,791	0,422	0,211	0,158
Silver	10,052	4,987	2,205	1,864	1,024	0,446	0,184	0,079

Table 26 Percentage deviation of exceedance rate from alpha

VaR	90	95	97,5	98	99	99,5	99,75	99,9	Mean	M Abs	M Pos	M Neg
alpha	10	5	2,5	2	1	0,5	0,25	0,1	alpha 10% to 1%			
BTC	1,6	12,2	34,3	42,5	47,1	65,5	10,3	-8,0	27,5	27,5	27,5	
ETH	5,6	31,0	63,1	67,1	53,7	73,8	113,9	33,7	44,1	44,1	44,1	
USD	-1,6	-1,6	4,4	0,5	5,1	20,2	50,2	31,4	1,4	2,6	3,3	-1,6
GBP	-1,4	4,8	2,9	6,1	20,2	23,9	20,2	87,8	6,5	7,1	8,5	-1,4
CHF	-0,3	-6,1	-6,9	-1,4	3,3	8,9	35,2	106,5	-2,3	3,6	3,3	-3,7
JPY	-5,9	-2,0	-4,6	-3,3	-8,0	12,7	-17,4	-6,1	-4,8	4,8		-4,8
CNY	-8,7	-15,4	-7,5	-7,1	4,8	22,6	58,1	77,9	-6,8	8,7	4,8	-9,7
Oil	4,9	8,5	24,2	24,4	45,4	73,4	112,5	179,6	21,5	21,5	21,5	
Gold	-3,5	-5,6	2,3	1,5	-20,9	-15,7	-15,7	58,1	-5,3	6,8	1,9	-10,0
Silver	0,5	-0,3	-11,8	-6,8	2,4	-10,8	-26,5	-21,3	-3,2	4,4	1,4	-6,3

Mean	-0,9	2,5	10,0	12,3	15,3	27,4	34,1	54,0	7,9			
M Abs	3,4	8,7	16,2	16,1	21,1	32,7	46,0	61,1		13,1		
M Pos	3,2	14,1	21,8	23,7	22,7	37,6	57,2	82,2			17,1	
M Neg	-3,6	-5,2	-7,7	-4,7	-14,5	-13,2	-19,9	-11,8				-7,1

Value-at-Risk forecasts based on growing time windows - MS(k)-GARCH(1,1) model

Constant - MS(k)-GARCH(1,1) model -- Skew Student-T distribution

One day-ahead Value-at-Risk forecasts with up to five regimes (i.e. the k optimised according to the BIC criterion is between 1 and 5).

Results from the estimation algorithm were obtained for all time series sections.

Table 27 Exceedance rates for selected Value-at-Risk quantiles

VaR	90	95	97,5	98	99	99,5	99,75	99,9
alpha	10	5	2,5	2	1	0,5	0,25	0,1
BTC	10,207	5,563	3,264	2,851	1,471	0,782	0,276	0,138
ETH	10,160	6,551	4,011	3,409	1,471	0,936	0,468	0,134
USD	9,839	4,938	2,478	1,990	1,070	0,582	0,376	0,131
GBP	9,745	5,220	2,666	2,084	1,239	0,620	0,300	0,169
CHF	10,045	4,769	2,422	2,047	1,033	0,601	0,357	0,207
JPY	9,407	4,863	2,328	1,915	0,920	0,563	0,207	0,094
CNY	9,231	4,151	2,234	1,819	1,008	0,593	0,395	0,158
Oil	10,319	5,537	2,908	2,545	1,398	0,923	0,531	0,252
Gold	9,647	4,665	2,478	2,082	0,817	0,422	0,211	0,158
Silver	9,974	4,882	2,283	1,942	1,024	0,420	0,184	0,079

Table 28 Percentage deviation of exceedance rate from alpha

VaR	90	95	97,5	98	99	99,5	99,75	99,9	Mean	M Abs	M Pos	M Neg
alpha	10	5	2,5	2	1	0,5	0,25	0,1	alpha 10% to 1%			
BTC	2,1	11,3	30,6	42,5	47,1	56,3	10,3	37,9	26,7	26,7	26,7	
ETH	1,6	31,0	60,4	70,5	47,1	87,2	87,2	33,7	42,1	42,1	42,1	
USD	-1,6	-1,2	-0,9	-0,5	7,0	16,4	50,2	31,4	0,6	2,2	7,0	-1,1
GBP	-2,6	4,4	6,6	4,2	23,9	23,9	20,2	69,0	7,3	8,3	9,8	-2,6
CHF	0,5	-4,6	-3,1	2,3	3,3	20,2	42,7	106,5	-0,3	2,8	2,0	-3,9
JPY	-5,9	-2,7	-6,9	-4,2	-8,0	12,7	-17,4	-6,1	-5,6	5,6		-5,6
CNY	-7,7	-17,0	-10,7	-9,1	0,8	18,6	58,1	58,1	-8,7	9,0	0,8	-11,1
Oil	3,2	10,7	16,3	27,2	39,8	84,6	112,5	151,7	19,5	19,5	19,5	
Gold	-3,5	-6,7	-0,9	4,1	-18,3	-15,7	-15,7	58,1	-5,1	6,7	4,1	-7,4
Silver	-0,3	-2,4	-8,7	-2,9	2,4	-16,0	-26,5	-21,3	-2,4	3,3	2,4	-3,5
Mean	-1,4	2,3	8,3	13,4	14,5	28,8	32,2	51,9	7,4			
M Abs	2,9	9,2	14,5	16,8	19,8	35,1	44,1	57,4		12,6		
M Pos	1,8	14,4	28,5	25,1	21,4	40,0	54,5	68,3			18,2	
M Neg	-3,6	-5,8	-5,2	-4,2	-13,1	-15,8	-19,9	-13,7				-6,4

Constant - MS(k)-GARCH(1,1) model -- Student-T distribution

One day-ahead Value-at-Risk forecasts with up to five regimes (i.e. the k optimised according to the BIC criterion is between 1 and 5).

Results from the estimation algorithm were obtained for all time series sections.

Table 29 Exceedance rates for selected Value-at-Risk quantiles

VaR	90	95	97,5	98	99	99,5	99,75	99,9
alpha	**10**	**5**	**2,5**	**2**	**1**	**0,5**	**0,25**	**0,1**
BTC	10,207	5,379	3,448	2,943	1,609	0,736	0,276	0,138
ETH	9,759	5,682	3,275	2,406	1,070	0,602	0,267	0,134
USD	9,557	4,375	2,347	1,765	1,033	0,526	0,319	0,150
GBP	9,388	4,600	2,065	1,784	0,976	0,451	0,207	0,131
CHF	10,364	5,051	2,572	2,122	1,220	0,638	0,338	0,225
JPY	9,670	5,257	2,722	2,234	1,051	0,620	0,357	0,131
CNY	9,211	4,052	2,194	1,779	0,988	0,553	0,376	0,178
Oil	10,626	5,817	3,523	2,880	1,790	1,119	0,755	0,336
Gold	10,200	5,271	3,005	2,319	1,107	0,501	0,264	0,158
Silver	10,446	5,591	2,730	2,283	1,312	0,604	0,315	0,105

Table 30 Percentage deviation of exceedance rate from alpha

VaR	90	95	97,5	98	99	99,5	99,75	99,9	Mean	M Abs	M Pos	M Neg
alpha	**10**	**5**	**2,5**	**2**	**1**	**0,5**	**0,25**	**0,1**	**alpha 10% to 1%**			
BTC	2,1	7,6	37,9	47,1	60,9	47,1	10,3	37,9	31,1	31,1	31,1	
ETH	-2,4	13,6	31,0	20,3	7,0	20,3	7,0	33,7	13,9	14,9	18,0	-2,4
USD	-4,4	-12,5	-6,1	-11,8	3,3	5,1	27,7	50,2	-6,3	7,6	3,3	-8,7
GBP	-6,1	-8,0	-17,4	-10,8	-2,4	-9,9	-17,4	31,4	-8,9	8,9		-8,9
CHF	3,6	1,0	2,9	6,1	22,0	27,7	35,2	125,3	7,1	7,1	7,1	
JPY	-3,3	5,1	8,9	11,7	5,1	23,9	42,7	31,4	5,5	6,8	7,7	-3,3
CNY	-7,9	-19,0	-12,2	-11,0	-1,2	10,7	50,2	77,9	-10,3	10,3		-10,3
Oil	6,3	16,3	40,9	44,0	79,0	123,7	202,0	235,6	37,3	37,3	37,3	
Gold	2,0	5,4	20,2	16,0	10,7	0,2	5,4	58,1	10,9	10,9	10,9	
Silver	4,5	11,8	9,2	14,2	31,2	20,7	26,0	5,0	14,2	14,2	14,2	

Mean	-0,6	2,1	11,5	12,6	21,6	27,0	38,9	68,7	9,45			
M Abs	4,3	10,0	18,7	19,3	22,3	28,9	42,4	68,7		14,91		
M Pos	3,7	8,7	21,6	22,8	27,4	31,1	45,2	68,7			16,83	
M Neg	-4,8	-13,2	-11,9	-11,2	-1,8	-9,9	-17,4					-8,57

Value-at-Risk forecasts based on growing time windows - MS(k)-GARCH(1,1) model

No mean equation - MS(k)-GARCH(1,1) model -- Skew Student-T distribution

One day-ahead Value-at-Risk forecasts with up to five regimes (i.e. the k optimised according to the BIC criterion is between 1 and 5).

Results from the estimation algorithm were obtained for all time series sections.

Table 31 Exceedance rates for selected Value-at-Risk quantiles

VaR	90	95	97,5	98	99	99,5	99,75	99,9
alpha	10	5	2,5	2	1	0,5	0,25	0,1
BTC	8,874	4,828	2,943	2,391	1,195	0,552	0,276	0,092
ETH	8,824	5,481	3,275	2,741	1,003	0,802	0,401	0,134
USD	10,008	5,107	2,610	2,028	1,089	0,582	0,357	0,131
GBP	9,895	5,351	2,666	2,084	1,239	0,601	0,282	0,169
CHF	10,514	5,051	2,554	2,103	1,051	0,526	0,357	0,188
JPY	9,407	5,051	2,403	1,971	0,976	0,563	0,225	0,094
CNY	9,330	4,072	2,273	1,838	1,008	0,593	0,395	0,158
Oil	9,927	5,285	2,768	2,405	1,370	0,867	0,503	0,224
Gold	8,935	4,270	2,214	1,687	0,712	0,343	0,211	0,158
Silver	9,291	4,514	2,021	1,706	0,840	0,341	0,157	0,079

Table 32 Percentage deviation of exceedance rate from alpha

VaR	90	95	97,5	98	99	99,5	99,75	99,9	Mean	M Abs	M Pos	M Neg
alpha	10	5	2,5	2	1	0,5	0,25	0,1	alpha 10% to 1%			
BTC	-11,3	-3,4	17,7	19,5	19,5	10,3	10,3	-8,0	8,4	14,3	18,9	-7,4
ETH	-11,8	9,6	31,0	37,0	0,3	60,4	60,4	33,7	13,2	17,9	19,5	-11,8
USD	0,1	2,1	4,4	1,4	8,9	16,4	42,7	31,4	3,4	3,4	3,4	
GBP	-1,1	7,0	6,6	4,2	23,9	20,2	12,7	69,0	8,1	8,6	10,4	-1,1
CHF	5,1	1,0	2,1	5,1	5,1	5,1	42,7	87,8	3,7	3,7	3,7	
JPY	-5,9	1,0	-3,9	-1,4	-2,4	12,7	-9,9	-6,1	-2,5	2,9	1,0	-3,4
CNY	-6,7	-18,6	-9,1	-8,1	0,8	18,6	58,1	58,1	-8,3	8,6	0,8	-10,6
Oil	-0,7	5,7	10,7	20,2	37,0	73,4	101,3	123,7	14,6	14,9	18,4	-0,7
Gold	-10,6	-14,6	-11,4	-15,7	-28,8	-31,5	-15,7	58,1	-16,2	16,2		-16,2
Silver	-7,1	-9,7	-19,2	-14,7	-16,0	-31,8	-37,0	-21,3	-13,3	13,3		-13,3

Mean	-5,0	-2,0	2,9	4,8	4,8	15,4	26,6	42,6	1,11			
M Abs	6,0	7,3	11,6	12,7	14,3	28,0	39,1	49,7		10,39		
M Pos	2,6	4,4	12,1	14,6	13,7	27,1	46,9	66,0			9,48	
M Neg	-6,9	-11,6	-10,9	-10,0	-15,7	-31,6	-20,8	-11,8				-11,0

No mean equation - MS(k)-GARCH(1,1) model -- Student-T distribution

One day-ahead Value-at-Risk forecasts with up to five regimes (i.e. the k optimised according to the BIC criterion is between 1 and 5).

Results from the estimation algorithm were obtained for all time series sections.

Table 33 Exceedance rates for selected Value-at-Risk quantiles

VaR	90	95	97,5	98	99	99,5	99,75	99,9
alpha	**10**	**5**	**2,5**	**2**	**1**	**0,5**	**0,25**	**0,1**
BTC	9,241	5,149	3,126	2,621	1,471	0,736	0,322	0,138
ETH	8,690	5,147	2,607	2,273	1,070	0,535	0,334	0,134
USD	9,876	4,431	2,328	1,802	1,014	0,526	0,338	0,150
GBP	9,444	4,788	2,140	1,765	0,976	0,469	0,207	0,131
CHF	10,965	5,389	2,760	2,178	1,202	0,657	0,376	0,225
JPY	9,688	5,370	2,779	2,216	1,070	0,620	0,357	0,131
CNY	9,192	4,111	2,214	1,739	0,988	0,553	0,395	0,158
Oil	10,291	5,705	3,328	2,796	1,734	1,035	0,727	0,336
Gold	9,278	4,797	2,688	2,188	1,054	0,501	0,237	0,185
Silver	9,921	5,354	2,651	2,231	1,286	0,604	0,315	0,105

Table 34 Percentage deviation of exceedance rate from alpha

VaR	90	95	97,5	98	99	99,5	99,75	99,9	Mean	M Abs	M Pos	M Neg
alpha	**10**	**5**	**2,5**	**2**	**1**	**0,5**	**0,25**	**0,1**	**alpha 10% to 1%**			
BTC	-7,6	3,0	25,1	31,0	47,1	47,1	28,7	37,9	19,7	22,8	26,6	-7,6
ETH	-13,1	2,9	4,3	13,6	7,0	7,0	33,7	33,7	2,9	8,2	7,0	-13,1
USD	-1,2	-11,4	-6,9	-9,9	1,4	5,1	35,2	50,2	-5,6	6,2	1,4	-7,3
GBP	-5,6	-4,2	-14,4	-11,8	-2,4	-6,1	-17,4	31,4	-7,7	7,7		-7,7
CHF	9,7	7,8	10,4	8,9	20,2	31,4	50,2	125,3	11,4	11,4	11,4	
JPY	-3,1	7,4	11,2	10,8	7,0	23,9	42,7	31,4	6,6	7,9	9,1	-3,1
CNY	-8,1	-17,8	-11,4	-13,0	-1,2	10,7	58,1	58,1	-10,3	10,3		-10,3
Oil	2,9	14,1	33,1	39,8	73,4	106,9	190,8	235,6	32,7	32,7	32,7	
Gold	-7,2	-4,1	7,5	9,4	5,4	0,2	-5,1	84,5	2,2	6,7	7,5	-5,6
Silver	-0,8	7,1	6,0	11,5	28,6	20,7	26,0	5,0	10,5	10,8	13,3	-0,8
Mean	-3,4	0,5	6,5	9,0	18,7	24,7	44,3	69,3	6,25			
M Abs	5,9	8,0	13,0	16,0	19,4	25,9	48,8	69,3		12,45		
M Pos	6,3	7,0	13,9	17,9	23,8	28,1	58,2	69,3			13,78	
M Neg	-5,8	-9,4	-10,9	-11,6	-1,8	-6,1	-11,2					-7,88

No mean equation - MS(k)-GARCH(1,1) model -- Gaussian distribution

One day-ahead Value-at-Risk forecasts with up to five regimes (i.e. the k optimised according to the BIC criterion is between 1 and 5).

Results from the estimation algorithm were obtained for all time series sections.

Table 35 Exceedance rates for selected Value-at-Risk quantiles

VaR	90	95	97,5	98	99	99,5	99,75	99,9
alpha	**10**	**5**	**2,5**	**2**	**1**	**0,5**	**0,25**	**0,1**
BTC	9,471	4,782	3,034	2,575	1,655	1,103	0,506	0,276
ETH	7,487	4,011	2,072	1,604	1,003	0,668	0,535	0,334
USD	8,900	4,243	2,385	2,009	1,258	0,676	0,526	0,300
GBP	8,750	4,656	2,272	1,878	1,127	0,620	0,300	0,169
CHF	8,280	4,225	2,610	2,178	1,352	0,807	0,526	0,338
JPY	9,313	5,464	3,098	2,497	1,220	0,620	0,451	0,225
CNY	8,440	3,874	2,313	1,957	1,146	0,672	0,573	0,297
Oil	9,983	5,677	3,356	2,880	1,790	1,091	0,755	0,391
Gold	8,672	4,481	2,530	2,161	1,212	0,580	0,422	0,316
Silver	9,003	4,856	2,756	2,310	1,444	0,761	0,446	0,315

Table 36 Percentage deviation of exceedance rate from alpha

VaR	90	95	97,5	98	99	99,5	99,75	99,9	Mean	M Abs	M Pos	M Neg
alpha	**10**	**5**	**2,5**	**2**	**1**	**0,5**	**0,25**	**0,1**	**alpha 10% to 1%**			
BTC	-5,3	-4,4	21,4	28,7	65,5	120,7	102,3	175,9	21,2	25,1	38,5	-4,8
ETH	-25,1	-19,8	-17,1	-19,8	0,3	33,7	113,9	234,2	-16,3	16,4	0,3	-20,5
USD	-11,0	-15,1	-4,6	0,5	25,8	35,2	110,3	200,4	-0,9	11,4	13,1	-10,3
GBP	-12,5	-6,9	-9,1	-6,1	12,7	23,9	20,2	69,0	-4,4	9,5	12,7	-8,7
CHF	-17,2	-15,5	4,4	8,9	35,2	61,5	110,3	238,0	3,2	16,2	16,2	-16,4
JPY	-6,9	9,3	23,9	24,9	22,0	23,9	80,2	125,3	14,6	17,4	20,0	-6,9
CNY	-15,6	-22,5	-7,5	-2,2	14,6	34,4	129,3	196,5	-6,6	12,5	14,6	-11,9
Oil	-0,2	13,5	34,2	44,0	79,0	118,1	202,0	291,5	34,1	34,2	42,7	-0,2
Gold	-13,3	-10,4	1,2	8,1	21,2	16,0	68,7	216,3	1,4	10,8	10,2	-11,8
Silver	-10,0	-2,9	10,2	15,5	44,4	52,2	78,5	215,0	11,4	16,6	23,4	-6,4

	90	95	97,5	98	99	99,5	99,75	99,9	Mean	M Abs	M Pos	M Neg
Mean	-11,7	-7,5	5,7	10,2	32,1	52,0	101,6	196,2	5,77			
M Abs	11,7	12,0	13,4	15,9	32,1	52,0	101,6	196,2		17,01		
M Pos		11,4	15,9	18,6	32,1	52,0	101,6	196,2			19,50	
M Neg	-11,7	-12,2	-9,6	-9,4								-10,7

Summary of key findings on the accuracy of Value-at-Risk forecasts

The following tables summarise major results, which give an indication of the effects of alternative formulations of the mean equation or the use of the MS-GARCH instead of the ordinary GARCH model on the Value-at-Risk forecast accuracy.

The (mean absolute) percentage deviations (i.e. forecast error MAPE) of the actual percentage exceedance frequencies from their target exceedance frequencies (i.e. alpha in %) are reported.

Measured by Mean Absolute Deviation (MAD) of 10.4%, the MS(k) GARCH(1,1) model (i.e. without fitting a mean equation) performs best for 90-99% VaR forecasts. On average, the target alpha is only exceeded or undercut by 10.4%. Measured against the 90% VaR (alpha = 10%), this would mean that the percentage exceedance frequency is on average 1.04 percentage points above or below the desired 10%.

Table 37 Percentage deviation of exceedance rate from alpha: All Series - Skew Student-T

	All Series - Skew Student-T : Mean Absolute Percentage Deviation									
VaR	**90,0**	**95,0**	**97,5**	**98,0**	**99,0**	**99,5**	**99,8**	**99,9**	**Mean**	**MAD**
alpha	**10,0**	**5,0**	**2,5**	**2,0**	**1,0**	**0,5**	**0,3**	**0,1**	**10-1**	**10-1**
ARMA(p,q)-GARCH(g,a)	4,6	9,3	15,9	16,5	20,7	27,0	38,3	55,7	0,0	13,4
ARMA(p,q)-MS(k)-GARCH(1,1)	3,5	9,3	16,1	16,4	21,9	33,9	42,6	57,4	0,0	13,5
AR(1)-MS(k)-GARCH(1,1)	3,4	8,7	16,2	16,1	21,1	32,7	46,0	61,1	0,0	13,1
const-MS(k)-GARCH(1,1)	2,9	9,2	14,5	16,8	19,8	35,1	44,1	57,4	0,0	12,6
0-MS(k)-GARCH(1,1)	6,0	7,3	11,6	12,7	14,3	28,0	39,1	49,7	0,0	10,4

Table 38 Percentage deviation of exceedance rate from alpha: All Series - Student-T

	All Series - Student-T : Mean Absolute Percentage Deviation									
VaR	**90,0**	**95,0**	**97,5**	**98,0**	**99,0**	**99,5**	**99,8**	**99,9**	**Mean**	**MAD**
alpha	**10,0**	**5,0**	**2,5**	**2,0**	**1,0**	**0,5**	**0,3**	**0,1**	**10-1**	**10-1**
ARMA(p,q)-GARCH(g,a)	10,6	17,7	25,4	25,9	33,0	36,0	39,8	67,7	0,0	22,5
ARMA(p,q)-MS(k)-GARCH(1,1)	4,7	11,0	19,8	20,2	23,2	28,7	43,5	71,5	0,0	15,8
AR(1)-MS(k)-GARCH(1,1)	4,5	10,7	19,4	18,8	23,9	27,6	40,1	74,0	0,0	15,5
const-MS(k)-GARCH(1,1)	4,3	10,0	18,7	19,3	22,3	28,9	42,4	68,7	0,0	14,9
0-MS(k)-GARCH(1,1)	5,9	8,0	13,0	16,0	19,4	25,9	48,8	69,3	0,0	12,5

Table 39 Percentage deviation of exceedance rate from alpha: All Series - Gaussian

	All Series - Gaussian : Mean Absolute Percentage Deviation									
VaR	**90,0**	**95,0**	**97,5**	**98,0**	**99,0**	**99,5**	**99,8**	**99,9**	**Mean**	**MAD**
alpha	**10,0**	**5,0**	**2,5**	**2,0**	**1,0**	**0,5**	**0,3**	**0,1**	**10-1**	**10-1**
ARMA(p,q)-GARCH(g,a)	13,9	9,0	25,6	32,6	79,1	141,2	239,7	465,6	0,0	32,0
ARMA(p,q)-MS(k)-GARCH(1,1)	9,7	10,8	14,5	16,0	35,1	57,7	102,4	211,1	0,0	17,2
AR(1)-MS(k)-GARCH(1,1)	10,2	11,3	14,4	17,5	35,4	64,2	110,1	203,1	0,0	17,8
const-MS(k)-GARCH(1,1)	10,0	10,6	13,5	14,9	34,1	58,9	100,0	200,4	0,0	16,6
0-MS(k)-GARCH(1,1)	11,7	12,0	13,4	15,9	32,1	52,0	101,6	196,2	0,0	17,0

Table 40 Percentage deviation of exceedance rate from alpha: Bitcoin returns - Skew Student-T

	USD / BTC - Skew Student-T : Percentage deviation of exceedance rate from alpha									
VaR	**90,0**	**95,0**	**97,5**	**98,0**	**99,0**	**99,5**	**99,8**	**99,9**	**Mean**	**MAD**
alpha	**10,0**	**5,0**	**2,5**	**2,0**	**1,0**	**0,5**	**0,3**	**0,1**	**10-1**	**10-1**
ARMA(p,q)-GARCH(g,a)	-1,1	3,9	6,7	5,7	-31,0	-54,0	-81,6	-54,0	-3,2	9,7
ARMA(p,q)-MS(k)-GARCH(1,1)	4,4	10,3	34,3	40,2	51,7	47,1	10,3	37,9	28,2	28,2
AR(1)-MS(k)-GARCH(1,1)	1,6	12,2	34,3	42,5	47,1	65,5	10,3	-8,0	27,5	27,5
const-MS(k)-GARCH(1,1)	2,1	11,3	30,6	42,5	47,1	56,3	10,3	37,9	26,7	26,7
0-MS(k)-GARCH(1,1)	-11,3	-3,4	17,7	19,5	19,5	10,3	10,3	-8,0	8,4	14,3

Table 41 Percentage deviation of exceedance rate from alpha: Bitcoin returns - Student-T

	USD / BTC - Student-T : Percentage deviation of exceedance rate from alpha									
VaR	**90,0**	**95,0**	**97,5**	**98,0**	**99,0**	**99,5**	**99,8**	**99,9**	**Mean**	**MAD**
alpha	**10,0**	**5,0**	**2,5**	**2,0**	**1,0**	**0,5**	**0,3**	**0,1**	**10-1**	**10-1**
ARMA(p,q)-GARCH(g,a)	-0,7	6,7	21,4	24,1	-8,0	-35,6	-44,8	-8,0	8,7	12,2
ARMA(p,q)-MS(k)-GARCH(1,1)	5,7	11,3	39,8	47,1	60,9	47,1	10,3	37,9	33,0	33,0
AR(1)-MS(k)-GARCH(1,1)	1,6	10,3	37,9	51,7	60,9	56,3	28,7	37,9	32,5	32,5
const-MS(k)-GARCH(1,1)	2,1	7,6	37,9	47,1	60,9	47,1	10,3	37,9	31,1	31,1
0-MS(k)-GARCH(1,1)	-7,6	3,0	25,1	31,0	47,1	47,1	28,7	37,9	19,7	22,8

Table 42 Percentage deviation of exceedance rate from alpha: Bitcoin returns - Gaussian

	USD / BTC - Gaussian : Percentage deviation of exceedance rate from alpha									
VaR	**90,0**	**95,0**	**97,5**	**98,0**	**99,0**	**99,5**	**99,8**	**99,9**	**Mean**	**MAD**
alpha	**10,0**	**5,0**	**2,5**	**2,0**	**1,0**	**0,5**	**0,3**	**0,1**	**10-1**	**10-1**
ARMA(p,q)-GARCH(g,a)	-25,1	-3,4	52,6	74,7	143,7	258,6	378,2	865,5	48,5	59,9
ARMA(p,q)-MS(k)-GARCH(1,1)	3,4	4,8	23,2	40,2	74,7	111,5	120,7	175,9	29,3	29,3
AR(1)-MS(k)-GARCH(1,1)	4,8	8,5	26,9	40,2	79,3	148,3	175,9	175,9	32,0	32,0
const-MS(k)-GARCH(1,1)	4,4	4,8	25,1	40,2	74,7	139,1	139,1	175,9	29,8	29,8
0-MS(k)-GARCH(1,1)	-5,3	-4,4	21,4	28,7	65,5	120,7	102,3	175,9	21,2	25,1

Table 43 Percentage deviation of exceedance rate from alpha: USD / EUR exchange rate returns - Skew Student-T

	USD / EUR - Skew Student-T : Percentage deviation of exceedance rate from alpha									
VaR	**90,0**	**95,0**	**97,5**	**98,0**	**99,0**	**99,5**	**99,8**	**99,9**	**Mean**	**MAD**
alpha	**10,0**	**5,0**	**2,5**	**2,0**	**1,0**	**0,5**	**0,3**	**0,1**	**10-1**	**10-1**
ARMA(p,q)-GARCH(g,a)	-0,9	-1,6	6,6	0,5	10,8	12,7	35,2	31,4	3,1	4,1
ARMA(p,q)-MS(k)-GARCH(1,1)	-1,8	-0,9	-0,9	-0,5	7,0	16,4	50,2	31,4	0,6	2,2
AR(1)-MS(k)-GARCH(1,1)	-1,6	-1,6	4,4	0,5	5,1	20,2	50,2	31,4	1,4	2,6
const-MS(k)-GARCH(1,1)	-1,6	-1,2	-0,9	-0,5	7,0	16,4	50,2	31,4	0,6	2,2
0-MS(k)-GARCH(1,1)	0,1	2,1	4,4	1,4	8,9	16,4	42,7	31,4	3,4	3,4

Table 44 Percentage deviation of exceedance rate from alpha: USD / EUR exchange rate returns - Student-T

	USD / EUR - Student-T : Percentage deviation of exceedance rate from alpha									
VaR	**90,0**	**95,0**	**97,5**	**98,0**	**99,0**	**99,5**	**99,8**	**99,9**	**Mean**	**MAD**
alpha	**10,0**	**5,0**	**2,5**	**2,0**	**1,0**	**0,5**	**0,3**	**0,1**	**10-1**	**10-1**
ARMA(p,q)-GARCH(g,a)	-4,4	-14,4	-12,1	-13,6	-2,4	-2,4	-2,4	31,4	-9,4	9,4
ARMA(p,q)-MS(k)-GARCH(1,1)	-4,6	-12,5	-6,1	-11,8	3,3	5,1	27,7	50,2	-6,3	7,7
AR(1)-MS(k)-GARCH(1,1)	-4,1	-11,4	-5,4	-8,9	5,1	5,1	27,7	50,2	-4,9	7,0
const-MS(k)-GARCH(1,1)	-4,4	-12,5	-6,1	-11,8	3,3	5,1	27,7	50,2	-6,3	7,6
0-MS(k)-GARCH(1,1)	-1,2	-11,4	-6,9	-9,9	1,4	5,1	35,2	50,2	-5,6	6,2

Table 45 Percentage deviation of exceedance rate from alpha: USD / EUR exchange rate returns - Gaussian

	USD / EUR - Gaussian : Percentage deviation of exceedance rate from alpha									
VaR	**90,0**	**95,0**	**97,5**	**98,0**	**99,0**	**99,5**	**99,8**	**99,9**	**Mean**	**MAD**
alpha	**10,0**	**5,0**	**2,5**	**2,0**	**1,0**	**0,5**	**0,3**	**0,1**	**10-1**	**10-1**
ARMA(p,q)-GARCH(g,a)	-11,8	-14,4	0,6	5,1	42,7	65,2	117,8	238,0	4,5	14,9
ARMA(p,q)-MS(k)-GARCH(1,1)	-14,4	-17,0	-4,6	1,4	22,0	31,4	87,8	200,4	-2,5	11,9
AR(1)-MS(k)-GARCH(1,1)	-14,4	-17,4	-3,1	4,2	25,8	35,2	87,8	238,0	-1,0	13,0
const-MS(k)-GARCH(1,1)	-14,2	-17,0	-3,9	3,3	23,9	31,4	87,8	181,6	-1,6	12,5
0-MS(k)-GARCH(1,1)	-11,0	-15,1	-4,6	0,5	25,8	35,2	110,3	200,4	-0,9	11,4

Value-at-Risk forecasts based on rolling time windows (300 trading days each) - ordinary GARCH(g,a) model

ARMA(p,q)- GARCH(g,a) model -- Skew Student-T distribution

One day-ahead Value-at-risk forecasts with maximum lags of two for g and a in GARCH(g,a) volatility equations and five for p and q in ARMA(p,q) mean equations:

For the CHF/EUR-, JPY/EUR- exchange rates and Oil- and Silver US dollar prices, the ordinary GARCH function did not produce a result for some time series segments. Due to the repeatedly interrupted estimation algorithm, the VaR forecasts were also not produced for the entire period of the time series in consideration.

Table 46 Exceedance rates for selected Value-at-Risk quantiles

VaR	90	95	97,5	98	99	99,5	99,75	99,9
alpha	10	5	2,5	2	1	0,5	0,25	0,1
BTC	9,922	5,351	2,805	2,286	1,039	0,519	0,260	0,104
ETH	10,353	6,340	3,050	2,568	1,043	0,562	0,321	0,080
USD	10,303	4,768	2,344	1,911	0,906	0,473	0,217	0,138
GBP	9,161	4,827	2,463	1,832	0,946	0,374	0,197	0,138
CHF	10,362	5,162	2,581	2,226	1,379	0,847	0,512	0,256
JPY	10,518	5,338	2,836	2,206	1,083	0,453	0,276	0,079
CNY	9,877	4,554	2,308	1,913	1,040	0,499	0,208	0,104
Oil	10,909	5,455	3,010	2,504	1,401	0,745	0,328	0,179
Gold	10,617	5,393	2,878	2,431	1,090	0,475	0,279	0,084
Silver	11,067	5,899	2,725	2,303	1,264	0,618	0,365	0,169

Table 47 Percentage deviation of exceedance rate from alpha

VaR	90	95	97,5	98	99	99,5	99,75	99,9	Mean	M Abs	M Pos	M Neg
alpha	10	5	2,5	2	1	0,5	0,25	0,1	alpha 10% to 1%			
BTC	-0,8	7,0	12,2	14,3	3,9	3,9	3,9	3,9	7,3	7,6	9,4	-0,8
ETH	3,5	26,8	22,0	28,4	4,3	12,4	28,4	-19,7	17,0	17,0	17,0	
USD	3,0	-4,6	-6,2	-4,5	-9,4	-5,4	-13,3	37,9	-4,3	5,5	3,0	-6,2
GBP	-8,4	-3,5	-1,5	-8,4	-5,4	-25,1	-21,2	37,9	-5,4	5,4		-5,4
CHF	3,6	3,2	3,2	11,3	37,9	69,4	104,9	156,1	11,9	11,9	11,9	
JPY	5,2	6,8	13,5	10,3	8,3	-9,4	10,3	-21,2	8,8	8,8	8,8	
CNY	-1,2	-8,9	-7,7	-4,3	4,0	-0,2	-16,8	4,0	-3,6	5,2	4,0	-5,5
Oil	9,1	9,1	20,4	25,2	40,1	49,0	31,1	78,8	20,8	20,8	20,8	
Gold	6,2	7,9	15,1	21,5	9,0	-5,0	11,8	-16,2	11,9	11,9	11,9	
Silver	10,7	18,0	9,0	15,2	26,4	23,6	46,1	68,5	15,8	15,8	15,8	
Mean	3,1	6,2	8,0	10,9	11,9	11,3	18,5	33,0	8,01			
M Abs	5,2	9,6	11,1	14,3	14,9	20,3	28,8	44,4		11,01		
M Pos	5,9	11,2	13,6	18,0	16,7	31,7	33,8	55,3			13,11	
M Neg	-3,5	-5,7	-5,1	-5,7	-7,4	-9,0	-17,1	-19,0				-5,48

Value-at-Risk forecasts based on rolling time windows (300 trading days each) - MS(k)-GARCH(1,1) model

No mean equation - MS(k)-GARCH(1,1) model -- Skew Student-T distribution

One day-ahead Value-at-Risk forecasts with up to five regimes (i.e. the k optimised according to the BIC criterion is between 1 and 5).

Results from the estimation algorithm were obtained for all time series sections.

Table 48 Exceedance rates for selected Value-at-Risk quantiles

VaR	90	95	97,5	98	99	99,5	99,75	99,9
alpha	10	5	2,5	2	1	0,5	0,25	0,1
BTC	10,234	5,766	3,740	3,065	1,610	0,935	0,571	0,364
ETH	9,872	6,019	3,531	2,970	1,525	1,204	0,803	0,321
USD	10,165	4,768	2,384	1,970	1,143	0,630	0,315	0,177
GBP	9,377	4,886	2,463	2,049	1,162	0,552	0,315	0,197
CHF	10,638	5,536	2,975	2,482	1,517	0,827	0,512	0,433
JPY	9,968	5,339	2,758	2,187	1,103	0,709	0,453	0,236
CNY	10,127	4,824	2,433	2,017	1,185	0,707	0,374	0,187
Oil	10,553	5,292	2,916	2,375	1,443	0,902	0,541	0,241
Gold	10,673	5,393	2,962	2,263	1,257	0,643	0,475	0,279
Silver	11,039	6,264	3,174	2,472	1,376	0,899	0,618	0,393

Table 49 Percentage deviation of exceedance rate from alpha

VaR	90	95	97,5	98	99	99,5	99,75	99,9	Mean	M Abs	M Pos	M Neg
alpha	10	5	2,5	2	1	0,5	0,25	0,1	alpha 10% to 1%			
BTC	2,3	15,3	49,6	53,2	61,0	87,0	128,6	263,6	36,3	36,3	36,3	
ETH	-1,3	20,4	41,3	48,5	52,5	140,8	221,0	221,0	32,3	32,8	40,7	-1,3
USD	1,7	-4,6	-4,6	-1,5	14,3	26,1	26,1	77,3	1,0	5,3	8,0	-3,6
GBP	-6,2	-2,3	-1,5	2,4	16,2	10,3	26,1	97,0	1,7	5,7	9,3	-3,3
CHF	6,4	10,7	19,0	24,1	51,7	65,5	104,9	333,4	22,4	22,4	22,4	
JPY	-0,3	6,8	10,3	9,3	10,3	41,8	81,2	136,4	7,3	7,4	9,2	-0,3
CNY	1,3	-3,5	-2,7	0,9	18,5	41,4	49,7	87,1	2,9	5,4	6,9	-3,1
Oil	5,5	5,8	16,7	18,8	44,3	80,4	116,5	140,5	18,2	18,2	18,2	
Gold	6,7	7,9	18,5	13,2	25,7	28,5	90,0	179,4	14,4	14,4	14,4	
Silver	10,4	25,3	27,0	23,6	37,6	79,8	147,2	293,3	24,8	24,8	24,8	
Mean	2,6	8,2	17,3	19,2	33,2	60,2	99,1	182,9	16,13			
M Abs	4,2	10,3	19,1	19,5	33,2	60,2	99,1	182,9		17,27		
M Pos	4,9	13,2	26,0	21,6	33,2	60,2	99,1	182,9			19,78	
M Neg	-2,6	-3,5	-2,9	-1,5								-2,63

Value-at-Risk forecasts based on rolling time windows (1000 trading days each) - ordinary GARCH(g,a) model

ARMA(p,q)- GARCH(g,a) model -- Skew Student-T distribution

One day-ahead Value-at-risk forecasts with maximum lags of two for g and a in GARCH(g,a) volatility equations and five for p and q in ARMA(p,q) mean equations:

For the USD/EUR-, GBP/EUR-, CHF/EUR-, JPY/EUR, CNY/EUR- exchange rates, the ordinary GARCH function did not produce a result for some time series segments. Due to the repeatedly interrupted estimation algorithm, the VaR forecasts were also not produced for the entire period of the time series in consideration.

Table 50 Exceedance rates for selected Value-at-Risk quantiles

VaR	90	95	97,5	98	99	99,5	99,75	99,9
alpha	10	5	2,5	2	1	0,5	0,25	0,1
BTC	9,878	4,898	2,857	1,959	0,898	0,327	0,082	0,082
ETH	10,806	6,410	3,480	2,381	1,282	0,733	0,366	0,000
USD	9,803	4,525	2,125	1,828	0,891	0,388	0,251	0,069
GBP	9,506	4,936	2,422	1,828	0,914	0,366	0,229	0,114
CHF	10,719	5,346	2,647	2,206	1,116	0,675	0,337	0,130
JPY	10,032	5,073	2,537	2,057	0,983	0,548	0,183	0,114
CNY	9,589	4,186	2,239	1,777	0,973	0,414	0,268	0,073
Oil	10,206	5,255	2,932	2,285	1,142	0,724	0,381	0,152
Gold	10,386	5,141	2,744	2,292	1,007	0,556	0,208	0,174
Silver	9,860	5,175	2,378	1,853	1,049	0,455	0,245	0,070

Table 51 Percentage deviation of exceedance rate from alpha

VaR	90	95	97,5	98	99	99,5	99,75	99,9	Mean	M Abs	M Pos	M Neg
alpha	10	5	2,5	2	1	0,5	0,25	0,1	alpha 10% to 1%			
BTC	-1,2	-2,0	14,3	-2,0	-10,2	-34,7	-67,3	-18,4	-0,2	6,0	14,3	-3,9
ETH	8,1	28,2	39,2	19,0	28,2	46,5	46,5	-100,0	24,5	24,5	24,5	
USD	-2,0	-9,5	-15,0	-8,6	-10,9	-22,3	0,5	-31,4	-9,2	9,2		-9,2
GBP	-4,9	-1,3	-3,1	-8,6	-8,6	-26,9	-8,6	14,3	-5,3	5,3		-5,3
CHF	7,2	6,9	5,9	10,3	11,6	35,0	35,0	29,8	8,4	8,4	8,4	
JPY	0,3	1,5	1,5	2,8	-1,7	9,7	-26,9	14,3	0,9	1,6	1,5	-1,7
CNY	-4,1	-16,3	-10,4	-11,2	-2,7	-17,3	7,1	-27,0	-8,9	8,9		-8,9
Oil	2,1	5,1	17,3	14,2	14,2	44,7	52,3	52,3	10,6	10,6	10,6	
Gold	3,9	2,8	9,8	14,6	0,7	11,1	-16,6	73,7	6,4	6,4	6,4	
Silver	-1,4	3,5	-4,9	-7,3	4,9	-9,1	-2,1	-30,1	-1,0	4,4	4,2	-4,5
Mean	0,8	1,9	5,4	2,3	2,6	3,7	2,0	-2,3	2,60			
M Abs	3,5	7,7	12,1	9,9	9,4	25,7	26,3	39,1		8,52		
M Pos	4,3	8,0	14,6	12,2	11,9	29,4	28,3	36,9			10,22	
M Neg	-2,7	-7,3	-8,4	-7,5	-6,8	-22,0	-24,3	-41,4				-6,54

Value-at-Risk forecasts based on rolling time windows (1000 trading days each) - MS(k)-GARCH(1,1) model

No mean equation - MS(k)-GARCH(1,1) model -- Skew Student-T distribution

One day-ahead Value-at-Risk forecasts with up to five regimes (i.e. the k optimised according to the BIC criterion is between 1 and 5).

Results from the estimation algorithm were obtained for all time series sections.

Table 52 Exceedance rates for selected Value-at-Risk quantiles

VaR	90	95	97,5	98	99	99,5	99,75	99,9
alpha	10	5	2,5	2	1	0,5	0,25	0,1
BTC	9,633	5,306	3,265	2,857	1,388	0,735	0,245	0,245
ETH	9,890	6,410	3,846	2,930	1,282	0,916	0,733	0,183
USD	9,621	4,479	2,331	1,851	0,868	0,526	0,297	0,091
GBP	9,621	5,096	2,377	1,828	1,028	0,526	0,274	0,183
CHF	10,923	5,439	2,765	2,080	1,257	0,686	0,411	0,229
JPY	9,575	4,867	2,559	1,988	1,005	0,594	0,343	0,206
CNY	9,783	4,332	2,239	1,801	0,973	0,560	0,292	0,097
Oil	9,977	5,560	3,085	2,628	1,371	0,838	0,533	0,190
Gold	10,038	4,828	2,709	2,292	1,007	0,556	0,382	0,278
Silver	9,615	5,175	2,587	1,958	1,119	0,524	0,245	0,070

Table 53 Percentage deviation of exceedance rate from alpha

VaR	90	95	97,5	98	99	99,5	99,75	99,9	Mean	M Abs	M Pos	M Neg
alpha	10	5	2,5	2	1	0,5	0,25	0,1	alpha 10% to 1%			
BTC	-3,7	6,1	30,6	42,9	38,8	46,9	-2,0	144,9	22,9	24,4	29,6	-3,7
ETH	-1,1	28,2	53,8	46,5	28,2	83,2	193,0	83,2	31,1	31,6	39,2	-1,1
USD	-3,8	-10,4	-6,8	-7,4	-13,2	5,1	18,8	-8,6	-8,3	8,3		-8,3
GBP	-3,8	1,9	-4,9	-8,6	2,8	5,1	9,7	82,8	-2,5	4,4	2,4	-5,8
CHF	9,2	8,8	10,6	4,0	25,7	37,1	64,5	128,5	11,7	11,7	11,7	
JPY	-4,3	-2,7	2,4	-0,6	0,5	18,8	37,1	105,7	-0,9	2,1	1,5	-2,5
CNY	-2,2	-13,4	-10,4	-10,0	-2,7	11,9	16,8	-2,7	-7,7	7,7		-7,7
Oil	-0,2	11,2	23,4	31,4	37,1	67,6	113,3	90,4	20,6	20,7	25,8	-0,2
Gold	0,4	-3,4	8,4	14,6	0,7	11,1	52,8	177,9	4,1	5,5	6,0	-3,4
Silver	-3,8	3,5	3,5	-2,1	11,9	4,9	-2,1	-30,1	2,6	5,0	6,3	-3,0
Mean	-1,3	3,0	11,1	11,1	13,0	29,2	50,2	77,2	7,36			
M Abs	3,2	9,0	15,5	16,8	16,2	29,2	51,0	85,5		12,13		
M Pos	4,8	10,0	19,0	27,9	18,2	29,2	63,3	116,2			15,96	
M Neg	-2,9	-7,5	-7,4	-5,7	-7,9		-2,1	-13,8				-6,27

Descriptive Statistics

The following tables show means, standard deviations, minima, first quartiles, medians, third quartiles and maxima of the time series used. The descriptive statistics are given for different time periods, with the shortest time series in a compilation defining the period.

Table 54 Descriptive statistics for the period from 13 January 2000 to 3 January 2020

	USD/EUR	JPY/EUR	CNY/EUR	GBP/EUR	CHF/EUR
Mean	0,0016	0,0020	-0,0018	0,0060	-0,0078
Std. Dev.	0,6136	0,7364	0,6000	0,5093	0,4260
Minimum	-4,7354	-6,1232	-4,4871	-2,9686	-15,5539
1st Quartile	-0,3322	-0,3750	-0,3267	-0,2739	-0,1428
Median	0,0082	0,0237	0,0014	0,0000	0,0000
3rd Quartile	0,3447	0,4042	0,3252	0,2783	0,1381
Maximum	4,2041	5,3963	4,2050	5,2826	7,9967

Table 55 Descriptive statistics for the period from 3 January 2006 to 3 January 2020

	USD/Gold	USD/Silver	USD/Oil	USD/EUR
Mean	0,0302	0,0192	0,0046	-0,0018
Std. Dev.	1,2028	2,0708	2,0818	0,5912
Minimum	-8,9535	-20,5324	-10,9455	-4,7354
1st Quartile	-0,5592	-0,8958	-1,0337	-0,3077
Median	0,0452	0,0772	0,0605	0,0075
3rd Quartile	0,6941	1,0998	1,0512	0,3169
Maximum	10,4371	13,9472	13,6392	4,0377

Table 56 Descriptive statistics for the period from 1 October 2013 to 3 January 2020

	USD/BTC	USD/Gold	USD/Silver	USD/Oil	USD/EUR
Mean	0,2608	0,0128	-0,0104	-0,0292	-0,0127
Std. Dev.	5,0656	0,9781	1,5004	2,0425	0,5040
Minimum	-28,4480	-3,6556	-7,1942	-7,9309	-3,6820
1st Quartile	-1,5588	-0,5349	-0,7536	-1,0488	-0,2798
Median	0,1638	-0,0024	0,0000	0,0638	-0,0088
3rd Quartile	2,3628	0,5477	0,7792	1,0047	0,2528
Maximum	30,6376	7,5871	6,3932	13,6392	2,4664

Table 57 Descriptive statistics for the period from 10 August 2015 to 3 January 2020

	USD/BTC	USD/ETH	USD/Gold	USD/Silver	USD/Oil	USD/EUR
Mean	0,3048	0,4862	0,0334	0,0155	0,0281	0,0016
Std. Dev.	4,7750	8,1631	0,9870	1,4859	2,1525	0,4933
Minimum	-26,8290	-55,2192	-3,6556	-7,1942	-7,9309	-2,8771
1st Quartile	-1,4573	-2,9211	-0,5503	-0,7657	-1,0796	-0,2814
Median	0,2502	0,0000	0,0035	0,0566	0,1861	-0,0088
3rd Quartile	2,3767	3,6261	0,5725	0,8304	1,1355	0,2613
Maximum	23,6982	43,2697	7,5871	5,5707	13,6392	2,4664

Autocorrelation Coefficients

The following tables provide information on the autocorrelation structures of the exchange rate returns for the time series period 13 January 2000 to 3 January 2020:

Table 58 Autocorrelation Function (ACF)

ACF	USD/EUR	JPY/EUR	CNY/EUR	GBP/EUR	CHF/EUR
1	-0,0063	-0,0097	-0,0138	0,0271	0,0553
2	-0,0161	-0,0094	-0,0090	-0,0327	-0,0261
3	0,0087	0,0090	0,0103	-0,0072	-0,0215
4	0,0052	0,0042	0,0052	-0,0213	-0,0049
5	0,0064	-0,0085	0,0073	0,0035	0,0141
6	-0,0046	-0,0243	0,0007	0,0253	0,0095
7	0,0207	0,0037	0,0128	-0,0055	-0,0401

Table 59 Partial Autocorrelation Function (PACF)

PACF	USD/EUR	JPY/EUR	CNY/EUR	GBP/EUR	CHF/EUR
1	-0,0063	-0,0097	-0,0138	0,0271	0,0553
2	-0,0162	-0,0095	-0,0092	-0,0335	-0,0293
3	0,0085	0,0089	0,0100	-0,0054	-0,0185
4	0,0051	0,0043	0,0054	-0,0221	-0,0034
5	0,0067	-0,0083	0,0077	0,0043	0,0135
6	-0,0044	-0,0244	0,0009	0,0237	0,0074
7	0,0208	0,0030	0,0128	-0,0069	-0,0406

Table 60 Ljung-Box test statistics and p-values

	Lag	USD/EUR	JPY/EUR	CNY/EUR	GBP/EUR	CHF/EUR
LB	1	0,2	0,5	1,0	3,8	15,6
p		0,653	0,490	0,323	0,052	0,000
LB	2	1,5	0,9	1,4	9,2	19,1
p		0,464	0,629	0,499	0,010	0,000
LB	3	1,9	1,3	1,9	9,5	21,5
p		0,588	0,718	0,587	0,023	0,000
LB	4	2,1	1,4	2,1	11,8	21,6
p		0,724	0,838	0,723	0,019	0,000
LB	5	2,3	1,8	2,3	11,9	22,6
p		0,810	0,875	0,800	0,036	0,000
LB	6	2,4	4,8	2,3	15,2	23,1
p		0,882	0,567	0,885	0,019	0,001
LB	7	4,6	4,9	3,2	15,3	31,3
p		0,711	0,673	0,868	0,032	0,000

The following tables provide information on the autocorrelation structures of the exchange rate returns for the time series period 3 January 2006 to 3 January 2020:

Table 61 Autocorrelation Function (ACF)

ACF	USD/Gold	USD/Silver	USD/Oil	USD/EUR
1	-0,0438	-0,0404	-0,0669	-0,0070
2	-0,0323	-0,0157	0,0107	-0,0035
3	0,0151	0,0198	0,0069	0,0113
4	0,0061	-0,0121	0,0255	0,0024
5	0,0292	0,0144	-0,0285	0,0116
6	-0,0378	-0,0031	0,0136	-0,0100
7	-0,0036	-0,0055	-0,0048	0,0075

Table 62 Partial Autocorrelation Function (PACF)

PACF	USD/Gold	USD/Silver	USD/Oil	USD/EUR
1	-0,0438	-0,0404	-0,0669	-0,0070
2	-0,0343	-0,0174	0,0062	-0,0036
3	0,0122	0,0185	0,0081	0,0112
4	0,0063	-0,0108	0,0266	0,0025
5	0,0307	0,0142	-0,0253	0,0117
6	-0,0350	-0,0027	0,0096	-0,0100
7	-0,0051	-0,0049	-0,0033	0,0074

Table 63 Ljung-Box test statistics and p-values

	Lag	USD/Gold	USD/Silver	USD/Oil	USD/EUR
LB	1	6,8	5,8	16,0	0,2
p		0,009	0,016	0,000	0,674
LB	2	10,6	6,7	16,4	0,2
p		0,005	0,035	0,000	0,895
LB	3	11,4	8,1	16,5	0,7
p		0,010	0,044	0,001	0,879
LB	4	11,5	8,6	18,9	0,7
p		0,021	0,071	0,001	0,952
LB	5	14,5	9,4	21,7	1,2
p		0,012	0,095	0,001	0,947
LB	6	19,7	9,4	22,4	1,5
p		0,003	0,152	0,001	0,957
LB	7	19,7	9,5	22,5	1,7
p		0,006	0,218	0,002	0,973

The following tables provide information on the autocorrelation structures of the exchange rate returns for the time series period 10 August 2015 to 3 January 2020:

Table 64 Autocorrelation Function (ACF)

ACF	USD/BTC	USD/ETH	USD/Gold	USD/Silver	USD/Oil	USD/EUR
1	-0,0148	0,0575	-0,1490	-0,0692	-0,0373	-0,0307
2	0,0430	0,0026	-0,0109	-0,0094	0,0267	-0,0389
3	0,0297	0,0183	-0,0250	0,0218	0,0040	-0,0484
4	0,0097	-0,0019	0,0473	-0,0410	-0,0142	0,0616
5	0,0232	0,0022	-0,0375	-0,0025	-0,0105	-0,0614
6	0,0106	0,0182	0,0316	0,0323	0,0221	-0,0185
7	-0,0263	0,0399	-0,0240	-0,0449	-0,0007	0,0465

Table 65 Partial Autocorrelation Function (PACF)

PACF	USD/BTC	USD/ETH	USD/Gold	USD/Silver	USD/Oil	USD/EUR
1	-0,0148	0,0575	-0,1490	-0,0692	-0,0373	-0,0307
2	0,0428	-0,0007	-0,0339	-0,0143	0,0253	-0,0399
3	0,0310	0,0183	-0,0325	0,0202	0,0059	-0,0510
4	0,0087	-0,0040	0,0391	-0,0384	-0,0145	0,0570
5	0,0209	0,0025	-0,0263	-0,0076	-0,0118	-0,0621
6	0,0097	0,0177	0,0241	0,0305	0,0221	-0,0201
7	-0,0285	0,0381	-0,0156	-0,0395	0,0016	0,0468

Table 66 Ljung-Box test statistics and p-values

	Lag	USD/BTC	USD/ETH	USD/Gold	USD/Silver	USD/Oil	USD/EUR
LB	1	0,2	3,6	23,9	5,2	1,5	1,0
p		0,628	0,059	0,000	0,023	0,222	0,315
LB	2	2,2	3,6	24,0	5,2	2,3	2,6
p		0,328	0,168	0,000	0,072	0,323	0,267
LB	3	3,2	3,9	24,7	5,8	2,3	5,2
p		0,365	0,270	0,000	0,124	0,517	0,160
LB	4	3,3	3,9	27,1	7,6	2,5	9,2
p		0,512	0,416	0,000	0,109	0,646	0,055
LB	5	3,9	3,9	28,6	7,6	2,6	13,3
p		0,570	0,559	0,000	0,181	0,760	0,021
LB	6	4,0	4,3	29,7	8,7	3,1	13,7
p		0,679	0,637	0,000	0,191	0,792	0,033
LB	7	4,7	6,0	30,3	10,9	3,1	16,0
p		0,693	0,538	0,000	0,144	0,872	0,025

Correlation Coefficients

The following tables show correlation coefficients of daily exchange rate returns:

Table 67 Euro exchange rates of major currencies: Correlation coefficients for the period from 13 January 2000 to 3 January 2020

/EUR	JPY	CNY	GBP	CHF
USD	0,60	0,97	0,45	0,18
JPY		0,59	0,22	0,34
CNY			0,46	0,18
GBP				0,09

Table 68 US dollar rates for gold, silver, crude oil and the euro: Correlation coefficients for the period from 3 January 2006 to 3 January 2020

USD/	Silver	Oil	EUR
Gold	0,75	0,19	0,16
Silver		0,30	0,20
Oil			0,14

Table 69 US dollar rates for the bitcoin, gold, silver, crude oil and the euro: Correlation coefficients for the period from 1 October 2013 to 3 January 2020

USD/	Gold	Silver	Oil	EUR
BTC	0,01	0,01	0,01	-0,02
Gold		0,60	0,01	0,11
Silver			0,14	0,13
Oil				0,04

Table 70 US dollar rates for bitcoin, ethereum, gold, silver, crude oil and the euro: Correlation coefficients for the period from 10 August 2015 to 3 January 2020

USD/	ETH	Gold	Silver	Oil	EUR
BTC	0,45	0,02	0,01	0,01	-0,03
ETH		0,06	0,04	-0,04	0,02
Gold			0,56	-0,03	0,14
Silver				0,12	0,15
Oil					0,01

Default Correlations

ARMA(p,q)- GARCH(g,a) model -- Skew Student-T distribution

Default correlations are calculated for the joint period of two time series in each case.

The following tables show the results for Value-at-Risk predictions with ARMA(p,q)-GARCH(g,a) models and assuming Skew Student-T distributed innovations. [53]

Table 71 Default correlations for 90%-VaR

USD/	ETH	Gold	Silver	Oil	EUR
BTC	0,49	0,02	0,02	0,00	-0,01
ETH		0,03	0,01	0,02	0,00
Gold			0,52	0,09	0,11
Silver				0,12	0,13
Oil					0,07

/EUR	JPY	CNY
USD	0,36	0,82
JPY		0,34

Table 72 Default correlations for 95%-VaR

USD/	ETH	Gold	Silver	Oil	EUR
BTC	0,45	-0,01	0,03	0,02	0,00
ETH		-0,01	0,02	0,01	0,01
Gold			0,41	0,07	0,06
Silver				0,11	0,10
Oil					0,11

/EUR	JPY	CNY
USD	0,31	0,79
JPY		0,30

Table 73 Default correlations for 99%-VaR

USD/	ETH	Gold	Silver	Oil	EUR
BTC	0,41	-0,01	-0,01	0,04	-0,01
ETH		-0,01	-0,01	0,05	-0,01
Gold			0,37	0,12	0,02
Silver				0,13	0,06
Oil					0,06

/EUR	JPY	CNY
USD	0,29	0,80
JPY		0,25

[53] An AR(p,q) model is fitted to the return series before a GARCH(g,a) model is fitted to the resulting residual series.

ARMA(p,q) - MS(k)-GARCH(1,1) model -- Skew Student-T distribution

Default correlations are calculated for the joint period of two time series in each case.

The following tables show the results for Value-at-Risk predictions with ARMA(p,q) - MS(k)-GARCH(1,1) models and assuming Skew Student-T distributed innovations.[54]

Table 74 Default correlations for 90%-VaR

USD/	ETH	Gold	Silver	Oil	EUR
BTC	0,44	0,01	0,03	0,00	0,00
ETH		0,02	-0,01	0,03	-0,01
Gold			0,51	0,10	0,10
Silver				0,15	0,12
Oil					0,07

/EUR	JPY	CNY	GBP	CHF
USD	0,37	0,81	0,30	0,12
JPY		0,34	0,17	0,07
CNY			0,27	0,10
GBP				0,03

Table 75 Default correlations for 95%-VaR

USD/	ETH	Gold	Silver	Oil	EUR
BTC	0,45	0,00	0,04	0,03	0,00
ETH		0,00	0,02	0,00	0,01
Gold			0,43	0,09	0,07
Silver				0,13	0,10
Oil					0,12

/EUR	JPY	CNY	GBP	CHF
USD	0,29	0,78	0,25	0,10
JPY		0,29	0,13	0,07
CNY			0,25	0,10
GBP				0,02

Table 76 Default correlations for 99%-VaR

USD/	ETH	Gold	Silver	Oil	EUR
BTC	0,52	-0,01	-0,01	0,02	-0,01
ETH		-0,01	0,06	0,04	-0,01
Gold			0,36	0,12	0,02
Silver				0,15	0,06
Oil					0,10

/EUR	JPY	CNY	GBP	CHF
USD	0,26	0,78	0,22	0,06
JPY		0,24	0,15	0,04
CNY			0,18	0,07
GBP				0,01

[54] A volatility equation is fitted to the residuals of an ARMA(p,q) model with p and q of up to 5 each and optimised according to the BIC criterion.

AR(1)- MS(k)-GARCH(1,1) model -- Skew Student-T distribution

Default correlations are calculated for the joint period of two time series in each case.

The following tables show the results for Value-at-Risk predictions with AR(1)- MS(k)-GARCH(1,1) models and assuming Skew Student-T distributed innovations. [55]

Table 77 Default correlations for 90%-VaR

USD/	ETH	Gold	Silver	Oil	EUR
BTC	0,45	0,02	0,04	0,00	-0,01
ETH		0,02	0,01	0,02	-0,01
Gold			0,52	0,09	0,11
Silver				0,13	0,11
Oil					0,08

/EUR	JPY	CNY	GBP	CHF
USD	0,37	0,82	0,29	0,14
JPY		0,34	0,17	0,22
CNY			0,26	0,13
GBP				0,07

Table 78 Default correlations for 95%-VaR

USD/	ETH	Gold	Silver	Oil	EUR
BTC	0,42	0,00	0,05	0,02	-0,02
ETH		0,01	0,03	0,01	0,01
Gold			0,41	0,10	0,07
Silver				0,12	0,10
Oil					0,12

/EUR	JPY	CNY	GBP	CHF
USD	0,29	0,80	0,25	0,10
JPY		0,29	0,13	0,17
CNY			0,24	0,12
GBP				0,07

Table 79 Default correlations for 99%-VaR

USD/	ETH	Gold	Silver	Oil	EUR
BTC	0,49	-0,01	0,04	0,02	-0,01
ETH		-0,01	0,15	0,04	-0,01
Gold			0,32	0,12	0,02
Silver				0,18	0,05
Oil					0,08

/EUR	JPY	CNY	GBP	CHF
USD	0,26	0,77	0,23	0,10
JPY		0,25	0,15	0,23
CNY			0,20	0,14
GBP				0,08

[55] An AR(1) model is fitted to the return series before an MS(k)-GARCH(1,1) model is fitted to the resulting residual series.

Constant- MS(k)-GARCH(1,1) model -- Skew Student-T distribution

Default correlations are calculated for the joint period of two time series in each case.

The following tables show the results for Value-at-Risk predictions with Constant- MS(k)-GARCH(1,1) models and assuming Skew Student-T distributed innovations.[56]

Table 80 Default correlations for 90%-VaR

USD/	ETH	Gold	Silver	Oil	EUR
BTC	0,45	0,02	0,04	0,00	0,00
ETH		0,02	0,01	0,04	-0,01
Gold			0,51	0,09	0,10
Silver				0,13	0,11
Oil					0,07

/EUR	JPY	CNY	GBP	CHF
USD	0,37	0,81	0,30	0,11
JPY		0,34	0,18	0,07
CNY			0,27	0,11
GBP				0,03

Table 81 Default correlations for 95%-VaR

USD/	ETH	Gold	Silver	Oil	EUR
BTC	0,43	0,00	0,04	0,02	-0,02
ETH		0,00	0,03	0,00	0,01
Gold			0,41	0,09	0,07
Silver				0,12	0,10
Oil					0,12

/EUR	JPY	CNY	GBP	CHF
USD	0,29	0,78	0,25	0,09
JPY		0,29	0,13	0,07
CNY			0,25	0,09
GBP				0,02

Table 82 Default correlations for 99%-VaR

USD/	ETH	Gold	Silver	Oil	EUR
BTC	0,50	-0,01	0,04	0,03	-0,01
ETH		-0,01	0,15	0,05	-0,01
Gold			0,34	0,12	0,02
Silver				0,16	0,05
Oil					0,10

/EUR	JPY	CNY	GBP	CHF
USD	0,26	0,78	0,22	0,04
JPY		0,24	0,15	0,02
CNY			0,18	0,04
GBP				0,01

[56] "Constant" means that the mean estimated over the constant is subtracted from the return series, before adjustment of MS(k) GARCH(1,1) models.

No mean equation - MS(k)-GARCH(1,1) model -- Skew Student-T distribution

Default correlations are calculated for the joint period of two time series in each case.

The following tables show the results for Value-at-Risk predictions with MS(k)-GARCH(1,1) models and assuming Skew Student-T distributed innovations.[57]

Table 83 Default correlations for 90%-VaR

USD/	ETH	Gold	Silver	Oil	EUR
BTC	0,46	0,03	0,04	-0,01	-0,02
ETH		0,03	0,01	0,02	-0,01
Gold			0,50	0,10	0,10
Silver				0,13	0,10
Oil					0,08

/EUR	JPY	CNY	GBP	CHF
USD	0,37	0,81	0,31	0,10
JPY		0,34	0,17	0,06
CNY			0,27	0,10
GBP				0,02

Table 84 Default correlations for 95%-VaR

USD/	ETH	Gold	Silver	Oil	EUR
BTC	0,42	-0,01	0,03	0,03	-0,01
ETH		0,01	0,02	0,01	0,02
Gold			0,43	0,09	0,05
Silver				0,13	0,09
Oil					0,11

/EUR	JPY	CNY	GBP	CHF
USD	0,30	0,80	0,26	0,09
JPY		0,29	0,13	0,08
CNY			0,24	0,09
GBP				0,01

Table 85 Default correlations for 99%-VaR

USD/	ETH	Gold	Silver	Oil	EUR
BTC	0,42	-0,01	-0,01	0,03	-0,01
ETH		-0,01	-0,01	0,05	-0,01
Gold			0,33	0,13	0,08
Silver				0,09	0,05
Oil					0,08

/EUR	JPY	CNY	GBP	CHF
USD	0,25	0,76	0,22	0,05
JPY		0,24	0,14	0,02
CNY			0,18	0,05
GBP				0,02

[57] The modelling of the dynamics of the expected value is dispensed with, so that a volatility equation is fitted directly to the return series.

Temporal Dynamics of Default Correlations

Default correlations were calculated for growing time windows. The first time window comprises 500 Value-at-Risk predictions up to 4 March 2008 (or 7 January 2016 for Bitcoin and 8 November 2017 for Ethereum). The time window was successively extended by one day until 3 January 2020.

The VaR forecasts were obtained using the MS(k)-GARCH(1,1) model, i.e. without fitting a mean equation. The Skew Student-T distribution was assumed for the residuals.

Figure 3 Default Correlations with USD / EUR -exchange rates for 90 % -VaR

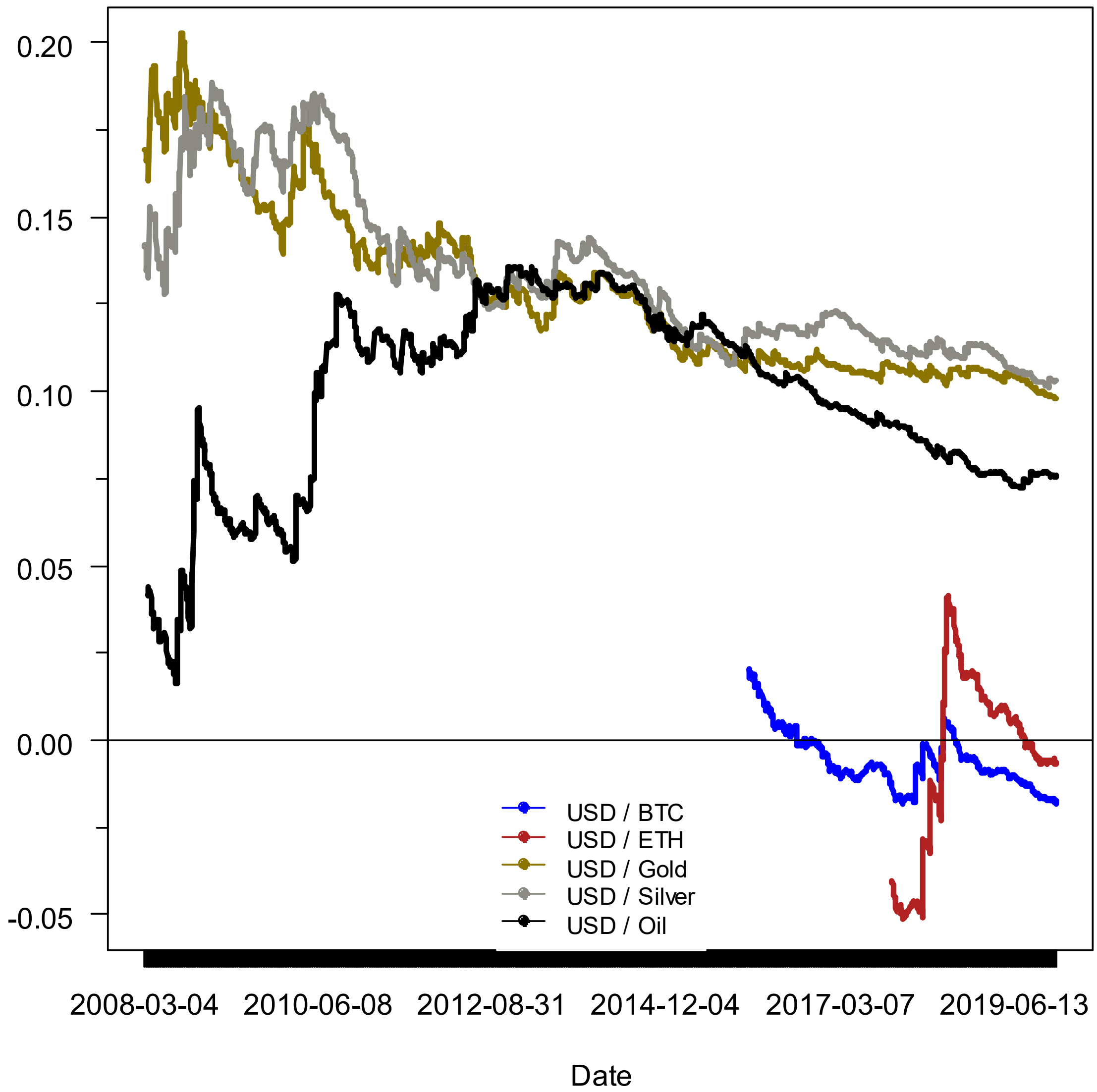


Figure 3 for example displays default correlations between 90% VaR forecasts for USD / EUR -exchange rate returns on the one hand and 90% VaR forecasts for Bitcoin or Ethereum, Gold, Silver and Crude Oil (Brent) price returns in US Dollars on the other.

Figure 4 Default Correlations with USD / EUR -exchange rates for 95 % -VaR

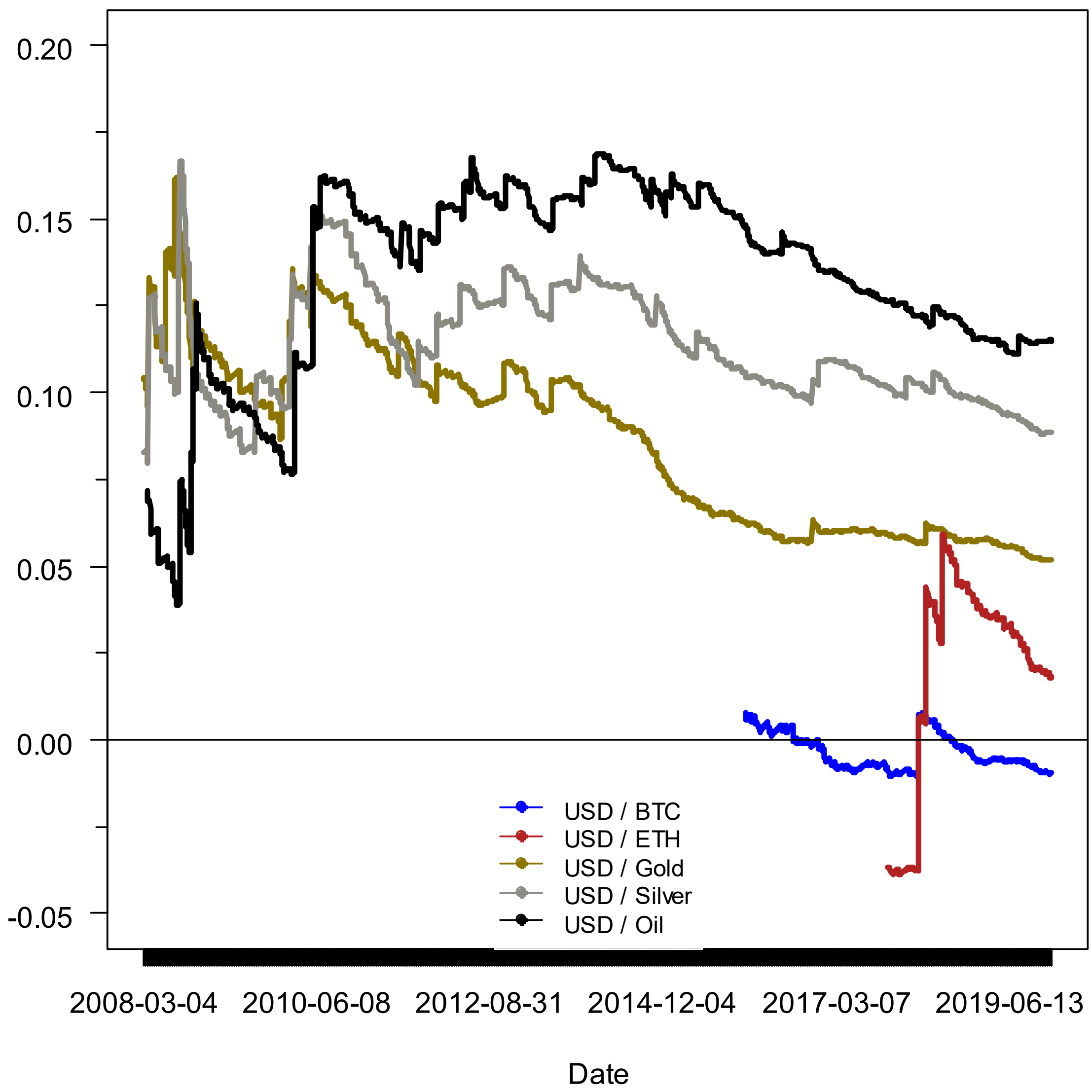


Figure 5 Default Correlations with USD / EUR -exchange rates for 99 % -VaR

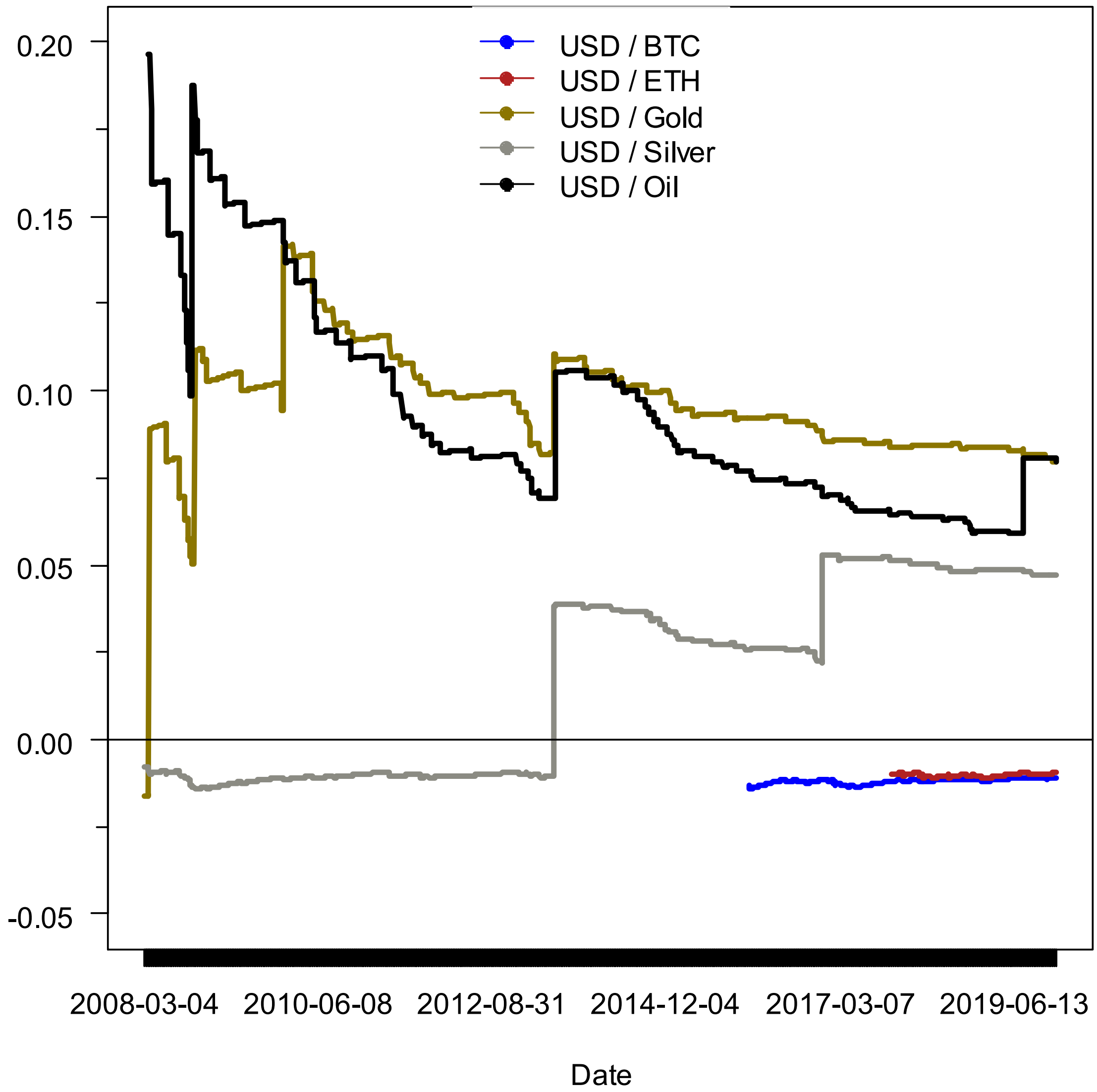


Figure 6 Default Correlations with USD / EUR -exchange rates for 90 % -VaR

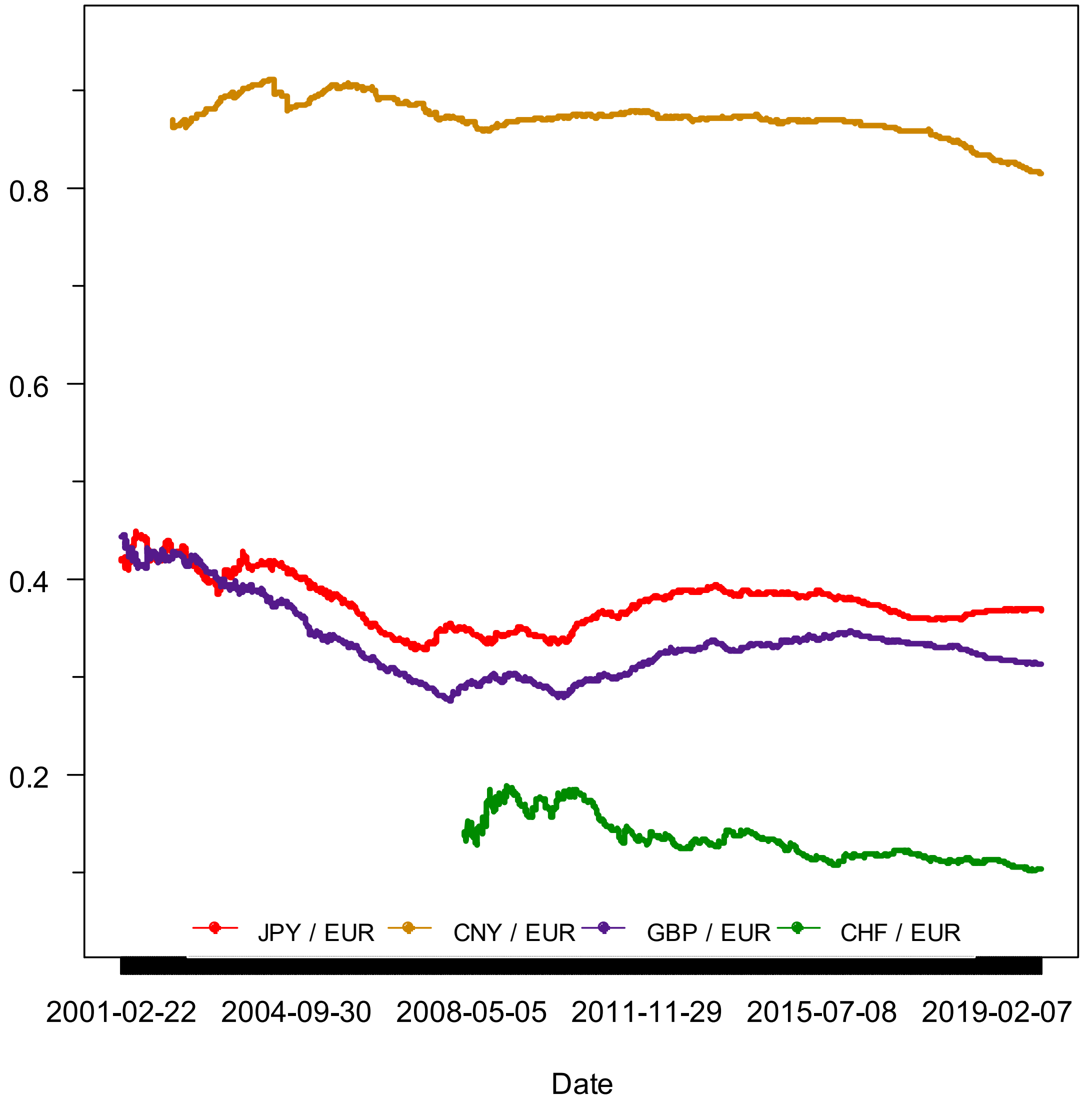


Figure 7 Default Correlations with USD / EUR -exchange rates for 95 % -VaR

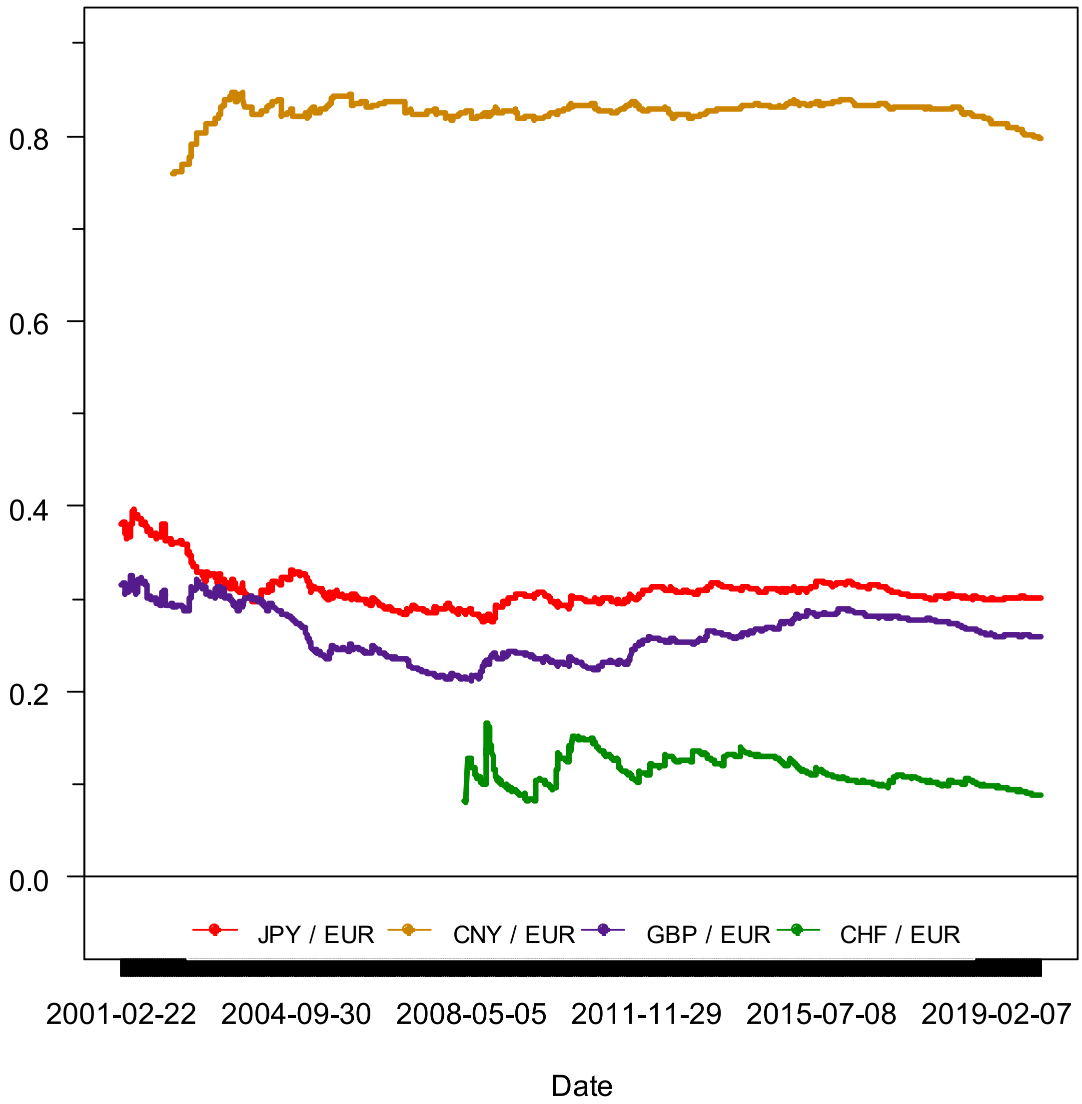
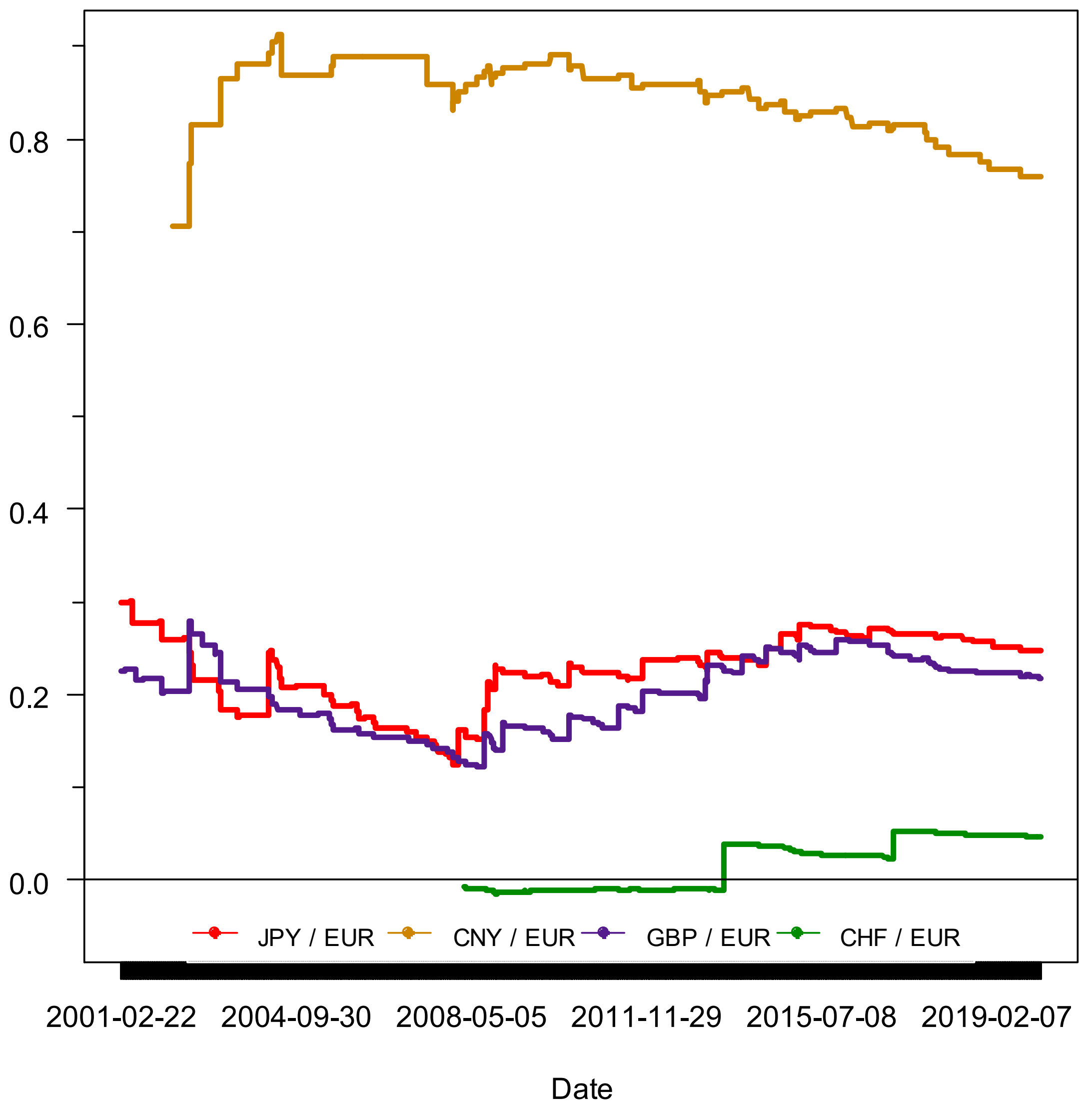


Figure 8 Default Correlations with USD / EUR -exchange rates for 99 % -VaR



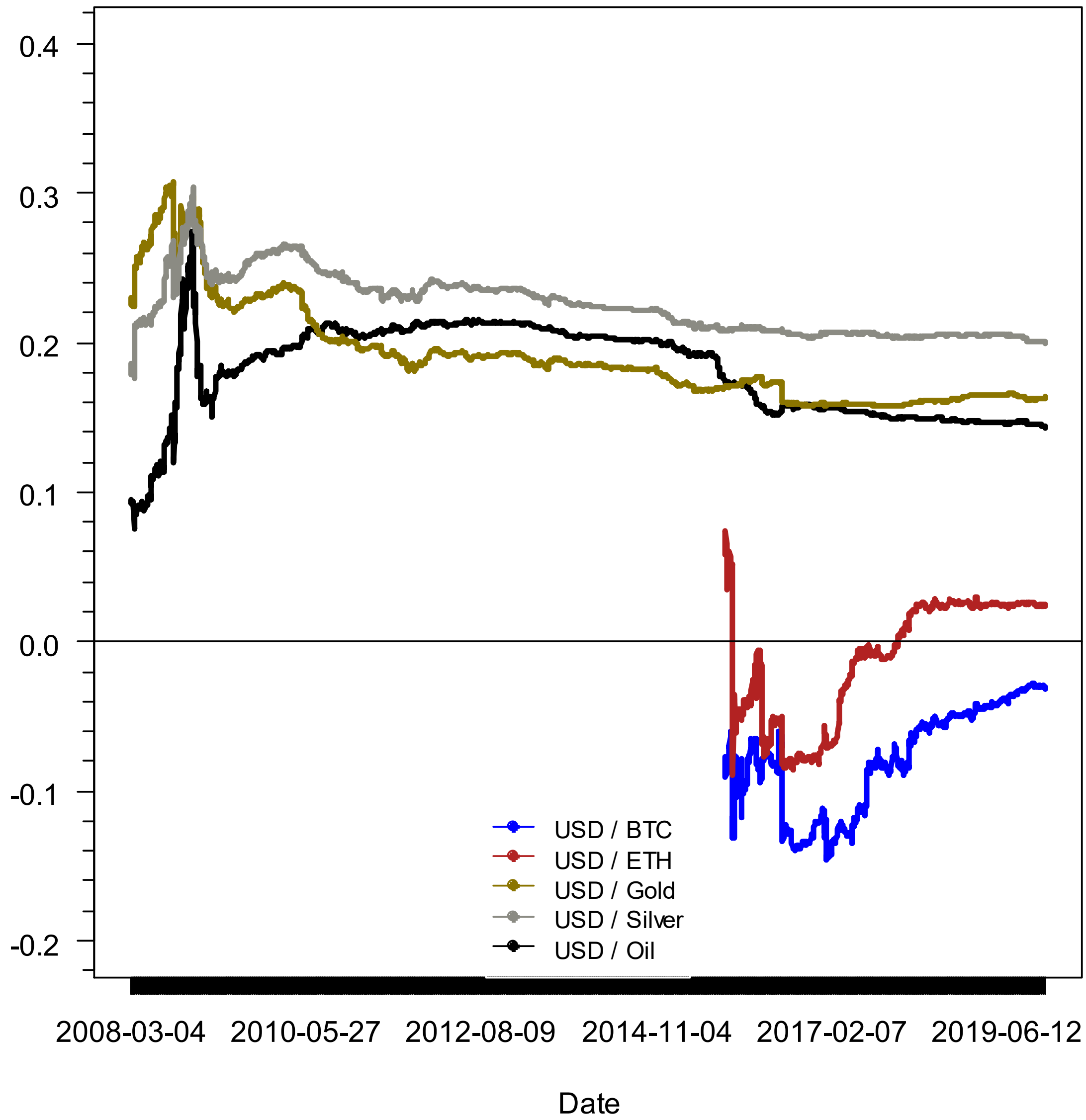
Dynamics of Correlation Coefficients

Results for growing time windows

In the literature it is considered to replace the default correlation by the ordinary correlation (i.e. Pearson's correlation coefficient).

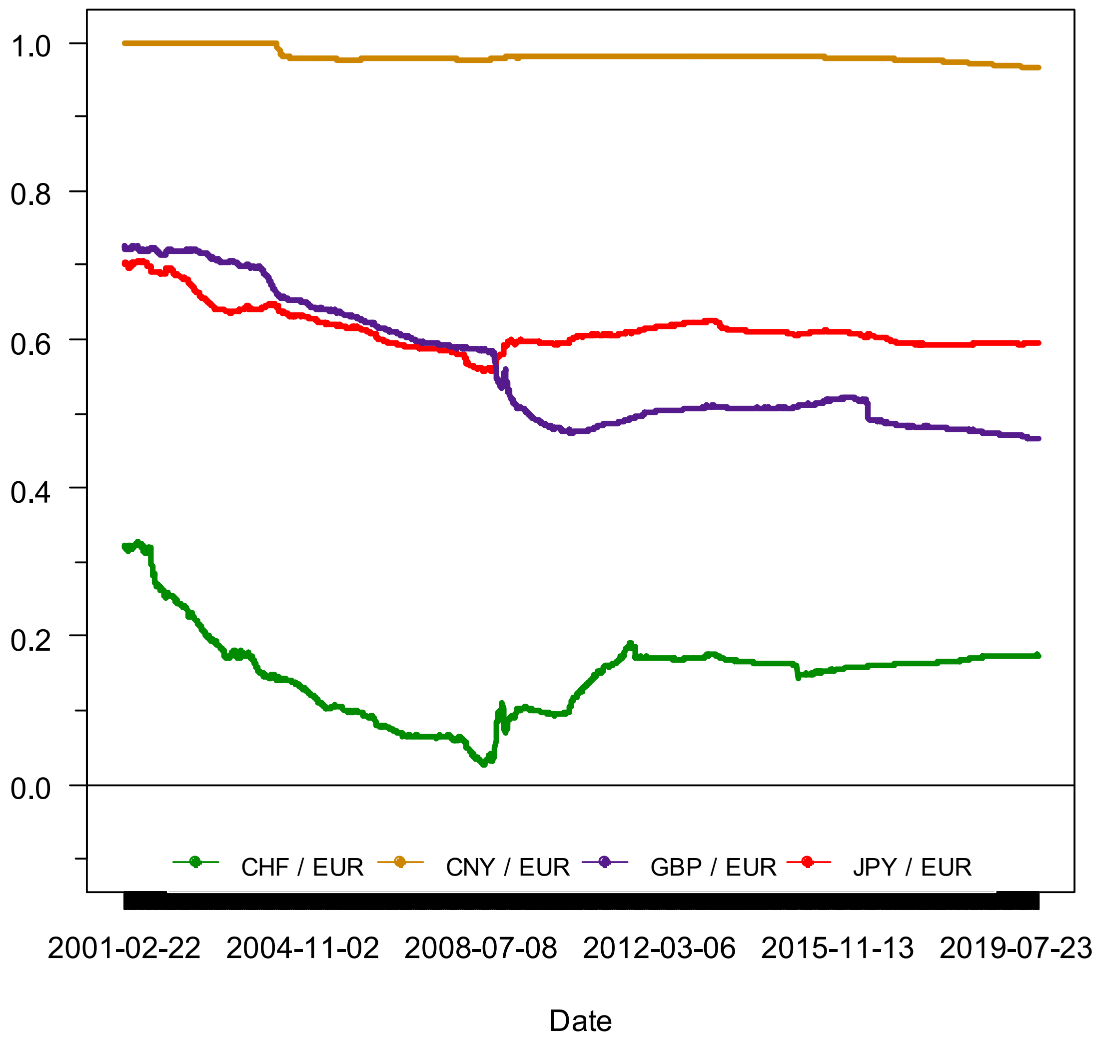
Pearson's correlation coefficients are plotted below, with one of the two time series being the USD / EUR exchange rate.

Figure 9 Pearson correlation coefficients with USD/EUR -exchange rates, growing time window



For comparison with the dynamics of the default correlation, therefore here are also calculated Pearson correlation coefficients for growing time windows and are also presented below for the period from 4 March 2008 to 3 January 2020.[58] I.e. for the calculation of a correlation coefficient shown for the date 4 March 2008, all pairwise return values up to this date are included. For Bitcoin and Ethereum, the first correlation coefficients are displayed for return values up to 24 September 2015. Earlier correlation coefficients are not presented for these two time series as they are based on less than 30 return values.

Figure 10 Pearson correlation coefficients with USD/EUR -exchange rates, growing time window



[58] While in the present study default correlations were calculated for time windows with at least 500 observations due to the rarity of VaR exceedances, the usual correlation coefficient gets by with significantly fewer - for example 30 - return values. This should also be relevant for considerations of approximating the default correlation by the ordinary Pearson's correlation coefficient when only few daily data are available for a new asset.

Results for rolling time windows

Correlation coefficients were calculated for a rolling time window covering 300 trading days, which are shown in the following figure. The dates again refer to the upper end of a time window.

Figure 11 Pearson correlation coefficients with USD / EUR -exchange rates, rolling time window (300 trading days each)

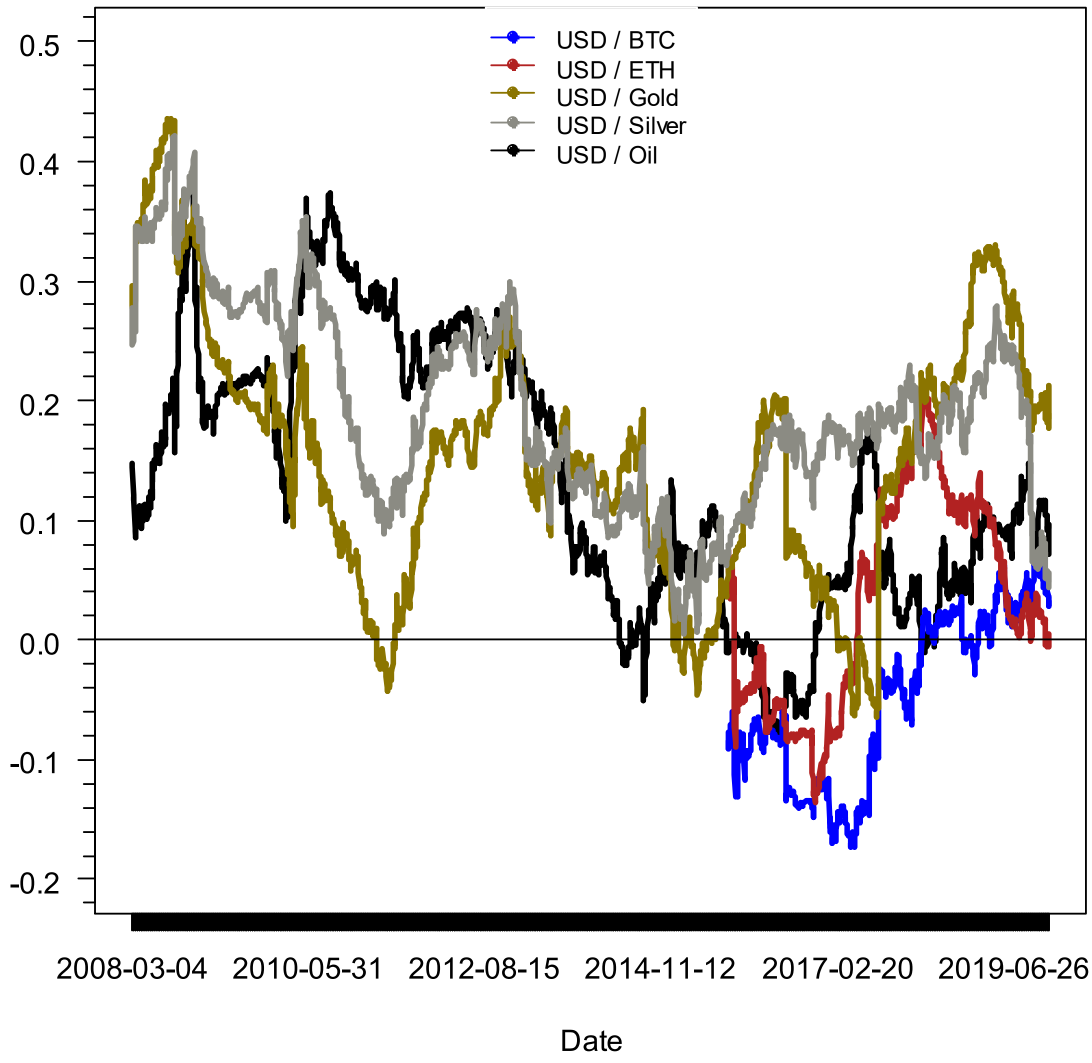
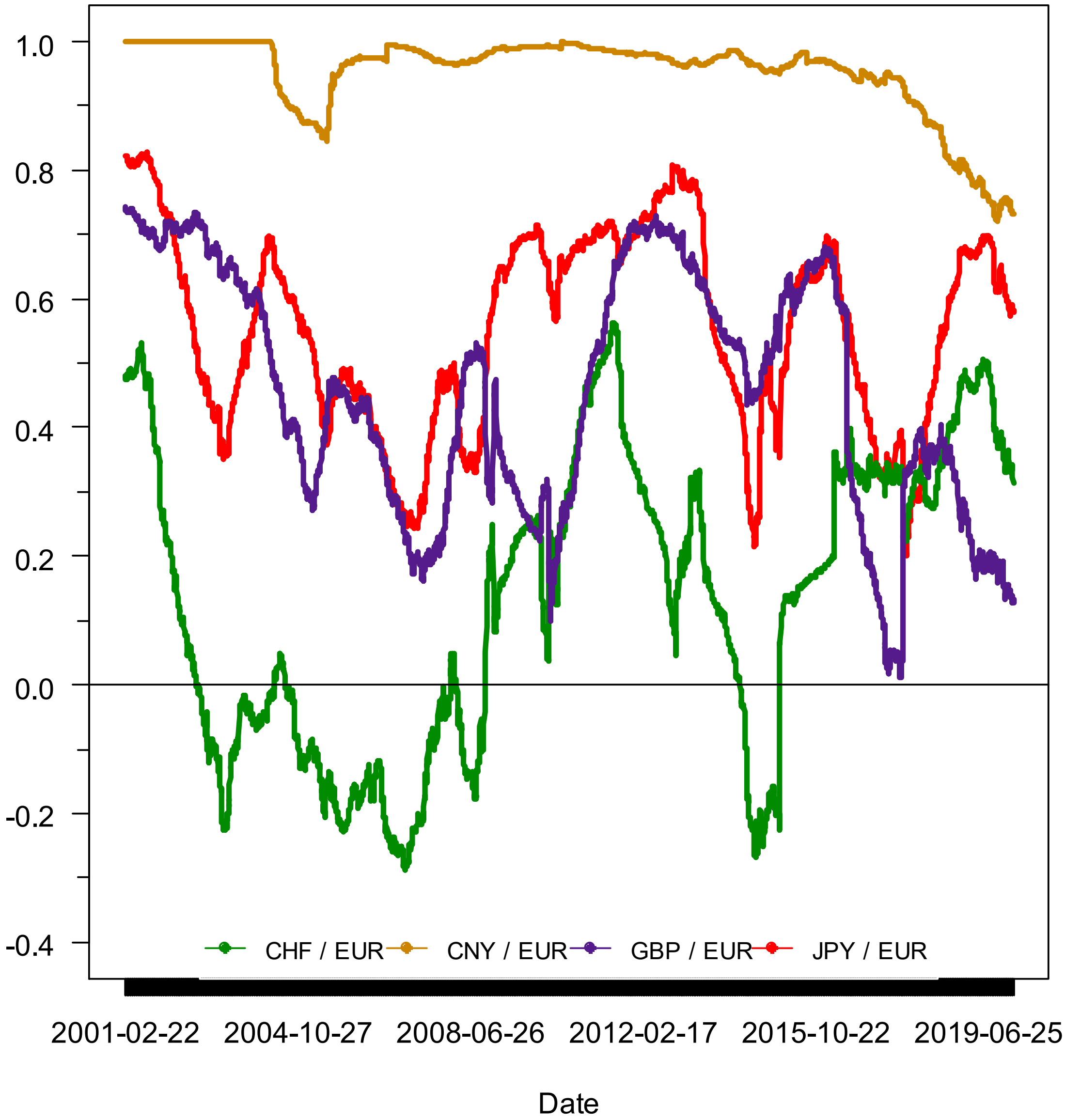


Figure 12 Pearson correlation coefficients with USD / EUR -exchange rates, rolling time window (300 trading days each)



Time Series

The following figures each show the exchange rate time series used in the study in their levels (top left) as well as on a logarithmic scale (top right), which were available in daily periodicity. In addition, the daily exchange rate changes (i.e. first differences, bottom left) as well as the exchange rate returns (i.e. first differences of the logarithmised exchange rates, bottom right) are presented.

Figure 13 Bitcoin-Price (in US-Dollar)

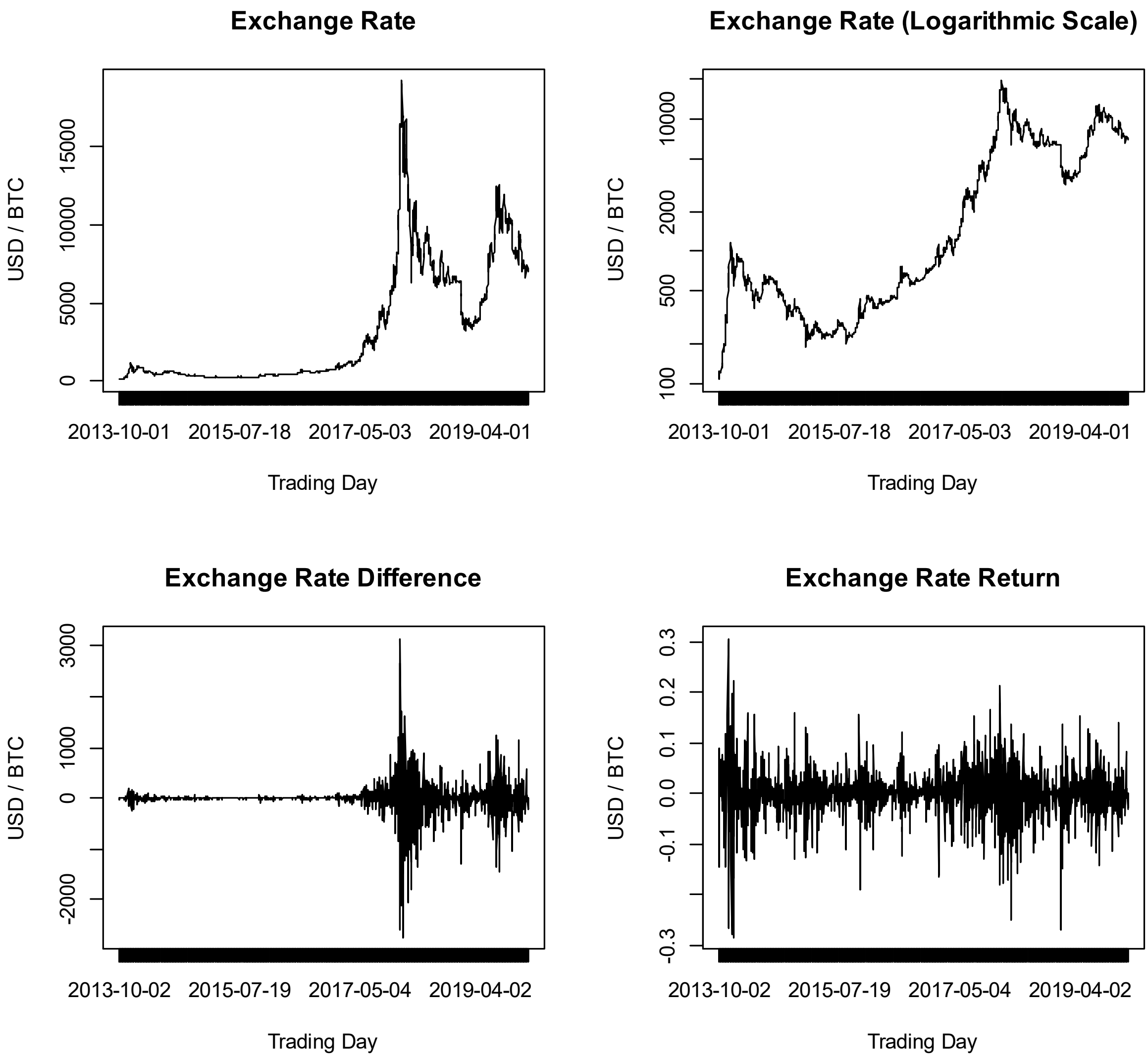


Figure 1 Ethereum-Price (in US-Dollar)

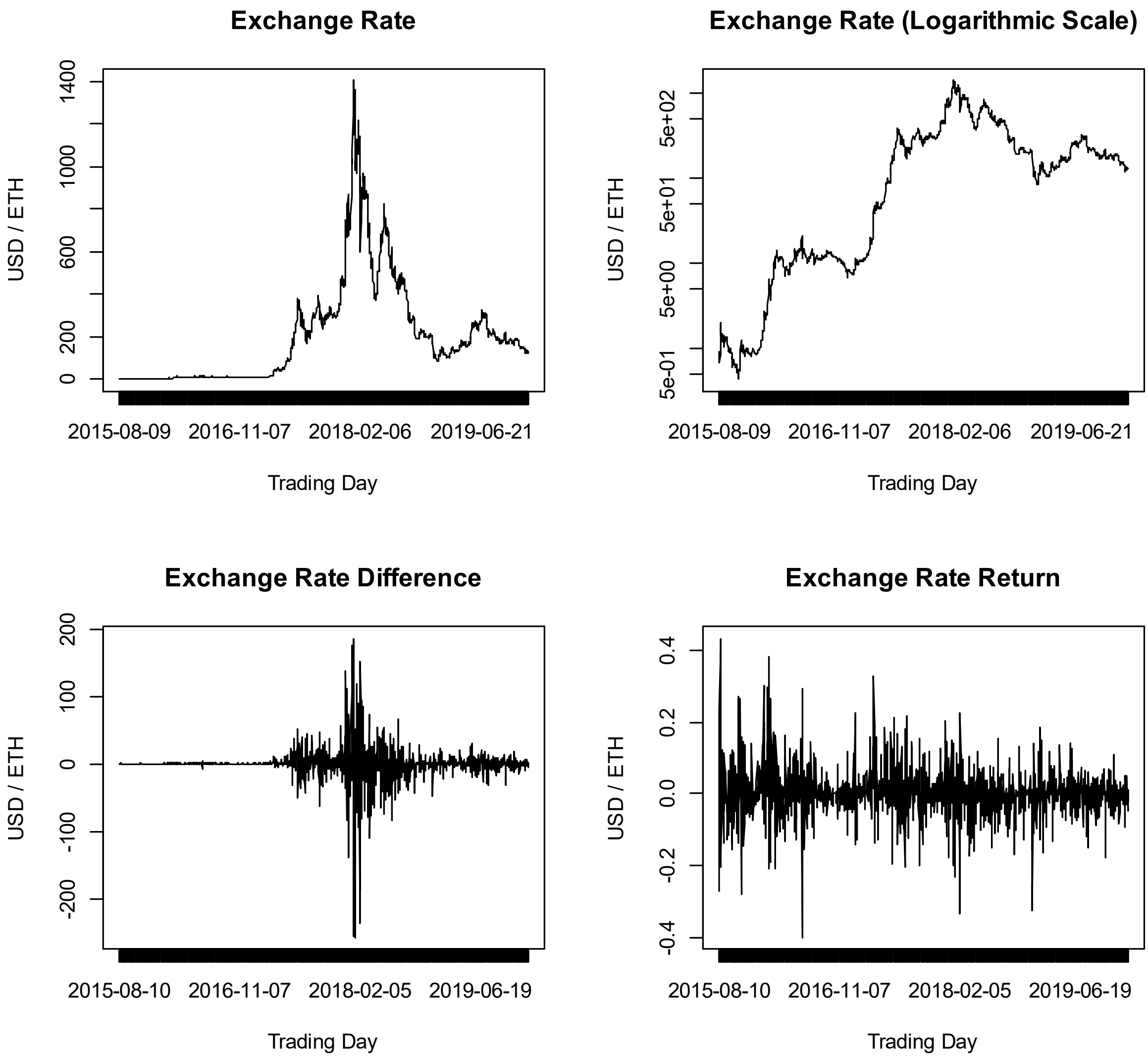


Figure 2 CHF / EUR – Exchange Rate

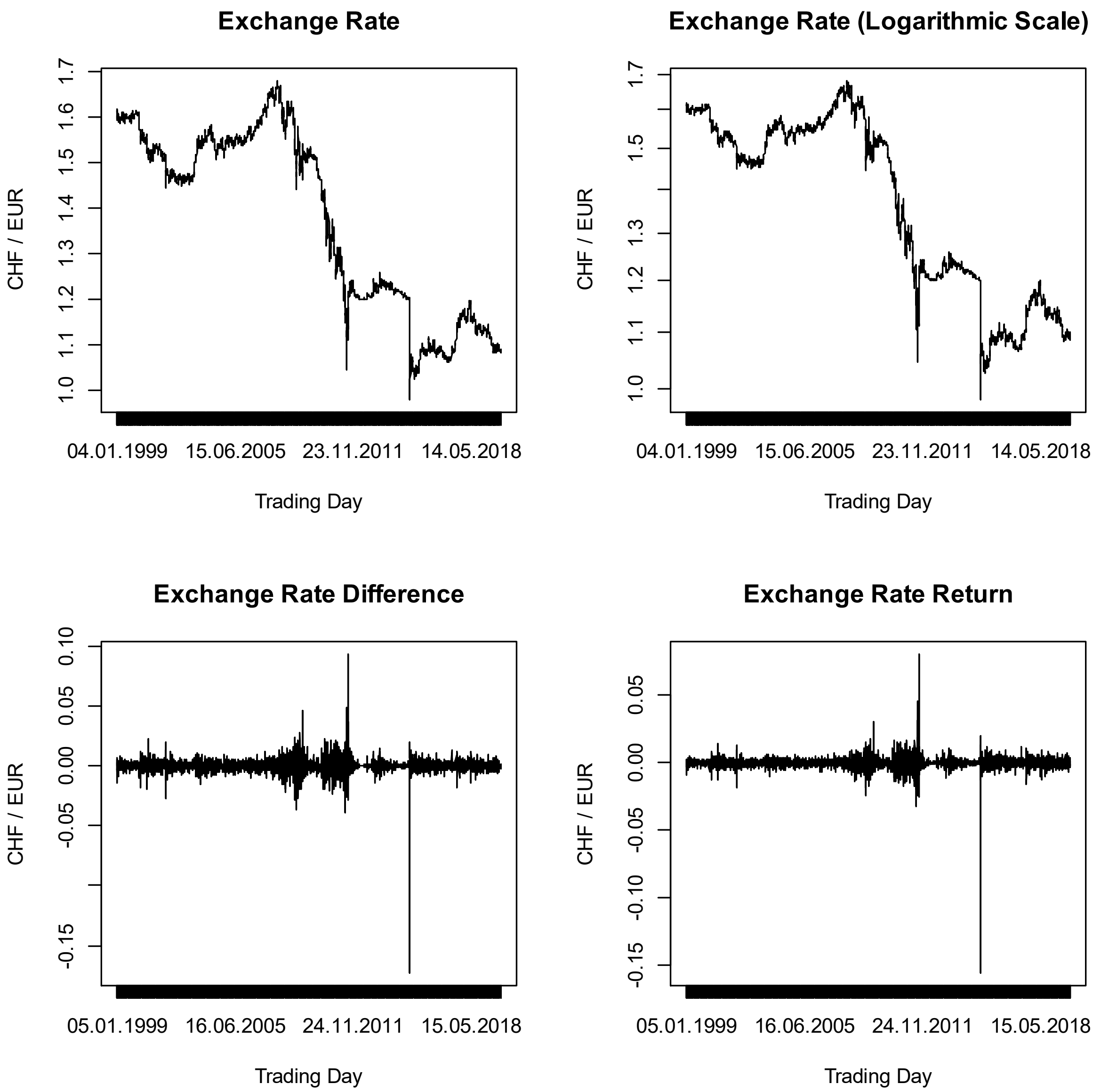


Figure 14 CNY / EUR – Exchange Rate

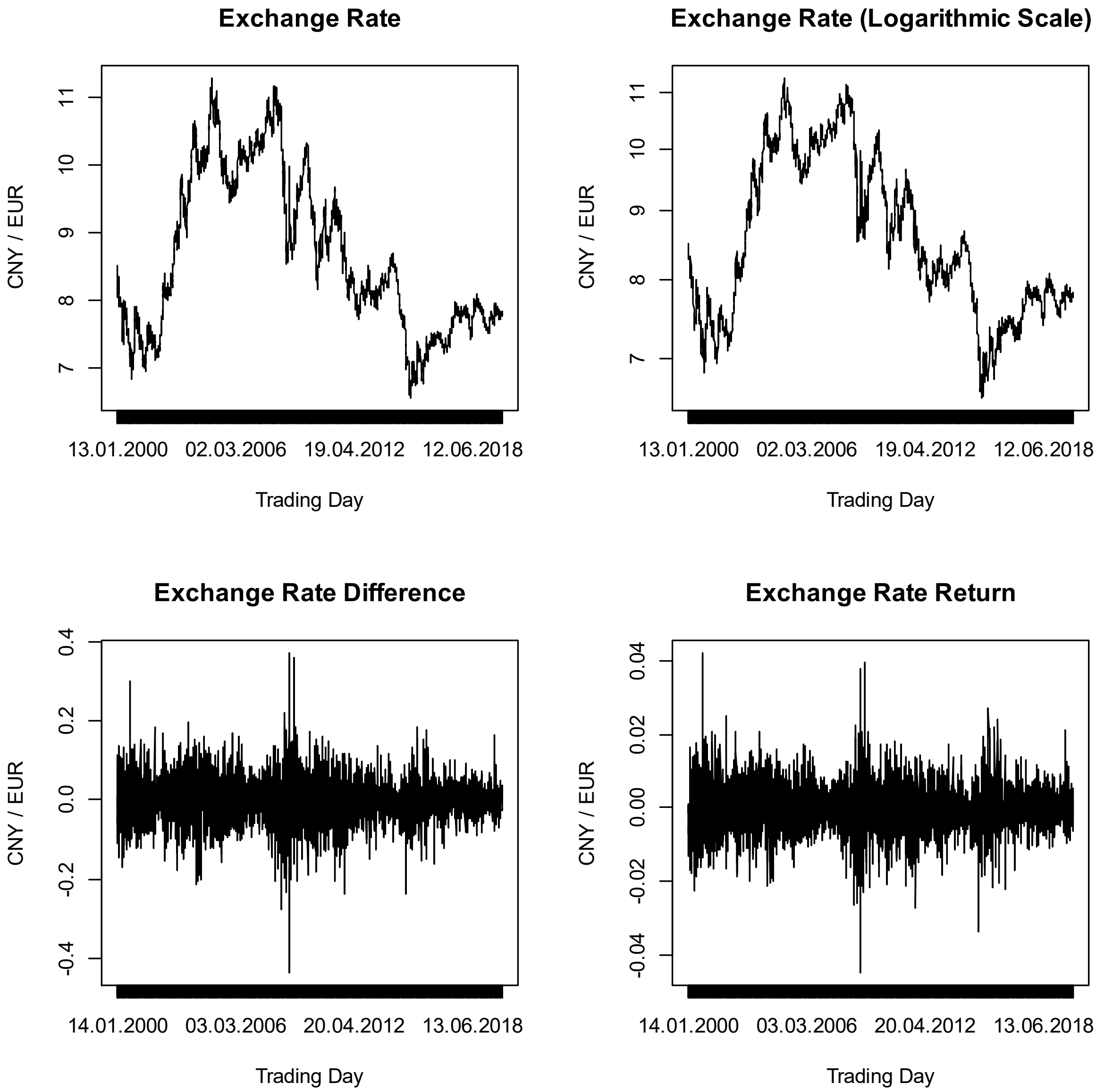


Figure 15 GBP / EUR – Exchange Rate

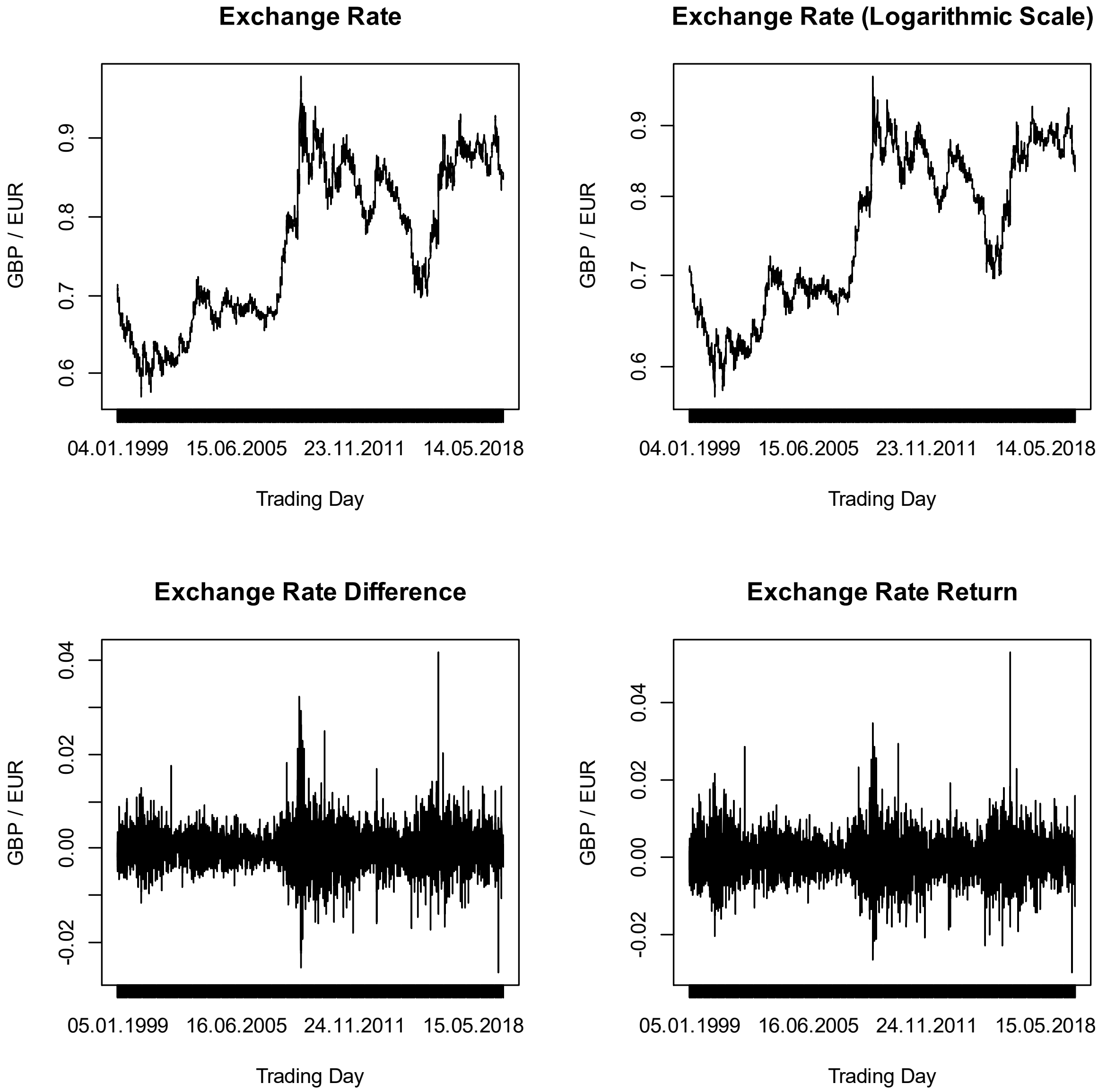


Figure 16 JPY / EUR – Exchange Rate

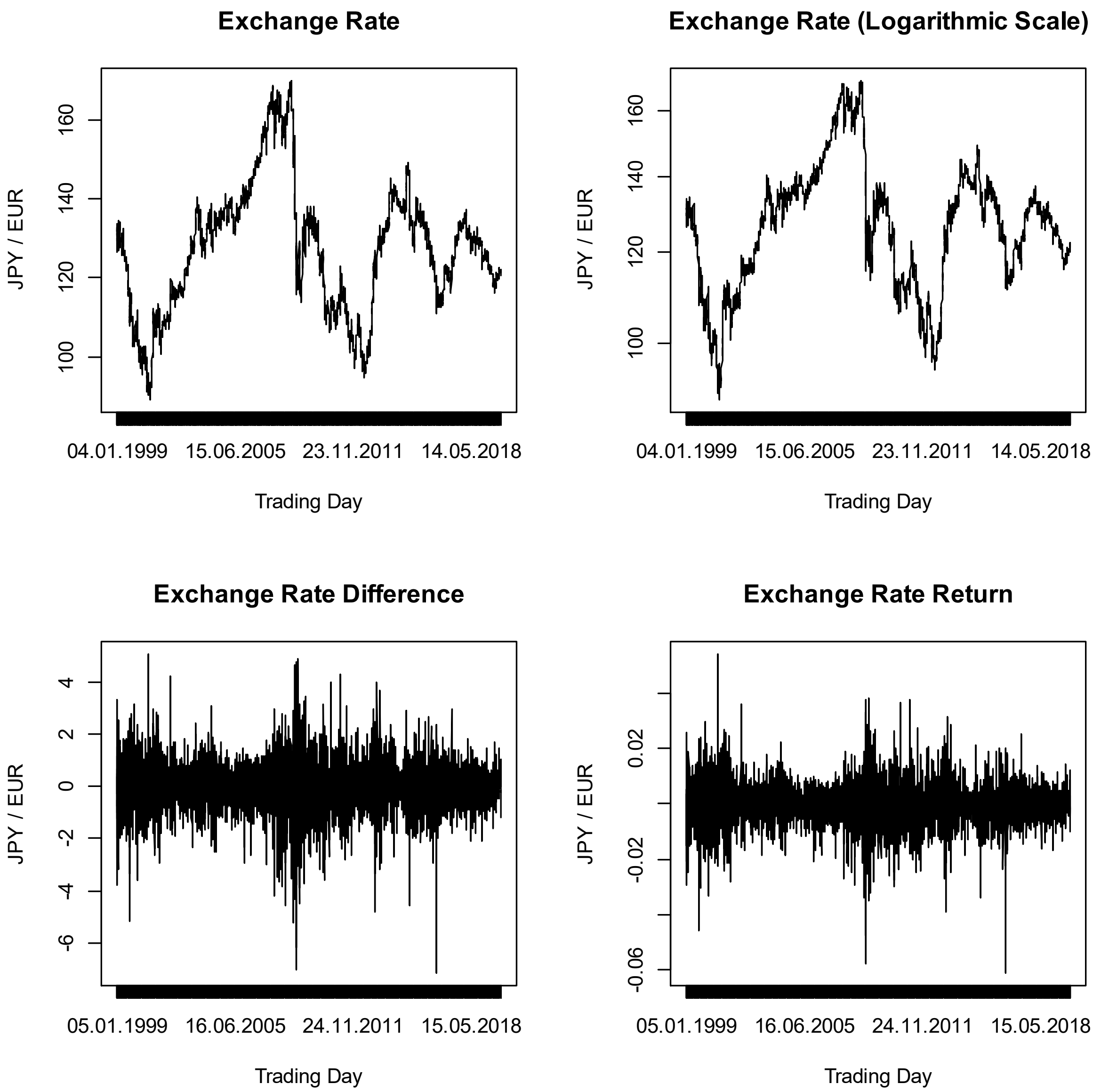


Figure 17 USD / EUR – Exchange Rate

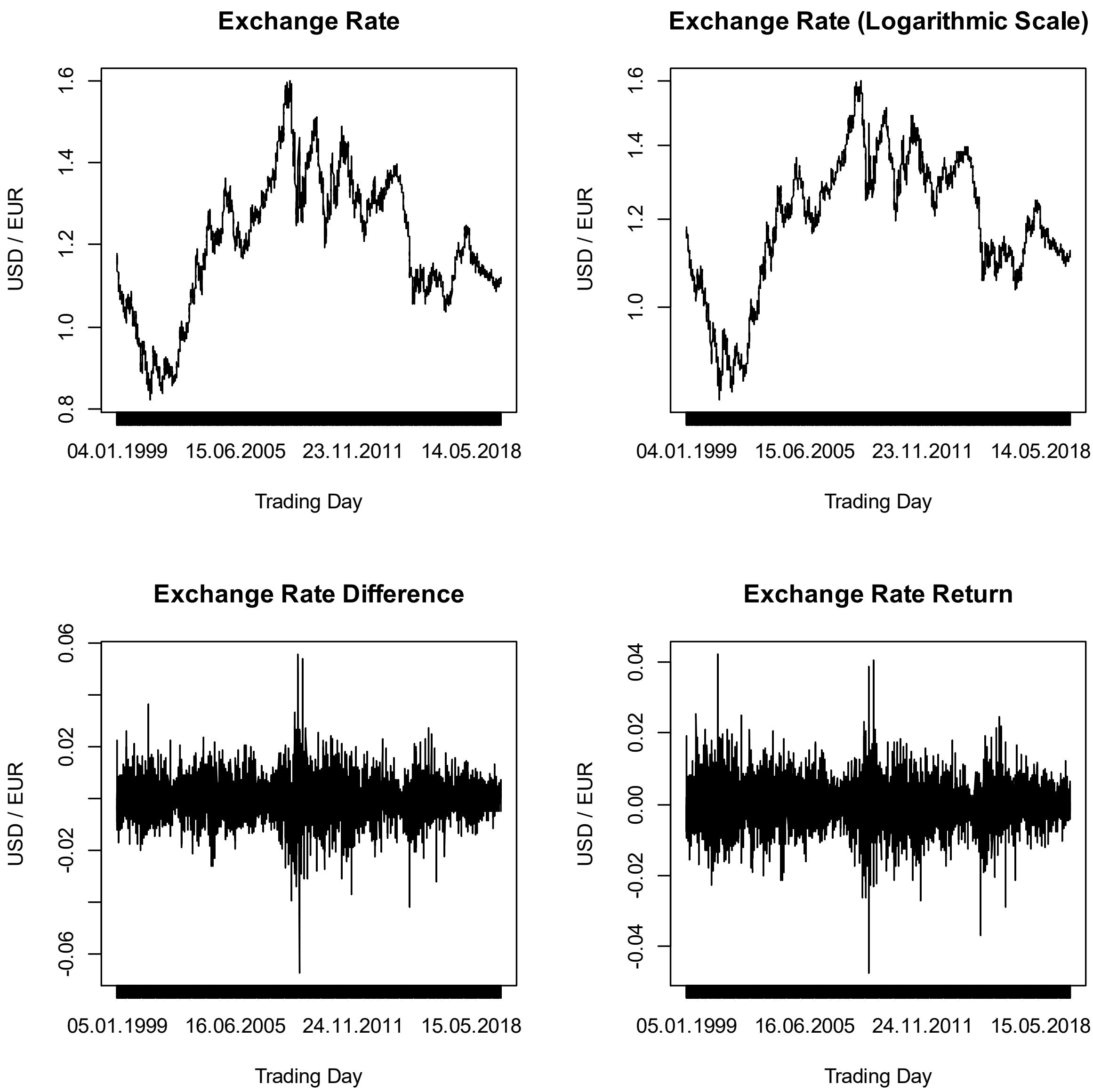


Figure 18 Gold Price in US Dollars per Fine Ounce

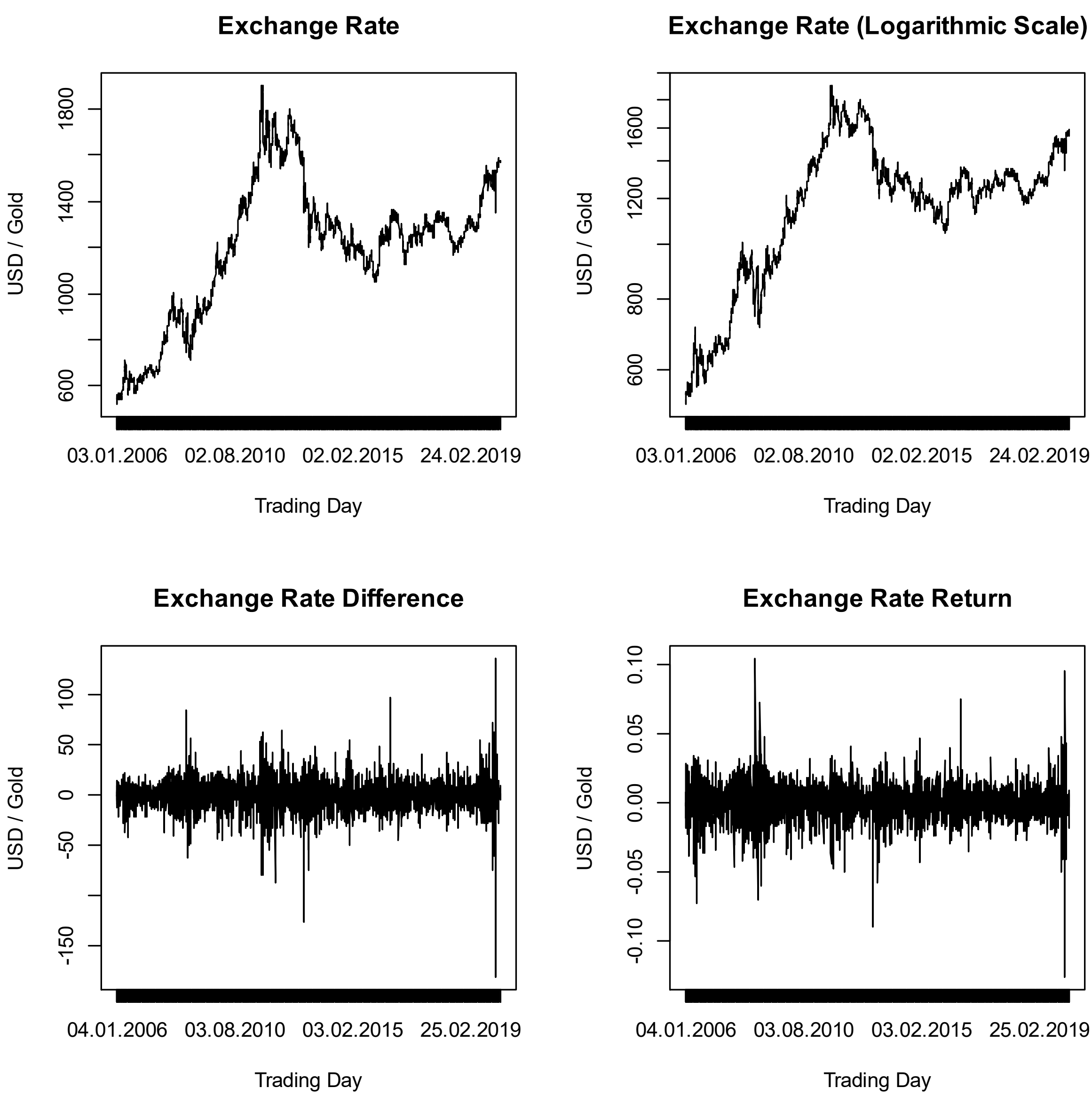


Figure 19 Silver Price in US Dollars per Fine Ounce

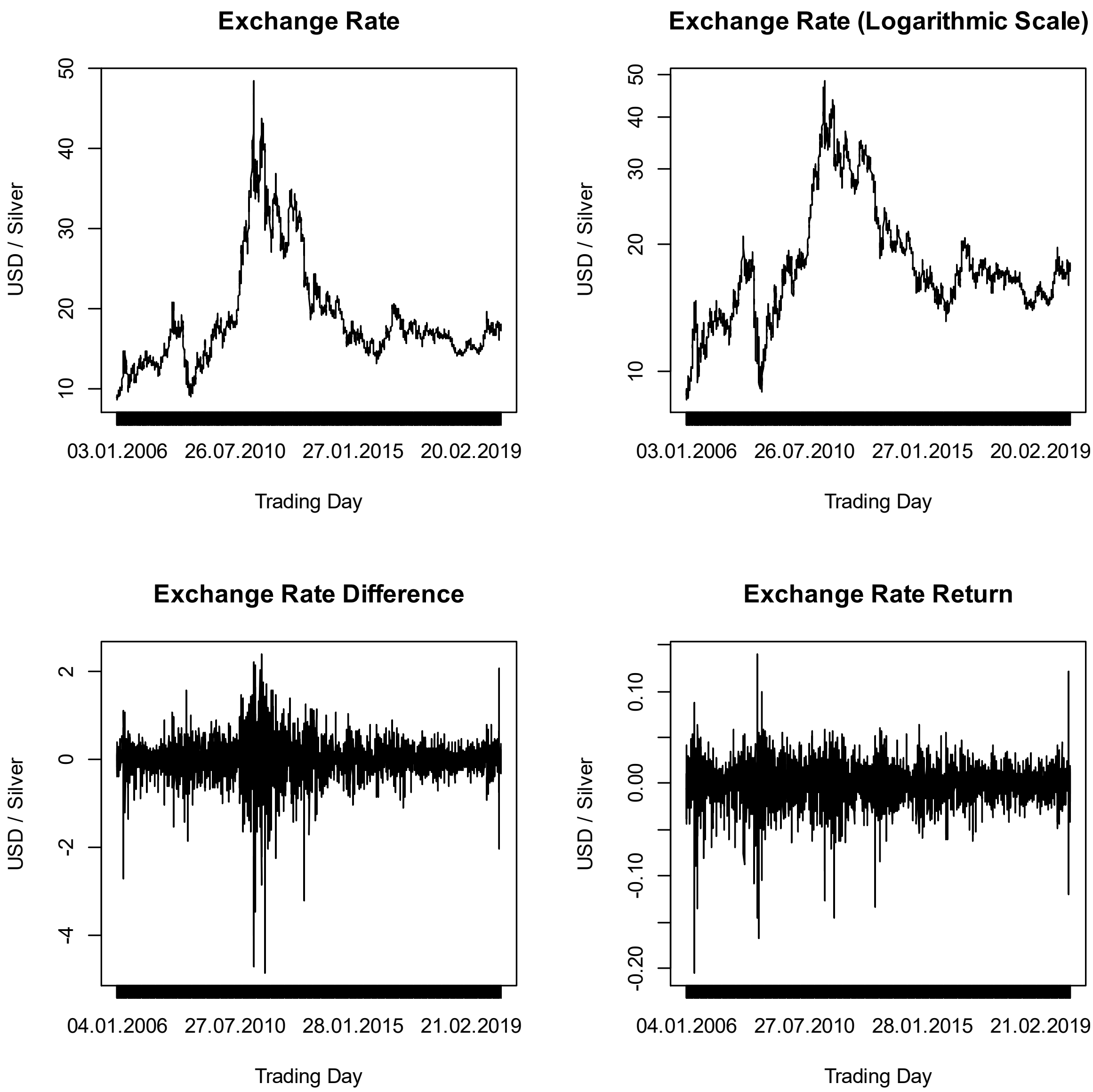
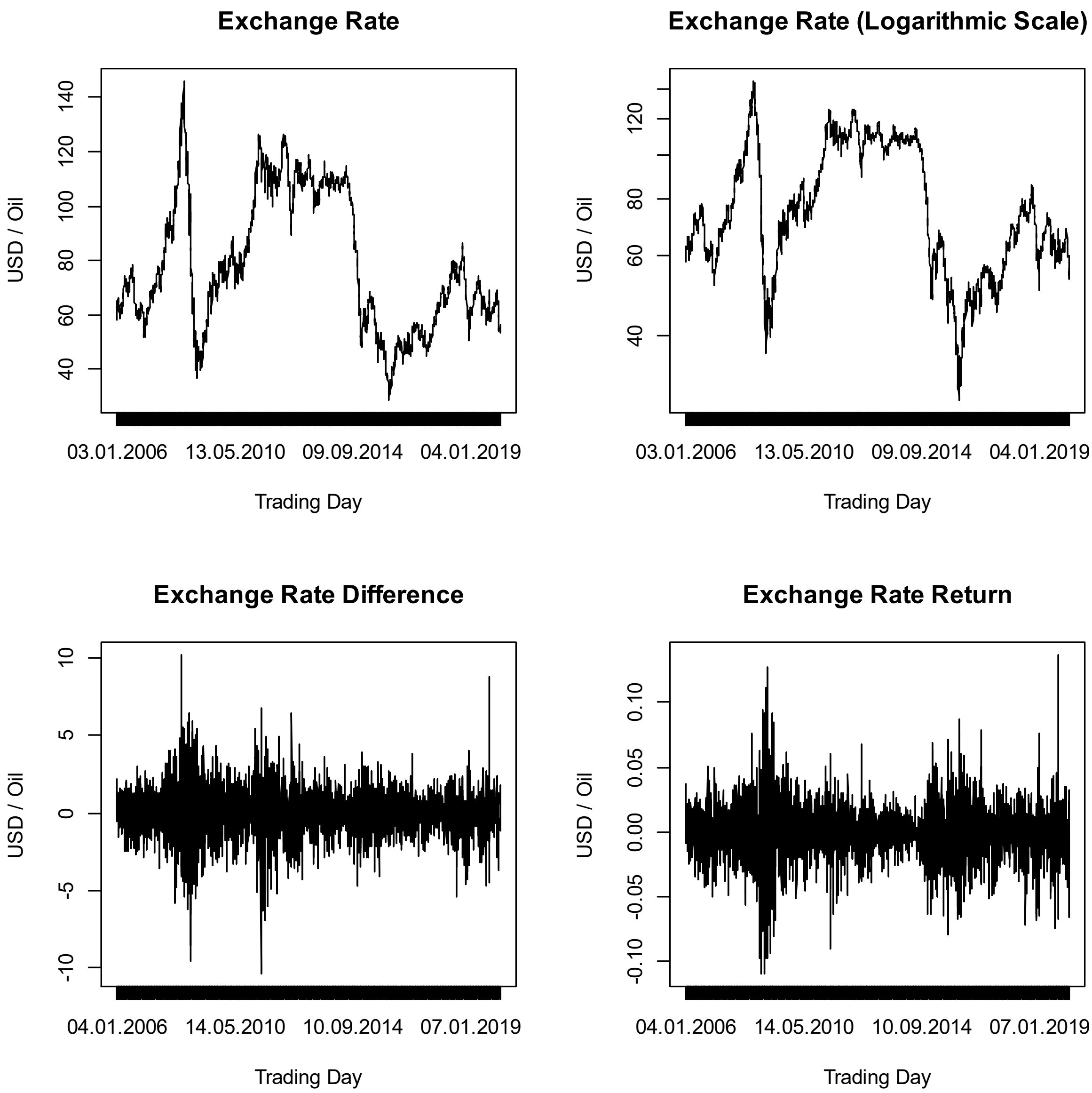


Figure 20 Oil Price in US Dollars per Barrel (Brent)



Acknowledgements

I would like to thank the participants of my lectures and seminars on “Time Series Analysis“ at the University of Kassel and the participants of the 26th International Conference "Forecasting Financial Markets" 2019 in Venice for valuable comments on my research work.

References

Alexander, Carol O. & Chibumba, A.M. (1997) *Multivariate Orthogonal Factor GARCH*. School of Mathematical Sciences, University of Sussex, Mimeo, UK.

Ali, Ghulam (2013) EGARCH, GJR-GARCH, TGARCH, AVGARCH, NGARCH, IGARCH and APARCH Models for Pathogens at Marine Recreational Sites. *Journal of Statistical and Econometric Methods*, 2 (3), 57-73. https://www.scienpress.com/Upload/JSEM/Vol%202_3_6.pdf

Ardia, D., Bluteau, K., Boudt, K. & Catania, L. (2018) Forecasting risk with Markov-switching GARCH models: A large-scale performance study. *International Journal of Forecasting*, 34 (4), 733-747. https://doi.org/10.1016/j.ijforecast.2018.05.004

Ardia, D., Bluteau, K., Boudt, K., Catania, L. & Trottier, D.-A. (2019a) Markov-Switching GARCH Models in R: The MSGARCH Package. *Journal of Statistical Software*, 91 (4), 1-38. https://doi.org/10.18637/jss.v091.i04

Ardia, D., Bluteau, K. & Rüede, M. (2019b) Regime changes in Bitcoin GARCH volatility dynamics. Finance Research Letters, 29, 266-271. https://doi.org/10.1016/j.frl.2018.08.009

Artzner, P., Delbaen, F., Eber, J.-M. & Heath, D. (1999) Coherent Measures of Risk. *Mathematical Finance*, 9 (3), 203-228. https://doi.org/10.1111/1467-9965.00068

Baillie, Richard T. (1996) Long memory processes and fractional Integration in econometrics. *Journal of Econometrics*, 73 (1), 5-59. http://long-memory.com/Baillie1996.pdf

Baillie, Richard T., Bollerslev, Tim & Mikkelsen, Hans Ole (1996) Fractionally integrated generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics*, 74 (1), 3-30. https://doi.org/10.1016/S0304-4076(95)01749-6

Basel Committee on Banking Supervision (2019) *Minimum capital requirements for market risk*. ISBN 978-92-9259-237-0 (online). https://www.bis.org/bcbs/publ/d457.pdf

Bollerslev, Tim (1986) Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics*, 31 (3), 307-327. https://doi.org/10.1016/0304-4076(86)90063-1

Bollerslev, Tim (1990) Modelling the coherence in short-run nominal exchange rates: A multivariate generalized ARCH model. *Review of Economics and Statistics*, 72 (3), 498-505. https://doi.org/10.2307/2109358

Bollerslev, Tim (1987) A Conditionally Heteroskedastic Time Series Model for Speculative Prices and Rates of Return. *The Review of Economics and Statistics*, 69 (3), 542-547. https://doi.org/10.2307/1925546

Bollerslev, T., Engle, R.F. & Wooldridge, J. (1988) A Capital Asset Pricing Model with Time Varying Covariances. *Journal of Political Economy*, 96, 116-131. http://dx.doi.org/10.1086/261527

Bollerslev, Tim & Wooldridge, Jeffrey M. (1992) Quasi-Maximum Likelihood Estimation and Inference in Dynamic Models with Time-Varying Covariances. *Econometric Reviews*, 11 (2), 143-172. https://doi.org/10.1080/07474939208800229

Boubaker, H. & Sghaier, N. (2016) Markov-Switching Time-Varying Copula Modeling of Dependence Structure between Oil and GCC Stock Markets. *Open Journal of Statistics*, 6 (4), 565-589. http://dx.doi.org/10.4236/ojs.2016.64048

Box, G.E.P. & Jenkins, G.M. (1970) *Time Series Analysis: Forecasting and Control*. Holden-Day, San Francisco.

Box, G. (2013). Box and Jenkins: Time Series Analysis, Forecasting and Control. *A Very British Affair*. Palgrave Advanced Texts in Econometrics. Palgrave Macmillan, London. https://doi.org/10.1057/9781137291264_6

Brooks, Chris, Burke, Simon P. & Persand, Gita (2003) Multivariate GARCH models: software choice and estimation issues. *Journal of Applied Econometrics*, 18 (6), 725-734. https://doi.org/10.1002/jae.717

Brownlees, C. T., Engle, R. F. & Kelly, B. T. (2011) A Practical Guide to Volatility Forecasting through Calm and Storm. *Risk Management eJournal*. http://dx.doi.org/10.2139/ssrn.1502915

Christoffersen, P. F. (1998) Evaluating Interval Forecasts. *International Economic Review*, 39 (4), 841-862. https://doi.org/10.2307/2527341

Colucci, Stefano (2018) *On Estimating Bitcoin Value at Risk: A Comparative Analysis*. University of Rome III - Department of Business Studies. http://dx.doi.org/10.2139/ssrn.3236813

Dhamija, A.K. & Bhalla, V.K. (2010) Financial Time Series Forecasting: Comparison of Neutral Networks and ARCH Models. *International Research Journal of Finance and Economics*, 49 (1), 194-212.

Dickey, David A. & Fuller, Wayne A. (1979) Distribution of the Estimators for Autoregressive Time Series With a Unit Root. *Journal of the American Statistical Association*, 74 (366), 427-431. https://doi.org/10.2307/2286348

Duan, Jin-Chuan (1995) The GARCH Option Pricing Model. *Mathematical Finance*, 5 (1), 13-32. https://doi.org/10.1111/j.1467-9965.1995.tb00099.x

Duffie, Darell & Pan, Jun (1997) An Overview of Value at Risk. *The Journal of Derivatives*, 4 (3), 7-49. https://doi.org/10.3905/jod.1997.407971

Engle, Robert F. (1982) Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation. *Econometrica*, 50 (4), 987-1007. http://www.jstor.org/stable/1912773

Engle, R. (2001) *GARCH 101: An Introduction to the Use of ARCH/GARCH models in Applied Econometrics.* NYU Working Paper No. FIN-01-030. https://ssrn.com/abstract=1294571

Engle, Robert F. & Bollerslev, Tim (1986) Modelling the persistence of conditional variances. *Econometric Reviews*, 5 (1), 1-50. https://doi.org/10.1080/07474938608800095

Engle, Robert F. & Manganelli, Simone (2004) CAViaR Conditional Autoregressive Value at Risk by Regression Quantiles, *Journal of Business & Economic Statistics*, 22 (4), 367-381. https://doi.org/10.1198/073500104000000370

Engle, Robert F. & Kroner, Kenneth F. (1995) Multivariate Simultaneous Generalized ARCH. *Econometric Theory*, 11 (1), 122-150. https://doi.org/10.1017/S0266466600009063

Engle, R. & Sheppard, K. (2001) Theoretical and Empirical properties of Dynamic Conditional Correlation Multivariate GARCH. NBER Working Paper 8554, National Bureau of Economic Research, Cambridge, MA. http://www.nber.org/papers/w8554

Federal Register (1996) Risk-Based Capital Standards: Market Risk. *Rules and Regulations*. 61 (174), 47358-47378. https://www.govinfo.gov/content/pkg/FR-1996-09-06/pdf/96-22546.pdf

Glosten, L. R., Jagannathan, R. & Runkle, D. E. (1993) On the Relation between the Expected Value and the Volatility of the Nominal Excess Return on Stocks. *The Journal of Finance*, 48 (5), 1779-1801. http://dx.doi.org/10.1111/j.1540-6261.1993.tb05128.x

Gray, S.F. (1996) Modeling the conditional distribution of interest rates as a regime-switching process. *Journal of Financial Economics*, 42(1), 27–62.https://doi.org/10.1016/0304-405x(96)00875-6

Greenspan, Alan (1996) *Remarks at the Financial Markets Conference of the Federal Reserve Bank of Atlanta*. Coral Gables, Florida https://fraser.stlouisfed.org/title/statements-speeches-alan-greenspan-452/remarks-financial-markets-conference-federal-reserve-bank-atlanta-coral-gables-florida-8561

Haas, M., Mittnik, S. & Paolella, M.S. (2004a) A New Approach to Markov-Switching GARCH Models. *Journal of Financial Econometrics*, 2(4), 493–530. https://doi.org/10.1093/jjfinec/nbh020

Haas, M., Mittnik, S. & Paolella, M.S. (2004b) Mixed Normal Conditional Heteroskedasticity. *Journal of Financial Econometrics*, 2(2), 211–250. https://doi.org/10.1093/jjfinec/nbh009

Hair, Joseph F., Black, William C., Babin, Barry J. & Anderson, Rolph E. (2010) *Multivariate Data Analysis*, Seventh Ed. Pearson Prentice Hall. https://vdoc.pub/download/multivariate-data-analysis-7th-edition-7src43i6uep0

Hansen, Bruce E. (1994) Autoregressive Conditional Density Estimation. *International Economic Review*, 35 (3), 705-730. https://doi.org/10.2307/2527081

Hart, O.D. & Jaffee, D.M. (1974) On the Application of Portfolio Theory to Depositary Financial Intermediaries. *The Review of Economic Studies*, 41 (1), 129-147. https://doi.org/10.2307/2296404

Hassler, Uwe (2003) Zeitabhängige Volatilität und instationäre Zeitreihen: Zum Nobelpreis an Robert F. Engle und Clive W. J. Granger. *Wirtschaftsdienst*, 83 (12), 811-816. Springer, Heidelberg. ISSN 0043-6275. https://www.econstor.eu/bitstream/10419/42107/1/376615559.pdf

Heracleous, Maria S. (2003) *Volatility Modeling Using the Student's t Distribution*. Dissertation, Faculty of the Virginia Polytechnic Institute and State University. Blacksburg, Virginia.

Heracleous, Maria S. (2007) *Sample Kurtosis, GARCH-t and the Degreees of Freedom Issue*. European University Institute EUI Working Paper ECO 2007/60. ISSN 1725-6704. http://hdl.handle.net/1814/7693

Koether, Paul (2005) *GARCH-like Models with Dynamic Crash-Probabilities*. Dissertation am Fachbereich Mathematik der Technischen Universität Kaiserslautern. https://kluedo.ub.uni-kl.de/frontdoor/deliver/index/docId/1671/file/doc.pdf

Kwon, J.H. (2021) On the factors of Bitcoin’s value at risk. *Financial Innovation*, 7 (87). https://doi.org/10.1186/s40854-021-00297-3

Laurent, S., Rombouts, J. V., & Violante, F. (2012) On the forecasting accuracy of multivariate GARCH models. Journal of Applied Econometrics, 27 (6), 934-955. https://doi.org/10.1002/jae.1248

Liu, Jinjing (2016): *A New Tail-based Correlation*. McGill University.

Liu, Jinjing (2021): *A New Tail-Based Correlation Measure and Its Application in Global Equity Markets* http://dx.doi.org/10.2139/ssrn.3457532

Liu, S., Qi, H., Shi, J. & Xie, Y. A. (2015) Inferring Default Correlation from Equity Return Correlation. *European Financial Management*, 21(2), 333-359. https://doi.org/10.1111/j.1468-036X.2013.12016.x

Lopez, Jose A. (2002) *The Empirical Relationship between Average Asset Correlation, Firm Probability of Default and Asset Size*. FRB of San Francisco Working Paper No. 2002-05, Federal Reserve bank of San Francisco. http://dx.doi.org/10.2139/ssrn.713721

Nakamoto, Satoshi (2009) *Bitcoin: A Peer-to-Peer Electronic Cash System*. https://nakamotoinstitute.org/static/docs/bitcoin.pdf

Nakatani, Tomoaki & Teräsvirta, Timo (2008) Positivity constraints on the conditional variances in the family of conditional correlation GARCH models. *Finance Research Letters*, 5 (2), 88-95. https://doi.org/10.1016/j.frl.2008.02.001

Nelson, Daniel B. (1991) Conditional Heteroscedasticity in Asset Returns: A New Approach. *Econometrica*, 59 (2), 347-370. https://doi.org/10.2307/2938260

Paschke, Raphael (2009) *Banking Regulation with Value-at-Risk*. Dissertation, Universität Mannheim. https://madoc.bib.uni-mannheim.de/2655

Pyle, David H. (1971) On the theory of financial intermediation. *Journal of Finance*, 26 (3): 737-747. https://doi.org/10.1111/j.1540-6261.1971.tb01727.x

Read, Oliver (1998) *Parametrische Modelle zur Ermittlung des Value-at-Risk*. Dissertation, Wirtschafts- und Sozialwissenschaftliche Fakultät, Universität zu Köln.

Schoffer, Olaf (2003) *Modellierung von Kapitalmarktrenditen mittels asymmetrischer GARCH-Modelle*. Dissertation, Fachbereich Statistik, Technische Universität Dortmund. https://doi.org/10.17877/DE290R-14164

De Servigny, Arnaud & Renault, Olivier (2002) *Default correlation: Empirical evidence*. Imperial College London.

Sukono, F., Wahyuddin, W., Nurhasanah, N., Saputra, J. & Hidayat, Y. (2019) Modeling of Modified Value-At-Risk for the Skew Student-T Distribution. Opción, 35 (89), 932-957.

Tayefi, Maryam & Ramanathan, T.V. (2012) An Overview of FIGARCH and Related Time Series Models. *Austrian Journal of Statistics*, 41 (3), 175–196. http://www.stat.tugraz.at/AJS/ausg123/123Tayefi.pdf

Trottier, D.A. & Ardia, D. (2016) Moments of standardized Fernandez-Steel skewed distributions: Applications to the estimation of GARCH-type models. *Finance Research Letters*, 18, 311–316. https://doi.org/10.1016/j.frl.2016.05.006

Yoon, Seong-Min & Kang, Sang-Hoon (2007) A Skew Student-T Value-at-Risk Approach for Long Memory Volatility Processes in Japanese Financial Markets.*Journal of International Economic Studies*, 11 (1), 211-242. http://dx.doi.org/10.2139/ssrn.3078326

Zakoian, Jean-Michel (1994) Threshold heteroscedastic models. *Journal of Economic Dynamics and Control*, 18 (5), 931-955. https://doi.org/10.1016/0165-1889(94)90039-6

Zeng B. & Zhang, J. (2002) *Measuring credit correlations: equity correlations are not enough !* Working Paper, KMV corporation.

Zhang, J., Zhu, F. & Lee, J. (2008) *Asset Correlation, Realized Default Correlation, and Portfolio Credit Risk*. Moody's KMV Company. https://www.moodysanalytics.com/-/media/whitepaper/before-2011/03-03-08-asset-correlation-realized-default-correlation-and-portfolio-credit-risk.pdf

Zhou, Chunsheng (1997) *Default Correlation: An Analytical Result*. Working Paper, Finance and Economics Discussion Series (FEDS), Federal Reserve Board. https://www.federalreserve.gov/econres/feds/default-correlation-an-analytical-result.htm