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Economic Hysteresis with Multiple Inputs
– a Simplified Treatment

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Abstract

Hysteresis in economics is up to now usually based on a representation of a system with only a single input variable, which has a persistent effect on an economic outcome (i.e. the output variable). However, in general there is more than one factor influencing economic decision problems, why the description of the path-dependency in relation to only one input variable may (possibly) be insufficient. The multidimensional path-dependence phenomenon is addressed (in mathematics and physics) by a vector-hysteresis system, with an input vector of two ore more variables. Unfortunately, these models are quite complicated for practical purposes in economics. However, since standard economic decisions are based on comparing economic values of alternatives (e.g. present values of investments), this can be used to reduce the dimensions of the hysteresis system. This paper outlines how the influence of several original input variables (e.g. price level and interest rate) is captured by the resulting variations of the present value of an investment. This economic value then can be used as a single signal/input variable of a modified hysteresis system. Since this system is dimensionally reduced to the standard hysteresis case with only a single input variable, the standard aggregation procedure for a situation with heterogeneous agents can be applied again.

JEL Classifications: C61.

Keywords:
vector-hysteresis; sunk-cost hysteresis; path-dependence; non-ideal relay; Mayergoyz/Preisach-model.

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Economic Hysteresis with Multiple Inputs – a Simplified Treatment

1. Introduction

'Hysteresis' – originally stemming from magnetism – describes a permanent effect of a temporary stimulus. Relations between economic variables are often characterized by circumstances where initial conditions and the past realizations of economic variables matter. I.e. past (transient) disturbances of the relevant economic determinants (technically: the input or forcing variables) and past states of the economic system do often have a permanent influence on the current economic outcome (the output or dependent variable).[1] Hysteresis in economics is typically based on sunk adjustment costs, which may occur either on the demand (as in the case of hiring and firing costs affecting labour demand) or on the supply side of the market (as e.g. entry or exit costs on international export markets).[2]

Analyzing hysteresis usually starts with describing the path-dependent behavioural pattern at the micro level of a single unit: e.g. the hysteretic supply-pattern of a single firm, being active on a market or not – under consideration of having spent sunk costs in the past. At the micro level the path-dependent switching of the activity status occurs at firm-specific triggers. However, aggregation over a multiplicity of heterogeneous agents/firms is not straightforward. The path-dependent pattern of the aggregate economic system (as may be known from the magnetic hysteresis-loop of an entire piece of iron compared to single iron crystals) is not characterised by discontinuous switches (between activity and inactivity), but by a smooth/continuous transition between different "branches" of the input-output-relation, which occurs every time the direction of the movement of the forcing variable alters.

The standard model representation of hysteresis in economics is based on a system with one input and one output variable, e.g. price level as an input and production activity as the output variable to describe supply hysteresis. However, the standard situation in economic relations is based on a multitude of input factors having an influence on the output – which may be

1 Cross/Allan, 1988, p. 26. The terms 'input' and 'output' are used to address the processing of the system and *not* used in a narrow economic sense (as e.g. factor input and production output).

2 In labour economics hysteresis was introduced by Phelps (1972), Sachs (1986), Blanchard/Summers (1986), and Lindbeck/Snower (1986), and in foreign trade theory by Kemp/Wan (1974), Baldwin (1989), Baldwin/Krugman (1989), and Dixit (1989). For an overview of applications and different concepts of hysteresis in economics see the surveys of Cross (1993), Cross et al. (2009), Göcke (2002) and Belke et al. (2014).

path-dependent. Thus, being able to model only path-dependent hysteretic relations related to a single (one-dimensional) input variable, applying 2D-representations if the output variable is included, restricts the application of the concept of hysteresis to more general economic problems. Modelling vector-hysteresis based on more-dimensional inputs was already addressed in the mathematics and physics (see e.g. Mayergoyz, 1988). Especially the properties of the non-ideal relay with 2 inputs and the consequences for the aggregation procedure were analyzed by Belbas/Kim (2010). While the application of multidimensional vector-hysteresis is quite tractable for single agents on a microeconomic level – since it shows a simple pattern of a non-ideal relay– the application of an adequate aggregation procedure becomes complicated if the agents are heterogeneous.

The aim of this paper is to show, that this complex multi-dimensional aggregation procedure can be avoided in a standard situation of hysteresis in economics. This is due to the fact, that for economic problems typically – in order to decide about the optimal strategy – a comparison of the *economic values* of feasible alternatives is executed. And the economic value is a scalar (generally measured in currency units)! Thus, even though there is more than one determinant behind the optimal decision, the effects of this multitude of inputs/determinants can be described by the consequences on the relevant economic value – as a single variable! E.g. this paper presents an example where the *present value* of an investment as a function of two inputs, the future revenue/price level (p_t) and the interest rate (i_t), is used as a "reduced" single signal capturing the effects of both original input variables on the economic decision problem. And referring to the present value as a single (one-dimensional) signal, a modified version of the standard aggregation procedure can be applied.

The structure of the paper is as follows. In the next section (2.) a simple model of microeconomic sunk cost hysteresis is presented to show the non-ideal relay property of the supply of a single firm. This path-dependent pattern is explicitly outlined in relation to price level and to the interest rate as different factors having an influence on this decision. Section 3 gives an intuition of the problems coming about via a vector-hysteresis situation with more than one input variable, resulting in (for practical purposes too) complicated aggregation procedures if the firms are heterogeneous. The following two chapters (4 and 5) present a simplification of the problem by using the present value as a scalar signal instead of the two-dimensional input vector comprising the price level and the interest rate. Moreover, in a second step (in section 5) a procedure based on the so called '*play*-operator' is presented in order to filter out a third factor of influence, the option value effect caused by stochastic uncertainty about future profits. The modified version of the standard aggregation procedure based on the (filtered) present value as a single input signal related to the aggregate output is outlined in section 6. Section 7 concludes.

2. Sunk cost hysteresis and non-ideal relays with a one-dimensional input

2.1 Sunk cost hysteresis in situations with no uncertainty

A change in the level of the relevant forcing variables typically induces a change of the economic behaviour/outcome. However, a change back to the initial levels of the forcing variables does in the case of hysteresis not induce a complete change back to the initial outcome. A typical mechanism behind this path-dependency is driven by *sunk adjustment costs* (Baldwin, 1989, 1990). A firm which is previously not selling on a market and intends to enter has to bear market entry costs, e.g. for setting up distribution networks or for advertising. These costs are firm and market specific and cannot be regained after market entry. Ex-post, these market entry 'investments' are sunk. A market entry is only profitable, if both, variable costs and sunk entry costs, are covered by revenues. If a temporary high market price in the past has led to a market entry, a subsequent price decrease back to the initial level will not induce a market exit – as long as the variable costs are covered. Summarizing, the same price level may result in different states of the firm's activity, depending on the history of its activity. In the following a simple model of this microeconomic supply pattern is presented.

A price-taking firm j decides in period t whether or not to supply one unit of a product ($x_{j,t} \in \{0,1\}$). Selling the product, the firm receives the market price p_t as a unit revenue. Two different components of costs have to be paid. Based on using capital as an input factor, the interest rate i_t has to be paid on the firm's capital stock K_j (≥ 0). Additionally, if the firm has not produced in the preceding period, it has to pay the starting costs H_j (≥ 0). The value created by H_j is completely firm specific and decays immediately as soon as the firm does not produce and sell. Thus, H_j represents the adjustment sunk costs. If the firm was inactive in the preceding period ($x_{j,t-1} = 0$) it has to pay both components. If it has been active in the preceding period ($x_{j,t-1} = 1$) only the interest costs are relevant. In the case of a market entry in period t, the profit $R_{j,t}$ in t and in the subsequent periods $t+\tau$ is given by:

(1) $$R_{j,t} = p_t - i_t \cdot K_j - H_j \qquad \text{and} \qquad R_{j,t+\tau} = p_{t+\tau} - i_{t+\tau} \cdot K_j \qquad (\tau > 0)$$

As a simple example we assume the firm is expecting a constant level of the price and the interest rate for the whole *infinite future* ($p_t = p_{t+\tau}$ and $i_t = i_{t+\tau}$, for all $\tau > 0$), that is anticipated with certainty. Thus, in the case of activity the (expected) present value of future revenues as an annuity is:

$$(2) \qquad V_t \equiv \frac{p_t}{i_t}$$

Under certainty the present value of revenues has to cover (at least) the value of the capital stock K_j plus the sunk entry costs ($V_t > K_j + H_j$) to make an entry a profitable investment. Solving ($V_t = K_j + H_j$) leads to the firm's entry trigger price α_j under certainty:

$$(3) \qquad \alpha_j = i_t \cdot (K_j + H_j) \qquad\qquad \text{entry if } p_t > \alpha_j$$

Therefore, the price has to cover at least the interest costs on both, capital stock K_j and sunk entry costs H_j.

If the firm was active in the preceding period ($x_{j,t\text{-}1} = 1$) it will leave the market and sell the production capital K_j, if the price is too low. However, it has to pay sunk exit cost F_j (≥ 0), e.g. for writing off firm-specific parts of the capital stock, or for severance payments. Thus, an exit of the firm is optimal if ($V_t < K_j - F_j$), and the exit trigger price β_j is:

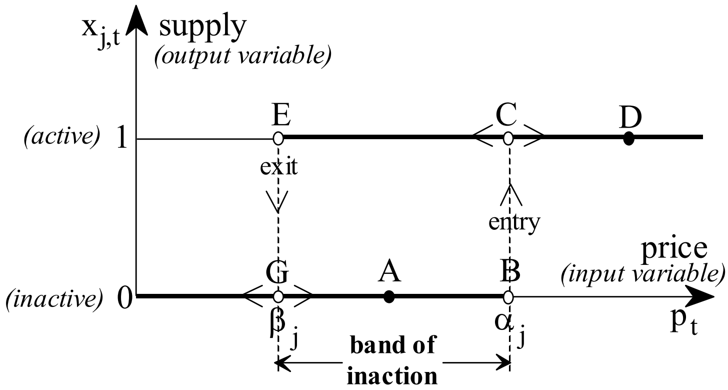
$$(4) \qquad \beta_j = i_t \cdot (K_j - F_j) \qquad\qquad \text{exit if } p_t < \beta_j$$

In the following (for reasons of a simple graphical representation) we assume ($K_j \geq F_j$), so that the exit trigger price is non-negative. For a constant interest rate, an (unexpected) change in the (current and future) price p_t results in a supply-pattern of the price-taking firm j which is described by a so called 'non-ideal relay':[3]

$$(5) \qquad x_{j,t} = \begin{cases} 1 & \text{if } \ (x_{j,t\text{-}1} = 0 \ \wedge \ p_t > \alpha_j) \\ 1 & \text{if } \ (x_{j,t\text{-}1} = 1 \ \wedge \ p_t \geq \beta_j) \\ 0 & \text{if } \ (x_{j,t\text{-}1} = 0 \ \wedge \ p_t \leq \alpha_j) \\ 0 & \text{if } \ (x_{j,t\text{-}1} = 1 \ \wedge \ p_t < \beta_j) \end{cases} \qquad\qquad \text{with } \alpha_j \geq \beta_j$$

[3] See Krasnosel'skii/Pokrovskii (1989, p. 263, and p. 271) and Brokate/Sprekels (1996, pp. 23 f.) for a general description of relay-hysteresis.

Fig. 1: Supply according to a 'non-ideal relay' related to the price as the input variable



A non-ideal relay describes a path-dependent multiple-equilibria characteristic. E.g., starting in an inactivity situation at point A (Fig. 1) a price increase exceeding the trigger α_j induces a market entry, i.e. a "jump" from the (x_j=0)-inactivity-line to the (x_j=1)-activity-line (point C). A later price decrease is resulting in a market exit (point E), only if the price falls below the exit trigger β_j. A switch from one equilibrium-branch to the other takes place when the triggers are passed – otherwise the activity status remains the same. Therefore, the area GB (or CE) can be described as a *'band of inaction'* or 'hysteresis-band' (Baldwin, 1989, pp. 7 f.; Baldwin/Lyons, 1989, p. 11.). Dependent on the past, two different equilibria are possible: The current level of the input variable (price) does not unambiguously determine the current state of the output/dependent variable (firm's activity). If a *temporary* change of the input variable results in a switch between these equilibria, a *permanent* effect on the output variable remains (called '*remanence*'). This after-effect is the constituting feature of hysteresis.

Up to now, the price level was implicitly assumed as the single input variable of the system, and the entry trigger condition ($V_t = K_j + H_j$) was solved for the price in order to derive price triggers. However, if alternatively the interest rate is assumed to be the single input variable, the entry condition can be solved for the interest rate, and an entry trigger interest rate A_j can be calculated:

(6) $$A_j = \frac{p_t}{K_j + H_j} \qquad \text{entry if } i_t < A_j$$

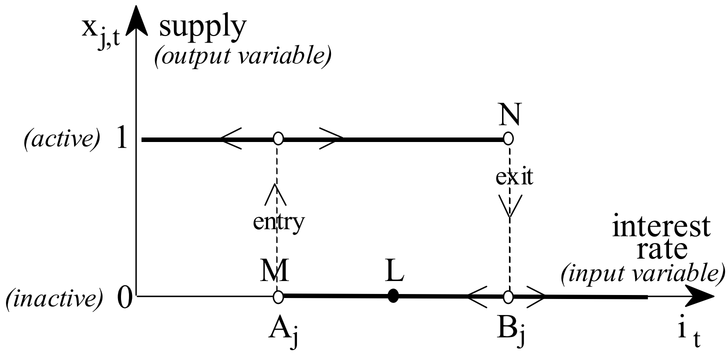
A low interest rate results in low capital costs for K_j and H_j and in a high present value of future profits making an entry profitable. A similar calculation could be done for the exit condition, determining the exit trigger rate B_j for a situation with a high interest rate:

$$B_j = \frac{p_t}{K_j - F_j} \qquad \text{exit if } i_t > B_j \tag{7}$$

Thus, for a constant price level, an (unexpected) change of the interest rate i_t analogously results in a 'non-ideal relay', however, now with respect to the interest rate as the (single) input variable:

$$x_{j,t} = \begin{cases} 1 & \text{if } (x_{j,t-1} = 0 \wedge i_t < A_j) \\ 1 & \text{if } (x_{j,t-1} = 1 \wedge i_t \leq B_j) \\ 0 & \text{if } (x_{j,t-1} = 0 \wedge i_t \geq A_j) \\ 0 & \text{if } (x_{j,t-1} = 1 \wedge i_t > B_j) \end{cases} \qquad \text{with } A_j \leq B_j \tag{8}$$

Fig. 2: 'Non-ideal relay' related to the interest rate as the input variable



Again there is a difference between an entry and exit trigger level of the input variable resulting in a range where path-dependent multiple equilibria exist (i.e. an interest rate related "band-of-inaction" between A_j and B_j). This can be demonstrated by a loop starting in point L, and then a decreasing interest rate, reducing interest costs making an entry profitable at point M, while a later increasing interest rate will only result in an exit (point N), if the exit trigger rate B_j is exceeded.

2.2 Non-ideal relay in a situation with stochastic input changes

A firm's market entry decision leading to sunk costs can be understood as an irreversible investment. Thus, in a situation with uncertainty, e.g. due to expected stochastic changes in future prices, a real option approach applies (Pindyck, 1988,1991; Dixit, 1989; Bentolila/Bertola, 1990; Belke/Göcke, 1999). A currently inactive firm has to decide whether to enter the market now or not, including the option to enter later. The option to decide on the entry in the future limits the risk by a "wait-and-see" strategy. A firm staying passive can avoid future losses if the stochastic future price will be unfavourable. An instantaneous entry kills the option to "wait-and-see" and to enter later if the future price movement will be favourable. Thus in addition to the sunk costs, the option value of waiting has to be covered in order to trigger an entry.

The option value effects on the trigger levels will be demonstrated with a very simplistic example: assume a single non-recurring stochastic change of the price, which can be either positive (+ε) or negative (–ε) (and $\varepsilon \geq 0$), with a probability of ½ for each of both realisations: $p_{t+1} = p_t \pm \varepsilon$ and $E_t(p_{t+1}) = p_t$. From period t+1 on, the firm will decide under certainty again. Instead of deciding to enter now or not never, in a situation with uncertainty the option to wait and make the entry decision in the future has to be taken into account. If the future price level turns out to be favourable (+ε) the firm can still enter the market in the next period. However, by staying passive, future losses can be avoided if the price change will be negative (–ε). Waiting and staying inactive implies zero profits in the current period t. Conditional on a (+ε)-realisation, the firm will use its option to enter in t+1, causing discounted entry costs and gaining the annuity $(p_t + \varepsilon - i_t \cdot K_j)$. In case of a (–ε)-realisation the firm will remain passive in the future. Consequently, the expected present value of the wait-and-see strategy is given by $E(WN_{j,t})$ in eq. (9). Corresponding to eqs. (1) and (2) the expected present value of an immediate entry (without a re-exit) is $E(N_{j,t})$:

(9) $$E(WN_{j,t}) = \frac{1}{2} \cdot \frac{1}{1+i_t} \cdot \left(\frac{p_t + \varepsilon}{i_t} - H_j \right) \quad \text{and} \quad E(N_{j,t}) = \frac{p_t}{i_t} - K_j - H_j$$

The entry-trigger price level α_j in the case of uncertainty is determined by a situation of indifference between immediate entry and wait-and-see, i.e if $E(WN_{j,t}) = E_t(N_{j,t})$:

(10) $$\alpha_j = i_t \cdot (K_j + H_j) + \frac{\varepsilon}{1 + 2 \cdot i_t} \qquad \text{in period t: entry if } p_t > \alpha_j$$

The decision problem of a currently active firm, deciding to leave the market now or to stay active, with an option to exit later if an unfavourable (–ε) price change will occur, is analogous. Here, the option value of waiting is based on avoiding sunk exit costs if the future price level turns out to be better. To remain active and to wait in period t results in a current profit of $(p_t - i_t \cdot K_j)$. Conditional on a (–ε)-realisation, the firm will use its option to exit in t+1 causing discounted exit costs. For a (+ε)-realisation the firm will continue to stay in the market with a future annuity of $(p_t + \varepsilon - i_t \cdot K_j)$. The resulting expected present value of the wait-and-see strategy is $E(WX_{j,t})$, and the expected present value of an immediate exit (without a re-entry) is $E(X_{j,t})$:

(11) $$E(WX_{j,t}) = \frac{1}{1+i_t} \cdot \left(p_t - i_t \cdot K_j + \frac{p_t + \varepsilon - i_t \cdot K_j}{2 \cdot i_t} - \frac{F_j}{2} \right) \qquad \text{and} \qquad E_t(X_{j,t}) = -F_j$$

The exit-trigger price level β_j in the case of uncertainty can be calculated for a situation of indifference between immediate exit in t and wait-and-see, i.e $E(WX_{j,t}) = E_t(X_{j,t})$:

(12) $$\beta_j = i_t \cdot (K_j - F_j) - \frac{\varepsilon}{1 + 2 \cdot i_t} \qquad \text{in period t: exit if } p_t < \beta_j$$

Thus, the entry trigger price α_j is under uncertainty augmented by the term $[+\varepsilon/(1+2 \cdot i)]$, and the option value effect for the exit trigger price β_j is of the same size, but negative. As a consequence, the ('band-of-inaction')-range between both triggers β_j and α_j where two path-dependent equilibria are possible is widened by uncertainty. However, when considering a situation with uncertainty, the qualitative non-ideal-relay property of microeconomic hysteresis has not changed.

A qualitatively similar widening effect on the 'band of inaction' would result, if uncertainty is not based on stochastic price changes, but on stochastic future changes of the interest rate. Moreover, the same consequence of widening the ('band-of-inaction')-range results due to uncertainty induced option value effects for the distance between the entry and exit trigger interest rate A_j and B_j. We refrain from presenting the explicit calculations for all these analogous cases, as explicitly modelling option value effects on the width of the non-ideal relay is not the focus of this paper.[4]

3. Consequences for the case of a non-ideal relay with two inputs

Up to now, the non-ideal relay characteristic was derived for only one-dimensional changes of the inputs – of either the price or the interest rate. However, both input variables may alter at the same time. Actually, the decision of the firm is simultaneously depending on both of the two input variables, price and interest rate. Thus, we have an example of so called *'vector hysteresis'* with a two-dimensional input vector, i.e. the non-ideal relay is actually two-dimensional with respect to the inputs (or even 3D, if the output dimension is included).

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4 For a comprehensive treatment of uncertainty effects see Dixit/Pindyck (1994).

Since more than one factor of influence is standard in economic relations, being able to model only path-dependent hysteretic relations related to a one-dimensional input variable (and if output is included, only applying 2D-representations of the entire input-output system) would be a severe restriction, even in a simple standard case of sunk cost hysteresis as outlined above. Modelling vector-hysteresis based on more-dimensional input-vectors was already done in the theory of mathematics and physics (see e.g. Mayergoyz, 1988). Especially the properties of the non-ideal relay with a 2-dimensional input-vector and the resulting consequences for the aggregation procedure were analyzed by Belbas/Kim (2010). In the simple case of a single scalar-valued input, the value of the output variable is switched when the input signal crosses one or the other of two threshold values (e.g. for the price-based relay, α_j and β_j). The adequate aggregation procedure for a multitude of heterogeneous individual non-ideal relays j, each relay having a different pair of threshold parameters (α_j,β_j) is done by a continuous superposition of this collection of relays: i.e. the so-called '*Mayergoyz/Preisach'*-procedure, which will be explicitly outlined later in this paper in section 6.

In the single input case the two thresholds (α_j,β_j) are scalars. However, in the generalized situation with two simultaneous inputs, the (α_j,β_j)-triggers are replaced by a pair of two threshold-*curves* on the two-variable-plane of both inputs. Analogous to an exit trigger scalar α_j in the 2-dimensional input-vector case there is a entry trigger curve $\gamma_{0,j}(p_t,i_t)$ and another curve $\gamma_{1,j}(p_t,i_t)$ as an analogy/generalisation of the exit trigger β_j. Having to deal with (heterogeneous) individual pairs of trigger curves instead of heterogeneous trigger scalars makes the aggregation procedure more complicated compared to the single input situation. As it is shown later on in section 6, an aggregated system has a different hysteresis pattern. For the single-input case the aggregate system shows a permanent ('remanence') effect resulting for every extremum of this input variable, via smooth/continuous switches between different branches [and not by a discontinuous (0-1)-"jump" as for relay-hysteresis]. Of course, it is not straightforward to identify an "extremum" for a pair of curves in the two-inputs-case. Moreover, the memory characteristics of the aggregate system (e.g. the so called *'wiping-out'* property of subsequent even "more extreme" inputs) is different, as it is shown by Belbas/Kim (2010) for the two-inputs-generalisation of the Mayergoyz/Preisach-model.

Indeed, in the following we will outline that the complex aggregation procedure for two-dimensional input-vector could be avoided in a typical economic situation. This is due to the fact, that comparing the *economic values* of feasible alternatives typically enables the decision about the optimal strategy for economic problems. Thus, even if there is more than one determinant affecting the optimal decision, the effects of this multitude of inputs can be described by the consequences on the economic value (measured in units of currency). In the following the *present value* – as a function of revenues (p_t) and interest rate (i_t) – works as "reduced" single signal capturing the effects of both original input variables on the economic decision problem. And with reference to this single signal a modified version of the standard aggregation procedure can be applied. In the next two chapters the "reduced" non-ideal relay system with the present value as the only input variable is presented. Moreover, in a second step, a filtering procedure based on the so called '*play*-operator' is presented for capturing a third factor of influence, the option value effect caused by stochastic uncertainty about future profits.

4. Non-ideal relay related to the present value as a signal

As usual in economic problems, values are compared in order to receive an optimal decision (in our example the present value of the annuity of profits compared with sunk entry or exit costs), the relevant signal function in many (sunk cost) investment decisions can be the same single (!) present value function, though two (!) trigger functions are addressed.

The immediate entry trigger condition $[E(WN_{j,t})=E_t(N_{j,t})]$ can be rearranged to receive:

$$\text{(13)} \qquad \frac{p_t}{i_t} = K_j + H_j + \frac{\varepsilon}{i_t \cdot (1+2 \cdot i_t)}$$

with $V_t \equiv \frac{p_t}{i_t}$ and $OE \equiv \frac{\varepsilon}{i_t \cdot (1+2 \cdot i_t)}$ in period t: entry if $V_t > K_j + H_j + OE$

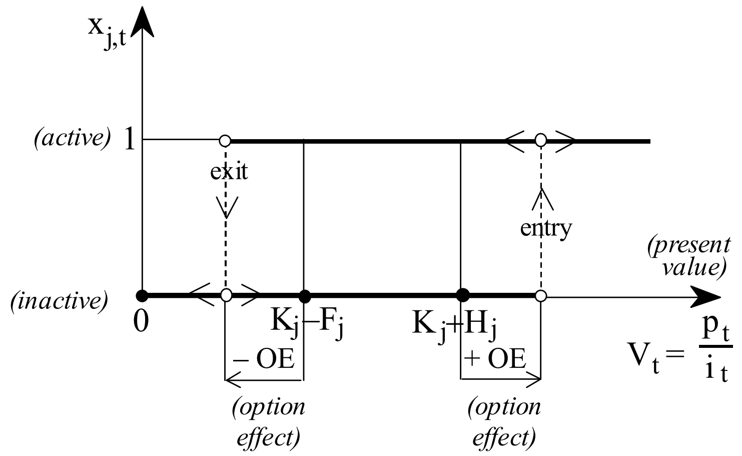
An entry in the current period is profitable if the present value of the revenues $V_{j,t}$ covers the capital K_j, plus sunk entry costs H_j, plus an option value effect OE. Analogously, the immediate exit trigger condition $[E(WX_{j,t})=-F_j]$ is:

$$\text{(14)} \qquad V_t = K_j - F_j - \frac{\varepsilon}{i_t \cdot (1+2 \cdot i_t)}$$

in period t: exit if $V_t < K_j - F_j - OE$

Using the present value as a function of price p_t and interest rate i_t, the two dimensional input problem is simplified and reduced to a one-dimensional non-ideal relay, again (see Fig. 3).

Fig. 3: ‘Non-ideal relay’ related to the present value of revenues



5. Play-operator applied to receive a simple single signal function

Above it was shown how the present value as a single variable is able to represent both trigger functions, for the entry and the exit trigger. However, the option value effect OE is still obstructing a simple representation, since there is still an interest rate effect on the RHS of eqs. (13) and (14) included in OE. These immediate entry/exit trigger conditions can be rearranged:

(15) $V_t - OE = K_j + H_j$ entry in t if $V_t - OE > K_j + H_j$

$V_t + OE = K_j - F_j$ exit in t if $V_t + OE < K_j - F_j$

The band of inaction related to the present value is:

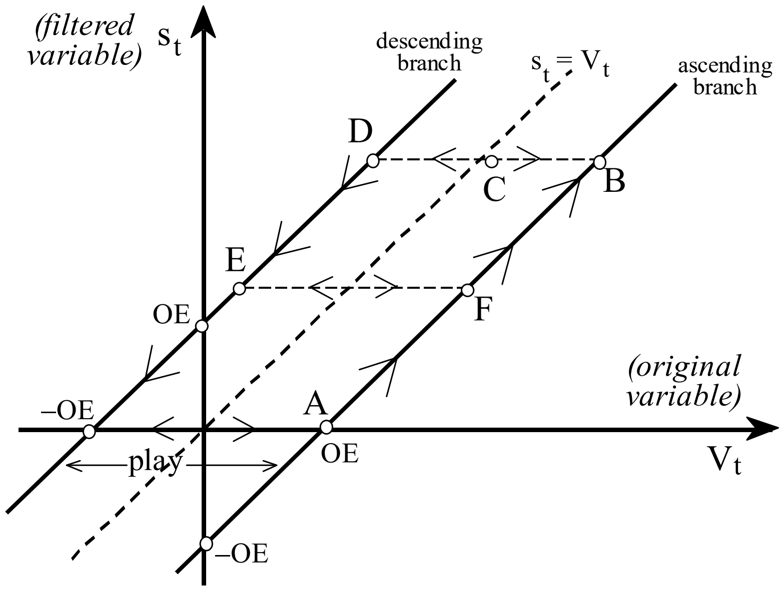
(16) $-F_j + K_j \; < \; V_t - OE \; \leq \; V_t + OE \; < \; K_j + H_j$

Thus in the case of a decreasing present value the relevant signal function comprising price and interest rate effects is $(V_t - OE)$, while it is $(V_t + OE)$ in the case of an increasing present value. Actually this signal function can be interpreted as a *play-(hysteresis-)operator* procedure on V_t with a play-width of $(2\cdot OE)$.[5] The play-operator is filtering out small changes in the original variable V_t, and only large changes exceeding the play area are transmitted to the resulting filtered variable s_t. The signal function s_t is derived from the present value V_t by the following play-operator:

$$(17)\qquad s_t = \begin{cases} V_t - OE & \text{if } \ (V_t \geq V_{t-1} \ \wedge \ V_t > s_{t-1} + OE) \ \ \textit{(ascending branch)} \\ s_{t-1} & \text{if } \ (s_{t-1} - OE \leq V_t \leq s_{t-1} + OE) \ \ \textit{(play area)} \\ V_t + OE & \text{if } \ (V_t \leq V_{t-1} \ \wedge \ V_t < s_{t-1} - OE) \ \ \textit{(descending branch)} \end{cases}$$

For changes in the current present value V_t the 'play-hysteresis' loop of the signal s_t as depicted in Fig. 4 results. If – starting from the origin – the present value V_t increases, the signal s_t at first does not react due to the option value effect OE. If a threshold is exceeded (OE at point A), the signal reacts according to line AFB along the 'ascending branch'. If V_t, and thus s_t, rises up to point B and falls later on, again the signal shows no reaction, but 'play' occurs along the line BCD. Only when a threshold is passed in point D, the signal decreases (to point E) with a lower V_t. As in the case of a non-ideal relay, for a particular input area no reaction of the output variable occurs.

Fig. 4: Play-operator deriving the filtered signal s_t from the original V_t



To be more precise, the option effect $OE \equiv \varepsilon / [i_t \cdot (1+2 \cdot i_t)]$ varies if the interest rate i_t is changing. If, e.g., the interest rate is increasing, OE is decreasing. This ceteris paribus results in a narrowing of the play-width, with a move of the 'ascending branch' to the left (closer to the ($s_t = V_t$)-line, and of the 'descending branch' to the right. If a position on the ascending branch was preexisting, the narrowing effect on the signal c.p. results in an increase of s_t,

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[5] A play-operator describes an effect which is well known as mechanical play or *'backlash'*. Car drivers know this if the steering wheel has to be turned by a small angle before the tires actually respond. In fact, play is another type of hysteresis, generalising the non-ideal-relay, since for play a continuous loop is resulting,

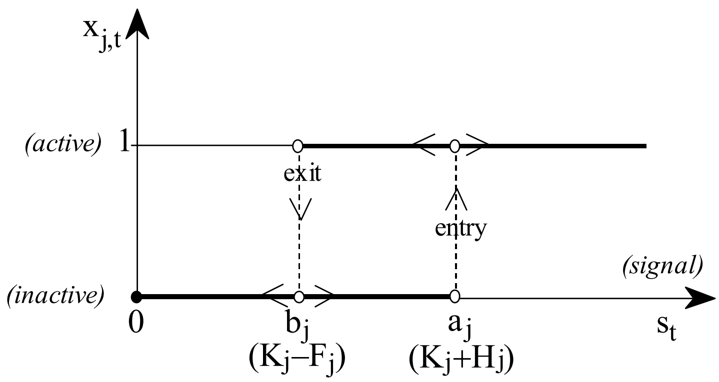
since the ascending branch moves to the left. If a position on the descending branch was preexisting, the narrowing effect is c.p. moving s_t downwards. However, in addition to the narrowing effect, an increase of the interest rate is ceteris paribus diminishing the present value V_t by nearly the same proportion.[6]

Using the filtered signal s_t instead of the original present value V_t eliminates the option value effect from the non-ideal relay representation of the firm's optimal decision (see Fig. 5). These trigger signals are defied as:

$$a_j \equiv K_j + H_j \qquad \text{entry in t if } s_t > a_j \tag{18}$$

$$b_j \equiv K_j - F_j \qquad \text{exit in t if } s_t < b_j$$

Fig. 5: 'Non-ideal relay' related to the signal (where OE is filtered out)



In the simple case of no (anticipated) uncertainty (i.e. $\varepsilon = 0 \Rightarrow OE = 0$), with a zero "play-width" the signal s_t is similar to the original variable V_t. Thus, the simple case without uncertainty is in the following implicitly included as a special/border case of the more general presentation.

6. Aggregation and macroeconomic hysteresis

In the following the application of the Mayergoyz (1986)-Preisach (1935)-procedure is outlined – which is based on the explicit aggregation of non-ideal relay agents (j = 1, ..., n; n ≫

while the non-ideal relay is based on discontinuous "jumps". See Krasnosel'skii/Pokrovskii (1989)., pp. 6 ff. and Brokate/Sprekels (1996), p. 24 and p. 42 for a general treatment of the play-operator.

6 For a more elaborated treatment of a situation with a variable play-width, see Belke/Göcke (2001, 2005).

0) each of them having different entry/exit triggers due to a heterogeneity in the firms' cost structure.[7]

Every potentially active firm j is characterized by a set of entry/exit triggers (a_j/b_j). In an a_j/b_j-diagram (see Fig. 6), the firms are represented by points in a triangle area above the 45°-line (since $a_j \geq b_j$). The aggregation procedure can be performed without any serious restriction of the heterogeneity of the distribution of the firms over the triangle area (i.e. of the cost structure of the firms). Points on the 45°-line describe non-hysteretic firms ($H_j = 0 \wedge F_j = 0 \Rightarrow a_j = b_j$) – the horizontal distance from the origin given by K_j. Firms with a position above the 45°-($a = b$)-line are characterized by a non-ideal relay supply – the distance from the ($a = b$)-line measured in vertical direction determined by ($+H_j$), and in horizontal direction by ($–F_j$).

To avoid a long description of the past development, a situation with a zero initial signal level ($s_t = 0$) is assumed, implying no firm is initially active. Now, a rising signal results in market entries by firms with the lowest costs – i.e. the lowest entry triggers a_j. Aggregate supply increases, as traced in Fig. 6 (a), with a growing space of the hatched triangle S_t^+ representing the active firms which have entered the market (and S_t^- representing the inactive firms). For a rising signal level, the S_t^+-expansion is indicated by an upward shift of the horizontal borderline. The corresponding aggregate macro reaction is depicted by the path OAB in Fig. 7.

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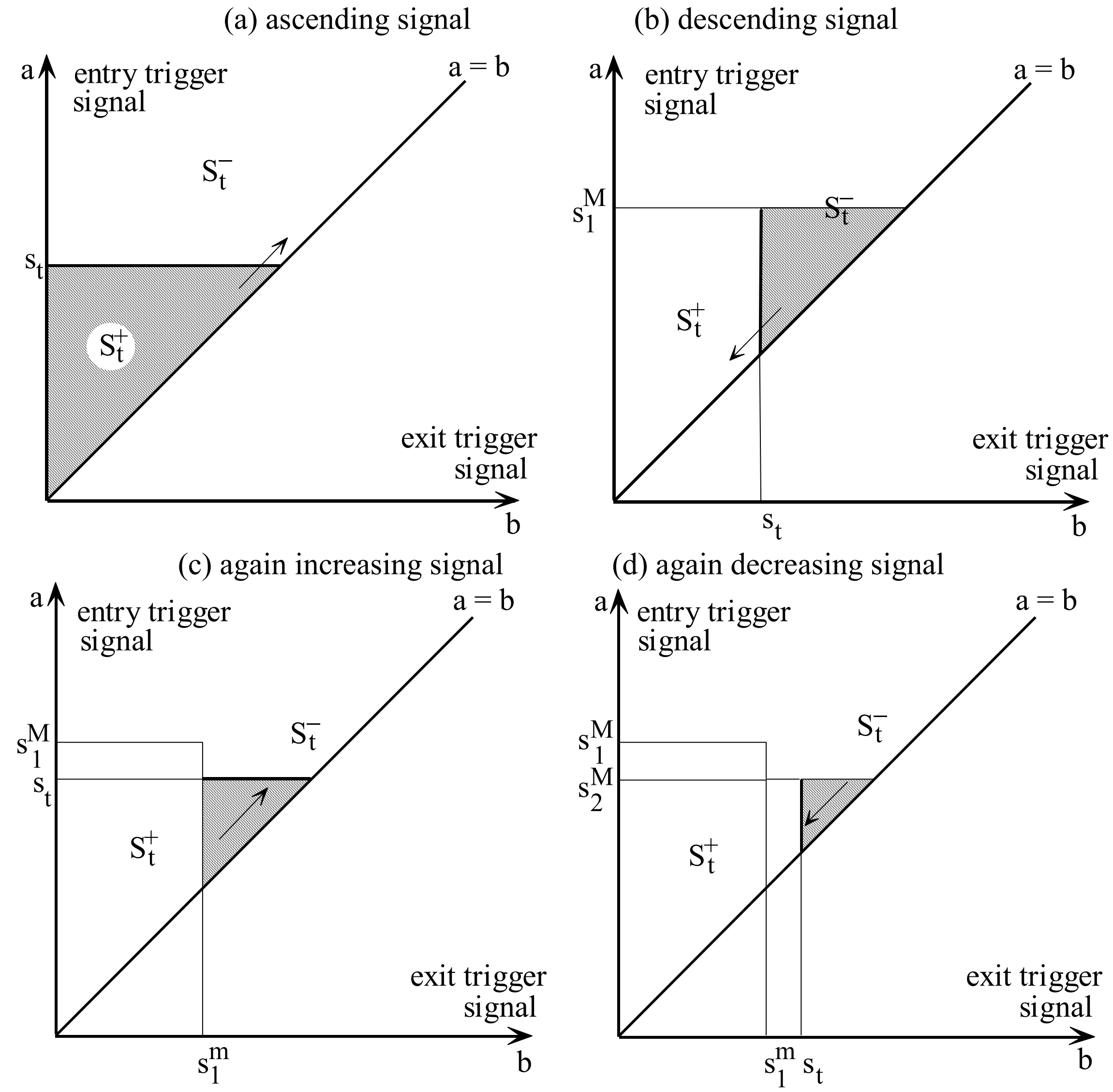
7 This procedure was introduced to economics by Amable et al. (1991) and Cross (1993). See e.g. Cross (1994), Göcke (1994), Piscitelli et al. (2000), and Mota/Vasconcelos (2012) for applications of the Preisach-Mayergoyz-model in foreign trade and in labour market economics.

Fig. 6: Application of the Mayergoyz/Preisach procedure – active firms under a volatile signal

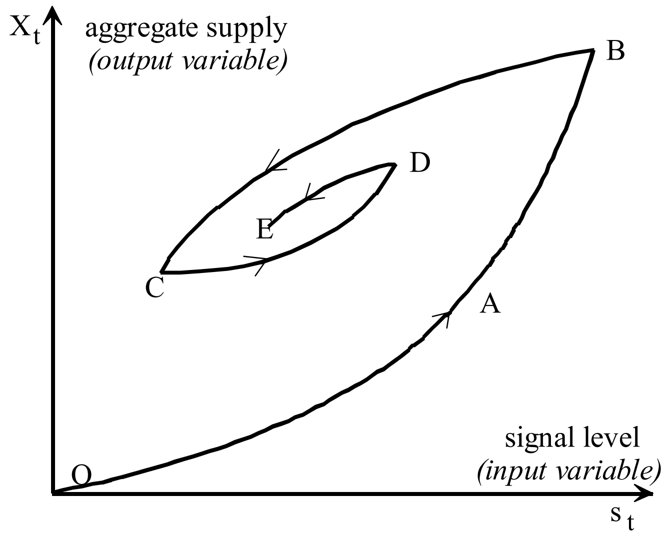


In Fig. 6 (b) a subsequent decrease of the signal is traced: s_t falls from the highest value, the (local) maximum s_1^M. Therefore, area S_t^+, representing active firms, now shrinks, since firms that have recently entered, leave the market as the signal falls below their exit trigger b_j. For a decreasing signal, the activity changes (hatched area) are illustrated by a left vertical shift of the S_t^--S_t^+-borderline. In Fig. 7 the corresponding path is BC.

If, after reaching the local minimum s_1^m the signal rises again, area S_t^+ again expands, depicted in Fig. 6 (c) by an upward shift of the right-horizontal part of the borderline. The corresponding macro reaction is path CD in Fig. 7. The result of the subsequent shifts is a "staircase-shape" of the border between the two parts of the triangle. If the recently reached (local) maximum is lower than the highest maximum s_1^M, a staircase step in the borderline remains – characterised by the coordinates ($a = s_1^M / b = s_1^m$). If the signal level had continued to increase and had passed the original maximum, the a-coordinate of the "s_1^M-step" would have been *"wiped out"* and replaced (Mayergoyz, 1986, p. 605). However, if (as traced in Fig. 6 (c)) the new local maximum is lower than the "old" s_1^M, this "old" maximum remains and the new local maximum becomes the second highest, labelled s_2^M.

Fig. 6 (d) illustrates a subsequent decrease in the signal. The borderline is changed by a shift to the left of the lower vertical part (path DE in Fig. 7). If s_t does not fall below s_1^m the new local minimum s_2^m is now the second lowest minimum. If the input were to fall under the "old" s_1^m, the b-coordinate of the corresponding staircase-step would be eliminated. If subsequent local maxima and minima are not as "extreme" as the preceding extrema, a new corner in the staircase border is created. However, local maxima which are higher than preceding maxima will erase the a-coordinate of the corresponding corners; subsequent local minima will '*wipe-out'* the b-coordinate of corners corresponding to higher preceding minima (Amable et al., 1991, pp. 11 ff.).

Fig. 7: The continuous macroeconomic hysteresis loop



Summarizing, the aggregate system shows a memory of non-erased (*'non-dominated'*) past input signal extrema – graphically represented by the "staircases" in the borderline of the area S_t^+ of active firms. Aggregation leads to a stronger pattern of hysteresis: For the aggregate loop a branch-to-branch transition occurs with *every local extremum* in the path of the input variable, while at the micro/firm level a passing of triggers is necessary, in order to induce permanent remanence effect. Therefore, this type of macro-hysteresis is called '*strong'* hysteresis (e.g. Amable et al.,1991, 1994; see Brokate/Sprekels,1996, pp. 22 ff., for typical characteristics of the macro hysteresis-loop). The *distribution* of the heterogeneous firms in the ($a_j \geq b_j$)-triangle is important for the results. A continuous distribution of the firms in the ($a_j \geq b_j$)-triangle implies a *continuous* macro loop as depicted in Fig. 7. The exact density of the (a_j,b_j)-distribution determines the curvature of the macro loop branches: the more they are clustered in a specific area (i.e. the less heterogeneous the firms), the more "curved" are the macro branches. In the borderline case of a multiplicity of similar/homogenous firms the macro loop degenerates to a non-ideal relay.

7. Conclusion

The purpose of this paper was to show how an economic hysteresis system based on persistence effects related to more than one input variable (i.e. a vector-hysteresis system), can be reduced to a simple system based on present values as the single input variable. This was shown for a standard example of sunk-cost hysteresis, where the firm's investment decision is based on (expected) future revenues. On a microeconomic level of a single firm, it was shown that (non-ideal relay) hysteresis can be derived related to the price level, but in an analogous way related to the interest rate as well. However, the present value – as a function based on the (future) levels of the price level and the interest rate – captures the effects of both original input variables on the decision problem. For a situation with uncertainty (due to the stochastic nature of future revenues), option value effects of waiting have additionally to be considered: These option value effects are widening the 'band-of-inaction' area of path-dependent multiple equilibria. However, if the present value is filtered by a play-operator procedure, these option value effects can be filtered out in order to derive a single signal/input variable capturing the dynamic effects of the present value. In a last step it is shown, how the usual Mayergoz/Preisach single-input-procedure for heterogeneous agents could be applied again, based on this reduced and filtered present value signal. Thus, for a typical economic decision situation, a case with multiple input (vector-)hysteresis was simplified to the standard single input variable case again.

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