Doktorarbeit

Finite dimensional Nichols algebras of diagonal type over fields of positive characteristic

zur Erlangung des Doktorgrades der Naturwissenschaften
dem Fachbereich Mathematik und Informatik
der Philips-Universität Marburg vorgelegt von
Jing wang aus China

Kandidatin:
Jing Wang

Doktorvater:
Prof. Dr. István Heckenberger

Abstract

In this thesis we classify the Nichols algebras of diagonal type with a finite root system over fields of positive characteristic. The thesis consists of five chapters. At the very beginning we give an introduction for the entire thesis. The notations of the Nichols algebra $\mathcal{B}(V)$ generated by a braided vector space V, the root system of $\mathcal{B}(V)$, and the Weyl groupoid of $\mathcal{B}(V)$ are introduced in Chapter 1 and so are some general results. In Chapter 2 we construct a semi-Cartan graph for Nichols algebras of diagonal type.

In Chapter 3 we obtain the main properties of the semi-Cartan graph which are needed for our main classification theorem in the next chapter. In the rank two case, we characterize finite Cartan graphs in terms of certain integer sequences in Theorem 3.1.14. In the rank three case, we obtain the possible reflections in the finite Cartan graph by analysing the possible 55 finite root systems in [15, Theorem 4.1]. To illustrate the possible reflections we give the definitions of the good A_3, B_3, and C_3 neighborhoods. We obtain that any finite connected indecomposable Cartan graph of rank three contains a point which has at least one of the good A_3, B_3, and C_3 neighborhoods. The results are in Theorem 3.1.14 and Theorem 3.2.6 for rank two and rank three cases, respectively. Both theorems simplify substantially the calculations in Chapter 4.

In Chapter 4 we formulate the main classification result of this thesis. All rank two and rank three Nichols algebras of diagonal type with finite root systems over fields of positive characteristic are classified in Theorem 4.1.1. To simplify the classification result, in Chapter 5 we list all Dynkin diagrams of braided vector space V of diagonal type such that Nichols algebra $\mathcal{B}(V)$ has a finite root system. Table 5.9 and Tables (5.6-5.8) are given for rank two and rank three cases, respectively. In order to have a better understanding of the reflections of such Nichols algebras we give Tables 5.10 and 5.11, which include all exchange graphs of the corresponding Cartan graphs in Theorem 4.1.1.

Keywords: Hopf algebra; Nichols algebra; Cartan graph; Weyl groupoid; root system.

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Introduction

The theory of Nichols algebras is dominated and motivated by Hopf algebra theory. The structure of a Nichols algebra turns out to be very important in Hopf algebras and quantum groups [3, 7, 41]. In any area of mathematics the classification of all objects of interest is important. In Hopf algebra theory, the classification of all finite dimensional Hopf algebras is a tough question. A nice survey on this task is [1]. For non-semisimple Hopf algebras, only the class of pointed Hopf algebras over an algebraically closed field of characteristic 0 is known. A Hopf algebra H over a field $\Bbbk$ is *pointed* if all its simple left or right comodules are one-dimensional. The *coradical* H_0 of H is the sum of all its simple subcoalgebras. Thus H is pointed if and only if the coradical of the Hopf algebra is a group algebra. In particular, group algebras, universal enveloping algebras of Lie algebras, and quantized Kac-Moody algebras are all pointed. Assume that H is a pointed Hopf algebra with coradical H_0. N. Andruskiewitsch and H.-J. Schneider classified pointed Hopf algebras with certain finiteness properties by the lifting method [5, 7]. Up to now the lifting method seems to be the most powerful method to understand pointed Hopf algebras and it works in principle for more general Hopf algebras than pointed. In general, the idea of the lifting method has three parts:

$(L1)$ Find an invariant R for H and attach it to H.

$(L2)$ Classify the invariant R.

$(L3)$ Recover H from R and H_0.

Let $\triangle_H$ denote the comultiplication of H. To describe a sensible invariant of H, we define the *coradical filtration* of H by

$$H_0 \subset H_1 \subset \dots H_n \subset H_{n+1} \dots$$

such that $H = \cup_{n\geq 0} H_n$, where $H_{n+1} := \{x \in H | \triangle_H(x) \in H_n \otimes H + H \otimes H_n\}$ are all subalgebras of H, $n \in \mathbb{N}_0$. Let

$$\mathrm{gr} H = H_0 \oplus (\bigoplus_{n\geq 1} H_n \diagup H_{n-1})$$

be the graded Hopf algebra associated to the coradical filtration. There is a graded projection $\pi : \mathrm{gr}H \to H_0$ and a retraction of the inclusion $\iota : H_0 \to \mathrm{gr}H$ such that $\pi\iota = \mathrm{id}_{H_0}$. Let $R := \{a \in \mathrm{gr}H | (\mathrm{id} \otimes \pi)\triangle_{\mathrm{gr}H}(a) = a \otimes 1\}$ is the kernel of π and also the algebra of coinvariants of π. By analogy with elementary group theory, this usual Hopf algebra $\mathrm{gr}H$ could be reconstructed as a biproduct (or bosonization) $\mathrm{gr}H \cong R\#H_0$ from R and H_0. In fact, the algebra R is a graded *braided Hopf algebra* in the *braided monoidal category* ${}^{H_0}_{H_0}\mathcal{YD}$ (Definitions in Section 1). Besides, R inherits the grading from $\mathrm{gr}H$: $R = R(0) \oplus (\bigoplus_{n>0} R(n))$, where $R(0) = H_0 \cap R = \Bbbk 1$ and $R(n) = (H_n / H_{n-1}) \cap R$. Notice that $R(1)$ is the space of primitive elements of R. The graded Hopf algebra R is an invariant of H and is called the *diagram* of H_0. Let $V := R(1)$. Then (V, c) is a *braided vector space* (Definition 1.1.1), where c is the braiding in ${}^{H_0}_{H_0}\mathcal{YD}$

$$c(v \otimes w) = v_{(-1)}.w \otimes v_{(0)}$$

for all $v, w \in V$. The subalgebra of R generated by (V, c) is termed the *Nichols algebra* generated by (V, c). The dimension of the space V is the *rank* of the Nichols algebra. Normally, a Nichols algebra is a graded braided Hopf algebra $R := \oplus_{n\geq 0} R(n)$ satisfying the following properties.

$(N1)$ $R(0) = \Bbbk 1$.

$(N2)$ $R(1) = P(R) = \{x \in R | \triangle(x) = x \otimes 1 + 1 \otimes x\}$, $P(R)$ is the space of the primitive elements of R.

$(N3)$ R is generated by $P(R)$ over $\Bbbk$.

Nichols algebras are named after W. Nichols, who first introduced this structure in the paper "Bialgebras of type one" [39] in 1978, where he studied certain pointed Hopf algebras. In Hopf algebra language, the bialgebras of type one is a biproduct (or bosonization) $\mathcal{B}(V)\#H$, where $\mathcal{B}(V)$ is the Nichols algebra of a braided vector space V and H is a group algebra. The structure of a Nichols algebra was rediscovered later by several authors in many different ways. For example, S.L. Woronowicz rediscovered this structure in his approach to "quantum differential calculus" [47, 48]. M. Rosso and G. Lusztig defined and used it to present quantum groups in a different language [40, 37]. Nichols algebras have been also redescribed independently by S. Majid [38]. These several definitions of Nichols algebras are equivalent.

The second step of the lifting method is to determine all braided vector space V such that the Nichols algebra $\mathcal{B}(V)$ is finite dimensional. There are several suitable classes of braided vector spaces. Several authors obtained the classification result for infinite and finite dimensional Nichols algebra of Cartan type, see [6, 23, 40].

We say that a Nichols algebra $\mathcal{B}(V)$ is termed of *diagonal type* if the braiding c is of diagonal type. It means that (V, c) admits a basis $\{x_i | 1 \leq i \leq \theta\}$ such that $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$ for all $i, j \in \{1, 2, \dots, \theta\}$ for certain $q_{ij} \in \Bbbk^*$. The matrix $(q_{ij})_{1 \leq i,j \leq \theta}$ is termed of the *braiding matrix* of V. N. Andruskiewitsch stated the following question.

Question 0.0.1. (N. Andruskiewitsch [1]) *Given a braiding matrix $(q_{ij})_{1\leq i,j\leq\theta}$ whose entries are roots of 1, when $\mathcal{B}(V)$ is finite-dimensional, where V is a vector space with basis $x_1, \dots, x_\theta$ and braiding $c(x_i \otimes x_j) = q_{ij}(x_j \otimes x_i)$? If so, compute $\dim_\Bbbk \mathcal{B}(V)$, and give a "nice" presentation by generators and relations.*

I. Heckenberger classified all finite dimensional Nichols algebra of diagonal type over fields of characteristic 0 in a series of papers [22, 24, 21, 27]. The explicit presentations by generators and relations of such Nichols algebras were given [10, 9]. With the results, N. Andruskiewitsch and H.-J. Schneider [8] succeeded with a classification theorem of all the finite-dimensional pointed Hopf algebras under some technical assumptions.

Based on such successful applications, the analysis to Nichols algebras over arbitrary fields is crucial and has also potential applications. Besides that, Nichols algebras have interesting applications to other research fields such as Kac-Moody Lie superalgebras [2, Example 3.2] and conformal field theory [42, 43, 44]. Since the Nichols algebras generated by braided vector spaces are defined over arbitrary fields, it is natural and desirable to analyze Nichols algebras of diagonal type for arbitrary fields. Towards this direction, the authors [11] discovered a combinatorial formula to study the relations in Nichols algebras and found new examples of Nichols algebras. In this thesis, we concentrate on Nichols algebras of diagonal type over fields of positive characteristic.

The crucial theoretical tools of the existing classification were the root system and the Weyl groupoid associated to a Nichols algebra of diagonal type, see [23, 30]. V. Kharchenko [35, Theorem 2] defined a Poincaré–Birkhoff–Witt basis (the height of the basis is restricted) for any Hopf algebra R generated by skew-primitive elements and group-like elements, where the conjugation action of the group-like elements on space of skew-primitive elements is diagonalizable. Since Nichols algebras have a natural $\mathbb{Z}^\theta$-grading, the root system of Nichols algebras of diagonal type is naturally defined as the degrees of Poincaré–Birkhoff–Witt basis. Under some finiteness conditions (see Definition 2.1.5) the transformations of Nichols algebras of diagonal type are introduced in [23]. This gave rise to a structure of Weyl groupoid associated to Nichols algebras of diagonal type. The Weyl groupoid plays a similar role as the Weyl group does for ordinary root systems in Kac-Moody algebras. Based on these results, the abstract combinatorial theory of Weyl groupoids and generalized root systems were initiated in [33, 13]. Later, the theory of root systems and Weyl groupoids was extended to more general Nichols algebras in [4, 28, 30]. M.Cuntz and I. Heckenberger [15, 16] classified all the finite Weyl groupoids.

In this thesis, we manage to classify the finite dimensional Nichols algebras of diagonal type over fields of arbitrary characteristic. The complete result can be given by doing the induction on the rank of Nichols algebras. Assume that the characteristic of the field $\Bbbk$ is $p > 0$. If $V = \Bbbk x$ is one-dimensional braided vector space over the field $\Bbbk$ and the braiding is $c(x \otimes x) = qx \otimes x$, then the Nichols algebra $\mathcal{B}(V) \simeq \Bbbk[x]$, the polynomial algebra in one variable, when q is not a root of 1; If $q = 1$ then $\mathcal{B}(V) \simeq \Bbbk[x]/(x^p)$; If q is a primitive *m*th root of 1, $p \nmid m$, $\gcd(m, p) = 1$ and $2 \leq m < \infty$ then $\mathcal{B}(V) \simeq \Bbbk[x]/(x^m)$. We give the result for rank two and rank three cases in this thesis. We believe that the higher rank will be soon known by the similar method of rank three. We always assume that the braiding matrix $(q_{ij})_{i,j\in I}$ of a braided vector space is indecomposable. Otherwise, we can refer to the lower rank of Nichols algebras by the following proposition proved in [18].

Proposition 0.0.2. *Assume that I, I_1, and I_2 are non-empty disjoint sets and there is a decomposition $I = I_1 \cup I_2$ such that $q_{ij}q_{ji} = 1$ for all $i \in I_1$, $j \in I_2$. Let $V = V' \oplus V''$ be the corresponding decomposition into Yetter–Drinfel'd modules. Then $\mathcal{B}(V) \simeq \mathcal{B}(V') \otimes \mathcal{B}(V'')$ as $\mathbb{Z}^\theta$-graded objects in ${}^H_H\mathcal{YD}$, where $\theta = |I|$.*

The structure of the present thesis is the following. In Chapter 1 we present the fundamental notations and some general results. The relations between Yetter–Drinfel'd modules and braided vector spaces are recalled in Section 1.1. The Nichols algebra $\mathcal{B}(V)$ generated by a Yetter–Drinfel'd module V is introduced in Section 1.2. Notice that there are several alternative description of $\mathcal{B}(V)$ and we choose an appropriate definition for this thesis. In Section 1.4 we obtain that there is a unique root system for any decomposable Nichols algebra. Since V. Kharchenko's [35, Theorem 2] gave a $\mathbb{Z}^n$-grading Poincaré–Birkhoff–Witt basis for Nichols algebras of diagonal type, all Nichols algebras of diagonal type are decomposable. This naturally gives a root system for Nichols algebras of diagonal type, see Definition 1.4.2. In Section 1.5 the "Weyl groupoid" of a Nichols algebra of diagonal type $\mathcal{B}(V)$ is introduced, which is a generalization of the Weyl group of a Kac-Moody Lie algebra. We start with a family of Cartan matrices and their reflections, named "Cartan graph". For any Cartan graph $\mathcal{C}$ we can define a root system of type $\mathcal{C}$, see Definition 1.5.5. Notice that there exists a unique root system for any finite Cartan graph, see Remark 1.5.12. It says that if $\mathcal{C} = \mathcal{C}(I, \mathcal{X}, r, (A^X)_{X\in\mathcal{X}})$ is a finite Cartan graph then $\mathcal{R}^{re} = \mathcal{R}^{re}(\mathcal{C}, (\boldsymbol{\Delta}^{X\,\mathrm{re}})_{X\in\mathcal{X}})$ is the unique root system of type $\mathcal{C}$.

Chapter 2 is devoted to construct a Cartan graph for Nichols algebras of diagonal type. Under a finiteness assumption (Definition 2.1.5), we define the "reflections" for Nichols algebras (or transformations of Nichols algebras). Let M be a θ-tuple of one-dimensional Yetter–Drinfel'd modules over an abelian group G. In Section 2.1, Theorem 2.1.9 associates a semi-Cartan graph $\mathcal{C}(M)$ to the tuple M.

In Chapter 3 we obtain the properties of Nichols algebra $\mathcal{B}(M)$ of diagonal type by analysing the semi-Cartan graph attached to M. Theorems 3.1.14 and 3.2.6 illustrate some properties of such Cartan graphs of rank two and rank three cases, respectively. In the rank two case, Theorem 3.1.14 characterizes finite connected Cartan graphs in terms of certain integer sequences. In Section 3.2 Theorem 3.2.6 illustrates that any finite connected indecomposable Cartan graph of rank three contains a point which has at least one of the good neighborhoods. The proof is based on the classification result of finite irreducible root systems of type $\mathcal{C}(M)$ in [15, Theorem 4.1]. It says that if $\mathcal{C}(M) = \mathcal{C}(I, \mathcal{X}, r, (A^X)_{X\in\mathcal{X}})$ is a rank 3 finite connected indecomposable Cartan graph then there exists a point $X \in \mathcal{X}$ satisfying that the set $\mathbf{\Delta}^{X\,\mathrm{re}} \cap \mathbb{N}_0^I$ is in the list of [15, Appendix A] up to a permutation of I. In Appendix A we list the GAP algorithms needed for the proof of Theorem 3.2.6. Both theorems simplify substantially the calculations in Chapter 4. Indeed, by using the theorems these calculations can be done by hand within a very short time, in contrast to the calculations based on the definition of the Weyl groupoid $\mathcal{W}$ or using a longest element of $\mathcal{W}$.

In Chapter 4 we formulate the main classification result of this thesis. All rank two and rank three Nichols algebras of diagonal type with a finite root system over fields of positive characteristic are classified in Theorem 4.1.1. The proof is given in Section 4.2 by using the characterization of finite Cartan graphs from Section 3. Since many subcases have to be considered, this is a large chapter.

To simplify the classification result, we list all Dynkin diagrams of braided vector space V of diagonal type such that Nichols algebra $\mathcal{B}(V)$ has a finite root system in Chapter 5. Table 5.9 and Tables (5.6-5.8) are given for rank two and rank three cases, respectively. Tables 5.10 and 5.11 illustrate all exchange graphs of the corresponding Cartan graphs in Theorem 4.1.1.

Throughout the paper $\Bbbk$ denotes a field of characteristic $p > 0$. Let $\Bbbk^* = \Bbbk \setminus \{0\}$. All vector spaces, tensor products, and Hopf algebras are considered over $\Bbbk$. The set of natural numbers not including 0 is denoted by $\mathbb{N}$ and we write $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}$, let G'_n denote the set of primitive n-th roots of unity in $\Bbbk$, that is $G'_n = \{q \in \Bbbk^* |\ q^n = 1, q^k \neq 1 \text{ for all } 1 \leq k < n\}$. For a Hopf algebra H with an antipode κ, the comultiplication $\triangle_H(a) = a_{(1)} \otimes a_{(2)}$ is written in the Sweedler's notation but dropping always the summation symbol. It allows us to write $(\triangle_H \otimes \mathrm{id})\triangle_H(a) = (\mathrm{id} \otimes \triangle_H)\triangle_H(a) = a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$. The Sweedler notation can be generalized to left comodules V via $\delta(v) = v_{(-1)} \otimes v_{(0)}$ for all $v \in V$.

If there are no confusions caused, then write the symbols without subscripts. If G is a group, then write $\Bbbk G$ the group algebra with product and unit induced by the product and the identity element of G, and the comultiplication $\triangle$ and counit ϵ by $\triangle(g) = g \otimes g$ and $\epsilon(g) = 1$ for all $g \in G$. Let $\kappa(g) = g^{-1}$ denote the antipode of the group algebra $\Bbbk G$.

The content of the present thesis covers the results from [32] and [46].

Acknowledgments

I would like to thank all the people who have made this work possible.

First of all, I would like to express my utmost gratitude to my supervisor, Prof. István Heckenberger. As a PhD supervisor, he has a great personality charm and always keep an open mind to different ideas. During my four years PhD he has provided me with invaluable support. He has always been generous with his time, teaching me with great patience and precision. He has guided me along the path from the study of Mathematics to the creative research in Mathematics. This is probably the most difficult part of becoming a mathematician. He not only invited some experts to our group but also provided me a lot opportunities to attend conferences and summer schools. So that I had the chance to communicate with the experts in my area.

I would like to thank Prof. Dr. Hans-Jürgen Schneider, who gave me advices in the very beginning of my PhD. Many thanks to Prof. Nicolás Andruskiewitsch, who held wonderful conferences and also organized other workshops that I was lucky to attend. Many thanks to Dr. Leandro Vendramin, who encouraged me a lot during his visit to our group in Marburg university. In the last four years I have the privilege of meeting them several times, in various conferences and summer schools, and I have learnt a lot from them.

I would like to express my sincere thanks to the committee. Many thanks to Prof. Volkmar Welker for proofreading the thesis. He followed the work of my thesis from the beginning, when I gave my first talk. I would like to thank Dr. Iván Ezequiel Angiono from Córdoba as my second referee.

Finally, I wish to thank my colleagues in the department of Mathematics, Andreas, Bastian, Eric, Karolina and Naqeeb. I thank my family and my friends for their constant support. In particular, my parents for the love, the encouragement and the motivation they always give me. Without them, I would not have been able to complete this work. I give my gratitude to the financial support of China Scholarship Council (CSC).

Chapter 1

Preliminaries

In this chapter we introduce notations and recall some related results which are useful in the following chapters. For more details on these topics we refer to [13].

1.1 Yetter–Drinfel'd modules

In this section, we recall Yetter–Drinfel'd modules, braided vector spaces, and their relations. For further details on these topics we refer to [1, 2, 3].

Let $\theta \in \mathbb{N}$, $I = \{1, \ldots, \theta\}$, and G be a group. Let V a θ-dimensional vector space over $\Bbbk$. We start from recalling the main object of our thesis.

Definition 1.1.1. The pair (V, c) is called a *braided vector space*, if $c \in \mathrm{Aut}(V \otimes V)$ is a solution of the braid equation, that is

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c). \tag{1.1}$$

A braided vector space (V, c) is termed *of diagonal type* if V admits a basis $\{x_i | i \in I\}$ such that for all $i, j \in I$ one has

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i \quad \textit{for some} \quad q_{ij} \in \Bbbk^* \tag{1.2}$$

The matrix $(q_{ij})_{i,j \in I}$ is termed of the *braiding matrix* of V. We say that the braiding matrix $(q_{ij})_{i,j \in I}$ is *indecomposable* if for any $i \neq j$ there exists a sequence $i_1 = i, i_2, \ldots, i_t = j$ of elements of I such that $q_{i_s i_{s+1}} q_{i_{s+1} i_s} \neq 1$ for $1 \leq s < t$. In this thesis, we mainly concern the braided vector spaces with indecomposable braiding matrices.

Definition 1.1.2. Let H be a Hopf algebra. A *Yetter–Drinfel'd module* V over H is a left H-module with left action $. : H \otimes V \longrightarrow V$ and a left H-comodule with left coaction $\delta : V \longrightarrow H \otimes V$ satisfying the compatibility condition

$$\delta(h.v) = h_{(1)}v_{(-1)}\kappa(h_{(3)}) \otimes h_{(2)}.v_{(0)}, h \in H, v \in V, \tag{1.3}$$

where κ is the antipode of H. We say that V is *of diagonal type* if $H = \Bbbk G$, G is an abelian group, and V is a direct sum of one-dimensional Yetter–Drinfel'd modules over the group algebra $\Bbbk G$.

We denote by ${}^H_H\mathcal{YD}$ the category of Yetter–Drinfel'd modules over H, where morphisms preserve both the action and the coaction of H. The category ${}^H_H\mathcal{YD}$ is braided with braiding

$$c_{V,W}(v \otimes w) = v_{(-1)}.w \otimes v_{(0)} \tag{1.4}$$

for all $V, W \in {}^H_H\mathcal{YD}$, $v \in V$, and $w \in W$. Actually, the category ${}^H_H\mathcal{YD}$ ia a braided monoidal category, where the monoidal structure is given by the tensor product over $\Bbbk$. Then any Yetter–Drinfel'd module $V \in {}^H_H\mathcal{YD}$ over H admits a braiding $c_{V,V}$ and hence $(V, c_{V,V})$ is a braided vector space. Conversely, any braided vector space can be realized as a Yetter–Drinfel'd module over some Hopf algebras if and only if the braiding is rigid [45, Section 2]. Notice that Yetter–Drinfel'd module structures on V with different Hopf algebras can give the same braiding and not all braidings of V are induced by the above Equation (1.4).

If $H = \Bbbk G$ then we write ${}^G_G\mathcal{YD}$ for the category of Yetter–Drinfel'd modules over $\Bbbk G$ and say that $V \in {}^G_G\mathcal{YD}$ is a Yetter–Drinfel'd module over G. Notice that if $V \in {}^G_G\mathcal{YD}$ is of diagonal type then $(V, c_{V,V})$ is a braided vector space of diagonal type. Indeed, assume that G is abelian and $V = \oplus_{i=1}^n \Bbbk x_i \in {}^G_G\mathcal{YD}$, where $\Bbbk x_i \in {}^G_G\mathcal{YD}$ and $\{x_i | i \in I\}$ is a basis of V. Then we obtain that there is a sequence of scalars $q_{ij} \in \Bbbk^*$ such that $g_i.x_j = q_{ij}x_j$ for all $j \in I$. Then the braiding defined in Equation (1.4) becomes $c(x_i \otimes x_j) = q_{ij}x_j \otimes x_i$ for all $i, j \in I$. Hence the braiding c is of diagonal type. Up to a permutation of I the matrix $(q_{ij})_{i,j\in I}$ does not depend on the choice of the basis $\{x_i | i \in I\}$. This fact is proven in [6].

There are some more relations between Yetter–Drinfel'd modules and braided vector spaces.

Remark 1.1.3. (i) If $V \in {}^G_G\mathcal{YD}$ then V is a G-graded $\Bbbk G$ module. Indeed, for all $g \in G$ and $v \in V$ there is $v_g \in G$ such that $\delta(v) = \sum_{g\in G} g \otimes v_g$ since (V, δ) is a $\Bbbk G$-comodule. Then by the coassociativity $(\delta \otimes \mathrm{id})\delta(v) = (\mathrm{id} \otimes \delta)\delta(v)$ we get

$$\sum_{g\in G} g \otimes g \otimes v_g = \sum_{g\in G} \delta(g) \otimes v_g = \sum_{g\in G} g \otimes \delta(v_g).$$

Hence $\delta(v_g) = g \otimes v_g$ for all $g \in G$ and $v \in V$. On the other hand, by the counitary property $v = (\epsilon \otimes \mathrm{id})\delta(v)$ we get that $v = \sum_{g\in G} \epsilon(g)v_g = \sum_{g\in G} v_g$ for all $v \in V$. Then $V = \oplus_{g\in G}V_g$, where $V_g = \{v \in V|\delta(v) = g \otimes v\}$. By Equation (1.3) we get $\delta(g.w) = ghg^{-1} \otimes g.w$ and hence $g.V_h \subset V_{ghg^{-1}}$ for all $g \in G$.

(ii) Any braided vector space of diagonal type is also a Yetter–Drinfel'd module of diagonal type. Assume that (V, c) is a braided vector space of diagonal type with an indecomposable braiding matrix $(q_{ij})_{i,j\in I}$ of a basis $\{x_i|i \in I\}$. Let G_0 be an abelian group generated by elements $\{g_i|i \in I\}$. Define the left coaction and left action by

$$\delta(x_i) = g_i \otimes x_i \in G_0 \otimes V, \quad g_i.x_j = q_{ij}x_j \in V.$$

Then $V = \oplus_{i\in I}\Bbbk x_i$ and each $\Bbbk x_i$ is one-dimensional Yetter–Drinfel'd modules over G_0. Hence V is a Yetter–Drinfel'd module of diagonal type over G_0.

1.2 Nichols algebras of diagonal type

In this section we present the main object of this thesis.

Let (V, c) be a θ-dimensional braided vector space of diagonal type. In this section, we give definition of the Nichols algebra $\mathcal{B}(V)$ generated by (V, c). In order to do that, we introduce one more notion in the category ${}^H_H\mathcal{YD}$.

Definition 1.2.1. Let H be a Hopf algebra. A *braided Hopf algebra* in ${}^H_H\mathcal{YD}$ is a 6-tuple $\mathcal{B} = (\mathcal{B}, \mu, 1, \triangle, \epsilon, \kappa_\mathcal{B})$, where $(\mathcal{B}, \mu, 1)$ is an algebra in ${}^H_H\mathcal{YD}$ and $(\mathcal{B}, \triangle, \epsilon)$ is a coalgebra in ${}^H_H\mathcal{YD}$ satisfying the compatibility between $\mu, 1$ and $\triangle, \epsilon$, and $\kappa_\mathcal{B} : \mathcal{B} \to \mathcal{B}$ is a morphism in ${}^H_H\mathcal{YD}$ such that $\triangle$, ϵ and $\kappa_\mathcal{B}$ satisfy $\kappa_\mathcal{B}(b^{(1)})b^{(2)} = b^{(1)}\kappa_\mathcal{B}(b^{(2)}) = \epsilon(b)1$, where we define $\triangle(b) = b^{(1)} \otimes b^{(2)}$ as the coproduct of $\mathcal{B}$ to avoid the confusion.

Note that if $\mathcal{B} \in {}^H_H\mathcal{YD}$ and $\mathcal{B}$ is an algebra in ${}^H_H\mathcal{YD}$ then $\mathcal{B} \otimes \mathcal{B}$ is an algebra in ${}^H_H\mathcal{YD}$ with the product given by

$$(a \otimes b)(c \otimes d) = a(b_{(-1)}.c) \otimes b_{(0)}d, \tag{1.5}$$

for all $a, b, c, d \in \mathcal{B}$, where . denotes the left action of H on $\mathcal{B}$.

The *tensor algebra* $T(V)$ admits a natural structure of a Yetter–Drinfel'd module and an algebra structure in ${}^H_H\mathcal{YD}$. It is then a braided Hopf algebra in ${}^H_H\mathcal{YD}$ with coproduct $\triangle(v) = 1 \otimes v + v \otimes 1 \in T(V) \otimes T(V)$ and counit $\epsilon(v) = 0$ for all

$v \in V$ such that $\triangle$ and ϵ are the algebra morphisms. The antipode of $T(V)$ exists, see [7, Section 2.1]. Notice that the product defined by Equation (1.5) on $T(V)$ is the unique algebra structure such that $\triangle(v) = 1 \otimes v + v \otimes 1 \in T(V) \otimes T(V)$ for all $v \in V$. The coproduct can be extended from V to $T(V)$. For example, for all $v, w \in V$ one gets (we write the elements of $T(V)$ without the tensor product sign for brevity)

$$\begin{aligned}\triangle(vw) =& \triangle(v)\triangle(w)\\ =&(1 \otimes v + v \otimes 1)(1 \otimes w + w \otimes 1)\\ =&1 \otimes vw + v_{(-1)}.w \otimes v_{(0)} + v \otimes w + vw \otimes 1.\end{aligned} \tag{1.6}$$

Definition 1.2.2. The *Nichols algebra* generated by $V \in {}_H^H\mathcal{YD}$ is defined as the quotient

$$\mathcal{B}(V) = T(V)/\mathcal{I}(V) = (\oplus_{n=0}^{\infty} V^{\otimes n})/\mathcal{I}(V)$$

where $\mathcal{I}(V)$ is the unique maximal coideal among all coideals of $T(V)$ which are contained in $\oplus_{n \geq 2} V^{\otimes n}$. Nichols algebra $\mathcal{B}(V)$ is said to be *of diagonal type* if V is a Yetter–Drinfel'd module of diagonal type. The dimension of V is the *rank* of Nichols algebra $\mathcal{B}(V)$.

Let $(I_i)_{i \in I}$ be the family of all coideals of $T(V)$ contained in $\oplus_{n \geq 2} V^{\otimes n}$, i.e.

$$\triangle(I_i) \subset I_i \otimes T(V) + T(V) \otimes I_i.$$

Then the coideal $\mathcal{I}(V) := \sum_{i \in I} I_i$ is the largest element in $(I_i)_{i \in I}$. Hence $\mathcal{B}(V)$ is a braided Hopf algebra in ${}_H^H\mathcal{YD}$. As proved in [3, Proposition 3.2.12], Nichols algebra $\mathcal{B}(V)$ is the unique $\mathbb{N}_0$-graded braided Hopf algebra generated by V in ${}_H^H\mathcal{YD}$ with homogenous components $\mathcal{B}(V)(0) = \Bbbk$, $\mathcal{B}(V)(1) = V$, and $P(\mathcal{B}(V)) = V$, where $P(\mathcal{B}(V))$ is the space of primitive elements of $\mathcal{B}(V)$.

Here are some well-known examples of Nichols algebras.

Example 1.2.3. Here we assume that the characteristic of the ground field $\Bbbk$ is 0 and H is the group algebra of the trivial group, namely, $H = \Bbbk 1$. Then $1.v = v$ for all $v \in V$. Hence $V = V_1$. Besides, by Equation (1.6) for all $v, w \in V$ one gets

$$\begin{aligned}\triangle(vw - wv) =& \triangle(vw) - \triangle(wv)\\ =&1 \otimes (vw - wv) + (vw - wv) \otimes 1\end{aligned}$$

It is not hard to prove that $\mathcal{I}(V)$ is the ideal generated by $vw - wv$ for all $v, w \in V$. Hence the Nichols algebra generated by V is the symmetric algebra of V, $\mathrm{Sym}(V)$, where we take the quotient of $T(V)$ modulo the ideal generated by all elements of the form $v \otimes w - w \otimes v$, where $v, w \in V$.

Example 1.2.4. Let $H = \Bbbk\mathbb{Z}/(2)$ and let $g \in \mathbb{Z}/(2)$ be the unique non-zero element. Assume that $g.v = -v$ for all $v \in V$ and that $V = V_g$. Then

$$\begin{aligned}\triangle(v^2) =& \triangle(v)\triangle(v)\\ =&1 \otimes v^2 - v \otimes v + v \otimes v + v^2 \otimes 1\\ =&1 \otimes v^2 + v^2 \otimes 1\end{aligned}$$

for all $v \in V$ by Equation (1.6). Hence $\mathcal{I}(V)$ is the ideal generated by v^2 for all $v \in V$. Then the Nichols algebra generated by V is the exterior algebra of V. That is $\mathcal{B}(V) = \wedge(V)$.

Example 1.2.5. Assume that the characteristic of the ground field $\Bbbk$ is 0. Let $V = V(0) \oplus V(1)$ be a super vector space and let $c : V \otimes V \to V \otimes V$ be the super symmetry

$$c(v \otimes w) = (-1)^{ij} w \otimes v, \quad v \in V(i), \quad w \in V(j).$$

Clearly, V can be realized as a Yetter–Drinfel'd module over $\Bbbk\mathbb{Z}/(2)$. Then the Nichols algebra $\mathcal{B}(V)$ generated by V is the super-symmetric algebra of V. That is $\mathcal{B}(V) \backsimeq \mathrm{Sym}(V) \otimes \wedge(V)$.

We give one more characterization of Nichols algebras. The explicit formula for the comultiplication leads to the following alternative description of $\mathcal{B}(V)$. Let (V, c) a braided vector space and $c_i := \mathrm{id}_{V^{\otimes(i-1)}} \otimes c \otimes \mathrm{id}_{V^{\otimes(n-i-1)}} \in \mathrm{Aut}(V^{\otimes n})$. Let S_n be the symmetric group generated by $n-1$ generators $\{\tau_1, \dots, \tau_{n-1}\}$ and relations:

$$\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}, 1 \le i \le n-2,$$

$$\tau_i\tau_j = \tau_j\tau_i, |i-j| \ge 2,$$

$$\tau_i^2 = 1.$$

We denote by B_n the *Artin's braid group*, which is the quotient of the free group in $\sigma_1, \dots, \sigma_{n-1}$ by the relations:

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, 1 \le i \le n-2,$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| \ge 2.$$

Then for all $n \in \mathbb{N}_0$ there are representations of the braid groups $\rho_n : B_n \to \mathrm{Aut}(V^{\otimes n})$ given by $\rho_n(\sigma_i) = c_i$, for all $1 \le i \le n-1$.

We consider the canonical projections $B_n \twoheadrightarrow S_n$ sending σ_i to the transposition τ_i for all $1 \le i \le n-1$. There exists a unique section of sets $\mu : S_n \to B_n, \tau_i \mapsto \sigma_i$

called the Matsumoto section such that $\mu(xy) = \mu(x)\mu(y)$ for any $x, y \in S_n$ with $length(xy) = length(x) + length(y)$.

For any $n \in \mathbb{N}_0$ define a map

$$\psi_n := \Sigma_{\tau \in S_n} \rho_n(\mu(\tau)) \tag{1.7}$$

where ρ_n is the representation of B_n induced by c and μ is the Matsumoto section. The map ψ_n is called the *quantum symmetrizer.*

Definition 1.2.6. *Let $V \in {}^H_H\mathcal{YD}$. The* ***Nichols algebra*** *generated by V is*

$$\mathcal{B}(V) = \bigoplus_{n \geq 0} V^{\otimes n}/ker(\psi_n),$$

where ψ_n is the quantum symmetrizer defined by Equation (1.7).

As an example, we can calculate the quantum symmetrizer ψ_n for $n = 2, 3$:

$$\begin{aligned}\psi_2 =&\mathrm{id} + c,\\ \psi_3 =&\mathrm{id} + c_1 + c_2 + c_1c_2 + c_2c_1 + c_1c_2c_1.\end{aligned}$$

In general the kernels of the map ψ_n are hard to calculate explicitly. The previous description relations of $\mathcal{B}(V)$ does not mean the defining relations are known.

1.3 Lyndon words and Poincaré–Birkhoff–Witt basis for Nichols algebras of diagonal type

Fix a braided vector space (V, c) of diagonal type. Let $X = \{x_1, \dots, x_\theta\}$ be a basis of V as in Equation (1.2). Let $\mathbb{X}$ be the corresponding vocabulary of X

$$\mathbb{X} = \{u_1u_2 \cdots u_k | k \in \mathbb{N}_0, u_i \in X \textit{ for all } i \in \{1, 2, \dots, k\}\}.$$

Fix a total ordering $\leq$ on X and we consider the lexicographic ordering on $\mathbb{X}$: if $u, v \in \mathbb{X}$, then $u \leq v$ if and only if either $v = uu'$ for some $u' \in \mathbb{X}$ or $u = wxu'$, $v = wyv'$, where $x, y \in X$, $x < y$, and $w, u', v' \in \mathbb{X}$. Then the empty word 1 is the minimal element for the order $\leq$. This ordering is denoted also by $\leq$. We write $u < v$ if $u \leq v$ and $u \neq v$. This ordering is stable under left multiplication, but it is not stable under right multiplication.

Definition 1.3.1. A word $u \in \mathbb{X}$, $u \neq 1$, is Lyndon if u is smaller than any of its proper ends. That is, for any decomposition $u = u_1u_2$, $u_1, u_2 \in \mathbb{X}, u_1 \neq 1, u_2 \neq 1$, we have $u < u_2$. We denote $\mathcal{L}$ the set of Lyndon words.

Definition 1.3.2. Let $w \in \mathcal{L}$. The *Shirshov decomposition* of w is the decomposition $w = uv$ with $u, v \in \mathcal{L}$ such that v is the longest proper end of w which is Lyndon word. We denote $\mathrm{Sh}w = (u, v)$.

Theorem 1.3.3. *[36, Theorem 5.1.5] Any $w \in \mathcal{L}$ admits a unique decomposition $w = w_1 w_2 \cdots w_m$, where $m \in \mathbb{N}_0$, $w_1, w_2, \dots, w_m \in \mathcal{L}$ and $w_m \leq \cdots \leq w_1$. This is the Lyndon decomposition of $w \in \mathcal{L}$.*

The Lyndon decomposition says that, any word $w \in \mathcal{L}$ admits a unique decomposition as a product of non-increasing Lyndon words.

Example 1.3.4. Let $X = \{x_1, x_2\}$ and $x_1 < x_2$. All words of length at most two have the following order:

$$x_1 < x_1^2 < x_1x_2 < x_2 < x_2x_1 < x_2^2.$$

The words x_1, x_2, x_1x_2 are Lyndon words. But x_1^2, x_2x_1, x_2^2 are not Lyndon words. The set of Lyndon words of length equal to 3 is

$$\{x_1^2x_2, x_1x_2^2\}.$$

The word $w := x_1x_2x_1x_2^2 \in \mathcal{L}$ and the $\mathrm{Sh}w = (x_1x_2, x_1x_2^2)$.

We can assume that (V, c) is also a Yetter–Drinfel'd module of diagonal type over $\Bbbk G$ by Remark 1.1.3(ii). For each pair $x, y \in T(V)$ define the braided commutator as follows:

$$[x, y]_c = xy - (x_{(-1)}.y)x_{(0)}.$$

Define a map $[-]_c : \mathcal{L} \to T(V)$ inductively on $\mathcal{L}$ such that $[w]_c = [[u]_c, [v]_c]_c$ if $w \in \mathcal{L}$ and $\mathrm{Sh}(w) = (u, v)$ with $|w| \geq 2$. Let $\mathbb{X}_{>w}^{|w|} = \{w' \in \mathbb{X} | \, |w'| = |w|, \, w' > w\}$. Note that for any $w \in \mathcal{L}$, $[w]_c \in w + \mathbb{Z}[q_{ij}]\mathbb{X}_{>w}^{|w|}$.

Let $\alpha_1, \dots, \alpha_\theta$ be the canonical basis of $\mathbb{Z}^\theta$. Then the tensor algebra $T(V)$ is $\mathbb{N}_0^\theta$-graded with $\deg x_i = \alpha_i$ for all $i \in I$. Then $\mathcal{L}$ is also $\mathbb{N}_0^\theta$-graded and so is $\{[w]_c | w \in \mathcal{L}\}$. We can find a $\mathbb{N}_0^\theta$-graded Poincaré–Birkhoff–Witt basis for $T(V)$.

Theorem 1.3.5. *The set*

$$\{[u_1]_c^{k_1} \cdots [u_m]_c^{k_m} | m \in \mathbb{N}_0, u_i \in \mathcal{L}, u_1 > u_2 > \cdots > u_m, k_1, k_2, \dots, k_m \geq 0\}$$

forms a vector space basis of $T(V)$.

As in [10] we consider another order $\succ$ in $\mathbb{X}$: if $u, v \in \mathbb{X}$, then $u \succ v$ if either $|u| < |v|$, or $|u| = |v|$ and $u > v$ for the lexicographical order. Then the empty word 1 is also the maximal element for $\succ$. Note that the order $\succ$ is a total order and we call $\succ$ the *deg-less* order.

For the Nichols algebra $\mathcal{B}(V) = T(V)/\mathcal{I}(V)$ generated by (V, c), we define a canonical projection:

$$\pi : T(V) \to \mathcal{B}(V).$$

Define a set

$$G_I = \{u \in \mathbb{X} | u \notin \Bbbk\mathbb{X}_{\succ u} + \mathcal{I}(V)\}.$$

Consider the subset

$$S_I = G_I \cap \mathcal{L} \tag{1.8}$$

of $\mathcal{L}$ and define the height function

$$h_I(u) := \min\{t \in \mathbb{N} | u^t \in \Bbbk\mathbb{X}_{\succ u^t} + I(V)\}. \tag{1.9}$$

Then we can give a Poincaré–Birkhoff–Witt basis using the subset S_I and the height defined by Equation 1.9.

Theorem 1.3.6. *[35, Theorem 2] There is a subset S_I as in Equation 1.8 and for each $u \in S_I$ there exists a number $h_I(u) \in \mathbb{N} \cup \infty$ as in Equation 1.9 such that the following set is a Poincaré–Birkhoff–Witt basis of $\mathcal{B}(V)$:*

$$\{[u_1]_c^{n_1} \cdots [u_k]_c^{n_k} | k \in \mathbb{N}_0, u_1 > u_2 > \cdots > u_k \in S_I, 0 \leq n_i < h_I(u_i) \textit{ for all } i\}.$$

The proof of this theorem proceeds through several lemmas, which are given in [35]. In this thesis we are able to check the finiteness of the subset S_I.

1.4 Root systems

Our main purpose in this section is to define the root systems of Nichols algebras if Nichols algebras are decomposable, see Definition (1.4.1).

Let H be a Hopf algebra over $\Bbbk$ with bijective antipode. Let ${}^H_H\mathcal{YD}$ denote the category of Yetter–Drinfel'd modules over H and $\mathcal{F}^H_\theta$ the set of θ-tuples of finite-dimensional irreducible objects in ${}^H_H\mathcal{YD}$ for all $\theta \in \mathbb{N}$. Let $I = \{1, \ldots, \theta\}$ and $(\alpha_i)_{i \in I}$ be the standard basis of $\mathbb{N}_0^\theta$. Let $M = (M_1, \ldots, M_\theta) \in \mathcal{F}^H_\theta$. Let $\mathcal{B}(M)$ denote the Nichols algebra $\mathcal{B}(M_1 \oplus \cdots \oplus M_\theta)$ generated by M. Write $V_M := M_1 \oplus \cdots \oplus M_\theta \in {}^H_H\mathcal{YD}$. Let $\mathcal{X}^H_\theta$ be the set of θ-tuples of isomorphism classes of finite-dimensional irreducible objects in ${}^H_H\mathcal{YD}$. Write $[M] := ([M_1], \ldots, [M_\theta]) \in \mathcal{X}^H_\theta$.

Definition 1.4.1. [30, Definition 6.8] The Nichols algebra $\mathcal{B}(M)$ generated by M is called *decomposable* if there exists a totally ordered index set $(L, \leq)$ and a family $(W_l)_{l\in L}$ of finite-dimensional irreducible $\mathbb{N}_0^\theta$-graded objects in ${}^H_H\mathcal{YD}$ such that

$$\mathcal{B}(M) \simeq \bigotimes_{l\in L} \mathcal{B}(W_l). \tag{1.10}$$

Assume that $(x_i)_{i\in I}$ is a basis of $V_M \in {}^H_H\mathcal{YD}$. The tensor algebra $T(V_M)$ is $\mathbb{N}_0^\theta$-graded with $\deg x_i = \alpha_i$ for all $i \in I$. One can show that the ideal $\mathcal{I}(V_M)$ is homogeneous with respect to the $\mathbb{N}_0^\theta$-grading of $T(V_M)$ by using the natural projection maps to the homogeneous components $V_M^{\otimes n}$ of $T(V_M)$. Hence Nichols algebra $\mathcal{B}(M)$ is an $\mathbb{N}_0^\theta$-graded Hopf algebra in ${}^H_H\mathcal{YD}$. Then we can define a root system for any decomposable Nichols algebra.

Definition 1.4.2. Assume that $\mathcal{B}(M)$ is decomposable. For any decomposition (1.10), we define the set of *positive roots* $\boldsymbol{\Delta}_+^{[M]} \subset \mathbb{Z}^I$ and the set of *roots* $\boldsymbol{\Delta}^{[M]} \subset \mathbb{Z}^I$ of $\mathcal{B}(M)$ by

$$\boldsymbol{\Delta}_+^{[M]} = \{\deg(W_l) \,|\, l \in L\}, \qquad \boldsymbol{\Delta}^{[M]} = \boldsymbol{\Delta}_+^{[M]} \cup -\boldsymbol{\Delta}_+^{[M]}.$$

Remark 1.4.3. Note that the decomposability of a Nichols algebra $\mathcal{B}(M)$ is known under some assumptions on V_M. V. Kharchenko [35, Theorem 2] proved that the Nichols algebra $\mathcal{B}(M)$ has a Poincaré–Birkhoff–Witt basis if V_M is a Yetter–Drinfel'd module of diagonal type over a group algebra. Theorem 1.3.6 illustrated that a subset of the Lyndon words based on the basis of V_M is a Poincaré–Birkhoff–Witt basis of $\mathcal{B}(M)$. Hence for any Nichols algebra of diagonal type $\mathcal{B}(M)$ we can define the set of roots $\boldsymbol{\Delta}^{[M]}$ as the $\mathbb{N}_0^\theta$ degrees of the Poincaré–Birkhoff–Witt generators $[u]_c$, where $u \in S_I$ as in Equation (1.8). Further, the decomposition defined by Equation (1.10) is unique followed by [28, Lemma 4.5 4.7]. Actually, if $\mathcal{B}(M) \simeq \bigotimes_{l\in L} \mathcal{B}(W_l)$ and $\mathcal{B}(M) \simeq \bigotimes_{l'\in L'} \mathcal{B}(W'_l)$ for index sets $(L, \leq)$, $(L', \leq)$, and families $(W_l)_{l\in L}$ and $(W'_{l'})_{l'\in L'}$ satisfy the assumptions in Definition 1.4.1. Then there is a bijection $\phi : L \to L'$ such that $W_l \simeq W'_{\phi(l')}$ in ${}^H_H\mathcal{YD}$ for all $l \in L$. Hence the set of roots of $[M]$ does not depend on the choice of the decomposition (1.10).

1.5 Weyl groupoids

We start by recalling the notations of semi-Cartan graphs, root systems and Weyl groupoids. We mainly follow the terminology from [33], [12]. See also [32] and [46].

Let $\theta \in \mathbb{N}$ and $I = \{1, \dots, \theta\}$. In this section, let $(\alpha_i)_{i\in I}$ be the standard basis of $\mathbb{Z}^I$.

Definition 1.5.1. [34, §1.1] *A generalized Cartan matrix is a matrix* $A = (a_{ij})_{i,j\in I}$ *with integer entries such that*

- $a_{ii} = 2$ *and* $a_{jk} \leq 0$ *for any* $i, j, k \in I$ *with* $j \neq k$*,*
- *if* $a_{ij} = 0$ *for some* $i, j \in I$*, then* $a_{ji} = 0$*.*

A generalized Cartan matrix $A \in \mathbb{Z}^{I\times I}$ is *decomposable* if there exists a nonempty proper subset $I_1 \subset I$ such that $a_{ij} = 0$ for any $i \in I_1$ and $j \in I \setminus I_1$. We say that A is indecomposable if A is not decomposable.

Definition 1.5.2. Let $\mathcal{X}$ be a non-empty set and $A^X = (a^X_{ij})_{i,j\in I}$ a generalized Cartan matrix for all $X \in \mathcal{X}$. For any $i \in I$ let $r_i : \mathcal{X} \to \mathcal{X}$, $X \mapsto r(i, X)$, where $r : I \times \mathcal{X} \to \mathcal{X}$ is a map. The quadruple

$$\mathcal{C} = \mathcal{C}(I, \mathcal{X}, r, (A^X)_{X\in\mathcal{X}})$$

is called a *semi-Cartan graph* if $r_i^2 = \mathrm{id}_{\mathcal{X}}$ for all $i \in I$, and $a^X_{ij} = a^{r_i(X)}_{ij}$ for all $X \in \mathcal{X}$ and $i, j \in I$. We say that a semi-Cartan graph $\mathcal{C}$ is *indecomposable* if A^X is indecomposable for all $X \in \mathcal{X}$.

For the sake of simplicity, the elements of the set $\{r_i(X), i \in I\}$ are termed the *neighbors* of X for all $X \in \mathcal{X}$. The cardinality of I is termed the *rank* of $\mathcal{C}$ and the elements of $\mathcal{X}$ are the *points* of $\mathcal{C}$.

Definition 1.5.3. The *exchange graph* of $\mathcal{C}$ is a labeled non-oriented graph with vertices set $\mathcal{X}$ and edges set I, where two vertices X, Y are connected by an edge i if and only if $X \neq Y$ and $r_i(X) = Y$ (and $r_i(Y) = X$). We display one edge with several labels instead of several edges for simplification.

We say that $\mathcal{C}$ is *connected* if its exchange graph is connected.

For the remaining part of this section let $\mathcal{C} = \mathcal{C}(I, \mathcal{X}, r, (A^X)_{X\in\mathcal{X}})$ be a connected semi-Cartan graph.

Let $i \in I$. For all $X \in \mathcal{X}$ let

$$s_i^X \in \mathrm{Aut}(\mathbb{Z}^I), \;\; s_i^X \alpha_j = \alpha_j - a^X_{ij}\alpha_i. \tag{1.11}$$

for all $j \in I$. Let $\mathcal{D}(\mathcal{X}, I)$ be the category such that $\mathrm{Ob}\mathcal{D}(\mathcal{X}, I) = \mathcal{X}$ and morphisms $\mathrm{Hom}(X, Y) = \{(Y, f, X) | f \in \mathrm{End}(\mathbb{Z}^I)\}$ for $X, Y \in \mathcal{X}$ with the composition $(Z, g, Y) \circ (Y, f, X) = (Z, gf, X)$ for all $X, Y, Z \in \mathcal{X}$, $f, g \in \mathrm{End}(\mathbb{Z}^I)$. Let $\mathcal{W}(\mathcal{C})$ be the smallest subcategory of $\mathcal{D}(\mathcal{X}, I)$, where the morphisms are generated by $(r_i(X), s_i^X, X)$, with $i \in I$, $X \in \mathcal{X}$. From now on, we write s_i^X instead of

$(r_i(X), s_i^X, X)$, if no confusion is possible. Notice that all generators s_i^X are reflections and hence are invertible. Then $\mathcal{W}(\mathcal{C})$ is a groupoid.

For any category $\mathcal{D}$ and any object X in $\mathcal{D}$ let $\mathrm{Hom}(\mathcal{D}, X) = \cup_{Y \in \mathcal{D}} \mathrm{Hom}(Y, X)$. For all $X \in \mathcal{X}$, the set

$$\boldsymbol{\Delta}^{X\,\mathrm{re}} = \{\omega\alpha_i \in \mathbb{Z}^I | \omega \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), X)\} \tag{1.12}$$

is called the *set of real roots* of $\mathcal{C}$ at X. The elements of $\boldsymbol{\Delta}_+^{X\,\mathrm{re}} = \boldsymbol{\Delta}^{X\,\mathrm{re}} \cap \mathbb{N}_0^I$ are called *positive roots* and those of $\boldsymbol{\Delta}^{X\,\mathrm{re}} \cap -\mathbb{N}_0^I$ *negative roots*, denoted by $\boldsymbol{\Delta}_-^{X\,\mathrm{re}}$.

Definition 1.5.4. We say $\mathcal{C}$ is *finite* if the set $\boldsymbol{\Delta}^{X\,\mathrm{re}}$ is finite for all $X \in \mathcal{X}$.

Definition 1.5.5. We say that $\mathcal{R} = \mathcal{R}(\mathcal{C}, (\boldsymbol{\Delta}^X)_{X \in \mathcal{X}})$ is a *root system of type* $\mathcal{C}$ if for all $X \in \mathcal{X}$, the sets $\boldsymbol{\Delta}^X$ are the subsets of $\mathbb{Z}^I$ such that

$(R1)$ $\boldsymbol{\Delta}^X = (\boldsymbol{\Delta}^X \cap \mathbb{N}_0^I) \cup -(\boldsymbol{\Delta}^X \cap \mathbb{N}_0^I)$.

$(R2)$ $\boldsymbol{\Delta}^X \cap \mathbb{Z}\alpha_i = \{\alpha_i, -\alpha_i\}$ for all $i \in I$.

$(R3)$ $s_i^X(\boldsymbol{\Delta}^X) = \boldsymbol{\Delta}^{r_i(X)}$ for all $i \in I$.

$(R4)$ $(r_i r_j)^{m_{ij}^X}(X) = X$ for any $i, j \in I$ with $i \neq j$ where $m_{ij}^X = |\boldsymbol{\Delta}^X \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j)|$ is finite.

We say that $\mathcal{W}(R) := \mathcal{W}(\mathcal{C})$ is the groupoid of $\mathcal{R}$. As in [13, Definition 4.3] we say that $\mathcal{R}$ is *reducible* if there exist non-empty disjoint subsets of $I', I'' \subset I$ such that $I = I' \cup I''$ and $a_{ij} = 0$ for all $i \in I', j \in I''$ and

$$\boldsymbol{\Delta}^X = \Big(\boldsymbol{\Delta}^X \cap \sum_{i \in I'} \mathbb{Z}\alpha_i\Big) \cup \Big(\boldsymbol{\Delta}^X \cap \sum_{j \in I''} \mathbb{Z}\alpha_j\Big) \qquad \text{for all} \quad X \in \mathcal{X}.$$

In this case, we write $\mathcal{R} = \mathcal{R}|_{I_1} \oplus \mathcal{R}|_{I_2}$. If $\mathcal{R} \neq \mathcal{R}|_{I_1} \oplus \mathcal{R}|_{I_2}$ for all non-empty disjoint subsets $I_1, I_2 \subset I$, then $\mathcal{R}$ is termed *irreducible.*

Definition 1.5.6. Let $\mathcal{R} = \mathcal{R}(\mathcal{C}, (\boldsymbol{\Delta}^X)_{X \in \mathcal{X}})$ be a root system of type $\mathcal{C}$. We say that $\mathcal{R}$ is *finite* if $\boldsymbol{\Delta}^X$ is finite for all $X \in \mathcal{X}$.

Let $\mathcal{R} = \mathcal{R}(\mathcal{C}, (\boldsymbol{\Delta}^X)_{X \in \mathcal{X}})$ be a root system of type $\mathcal{C}$. We recall some properties of $\mathcal{R}$.

Lemma 1.5.7. *[15, Corollary 2.9] Let $X \in \mathcal{X}$, $k \in \mathbb{Z}$, and $i, j \in I$ such that $i \neq j$. Then $\alpha_j + k\alpha_i \in \boldsymbol{\Delta}^{X\,\mathrm{re}}$ if and only if $0 \leq k \leq -a_{ij}^X$*

Notice that the finiteness of $\mathcal{R}$ does not mean that $\mathcal{W}(\mathcal{R})$ is also finite, since the set $\mathcal{X}$ might be infinite.

Lemma 1.5.8. *[13, Lemma 2.11] Let $\mathcal{C} = \mathcal{C}(I, \mathcal{X}, r, (A^X)_{X\in\mathcal{X}})$ be a connected semi-Cartan graph and $\mathcal{R} = \mathcal{R}(\mathcal{C}, (\mathbf{\Delta}^X)_{X\in\mathcal{X}})$ is a root system of type $\mathcal{C}$. Then the following are equivalent.*

(1.) *$\mathcal{R}$ is finite.*

(2.) *$\mathbf{\Delta}^X$ is finite for some $X \in \mathcal{X}$.*

(3.) *$\mathcal{C}$ is finite.*

(4.) *$\mathcal{W}(\mathcal{R})$ is finite.*

Recall that $\mathcal{C}$ is a connected semi-Cartan graph. Then we get the following.

Proposition 1.5.9. *[13, Proposition 4.6] Let $\mathcal{R} = \mathcal{R}(\mathcal{C}, (\mathbf{\Delta}^X)_{X\in\mathcal{X}})$ be a root system of type $\mathcal{C}$. Then the following are equivalent.*

(1) *There exists $X \in \mathcal{X}$ such that A^X is indecomposable*

(2) *The semi-Cartan graph $\mathcal{C}$ is indecomposable.*

If $\mathcal{R}$ is finite then the semi-Cartan graph $\mathcal{C}$ is indecomposable if and only if the root system $\mathcal{R}$ is irreducible.

Definition 1.5.10. We say $\mathcal{C}$ is a *Cartan graph* if the following hold:

- For all $X \in \mathcal{X}$ the set $\mathbf{\Delta}^{X\,\mathrm{re}} = \mathbf{\Delta}_+^{X\,\mathrm{re}} \bigcup \mathbf{\Delta}_-^{X\,\mathrm{re}}$.
- If $l_{mn}^Y := |\mathbf{\Delta}^{Y\,\mathrm{re}} \cap (\mathbb{N}_0\alpha_m + \mathbb{N}_0\alpha_n)|$ is finite, then $(r_m r_n)^{l_{mn}^Y}(Y) = Y$, where $m, n \in I, Y \in \mathcal{X}$.

In this case, $\mathcal{W}(\mathcal{C})$ is called the *Weyl groupoid of $\mathcal{C}$*.

Notice that $\mathcal{C}$ is a Cartan graph if and only if $\mathcal{R}^{re} := \mathcal{R}(\mathcal{C}, (\mathbf{\Delta}^{X\,\mathrm{re}})_{X\in\mathcal{X}})$ is a root system of type $\mathcal{C}$. Indeed, we get that $s_i^X(\mathbf{\Delta}^{X\,\mathrm{re}}) = \mathbf{\Delta}^{r_i(X)\,\mathrm{re}}$ by Equation (1.12). For all $X \in \mathcal{X}$, we obtain that $\mathbf{\Delta}^{X\,\mathrm{re}} = \mathbf{\Delta}_+^{X\,\mathrm{re}} \cup \mathbf{\Delta}_-^{X\,\mathrm{re}}$, since $\omega s_i^{r_i(X)}(\alpha_i) = -\omega(\alpha_i)$ for any $\omega \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), X)$.

The following proposition implies that if $\mathcal{R}$ is a finite root system of type $\mathcal{C}$ then $\mathcal{R} = \mathcal{R}^{re}$, namely, all roots are real and $\mathcal{R}$ is uniquely determined by $\mathcal{C}$.

Proposition 1.5.11. *[13, Proposition 2.12] Let $\mathcal{R} = \mathcal{R}(\mathcal{C}, (\mathbf{\Delta}^X)_{X\in\mathcal{X}})$ be a root system of type $\mathcal{C}$. Let $X \in \mathcal{X}$, $m \in \mathbb{N}_0$, and $i_1, \dots, i_m \in I$ such that*

$$\omega = \mathrm{id}_X s_{i_1} s_{i_2} \cdots s_{i_m} \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), X)$$

and $\ell(\omega) = m$. *Then the elements*

$$\beta_n = \mathrm{id}_X s_{i_1} s_{i_2} \cdots s_{i_{n-1}}(\alpha_{i_n}) \in \boldsymbol{\Delta}^X \cap \mathbb{N}_0^I,$$

are pairwise different, where $n \in \{1, 2, \ldots, m\}$ *(and* $\beta_1 = \alpha_{i_1}$*). Here,*

$$\ell(\omega) = \min\{m \in \mathbb{N}_0 | \omega = \mathrm{id}_X s_{i_1} s_{i_2} \cdots s_{i_m}, i_1, i_2, \ldots, i_m \in I\}$$

is the length of $\omega \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}), X)$. *In particular, if* $\mathcal{R}$ *is finite and* $\omega \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}))$ *is the longest element, then*

$$\{\beta_n \,|\, 1 \leq n \leq \ell(\omega) = |\boldsymbol{\Delta}^X|/2\} = \boldsymbol{\Delta}^X \cap \mathbb{N}_0^I.$$

Remark 1.5.12. If $\mathcal{C}$ is a finite Cartan graph then $\mathcal{R}$ is finite and hence $\mathcal{R}^{re} = \mathcal{R}^{re}(\mathcal{C}, (\boldsymbol{\Delta}^{X\,\mathrm{re}})_{X \in \mathcal{X}})$ is the unique root system of type $\mathcal{C}$ by Proposition 1.5.11, that is, $\mathcal{R}$ is uniquely determined by $\mathcal{C}$.

There are many examples of Weyl groupoids. We give the following.

Example 1.5.13. Let $\mathcal{C} = \mathcal{C}(I, \mathcal{X}, R, (A^X)_{X \in \mathcal{X}})$ be a semi-Cartan graph, where $I = \{1, 2\}$, $\mathcal{X} = \{X_1, X_2, \ldots, X_6\}$, $A^{X_1} = A^{X_6} = \begin{pmatrix} 2 & -2 \\ -3 & 2 \end{pmatrix}$, $A^{X_2} = A^{X_5} = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$, $A^{X_3} = A^{X_4} = \begin{pmatrix} 2 & -6 \\ -1 & 2 \end{pmatrix}$, and the reflection maps R_1 and R_2 are defined by $\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} (1, 2, 3, 4, 5, 6) = \begin{pmatrix} 2 & 1 & 4 & 3 & 6 & 5 \\ 1 & 3 & 2 & 5 & 4 & 6 \end{pmatrix}$. That means the exchange graph of $\mathcal{C}$ is

$$X_1 \overset{1}{\rule[0.5ex]{2em}{0.4pt}} X_2 \overset{2}{\rule[0.5ex]{2em}{0.4pt}} X_3 \overset{1}{\rule[0.5ex]{2em}{0.4pt}} X_4 \overset{2}{\rule[0.5ex]{2em}{0.4pt}} X_5 \overset{1}{\rule[0.5ex]{2em}{0.4pt}} X_6.$$

Then for all $i \in I$ the sets of positive roots $\boldsymbol{\Delta}_+^{X_i\,\mathrm{re}}$ up to the permutation of I are

$$\begin{aligned}
\boldsymbol{\Delta}_+^{X_1\,\mathrm{re}} = \boldsymbol{\Delta}_+^{X_6\,\mathrm{re}} &= \{1, 2, 12, 12^2, 12^3, 1^22, 1^22^3, 1^22^5, 1^32^4, 1^32^5, 1^42^5, 1^42^7\} \\
\boldsymbol{\Delta}_+^{X_2\,\mathrm{re}} = \boldsymbol{\Delta}_+^{X_5\,\mathrm{re}} &= \{1, 2, 12, 1^22, 1^32^2, 1^42^3, 1^52^3, 1^52^4, 1^62^5, 1^72^5, 1^82^5, 1^{10}2^7\} \\
\boldsymbol{\Delta}_+^{X_3\,\mathrm{re}} = \boldsymbol{\Delta}_+^{X_4\,\mathrm{re}} &= \{1, 2, 12, 1^22, 1^32, 1^42, 1^52, 1^52^2, 1^62, 1^72^2, 1^82^3, 1^{10}2^3\},
\end{aligned}$$

where the word $1^{x_1}2^{x_2}$ is corresponding to $x_1\alpha_1 + x_2\alpha_2$, $x_1, x_2 \in \mathbb{N}_0$.

We will see that in the following chapters there are Nichols algebras of diagonal type with such Weyl groupoids appearing in Example 1.5.13 over fields of characteristic 7.

Chapter 2

Cartan graphs for Nichols algebras of diagonal type

In this Chapter we attach a Semi-Cartan graph to a tuple of finite-dimensional Yetter–Drinfel'd modules under some finiteness conditions.

2.1 Cartan graphs for Nichols algebras of diagonal type

Let G be an abelian group, $\theta \in \mathbb{N}$, and $I = \{1, 2, \dots, \theta\}$. Let $\mathcal{F}_\theta^G$ be the set of θ-tuples of finite-dimensional irreducible objects in ${}_G^G\mathcal{YD}$ and $\mathcal{X}_\theta^G$ be the set of θ-tuples of isomorphism classes of finite-dimensional irreducible objects in ${}_G^G\mathcal{YD}$. For any $(M_1, \dots, M_\theta) \in \mathcal{F}_\theta^G$, write $[M] := ([M_1], \dots, [M_\theta]) \in \mathcal{X}_\theta^G$ the corresponding isomorphism class of $(M_1, \dots, M_\theta)$.

In this section, we always assume that $V_M = \oplus_{i\in I}\Bbbk x_i \in {}_G^G\mathcal{YD}$ is a Yetter–Drinfel'd module of diagonal type over G, where $\{x_i | i \in I\}$ is a basis of V. Then by Remark 1.1.3(ii) there exists a matrix $(q_{ij})_{i,j\in I}$ such that $\delta(x_i) = g_i \otimes x_i$ and $g_i.x_j = q_{ij}x_j$ for all $i, j \in I$. We fix that $M = (\Bbbk x_1, \Bbbk x_2, \dots, \Bbbk x_\theta) \in \mathcal{F}_\theta^G$ is a tuple of one-dimensional Yetter–Drinfel'd over G and $[M] \in \mathcal{X}_\theta^G$. We say that the matrix $(q_{ij})_{i,j\in I}$ is the *braiding matrix of* M. Recall that the matrix is independent of the basis $\{x_i | i \in I\}$ up to permutation of I. We say $\mathcal{B}(V_M) = \mathcal{B}(\oplus_{i=1}^n \Bbbk x_i)$ is the Nichols algebra of the tuple M, denoted by $\mathcal{B}(M)$. The set of roots of Nichols algebra $\mathcal{B}(M)$ can be always defined by Remark 1.4.3 and it is denoted by $\boldsymbol{\Delta}^{[M]}$.

Definition 2.1.1. [5] The adjoint representation ad of a Nichols algebra $\mathcal{B}(V)$ is the linear map $\mathrm{ad}_c : V \to \mathrm{End}(\mathcal{B}(V))$

$$\mathrm{ad}_c x(y) = \mu(\mathrm{id} - c)(x \otimes y) = xy - (x_{(-1)}.y)x_{(0)}$$

for all $x \in V, y \in \mathcal{B}(V)$, where μ is the multiplication map of $\mathcal{B}(V)$ and c is defined by Equation (1.4).

In particular, the braided commutator ad_c of $\mathcal{B}(M)$ takes the form

$$\mathrm{ad}_c x_i(y) = x_i y - (g_i.y)x_i \textit{ for all } i \in I,\ y \in \mathcal{B}(M).$$

Definition 2.1.2. [28, Definition 6.4] Let $i \in I$. We say that M is *i-finite*, if for any $j \in I \setminus \{i\}$, $(\mathrm{ad}_c x_i)^m(x_j) = 0$ for some $m \in \mathbb{N}$.

Lemma 2.1.3. *[7, Lemma 3.7] For any $i, j \in I$ with $i \neq j$, the following are equivalent.*

(a) $(m+1)_{q_{ii}}(q_{ii}^m q_{ij} q_{ji} - 1) = 0$ *and* $(k+1)_{q_{ii}}(q_{ii}^k q_{ij} q_{ji} - 1) \neq 0$ *for all* $0 \leq k < m$.

(b) $(\mathrm{ad}_c x_i)^{m+1}(x_j) = 0$ *and* $(\mathrm{ad}_c x_i)^m(x_j) \neq 0$ *in* $\mathcal{B}(V)$.

Here $(n)_q := 1 + q + \cdots + q^{n-1}$, *which is* 0 *if and only if* $q^n = 1$ *for* $q \neq 1$ *or* $p|n$ *for* $q = 1$.

Hence we get the following from Lemma 2.1.3.

Lemma 2.1.4. *Let $i \in I$. Then $M = (\Bbbk x_j)_{j \in I}$ is i-finite if and only if for any $j \in I \setminus \{i\}$ there is a non-negative integer m satisfying $(m+1)_{q_{ii}}(q_{ii}^m q_{ij} q_{ji} - 1) = 0$.*

Let $i \in I$. Assume that M is i-finite. Let $(a_{ij}^M)_{j \in I} \in \mathbb{Z}^I$ and $R_i(M) = (R_i(M)_j)_{j \in I}$, where

$$a_{ij}^M = \begin{cases} 2 & \text{if } j = i, \\ -\max\{m \in \mathbb{N}_0 \mid (\mathrm{ad}_c x_i)^m(x_j) \neq 0\} & \text{if } j \neq i. \end{cases}$$

$$R_i(M)_i = \Bbbk y_i, \qquad R_i(M)_j = \Bbbk(\mathrm{ad}_c x_i)^{-a_{ij}^M}(x_j), \tag{2.1}$$

where $y_i \in (\Bbbk x_i)^* \setminus \{0\}$. If M is not i-finite, then let $R_i(M) = M$. Then $R_i(M)$ is a θ-tuple of one-dimensional Yetter–Drinfel'd modules over G.

Let

$$\mathcal{F}_\theta^G(M) = \{R_{i_1} \cdots R_{i_n}(M) \in \mathcal{F}_\theta^G | \, n \in \mathbb{N}_0, i_1, \dots, i_n \in I\}.$$

$$\mathcal{X}_\theta^G(M) = \{[R_{i_1} \cdots R_{i_n}(M)] \in \mathcal{X}_\theta^G | \, n \in \mathbb{N}_0, i_1, \dots, i_n \in I\}$$

Definition 2.1.5. We say that M *admits all reflections* if N is i-finite for all $N \in \mathcal{F}_\theta^G(M)$.

The reflections depend only on the braiding matrix $(q_{ij})_{i,j\in I}$. It is useful to introduce a new notion for braided vector spaces of diagonal type.

Definition 2.1.6. [25, Definition 4] Let V be a θ-dimensional braided vector space of diagonal type with the braiding matrix $(q_{ij})_{i,j\in I}$. The *Dynkin diagram* of V is a non-directed graph $\mathcal{D}$ with the following properties:

- there is a bijective map ϕ from I to the vertices of $\mathcal{D}$,
- for all $i \in I$ the vertex $\phi(i)$ is labeled by q_{ii},
- for all $i, j \in I$ with $i \neq j$, the number n_{ij} of edges between $\phi(i)$ and $\phi(j)$ is either 0 or 1. If $q_{ij}q_{ji} = 1$ then $n_{ij} = 0$, otherwise $n_{ij} = 1$ and the edge is labeled by $q_{ij}q_{ji}$.

We say that the *Dynkin diagram of* M is the Dynkin diagram of braided vector space $\oplus_{i\in I}M_i$. Notice that the Dynkin diagram of M is connected if the braiding matrix of M is indecomposable.

From the method in [25, Example 1], one can obtain the labels of the Dynkin diagram of $R_i(M) = (R_i(M)_j)_{j\in I}$. In more details, we have the following lemma.

Lemma 2.1.7. *Let $i \in I$. Assume that M is i-finite and let $a_{ij} := a_{ij}^M$ for all $j \in I$. Let $(q'_{jk})_{j,k\in I}$ be the braiding matrix of $R_i(M)$ with respect to $(y_j)_{j\in I}$. Then*

$$q'_{jj} = \begin{cases} q_{ii} & \text{if } j = i, \\ q_{jj} & \text{if } j \neq i,\ q_{ij}q_{ji} = q_{ii}^{a_{ij}}, \\ q_{ii}q_{jj}(q_{ij}q_{ji})^{-a_{ij}} & \text{if } j \neq i,\ q_{ii} \in G'_{1-a_{ij}}, \\ q_{jj}(q_{ij}q_{ji})^{-a_{ij}} & \text{if } j \neq i,\ q_{ii} = 1, \end{cases}$$

$$q'_{ij}q'_{ji} = \begin{cases} q_{ij}q_{ji} & \text{if } j \neq i,\ q_{ij}q_{ji} = q_{ii}^{a_{ij}}, \\ q_{ii}^2(q_{ij}q_{ji})^{-1} & \text{if } j \neq i,\ q_{ii} \in G'_{1-a_{ij}}, \\ (q_{ij}q_{ji})^{-1} & \text{if } j \neq i,\ q_{ii} = 1, \end{cases}$$

and

$$q'_{jk}q'_{kj} = \begin{cases} q_{jk}q_{kj} & \text{if } q_{ir}q_{ri} = q_{ii}^{a_{ir}},\ r \in \{j,k\}, \\ q_{jk}q_{kj}(q_{ik}q_{ki}q_{ii}^{-1})^{-a_{ij}} & \text{if } q_{ij}q_{ji} = q_{ii}^{a_{ij}},\ q_{ii} \in G'_{1-a_{ik}}, \\ q_{jk}q_{kj}(q_{ij}q_{ji})^{-a_{ik}}(q_{ik}q_{ki})^{-a_{ij}} & \text{if } q_{ii} = 1, \\ q_{jk}q_{kj}q_{ii}^2(q_{ij}q_{ji}q_{ik}q_{ki})^{-a_{ij}} & \text{if } q_{ii} \in G'_{1-a_{ik}},\ q_{ii} \in G'_{1-a_{ij}}. \end{cases}$$

for $j, k \neq i$, $j \neq k$.

Example 2.1.8. Assume that M is a braided vector space of diagonal type over fields of characteristic 7 and the Dynkin diagram of M is $\overset{1}{\circ}\overset{-\zeta}{\text{———}}\overset{-1}{\circ}$. Then the generalized Cartan matrix A^M of M is $\begin{pmatrix} 2 & -6 \\ -1 & 2 \end{pmatrix}$. By Lemma 2.1.7 the Dynkin diagrams of $R_1(M)$ and $R_2(M)$ are $\overset{1}{\circ}\overset{-\zeta^{-1}}{\text{———}}\overset{-1}{\circ}$ and $\overset{\zeta}{\circ}\overset{-\zeta^{-1}}{\text{———}}\overset{-1}{\circ}$, respectively.

If M admits all reflections, then we are able to construct a semi-Cartan graph $\mathcal{C}(M)$ of M by the above discussions.

Theorem 2.1.9. *Assume that M admits all reflections. For all $X \in \mathcal{X}_\theta^G(M)$ let*

$$[X]_\theta = \{Y \in \mathcal{X}_\theta^G(M) |\ Y \text{ and } X \text{ have the same Dynkin diagram}\}.$$

Let $\mathcal{Y}_\theta(M) = \{[X]_\theta |\ X \in \mathcal{X}_\theta^G(M)\}$ and $A^{[X]_\theta} = A^X$ for all $X \in \mathcal{X}_\theta^G(M)$. Let $t : I \times \mathcal{Y}_\theta(M) \to \mathcal{Y}_\theta(M)$, $(i, [X]_\theta) \mapsto [R_i(X)]_\theta$. Then the tuple

$$\mathcal{C}(M) = \{I, \mathcal{Y}_\theta(M), t, (A^Y)_{Y \in \mathcal{Y}_\theta(M)}\}$$

is a connected semi-Cartan graph. We say that $\mathcal{C}(M)$ is the semi-Cartan graph attached to M.

Proof. Since M admits all reflections, we obtain that A^X, where $X \in \mathcal{X}_\theta^G(M)$, is well-defined and all entries are finite. Moreover, if $a_{ij}^X = 0$ then $a_{ji}^X = 0$ by Lemma 2.1.4. Hence A^X is a well-defined generalized Cartan matrix for all $X \in \mathcal{X}_\theta^G(M)$. For any $X, Y \in \mathcal{X}_\theta^G(M)$, if X and Y have the same Dynkin diagram then $A^X = A^Y$ and hence $A^{[X]_\theta} = A^{[Y]_\theta}$. Then $A^{[X]_\theta}$ is well-defined for all $X \in \mathcal{X}_\theta^G(M)$. Hence $\{A^Y\}_{Y \in \mathcal{Y}_\theta(M)}$ is a family of generalized Cartan matrices. Besides, if N is i-finite then $a_{ij}^N = a_{ij}^{R_i(N)}$ and $R_i^2(N) = N$ for all $N \in \mathcal{X}_\theta^G(M)$ by [4, Theorem 3.12(2)]. Hence t_i is a reflection map for all $i \in I$. Then $\mathcal{C}(M)$ is a well-defined semi-Cartan graph. From the construction of the reflection R_i by Equation 2.1 we obtain that $\mathcal{C}(M)$ is connected. ☐

Therefore if M admits all reflections then we can attach a groupoid $\mathcal{W}(M) := \mathcal{W}(\mathcal{C}(M))$ to M.

Notice that the set of roots $\boldsymbol{\Delta}^{[M]}$ of M is defined, since $\mathcal{B}(M)$ is decomposable. If the set of roots $\boldsymbol{\Delta}^{[M]}$ is finite then we can check that M admits all reflections by [28, Corollary 6.12]. If M admits all reflections then we can define a root system $\mathcal{R}(M)(\mathcal{C}(M), (\boldsymbol{\Delta}^{[N]})_{N \in \mathcal{F}_\theta^G(M)})$ of type $\mathcal{C}(M)$ by the following theorem.

Theorem 2.1.10. *Assume that M admits all reflections. Then*

$$\mathcal{R}(M) := \mathcal{R}(M)(\mathcal{C}(M), (\boldsymbol{\Delta}^{[N]})_{N\in\mathcal{F}_\theta^G(M)})$$

is a root system of type $\mathcal{C}(M)$.

Proof. Since M is a θ-tuple of one-dimensional Yetter–Drinfel'd modules, we obtain that the Nichols algebra $\mathcal{B}(M)$ generated by V_M is decomposable and hence $\boldsymbol{\Delta}^{[M]}$ is defined. Then the claim is true by [28, Theorem 6.11]. ☐

Hence we get the following results by Lemma 1.5.8.

Corollary 2.1.11. *Assume that M admits all reflections. Then the following are equivalent.*

(1) $\boldsymbol{\Delta}^{[M]}$ *is finite.*

(2) $\mathcal{C}(M)$ *is a finite Cartan graph.*

(3) $\mathcal{W}(M)$ *is finite.*

(4) $\mathcal{R}(M)$ *is finite.*

In all cases, $\mathcal{R}(M) = \mathcal{R}(M)(\mathcal{C}(M), (\boldsymbol{\Delta}^{[N]})_{N\in\mathcal{F}_\theta^G(M)})$ is the unique root system of type $\mathcal{C}(M)$.

Hence if the set of roots $\boldsymbol{\Delta}^{[M]}$ of Nichols algebra $\mathcal{B}(M)$ is finite then $\mathcal{C}(M)$ is a finite Cartan graph and $\mathcal{R}(M) = \mathcal{R}(M)(\mathcal{C}(M), (\boldsymbol{\Delta}^{[N]})_{M\in\mathcal{F}_\theta^G(N)})$ is the unique finite root system of type $\mathcal{C}(M)$.

Chapter 3

Finite Cartan graphs

Assume that $M = (\Bbbk x_i)_{i\in I} \in \mathcal{F}_\theta^G$ is a tuple of one-dimensional Yetter–Drinfel'd modules over $\Bbbk G$ and that the semi-Cartan graph $\mathcal{C}(M) = \mathcal{C}(I, \mathcal{X}, r, (A^X)_{X\in\mathcal{X}})$ attached to M defined in Chapter 2 is connected and indecomposable. In this chapter we obtain the properties of the Nichols algebra $\mathcal{B}(M)$ of diagonal type by analysing $\mathcal{C}(M)$. The main results are Theorem 3.1.14 and Theorem 3.2.6, which are important to our further classification in Chapter 4. They simplify enormously the calculations needed to check that the Weyl groupoids of the Nichols algebras of our classification are finite.

In Section 3.1 we obatin Theorem 3.1.14, which characterizes finite connected Cartan graph $\mathcal{C}(M)$ of rank two in terms of certain integer sequences under some conditions. To prove this we introduce the characteristic sequence of $\mathcal{C}(M)$, which characterizes the reflections of $\mathcal{C}(M)$. Besides, we introduce a set of integer sequences $\mathcal{A}^+$, which has nice combinatorial properties. In Proposition 3.1.6 we prove that any integer sequence in $\mathcal{A}^+$ contains at least one of the subsequence forms

$$(1,1), (1,2,a), (2,1,b), (1,3,1,b)$$

or their transpose, where $1 \le a \le 3$ and $3 \le b \le 5$. If the Cartan graph $\mathcal{C}(M)$ is finite, then there exist subsequences of characteristic sequences of $\mathcal{C}(M)$ are the elements of $\mathcal{A}^+$.

Section 3.2 is denoted to obtain the properties of the Weyl groupoids of rank three connected indecomposable finite Cartan graph $\mathcal{C}(M)$. In Theorem 3.2.6, we prove that every finite connected indecomposable Cartan graph of rank three contains a point which has at least one of the good A_3, B_3, or C_3 neighborhoods, see Definitions (3.2.2-3.2.4). Our main reference is [15]. I. Heckenberger and M. Cuntz classified all finite irreducible root systems of type $\mathcal{C}(M)$ in [15]. The authors proved that if $\mathcal{C}(M)$ is finite then there exists a point $X \in \mathcal{X}$ satisfying that the set $\boldsymbol{\Delta}_+^{X\,\mathrm{re}}$ is in the list of [15, Appendix A] up to a permutation of I.

All the results presented in this chapter are independent of the field $\Bbbk$ and the notion of the Nichols algebras.

3.1 Rank 2 case

In this section we give a characterization of finite indecomposable Cartan graphs of rank two. Our main references are [12] and [32].

Definition 3.1.1. Let $\mathcal{A}^+$ denote the smallest subset of $\cup_{n\geq 2}\mathbb{N}_0^n$ such that

- $(0,0) \in \mathcal{A}^+$,
- if $(c_1,\dots,c_n) \in \mathcal{A}^+$ and $1 < i \leq n$, then $(c_1,\dots,c_{i-2},c_{i-1}+1,1,c_i+1,\dots,c_n) \in \mathcal{A}^+$.

Remark 3.1.2. We say that two consecutive entries of a sequence in $\mathcal{A}^+$ are neighbors. Note that our definition of $\mathcal{A}^+$ is different from the one from [14] and [12].

From the definition of $\mathcal{A}^+$, it is not hard to get the following Lemma.

Lemma 3.1.3. *Let $n \geq 2$ and $(c_1,\dots,c_n) \in \mathcal{A}^+$. Then $\Sigma_{i=1}^n c_i = 3n-6$.*

The definition of $\mathcal{A}^+$ implies the following interesting property.

Proposition 3.1.4. *Let $n \geq 2$. Enumerate the vertices of a convex n-gon by $1,\dots,n$ such that consecutive integers correspond to neighboring vertices. Let T_n be the set of triangulations of a convex n-gon with non-intersecting diagonals. Let $T = \cup_{n\geq 2}T_n$. For any triangulation $t \in T_n$ and any $i \in \{1,\dots,n\}$, let c_i be the number of triangles meeting at the i-th vertex. Then the map $\psi : T \to \mathcal{A}^+, t \mapsto (c_1,\dots,c_n)$ is a bijection.*

Proof. We proceed by induction on n. For $n=2$, a triangulation of a convex 2-gon is itself. Then $(c_1,c_2)=(0,0)$. Hence the claim is true for $n=2$. For $n\geq 3$, the definition of $\mathcal{A}^+$ corresponds bijectively to the construction of a triangulation of a convex $(n+1)$-gon by adding a new triangle between two consecutive vertices of a convex n-gon, but not at the edge between the first and the last vertex. By adding one triangle between two consecutive vertices of a convex n-gon, one increases the number of triangles at the two adjacent vertices and the number of triangles at the new vertex becomes 1. $\square$

Corollary 3.1.5. *Let $n \geq 2$ and let $(c_1,\dots,c_n) \in \mathcal{A}^+$.*

(1) *$(c_n, c_{n-1}, \dots, c_1) \in \mathcal{A}^+$ and $(c_2, c_3, \dots, c_n, c_1) \in \mathcal{A}^+$.*

(2) *If $n \geq 3$, then there is $1 < i < n$ satisfying $c_i = 1$. For any such i,*

$$(c_1, \dots, c_{i-2}, c_{i-1} - 1, c_{i+1} - 1, c_{i+2}, \dots, c_n) \in \mathcal{A}^+.$$

(3) *If $n \geq 3$, then $c_i \geq 1$ for all $1 \leq i \leq n$.*

(4) *If $c_i = 1$ and $c_{i+1} = 1$ for some $1 \leq i \leq n-1$, then $n = 3$ and $c = (1, 1, 1)$.*

Proof. (1) and (2) follow directly from the bijection between $\mathcal{A}^+$ and triangulations of convex n-gons in Proposition 3.1.4. (3) follows from the definition of $\mathcal{A}^+$. (4) follows from (2) and (3). □

Proposition 3.1.6. *Let $n \geq 3$. Then any sequence $(c_1, \dots, c_n) \in \mathcal{A}^+$ contains a subsequence $(c_k)_{i \leq k \leq j}$, where $1 \leq i \leq j \leq n$, of the form*

$$(1, 1), (1, 2, a), (2, 1, b), (1, 3, 1, b)$$

or their transpose, where $1 \leq a \leq 3$ and $3 \leq b \leq 5$.

Remark 3.1.7.

- We record that it is natural to exclude the cases $b = 1$ and $b = 2$ since $(1, 3, 1, 1)$ contains the subsequence $(1, 1)$ and $(1, 3, 1, 2)$ contains the transpose of $(2, 1, 3)$.
- The claim becomes false by omitting one of the sequences from the theorem. In Table 3.1 we list sequences in $\mathcal{A}^+$ which contain precisely one of the sequences in Proposition 3.1.6.

Proof. Let $c = (c_1, \dots, c_n) \in \mathcal{A}^+$ such that the claim does not hold for c. Then $n \geq 5$ and c has no subsequence $(2, 1, 2)$. Otherwise $c = (1, 2, 1, 2)$ or $c = (2, 1, 2, 1)$ by Corollary 3.1.5(2),(4). We define $E = \{\nu_{ij} \,|\, i, j \in \{1, 2\}\}$, where the sequences ν_{ij} are given by

$$\nu_{11} = (1),\ \nu_{12} = (2, 1),\ \nu_{21} = (1, 2),\ \nu_{22} = (1, 3, 1).$$

Now we decompose c by the following steps.
Replace all subsequences $(2, 1)$ by ν_{12}, then all subsequences $(1, 2)$ by ν_{21}, then all subsequences $(1, 3, 1)$ by ν_{22}, and finally all entries 1 by ν_{11}. By this construction, $(\nu_{11}, 3, \nu_{11})$ is not a subsequence of d. Hence we get a decomposition $d = (d_1, \dots, d_k)$, where $k \geq 2$, of c into subsequences of the form (a) and ν, where

subsequences	sequences in $\mathcal{A}^+$
$(1,1)$	$(1,1,1)$
$(1,2,1)$	$(1,2,1,2)$
$(1,2,2)$	$(1,2,2,2,2,2,1,6)$
$(1,2,3)$	$(1,2,3,1,6,1,2,3,1,6,1,2,3,1,6)$
$(2,1,3)$	$(2,1,3,4,2,1,3,4,2,1,3,4)$
$(2,1,4)$	$(2,1,4,2,1,4,2,1,4)$
$(2,1,5)$	$(2,1,5,1,2,4,2,1,5,1,2,4)$
$(1,3,1,3)$	$(1,3,1,3,1,3)$
$(1,3,1,4)$	$(1,3,1,4,1,3,1,4)$
$(1,3,1,5)$	$(1,3,1,5,1,3,1,5,1,3,1,5)$

Table 3.1: Sequences in $\mathcal{A}^+$ containing exactly one subsequence

$a \geq 2$ and $\nu \in E$.

Since $(1,1)$, $(1,2,a)$, $(2,1,b)$, $(1,3,1,b)$ and their transposes are not subsequences of c, where $1 \leq a \leq 3$, and $2 \leq b \leq 5$, we obtain the following conditions on the entries of d.

- No entry ν_{ij} of d, where $i,j \in \{1,2\}$, has 2 or ν_{kl} with $k,l \in \{1,2\}$ as a neighbor.
- If (ν_{21}, a) or (a, ν_{12}) is a subsequence of d, then $a \geq 4$.
- If (ν_{i2}, b) or (b, ν_{2i}) is a subsequence of d, where $i \in \{1,2\}$, then $b \geq 6$.

By applying Corollary 3.1.5(2) we get further reductions of d:

$$(\dots, d_{m-1}, \nu_{ij}, d_{m+1}, \dots) \to (\dots, d_{m-1} - i, d_{m+1} - j)$$

$$(\nu_{i2}, d_2, \dots) \to (\nu_{i1}, d_2 - 1, \dots)$$

where $i,j \in \{1,2\}$. Thus we can perform such reductions at all places in d, where an entry ν_{ij} with $i,j \in \{1,2\}$ appears. After decreasing them, we get $d_m \geq 2$, where $1 < m < k$. Indeed, we get the following conditions.

- If $d = (\dots, d_{m-1}, d_m, d_{m+1}, \dots)$, where $d_m \geq 6$, then d_m can be reduced at most by 4. Hence the value of d_m after reduction is at least 2.
- If $4 \leq d_m \leq 5$, then neither (ν_{i2}, d_m) nor (d_m, ν_{2i}) is a subsequence of d, where $i \in \{1,2\}$. Hence d can be reduced by at most 2.

- If $d_m = 3$, then $d_{m-1}, d_{m+1} \notin \{\nu_{12}, \nu_{21}, \nu_{22}\}$. Further, $(d_{m-1}, d_{m+1}) \neq (\nu_{11}, \nu_{11})$. Hence d_m decreases by at most 1.
- If $d_m = 2$, then it has no neighbour ν_{ij} with $i, j \in \{1, 2\}$. Hence d_m does not change.

Thus one can reduce c to a sequence $(c'_1, \ldots, c'_l)$ with $l \geq 1$, where $c'_m \geq 2$ for all $1 < m < l$ and $c'_1, c'_l \geq 1$. This is a contradiction to Corollary 3.1.5(2). ☐

Recall that (α_1, α_2) is the standard basis of $\mathbb{Z}^2$. We define a map

$$\eta : \mathbb{Z} \to \mathrm{SL}(2, \mathbb{Z}), \quad a \mapsto \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} \tag{3.1}$$

Lemma 3.1.8. *Let $n \in \mathbb{N}$ and $(c_k)_{1 \leq k \leq n} \in \mathbb{Z}^n$. For all $1 \leq k \leq n+1$, let $\beta_0 = -\alpha_2$ and $\beta_k = \eta(c_1) \cdots \eta(c_{k-1})(\alpha_1)$. Then the following hold.*

(1) *$\beta_{k+1} = c_k \beta_k - \beta_{k-1}$ for all $1 \leq k \leq n$.*

(2) *If $c_1 \geq 1$ and $c_k \geq 2$ for all $1 < k < n$, then $\beta_k \in \mathbb{N}_0^2$ for all $1 \leq k \leq n$ and $\beta_k - \beta_{k-1} \in \mathbb{N}_0^2 \setminus \{0\}$ for $1 < k \leq n$.*

Proof. (1) By definition, $\beta_1 = \alpha_1$ and $\beta_2 = \eta(c_1)(\alpha_1) = c_1\alpha_1 + \alpha_2$. Thus the claim holds for $k = 1$. Since $\eta(c_{k-1})(\alpha_2) = -\alpha_1$, then

$$\begin{aligned} \beta_{k+1} =& \eta(c_1) \cdots \eta(c_k)(\alpha_1) \\ =& \eta(c_1) \cdots \eta(c_{k-1})(c_k\alpha_1 + \alpha_2) \\ =& c_k\beta_k - \beta_{k-1} \end{aligned}$$

for all $k \geq 2$.

(2) For all $0 \leq k \leq n$, let $a_k, b_k \in \mathbb{Z}$ such that $\beta_k = a_k\alpha_1 + b_k\alpha_2$. By induction on k, we get the following.

- If $c_k \geq 2$ for $1 \leq k < n$, then

  $$a_k > b_k \geq 0, \quad a_k > a_{k-1}, \quad b_k > b_{k-1}, \quad a_k - b_k - (a_{k-1} - b_{k-1}) \geq 0$$

  for all $1 \leq k \leq n$.
- If $c_1 = 1$ and $c_k \geq 2$ for $2 \leq k < n$, then

  $$b_k \geq a_k > 0, \quad a_k \geq a_{k-1}, \quad b_k > b_{k-1}, \quad a_k - b_k - (a_{k-1} - b_{k-1}) < 0$$

  for all $2 \leq k \leq n$.

Thus $\beta_k \in \mathbb{N}_0^2$ for all $1 \le k \le n$ and $\beta_k - \beta_{k-1} \in \mathbb{N}_0^2 \setminus \{0\}$ for $1 < k \le n$. □

The following theorem will be used in the proof of Theorem 3.1.14. It was proven partially in [12].

Theorem 3.1.9. *Let* $n \ge 2$ *and* $(c_i)_{1\le i\le n} \in \mathbb{Z}^n$. *Then the following are equivalent.*

(1) $(c_i)_{1\le i\le n} \in \mathcal{A}^+$,

(2) $\eta(c_1)\cdots\eta(c_n) = -\mathrm{id}$ *and* $\beta_k = \eta(c_1)\cdots\eta(c_{k-1})(\alpha_1) \in \mathbb{N}_0^2$ *for all* $1 \le k \le n$.

Proof. $(1) \Rightarrow (2)$. We apply induction on n. If $n = 2$, then $(c_1, c_2) = (0, 0)$, $\eta(0)^2 = -\mathrm{id}$, $\beta_1 = \alpha_1$, and $\beta_2 = \alpha_2$. Assume that $n \ge 3$. By the definition of $\mathcal{A}^+$, there is a $(c'_1, \dots, c'_{n-1}) \in \mathcal{A}^+$ and $1 < i \le n-1$ such that

$$(c_1, \dots, c_n) = (c'_1, \dots, c'_{i-1} + 1, 1, c'_i + 1, c'_{i+1}, \dots, c'_{n-1}).$$

By calculation,

$$\eta(a)\eta(b) = \eta(a+1)\eta(1)\eta(b+1) \tag{3.2}$$

for all $a, b \in \mathbb{Z}$. Then $\eta(c_1)\cdots\eta(c_n) = \eta(c'_1)\cdots\eta(c'_{n-1}) = -\mathrm{id}$.
Let $\beta'_i = \eta(c'_1)\cdots\eta(c'_{i-1})(\alpha_1)$ for all $1 \le i \le n-1$. Then $\beta_k = \beta'_k$ for all $1 \le k < i$ and $\beta_k = \beta'_{k-1}$ for all $i+1 \le k \le n+1$. Finally

$$\begin{aligned}\beta_i =&\eta(c_1)\cdots\eta(c_{i-1})(\alpha_1)\\ =&\eta(c'_1)\cdots\eta(c'_{i-2})\eta(c'_{i-1}+1)(\alpha_1)\\ =&\eta(c'_1)\cdots\eta(c'_{i-2})(\eta(c'_{i-1})(\alpha_1)+\alpha_1)\\ =&\beta'_i + \beta'_{i-1} \in \mathbb{N}_0^2.\end{aligned}$$

Then (2) follows.

$(2) \Rightarrow (1)$. Again we proceed by induction. If $n = 2$, then

$$\eta(c_1)\eta(c_2) = \begin{pmatrix} c_1c_2 - 1 & -c_1 \\ c_2 & -1 \end{pmatrix} = -\mathrm{id}$$

implies that $(c_1, c_2) = (0, 0) \in \mathcal{A}^+$. Assume that $n \ge 3$. Set $\beta_0 = -\alpha_2$. One has $\beta_{k+1} = c_k\beta_k - \beta_{k-1}$ for all $1 \le k < n$. By assumption, the condition β_{k-1}, β_k, $\beta_{k+1} \in \mathbb{N}_0^2$ implies $c_k > 0$ for $2 \le k < n$ and $c_1 \ge 0$. If $c_1 = 0$ then $\beta_2 = \alpha_2$ and $\beta_3 = c_2\alpha_2 - \alpha_1 \notin \mathbb{N}_0^2$. Hence $c_k \ge 1$ for all $1 \le k < n$. Moreover, there is $1 < i < n$ satisfying $c_i = 1$. Indeed, $\beta_{n+1} = c_n\beta_n - \beta_{n-1} = (c_n - 1)\beta_n + (\beta_n - \beta_{n-1})$ by Lemma 3.1.8(1). Assume that $c_i \ge 2$ for all $1 < i < n$. Then $\beta_{n+1} \in \mathbb{N}_0^2$ if $c_n \ge 1$

and $-\beta_{n+1} \in \mathbb{N}_0^2 \setminus \{0, \alpha_1\}$ if $c_n \leq 0$ by Lemma 3.1.8(2), since $n \geq 3$. This is a contradiction to $\beta_{n+1} = \eta(c_1) \cdots \eta(c_n)(\alpha_1) = (-\mathrm{id})(\alpha_1) = -\alpha_1$.
Hence there is $(c'_1, \ldots, c'_{n-1}) \in \mathbb{Z}^{n-1}$ such that

$$(c_1, \ldots, c_n) = (c'_1, \ldots, c'_{i-1} + 1, 1, c'_i + 1, c'_{i+1}, \ldots, c'_{n-1}).$$

Then

$$\eta(c_1) \cdots \eta(c_n) = \eta(c'_1) \cdots \eta(c'_{n-1}) = -\mathrm{id}$$

by equation (3.2) and

$$\beta'_k = \eta(c'_1) \cdots \eta(c'_{k-1})(\alpha_1) \in \mathbb{N}_0^2$$

for all $1 \leq k \leq n-1$. Hence $(c'_1, \ldots, c'_{n-1}) \in \mathcal{A}^+$ by induction hypothesis. Then $(c_1, \ldots, c_n) \in \mathcal{A}^+$. ☐

Definition 3.1.10. Let $\mathcal{C} = \mathcal{C}(I, \mathcal{X}, r, (A^X)_{X \in \mathcal{X}})$ be a connected semi-Cartan graph of rank two and let $X \in \mathcal{X}$ and $i \in I$. The **characteristic sequence of $\mathcal{C}$ with respect to X and i** is the infinite sequence $(c_k^{X,i})_{k \geq 1}$ of non-negative integers, where

$$c_{2k+1}^{X,i} = -a_{ij}^{(r_j r_i)^k(X)} = -a_{ij}^{r_i(r_j r_i)^k(X)}$$

$$c_{2k+2}^{X,i} = -a_{ji}^{r_i(r_j r_i)^k(X)} = -a_{ji}^{(r_j r_i)^{k+1}(X)}$$

for all $k \geq 0$ and $j \in I \setminus \{i\}$.

By the definition of a characteristic sequence, we get the following remark.

Remark 3.1.11. Let $\mathcal{C} = \mathcal{C}(I, \mathcal{X}, r, (A^X)_{X \in \mathcal{X}})$ be a connected semi-Cartan graph of rank two and let $X \in \mathcal{X}$ and $i, j \in I$ with $i \neq j$. Let $(c_k)_{k \geq 1}$ be the characteristic sequence of $\mathcal{C}$ with respect to X and i.

- The characteristic sequence of $\mathcal{C}$ with respect to $r_i(X)$ and j is $(c_{k+1})_{k \geq 1}$.
- Suppose that $(r_j r_i)^n(X) = X$ for some $n \geq 1$. Then the characteristic sequence of $\mathcal{C}$ with respect to X and j is $(c_{2n+1-k})_{k \geq 1}$.

Definition 3.1.12. Let $\mathcal{C} = \mathcal{C}(I, \mathcal{X}, r, (A^X)_{X \in \mathcal{X}})$ be a connected semi-Cartan graph of rank two and let $X \in \mathcal{X}$ and $i \in I$. Let $(c_k)_{k \geq 1}$ be the characteristic sequence of $\mathcal{C}$ with respect to X and i. The **root sequence of $\mathcal{C}$ with respect to X and i** is the infinite sequence $(\beta_k)_{k \geq 1}$ of elements of $\mathbb{Z}^2$, where

$$\beta_k = \eta(c_1) \cdots \eta(c_{k-1})(\alpha_1)$$

for all $k \geq 1$. In particular, $\beta_1 = \alpha_1$.

Let $\mathcal{C} = \mathcal{C}(I = \{1, 2\}, \mathcal{X}, r, (A^X)_{X\in\mathcal{X}})$ be a semi-Cartan graph. For all $X \in \mathcal{X}$, the maps s_1^X, s_2^X are defined by equation (1.11). Recall that (α_1, α_2) is a basis of $\mathbb{Z}^2$ and η is a map defined by equation (3.1). Define a map

$$\tau : \mathbb{Z}^2 \to \mathbb{Z}^2, \;\; a\alpha_1 + b\alpha_2 \mapsto b\alpha_1 + a\alpha_2$$

for any $a, b \in \mathbb{Z}$. One obtains

$$s_1^X = \eta(-a_{12}^X)\tau, \;\; s_2^X = \tau\eta(-a_{21}^X) \tag{3.3}$$

for all $X \in \mathcal{X}$.

Lemma 3.1.13. *Let $\mathcal{C} = \mathcal{C}(I = \{1, 2\}, \mathcal{X}, r, (A^X)_{X\in\mathcal{X}})$ be a connected semi-Cartan graph of rank two and let $X \in \mathcal{X}$. Let $(\beta_k)_{k\geq 1}$ be the root sequence of $\mathcal{C}$ with respect to X and 1 and let $(\gamma_k)_{k\geq 1}$ be the root sequence of $\mathcal{C}$ with respect to X and 2. Then*

$$\begin{aligned} \beta_{2k+1} =& \mathrm{id}_X(s_1s_2)^k\alpha_1, & \beta_{2k+2} = \mathrm{id}_X(s_1s_2)^k s_1\alpha_2, \\ \tau\gamma_{2k+1} =& \mathrm{id}_X(s_2s_1)^k\alpha_2, & \tau\gamma_{2k+2} = \mathrm{id}_X(s_2s_1)^k s_2\alpha_1 \end{aligned}$$

for all $k \geq 0$. Hence $\boldsymbol{\Delta}^{X\,\mathrm{re}} = \{\pm\beta_k, \pm\tau\gamma_k | k \geq 1\}$.

Proof. Let $(c_k)_{k\geq 1}$ be the characteristic sequence with respect to X and $i = 1$. By equation (3.3) and the definition of the root sequence, one obtains that

$$\begin{aligned} \beta_{2k+1} =& \eta(c_1)\,\eta(c_2)\cdots\eta(c_{2k-1})\,\eta(c_{2k})(\alpha_1) \\ =& \eta(-a_{12}^X)\,\eta(-a_{21}^{r_1(X)})\cdots\eta(-a_{12}^{(r_2r_1)^{k-1}(X)})\,\eta(-a_{21}^{r_1(r_2r_1)^{k-1}(X)})\,(\alpha_1) \\ =& \eta(-a_{12}^X)\tau\,\tau\eta(-a_{21}^{r_1(X)})\cdots\eta(-a_{12}^{(r_2r_1)^{k-1}(X)})\tau\,\tau\eta(-a_{21}^{r_1(r_2r_1)^{k-1}(X)})(\alpha_1) \\ =& s_1^X s_2^{r_1(X)}\cdots s_1^{(r_2r_1)^{k-1}(X)} s_2^{r_1(r_2r_1)^{k-1}(X)}(\alpha_1) \\ =& \mathrm{id}_X(s_1s_2)^k\alpha_1, \\ \tau\,\gamma_{2k+1} =& \tau\,\eta(-a_{21}^X)\,\eta(-a_{12}^{r_2(X)})\cdots\eta(-a_{21}^{(r_1r_2)^{n-2}(X)})\,\eta(-a_{12}^{r_2(r_1r_2)^{k-1}(X)})(\alpha_1) \\ =& (\tau\,\eta(-a_{21}^X)\,\eta(-a_{12}^{r_2(X)})\tau)\,(\tau\cdots\tau) \\ & (\tau\eta(-a_{21}^{(r_1r_2)^{k-2}(X)})\,\eta(-a_{12}^{r_2(r_1r_2)^{k-1}(X)})\tau)\,\tau(\alpha_1) \\ =& \mathrm{id}_X(s_2s_1)^k\alpha_2. \end{aligned}$$

The claims $\beta_{2k+2} = \mathrm{id}_X(s_1s_2)^k s_1\alpha_2$, $\tau\gamma_{2k+2} = \mathrm{id}_X(s_2s_1)^k s_2\alpha_1$ hold by a similar argument.

Thus $\boldsymbol{\Delta}^{X\,\mathrm{re}} = \{\pm\beta_k, \pm\tau\gamma_k | k \geq 1\}$ follows from the definition of $\boldsymbol{\Delta}^{X\,\mathrm{re}}$. □

For a finite sequence $(v_1, \dots, v_n)$ of integers or vectors, where $n \geq 1$, let $(v_1, \dots, v_n)^\infty = (u_k)_{k\geq 1}$ be the sequence where $u_{mn+i} = v_i$ for all $1 \leq i \leq n$, $m \geq 0$.

Theorem 3.1.14. *Let $\mathcal{C} = \mathcal{C}(I = \{1,2\}, \mathcal{X}, r, (A^X)_{X\in\mathcal{X}})$ be a connected semi-Cartan graph of rank two such that $|\mathcal{X}|$ is finite. Let $X \in \mathcal{X}$ and let n be the smallest positive integer with $(r_2r_1)^n(X) = X$. Let $(c_k)_{k\geq 1}$ be the characteristic sequence of $\mathcal{C}$ with respect to X and 1, and let $l = 6n - \Sigma_{i=1}^{2n} c_i$. The following are equivalent.*

(1) *$\mathcal{C}$ is a finite Cartan graph.*

(2) *$l > 0$, $l \mid 12$, $(c_1, c_2, \dots, c_{12n/l}) \in \mathcal{A}^+$, and $(c_k)_{k\geq 1} = (c_1, c_2, \dots, c_{12n/l})^\infty$.*

In this case $12n/l = |\boldsymbol{\Delta}_+^{X\,\mathrm{re}}| = t_{12}^X$.

Proof. Let $(\beta_k)_{k\geq 1}$ be the root sequence of $\mathcal{C}$ with respect to X and 1 and $(\gamma_k)_{k\geq 1}$ the root sequence of $\mathcal{C}$ with respect to X and 2.

$(1) \Rightarrow (2)$. Let $q = t_{12}^X$. Then $\boldsymbol{\Delta}^{Y\,\mathrm{re}} \subset \mathbb{N}_0^2 \cup -\mathbb{N}_0^2$ for all $Y \in \mathcal{X}$ since $\mathcal{C}$ is a Cartan graph. By [33, Lemmas 3, 4] and Lemma 3.1.13, we have $\beta_k \in \mathbb{N}_0^2$ for all $1 \leq k \leq q$ and $\beta_q = \eta(c_1)\cdots\eta(c_{q-1})(\alpha_1) = \alpha_2$. By the same reason we obtain that $\eta(c_2)\cdots\eta(c_q)(\alpha_1) = \alpha_2$. Then we have

$$-\beta_{q+1} = -\eta(c_1)\cdots\eta(c_q)(\alpha_1) = -\eta(c_1)(\alpha_2) = \alpha_1.$$

Thus $-\eta(c_1)\cdots\eta(c_q) = \mathrm{id}$. Indeed, if we set $w := -\eta(c_1)\cdots\eta(c_q)$ and $w(\alpha_2) := a\alpha_1 + b\alpha_2$, then $b = 1$ since $\det(w) = 1$. If q is odd then $-w\tau \in \mathrm{Hom}(Y, X)$ by equation (3.1) and equation (3.3), where $Y = r_1(r_2r_1)^{(q-1)/2}(X)$. In the same way, one gets $-w \in \mathrm{Hom}((r_2r_1)^{q/2}(X), X)$ if q is even. Hence $w(\alpha_1), w(\alpha_2) \in \boldsymbol{\Delta}^{X\,\mathrm{re}}$. Then $a \geq 0$ since $w(\alpha_2) = \eta(c_1)\cdots\eta(c_{q-1})(\alpha_1) = \beta_q \in \boldsymbol{\Delta}^{X\,\mathrm{re}} \subset \mathbb{N}_0^2 \cup -\mathbb{N}_0^2$. Moreover, $a \leq 0$ since $w^{-1}(\alpha_2) = \alpha_2 - a\alpha_1 \in \mathbb{N}_0^2 \cup -\mathbb{N}_0^2$. Hence $a = 0$ and $w(\alpha_2) = \alpha_2$. Then $-\eta(c_1)\cdots\eta(c_q) = \mathrm{id}$. Hence $(c_1, \dots, c_q) \in \mathcal{A}^+$ by Theorem 3.1.9. Therefore $\Sigma_{i=1}^q c_i = 3q - 6$ by Lemma 3.1.3.
Further, we apply the first part of the proof for $r_1(X)$ and the label 2 instead of X and the label 1, respectively. Then $(c_2, \dots, c_{q+1}) \in \mathcal{A}^+$ by Remark 3.1.11 and $\Sigma_{i=2}^{q+1} c_i = 3q - 6$. Hence $c_{q+1} = c_1$. By induction, $(c_k)_{k\geq 1} = (c_1, c_2, \dots, c_q)^\infty$. In particular, we obtain that $\Sigma_{i=1}^{2qn} c_i = 2n(3q - 6) = q(\Sigma_{i=1}^{2n} c_i)$. Therefore $\Sigma_{i=1}^{2n} c_i = 6n - 12n/q$. Hence $q|12n$ and $l = 6n - \Sigma_{i=1}^{2n} c_i = 12n/q > 0$. Further, $n|q$ since $\mathcal{C}$ is a Cartan graph. Thus $l|12$.

$(2) \Rightarrow (1)$. Set $q = 12n/l$. Then $\eta(c_1)\cdots\eta(c_q) = -\mathrm{id}$ and $\beta_k \in \mathbb{N}_0^2$ for $1 \leq k \leq q$ by Theorem 3.1.9. Then $(c_q, c_{q-1}, \dots, c_1) \in \mathcal{A}^+$ by Corollary 3.1.5(1) since $(c_1, c_2, \dots, c_q) \in \mathcal{A}^+$. Since $l|12$, q is a multiple of n. Hence $(c_q, c_{q-1}, \dots, c_1)^\infty$

is the characteristic sequence of $\mathcal{C}$ with respect to X and 2. By Lemma 3.1.13, we get that $\gamma_k \in \mathbb{N}_0^2$ for all $1 \leq k \leq q$. Therefore, since $(c_k)_{k\geq 1} = (c_1, \dots, c_q)^\infty$, $\boldsymbol{\Delta}^{X\,\mathrm{re}} = \{\pm\beta_k, \pm\tau\gamma_k | 1 \leq k \leq q\} \subseteq \mathbb{N}_0^2 \cup -\mathbb{N}_0^2$ for all $X \in \mathcal{X}$ by Lemma 3.1.13. Hence $\mathcal{C}$ is finite.
By the definition of t_{12}^X and [33, Lemma 4], we obtain that $t_{12}^X = q = |\boldsymbol{\Delta}_+^{X\,\mathrm{re}}|$. Hence $n | t_{12}^X$ by assumption and $(r_2 r_1)^{t_{12}^X}(X) = X$. Therefore, $\mathcal{C}$ is a Cartan graph. □

3.2 Rank 3 case

In this section, we illustrate the properties of finite connected indecomposable Cartan graphs of rank three. The main result is Theorem 3.2.6, which is useful in Chapter 4. Our references are [15] and [46].

Let $\mathcal{C} = \mathcal{C}(I, \mathcal{X}, r, (A^X)_{X\in\mathcal{X}})$ be a semi-Cartan graph where $I = \{1, 2, 3\}$. Let $X \in \mathcal{X}$. Since the points of $\mathcal{C}$ could have many different neighborhoods, we define the following good neighborhoods in order to cover all the finite connected indecomposable Cartan graphs in such way that at least one point of $\mathcal{C}$ has one of the good neighborhoods.

We denote that

$$A_3 := \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix},\ B_3 := \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix},\ \text{and}\ C_3 := \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}.$$

Definition 3.2.1. Let $\mathcal{C} = \mathcal{C}(I, \mathcal{X}, r, (A^X)_{X\in\mathcal{X}})$ be a semi-Cartan graph such that $A^X = A^Y$ for all $X, Y \in \mathcal{X}$. Then we say that $\mathcal{C}$ is **standard**.

Definition 3.2.2. We say that X has a **good** A_3 **neighborhood** if there exists integer sequence (a, b, c, d) such that there is a permutation of I with respect to which $A^X = A_3$, $A^{r_1(X)} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -a \\ 0 & -1 & 2 \end{pmatrix}$, $A^{r_2(X)} = \begin{pmatrix} 2 & -1 & -b \\ -1 & 2 & -1 \\ -c & -1 & 2 \end{pmatrix}$,
$A^{r_3(X)} = \begin{pmatrix} 2 & -1 & 0 \\ -d & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$, and one of the following holds.

(1) $(a, b, c, d) \in \{(1, 0, 0, 1), (1, 1, 1, 1), (1, 1, 1, 2), (2, 1, 1, 2), (2, 1, 2, 2), (1, 1, 2, 3)\}$.

(2) $(a, b, c, d) = (2, 1, 2, 3)$, $a_{21}^{r_3 r_1(X)} = -3$.

Definition 3.2.3. We say that X has a **good** B_3 **neighborhood** if up to a permutation of I one of the following holds.

(1) $A^X = A^{r_1(X)} = A^{r_2(X)} = A^{r_3(X)} = B_3$ and $a_{23}^{r_1 r_3(X)} \in \{-1, -2\}$.

(2) $A^X = A^{r_1(X)} = A^{r_2(X)} = B_3$ and $A^{r_3(X)} = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$.

Definition 3.2.4. We say that X has a **good C_3 neighborhood** if there is a permutation of I with respect to which

$$A^X = A^{r_1(X)} = A^{r_2(X)} = C_3 \text{ and } A^{r_3(X)} = \begin{pmatrix} 2 & -1 & 0 \\ -a & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}, \text{ where } a \in \{1, 2\}.$$

Recall that for each finite Cartan graph $\mathcal{C}$ there is a unique root system $\mathcal{R}^{re} = \mathcal{R}^{re}(\mathcal{C}, (\boldsymbol{\Delta}^{X\,\mathrm{re}})_{X\in\mathcal{X}})$ of type $\mathcal{C}$ by Remark 1.5.12. To get our main theorem of this section we need the classification result of connected Cartan graphs of rank three such that $\mathcal{R}^{re}(\mathcal{C})$ is a finite irreducible root system of type $\mathcal{C}$. We recall this powerful classification theorem from [15, Theorem 4.1].

Theorem 3.2.5. *Let $\mathcal{C}$ be a connected semi-Cartan graph of rank three with $I = \{1, 2, 3\}$. Assume that $\mathcal{R}^{re}(\mathcal{C})$ is a finite irreducible root system of type $\mathcal{C}$. Then either $\mathcal{C}$ is standard or up to a permutation of I there exists a point $X \in \mathcal{X}$ such that $\boldsymbol{\Delta}_+^{X\,\mathrm{re}}$ is one of the 55 sets listed in the Algorithm A.0.1 in Appendix A. Moreover, $\boldsymbol{\Delta}_+^{X\,\mathrm{re}}$ with this property is uniquely determined.*

Based on Theorem 3.2.5 we are able to characterize the rank three finite Cartan graphs.

Theorem 3.2.6. *Let $\mathcal{C} = \mathcal{C}(I, \mathcal{X}, r, (A^X)_{X\in\mathcal{X}})$ be a rank three finite connected indecomposable Cartan graph. Then up to a permutation of I one of the following holds.*

P1. The Cartan graph $\mathcal{C}$ is standard and the generalized Cartan matrices are of type A_3, B_3, or C_3.

P2. There exists at least one point $Y \in \mathcal{X}$ which has a good A_3, B_3, or C_3 neighborhood.

Remark 3.2.7. There are finite connected indecomposable Cartan graphs of rank three which contain precisely one case of the good neighborhoods. This is not hard to check by computer algorithms.

Proof. Since $\mathcal{C}$ is a finite connected indecomposable Cartan graph, we obtain that $\mathcal{R}^{re} = \mathcal{R}(\mathcal{C}, (\boldsymbol{\Delta}^{X\,\mathrm{re}})_{X\in\mathcal{X}})$ is a finite root system of type $\mathcal{C}$ by Remark 1.5.12. Then $\mathcal{R}^{re}$ is irreducible by Proposition 1.5.9. Hence by Theorem 3.2.5 we obtain that either

$\mathcal{C}$ is standard or up to a permutation of I there exists a point $X \in \mathcal{X}$ such that the set $\boldsymbol{\Delta}_{+}^{X\,\mathrm{re}}$ is in the 55 sets listed in Appendix A. If $\mathcal{C}$ is standard then the generalized Cartan matrices are of finite type by [13, Theorem 3.3]. Otherwise, we consider each set in the list. For each point Y in the list, we calculate all neighbors $\{r_i(Y), i \in I\}$ of Y. Since the reflection s_i^Y maps $\boldsymbol{\Delta}_{+}^{Y\,\mathrm{re}} \setminus \{\alpha_i\}$ bijectively to $\boldsymbol{\Delta}_{+}^{r_i(Y)\,\mathrm{re}} \setminus \{\alpha_i\}$ for all $i \in I$, the Cartan matrices of all neighbors of Y can be obtained from $\boldsymbol{\Delta}_{+}^{Y\,\mathrm{re}}$, $Y \in \mathcal{X}$ by Lemma 1.5.7 If Y has a good A_3, B_3 or C_3 neighborhood, then the proof is done. Otherwise repeat the previous step to the neighbours of Y. Since $\mathcal{X}$ is finite, this algorithm terminates. The elementary calculations are done by computer algorithms and they are skipped here, see Appendix A for the complete computer algorithms. □

Chapter 4

The classification result

In this chapter we obtain the classification result of rank two and rank three Nichols algebras of diagonal type over fields of positive characteristic. In section 4.1 we give all braided vector spaces V of diagonal type such that the Nichols algebra $\mathcal{B}(V)$ has a finite root system in terms of Dynkin diagrams. The proof is given in Section 4.2, which uses the characterization of the finite Cartan graphs in Chapter 3.

4.1 The classification result

Let $\theta \in \{2,3\}$, and $I = \{1,2,\dots,\theta\}$. Assume that G is an abelian group.

Theorem 4.1.1. *Let $V \in {}^G_G\mathcal{YD}$ be a Yetter–Drinfel'd module of diagonal type over G. Assume that $V = \oplus_{i\in I}\Bbbk x_i \in {}^G_G\mathcal{YD}$, where $\{x_i|i \in I\}$ is a basis of V. Let $M = (\Bbbk x_1, \Bbbk x_2, \dots, \Bbbk x_\theta) \in \mathcal{F}^G_\theta$. Assume that the braiding matrix $(q_{ij})_{i,j\in I}$ of M is indecomposable. Let $\mathcal{D}$ be the Dynkin diagram of M. Then the set of roots $\boldsymbol{\Delta}^{[M]}$ of Nichols algebra $\mathcal{B}(V)$ is finite if and only if the following hold.*

- $(2a)$ *If $\theta = 2$, then the Dynkin diagram $\mathcal{D}$ appears in Tables 5.1, 5.2, 5.3, 5.4 and 5.5 for $p = 2, p = 3, p = 5, p = 7$, and $p > 7$, respectively.*
- $(2b)$ *If $\theta = 3$, then the Dynkin diagram $\mathcal{D}$ appears in Tables 5.6, 5.7, and 5.8 for $p = 2, p = 3$, and $p > 3$, respectively*

Remark 4.1.2. (i) Assume that M satisfies the assumptions of Theorem 4.1.1 and the set of roots $\boldsymbol{\Delta}^{[M]}$ is finite. Then we obtain that M admits all reflections from Chapter 2. Let $\mathcal{C}(M)$ be the semi-Cartan graph attached to M. The row of Table 5.9 and Tables (5.6)-(5.8) containing $\mathcal{D}$ consists precisely of the Dynkin diagrams of the points of $\mathcal{C}(M)$. Further, the corresponding row of Table 5.10 and Table 5.11 contains the exchange graph of $\mathcal{C}(M)$.

(ii) In order to illustrate the exchange graph of the semi-Cartan graph $\mathcal{C}(M)$ in Theorem 4.1.1, we use the following notations in Tables 5.1-5.10.

$a)$ For each row n of Tables (5.1)-(5.8), we enumerate l-th Dynkin diagram with $\mathcal{D}_{nl}$ column by column for all $l \geq 1$. For each Dynkin diagram, we enumerate the vertices from left to right with 1, 2, $\dots$, θ, respectively. If $\theta = 2$, then we write $\tau\mathcal{D}_{nl}$ for the graph $\mathcal{D}_{nl}$ where the two vertices of $\mathcal{D}_{nl}$ switch the positions. If $\theta = 3$, then for $i, j, k \in I$ we write $\tau_{ijk}\mathcal{D}_{nl}$ for the graph $\mathcal{D}_{nl}$ where the three vertices of $\mathcal{D}_{nl}$ change to i, j, k, respectively.

$b)$ We present the i-vertex of $\mathcal{D}$ with a bullet point $\bullet$ to describe the i-th reflection map R_i of the Dynkin diagram $\mathcal{D}$.

$c)$ We also use the notation $(2,1,6,1,2,3)^2 = (2,1,6,1,2,3,2,1,6,1,2,3)$ and $(3,1,5,1)^3 = (3,1,5,1,3,1,5,1,3,1,5,1)$ in Table 5.10.

We point out that Theorem 4.1.1 yields a classification of finite-dimensional Nichols algebra $\mathcal{B}(V)$ as a vector space over $\Bbbk$ by [25, Corollary 6].

Corollary 4.1.3. *Assume that V satisfies the setting in Theorem 4.1.1. Then the Nichols algebra $\mathcal{B}(V)$ is finite dimensional if and only if the Dynkin diagram $\mathcal{D}$ of V appears in one of the Tables (5.1- 5.8) and the labels of the vertices of $\mathcal{D}$ are roots of unity (including 1).*

4.2 The proof of Theorem 4.1.1

In this section, we give the proof of Theorem 4.1.1.

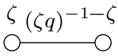
Proof. (I) The first step is to prove that if the Dynkin diagram $\mathcal{D}$ of M appears in one of the listed tables then the set of roots $\boldsymbol{\Delta}^{[M]}$ of Nichols algebra $\mathcal{B}(V)$ is finite.

Assume that $\mathcal{D}$ appears in row r of one of the Tables (5.1-5.8). Then M admits all reflections. Indeed, by Lemma 2.1.4 one obtains that M is i-finite for all $i \in I$. For $i \in I$, one can determine the Dynkin diagram of $R_i(M)$ by Lemma 2.1.7. One observes that it appears in the same row as $\mathcal{D}$. Do the same for all the Dynkin diagrams in the same row of $\mathcal{D}$. It implies that M admits all reflections by Lemma 2.1.4. Hence $\mathcal{C}(M)$ is a well-defined Semi-Cartan graph by Theorem 2.1.9. Assume that $\mathcal{C} = \mathcal{C}(I, \mathcal{X}, R, (A^X)_{X\in\mathcal{X}})$ is the semi-Cartan graph attached to M. Notice that $\mathcal{C}(M)$ is connected. Now, we identify the objects of $\mathcal{C}(M)$ with their Dynkin diagrams. The above calculations imply

that the exchange graph of $\mathcal{C}(M)$ appears in row r of Table 5.10 and Table 5.11 for rank 2 and rank 3, respectively.
If $\theta = 2$, then we calculate the smallest integer n with $(R_2R_1)^n(\mathcal{D}) = \mathcal{D}$ and we observe that the integer n appears in the third column and rth row of Table 5.9. Then we compute the characteristic sequence $(c_k)_{k\geq 1}$ with respect to the first Dynkin diagram in row r and the label 1. We observe that $(c_k)_{k\geq 1}$ is the infinite power of the sequence in the fifth column and rth row of Table 5.9. Further, we get the number $l = 6n - \Sigma_{i=1}^{2n} c_i$. It appears in the fourth column of Table 5.9. One checks that $l \mid 12$ and $(c_1, c_2, \dots, c_{12n/l}) \in \mathcal{A}^+$ by Corollary 3.1.5(2). Then $\mathcal{C}(M)$ is a finite Cartan graph by Theorem 3.1.14. Hence the set of roots $\boldsymbol{\Delta}^{[M]}$ is finite by Corollary 2.1.11.
If $\theta = 3$, then we obtain that $\mathcal{C}(M)$ is the same as the Cartan graph obtained from the Dynkin diagrams appearing in row s of Table 2 in [22], where s appears in the third column and rth row of Table 5.11. The detailed calculations are skipped here. Moreover, the Weyl groupoids of the above Cartan graphs from [22] are finite, see [22, Table 2]. Then $\boldsymbol{\Delta}^{[M]}$ is finite by Corollary 2.1.11.

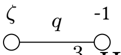
(II) The second step is to prove that if the set of roots $\boldsymbol{\Delta}^{[M]}$ of Nichols algebra $\mathcal{B}(V)$ is finite then the Dynkin diagram of M appears in the listed tables. If $\boldsymbol{\Delta}^{[M]}$ is finite then M admits all reflections by [28, Corollary 6.12]. Since $\boldsymbol{\Delta}^{[M]}$ is finite, $\mathcal{C}(M)$ is a finite Cartan graph by Corollary 2.1.11. Since the braiding matrix of M is indecomposable, then the generalized Cartan matrix A^X is indecomposable by the definition of A^X and Lemma 2.1.4. Since $\mathcal{C}(M)$ is connected, we obtain that $\mathcal{C}$ is indecomposable by Proposition 1.5.9. We can apply Theorem 3.1.14 and Theorem 3.2.6 to rank 2 and rank 3 Cartan graph $\mathcal{C}(M)$, respectively. From the above argument I it is enough to prove that the Dynkin diagram of at least one point in $\mathcal{C}(M)$ is contained in the listed Tables (5.1-5.8.

We separate the proof into $\theta = 2$ and $\theta = 3$.
(IIa) Consider the case $\theta = 2$.
Set $X = [M]_2$ and $m = t^X_{12}$. We may assume that $i = 1, j = 2$, but change the labels if necessary. Let $(c_k)_{k\geq 1}$ be the characteristic sequence of $\mathcal{C}(M)$ with respect to X and the label $i = 1$. Then we have $(c_1, c_2, \dots, c_m) \in \mathcal{A}^+$ and $(c_k)_{k\geq 1} = (c_1, c_2, \dots, c_m)^\infty$ by Theorem 3.1.14.
If $m = 2$ then $(c_1, c_2) = (0, 0)$. Hence $a^X_{12} = a^X_{21} = 0$. Then $q_{12}q_{21} = 1$ and hence the braiding matrix of X is decomposable, which is a contradiction. Hence $m > 2$. By Proposition 3.1.6 one of the subsequences $(1, 1)$, $(1, 2, a)$, $(2, 1, b)$, $(1, 3, 1, b)$ or their transpose, where $1 \leq a \leq 3$ and $3 \leq b \leq 5$, is a subsequence of $(c_1, c_2, \dots, c_m)$. Let n be the smallest integer with

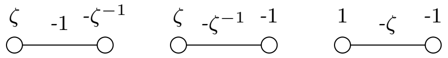
$(R_2R_1)^n(X) = X$. Then $n|m$ by Theorem 3.1.14. Since $c_{m+k} = c_k$ for all $k \in \mathbb{N}$, we have the freedom to assume any position in $(c_k)_{k\geq 1}$, where any of these subsequences is starting. Let $c_0 := c_m$ and $q_0 := q_{12}q_{21}$.
We proceed case by case:
Step 1. If $c_0 = c_1 = 1$, then $a^X_{12} = a^X_{21} = -1$. Hence $q_0 \neq 1$. We distinguish four cases: 1aa,1ab,1ba and 1bb.
Case 1aa. If $q_{11}q_0 = 1$ and $q_{22}q_0 = 1$, then $\mathcal{D} = \mathcal{D}_{21}$.
Case 1ab. If $q_{11}q_0 = 1$, $q_{22} = -1$, and $q_{22}q_0 \neq 1$, then $\mathcal{D} = \mathcal{D}_{31}$.
Case 1ba. If $q_{11} = -1$, $q_{22}q_0 = 1$, and $q_{11}q_0 \neq 1$, then $\mathcal{D} = \tau\mathcal{D}_{31}$.
Case 1bb. If $q_{11} = -1$, $q_{22} = -1$, and $q_0 \neq -1$, then $\mathcal{D} = \mathcal{D}_{32}$.
Step 2. Assume that $(c_0, c_1, c_2) = (1, 2, a')$, where $a' \in \{1, 2, 3\}$. Then we obtain that $a^X_{21} = -1$, $a^X_{12} = a^{r_1(X)}_{12} = -2$, and $a^{r_1(X)}_{21} = -a'$. We distinguish four cases: 2aa, 2ab, 2ba and 2bb.
Case 2aa. If ${q_{11}}^2q_0 = 1$ and $q_{22}q_0 = 1$, then $\mathcal{D} = \mathcal{D}_{41}$.
Case 2ab. If ${q_{11}}^2q_0 = 1$, $q_{22} = -1$, and $q_{22}q_0 \neq 1$, then $\mathcal{D} = \mathcal{D}_{51}$.
Case 2ba. Assume that $1+q_{11}+{q_{11}}^2 = 0$, $q_{22}q_0 = 1$, and ${q_{11}}^2q_0 \neq 1$. If $p = 3$ then $1 + q_{11} + {q_{11}}^2 = 0$ yields $q_{11} = 1$. If $q_{22} \neq -1$ then $\mathcal{D} = \mathcal{D}_{6',1}$ and if $q_{22} = -1$ then $\mathcal{D} = \mathcal{D}_{6''',1}$. Assume that $p \neq 3$. Set $\zeta := q_{11}$ and $q := q_{22}$. Then $q_0 = q^{-1} \notin \{1, \zeta^{-1}\}$ since $a^X_{12} = -2$, and $q_0 \neq \zeta$ since ${q_{11}}^2q_0 \neq 1$. Thus $\mathcal{D} = \mathcal{D}_{61}$ or $\mathcal{D}_{6'',1}$, $p \neq 2$.
Case 2bb. Consider the last case $1 + q_{11} + {q_{11}}^2 = 0$, $q_{22} = -1$ and $q_0 \notin \{1, -1, q_{11}, q^2_{11}\}$.
Case 2bba. If $p = 3$ then $q_{11} = 1$. Set $q := q_0$. By Lemma 2.1.7, the Dynkin diagrams of $r_1(X)$ and X are with $q \in \Bbbk^* \setminus \{-1, 1\}$. Then $a^{r_1(X)}_{21} \leq -2$ since $(-q^2)q^{-1} = -q \neq 1$, and $-q^2 \neq -1$.
Case $2bba1$. If $p = 3$ and $a' = -a^{r_1(X)}_{21} = 2$, then one gets $(-q^2)^2q^{-1} = 1$ or $1 + (-q^2) + (-q^2)^2 = 0$. If $(-q^2)^2q^{-1} = 1$ then $q = 1$, which is a contradiction. Hence $-q^2 = 1$ from the second equation since $p = 3$. Then $\mathcal{D} = \mathcal{D}_{9',2}$.
Case 2bba2. If $p = 3$ and $a' = -a^{r_1(X)}_{21} = 3$, then one has $(-q^2)^3q^{-1} = 1$ or $1 + (-q^2) + (-q^2)^2 + (-q^2)^3 = 0$. The first equation $(-q^2)^3q^{-1} = 1$ yields $(-q)^5 = 1$, hence $\mathcal{D} = \mathcal{D}_{16',2}$. If $1 + (-q^2) + (-q^2)^2 + (-q^2)^3 = 0$, then $(1 - q^2)(1 + q^4) = 0$ and hence $q \in G'_8$. Then $\mathcal{D} = \mathcal{D}_{13',2}$.
Case 2bbb. We now suppose that $p \neq 3$. Set $\zeta := q_{11}$ and $q := q_0$. Hence the Dynkin diagram of $r_1(X)$ is  with $\zeta \in G'_3$, $q \in \Bbbk^*\setminus\{1, -1, \zeta, \zeta^{-1}\}$.
Since $a' \in \{1, 2, 3\}$, we distinguish three cases: $2bbb1$, $2bbb2$ and $2bbb3$.
Case 2bbb1. Consider that $p \neq 3$ and $a^{r_1(X)}_{21} = -1$. then one gets $(-\zeta q^2)(\zeta q)^{-1} = 1$ or $1 + (-\zeta q^2) = 0$. If $(-\zeta q^2)(\zeta q)^{-1} = 1$, then $q = -1$, which is a con-

tradiction. If $1+(-\zeta q^2) = 0$ then $\zeta^2 = q^2$ and hence $q = -\zeta$. Then $\mathcal{D} = \mathcal{D}_{71}$.
Case 2bbb2. If the condition $p \neq 3$ and $a_{21}^{r_1(X)} = -2$ hold, then $(-\zeta q^2)^2(\zeta q)^{-1} = 1$ or $\sum_{i=0}^{2}(-\zeta q^2)^i = 0$.
Case 2bbb2a. Consider the equation $(-\zeta q^2)^2(\zeta q)^{-1} = 1$. Then $\zeta q^3 = 1$ and hence $q \in G'_9$ since $\zeta \in G'_3$ and $p \neq 3$. Hence $\mathcal{D} = \mathcal{D}_{10,2}$.
Case 2bbb2b. If $\sum_{i=0}^{2}(-\zeta q^2)^i = 0$, then $-\zeta q^2 \in \{\zeta, \zeta^{-1}\}$. Hence $q^2 = -1$ or $-q^2 = \zeta$.

Case 2bbb2b1. If $q^2 = -1$, then $p \neq 2$ and the Dynkin diagram of X is

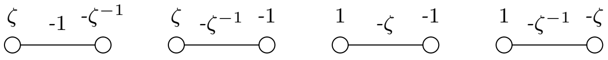


with $q \in G'_4$, $\zeta \in G'_3$. Set $\eta := \zeta^2 q^{-1}$. Then $\eta \in G'_{12}$, $\zeta = -\eta^2$, and $q = \eta^3$. Hence $\mathcal{D} = \mathcal{D}_{92}$.
Case 2bbb2b2. If $-q^2 = \zeta$, then $q \in G'_{12}$ and $p \neq 2$ since $q \neq \zeta^{-1}$. Hence $\mathcal{D} = \mathcal{D}_{81}$.
Case 2bbb3. Consider that $p \neq 3$ and $a_{21}^{r_1(X)} = -3$. Then $(-\zeta q^2)^3(\zeta q)^{-1} = 1$ or $\sum_{i=0}^{3}(-\zeta q^2)^i = 0$, $1 - \zeta q^2 \neq 0$.
Case 2bbb3a. Consider the equation $(-\zeta q^2)^3(\zeta q)^{-1} = 1$, that is $-q^5 = \zeta$. Hence $q = -\zeta^{-1}$ or $-q \in G'_{15}$, $p \neq 5$.
Case 2bbb3a1. If $-q \in G'_{15}$, $p \neq 3, 5$, and $\zeta = -q^5$, then $\mathcal{D} = \mathcal{D}_{16,2}$.
Case 2bbb3a2. If $q = -\zeta^{-1}$ then $p \neq 2$ since $q \neq \zeta^{-1}$. Hence the Dynkin diagrams of $r_1(X)$, X and $r_2(X)$, respectively, are

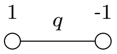


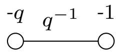
with $\zeta \in G'_3$. Then one obtains that $a_{12}^{r_2(X)} = 1 - p$. We distinguish four cases.
Case 2bbb3a2a. If $p = 5$, then $\mathcal{D} = \mathcal{D}_{16''2}$.
Case 2bbb3a2b. If $p = 7$, then $\mathcal{D} = \mathcal{D}_{18,2}$.
Case 2bbb3a2c. If $p = 6s + 1$ $(s \geq 2)$, then the Dynkin diagrams of $r_1(X)$, X, $r_2(X)$, $r_1r_2(X)$, $r_2r_1r_2(X)$, and $(r_1r_2)^2(X)$, respectively, are

with $\zeta \in G'_3$. Hence $n = 6$ in Theorem 3.1.14 and $(c_k)_{k\geq 0} = (2, 3, 2, 1, p - 1, 1, 2, 3, 2, 1, p - 1, 1)^\infty$. Then $l = 20 - 2p < 0$, which is a contradiction to Theorem 3.1.14.
Case 2bbb3a2d. If $p = 6s + 5$, where $s \geq 1$, then the Dynkin diagrams of $r_1(X)$, X, $r_2(X)$ and $r_1r_2(X)$, respectively, are



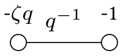
with $\zeta \in G'_3$. Then $n = 4$ and $(c_k)_{k\geq 0} = (2,3,2,1,p-1,1,p-1,1)^\infty$. Hence $l = 16 - 2p < 0$. Again, one gets a contradiction.
Case 2bbb3b. Consider the equation $0 = \sum_{i=0}^{3}(-\zeta q^2)^i = (1-\zeta q^2)(1+\zeta^2 q^4)$, where $\zeta q^2 \neq 1$. One gets $\zeta = -q^4$. If $p = 2$, then $\zeta^4 = q^4$ and hence $\zeta = q$, which is a contradiction to $\zeta q^2 \neq 1$. Otherwise $q \in G'_{24}$ and $\mathcal{D} = \mathcal{D}_{13,2}$.
Step 3. Now we change the label. It means that $(c_k)_{k\geq 1}$ is the characteristic sequence of $\mathcal{C}_s(M)$ with respect to X and the label 2.
Assume that $(c_0, c_1, c_2) = (2, 1, b')$, where $b' \in \{3,4,5\}$. Then we obtain that $a_{12}^X = -2$, $a_{21}^X = -1$ and $a_{12}^{r_2(X)} = -b'$. If ${q_{11}}^2 q_0 = 1$ and $q_{22} = -1$ then $a_{12}^{r_2(X)} = -2$, which is a contradiction. If $q_0 q_{22} = 1$ then $a_{12}^{r_2(X)} = a_{12}^X = -2$, which is again a contradiction. Hence we may assume that $1 + q_{11} + {q_{11}}^2 = 0$, $q_{22} = -1$ and $q_0 \notin \{1, -1, q_{11}^{-2}\}$. Since $a_{12}^X = -2$, we also obtain that $q_0 \neq q_{11}^{-1}$.
Case 3a. If $p = 3$, then by setting $q := q_0$, the Dynkin diagrams of X and $r_2(X)$, respectively, are





$$q \in \Bbbk^* \setminus \{-1, 1\}. \tag{4.1}$$

Case 3a1. If $p = 3$ and $b' = 3$, then one gets $(-q)^3 q^{-1} = 1$ or $\sum_{i=0}^{3}(-q)^i = 0$. Since $q \neq 1$, both equations imply that $q^2 = -1$. Hence $\mathcal{D} = \mathcal{D}_{9',2}$.
Case 3a2. If $p = 3$ and $b' = 4$, then one has $(-q)^4 q^{-1} = 1$ or $\sum_{i=0}^{4}(-q)^i = 0$. If $(-q)^4 q^{-1} = 1$ then $q = 1$, which is a contradiction to (4.1). If $\sum_{i=0}^{4}(-q)^i = 0$ then $q^5 = -1$ and $\mathcal{D} = \mathcal{D}_{16',2}$.
Case 3a3. If $p = 3$ and $b' = 5$, then one obtains $(-q)^5 q^{-1} = 1$ or $\sum_{i=0}^{5}(-q)^i = 0$. If $(-q)^5 q^{-1} = 1$ then $q \in G'_8$ and $\mathcal{D} = \mathcal{D}_{13',2}$. Consider the equation $0 = \sum_{i=0}^{5}(-q)^i = (1-q)(1+q^2+q^4)$. Then $q^2 = 1$ since $p = 3$ and $q \neq 1$, which is a contradiction to (4.1).
Case 3b. We now consider the cases in which the condition $p \neq 3$ holds. Set $\zeta := q_{11}$ and $q := q_0$. The Dynkin diagram of $r_2(X)$ is



$$\zeta \in G'_3 \quad q \in \Bbbk^* \setminus \{1, -1, \zeta, \zeta^{-1}\}. \tag{4.2}$$

Case 3b1. If $p \neq 3$ and $b' = 3$, then one gets $(-\zeta q)^3 q^{-1} = 1$ or $\sum_{i=0}^{3}(-\zeta q)^i = 0$. If $(-\zeta q)^3 q^{-1} = 1$ then $q \in G'_4$, $p \neq 2$ and $\mathcal{D} = \mathcal{D}_{92}$. If $0 = \sum_{i=0}^{3}(-\zeta q)^i = (1-\zeta q)(1+(\zeta q)^2)$, then $(\zeta q)^2 = -1$ and $p \neq 2$ since $q \neq \zeta^{-1}$. Hence $\mathcal{D} = \mathcal{D}_{81}$.
Case 3b2. If $p \neq 3$ and $b' = 4$, then one gets $(-\zeta q)^4 q^{-1} = 1$ or $\sum_{i=0}^{4}(-\zeta q)^i =$

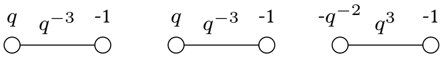
0. If $(-\zeta q)^4 q^{-1} = 1$ then $\zeta = q^{-3}$. Since $q \notin G'_3$, one obtains $q \in G'_9$ and $\mathcal{D} = \mathcal{D}_{10,2}$. The equation $\sum_{i=0}^{4}(-\zeta q)^i = 0$ gives $\zeta = -q^5$. Since $\zeta \in G'_3$, one gets $-q \in G'_3$, $\zeta = -q^{-1}$, $p = 5$ or $-q \in G'_{15}$, $p \neq 3, 5$. If $-q \in G'_3$ then $\mathcal{D} = \mathcal{D}_{16'',2}$ and if $-q \in G'_{15}$ then $\mathcal{D} = \mathcal{D}_{16,2}$.

Case 3b3. If $p \neq 3$ and $b' = 5$, then one gets $(-\zeta q)^5 q^{-1} = 1$ or $\sum_{i=0}^{5}(-\zeta q)^i = 0$, $-\zeta q \neq 1$. If $(-\zeta q)^5 q^{-1} = 1$ and $p = 2$ then $q\zeta^{-1} = 1$, which is a contradiction to (4.2). If $(-\zeta q)^5 q^{-1} = 1$ and $p \neq 2$ then $\zeta = -q^4$ and $\mathcal{D} = \mathcal{D}_{13,2}$. Consider $0 = \sum_{i=0}^{5}(-\zeta q)^i = (1 - \zeta q)(1 + (-\zeta q)^2 + (-\zeta q)^4)$. Since $q \notin \{1, -1, \zeta, \zeta^{-1}\}$, one gets $p \neq 2$ and $q^3 = -1$. Since $-\zeta q \neq 1$, one gets $q = -\zeta$ and $a_{12}^{r_2(X)} = -2$, which is a contradiction.

Step 4. Again we use the same labeling as in steps 1 and 2. Assume that $(c_0, c_1, c_2, c_3) = (1, 3, 1, c')$, where $c' \in \{3, 4, 5\}$. Then $a_{21}^X = -1$ and $a_{12}^X = -3$. We distinguish four cases: 4aa, 4ab, 4ba and 4bb.

Case 4aa. If ${q_{11}}^3 q_0 = 1$ and $q_{22} q_0 = 1$, then $\mathcal{D} = \mathcal{D}_{11,1}$.

Case 4ab. Set $q := q_{11}$. If ${q_{11}}^3 q_0 = 1$ and $q_{22} = -1$, then the Dynkin diagrams of X, $r_1(X)$ and $r_2 r_1(X)$, respectively, are



$q \in \Bbbk^* \setminus \{1, -1\}, \quad q \notin G'_3.$

Then $-a_{12}^{r_2 r_1(X)} = c' \in \{3, 4, 5\}$. Hence we distinguish three cases: 4ab1, 4ab2 and 4ab3.

Case 4ab1. If $c' = 3$, then $(-q^{-2})^3 q^3 = 1$ or $\sum_{i=0}^{3}(-q^{-2})^i = 0, 1 - q^{-2} \neq 0$. If $(-q^{-2})^3 q^3 = 1$ then $\mathcal{D} = \mathcal{D}_{11,1}$, where $q^3 = -1$. If $\sum_{i=0}^{3}(-q^{-2})^i = 0$, then $\mathcal{D} = \mathcal{D}_{12,1}$.

Case 4ab2. If $c' = 4$, then $(-q^{-2})^4 q^3 = 1$ or $\sum_{i=0}^{4}(-q^{-2})^i = 0$.

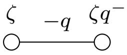
Case 4ab2a. The equation $(-q^{-2})^4 q^3 = 1$ gives $q^5 = 1$ and $p \neq 5$, since $q \neq 1$. Hence $\mathcal{D} = \mathcal{D}_{14,1}$.

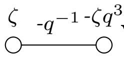
Case 4ab2b. Consider the equation $\sum_{i=0}^{4}(-q^{-2})^i = 0$. One gets $q^{10} = -1$. If $p = 2$ then $\mathcal{D} = \mathcal{D}_{14,1}$. If $p = 5$ then $q^2 = -1$ and $\mathcal{D} = \mathcal{D}_{15',1}$. If $p \neq 2, 5$, then $q \in G'_{20}$ and $\mathcal{D} = \mathcal{D}_{15,1}$.

Case 4ab3. If $c' = 5$, then $(-q^{-2})^5 q^3 = 1$ or $\sum_{i=0}^{5}(-q^{-2})^i = 0$.

Case 4ab3a. Consider the equation $(-q^{-2})^5 q^3 = 1$, which gives $-q^7 = 1$. Since $q \neq -1$, one gets $p \neq 7$ and $-q \in G'_7$. Hence $\mathcal{D} = \mathcal{D}_{17,1}$.

Case 4ab3b. Consider the equation $0 = \sum_{i=0}^{5}(-q^{-2})^i = (1 - q^{-2})(1 + q^{-4} + q^{-8})$. Since $q^2 \neq 1$, one gets $1 + q^{-4} + q^{-8} = 0$. If $p = 3$ then $q \in G'_4$ since $q^2 \neq 1$. Hence $a_{12}^{r_2 r_1(X)} = -2$, which is a contradiction. Then $p \neq 3$ and $q^{-4} \in G'_3$. Since $c' = 5$, one gets $q \in G'_6$ or $q \in G'_{12}$. If $q \in G'_6$, then $a_{12}^{r_2 r_1(X)} = -3$, which is a contradiction. If $q \in G'_{12}$, then $a_{12}^{r_2 r_1(X)} = -2$,

which is again a contradiction.
Case 4ba. The conditions $1 + q_{11} + {q_{11}}^2 + {q_{11}}^3 = 0$, $q_{11} \neq -1$ and $q_{22}q_0 = 1$ hold. Then $q_{11} \in G'_4$ and $p \neq 2$ since $a^X_{12} = -3$. Set $\zeta := q_{11}$ and $q = q_{22}$. The Dynkin diagram of $r_1(X)$ is 

with $\zeta \in G'_4$ and $q \in \Bbbk^* \setminus \{1, -1, \zeta, \zeta^{-1}\}$. Since $-a^{r_1(X)}_{21} = c_2 = 1$, one gets $(\zeta q^{-2})(-q) = 1$ or $\zeta q^{-2} = -1$. If $(\zeta q^{-2})(-q) = 1$ then $q = -\zeta$, which is a contradiction. If $\zeta q^{-2} = -1$ then $q \in G'_8$ and $\mathcal{D} = \mathcal{D}_{12,3}$.
Case 4bb. Consider the last case: $1 + q_{11} + q^2_{11} + q^3_{11} = 0$, $q_{22} = -1$ and $q_{11} \neq -1$. Then $q^2_{11} = -1$ and $p \neq 2$. Set $\zeta := q_{11}$ and $q = q_0$. The Dynkin diagram of $r_1(X)$ is 

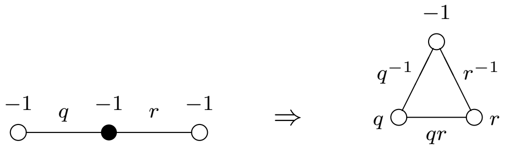
with $q \in \Bbbk^* \setminus \{1, -1, \zeta, \zeta^{-1}\}$ and $\zeta \in G'_4$. Since $-a^{r_1(X)}_{21} = c_2 = 1$, one has $(-q^{-1})(-\zeta q^3) = 1$ or $\zeta q^3 = 1$.
Case 4bb1. If $\zeta q^3 = 1$, then $\zeta = q^{-3} \in G'_4$. If $p = 3$, then $\zeta = q$, which is a contradiction. Hence $p \neq 3$ and $\mathcal{D} = \tau\mathcal{D}_{85}$.
Case 4bb2. The condition $(-q^{-1})(-\zeta q^3) = 1$ holds. Then $\zeta = q^{-2} \in G'_4$. Hence $q \in G'_8$ and $\mathcal{D} = \mathcal{D}_{12,2}$.
By checking all cases in Proposition 3.1.6, the proof of Theorem 4.1.1 is completed.

(IIb) Consider the case $\theta = 3$.
Since the matrix $(q_{ij})_{i,j \in I}$ is indecomposable, the finite Cartan graph $\mathcal{C}(M)$ is indecomposable. Set $X = [M]_3$, $q_1 = q_{12}q_{21}$, $q_2 = q_{23}q_{32}$, and $q_3 = q_{31}q_{13}$. We write the Cartan matrix $A^X := (a_{ij})_{i,j \in I}$. Since $\mathcal{C}(M)$ is a finite indecomposable Cartan graph, by Theorem 3.2.6 we are free to assume that Cartan matrix A^X is one of the following cases: (a), (b) and (c).
Case (a). Assume that either X has a good A_3 neighborhood or $\mathcal{C}$ is standard, $A^X = A_3$ for all $X \in \mathcal{X}$. Let

$$(a, b, c, d) := (-a^{r_1(X)}_{23}, -a^{r_2(X)}_{13}, -a^{r_2(X)}_{31}, -a^{r_3(X)}_{21})$$

be the sequence in Definition 3.2.2. By applying Lemma 2.1.4 to $a_{13} = 0$, $a_{12} = -1$, and $a_{23} = -1$, we get $q_3 = 1$, $q_1 \neq 1$ and $q_2 \neq 1$, respectively. Since A^X is of A_3 type, we distinguish subcases a_1, a_2, a_3, a_4, a_5, a_6 and a_7 by Lemma 2.1.4.
Case a_1. Consider the case $(2)_{q_{11}} = (2)_{q_{22}} = (2)_{q_{33}} = 0$. Set $q_1 := q$ and $q_2 := r$. Then we get $q_{11} = q_{22} = q_{33} = -1$ for $p \neq 2$ or $q_{11} = q_{22} = q_{33} = 1$ for $p = 2$. Hence we distinguish subcases a_{1a} and a_{1b}.
case a_{1a}. Consider the case $q_{11} = q_{22} = q_{33} = -1$, $p \neq 2$. If $qr = 1$ and $q \neq -1$, then $\mathcal{D} = \mathcal{D}_{82}$, $p \neq 2$. If $qr = 1$ and $q = -1$, then $\mathcal{C}$ is standard

and $\mathcal{D} = \mathcal{D}_{11}$, where $q = -1$, and $p \neq 2$. If $qr \neq 1$, then X has a good A_3 neighborhood and by Lemma 2.1.7 we get the reflection $r_2(X)$ of X

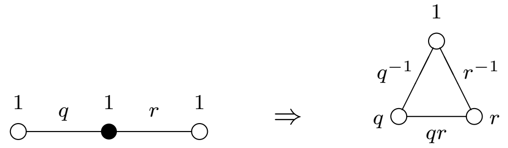


with $q \neq 1$, $r \neq 1$, $qr \neq 1$, and $p \neq 2$. Since X has a good A_3 neighborhood, we obtain that $a_{13}^{r_2(X)} = -1$ and $a_{31}^{r_2(X)} \in \{-1, -2\}$. Then we distinguish subcases a_{1a1} and a_{1a2}.

Case a_{1a1}. Consider the case $a_{13}^{r_2(X)} = a_{31}^{r_2(X)} = -1$. We obtain $(2)_q(q^2r - 1) = 0$ and $(2)_r(qr^2 - 1) = 0$ by Lemma 2.1.4. If $q = -1$, then $r \neq -1$ by $qr \neq 1$. If $q = -1$ and $qr^2 - 1 = 0$, then $r \in G'_4$ and hence $\mathcal{D} = \tau_{321}\mathcal{D}_{62}$, where $q \in G'_4$ and $p \neq 2$. If $q^2r = 1$ and $r = -1$, then $\mathcal{D} = \mathcal{D}_{62}$, where $q \in G'_4$ and $p \neq 2$. If $q^2r = qr^2 = 1$, then $q = r \in G'_3$, $p \neq 3$ and hence $\mathcal{D} = \mathcal{D}_{15,2}, p \neq 2, 3$.

Case a_{1a2}. Consider the case $a_{13}^{r_2(X)} = -1$ and $a_{31}^{r_2(X)} = -2$. Then $(2)_q(q^2r - 1) = 0$, $(3)_r(qr^3 - 1) = 0$ and $(2)_r(qr^2 - 1) \neq 0$. Hence $r \neq -1$ and $qr^2 \neq 1$. If $q = -1$, $r \in G'_3$ and $p \neq 3$, then $\mathcal{D} = \mathcal{D}_{17,1}, p \neq 2, 3$. If $q = -1$ and $qr^3 = 1$, then $r \in G'_6$, $p \neq 3$ and hence $\mathcal{D} = \tau_{321}\mathcal{D}_{72}$, where $q \in G'_6$ and $p \neq 2, 3$. If $q^2r = r^3 = 1$, then $-q = r \in G'_3$, $p \neq 3$ by $qr^2 \neq 1$. Hence $\mathcal{D} = \tau_{321}\mathcal{D}_{17,3}, p \neq 2, 3$. If $q^2r = qr^3 = 1$, then $r^2 = q$, where $r \in G'_5, p \neq 5$. Hence by Lemmas 2.1.4 and 2.1.7 we get the sequence $(a, b, c, d) = (2, 1, 2, 3)$. Further $a_{21}^{r_3r_1(X)} = -4$, which is a contradiction.

case a_{1b}. Consider the case $q_{11} = q_{22} = q_{33} = 1, p = 2$. If $qr = 1$, then $\mathcal{D} = \mathcal{D}_{82}, p = 2$. If $qr \neq 1$, then the Dynkin diagrams of X and $r_2(X)$ are



with $q \neq 1$, $r \neq 1$, $qr \neq 1$, and $p = 2$. Since X has a good A_3 neighborhood, we obtain that $a_{13}^{r_2(X)} = -1$ and $a_{31}^{r_2(X)} \in \{-1, -2\}$. Then we have subcases a_{1b1} and a_{1b2}.

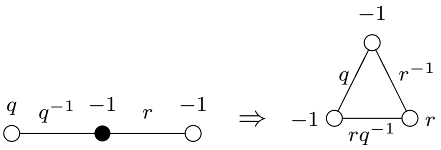
Case a_{1b1}. Consider the case $a_{13}^{r_2(X)} = a_{31}^{r_2(X)} = -1$. Since $q \neq 1$ and $r \neq 1$, we obtain $q^2r = qr^2 = 1$ by Lemma 2.1.4. Then $q = r \in G'_3$ and hence $\mathcal{D} = \mathcal{D}_{15,2}, p = 2$.

Case a_{1b2}. Consider the case $a_{13}^{r_2(X)} = -1$ and $a_{31}^{r_2(X)} = -2$. Then we get

$q^2r = 1$, $(3)_r(qr^3 - 1) = 0$, and $qr^2 \neq 1$. If $q^2r = r^3 = 1$, then $q = r \in G'_3$ and hence $qr^2 = 1$, which is a contradiction. Then we obtain $q^2r = qr^3 = 1$, then $r^2 = q$, where $r \in G'_5$, $p = 2$. Hence by Lemmas 2.1.4 and 2.1.7 we get the sequence $(a, b, c, d) = (2, 1, 2, 3)$ and $a_{21}^{r_3r_1(X)} = -4$ in Definition 3.2.2, which is again a contradiction.

Case a_2. Consider the case $(2)_{q_{22}} = (2)_{q_{33}} = 0$, $q_{11}q_1 - 1 = 0$, and $(2)_{q_{11}} \neq 0$. Set $q_{11} := q$ and $q_2 := r$. Then we get $q \neq -1$ for $p \neq 2$ and $q \neq 1$ for $p = 2$. Hence we obtain $q_{22} = q_{33} = -1$, $qq_1 = 1$, $q \neq -1$ for $p \neq 2$ or $q_{22} = q_{33} = 1$, $qq_1 = 1$, $q \neq 1$ for $p = 2$. We distinguish two subcases a_{2a} and a_{2b}.

Case a_{2a} Consider the case $q_{22} = q_{33} = -1$, $qq_1 = 1$, $q \neq -1$, and $p \neq 2$. If $r = q \neq -1$, then $\mathcal{C}$ is standard and $\mathcal{D} = \tau_{321}\mathcal{D}_{42}$, $p \neq 2$. If $r = -1$, $q \neq r$, and $q \notin G'_4$, then $\mathcal{D} = \mathcal{D}_{91}$, where $r = -1$, $q \notin G'_4$ and $p \neq 2$. If $r = -1$, $q \neq r$, and $q \in G'_4$, then $\mathcal{D} = \mathcal{D}_{10,2}$, where $q \in G'_4$ and $p \neq 2$. Now we consider the case $r \neq -1$ and $q \neq r$. Then X has a good A_3 neighborhood and by Lemma 2.1.7 we get the following reflection $r_2(X)$ of X

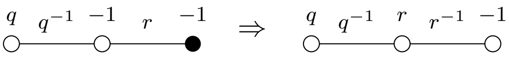


with $q \neq 1$, $r \notin \{-1, 1, q\}$, and $p \neq 2$. Since X has a good A_3 neighborhood, we get $a_{31}^{r_2(X)} \in \{-1, -2\}$. Then we get either $(2)_r(r^2q^{-1} - 1) = 0$ or $(3)_r(r^3q^{-1} - 1) = 0$, $(2)_r(r^2q^{-1} - 1) \neq 0$. Hence we can distinguish three subcases a_{2a1}, a_{2a2} and a_{2a3}.

Case a_{2a1}. Consider the case $(2)_r(r^2q^{-1} - 1) = 0$. Since $r \neq -1$ and $p \neq 2$, we obtain that $r^2q^{-1} = 1$ and hence $\mathcal{D} = \tau_{321}\mathcal{D}_{62}$, $p \neq 2$.

Case a_{2a2}. Consider the case $r^3q^{-1} = 1$ and $(2)_r(r^2q^{-1} - 1) \neq 0$. Then we get $q = r^3 \neq 1$ and $r \notin \{1, -1\}$. Hence $\mathcal{D} = \tau_{321}\mathcal{D}_{72}$, $p \neq 2$.

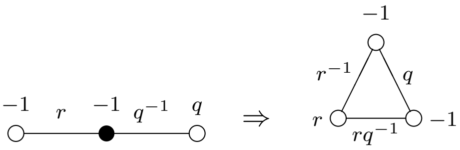
Case a_{2a3}. Consider the case $r \in G'_3$, $(2)_r(r^2q^{-1} - 1) \neq 0$, and $p \neq 3$. Then $qr \neq 1$. By Lemma 2.1.7 we get the following reflection $r_3(X)$ of X



with $q \neq 1$, $r \notin \{-1, 1, q, q^{-1}\}$. By Lemmas 2.1.4 and 2.1.7 we obtain that $(a, b, c, d) = (1, 1, 2, 2)$, which is a contradiction.

Case a_{2b}. Consider the case $q_{22} = q_{33} = 1$, $qq_1 = 1$, $q \neq 1$ for $p = 2$. If $r = q$, then $\mathcal{D} = \tau_{321}\mathcal{D}_{42}$, $p = 2$. If $r \neq q$, then we get $a_{31}^{r_2(X)} \in \{-1, -2\}$. Hence

we distinguish subcases a_{2b1}, a_{2b2}, and a_{2b3}.
Case a_{2b1}. If $a_{31}^{r_2(X)} = -1$, then $r^2q^{-1} = 1$ and hence $\mathcal{D} = \tau_{321}\mathcal{D}_{62}$, $p = 2$.
Case a_{2b2}. Consider the case $a_{31}^{r_2(X)} = -2$. If $r \in G'_3$, then we get $(a, b, c, d) = (1, 1, 2, 2)$, which is a contradiction. If $r^3q^{-1} = 1$, then $q = r^3 \neq 1$ and hence $\mathcal{D} = \tau_{321}\mathcal{D}_{72}$, $p = 2$.
Case a_3. Consider the case $(2)_{q_{11}} = (2)_{q_{22}} = 0$, $q_{33}q_2 - 1 = 0$, and $(2)_{q_{33}} \neq 0$. Let $q_{33} := q$ and $q_1 := r$. Then by $(2)_q \neq 0$ we get $q \neq -1$ for $p \neq 2$ and $q \neq 1$ for $p = 2$. Hence we obtain $q_{11} = q_{22} = -1$, $qq_2 = 1$, $q \neq -1$ for $p \neq 2$ or $q_{11} = q_{22} = 1$, $qq_2 = 1$, $q \neq 1$ for $p = 2$. We distinguish two subcases a_{3a} and a_{3b}.
Case a_{3a} Consider the case $q_{11} = q_{22} = -1$, $qq_2 = 1$, $q \neq -1$, and $p \neq 2$. If $r = q \neq -1$, then $\mathcal{C}$ is standard and $\mathcal{D} = \mathcal{D}_{42}$, $p \neq 2$. If $r = -1$, $q \neq r$, and $q \notin G'_4$, then $\mathcal{D} = \tau_{321}\mathcal{D}_{91}$, where $r = -1$, $q \notin G'_4$ and $p \neq 2$. If $r = -1$, $q \neq r$, and $q \in G'_4$, then $\mathcal{D} = \tau_{321}\mathcal{D}_{10,2}$, where $q \in G'_4$ and $p \neq 2$. Now we consider the case $r \neq -1$ and $q \neq r$. Then X has a good A_3 neighborhood and by Lemma 2.1.7 we get the following reflection $r_2(X)$ of X



with $q \neq 1$, $r \notin \{-1, 1, q\}$, and $p \neq 2$. Since X has a good A_3 neighborhood, we get $a_{31}^{r_2(X)} = -1$ and hence $r^2q^{-1} = 1$. Then $\mathcal{D} = \mathcal{D}_{62}$, $p \neq 2$.
Case a_{3b}. Consider the case $q_{11} = q_{22} = 1$, $qq_2 = 1$, $q \neq 1$ for $p = 2$. If $r = q$, then $\mathcal{D} = \mathcal{D}_{42}$, $p = 2$. If $r \neq q$, then we get $a_{31}^{r_2(X)} = -1$. Hence $r^2q^{-1} = 1$ and $\mathcal{D} = \mathcal{D}_{62}$, $p = 2$.
Case a_4. Consider the case $(2)_{q_{11}} = (2)_{q_{33}} = 0$, $q_{22}q_1 - 1 = q_{22}q_2 - 1 = 0$, and $(2)_{q_{22}} \neq 0$. Let $q_{22} := q$. Then by $(2)_q \neq 0$ we get $q \neq -1$ for $p \neq 2$ and $q \neq 1$ for $p = 2$. Hence we obtain $q_{11} = q_{33} = -1$, $qq_1 = qq_2 = 1$, $q \neq -1$ for $p \neq 2$ or $q_{11} = q_{33} = 1$, $qq_1 = qq_2 = 1$, $q \neq 1$ for $p = 2$. If $p \neq 2$, then $\mathcal{C}$ is standard and $\mathcal{D} = \mathcal{D}_{82}$, $p \neq 2$. If $p = 2$, then $\mathcal{C}$ is standard and $\mathcal{D} = \mathcal{D}_{82}$, $p = 2$.
Case a_5. Consider the case $(2)_{q_{11}} = 0$, $q_{22}q_1 - 1 = q_{22}q_2 - 1 = q_{33}q_2 - 1 = 0$, $(2)_{q_{22}} \neq 0$, and $(2)_{q_{33}} \neq 0$. Set $q_{33} := q$. If $q_{11} = 1$ and $p = 2$, then $\mathcal{C}$ is standard and $\mathcal{D} = \mathcal{D}_{41}$, $p = 2$. If $q_{11} = -1$ and $p \neq 2$, then $q \neq -1$, $\mathcal{C}$ is standard, and $\mathcal{D} = \mathcal{D}_{41}$, $p \neq 2$.
Case a_6. Consider the case $(2)_{q_{33}} = 0$, $q_{11}q_1 - 1 = q_{22}q_1 - 1 = q_{22}q_2 - 1 = 0$, $(2)_{q_{11}} \neq 0$, and $(2)_{q_{22}} \neq 0$. Set $q_{11} := q$. Then $q \neq -1$. If $q_{33} = 1$ and $p = 2$,

then $\mathcal{C}$ is standard and $\mathcal{D} = \tau_{321}\mathcal{D}_{41}$, $p = 2$. If $q_{33} = -1$ and $p \neq 2$, then $q \neq -1$ and $\mathcal{D} = \tau_{321}\mathcal{D}_{41}$, $p \neq 2$.

Case a_7. Consider the case $(2)_{q_{22}} = 0$, $q_{11}q_1 = q_{33}q_2 = 1$, $(2)_{q_{11}} \neq 0$, and $(2)_{q_{33}} \neq 0$. Set $q_{11} := q$ and $q_{33} := r$. We distinguish subcases a_{7a} and a_{7b}.

Case a_{7a}. Consider the case $q_{22} = -1$, $qq_1 = rq_2 = 1$, and $p \neq 2$. If $qr = 1$, then $q \neq -1$ and $\mathcal{D} = \mathcal{D}_{81}$, $p \neq 2$. If $qr \neq 1$, $q = r \in G'_3$, and $p \neq 3$, then $\mathcal{D} = \mathcal{D}_{11,1}$, $p \neq 2, 3$. If $qr \neq 1$, $q = r \notin G'_3$ and $p \neq 3$, then $\mathcal{D} = \mathcal{D}_{10,1}$, $p \neq 2, 3$. If $qr \neq 1$, $q \neq r$, $qr^2 \neq 1$, and $rq^2 \neq 1$, then $\mathcal{D} = \mathcal{D}_{91}$, $p \neq 2$. If $qr \neq 1$, $q \neq r$, and $rq^2 = 1$, then $q \notin G'_3$ for $p \neq 3$. Hence $\mathcal{D} = \mathcal{D}_{10,2}$, $p \neq 2$. If $qr \neq 1$, $q \neq r$, and $qr^2 = 1$, then $r \notin G'_3$ for $p \neq 3$, $r^2 \neq 1$. Hence $\mathcal{D} = \tau_{321}\mathcal{D}_{10,2}$, $p \neq 2$.

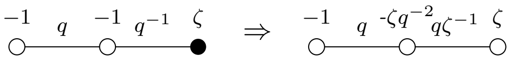
Case a_{7b}. Consider the case $q_{22} = 1$, $qq_1 = rq_2 = 1$, and $p = 2$. If $qr = 1$, then $\mathcal{D} = \mathcal{D}_{81}$, $p = 2$. If $qr \neq 1$ and $q = r \in G'_3$, then $\mathcal{D} = \mathcal{D}_{11,1}$, $p = 2$. If $qr \neq 1$ and $q = r \notin G'_3$, then $\mathcal{D} = \mathcal{D}_{10,1}$, $p = 2$. If $qr \neq 1$, $q \neq r$, $qr^2 \neq 1$, and $rq^2 \neq 1$, then $\mathcal{D} = \mathcal{D}_{91}$, $p = 2$. If $qr \neq 1$, $q \neq r$, and $rq^2 = 1$, then $q \notin G'_3$ and $\mathcal{D} = \mathcal{D}_{10,2}$, $p = 2$. If $qr \neq 1$, $q \neq r$, and $qr^2 = 1$, then $r \notin G'_3$, and $\mathcal{D} = \tau_{321}\mathcal{D}_{10,2}$, $p = 2$.

Case a_8. Consider the case $q_{11}q_1 = q_{22}q_1 = q_{22}q_2 = q_{33}q_2 = 1$, $(2)_{q_{11}} \neq 0$, $(2)_{q_{22}} \neq 0$, and $(2)_{q_{33}} \neq 0$. Then $\mathcal{D} = \mathcal{D}_{11}$, $q \neq -1$.

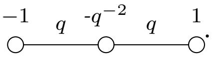
Case (b). Assume that either X has a good B_3 neighborhood or $\mathcal{C}$ is standard, $A^X = B_3$ for all $X \in \mathcal{X}$. Since A^X is of B_3 type, we obtain that $q_3 = 1$, $q_1 \neq 1$ and $q_2 \neq 1$ by Lemma 2.1.4. By Lemma 2.1.7 we get $q_1q_2 = 1$ if $(2)_{q_{22}} = 0$, since $a_{13}^{r_2(X)} = 0$. Further, we distinguish subcases b_1, b_2, b_3, b_4, b_5, b_6, b_7 and b_8.

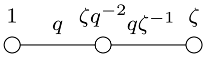
Case b_1. Consider the case $(2)_{q_{11}} = (2)_{q_{22}} = (3)_{q_{33}} = 0$ and $(2)_{q_{33}}(q_{33}q_2 - 1) \neq 0$. Let $q_1 := q$. Then $q_2 = q^{-1}$ and $q_{33} \neq q$. We distinguish subcases b_{1a}, b_{1b} and b_{1c}.

case b_{1a}. Consider the case $q_{11} = q_{22} = -1$, $q_{33} := \zeta \in G'_3$, and $p \neq 2, 3$. By Lemma 2.1.7 we get the reflection $r_3(X)$ of X



with $\zeta \in G'_3$, $q \neq \zeta$, and $p \neq 2, 3$. Since $a_{23}^{r_3(X)} = -1$ by Definition 3.2.3, we get $((-\zeta q^{-2})q\zeta^{-1} - 1)(\zeta q^{-2} - 1) = 0$ and hence $q = -1$, $q = \zeta^{-1}$, or $q = -\zeta^{-1}$. If $q = \zeta^{-1}$, then $\mathcal{C}$ is standard and $\mathcal{D} = \mathcal{D}_{52}$, where $q \in G'_3$, $p \neq 2, 3$. If $q = -\zeta^{-1}$, then $\mathcal{C}$ is standard and $\mathcal{D} = \mathcal{D}_{14,2}$, $p \neq 2, 3$. If $q = -1$, then $a_{21}^{r_3(X)} = -3$, which is a contradiction.

Case b_{1b}. Consider the case $q_{11} = q_{22} = -1$, $q_{33} = 1$, and $p = 3$. Then the Dynkin diagram of $r_3(X)$ is . By $a_{23}^{r_3(X)} = -1$ we get $q = -1$. Hence $\mathcal{C}$ is standard and $\mathcal{D} = \mathcal{D}_{12',1}$.

Case b_{1c}. Consider the case $q_{11} = q_{22} = 1$, $q_{33} := \zeta \in G'_3$, and $p = 2$. Then the Dynkin diagram of $r_3(X)$ is  with $\zeta \in G'_3$, $q \neq \zeta$, and $p = 2$. The condition $a_{23}^{r_3(X)} = -1$ gives $q = \zeta^{-1}$. Hence $\mathcal{C}$ is standard and $\mathcal{D} = \mathcal{D}_{52}$, where $q \in G'_3$ and $p = 2$.

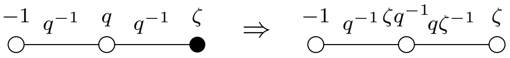
Case b_2. Consider the case $(2)_{q_{11}} = (2)_{q_{22}} = 0$, $q_{33}^2 q_2 - 1 = 0$, $(3)_{q_{33}} \neq 0$, and $(2)_{q_{33}}(q_{33}q_2 - 1) \neq 0$. Let $q_{33} := q$. Then $q_2 = q^{-2}$ and hence $q_1 = q^2$ by $q_1 q_2 = 1$. Hence we distinguish two cases b_{2a} and b_{2b}.

case b_{2a}. Consider the case $q_{11} = q_{22} = -1$ and $p \neq 2$. If $q^2 \neq -1$ and $p \neq 3$, then $\mathcal{C}$ is standard and $\mathcal{D} = \mathcal{D}_{52}$, where $q \notin G'_3$, $p \neq 2, 3$. If $q^2 \neq -1$ and $p = 3$, then $\mathcal{D} = \mathcal{D}_{52}$, $p = 3$. If $q^2 = -1$, then $\mathcal{C}$ is standard and $\mathcal{D} = \mathcal{D}_{21}$, where $q \in G'_4$, $p \neq 2$.

Case b_{2b}. Consider the case $q_{11} = q_{22} = 1$ and $p = 2$. Then $\mathcal{C}$ is standard and $\mathcal{D} = \mathcal{D}_{52}$, where $q \notin G'_3$, $p = 2$.

Case b_3. Consider the case $(2)_{q_{11}} = (3)_{q_{33}} = 0$, $q_{22}q_1 - 1 = 0$, $q_{22}q_2 - 1 = 0$, $(2)_{q_{22}} \neq 0$, and $(2)_{q_{33}}(q_{33}q_2 - 1) \neq 0$. Let $q_{22} := q$ and $q_{33} := \zeta$. Then $q_1 = q_2 = q^{-1}$ and $q \neq \zeta$. We distinguish subcases b_{3a}, b_{3b}, and b_{3c}.

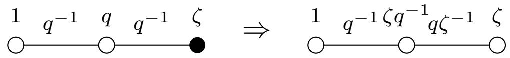
Case b_{3a}. Consider the case $q_{11} = -1$, $\zeta \in G'_3$ and $p \neq 2, 3$. By Lemma 2.1.7 we get the Dynkin diagrams of X and $r_3(X)$

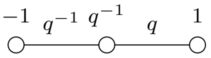


with $\zeta \in G'_3$, $q \notin \{-1, 1, \zeta\}$, and $p \neq 2, 3$. We get $a_{21}^{r_3(X)} \in \{-1, -2\}$ by Definition 3.2.3. Hence $((\zeta q^{-1}) + 1)(\zeta q^{-2} - 1)(q^{-3} - 1)(\zeta^2 q^{-3} - 1) = 0$.

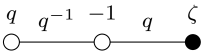
Then we obtain $q \in \{\zeta^{-1}, -\zeta^{-1}, -\zeta\}$ or $q^{-3} = \zeta$. If $q = \zeta^{-1}$, then $\mathcal{D} = \mathcal{D}_{51}$, where $q \in G'_3$, $p \neq 2, 3$. If $q = -\zeta^{-1}$, then $\mathcal{C}$ is standard and $\mathcal{D} = \mathcal{D}_{14,1}$, $p \neq 2, 3$. If $q = -\zeta$, then by Lemmas 2.1.4 and 2.1.7 we get $a_{23}^{r_1 r_3(X)}$ $a_{23}^{r_1 r_3(X)} = -3 \notin \{-1, -2\}$, which is a contradiction. If $q^{-3} = \zeta \in G'_3$, then $a_{23}^{r_1 r_3(X)} = -8 \notin \{-1, -2\}$, which is again a contradiction.

Case b_{3b}. Consider the case $q_{11} = 1$, $\zeta \in G'_3$ and $p = 2$. By Lemma 2.1.7 we get the reflection $r_3(X)$ of X is

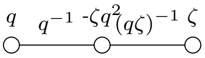


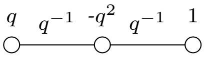
with $\zeta \in G'_3$, $q \neq \zeta$, and $p = 2$. We get $q = \zeta^{-1}$ or $q^{-3} = \zeta$ by $a_{21}^{r_3(X)} \in \{-1, -2\}$. If $q = \zeta^{-1}$, then $\mathcal{C}$ is standard and $\mathcal{D} = \mathcal{D}_{51}$, where $q \in G'_3, p = 2$. If $q^{-3} = \zeta \in G'_3$, then by Lemmas 2.1.4 and 2.1.7 we get $a_{23}^{r_1 r_3(X)} = -8 \notin \{-1, -2\}$, which is a contradiction.
Case b_{3c}. Consider the case $q_{11} = -1, \zeta = 1$ and $p = 3$. By Lemma 2.1.7 the Dynkin diagrams of $r_3(X)$ is 

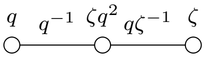
with $q \notin G'_2$, and $p = 3$. Then $a_{21}^{r_3(X)} \notin \{-1, -2\}$, which is a contradiction.
Case b_4. Consider the case $(2)_{q_{22}} = (3)_{q_{33}} = 0$, $q_{11}q_1 - 1 = 0$, $(2)_{q_{11}} \neq 0$, and $(2)_{q_{33}}(q_{33}q_2 - 1) \neq 0$. Let $q_{11} := q$ and $q_{33} := \zeta$. Then $q_1 = q^{-1}$ and $q_2 = q$ by $q_1 q_2 = 1$. Hence $q \neq \zeta^{-1}$. We distinguish three subcases b_{4a}, b_{4b}, and b_{4c}.
case b_{4a}. Consider the case $q_{22} = -1, \zeta \in G'_3, qq_1 - 1 = 0$, and $p \neq 2, 3$. The Dynkin diagrams of X and $r_3(X)$ are



$$\Rightarrow$$



with $\zeta \in G'_3$, $q \notin \{1, -1, \zeta^{-1}\}$, and $p \neq 2, 3$. We get $a_{23}^{r_3(X)} = -1$ by Definition 3.2.3. Then $(\zeta q^2 - 1)(\zeta q^2(\zeta q)^{-1} + 1) = 0$. Hence $q = -\zeta$ or $q = \zeta$. If $q = -\zeta$ then $\mathcal{C}$ is standard and $\mathcal{D} = \mathcal{D}_{14,3}, p \neq 2, 3$. If $q = \zeta$ then $\mathcal{C}$ is standard and $\mathcal{D} = \mathcal{D}_{53}$, where $q := -\zeta^{-1}$, $\zeta \in G'_3$, and $p \neq 2, 3$.
Case b_{4b}. Consider the case $q_{22} = -1$, $\zeta = 1$, and $p = 3$. Then the Dynkin diagram of $r_3(X)$ is 

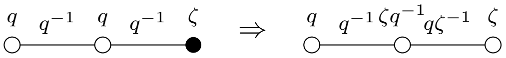
with $q \notin G'_2$, $p = 3$. Then $a_{23}^{r_3(X)} \neq -1$, which is a contradiction.
Case b_{4c}. Consider the case $q_{22} = 1$, $q_{33} := \zeta \in G'_3$, and $p = 2$. Then the Dynkin diagram of $r_3(X)$ is 

. The condition $a_{23}^{r_3(X)} = -1$ gives that $q = \zeta$. Then $\mathcal{C}$ is standard and $\mathcal{D} = \mathcal{D}_{53}$, where $q := \zeta^{-1}$, $\zeta \in G'_3$, and $p = 2$.
Case b_5. Consider the case $(2)_{q_{11}} = 0$, $q_{22}q_1 = 1$, $q_{22}q_2 = 1$, $q_{33}^2 q_2 = 1$, $(2)_{q_{22}} \neq 0$, and $(3)_{q_{33}} \neq 0$. Let $q_{33} := q$. Then $q_2 = q^{-2}$, $q_{22} = q^2$ and hence $q_1 = q^{-2}$. We obtain that $q^2 \neq 1, -1$, and $q^3 \neq 1$. If $p = 3$ and $q_{11} = -1$, then $\mathcal{C}$ is standard and $\mathcal{D} = \mathcal{D}_{51}$, $p = 3$. If $p \neq 2, 3$ and $q_{11} = -1$, then $\mathcal{D} = \mathcal{D}_{51}$, where $q \notin G'_3$ and $p \neq 2, 3$. If $p = 2$ and $q_{11} = 1$, then $\mathcal{D} = \mathcal{D}_{51}$, where $q \notin G'_3$ and $p = 2$.
Case b_6. Consider the case $(2)_{q_{22}} = 0$, $q_{11}q_1 = 1$, $q_{33}^2 q_2 = 1$, $(2)_{q_{11}} \neq 0$, and $(3)_{q_{33}} \neq 0$. Let $q_{33} := q$. Then $q_2 = q^{-2}$ and $q_1 = q^2$ by $q_1 q_2 = 1$. Hence $q_{11} = q^{-2}$. If $p = 3$ and $q_{22} = -1$, then $\mathcal{D} = \mathcal{D}_{53}$, $p = 3$. If $p \neq 2, 3$ and

$q_{22} = -1$, then $\mathcal{D} = \mathcal{D}_{53}$, where $q \notin G'_3$ and $p \neq 2, 3$. If $p = 2$ and $q_{22} = 1$, then $\mathcal{D} = \mathcal{D}_{53}$, where $q \notin G'_3$ and $p = 2$.

Case b_7. Consider the case $(3)_{q_{33}} = 0$, $q_{11}q_1 = 1$, $q_{22}q_1 = 1$, $q_{22}q_2 = 1$, $(2)_{q_{11}} \neq 0$, $(2)_{q_{22}} \neq 0$ and $(2)_{q_{33}}(q_{33}q_2 - 1) \neq 0$. Let $q_{22} := q$ and $q_{33} := \zeta$. Then $q_2 = q_1 = q^{-1}$, $q_{11} = q$, and $q \neq \zeta$. Hence we distinguish two subcases b_{7a} and b_{7b}.

Case b_{7a}. Consider the case $\zeta \in G'_3$ and $p \neq 3$. Then by Lemma 2.1.7 the reflection $r_3(X)$ of X is



with $\zeta \in G'_3$, $q \notin \{1, -1, \zeta\}$, and $p \neq 3$. We get $a_{21}^{r_3(X)} \in \{-1, -2\}$ from Definition3.2.3. Hence $((\zeta q^{-1}) + 1)(\zeta q^{-2} - 1)(q^{-3} - 1)(\zeta^2 q^{-3} - 1) = 0$. Then $q \in \{\zeta^{-1}, -\zeta^{-1}, -\zeta\}$ or $q^{-3} = \zeta$. If $q = \zeta^{-1}$, then $\mathcal{C}$ is standard and $\mathcal{D} = \mathcal{D}_{21}$, where $q \in G'_3$, $p \neq 3$. If $q = -\zeta^{-1}$, then $\mathcal{C}$ is standard and $\mathcal{D} = \mathcal{D}_{12,1}$. If $q = -\zeta$, then X has a good B_3 neighborhood and $\mathcal{D} = \tau_{321}\mathcal{D}_{16,5}$. If $q^{-3} = \zeta$, then $\mathcal{D} = \mathcal{D}_{18,1}$.

Case b_{7b}. Consider the case $p = 3$ and $\zeta = 1$. Since $(2)_q \neq 0$ and $q \neq 1$, we get $a_{21}^{r_3(X)} \notin \{-1, -2\}$, which is a contradiction.

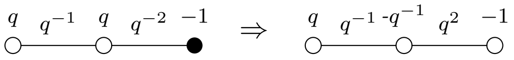
Case b_8. Consider the case $q_{11}q_1 = q_{22}q_1 = q_{22}q_2 = q_{33}^2 q_2 = 1$, $(2)_{q_{11}} \neq 0$, $(2)_{q_{22}} \neq 0$, and $(3)_{q_{33}} \neq 0$. If $p \neq 2, 3$, then $\mathcal{C}$ is standard and $\mathcal{D} = \mathcal{D}_{21}$, where $q \notin G'_3 \cup G'_4$ and $p \neq 2, 3$. If $p = 3$, then $\mathcal{D} = \mathcal{D}_{21}$, where $q \notin G'_4$ and $p = 3$. If $p = 2$, then $\mathcal{D} = \mathcal{D}_{21}$, where $q \notin G'_3$ and $p = 2$.

Case (c). Assume that either A^X has a good C_3 neighborhood or $\mathcal{C}$ is standard, $A^X = C_3$ for all $X \in \mathcal{X}$. Since A^X is of C_3 type, we obtain that $q_3 = 1$, $q_1 \neq 1$ and $q_2 \neq 1$ by Lemma 2.1.4. Since $a_{23} = -2$, we get $(2)_{q_{22}} \neq 0$ and hence $q_{22}q_1 = 0$ by $a_{21} = -1$. Hence $q_{22} \neq 1$. Since $a_{23}^{r_1(X)} = -2$, we get $(2)_{q_{11}} \neq 0$ and hence $q_{11}q_1 - 1 = 0$ by $a_{12} = -1$. If $(3)_{q_{22}} = 0$, then $q_1 q_2 = 1$ by Lemma 2.1.4. Hence, by Lemma 2.1.4 we distinguish the following cases c_1, c_2, c_3, and c_4.

Case c_1. Consider the case $q_{11}q_1 = q_{22}q_1 = q_{33}q_2 = q_{22}^2 q_2 = 1$. Let $q_{22} := q$. Then $q^2 \neq 1$ and $\mathcal{C}$ is standard and hence $\mathcal{D} = \mathcal{D}_{31}$.

Case c_2. Consider the case $q_{11}q_1 - 1 = q_{22}q_1 - 1 = q_{33}q_2 - 1 = 0$, $(3)_{q_{22}} = 0$, and ${q_{22}}^2 q_2 - 1 \neq 0$. Since $q_{22} \neq 1$, we get $q_{22} := \zeta \in G'_3$ and $p \neq 3$. Then we get $q_1 = \zeta^{-1}$, $q_2 = \zeta$ and hence ${q_{22}}^2 q_2 - 1 = 0$, which is a contradiction.

Case c_3. Consider the case $q_{11}q_1 - 1 = q_{22}q_1 - 1 = q_{22}^2 q_2 - 1 = 0$, $(2)_{q_{33}} = 0$, and $q_{33}q_2 - 1 \neq 0$. Let $q_{11} := q$. If $p \neq 2$, then by Lemma 2.1.7 we get the reflection of X



with $q \notin G_2' \cup G_4'$. We obtain $a_{21}^{r_3(X)} \in \{-1,-2\}$ from Definition 3.2.4. Since $q^2 \notin G_2' \cup G_4'$, we get $a_{21}^{r_3(X)} = -2$ and hence $q^{-3} = -1$ or $q^{-3} = 1$. If $q^{-3} = -1$ and $p \neq 3$, then $\mathcal{C}$ is standard and $\mathcal{D} = \mathcal{D}_{13,1}$, with $\zeta \in G_6'$. If $q^{-3} = 1$ and $p \neq 3$, then $\mathcal{D} = \mathcal{D}_{13,1}$, with $\zeta \in G_3'$. If $p = 2$, then we get $q^{-3} = 1$ by $a_{21}^{r_3(X)} \in \{-1,-2\}$ and then $\mathcal{D} = \mathcal{D}_{13',1}$.

Case c_4. Consider the case $q_{11}q_1 - 1 = q_{22}q_1 - 1 = 0$, $(2)_{q_{33}} = (3)_{q_{22}} = 0$, $q_{22}^2 q_2 - 1 \neq 0$, and $q_{33}q_2 - 1 \neq 0$. Since $(3)_{q_{22}} = 0$ and $q_{22} \neq 1$, we get $q_{22} := \zeta \in G_3'$, $q_2 = \zeta$, and $p \neq 3$. Hence $q_{22}^2 q_2 - 1 = 0$, which is a contradiction.

□

Chapter 5

Generalized Dynkin diagrams

In this section, we give all the Nichols algebras of diagonal type of rank 2 and rank 3 with finite root systems as well as their exchange graphs.

5.1 Rank 2 Nichols algebras

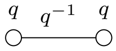
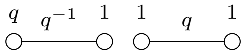
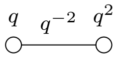
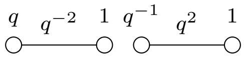
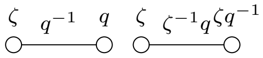
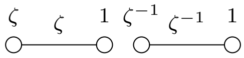
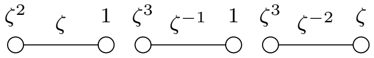
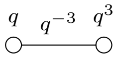
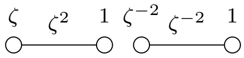
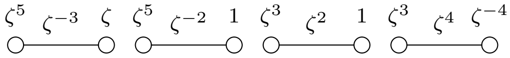
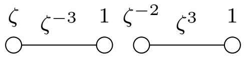
	Dynkin diagrams	fixed parameters
2		$q \in \Bbbk^* \setminus \{1\}$
3		$q \in \Bbbk^* \setminus \{1\}$
4		$q \in \Bbbk^* \setminus \{1\}$
5		$q \in \Bbbk^* \setminus \{1\}$
6		$\zeta \in G'_3$, $q \in \Bbbk^* \setminus \{1, \zeta, \zeta^2\}$
7		$\zeta \in G'_3$
10		$\zeta \in G'_9$
11		$q \in \Bbbk^* \setminus \{1\}, q \notin G'_3$
14		$\zeta \in G'_5$
16		$\zeta \in G'_{15}$
17		$\zeta \in G'_7$

Table 5.1: Dynkin diagrams of rank 2 in characteristic $p = 2$

	Dynkin diagrams	fixed parameters
2	$\overset{q}{\circ}\overset{q^{-1}}{\text{——}}\overset{q}{\circ}$	$q \in \Bbbk^* \setminus \{1\}$
3	$\overset{q}{\circ}\overset{q^{-1}}{\text{——}}\overset{-1}{\circ} \quad \overset{-1}{\circ}\overset{q}{\text{——}}\overset{-1}{\circ}$	$q \in \Bbbk^* \setminus \{-1, 1\}$
4	$\overset{q}{\circ}\overset{q^{-2}}{\text{——}}\overset{q^2}{\circ}$	$q \in \Bbbk^* \setminus \{-1, 1\}$
5	$\overset{q}{\circ}\overset{q^{-2}}{\text{——}}\overset{-1}{\circ} \quad \overset{-q^{-1}}{\circ}\overset{q^2}{\text{——}}\overset{-1}{\circ}$	$q \in \Bbbk^* \setminus \{-1, 1\}, q \notin G'_4$
$6'$	$\overset{1}{\circ}\overset{q}{\text{——}}\overset{q^{-1}}{\circ} \quad \overset{1}{\circ}\overset{q^{-1}}{\text{——}}\overset{q}{\circ}$	$q \in \Bbbk^* \setminus \{1, -1\}$
$6'''$	$\overset{1}{\circ}\overset{-1}{\text{——}}\overset{-1}{\circ}$	
$9'$	$\overset{\zeta}{\circ}\overset{\zeta}{\text{——}}\overset{-1}{\circ} \quad \overset{1}{\circ}\overset{-\zeta}{\text{——}}\overset{-1}{\circ} \quad \overset{1}{\circ}\overset{\zeta}{\text{——}}\overset{1}{\circ}$	$\zeta \in G'_4$
11	$\overset{q}{\circ}\overset{q^{-3}}{\text{——}}\overset{q^3}{\circ}$	$q \in \Bbbk^* \setminus \{-1, 1\}$
12	$\overset{\zeta}{\circ}\overset{-\zeta}{\text{——}}\overset{-1}{\circ} \quad \overset{\zeta^2}{\circ}\overset{-\zeta^{-1}}{\text{——}}\overset{-1}{\circ} \quad \overset{\zeta^2}{\circ}\overset{\zeta}{\text{——}}\overset{\zeta^{-1}}{\circ}$	$\zeta \in G'_8$
$13'$	$\overset{-\zeta}{\circ}\overset{\zeta^{-1}}{\text{——}}\overset{-1}{\circ} \quad \overset{1}{\circ}\overset{\zeta}{\text{——}}\overset{-1}{\circ} \quad \overset{1}{\circ}\overset{\zeta^{-1}}{\text{——}}\overset{-\zeta^2}{\circ} \quad \overset{-\zeta^{-1}}{\circ}\overset{-\zeta}{\text{——}}\overset{-\zeta^2}{\circ}$	$\zeta \in G'_8$
14	$\overset{\zeta}{\circ}\overset{\zeta^2}{\text{——}}\overset{-1}{\circ} \quad \overset{-\zeta^{-2}}{\circ}\overset{\zeta^{-2}}{\text{——}}\overset{-1}{\circ}$	$\zeta \in G'_5$
15	$\overset{\zeta}{\circ}\overset{\zeta^{-3}}{\text{——}}\overset{-1}{\circ} \quad \overset{-\zeta^{-2}}{\circ}\overset{\zeta^3}{\text{——}}\overset{-1}{\circ} \quad \overset{-\zeta^{-2}}{\circ}\overset{-\zeta^3}{\text{——}}\overset{-1}{\circ} \quad \overset{-\zeta}{\circ}\overset{-\zeta^{-3}}{\text{——}}\overset{-1}{\circ}$	$\zeta \in G'_{20}$
$16'$	$\overset{1}{\circ}\overset{-\zeta^{-1}}{\text{——}}\overset{-\zeta^2}{\circ} \quad \overset{1}{\circ}\overset{-\zeta}{\text{——}}\overset{-1}{\circ} \quad \overset{\zeta}{\circ}\overset{-\zeta^{-1}}{\text{——}}\overset{-1}{\circ} \quad \overset{\zeta}{\circ}\overset{-\zeta^3}{\text{——}}\overset{-\zeta^{-3}}{\circ}$	$\zeta \in G'_5$
17	$\overset{-\zeta}{\circ}\overset{-\zeta^{-3}}{\text{——}}\overset{-1}{\circ} \quad \overset{-\zeta^{-2}}{\circ}\overset{-\zeta^3}{\text{——}}\overset{-1}{\circ}$	$\zeta \in G'_7$

Table 5.2: Dynkin diagrams of rank 2 in characteristic $p = 3$

	Dynkin diagrams	fixed parameters
2	$\overset{q}{\circ}\overset{q^{-1}}{\text{———}}\overset{q}{\circ}$	$q \in \Bbbk^* \setminus \{1\}$
3	$\overset{q}{\circ}\overset{q^{-1}}{\text{———}}\overset{-1}{\circ}$ $\overset{-1}{\circ}\overset{q}{\text{———}}\overset{-1}{\circ}$	$q \in \Bbbk^* \setminus \{-1, 1\}$
4	$\overset{q}{\circ}\overset{q^{-2}}{\text{———}}\overset{q^2}{\circ}$	$q \in \Bbbk^* \setminus \{-1, 1\}$
5	$\overset{q}{\circ}\overset{q^{-2}}{\text{———}}\overset{-1}{\circ}$ $\overset{-q^{-1}}{\circ}\overset{q^2}{\text{———}}\overset{-1}{\circ}$	$q \in \Bbbk^* \setminus \{-1, 1\}, q \notin G'_4$
6	$\overset{\zeta}{\circ}\overset{q^{-1}}{\text{———}}\overset{q}{\circ}$ $\overset{\zeta}{\circ}\overset{\zeta^{-1}q}{\text{———}}\overset{\zeta q^{-1}}{\circ}$	$\zeta \in G'_3, q\zeta \neq -1$ $q \in \Bbbk^* \setminus \{1, \zeta, \zeta^2\}$
$6''$	$\overset{\zeta}{\circ}\overset{-\zeta}{\text{———}}\overset{-\zeta^{-1}}{\circ}$	$\zeta \in G'_3$
7	$\overset{\zeta}{\circ}\overset{-\zeta}{\text{———}}\overset{-1}{\circ}$ $\overset{\zeta^{-1}}{\circ}\overset{-\zeta^{-1}}{\text{———}}\overset{-1}{\circ}$	$\zeta \in G'_3$
8	$\overset{-\zeta^2}{\circ}\overset{\zeta}{\text{———}}\overset{-1}{\circ}$ $\overset{-1}{\circ}\overset{-\zeta^{-1}}{\text{———}}\overset{-\zeta^{-2}}{\circ}$ $\overset{-1}{\circ}\overset{\zeta^{-1}}{\text{———}}\overset{\zeta^3}{\circ}$ $\overset{-\zeta^2}{\circ}\overset{\zeta^3}{\text{———}}\overset{-\zeta^{-2}}{\circ}$ $\overset{-1}{\circ}\overset{-\zeta}{\text{———}}\overset{\zeta^3}{\circ}$	$\zeta \in G'_{12}$
9	$\overset{-\zeta^{-1}}{\circ}\overset{-\zeta^3}{\text{———}}\overset{-1}{\circ}$ $\overset{-\zeta^2}{\circ}\overset{\zeta^3}{\text{———}}\overset{-1}{\circ}$ $\overset{-\zeta^2}{\circ}\overset{\zeta}{\text{———}}\overset{-\zeta^2}{\circ}$	$\zeta \in G'_{12}$
10	$\overset{-\zeta^2}{\circ}\overset{\zeta}{\text{———}}\overset{-1}{\circ}$ $\overset{\zeta^3}{\circ}\overset{\zeta^{-1}}{\text{———}}\overset{-1}{\circ}$ $\overset{\zeta^3}{\circ}\overset{\zeta^{-2}}{\text{———}}\overset{-\zeta}{\circ}$	$\zeta \in G'_9$
11	$\overset{q}{\circ}\overset{q^{-3}}{\text{———}}\overset{q^3}{\circ}$	$q \in \Bbbk^* \setminus \{-1, 1\}, q \notin G'_3$
12	$\overset{\zeta}{\circ}\overset{-\zeta}{\text{———}}\overset{-1}{\circ}$ $\overset{\zeta^2}{\circ}\overset{-\zeta^{-1}}{\text{———}}\overset{-1}{\circ}$ $\overset{\zeta^2}{\circ}\overset{\zeta}{\text{———}}\overset{\zeta^{-1}}{\circ}$	$\zeta \in G'_8$
13	$\overset{\zeta}{\circ}\overset{\zeta^{-5}}{\text{———}}\overset{-1}{\circ}$ $\overset{-\zeta^{-4}}{\circ}\overset{\zeta^5}{\text{———}}\overset{-1}{\circ}$ $\overset{-\zeta^{-4}}{\circ}\overset{-\zeta^{-1}}{\text{———}}\overset{\zeta^6}{\circ}$ $\overset{\zeta^{-1}}{\circ}\overset{\zeta}{\text{———}}\overset{\zeta^6}{\circ}$	$\zeta \in G'_{24}$
$15'$	$\overset{\zeta}{\circ}\overset{\zeta}{\text{———}}\overset{-1}{\circ}$ $\overset{1}{\circ}\overset{-\zeta}{\text{———}}\overset{-1}{\circ}$ $\overset{1}{\circ}\overset{\zeta}{\text{———}}\overset{-1}{\circ}$ $\overset{-\zeta}{\circ}\overset{-\zeta}{\text{———}}\overset{-1}{\circ}$	$\zeta \in G'_4$
$16''$	$\overset{\zeta^2}{\circ}\overset{-1}{\text{———}}\overset{-\zeta}{\circ}$ $\overset{\zeta^2}{\circ}\overset{-\zeta}{\text{———}}\overset{-1}{\circ}$ $\overset{1}{\circ}\overset{-\zeta^{-1}}{\text{———}}\overset{-1}{\circ}$ $\overset{1}{\circ}\overset{-\zeta}{\text{———}}\overset{-\zeta^{-1}}{\circ}$	$\zeta \in G'_3$
17	$\overset{-\zeta}{\circ}\overset{-\zeta^{-3}}{\text{———}}\overset{-1}{\circ}$ $\overset{-\zeta^{-2}}{\circ}\overset{-\zeta^3}{\text{———}}\overset{-1}{\circ}$	$\zeta \in G'_7$

Table 5.3: Dynkin diagrams of rank 2 in characteristic $p = 5$

	Dynkin diagrams	fixed parameters
2	$\overset{q}{\circ}\overset{q^{-1}}{\rule[2pt]{2em}{0.4pt}}\overset{q}{\circ}$	$q \in \Bbbk^* \setminus \{1\}$
3	$\overset{q}{\circ}\overset{q^{-1}}{\rule[2pt]{2em}{0.4pt}}\overset{-1}{\circ}\quad\overset{-1}{\circ}\overset{q}{\rule[2pt]{2em}{0.4pt}}\overset{-1}{\circ}$	$q \in \Bbbk^* \setminus \{-1, 1\}$
4	$\overset{q}{\circ}\overset{q^{-2}}{\rule[2pt]{2em}{0.4pt}}\overset{q^2}{\circ}$	$q \in \Bbbk^* \setminus \{-1, 1\}$
5	$\overset{q}{\circ}\overset{q^{-2}}{\rule[2pt]{2em}{0.4pt}}\overset{-1}{\circ}\quad\overset{-q^{-1}}{\circ}\overset{q^2}{\rule[2pt]{2em}{0.4pt}}\overset{-1}{\circ}$	$q \in \Bbbk^* \setminus \{-1, 1\}, q \notin G'_4$
6	$\overset{\zeta}{\circ}\overset{q^{-1}}{\rule[2pt]{2em}{0.4pt}}\overset{q}{\circ}\quad\overset{\zeta}{\circ}\overset{\zeta^{-1}q}{\rule[2pt]{2em}{0.4pt}}\overset{\zeta q^{-1}}{\circ}$	$\zeta \in G'_3$, $q\zeta \neq -1$ $q \in \Bbbk^* \setminus \{1, \zeta, \zeta^2\}$
$6''$	$\overset{\zeta}{\circ}\overset{-\zeta}{\rule[2pt]{2em}{0.4pt}}\overset{-\zeta^{-1}}{\circ}$	$\zeta \in G'_3$
7	$\overset{\zeta}{\circ}\overset{-\zeta}{\rule[2pt]{2em}{0.4pt}}\overset{-1}{\circ}\quad\overset{\zeta^{-1}}{\circ}\overset{-\zeta^{-1}}{\rule[2pt]{2em}{0.4pt}}\overset{-1}{\circ}$	$\zeta \in G'_3$
8	$\overset{-\zeta^2}{\circ}\overset{\zeta}{\rule[2pt]{2em}{0.4pt}}\overset{-1}{\circ}\quad\overset{-1}{\circ}\overset{-\zeta^{-1}}{\rule[2pt]{2em}{0.4pt}}\overset{-\zeta^{-2}}{\circ}\quad\overset{-1}{\circ}\overset{\zeta^{-1}}{\rule[2pt]{2em}{0.4pt}}\overset{\zeta^3}{\circ}$ $\overset{-\zeta^2}{\circ}\overset{\zeta^3}{\rule[2pt]{2em}{0.4pt}}\overset{-\zeta^{-2}}{\circ}\quad\overset{-1}{\circ}\overset{-\zeta}{\rule[2pt]{2em}{0.4pt}}\overset{\zeta^3}{\circ}$	$\zeta \in G'_{12}$
9	$\overset{-\zeta^{-1}}{\circ}\overset{-\zeta^3}{\rule[2pt]{2em}{0.4pt}}\overset{-1}{\circ}\quad\overset{-\zeta^2}{\circ}\overset{\zeta^3}{\rule[2pt]{2em}{0.4pt}}\overset{-1}{\circ}\quad\overset{-\zeta^2}{\circ}\overset{\zeta}{\rule[2pt]{2em}{0.4pt}}\overset{-\zeta^2}{\circ}$	$\zeta \in G'_{12}$
10	$\overset{-\zeta^2}{\circ}\overset{\zeta}{\rule[2pt]{2em}{0.4pt}}\overset{-1}{\circ}\quad\overset{\zeta^3}{\circ}\overset{\zeta^{-1}}{\rule[2pt]{2em}{0.4pt}}\overset{-1}{\circ}\quad\overset{\zeta^3}{\circ}\overset{\zeta^{-2}}{\rule[2pt]{2em}{0.4pt}}\overset{-\zeta}{\circ}$	$\zeta \in G'_9$
11	$\overset{q}{\circ}\overset{q^{-3}}{\rule[2pt]{2em}{0.4pt}}\overset{q^3}{\circ}$	$q \in \Bbbk^* \setminus \{-1, 1\}, q \notin G'_3$
12	$\overset{\zeta}{\circ}\overset{-\zeta}{\rule[2pt]{2em}{0.4pt}}\overset{-1}{\circ}\quad\overset{\zeta^2}{\circ}\overset{-\zeta^{-1}}{\rule[2pt]{2em}{0.4pt}}\overset{-1}{\circ}\quad\overset{\zeta^2}{\circ}\overset{\zeta}{\rule[2pt]{2em}{0.4pt}}\overset{\zeta^{-1}}{\circ}$	$\zeta \in G'_8$
13	$\overset{\zeta}{\circ}\overset{\zeta^{-5}}{\rule[2pt]{2em}{0.4pt}}\overset{-1}{\circ}\quad\overset{-\zeta^{-4}}{\circ}\overset{\zeta^5}{\rule[2pt]{2em}{0.4pt}}\overset{-1}{\circ}\quad\overset{-\zeta^{-4}}{\circ}\overset{-\zeta^{-1}}{\rule[2pt]{2em}{0.4pt}}\overset{\zeta^6}{\circ}\quad\overset{\zeta^{-1}}{\circ}\overset{\zeta}{\rule[2pt]{2em}{0.4pt}}\overset{\zeta^6}{\circ}$	$\zeta \in G'_{24}$
14	$\overset{\zeta}{\circ}\overset{\zeta^2}{\rule[2pt]{2em}{0.4pt}}\overset{-1}{\circ}\quad\overset{-\zeta^{-2}}{\circ}\overset{\zeta^{-2}}{\rule[2pt]{2em}{0.4pt}}\overset{-1}{\circ}$	$\zeta \in G'_5$
15	$\overset{\zeta}{\circ}\overset{\zeta^{-3}}{\rule[2pt]{2em}{0.4pt}}\overset{-1}{\circ}\quad\overset{-\zeta^{-2}}{\circ}\overset{\zeta^3}{\rule[2pt]{2em}{0.4pt}}\overset{-1}{\circ}\quad\overset{-\zeta^{-2}}{\circ}\overset{-\zeta^3}{\rule[2pt]{2em}{0.4pt}}\overset{-1}{\circ}\quad\overset{-\zeta}{\circ}\overset{-\zeta^{-3}}{\rule[2pt]{2em}{0.4pt}}\overset{-1}{\circ}$	$\zeta \in G'_{20}$
16	$\overset{\zeta^5}{\circ}\overset{-\zeta^{-3}}{\rule[2pt]{2em}{0.4pt}}\overset{-\zeta}{\circ}\quad\overset{\zeta^5}{\circ}\overset{-\zeta^{-2}}{\rule[2pt]{2em}{0.4pt}}\overset{-1}{\circ}\quad\overset{\zeta^3}{\circ}\overset{-\zeta^2}{\rule[2pt]{2em}{0.4pt}}\overset{-1}{\circ}\quad\overset{\zeta^3}{\circ}\overset{-\zeta^4}{\rule[2pt]{2em}{0.4pt}}\overset{-\zeta^{-4}}{\circ}$	$\zeta \in G'_{15}$
18	$\overset{\zeta^{-1}}{\circ}\overset{-1}{\rule[2pt]{2em}{0.4pt}}\overset{-\zeta}{\circ}\quad\overset{1}{\circ}\overset{-\zeta^{-1}}{\rule[2pt]{2em}{0.4pt}}\overset{-1}{\circ}\quad\overset{\zeta}{\circ}\overset{-\zeta^{-1}}{\rule[2pt]{2em}{0.4pt}}\overset{-1}{\circ}$ $\overset{\zeta^{-1}}{\circ}\overset{-\zeta}{\rule[2pt]{2em}{0.4pt}}\overset{-1}{\circ}\quad\overset{1}{\circ}\overset{-\zeta}{\rule[2pt]{2em}{0.4pt}}\overset{-1}{\circ}\quad\overset{\zeta}{\circ}\overset{-1}{\rule[2pt]{2em}{0.4pt}}\overset{-\zeta^{-1}}{\circ}$	$\zeta \in G'_3$

Table 5.4: Dynkin diagrams of rank 2 in characteristic $p = 7$

	Dynkin diagrams	fixed parameters
2	$\overset{q}{\circ}\overset{q^{-1}}{\text{———}}\overset{q}{\circ}$	$q \in \Bbbk^* \setminus \{1\}$
3	$\overset{q}{\circ}\overset{q^{-1}}{\text{———}}\overset{-1}{\circ} \quad \overset{-1}{\circ}\overset{q}{\text{———}}\overset{-1}{\circ}$	$q \in \Bbbk^* \setminus \{-1,1\}$
4	$\overset{q}{\circ}\overset{q^{-2}}{\text{———}}\overset{q^2}{\circ}$	$q \in \Bbbk^* \setminus \{-1,1\}$
5	$\overset{q}{\circ}\overset{q^{-2}}{\text{———}}\overset{-1}{\circ} \quad \overset{-q^{-1}}{\circ}\overset{q^2}{\text{———}}\overset{-1}{\circ}$	$q \in \Bbbk^* \setminus \{-1,1\}$ $q \notin G'_4$
6	$\overset{\zeta}{\circ}\overset{q^{-1}}{\text{———}}\overset{q}{\circ} \quad \overset{\zeta}{\circ}\overset{\zeta^{-1}q}{\text{———}}\overset{\zeta q^{-1}}{\circ}$	$\zeta \in G'_3$, $q\zeta \neq -1$ $q \in \Bbbk^* \setminus \{1,\zeta,\zeta^2\}$
$6''$	$\overset{\zeta}{\circ}\overset{-\zeta}{\text{———}}\overset{-\zeta^{-1}}{\circ}$	$\zeta \in G'_3$
7	$\overset{\zeta}{\circ}\overset{-\zeta}{\text{———}}\overset{-1}{\circ} \quad \overset{\zeta^{-1}}{\circ}\overset{-\zeta^{-1}}{\text{———}}\overset{-1}{\circ}$	$\zeta \in G'_3$
8	$\overset{-\zeta^2}{\circ}\overset{\zeta}{\text{———}}\overset{-1}{\circ} \quad \overset{-1}{\circ}\overset{-\zeta^{-1}}{\text{———}}\overset{-\zeta^{-2}}{\circ} \quad \overset{-1}{\circ}\overset{\zeta^{-1}}{\text{———}}\overset{\zeta^3}{\circ}$ $\overset{-\zeta^2}{\circ}\overset{\zeta^3}{\text{———}}\overset{-\zeta^{-2}}{\circ} \quad \overset{-1}{\circ}\overset{-\zeta}{\text{———}}\overset{\zeta^3}{\circ}$	$\zeta \in G'_{12}$
9	$\overset{-\zeta^{-1}}{\circ}\overset{-\zeta^3}{\text{———}}\overset{-1}{\circ} \quad \overset{-\zeta^2}{\circ}\overset{\zeta^3}{\text{———}}\overset{-1}{\circ} \quad \overset{-\zeta^2}{\circ}\overset{\zeta}{\text{———}}\overset{-\zeta^2}{\circ}$	$\zeta \in G'_{12}$
10	$\overset{-\zeta^2}{\circ}\overset{\zeta}{\text{———}}\overset{-1}{\circ} \quad \overset{\zeta^3}{\circ}\overset{\zeta^{-1}}{\text{———}}\overset{-1}{\circ} \quad \overset{\zeta^3}{\circ}\overset{\zeta^{-2}}{\text{———}}\overset{-\zeta}{\circ}$	$\zeta \in G'_9$
11	$\overset{q}{\circ}\overset{q^{-3}}{\text{———}}\overset{q^3}{\circ}$	$q \in \Bbbk^* \setminus \{-1,1\}$ $q \notin G'_3$
12	$\overset{\zeta}{\circ}\overset{-\zeta}{\text{———}}\overset{-1}{\circ} \quad \overset{\zeta^2}{\circ}\overset{-\zeta^{-1}}{\text{———}}\overset{-1}{\circ} \quad \overset{\zeta^2}{\circ}\overset{\zeta}{\text{———}}\overset{\zeta^{-1}}{\circ}$	$\zeta \in G'_8$
13	$\overset{\zeta}{\circ}\overset{\zeta^{-5}}{\text{———}}\overset{-1}{\circ} \quad \overset{-\zeta^{-4}}{\circ}\overset{\zeta^5}{\text{———}}\overset{-1}{\circ} \quad \overset{-\zeta^{-4}}{\circ}\overset{-\zeta^{-1}}{\text{———}}\overset{\zeta^6}{\circ} \quad \overset{\zeta^{-1}}{\circ}\overset{\zeta}{\text{———}}\overset{\zeta^6}{\circ}$	$\zeta \in G'_{24}$
14	$\overset{\zeta}{\circ}\overset{\zeta^2}{\text{———}}\overset{-1}{\circ} \quad \overset{-\zeta^{-2}}{\circ}\overset{\zeta^{-2}}{\text{———}}\overset{-1}{\circ}$	$\zeta \in G'_5$
15	$\overset{\zeta}{\circ}\overset{\zeta^{-3}}{\text{———}}\overset{-1}{\circ} \quad \overset{-\zeta^{-2}}{\circ}\overset{\zeta^3}{\text{———}}\overset{-1}{\circ} \quad \overset{-\zeta^{-2}}{\circ}\overset{-\zeta^3}{\text{———}}\overset{-1}{\circ} \quad \overset{-\zeta}{\circ}\overset{-\zeta^{-3}}{\text{———}}\overset{-1}{\circ}$	$\zeta \in G'_{20}$
16	$\overset{\zeta^5}{\circ}\overset{-\zeta^{-3}}{\text{———}}\overset{-\zeta}{\circ} \quad \overset{\zeta^5}{\circ}\overset{-\zeta^{-2}}{\text{———}}\overset{-1}{\circ} \quad \overset{\zeta^3}{\circ}\overset{-\zeta^2}{\text{———}}\overset{-1}{\circ} \quad \overset{\zeta^3}{\circ}\overset{-\zeta^4}{\text{———}}\overset{-\zeta^{-4}}{\circ}$	$\zeta \in G'_{15}$
17	$\overset{-\zeta}{\circ}\overset{-\zeta^{-3}}{\text{———}}\overset{-1}{\circ} \quad \overset{-\zeta^{-2}}{\circ}\overset{-\zeta^3}{\text{———}}\overset{-1}{\circ}$	$\zeta \in G'_7$

Table 5.5: Dynkin diagrams of rank 2 in characteristic $p > 7$

5.2 Rank 3 Nichols algebras

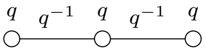
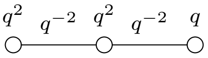
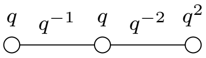
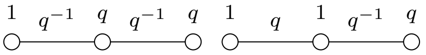
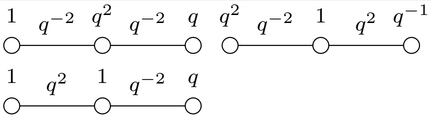
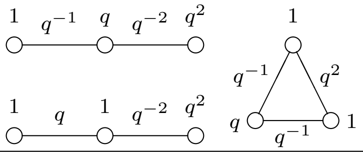
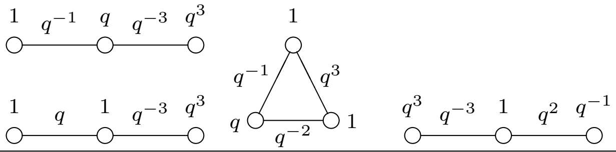
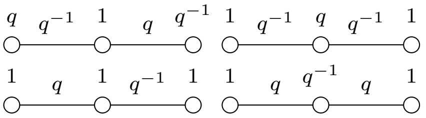
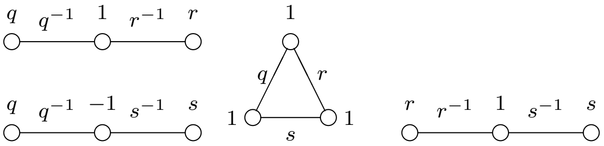
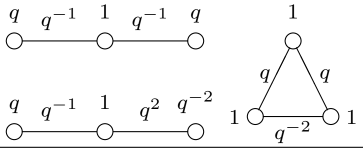
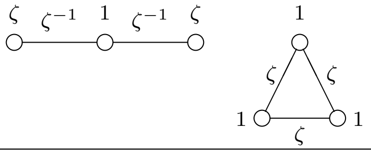
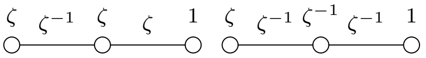
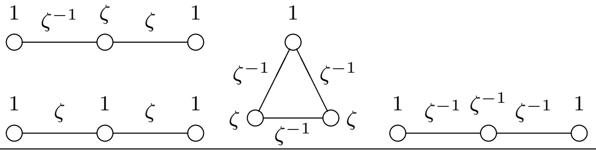
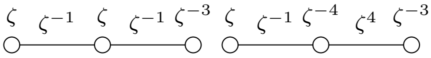
row	Dynkin diagrams	fixed parameters
1		$q \in \Bbbk^* \setminus \{1\}$
2		$q \in \Bbbk^* \setminus \{1\}$
3		$q \in \Bbbk^* \setminus \{1\}$
4		$q \in \Bbbk^* \setminus \{1\}$
5		$q \in \Bbbk^* \setminus \{1\}$
6		$q \in \Bbbk^* \setminus \{1\}$
7		$q \in \Bbbk^* \setminus \{1\}$, $q \notin G'_3$
8		$q \in \Bbbk^* \setminus \{1\}$
9		$q, r, s \in \Bbbk^* \setminus \{1\}$, $qrs = 1, q \neq r$, $q \neq s, r \neq s$
10		$q \in \Bbbk^* \setminus \{1\}$, $q \notin G'_3$
11		$\zeta \in G'_3$
13’		$\zeta \in G'_3$
15		$\zeta \in G'_3$
18		$\zeta \in G'_9$

Table 5.6: Dynkin diagrams of rank 3 in characteristic $p = 2$

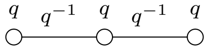
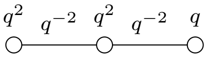
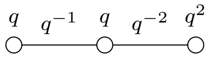
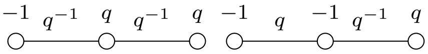
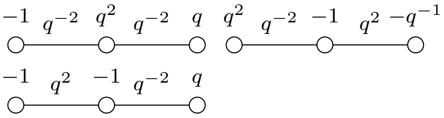
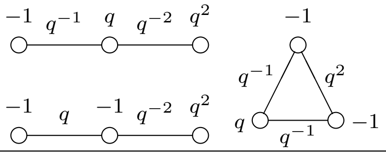
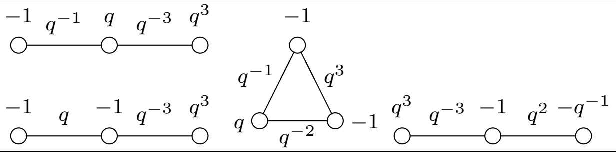
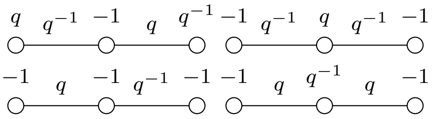
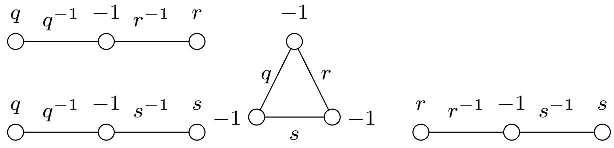
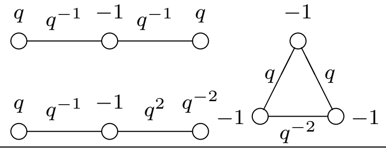
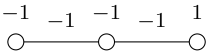
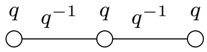
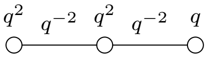
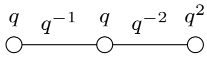
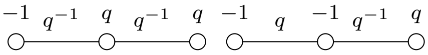
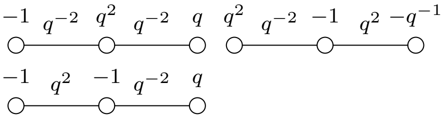
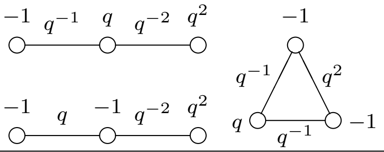
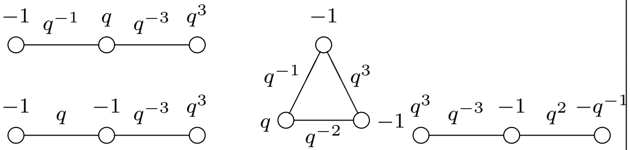
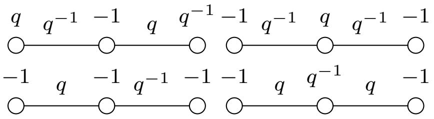
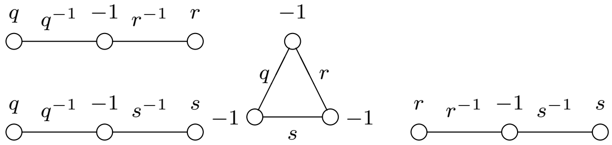
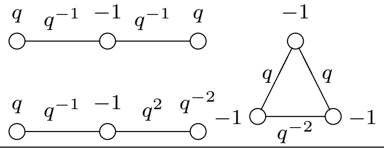
row	Dynkin diagrams	fixed parameters
1		$q \in \Bbbk^* \setminus \{1\}$
2		$q \in \Bbbk^* \setminus \{-1,1\}$
3		$q \in \Bbbk^* \setminus \{-1,1\}$
4		$q \in \Bbbk^* \setminus \{-1,1\}$
5		$q \in \Bbbk^* \setminus \{-1,1\}$, $q \notin G'_4$
6		$q \in \Bbbk^* \setminus \{-1,1\}$
7		$q \in \Bbbk^* \setminus \{-1,1\}$
8		$q \in \Bbbk^* \setminus \{-1,1\}$
9		$q, r, s \in \Bbbk^* \setminus \{1\}$, $qrs = 1, q \neq r$, $q \neq s, r \neq s$
10		$q \in \Bbbk^* \setminus \{-1,1\}$
12’		

Table 5.7: Dynkin diagrams of rank 3 in characteristic $p = 3$

row	Dynkin diagrams	fixed parameters
1		$q \in \Bbbk^* \setminus \{1\}$
2		$q \in \Bbbk^* \setminus \{-1,1\}$
3		$q \in \Bbbk^* \setminus \{-1,1\}$
4		$q \in \Bbbk^* \setminus \{-1,1\}$
5		$q \in \Bbbk^* \setminus \{-1,1\}$ $q \notin G'_4$
6		$q \in \Bbbk^* \setminus \{-1,1\}$
7		$q \in \Bbbk^* \setminus \{-1,1\}$ $q \notin G'_3$
8		$q \in \Bbbk^* \setminus \{-1,1\}$
9		$q, r, s \in \Bbbk^* \setminus \{1\}$, $qrs = 1, q \neq r$, $q \neq s, r \neq s$
10		$q \in \Bbbk^* \setminus \{-1,1\}$ $q \notin G'_3$

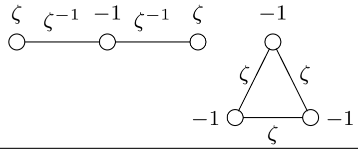
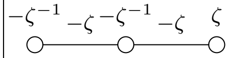
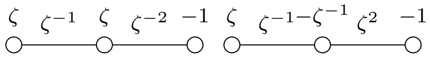
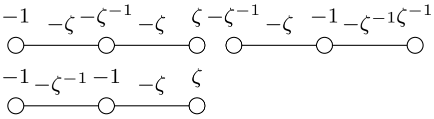
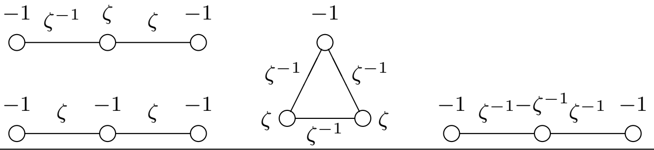
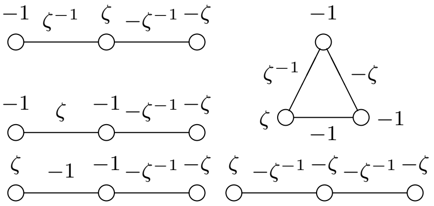
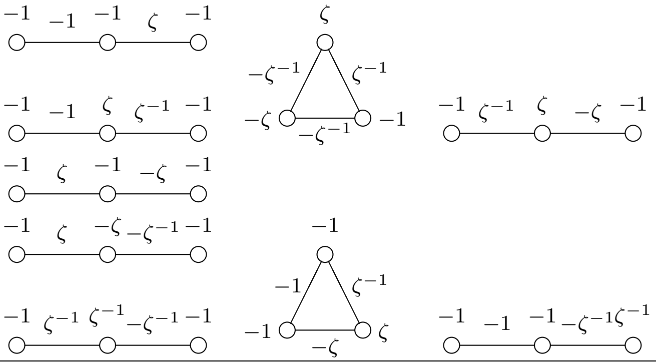
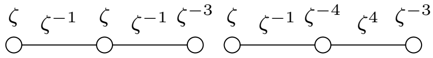
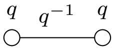
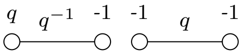
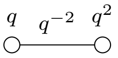
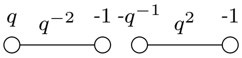
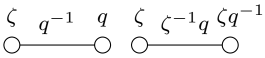
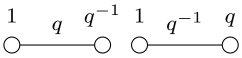
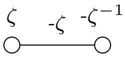
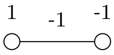
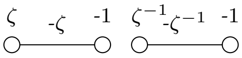
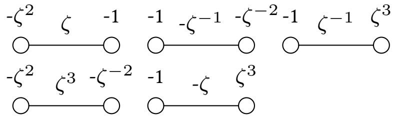
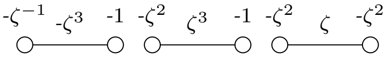
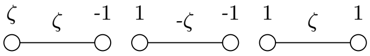
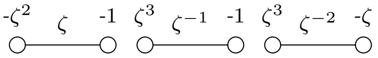
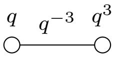
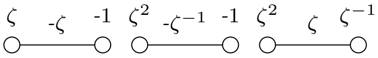
row	Dynkin diagrams	fixed parameters
11		$\zeta \in G'_3$
12		$\zeta \in G'_3$
13		$\zeta \in G'_3 \cup G'_6$
14		$\zeta \in G'_3$
15		$\zeta \in G'_3$
16		$\zeta \in G'_3$
17		$\zeta \in G'_3$
18		$\zeta \in G'_9$

Table 5.8: Dynkin diagrams of rank 3 in characteristic $p > 3$

	Dynkin diagrams	fixed parameters	char $\Bbbk$
2		$q \in \Bbbk^* \setminus \{1\}$	
3		$q \in \Bbbk^* \setminus \{-1, 1\}$	
4		$q \in \Bbbk^* \setminus \{-1, 1\}$	
5		$q \in \Bbbk^* \setminus \{-1, 1\}$ $q \notin G'_4$	
6		$\zeta \in G'_3,\, q\zeta \neq -1$ $q \in \Bbbk^* \setminus \{1, \zeta, \zeta^2\}$	$p \neq 3$
$6'$		$q \in \Bbbk^* \setminus \{1, -1\}$	$p = 3$
$6''$		$\zeta \in G'_3$	$p \neq 2, 3$
$6'''$			$p = 3$
7		$\zeta \in G'_3$	$p \neq 3$
8		$\zeta \in G'_{12}$	$p \neq 2, 3$
9		$\zeta \in G'_{12}$	$p \neq 2, 3$
$9'$		$\zeta \in G'_4$	$p = 3$
10		$\zeta \in G'_9$	$p \neq 3$
11		$q \in \Bbbk^* \setminus \{-1, 1\}$ $q \notin G'_3$	
12		$\zeta \in G'_8$	$p \neq 2$

Tab.6	Dynkin diagrams	fixed parameters	char $\Bbbk$
13	$\overset{\zeta}{\circ}\overset{\zeta^{-5}}{\rule[3pt]{2em}{0.4pt}}\overset{-1}{\circ}$ $\overset{-\zeta^{-4}}{\circ}\overset{\zeta^{5}}{\rule[3pt]{2em}{0.4pt}}\overset{-1}{\circ}$ $\overset{-\zeta^{-4}}{\circ}\overset{-\zeta^{-1}}{\rule[3pt]{2em}{0.4pt}}\overset{\zeta^{6}}{\circ}$ $\overset{\zeta^{-1}}{\circ}\overset{\zeta}{\rule[3pt]{2em}{0.4pt}}\overset{\zeta^{6}}{\circ}$	$\zeta \in G'_{24}$	$p \neq 2, 3$
$13'$	$\overset{-\zeta}{\circ}\overset{\zeta^{-1}}{\rule[3pt]{2em}{0.4pt}}\overset{-1}{\circ}$ $\overset{1}{\circ}\overset{\zeta}{\rule[3pt]{2em}{0.4pt}}\overset{-1}{\circ}$ $\overset{1}{\circ}\overset{\zeta^{-1}}{\rule[3pt]{2em}{0.4pt}}\overset{-\zeta^{2}}{\circ}$ $\overset{-\zeta^{-1}}{\circ}\overset{-\zeta}{\rule[3pt]{2em}{0.4pt}}\overset{-\zeta^{2}}{\circ}$	$\zeta \in G'_{8}$	$p = 3$
14	$\overset{\zeta}{\circ}\overset{\zeta^{2}}{\rule[3pt]{2em}{0.4pt}}\overset{-1}{\circ}$ $\overset{-\zeta^{-2}}{\circ}\overset{\zeta^{-2}}{\rule[3pt]{2em}{0.4pt}}\overset{-1}{\circ}$	$\zeta \in G'_{5}$	$p \neq 5$
15	$\overset{\zeta}{\circ}\overset{\zeta^{-3}}{\rule[3pt]{2em}{0.4pt}}\overset{-1}{\circ}$ $\overset{-\zeta^{-2}}{\circ}\overset{\zeta^{3}}{\rule[3pt]{2em}{0.4pt}}\overset{-1}{\circ}$ $\overset{-\zeta^{-2}}{\circ}\overset{-\zeta^{3}}{\rule[3pt]{2em}{0.4pt}}\overset{-1}{\circ}$ $\overset{-\zeta}{\circ}\overset{-\zeta^{-3}}{\rule[3pt]{2em}{0.4pt}}\overset{-1}{\circ}$	$\zeta \in G'_{20}$	$p \neq 2, 5$
$15'$	$\overset{\zeta}{\circ}\overset{\zeta}{\rule[3pt]{2em}{0.4pt}}\overset{-1}{\circ}$ $\overset{1}{\circ}\overset{-\zeta}{\rule[3pt]{2em}{0.4pt}}\overset{-1}{\circ}$ $\overset{1}{\circ}\overset{\zeta}{\rule[3pt]{2em}{0.4pt}}\overset{-1}{\circ}$ $\overset{-\zeta}{\circ}\overset{-\zeta}{\rule[3pt]{2em}{0.4pt}}\overset{-1}{\circ}$	$\zeta \in G'_{4}$	$p = 5$
16	$\overset{\zeta^{5}}{\circ}\overset{-\zeta^{-3}}{\rule[3pt]{2em}{0.4pt}}\overset{-\zeta}{\circ}$ $\overset{\zeta^{5}}{\circ}\overset{-\zeta^{-2}}{\rule[3pt]{2em}{0.4pt}}\overset{-1}{\circ}$ $\overset{\zeta^{3}}{\circ}\overset{-\zeta^{2}}{\rule[3pt]{2em}{0.4pt}}\overset{-1}{\circ}$ $\overset{\zeta^{3}}{\circ}\overset{-\zeta^{4}}{\rule[3pt]{2em}{0.4pt}}\overset{-\zeta^{-4}}{\circ}$	$\zeta \in G'_{15}$	$p \neq 3, 5$
$16'$	$\overset{1}{\circ}\overset{-\zeta^{-1}}{\rule[3pt]{2em}{0.4pt}}\overset{-\zeta^{2}}{\circ}$ $\overset{1}{\circ}\overset{-\zeta}{\rule[3pt]{2em}{0.4pt}}\overset{-1}{\circ}$ $\overset{\zeta}{\circ}\overset{-\zeta^{-1}}{\rule[3pt]{2em}{0.4pt}}\overset{-1}{\circ}$ $\overset{\zeta}{\circ}\overset{-\zeta^{3}}{\rule[3pt]{2em}{0.4pt}}\overset{-\zeta^{-3}}{\circ}$	$\zeta \in G'_{5}$	$p = 3$
$16''$	$\overset{\zeta^{-1}}{\circ}\overset{-1}{\rule[3pt]{2em}{0.4pt}}\overset{-\zeta}{\circ}$ $\overset{\zeta^{-1}}{\circ}\overset{-\zeta}{\rule[3pt]{2em}{0.4pt}}\overset{-1}{\circ}$ $\overset{1}{\circ}\overset{-\zeta^{-1}}{\rule[3pt]{2em}{0.4pt}}\overset{-1}{\circ}$ $\overset{1}{\circ}\overset{-\zeta}{\rule[3pt]{2em}{0.4pt}}\overset{-\zeta^{-1}}{\circ}$	$\zeta \in G'_{3}$	$p = 5$
17	$\overset{-\zeta}{\circ}\overset{-\zeta^{-3}}{\rule[3pt]{2em}{0.4pt}}\overset{-1}{\circ}$ $\overset{-\zeta^{-2}}{\circ}\overset{-\zeta^{3}}{\rule[3pt]{2em}{0.4pt}}\overset{-1}{\circ}$	$\zeta \in G'_{7}$	$p \neq 7$
18	$\overset{\zeta^{-1}}{\circ}\overset{-1}{\rule[3pt]{2em}{0.4pt}}\overset{-\zeta}{\circ}$ $\overset{1}{\circ}\overset{-\zeta}{\rule[3pt]{2em}{0.4pt}}\overset{-1}{\circ}$ $\overset{\zeta}{\circ}\overset{-\zeta^{-1}}{\rule[3pt]{2em}{0.4pt}}\overset{-1}{\circ}$ $\overset{\zeta^{-1}}{\circ}\overset{-\zeta}{\rule[3pt]{2em}{0.4pt}}\overset{-1}{\circ}$ $\overset{1}{\circ}\overset{-\zeta^{-1}}{\rule[3pt]{2em}{0.4pt}}\overset{-1}{\circ}$ $\overset{\zeta}{\circ}\overset{-1}{\rule[3pt]{2em}{0.4pt}}\overset{-\zeta^{-1}}{\circ}$	$\zeta \in G'_{3}$	$p = 7$

Table 5.9: All Dynkin diagrams of rank 2 in Theorem 4.1.1

5.3 The exchange graphs in Theorem 4.1.1

	exchange graphs	n	l	sequences in $\mathcal{A}^+$	char $\Bbbk$
2	$\mathcal{D}_{21}$	1	4	$(1,1,1)$	
3	$\mathcal{D}_{31} \stackrel{2}{\text{——}} \mathcal{D}_{32} \stackrel{1}{\text{——}} \tau\mathcal{D}_{31}$	3	12	$(1,1,1)$	
4	$\mathcal{D}_{41}$	1	3	$(2,1,2,1)$	
5	$\mathcal{D}_{51} \stackrel{2}{\text{——}} \mathcal{D}_{52}$	2	6	$(2,1,2,1)$	
6	$\mathcal{D}_{61} \stackrel{1}{\text{——}} \mathcal{D}_{62}$	2	6	$(2,1,2,1)$	$p \neq 3$
$6'$	$\mathcal{D}_{6',1} \stackrel{1}{\text{——}} \mathcal{D}_{6',2}$	2	6	$(2,1,2,1)$	$p = 3$
$6''$	$\mathcal{D}_{6'',1}$	1	3	$(2,1,2,1)$	$p \neq 2,3$
$6'''$	$\mathcal{D}_{6''',1}$	1	3	$(2,1,2,1)$	$p = 3$
7	$\mathcal{D}_{71} \stackrel{2}{\text{——}} \mathcal{D}_{72}$	2	6	$(2,1,2,1)$	$p \neq 3$
8	$\begin{array}{ccccccccc} \mathcal{D}_{81} & \stackrel{1}{\text{——}} & \mathcal{D}_{82} & \stackrel{2}{\text{——}} & \mathcal{D}_{83} & \stackrel{1}{\text{——}} & \mathcal{D}_{84} & \stackrel{2}{\text{——}} & \mathcal{D}_{85} \\ 2 \vert & & & & & & & & \vert 1 \\ \tau\mathcal{D}_{85} & \stackrel{1}{\text{——}} & \tau\mathcal{D}_{84} & \stackrel{2}{\text{——}} & \tau\mathcal{D}_{83} & \stackrel{1}{\text{——}} & \tau\mathcal{D}_{82} & \stackrel{2}{\text{——}} & \tau\mathcal{D}_{81} \end{array}$	5	12	$(2,2,1,3,1)$	$p \neq 2,3$
9	$\mathcal{D}_{91} \stackrel{2}{\text{——}} \mathcal{D}_{92} \stackrel{1}{\text{——}} \mathcal{D}_{93} \stackrel{2}{\text{——}} \tau\mathcal{D}_{92} \stackrel{1}{\text{——}} \tau\mathcal{D}_{91}$	5	12	$(3,1,2,2,1)$	$p \neq 2,3$
$9'$	$\mathcal{D}_{9',1} \stackrel{2}{\text{——}} \mathcal{D}_{9',2} \stackrel{1}{\text{——}} \mathcal{D}_{9',3} \stackrel{2}{\text{——}} \tau\mathcal{D}_{9',2} \stackrel{1}{\text{——}} \tau\mathcal{D}_{9',1}$	5	12	$(3,1,2,2,1)$	$p = 3$
10	$\mathcal{D}_{10,1} \stackrel{2}{\text{——}} \mathcal{D}_{10,2} \stackrel{1}{\text{——}} \mathcal{D}_{10,3}$	3	6	$(4,1,2,2,2,1)$	$p \neq 3$
11	$\mathcal{D}_{11,1}$	1	2	$(3,1,3,1,3,1)$	
12	$\mathcal{D}_{12,1} \stackrel{2}{\text{——}} \mathcal{D}_{12,2} \stackrel{1}{\text{——}} \mathcal{D}_{12,3}$	3	6	$(3,1,3,1,3,1)$	$p \neq 2$
13	$\mathcal{D}_{13,1} \stackrel{2}{\text{——}} \mathcal{D}_{13,2} \stackrel{1}{\text{——}} \mathcal{D}_{13,3} \stackrel{2}{\text{——}} \mathcal{D}_{13,4}$	4	6	$(5,1,2,3,1,3,2,1)$	$p \neq 2,3$
$13'$	$\mathcal{D}_{13',1} \stackrel{2}{\text{——}} \mathcal{D}_{13',2} \stackrel{1}{\text{——}} \mathcal{D}_{13',3} \stackrel{2}{\text{——}} \mathcal{D}_{13',4}$	4	6	$(5,1,2,3,1,3,2,1)$	$p = 3$
14	$\mathcal{D}_{14,1} \stackrel{2}{\text{——}} \mathcal{D}_{14,2}$	2	3	$(3,1,4,1,3,1,4,1)$	$p \neq 5$
15	$\mathcal{D}_{15,1} \stackrel{2}{\text{——}} \mathcal{D}_{15,2} \stackrel{1}{\text{——}} \mathcal{D}_{15,3} \stackrel{2}{\text{——}} \mathcal{D}_{15,4}$	4	6	$(3,1,4,1,3,1,4,1)$	$p \neq 2,5$
$15'$	$\mathcal{D}_{15',1} \stackrel{2}{\text{——}} \mathcal{D}_{15',2} \stackrel{1}{\text{——}} \mathcal{D}_{15',3} \stackrel{2}{\text{——}} \mathcal{D}_{15',4}$	4	6	$(3,1,4,1,3,1,4,1)$	$p = 5$
16	$\mathcal{D}_{16,1} \stackrel{1}{\text{——}} \mathcal{D}_{16,2} \stackrel{2}{\text{——}} \mathcal{D}_{16,3} \stackrel{1}{\text{——}} \mathcal{D}_{16,4}$	4	6	$(2,1,4,1,4,1,2,3)$	$p \neq 3,5$
$16'$	$\mathcal{D}_{16',1} \stackrel{1}{\text{——}} \mathcal{D}_{16',2} \stackrel{2}{\text{——}} \mathcal{D}_{16',3} \stackrel{1}{\text{——}} \mathcal{D}_{16',4}$	4	6	$(2,1,4,1,4,1,2,3)$	$p = 3$
$16''$	$\mathcal{D}_{16'',1} \stackrel{1}{\text{——}} \mathcal{D}_{16'',2} \stackrel{2}{\text{——}} \mathcal{D}_{16'',3} \stackrel{1}{\text{——}} \mathcal{D}_{16'',4}$	4	6	$(2,1,4,1,4,1,2,3)$	$p = 5$
17	$\mathcal{D}_{17,1} \stackrel{2}{\text{——}} \mathcal{D}_{17,2}$	2	2	$(3,1,5,1)^3$	$p \neq 7$
18	$\begin{array}{ccccc} \mathcal{D}_{18,1} & \stackrel{1}{\text{——}} & \mathcal{D}_{18,2} & \stackrel{2}{\text{——}} & \mathcal{D}_{18,3} \\ & & & & \vert 1 \\ \mathcal{D}_{18,6} & \stackrel{1}{\text{——}} & \mathcal{D}_{18,5} & \stackrel{2}{\text{——}} & \mathcal{D}_{18,4} \end{array}$	6	6	$(2,1,6,1,2,3)^2$	$p = 7$

Table 5.10: The exchange graphs of $\mathcal{C}(M)$ of rank 2 in Theorem 4.1.1.

	exchange graphs	$\lvert \boldsymbol{\Delta}_+^{[M]} \rvert$	char $\Bbbk$
1	$\mathcal{D}_{11}$	1	$p > 0$
2	$\mathcal{D}_{21}$	2	$p > 0$
3	$\mathcal{D}_{31}$	3	$p > 0$
4	$\begin{tikzcd} \mathcal{D}_{41} \arrow[r, dash, "1"] & \mathcal{D}_{42} \arrow[r, dash, "2"] & \tau_{321}\mathcal{D}_{42} \arrow[r, dash, "3"] & \tau_{321}\mathcal{D}_{41} \end{tikzcd}$	4	$p > 0$
5	$\begin{tikzcd} \mathcal{D}_{51} \arrow[r, dash, "1"] & \mathcal{D}_{52} \arrow[r, dash, "2"] & \mathcal{D}_{53} \end{tikzcd}$	5	$p > 0$
6	$\begin{tikzcd} \mathcal{D}_{61} \arrow[r, dash, "1"] & \mathcal{D}_{62} \arrow[r, dash, "2"] & \mathcal{D}_{63} \arrow[r, dash, "3"] & \tau_{132}\mathcal{D}_{62} \arrow[r, dash, "1"] & \tau_{132}\mathcal{D}_{61} \end{tikzcd}$	6	$p > 0$
7	$\begin{tikzcd} \mathcal{D}_{71} \arrow[r, dash, "1"] & \mathcal{D}_{72} \arrow[r, dash, "2"] & \mathcal{D}_{73} \arrow[r, dash, "3"] & \tau_{231}\mathcal{D}_{74} \end{tikzcd}$	7	$p > 0$
8	$\begin{tikzcd} \mathcal{D}_{81} \arrow[r, dash, "2"] & \mathcal{D}_{82} \arrow[r, dash, "1"] \arrow[d, dash, "3"'] & \mathcal{D}_{83} \arrow[r, dash, "3"] & \tau_{321}\mathcal{D}_{82} \arrow[r, dash, "2"] & \tau_{321}\mathcal{D}_{81} \\ & \mathcal{D}_{84} \arrow[urr, dash, "1"] & & & \end{tikzcd}$	8	$p > 0$
9	$\begin{tikzcd} \mathcal{D}_{91} \arrow[r, dash, "2"] & \mathcal{D}_{93} \arrow[r, dash, "1"] \arrow[d, dash, "3"'] & \tau_{213}\mathcal{D}_{92} \\ & \tau_{231}\mathcal{D}_{94} & \end{tikzcd}$	9	$p > 0$
10	$\begin{tikzcd} \mathcal{D}_{10,1} \arrow[r, dash, "2"] & \mathcal{D}_{10,3} \arrow[r, dash, "1"] \arrow[d, dash, "3"'] & \tau_{213}\mathcal{D}_{10,2} \\ & \tau_{231}\mathcal{D}_{10,2} & \end{tikzcd}$	10	$p > 0$
11	$\begin{tikzcd} \mathcal{D}_{11,1} \arrow[r, dash, "2"] & \mathcal{D}_{11,2} \arrow[r, dash, "1"] \arrow[d, dash, "3"'] & \tau_{213}\mathcal{D}_{11,1} \\ & \tau_{231}\mathcal{D}_{11,1} & \end{tikzcd}$	11	$p \neq 3$
12	$\mathcal{D}_{12,1}$	12	$p \neq 2, 3$
12'	$\mathcal{D}_{12',1}$	12	$p = 3$
13	$\begin{tikzcd} \mathcal{D}_{13,1} \arrow[r, dash, "3"] & \mathcal{D}_{13,2} \end{tikzcd}$	13	$p \neq 2, 3$
13′	$\begin{tikzcd} \mathcal{D}_{13',1} \arrow[r, dash, "2"] & \mathcal{D}_{13',2} \end{tikzcd}$	13	$p = 2$
14	$\begin{tikzcd} \mathcal{D}_{14,1} \arrow[r, dash, "1"] & \mathcal{D}_{14,2} \arrow[r, dash, "2"] & \mathcal{D}_{14,3} \end{tikzcd}$	14	$p \neq 2, 3$
15	$\begin{tikzcd} \mathcal{D}_{15,1} \arrow[r, dash, "1"] \arrow[d, dash, "3"] & \mathcal{D}_{15,2} \arrow[r, dash, "2"] \arrow[d, dash, "3"'] & \mathcal{D}_{15,3} \\ \mathcal{D}_{15,4} \arrow[r, dash, "1"] & \tau_{321}\mathcal{D}_{15,1} & \end{tikzcd}$	15	$p \neq 3$
16	$\begin{tikzcd} \mathcal{D}_{16,1} \arrow[r, dash, "1"] \arrow[d, dash, "2"] & \mathcal{D}_{16,2} \arrow[r, dash, "2"] & \mathcal{D}_{16,4} \arrow[r, dash, "3"] \arrow[dr, dash, "1"] & \tau_{132}\mathcal{D}_{16,3} \arrow[r, dash, "1"] & \tau_{132}\mathcal{D}_{16,5} \\ \tau_{213}\mathcal{D}_{16,4} \arrow[r, dash, "1"] \arrow[d, dash, "3"] & \tau_{213}\mathcal{D}_{16,2} \arrow[rr, dash, "2"] & & \tau_{213}\mathcal{D}_{16,1} & \\ \tau_{231}\mathcal{D}_{16,3} \arrow[r, dash, "2"] & \tau_{132}\mathcal{D}_{16,5} & & & \end{tikzcd}$	16	$p \neq 2, 3$
17	$\begin{tikzcd} \mathcal{D}_{17,1} \arrow[r, dash, "3"] \arrow[dr, dash, "2"'] & \mathcal{D}_{17,2} \arrow[r, dash, "2"] & \mathcal{D}_{17,6} \arrow[r, dash, "3"] & \tau_{231}\mathcal{D}_{17,3} \arrow[r, dash, "1"] \arrow[dl, dash, "2"] & \tau_{231}\mathcal{D}_{17,4} \arrow[d, dash, "2"] \\ \tau_{213}\mathcal{D}_{17,9} \arrow[r, dash, "1"] & \mathcal{D}_{17,7} \arrow[r, dash, "3"] & \tau_{231}\mathcal{D}_{17,8} \arrow[rr, dash, "1"] & & \tau_{231}\mathcal{D}_{17,5} \end{tikzcd}$	17	$p \neq 2, 3$
18	$\begin{tikzcd} \mathcal{D}_{18,1} \arrow[r, dash, "3"] & \mathcal{D}_{18,2} \end{tikzcd}$	18	$p \neq 3$

Table 5.11: The exchange graphs of $\mathcal{C}(M)$ of rank 3 in Theorem 4.1.1.

Appendix A

In this Appendix we use computer algorithms to complete the proof of Theorem 3.2.6. We use the same notation as in subsection 3.2.
Here we use GAP to run the code:"Read(".../Outputgoodneighborhoods.txt") ". The output is listed in the file "Result.txt ".
In the file "Outputgoodneighborhoods.txt ", the input is the 55 rank 3 irreducible root systems in [15, Appendix A]. The roots $x\alpha_3 + y\alpha_2 + z\alpha_1$ are denoted by vectors $[x, y, z]$, where $x, y, z \in \mathbb{N}$.

Algorithm A.0.1. File Outputgoodneighborhoods.txt
Input: all finite irreducible root system of type $\mathcal{C}$.
Output: the good neighborhoods contained in every irreducible root system of type $\mathcal{C}$.

In order to simplify the output of the Algorithm, in the file "Result.txt " we use the following notations.

(1) Assume that X has a good A_3 neighborhood. If $(a, b, c, d) = (1, 0, 0, 1)$, $(a, b, c, d) = (1, 1, 1, 1)$, $(a, b, c, d) = (1, 1, 1, 2)$, $(a, b, c, d) = (2, 1, 1, 2)$, $(a, b, c, d) = (2, 1, 2, 2)$, $(a, b, c, d) = (1, 1, 2, 3)$, or $(a, b, c, d) = (2, 1, 2, 3)$ then we say that X has a good neighborhood $(A_3, 1)$, $(A_3, 2)$, $(A_3, 3)$, $(A_3, 4)$, $(A_3, 5)$, $(A_3, 6)$, or $(A_3, 7)$, respectively.

(2) If X has a good B_3 neighborhood and there exists $a, b \in \{1, 2\}$ such that $-a_{23}^{r_1 r_3(X)} = a$ and $-a_{21}^{r_3(X)} = b$, then we say that X has a good neighborhood $(B_3, [a, b])$.

(3) If X has a good C_3 neighborhood and there exists $a \in \{1, 2\}$ such that $-a_{21}^{r_3(X)} = a$, then we say that X has a good neighborhood (C_3, a).

File ”Outputgoodneighborhoods.txt ”:

```
rk:=3;
# Nr. 1
 RS1:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[0,1,1],[1,1,1]];
# Nr. 2
 RS2:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[0,1,1],[1,1,1]];
# Nr. 3
 RS3:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0], [1,1,1],
       [2,1,1]];
# Nr. 4
 RS4:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[1,1,1],
       [2,1,1],[2,1,2]];
# Nr. 5
 RS5:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[0,1,1],[2,1,0],[1,1,1],
      [2,1,1],[2,2,1]];
# Nr. 6
 RS6:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
      [1,1,1],[2,1,1],[3,1,1]];
# Nr. 7
 RS7:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[0,1,1],[2,1,0],
       [1,1,1],[2,1,1],[2,2,1]];
# Nr. 8
 RS8:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
       [1,1,1],[2,1,1],[3,1,1],[3,2,1]];
# Nr. 9
 RS9:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[1,1,1],
       [3,1,0],[2,1,1],[3,1,1],[3,2,1],[4,2,1]];
# Nr. 10
 RS10:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
        [1,1,1],[2,1,1],[3,1,1],[2,2,1],[3,2,1]];
# Nr. 11
 RS11:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[1,1,1],
        [3,1,0],[2,1,1],[3,2,0],[3,1,1],[3,2,1],[4,2,1]];
# Nr. 12
 RS12:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
        [1,1,1],[3,1,0],[2,1,1],[3,1,1],[4,1,1],[4,2,1]];
# Nr. 13
```

```
 RS13:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
        [1,1,1],[2,1,1],[3,1,1],[2,2,1],[3,2,1],[4,2,1]];
# Nr. 14
 RS14:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[1,1,1],
        [1,0,2],[2,1,1],[1,1,2],[2,1,2],[3,1,2],[3,2,2]];
# Nr. 15
 RS15:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
        [1,1,1],[3,1,0],[2,1,1],[3,1,1],[4,1,1],[3,2,1],[4,2,1]];
# Nr. 16
 RS16:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
        [1,1,1],[3,1,0],[2,1,1],[3,1,1],[4,1,1],[3,2,1],[4,2,1],
        [5,2,1]];
# Nr. 17
 RS17:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
        [1,1,1],[3,1,0],[2,1,1],[3,2,0],[3,1,1],[4,1,1],[3,2,1],
        [4,2,1],[5,2,1]];
# Nr. 18
 RS18:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[0,1,1],[2,1,0],[1,1,1],
        [3,1,0],[2,1,1],[1,2,1],[3,1,1],[2,2,1],[3,2,1],[4,2,1],
        [4,3,1],[4,3,2]];
# Nr. 19
 RS19:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
        [1,1,1],[3,1,0],[2,1,1],[4,1,0],[3,1,1],[4,1,1],[5,1,1],
        [4,2,1],[5,2,1],[6,2,1]];
# Nr. 20
 RS20:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
        [1,1,1],[3,1,0],[2,1,1],[3,2,0],[3,1,1],[4,1,1],[3,2,1],
        [4,2,1],[5,2,1],[5,2,2]];
# Nr. 21
 RS21:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[1,1,1],
        [3,1,0],[2,1,1],[3,1,1],[2,2,1],[3,2,1],[4,2,1],[5,2,1],
        [5,3,1],[5,3,2],[6,3,2]];
# Nr. 22
 RS22:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
        [1,1,1],[3,1,0],[2,1,1],[3,2,0],[3,1,1],[4,1,1],[3,2,1],
        [4,2,1],[5,2,1],[5,3,1],[6,3,1]];
# Nr. 23
 RS23:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[0,1,1],[2,1,0],
```

```
        [1,1,1],[3,1,0],[2,1,1],[1,2,1],[3,1,1],[2,2,1],[3,2,1],
        [4,2,1],[3,3,1],[4,3,1],[4,3,2]];
# Nr. 24
 RS24:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[1,1,1],
        [3,1,0],[2,1,1],[4,1,0],[3,1,1],[4,1,1],[3,2,1],[4,2,1],
        [5,2,1],[6,2,1],[6,3,1],[7,3,1],[7,3,2]];
# Nr. 25
 RS25:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[0,1,1],[2,1,0],[1,1,1],
        [3,1,0],[2,1,1],[4,1,0],[3,1,1],[2,2,1],[4,1,1],[3,2,1],
        [4,2,1],[5,2,1],[6,2,1],[6,3,1],[6,3,2]];
# Nr. 26
 RS26:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[0,1,1],[2,1,0],
        [1,2,0],[1,1,1],[3,1,0],[2,1,1],[1,2,1],[3,1,1],[2,2,1],
        [3,2,1],[4,2,1],[3,3,1],[4,3,1],[4,3,2]];
# Nr. 27
 RS27:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[0,1,1],[2,1,0],
        [1,1,1],[3,1,0],[2,1,1],[1,2,1],[3,2,0],[3,1,1],[2,2,1],
        [3,2,1],[4,2,1],[3,3,1],[4,3,1],[4,3,2]];
# Nr. 28
 RS28:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[0,1,1],[2,1,0],[1,1,1],
        [3,1,0],[2,1,1],[3,2,0],[3,1,1],[2,2,1],[3,2,1],[4,2,1],
        [3,3,1],[4,3,1],[5,3,1],[6,3,1],[6,4,1]];
# Nr. 29
 RS29:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[1,1,1],
        [3,1,0],[2,1,1],[4,1,0],[3,2,0],[3,1,1],[4,1,1],[3,2,1],
        [5,2,0],[4,2,1],[5,2,1],[6,2,1],[6,3,1],[7,3,1]];
# Nr. 30
 RS30:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[1,1,1],
        [3,1,0],[2,1,1],[4,1,0],[3,2,0],[3,1,1],[4,1,1],[3,2,1],
        [4,2,1],[5,2,1],[6,2,1],[6,3,1],[7,3,1],[7,3,2]];
# Nr. 31
 RS31:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
        [1,1,1],[3,1,0],[2,1,1],[3,2,0],[3,1,1],[2,2,1],[4,1,1],
        [3,2,1],[4,2,1],[5,2,1],[4,3,1],[5,3,1],[6,3,2]];
# Nr. 32
 RS32:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[1,1,1],
        [3,1,0],[2,1,1],[4,1,0],[3,2,0],[3,1,1],[4,1,1],[3,2,1],
        [5,2,0],[4,2,1],[5,2,1],[6,2,1],[6,3,1],[7,3,1],[7,3,2]];
```

```
# Nr. 33
 RS33:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
        [1,1,1],[3,1,0],[2,1,1],[3,2,0],[3,1,1],[2,2,1],[4,1,1],
        [3,2,1],[4,2,1],[5,2,1],[4,3,1],[5,3,1],[6,3,1],[6,3,2]];
# Nr. 34
 RS34:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
        [1,1,1],[3,1,0],[2,1,1],[3,2,0],[3,1,1],[4,1,1],[3,2,1],
        [4,2,1],[5,2,1],[5,3,1],[5,2,2],[6,3,1],[6,3,2],[7,3,2]];
# Nr. 35
 RS35:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
        [1,1,1],[3,1,0],[2,1,1],[3,2,0],[3,1,1],[2,2,1],[4,1,1],
        [3,2,1],[4,2,1],[5,2,1],[4,3,1],[5,3,1],[5,2,2],[5,3,2],
        [6,3,2]];
# Nr. 36
 RS36:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
        [1,1,1],[3,1,0],[2,1,1],[4,1,0],[3,2,0],[3,1,1],[4,1,1],
        [3,2,1],[5,2,0],[5,1,1],[4,2,1],[5,2,1],[6,2,1],[7,2,1],
        [6,3,1],[7,3,1],[8,3,1],[8,3,2]];
# Nr. 37
 RS37:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
        [1,1,1],[3,1,0],[2,1,1],[4,1,0],[3,1,1],[4,1,1],[3,2,1],
        [5,1,1],[4,2,1],[5,2,1],[6,2,1],[7,2,1],[6,3,1],[7,3,1],
        [8,3,1],[7,3,2],[8,3,2],[9,3,2]];
# Nr. 38
 RS38:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
        [1,2,0],[1,1,1],[3,1,0],[2,1,1],[1,2,1],[3,1,1],[2,2,1],
        [4,1,1],[3,2,1],[4,2,1],[3,3,1],[3,2,2],[4,3,1],[5,3,1],
        [4,3,2],[5,3,2],[6,3,2],[7,4,2]];
# Nr. 39
 RS39:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
        [1,1,1],[3,1,0],[2,1,1],[3,2,0],[3,1,1],[2,2,1],[4,1,1],
        [3,2,1],[4,2,1],[5,2,1],[4,3,1],[5,3,1],[5,2,2],[6,3,1],
        [5,3,2],[6,3,2],[7,3,2],[7,4,2]];
# Nr. 40
 RS40:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
        [1,1,1],[3,1,0],[2,1,1],[4,1,0],[3,2,0],[3,1,1],[4,1,1],
        [3,2,1],[5,2,0],[5,1,1],[4,2,1],[5,2,1],[6,2,1],[7,2,1],
        [6,3,1],[7,3,1],[8,3,1],[7,3,2],[8,3,2]];
```

```
# Nr. 41
 RS41:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
        [1,1,1],[3,1,0],[2,1,1],[4,1,0],[3,2,0],[3,1,1],[4,1,1],
        [3,2,1],[5,1,1],[4,2,1],[5,2,1],[6,2,1],[7,2,1],[6,3,1],
        [7,3,1],[8,3,1],[7,3,2],[8,3,2],[9,3,2]];
# Nr. 42
 RS42:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
        [1,1,1],[3,1,0],[2,1,1],[4,1,0],[3,2,0],[3,1,1],[4,1,1],
        [3,2,1],[5,2,0],[5,1,1],[4,2,1],[5,2,1],[6,2,1],[7,2,1],
        [6,3,1],[7,3,1],[8,3,1],[7,3,2],[8,3,2],[9,3,2]];
# Nr. 43
 RS43:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
        [1,1,1],[3,1,0],[2,1,1],[4,1,0],[3,2,0],[3,1,1],[4,1,1],
        [3,2,1],[5,1,1],[4,2,1],[5,2,1],[6,2,1],[5,2,2],[7,2,1],
        [6,3,1],[7,3,1],[8,3,1],[7,3,2],[8,3,2],[9,3,2]];
# Nr. 44
 RS44:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
        [1,1,1],[3,1,0],[2,1,1],[4,1,0],[3,2,0],[3,1,1],[4,1,1],
        [3,2,1],[5,1,1],[4,2,1],[5,2,1],[6,2,1],[7,2,1],[6,3,1],
        [7,3,1],[7,2,2],[8,3,1],[7,3,2],[8,3,2],[9,3,2]];
# Nr. 45
 RS45:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
        [1,1,1],[3,1,0],[2,1,1],[4,1,0],[3,2,0],[3,1,1],[4,1,1],
        [3,2,1],[5,2,0],[5,1,1],[4,2,1],[5,2,1],[6,2,1],[5,2,2],
        [7,2,1],[6,3,1],[7,3,1],[8,3,1],[7,3,2],[8,3,2],[9,3,2]];
# Nr. 46
 RS46:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
        [1,1,1],[3,1,0],[2,1,1],[4,1,0],[3,2,0],[3,1,1],[4,1,1],
        [3,2,1],[5,1,1],[4,2,1],[5,2,1],[6,2,1],[5,2,2],[7,2,1],
        [6,3,1],[7,3,1],[8,3,1],[7,3,2],[8,3,2],[9,3,2],[9,4,2]];
# Nr. 47
 RS47:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
        [1,1,1],[3,1,0],[2,1,1],[4,1,0],[3,2,0],[3,1,1],[4,1,1],
        [3,2,1],[5,1,1],[4,2,1],[5,2,1],[6,2,1],[5,2,2],[7,2,1],
        [6,3,1],[7,3,1],[8,3,1],[7,3,2],[8,3,2],[9,3,2],[11,4,2]];
# Nr. 48
 RS48:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
        [1,1,1],[3,1,0],[2,1,1],[4,1,0],[3,2,0],[3,1,1],[4,1,1],
```

```
        [3,2,1],[5,2,0],[5,1,1],[4,2,1],[5,2,1],[6,2,1],[5,2,2],
        [7,2,1],[6,3,1],[7,3,1],[7,2,2],[8,3,1],[7,3,2],[8,3,2],
        [9,3,2]];
# Nr. 49
 RS49:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
       [1,1,1],[3,1,0],[2,1,1],[4,1,0],[3,2,0],[3,1,1],[4,1,1],
       [3,2,1],[5,2,0],[5,1,1],[4,2,1],[5,2,1],[6,2,1],[5,2,2],
       [7,2,1],[6,3,1],[7,3,1],[8,3,1],[7,3,2],[8,3,2],[9,3,2],
       [9,4,2]];
# Nr. 50
 RS50:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
        [1,1,1],[3,1,0],[2,1,1],[4,1,0],[3,2,0],[3,1,1],[4,1,1],
        [3,2,1],[5,2,0],[5,1,1],[4,2,1],[5,2,1],[6,2,1],[5,2,2],
        [7,2,1],[6,3,1],[7,3,1],[8,3,1],[7,3,2],[8,3,2],[9,3,2],
        [11,4,2]];
# Nr. 51
 RS51:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
        [1,1,1],[3,1,0],[2,1,1],[4,1,0],[3,2,0],[3,1,1],[4,1,1],
        [3,2,1],[5,2,0],[5,1,1],[4,2,1],[5,2,1],[6,2,1],[5,2,2],
        [7,2,1],[6,3,1],[7,3,1],[7,2,2],[8,3,1],[7,3,2],[8,3,2],
        [9,3,2],[9,4,2]];
# Nr. 52
 RS52:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
        [1,1,1],[3,1,0],[2,1,1],[4,1,0],[3,1,1],[5,1,0],[4,1,1],
        [6,1,0],[5,1,1],[4,2,1],[6,1,1],[5,2,1],[7,1,1],[6,2,1],
        [7,2,1],[8,2,1],[9,2,1],[10,2,1],[9,3,1],[10,3,1],[11,3,1],
        [10,3,2],[11,3,2],[12,3,2]];
# Nr. 53
 RS53:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
        [1,1,1],[3,1,0],[2,1,1],[4,1,0],[3,2,0],[3,1,1],[4,1,1],
        [3,2,1],[5,2,0],[5,1,1],[4,2,1],[5,2,1],[6,2,1],[5,2,2],
        [7,2,1],[6,3,1],[7,3,1],[7,2,2],[8,3,1],[7,3,2],[8,3,2],
        [9,3,2],[9,4,2],[11,4,2]];
# Nr. 54
 RS54:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
        [1,1,1],[3,1,0],[2,1,1],[4,1,0],[3,2,0],[3,1,1],[4,1,1],
        [3,2,1],[5,2,0],[5,1,1],[4,2,1],[5,2,1],[6,2,1],[5,3,1],
        [7,2,1],[6,3,1],[7,3,1],[8,3,1],[7,3,2],[8,4,1],[8,3,2],
```

```
        [9,4,1],[9,3,2],[9,4,2],[11,4,2],[11,5,2],[12,5,2]];
# Nr. 55
 RS55:=[[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[2,1,0],[2,0,1],
        [1,1,1],[3,1,0],[2,1,1],[4,1,0],[3,2,0],[3,1,1],[4,1,1],
        [3,2,1],[5,2,0],[5,1,1],[4,2,1],[5,2,1],[6,2,1],[5,3,1],
        [7,2,1],[6,3,1],[7,3,1],[8,3,1],[7,3,2],[9,3,1],[8,4,1],
        [8,3,2],[9,4,1],[9,3,2],[10,4,1],[9,4,2],[11,4,2],[11,5,2],
        [12,5,2],[13,5,2]];

rsl:=[RS1,RS2,RS3,RS4,RS5,RS6,RS7,RS8,RS9,RS10,RS11,RS12,RS13,RS14,
      RS15,RS16,RS17,RS18,RS19,RS20,RS21,RS22,RS23,RS24,RS25,RS26,
  RS27,RS28,RS29,RS30,RS31,RS32,RS33,RS34,RS35,RS36,RS37,RS38,
  RS39,RS40,RS41,RS42,RS43,RS44,RS45,RS46,RS47,RS48,RS49,RS50,
      RS51,RS52,RS53,RS54,RS55];

simpleroots:=IdentityMat(rk);

cmpvect :=function( vec1, vec2 )
# compares two vectors with integer entries of the same length
# lexicographically
# returns true if vec1<vec2 and false if vec1 > vec2 or vec1=vec2

  local i;
  for i in [1..Length(vec1)] do
    if vec1[i]<vec2[i] then return true; fi;
    if vec1[i]>vec2[i] then return false; fi;
  od;
  return false;
end;;

cmentry :=function( rootsystem , i , j )
# calculates the i,j-entry of the Cartan matrix of rootsystem
  local k,root;

  if i=j then return 2;
  else
    k:=0; root:=ShallowCopy(simpleroots[j]);
    while root in rootsystem do
```

```
      k:=k+1; root[i]:=k;
    od;
    return 1-k;
  fi;
end;;


cm:=function( rootsystem )
# calculates the Cartan matrix of rootsystem
  local i,j,cmat,crow;

  cmat:=[];
  for i in [1..rk] do
    crow:=[];
    for j in [1..rk] do
      Add(crow,cmentry(rootsystem,i,j));
    od;
    Add(cmat,crow);
  od;
  return cmat;
end;;


ref:=function( rootsystem , i )
# calculates the i-th reflection of rootsystem
  local j,k,si,sirow,newroots;

  si:=[];
  for j in [1..rk] do
    if i=j then
      sirow:=[];
      for k in [1..rk] do
        if k=i then Add(sirow,-1);
        else Add(sirow,-cmentry(rootsystem,i,k));
        fi;
      od;
      Add(si,sirow);
    else
```

```
      Add(si,ShallowCopy(simpleroots[j]));
    fi;
  od;
 newroots:=[];
 for k in [1..Length(rootsystem)] do
   if rootsystem[k]=simpleroots[i] then Add(newroots,simpleroots[i]);
    else Add(newroots,si*rootsystem[k]);
    fi;
  od;
  return(newroots);
end;;

IsTypeA3:=function ( mat )
# determines whether the Cartan matrix mat is of type A3;
# assumes that mat is 3x3
  local i,j,k;

  i:=0;
  if mat[1][2]=0 then i:=1; j:=3; k:=2; fi;
  if mat[1][3]=0 then i:=1; j:=2; k:=3; fi;
  if mat[2][3]=0 then i:=2; j:=1; k:=3; fi;
  if i=0 then return false; fi;
  if not(mat[i][i]=2) then return false; fi;
  if not(mat[i][j]=-1) then return false; fi;
  if not(mat[j][i]=-1) then return false; fi;
  if not(mat[j][j]=2) then return false; fi;
  if not(mat[j][k]=-1) then return false; fi;
  if not(mat[k][i]=0) then return false; fi;
  if not(mat[k][j]=-1) then return false; fi;
  if not(mat[k][k]=2) then return false; fi;

  return true;
end;;

IsTypeB3:=function ( mat )
# determines whether the Cartan matrix mat is of type B3;
# assumes that mat is 3x3
  local i,j,k;
```

```
  k:=0;
  if mat[1][2]=-2 then k:=1; j:=2; fi;
  if mat[1][3]=-2 then k:=1; j:=3; fi;
  if mat[2][1]=-2 then k:=2; j:=1; fi;
  if mat[2][3]=-2 then k:=2; j:=3; fi;
  if mat[3][1]=-2 then k:=3; j:=1; fi;
  if mat[3][2]=-2 then k:=3; j:=2; fi;
  if k=0 then return false; fi;
  i:=6-k-j;

  if not(mat[i][i]=2) then return false; fi;
  if not(mat[i][j]=-1) then return false; fi;
  if not(mat[i][k]=0) then return false; fi;
  if not(mat[j][i]=-1) then return false; fi;
  if not(mat[j][j]=2) then return false; fi;
  if not(mat[j][k]=-1) then return false; fi;
  if not(mat[k][i]=0) then return false; fi;
  if not(mat[k][k]=2) then return false; fi;

  return true;
end;;


IsTypeC3:=function ( mat )
# determines whether the Cartan matrix mat is of type C3;
# assumes that mat is 3x3
  local i,j,k;

  k:=0;
  if mat[1][2]=-2 then j:=1; k:=2; fi;
  if mat[1][3]=-2 then j:=1; k:=3; fi;
  if mat[2][1]=-2 then j:=2; k:=1; fi;
  if mat[2][3]=-2 then j:=2; k:=3; fi;
  if mat[3][1]=-2 then j:=3; k:=1; fi;
  if mat[3][2]=-2 then j:=3; k:=2; fi;
  if k=0 then return false; fi;
  i:=6-k-j;
```

```
  if not(mat[i][i]=2) then return false; fi;
  if not(mat[i][j]=-1) then return false; fi;
  if not(mat[i][k]=0) then return false; fi;
  if not(mat[j][i]=-1) then return false; fi;
  if not(mat[j][j]=2) then return false; fi;
  if not(mat[k][i]=0) then return false; fi;
  if not(mat[k][j]=-1) then return false; fi;
  if not(mat[k][k]=2) then return false; fi;

  return true;
end;;


mapr := function ( rslist )
# determines the map r of the Cartan graph,
# where rslist is the list of all root
# systems in all objects all of these root systems are
# different and sorted
  local i,r,rrow,rs,rs0;

  r:=[];
  for i in [1..rk] do
    rrow:=[];
    for rs in [1..Length(rslist)] do
      rs0:=ref(rslist[rs],i);
      Sort(rs0,cmpvect);
      Add(rrow,Position(rslist,rs0));
    od;
    Add(r,rrow);
  od;
  return r;
end;;


goodA3point :=function (p,lrs,rmap)
# lrs is the list of root systems in the different points of
# a Cartan graph,
```

```
# rmap is the map r of the Cartan graph
# p is a point
# Question: Is p a good point in the following sense:
# A^p is of type A3 with a_{13}=0;
# Let a=-a^{r1(p)}_{23}, b=-a^{r2(p)}_{13}, c=-a^{r2(p)}_{31},
# d=-a^{r3(p)}_{21}.
# Is then (a,b,c,d) one of (1,0,0,1),(1,1,1,1),(1,1,1,2),
# (2,1,1,2),(1,1,2,2),(2,1,2,2),(1,1,2,3),(2,1,2,3)?
# If yes, check also:
#  if (a,b,c,d)=(1,1,2,2),
#  is then A^{r1 r2(p)}= [2 -1 -1][-1 2 0][-2 0 2]?
# If yes, return the list [k1,k2,k3,k4], where
# [k1,k2,k3] are the corresponding coordinates
# and k4 is the index in 1..9 of the type of the point.
# If not, return [].

  local pcm,pcm1,pcm2,pcm3,p1,p2,p3,k1,k2,k3,a1,a2,a3,a4;

  pcm:=cm(lrs[p]);
  if not(IsTypeA3(pcm)) then return []; fi;
  if pcm[1][2]=0 then k1:=1; k2:=3; k3:=2; fi;
  if pcm[1][3]=0 then k1:=1; k2:=2; k3:=3; fi;
  if pcm[2][3]=0 then k1:=2; k2:=1; k3:=3; fi;
  p1:=rmap[k1][p]; p2:=rmap[k2][p]; p3:=rmap[k3][p];
  pcm1:=cm(lrs[p1]); pcm2:=cm(lrs[p2]); pcm3:=cm(lrs[p3]);
  a1:=-pcm1[k2][k3]; a2:=-pcm2[k1][k3];
  a3:=-pcm2[k3][k1]; a4:=-pcm3[k2][k1];
  if a1>a4 or (a1=a4 and a2>a3) then
    a4:=k1; k1:=k3; k3:=a4;
    a4:=p1; p1:=p3; p3:=a4;
    a4:=pcm1; pcm1:=pcm3; pcm3:=a4;
    a1:=-pcm1[k2][k3]; a2:=-pcm2[k1][k3];
    a3:=-pcm2[k3][k1]; a4:=-pcm3[k2][k1];
  fi;
  if a1=1 and a2=0 and a3=0 and a4=1 then return [k1,k2,k3,1]; fi;
  if a1=1 and a2=1 and a3=1 and a4=1 then return [k1,k2,k3,2]; fi;
  if a1=1 and a2=1 and a3=1 and a4=2 then return [k1,k2,k3,3]; fi;
  if a1=2 and a2=1 and a3=1 and a4=2 then return [k1,k2,k3,4]; fi;
```

```
  if a1=2 and a2=1 and a3=2 and a4=2 then return [k1,k2,k3,5]; fi;
  if a1=1 and a2=1 and a3=2 and a4=3 then return [k1,k2,k3,6]; fi;
  if a1=2 and a2=1 and a3=2 and a4=3 then
    pcm1:=cm(lrs[rmap[k3][p1]]);
    if pcm1[k2][k1]=-3 then return [k1,k2,k3,7]; fi;
    return [];
  fi;
  return [];
end;;

goodB3point :=function (p,lrs,rmap)
# lrs is the list of root systems in the different points of
# a Cartan graph,
# rmap is the map r of the Cartan graph
# p is a point
# Question: Is p a good point in the following sense:
# A^p is of type B3 with a_{32}=-2;
# Let a=-a^{r1(p)}_{23}, b=-a^{r2(p)}_{13}, c=-a^{r2(p)}_{32},
# d=-a^{r3(p)}_{21}, e=-a^{r3(p)}_{23}
# Is then a=1, b=0, c=2, e=1, and d one of 1, 2?
# If yes, return the list [k1,k2,k3,k4], where
# [k1,k2,k3] are the corresponding coordinates
# and k4 is the index in 1..2 of the type of the point.
# If not, return [].
  local pcm,pcm1,pcm2,pcm3,pcm4,p1,p2,p3,p4,k1,k2,k3,k4,k5;

  pcm:=cm(lrs[p]);
  if not(IsTypeB3(pcm)) then return []; fi;
  if pcm[1][2]=-2 then k1:=3; k2:=2; k3:=1; fi;
  if pcm[1][3]=-2 then k1:=2; k2:=3; k3:=1; fi;
  if pcm[2][1]=-2 then k1:=3; k2:=1; k3:=2; fi;
  if pcm[2][3]=-2 then k1:=1; k2:=3; k3:=2; fi;
  if pcm[3][1]=-2 then k1:=2; k2:=1; k3:=3; fi;
  if pcm[3][2]=-2 then k1:=1; k2:=2; k3:=3; fi;
  p1:=rmap[k1][p]; p2:=rmap[k2][p]; p3:=rmap[k3][p];
  p4:=rmap[k1][p3]; pcm1:=cm(lrs[p1]); pcm2:=cm(lrs[p2]);
  pcm3:=cm(lrs[p3]); pcm4:=cm(lrs[p4]);
  if not (pcm1[k2][k3]=-1) then return []; fi;
```

```
  if not (pcm2[k1][k3]=0) then return []; fi;
  if not (pcm2[k3][k2]=-2) then return []; fi;
  if not (pcm3[k2][k3]=-1) then return []; fi;
  k4:=-pcm3[k2][k1];  k5:=-pcm4[k2][k3];
  if k4=1 and k5=1 then return [k1,k2,k3,[k4,k5]]; fi;
  if k4=1 and k5=2 then return [k1,k2,k3,[k4,k5]]; fi;
  if k4=2 and k5=1 then return [k1,k2,k3,[k4,k5]]; fi;
  if k4=2 and k5=2 then return [k1,k2,k3,[k4,k5]]; fi;
  return [];
end;;

goodC3point :=function (p,lrs,rmap)
# lrs is the list of root systems in the different points of
# a Cartan graph,
# rmap is the map r of the Cartan graph
# p is a point
# Question: Is p a good point in the following sense:
# A^p is of type B3 with a_{23}=-2;
# Let a=-a^{r1(p)}_{23}, b=-a^{r2(p)}_{13}, c=-a^{r2(p)}_{32},
# d=-a^{r3(p)}_{21}, e=-a^{r3(p)}_{23}
# Is then a=2, b=0, c=1, e=2, and d one of 1,2?
# If yes, return the list [k1,k2,k3,k4], where
# [k1,k2,k3] are the corresponding coordinates and k4=d.
# If not, return [].
  local pcm,pcm1,pcm2,pcm3,p1,p2,p3,k1,k2,k3,k4;

  pcm:=cm(lrs[p]);
  if not(IsTypeC3(pcm)) then return []; fi;
  if pcm[1][2]=-2 then k1:=3; k2:=1; k3:=2; fi;
  if pcm[1][3]=-2 then k1:=2; k2:=1; k3:=3; fi;
  if pcm[2][1]=-2 then k1:=3; k2:=2; k3:=1; fi;
  if pcm[2][3]=-2 then k1:=1; k2:=2; k3:=3; fi;
  if pcm[3][1]=-2 then k1:=2; k2:=3; k3:=1; fi;
  if pcm[3][2]=-2 then k1:=1; k2:=3; k3:=2; fi;
  p1:=rmap[k1][p]; p2:=rmap[k2][p]; p3:=rmap[k3][p];
  pcm1:=cm(lrs[p1]);pcm2:=cm(lrs[p2]);pcm3:=cm(lrs[p3]);
  if not (pcm1[k2][k3]=-2) then return []; fi;
  if not (pcm2[k1][k3]=0) then return []; fi;
```

```
  if not (pcm2[k3][k2]=-1) then return []; fi;
  if not (pcm3[k2][k3]=-2) then return []; fi;
  k4:=-pcm3[k2][k1];
  if k4=1 or k4=2 then return [k1,k2,k3,k4]; fi;
  return [];
end;;

bc:=0;
be:=0;
fwb:=0;

a3l:=[];
b3l:=[];
c3l:=[];

for r0 in [1..Length(rsl)] do

Print("\nNumber ",r0,"\n");

a3:=0; b3:=0; c3:=0;
rs:=rsl[r0];
Sort(rs,cmpvect);
rslist:=[rs];
allrs:=[rs];

while not(rslist=[]) do
#  Print("Calculate reflections of ",rslist[1],"\n");
  for i in [1..rk] do
    rs0:=ref(rslist[1],i);
    Sort(rs0,cmpvect);
    if not(rs0 in allrs) then
      Add(rslist,rs0);
      Add(allrs,rs0);
#      Print("\n New basis: \n",rs0,"\n");
    fi;
  od;
  Remove(rslist,1);
od;
```

```
rr:=mapr(allrs);
# Print("\n The map r: ",rr,"\n");

for i in [1..Length(allrs)] do
#check whether allrs[i] has a good A3 or B3 or C3 neighborhood
  gp:=goodA3point(i,allrs,rr);
  if not gp=[] then Print(i,"(A3,",gp[4],") "); fi;
  gp:=goodB3point(i,allrs,rr);
  if not gp=[] then Print(i,"(B3,",gp[4],") "); fi;
  gp:=goodC3point(i,allrs,rr);
  if not gp=[] then Print(i,"(C3,",gp[4],") "); fi;
od;
Print("\n");
od;
```

File ”Result.txt ”

```
Root system 1
(A3,1)

Root system 2
(A3,2)

Root system 3
(A3,3)

Root system 4
(C3,1)

Root system 5
(B3,[1,1])

Root system 6
(A3,4)

Root system 7
(B3,[1,1])

Root system 8
(A3,5)

Root system 9
(A3,6)

Root system 10
(A3,5) (C3,2)

Root system 11
(A3,6)

Root system 12
(A3,4) (B3,[1,2])
```

```
Root system 13
(C3,2)

Root system 14
(B3,[2,1])

Root system 15
(A3,4) (A3,5)

Root system 16
(A3,5)

Root system 17
(A3,5) (A3,6) (A3,7)

Root system 18
(B3,[1,2]) (B3,[2,1])

Root system 19
(A3,4)

Root system 20
(A3,6)(A3,7)

Root system 21
(B3,[2,2])

Root system 22
(A3,5) (A3,7)

Root system 23
(B3,[2,2])

Root system 24
(B3,[2,2])

Root system 25
(B3,[1,2])
```

```
Root system 26
(B3,[2,2])

Root system 27
(A3,6) (B3,[2,2])

Root system 28
(C3,2)

Root system 29
(A3,4) (A3,6)

Root system 30
(A3,6) (B3,[2,2])

Root system 31
(A3,5) (A3,6) (A3,7) (B3,[2,2]) (C3,2)

Root system 32
(A3,6)

Root system 33
(A3,5) (A3,6) (A3,7) (B3,[2,2]) (C3,2)

Root system 34
(A3,7)

Root system 35
(A3,6) (A3,7)

Root system 36
(A3,5) (A3,6) (A3,7) (B3,[1,2])

Root system 37
(A3,5)

Root system 38
```

```
(A3,4) (B3,[2,2])

Root system 39
(A3,7)

Root system 40
(A3,5) (A3,6) (A3,7)

Root system 41
(A3,5) (A3,6)

Root system 42
(A3,5) (A3,6)

Root system 43
(A3,5) (A3,6)

Root system 44
(A3,5) (A3,6)

Root system 45
(A3,5) (A3,6)

Root system 46
(A3,5) (A3,6) (C3,2)

Root system 47
(A3,5) (A3,6)

Root system 48
(A3,5) (A3,6)

Root system 49
(A3,5) (A3,6) (C3,2)

Root system 50
(A3,5) (A3,6) (C3,2)
```

```
Root system 51
(A3,5) (A3,6) (C3,2)

Root system 52
(A3,4)

Root system 53
(A3,6) (C3,2)

Root system 54
(A3,5) (A3,6)

Root system 55
(A3,5)
```

Zusammenfassung

Die Theorie der Nichols-Algebren wird durch die Theorie von Hopfalgebren beherrscht und motiviert [3, 7, 41]. In jedem Teilgebiet der Mathematik ist die vollständige Klassifikation der Objekte, die von Interesse sind, wichtig. In der Theorie der Hopfalgebren ist die Klassifikation der endlich dimensionalen Hopfalgebren ein schwieriges Problem. Eine gute Übersicht ist zum Beispiel [1]. Für nicht halbeinfache Hopfalgebren, ist nur die Klasse der punktierten Hopfalgebren über einen algebraisch abgeschlossenen Körper $\Bbbk$ der Charakteristisch 0 bekannt. N. Andruskiewitsch und H.-J. Schneider klassifizierten die punktierten Hopfalgebren mit bestimmten Endlichkeitseigenschaften durch die "lifting Methode" [5, 7]. Bis jetzt scheint diese Methode die leistungsfähigste Methode für das Verständnis von punktierten Hopfalgebren zu sein und diese Methode funktioniert im Prinzip für allgemeinere Hopfalgebren. Nichols-Algebren spielen eine wichtige Rolle in dieser Methode. In der Tat ist der zweite Schritt dieser Methode alle gezopften Vektorräume so V zu finden, so dass die Nichols-Algebra $\mathcal{B}(V)$ endlichdimensional ist. Es gibt mehrere geeignete Klassen von gezopften Vektorräumen.

Mehrere Autoren klassifizierten die unendlich und endlich dimensionalen Nichols-Algebren von Cartan-Typ [6, 23, 40]. Wir sagen, dass eine Nichols-Algebra $\mathcal{B}(V)$ von *diagonalem Typ* ist, wenn die Verzopfung c von diagonalem Typ ist. Das bedeutet, dass (V, c) eine Basis $\{x_i | 1 \leq i \leq \theta\}$ hat, so dass $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$ für alle $i, j \in \{1, 2, \dots, \theta\}$ für gewisse $q_{ij} \in \Bbbk^*$. Die Matrix $(q_{ij})_{1 \leq i,j \leq \theta}$ wird als die *Verzopfungsmatrix* von V bezeichnet.

N. Andruskiewitsch stellte die folgende Frage.

Question 5.0.2. (N. Andruskiewitsch [1]) *Given a braiding matrix* $(q_{ij})_{1 \leq i,j \leq \theta}$ *whose entries are roots of* 1*, when* $\mathcal{B}(V)$ *is finite-dimensional, where* V *is a vector space with basis* $x_1, \dots, x_\theta$ *and braiding* $c(x_i \otimes x_j) = q_{ij}(x_j \otimes x_i)$*? If so, compute* $\dim_{\Bbbk} \mathcal{B}(V)$*, and give a "nice" presentation by generators and relations.*

I. Heckenberger klassifizierte endlich dimensionale Nichols-Algebren von diagonalem Typ über Körpern von Charakteristik 0 in den Arbeiten [22, 24, 21, 27]. Die expliziten Darstellungen mit Erzeugern und Relationen solcher Nichols-Algebren wurden in [10, 9] gegeben. Mit diesen Ergebnissen, haben N. Andruskiewitsch und H.-J. Schneider [8] alle endlich dimensionalen und punktierten Hopfalgebren unter einigen technischen Annahmen klassifizierte.

Auf der Basis dieser erfolgreichen Anwendungen, ist die Analyse von Nichols-Algebren, über Körpern von positiver Charakteristik, entscheidend und hat auch potenzielle Applikationen. Außerdem haben Nichols-Algebren interessante Anwendungen auf anderen Gebieten wie Kac-Moody Lie-superalgebras [2, Example 3.2] und konforme Feldtheorie [42, 43, 44]. Da die Nichols-Algebren durch gezopfte Vektorräume erzeugt über Körpern von positiver Charakteristik definiert sind, ist es natürlich und wünschenswert Nichols-Algebren von diagonalem Typ über Körpern von positiver Charakteristik zu analysieren. Auf dem Weg zu dieser Richtung, entdeckten die Autoren [11] eine kombinatorische Formel, um die Relationen in Nichols-Algebren zu studieren und fanden neue Beispiele für Nichols-Algebren. In dieser Arbeit konzentrieren wir uns auf Nichols-Algebren diagonaler Art über Körpern positiver Charakteristik.

Die entscheidenden theoretischen Werkzeuge der bestehenden Klassifikation waren das Wurzelsystem und das Weyl-Gruppoid zu einer Nichols-Algebra von diagonalem Typ [23, 30]. V. Kharchenko [35, Theorem 2] definiert eine Poincaré–Birkhoff–Witt Basis (die Höhe der Basis ist beschränkt) für eine Hopfalgebra R, die durch schief-primitive Elemente und gruppenartige Elemente, in der Wirkung durch Konjugation der gruppenartigen Elemente auf dem Raum von schief-primitiven Elementen diagonalisierbar ist. Da Nichols-Algebren eine natürliche $\mathbb{Z}^\theta$-Graduierung haben, wird das Wurzelsystem von Nichols-Algebren von diagonalem Typ durch die Grade der Poincaré–Birkhoff–Witt Basis definiert. Unter bestimmten Endlichkeitsbedingungen (Definition 2.1.5) werden die Transformationen von Nichols-Algebren von diagonalem Typ eingeführt in [23]. Dies führte zu einer Struktur von Weyl-Gruppoiden assoziiert zu Nichols-Algebren von diagonalem Typ. Das Weyl-Gruppoide spielt eine ähnliche Rolle wie die Weyl-Gruppe für normale Wurzelsysteme in Kac-Moody-Algebren. Basierend auf diesen Ergebnissen, wurde die abstrakte kombinatorische Theorie von Weyl Gruppoiden und verallgemeinerten Wurzelsystemen in [33, 13] eingeleitet. Später wurde die Theorie der Wurzelsysteme und Weyl-Gruppoide auf allgemeinere Nichols-Algebren erweitert [4, 28, 30]. M.Cuntz und I. Heckenberger [15, 16] haben alle endlichen Weyl-Gruppoide klassifizierten.

In dieser Arbeit gelingt es uns, die endlich dimensionalen Nichols-Algebren von diagonalem Typ über Körpern von positiver Charakteristik zu klassifizieren. Das vollständige Ergebnis kann durch Induktion über den Rang von Nichols-Algebren gegeben werden. Es sei angenommen, dass die Charakteristik p des Körpers $\Bbbk$ positive ist. Wenn $V = \Bbbk x$ ein ein-dimensionaler gezopter Vektorraum über dem Körper $\Bbbk$ ist und die Verzopfung $c(x \otimes x) = qx \otimes x$ ist, wobei q keine Einheitswurzel ist, dann ist die Nichols-Algebra $\mathcal{B}(V) \simeq \Bbbk[x]$, die Polynomalgebra in einer Variablen; Wenn $q = 1$ dann $\mathcal{B}(V) \simeq \Bbbk[x]/(x^p)$; Wenn q eine primitive m-te Einheitswurzel von 1, $p \nmid m$, gcd $(m, p) = 1$ und $2 \leq m < \infty$ dann gilt $\mathcal{B}(V) \simeq \Bbbk[x]/(x^m)$. Wir geben das Ergebnis für Rang zwei und Rang drei in dieser Arbeit an.

In dieser Arbeit klassifizieren wir die Nichols-Algebren von diagonalem Typ mit einen endlichen Wurzelsystem über Körpern positive Charakteristik. Die Arbeit besteht aus fünf Kapiteln. Die Bezeichnung der Nichols-Algebra $\mathcal{B}(V)$ erzeugt durch einen verzopften Vektorraum V, das Wurzelsystem von $\mathcal{B}(V)$, und das Weyl-Gruppoid von $\mathcal{B}(V)$ werden in Kapitel 1 eingeführt und ebenso einige allgemeine Ergebnisse. Sei G eine abelsche Gruppe, $\theta \in \mathbb{N}$ und $M = (\Bbbk x_1, \Bbbk x_2, \dots, \Bbbk x_\theta) \in \mathcal{F}_\theta^G$ ein Tupel von ein-dimensionalen Yetter–Drinfel'd moduln über der Gruppenalgebra $\Bbbk G$. Sei $\mathcal{B}(M) := \mathcal{B}(\oplus_{i=1}^n \Bbbk x_i)$ die Nichols-Algebra des Tupels M. In Kapitel 2 konstruieren wir ein semi-Cartan Graph für Nichols-Algebra $\mathcal{B}(M)$ von diagonalem Typ, wenn das Tupel M alle Spiegelungen zulässt.

Satz 5.0.3. Es sei angenommen, dass M alle Spiegelungen zulässt. Für alle $X \in \mathcal{X}_\theta^G(M)$, sei

$$[X]_\theta = \{Y \in \mathcal{X}_\theta^G(M) | \, \text{Y und X haben das gleiche Dynkin Diagramm}\}.$$

Sei $\mathcal{Y}_\theta(M) = \{[X]_\theta | \, X \in \mathcal{X}_\theta^G(M)\}$ und $A^{[X]_\theta} = A^X$ for all $X \in \mathcal{X}_\theta^G(M)$. Wir definieren $t : I \times \mathcal{Y}_\theta(M) \to \mathcal{Y}_\theta(M)$, $(i, [X]_\theta) \mapsto [R_i(X)]_\theta$. Dann wird das Tupel

$$\mathcal{C}(M) = \{I, \mathcal{Y}_\theta(M), t, (A^Y)_{Y \in \mathcal{Y}_\theta(M)}\}$$

ein zusammenhängender semi-Cartan Graph. Wir sagen, dass $\mathcal{C}(M)$ der semi-Cartan graph von M ist.

In Kapitel 3 erhalten wir die wichtigsten Eigenschaften des semi-Cartan Diagramm, die für unseren Klassifikationssatz im nächsten Kapitel benötigt werden. Im Rang 2 charakterisiert Satz 3.1.14 endliche Cartan Graphen mit Bezug auf bestimmten Zahlenfolgen.

Satz 5.0.4. Angenommen $\mathcal{C} = \mathcal{C}(I = \{1, 2\}, \mathcal{X}, r, (A^X)_{X \in \mathcal{X}})$ ist ein zusammenhängender semi-Cartan Graph von Rang zwei, so dass $|\mathcal{X}|$ endlich ist. Sei $X \in \mathcal{X}$ und n die kleinste positive Zahl mit $(r_2 r_1)^n(X) = X$. Lassen wir $(c_k)_{k \geq 1}$ die charakteristische Sequenz von $\mathcal{C}$ in Bezug auf X und 1. Sei $l = 6n - \Sigma_{i=1}^{2n} c_i$. Dann die folgenden Eigenschaften sind äquivalent.

(1) $\mathcal{C}$ ist ein endlicher Cartan Graph.

(2) $l > 0, l \mid 12, (c_1, c_2, \ldots, c_{12n/l}) \in \mathcal{A}^+$, und $(c_k)_{k\geq 1} = (c_1, c_2, \ldots, c_{12n/l})^\infty$.

Im Rang 3 erhalten wir die möglichen Spiegelungen in endlichen Cartan Graphen durch die 55 endlichen Wurzelsysteme in [15, Theorem 4.1]. Um die möglichen Reflexionen zu illustrieren geben wir die Definitionen der guten A_3, B_3 und C_3 Nachbarschaften.

Es sei angenommen, dass

$$A_3 := \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, B_3 := \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}, \text{and } C_3 := \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}.$$

Definition 5.0.5. Sei $\mathcal{C} = \mathcal{C}(I, \mathcal{X}, r, (A^X)_{X\in\mathcal{X}})$ ein semi-Cartan Graph, so dass $A^X = A^Y$ für alle $X, Y \in \mathcal{X}$ sein. Dann sagen wir, dass $\mathcal{C}$ ist **Standard-Typ**.

Definition 5.0.6. Wir sagen, dass X eine **gute** A_3 **Nachbarschaft** hat, wenn eine Integer-Sequenz (a, b, c, d) existiert, so dass bis auf Permutation von I,

$$A^X = A_3, A^{r_1(X)} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -a \\ 0 & -1 & 2 \end{pmatrix}, A^{r_2(X)} = \begin{pmatrix} 2 & -1 & -b \\ -1 & 2 & -1 \\ -c & -1 & 2 \end{pmatrix},$$

$$A^{r_3(X)} = \begin{pmatrix} 2 & -1 & 0 \\ -d & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \text{ und eine der folgenden Eigenschaften zutrifft.}$$

(1) $(a, b, c, d) \in \{(1, 0, 0, 1), (1, 1, 1, 1), (1, 1, 1, 2), (2, 1, 1, 2), (2, 1, 2, 2), (1, 1, 2, 3)\}$.

(2) $(a, b, c, d) = (2, 1, 2, 3), a_{21}^{r_3r_1(X)} = -3$.

Definition 5.0.7. Wir sagen, dass X eine **gute** B_3 **Nachbarschaft** hat, wenn bis auf Permutation von I eine der folgenden Eigenschaften zutrifft.

(1) $A^X = A^{r_1(X)} = A^{r_2(X)} = A^{r_3(X)} = B_3$ and $a_{23}^{r_1r_3(X)} \in \{-1, -2\}$.

(2) $A^X = A^{r_1(X)} = A^{r_2(X)} = B_3$ and $A^{r_3(X)} = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$.

Definition 5.0.8. Wir sagen, dass X eine **gute** C_3 **Nachbarschaft** hat, wenn bis auf Permutation von I eine der folgenden Eigenschaften zutrifft.

$$A^X = A^{r_1(X)} = A^{r_2(X)} = C_3 \text{ und } A^{r_3(X)} = \begin{pmatrix} 2 & -1 & 0 \\ -a & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}, \text{ wobei } a \in \{1, 2\}.$$

Wir erhalten, dass jeder endliche zusammenhängende unzerlegbare Cartan Graph von Rang 3 entweder Standard-Typ ist oder einen Punkt enthält, der mindestens eine der guten A_3, B_3 und C_3 Nachbarschaften hat.

Satz 5.0.9. Sei $\mathcal{C} = \mathcal{C}(I, \mathcal{X}, r, (A^X)_{X\in\mathcal{X}})$ ein endlicher Rang 3 zusammenhängende unzerlegbarer Cartan Graph. Dann gilt Bis auf Permutation von I, eine der folgenden Eigenschaften zutrifft.

P1. Die Cartan Graph $\mathcal{C}$ ist Standard-Typ und die verallgemeinerten Cartan Matrizen sind vom Typ A_3, B_3, order C_3.

P2. Es gibt mindestens einen Punkt $Y \in \mathcal{X}$, der eine gute A_3, B_3, order C_3 Nachbarschaft hat.

Die Ergebnisse sind in den Sätze 3.1.14 und 3.2.6 für Rang 2 und Rang 3 gegeben. Beide Sätze vereinfachen wesentlich die Berechnungen in Kapitel 4. In Kapitel 4 formulieren wir das Hauptklassifizierungsergebnis dieser Arbeit. Alle Rang 2 und Rang 3 Nichols-Algebren von diagonalem Typ mit einem endlichen Wurzelsystem über Körpern von positiver Charakteristik sind in Satz 4.1.1 klassifiziert. Um das Klassifizierungsergebnis zu vereinfachen, benutzen wir Dynkin Diagramme von verzopten Vektorräumen V von diagonalem Typ, derart dass die Nichols-Algebra $\mathcal{B}(V)$ ein endliches Wurzelsystem in Kapitel 5 hat. Tabelle 5.9 und Tabellen (5.6 - 5.8) sind für Rang 2 und Rang 3 gegeben. Tabellen 5.10 und 5.11 illustrieren alle Austausch-Graphen der entsprechenden Cartan Graphen im Satz 4.1.1.

Satz 5.0.10. Sei $V \in {}^G_G\mathcal{YD}$ ein Yetter–Drinfel'd Modul von diagonalem Typ über G. Nehmen wir an, $V = \oplus_{i\in I}\Bbbk x_i \in {}^G_G\mathcal{YD}$, wobei $\{x_i | i \in I\}$ die Basis von V ist. Sei $M = (\Bbbk x_1, \Bbbk x_2, \dots, \Bbbk x_\theta) \in \mathcal{F}^G_\theta$. Es sei angenommen, dass das Verzopfungsmatrix $(q_{ij})_{i,j\in I}$ von M unverlegbar ist. Sei $\mathcal{D}$ das Dynkin Diagramm von M. Dann ist die Menge von Wurzeln $\boldsymbol{\Delta}^{[M]}$ von Nichols-Algebra $\mathcal{B}(V)$ endlich, genau dann wenn die folgenden Eigenschaften gelten.

$(2a)$ Wenn $\theta = 2$, dann tritt das Dynkin Diagramm $\mathcal{D}$ in den Tabellen 5.1, 5.2, 5.3, 5.4 und 5.5 für $p = 2$, $p = 3$, $p = 5$, $p = 7$, beziehungsweise $p > 7$, auf.

$(2b)$ Wenn $\theta = 3$, dann tritt das Dynkin Diagramm $\mathcal{D}$ in den Tabellen 5.6, 5.7, and 5.8 für $p = 2$, $p = 3$, beziehungsweise $p > 3$, auf.

Korollar 5.0.11. Angenommen V erfüllt die Voraussetzung aus Satz 4.1.1. Dann ist die Nichols-Algebra $\mathcal{B}(V)$ endlich dimensional genau dann wenn das Dynkin diagramm $\mathcal{D}$ von V in einer der Tabellen (5.1- 5.8) auftritt und die Labels von den Knoten von $\mathcal{D}$ Einheitswurzeln sind.

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Erklärung

Ich versichere, dass ich meine Dissertation

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selbständig, ohne unerlaubte Hilfe angefertigt und mich dabei keiner anderen als der von mir ausdrücklich bezeichneten Quellen und Hilfen bedient habe.

Die Dissertation wurde in der jetzigen oder einer ähnlichen Form noch bei keiner anderen Hochschule eingereicht und hat noch keinen sonstigen Prüfungszwecken gedient.

Mrburg, den 02. Mai. 2016

Jing Wang

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Jing Wang– Curriculum Vitae

Address Philipps-Universität Marburg
Fachbereich Mathematik und Informatik
Hans-Meerwein-Straße
(Paketpost: 35043 Marburg)
Email wangjingmath@gmail.com
Nationality Chinese

Education

2012- now **PhD**, Algebra - Philipps-Universität Marburg, Germany

Advisor Prof. Dr. István Heckenberger
Dissertation:
The classification of Nichols algebras of diagonal type over fields of positive characteristic

2010-2012 Master of Science, Department of Mathematics, - Zhejiang University, VR China

Advisor Prof. Dr. Zhixiang Wu
Thesis: Left symmetric algebras from DNA insertion

2006-2010 Bachelor of Science, Mathematics and Applied Mathematics, - Shandong University, VR China

Published papers

2015 Wang, J. Rank three Nichols algebras of diagonal type over arbitrary fields. arXiv:1505.07358
to appear in Israel Journal of Mathematics.

2014 Heckenberger, I. and Wang, J.: Rank 2 Nichols algebras of diagonal type over fields of positive characteristic. arXiv:1407.6817.
SIGMA 11 (2015), 011, 24 pages.

2012 Wu, Zhixiang and Wang, J. : Left symmetric algebras from DNA insertion
Zhejiang university.
arXiv:1605.03837.

Conferences and Workshops

Sep.2015 2015 annual meeting of the Deutschen Mathematiker-Vereinigung (DMV), Hamburg - DMV
Talk Nichols algebras of diagonal type over arbitrary fields

Mar.2015 Colloquium on Algebras and Representations -Cordoba, Argentina - Quantum 15
Talk Rank three Nichols algebras of diagonal type over fields of positive characteristic

Sep.2014 The 2nd Heidelberg Laureate Forum - The 2nd HLF, Heidelberg, Germany - The 2nd HLF

Aug.2014 Conformal Field Theory and Nichols Algebras- Rauischholzhausen- Summer School 2014
Talk Rank 2 Nichols algebras of diagonal type over fields of positive characteristic

Aug.2014 International Congress of Mathematicians- Seoul, Korea 2014- Seoul ICM 2014

Mar.2014 Colloquium on Algebra and Representations -Santa Maria, Brazil- Quantum 14

Talk Rank 2 Nichols algebras of diagonal type over fields of positive characteristic

Mar.2014 Structures on Tensor Categories and Topological Field Theories-Erlangen, Germany- Emerging Fields Programme

Research experience

Sep.2012 - current Philipps-Universität Marburg, Germany
The classification of Nichols algebras of diagonal type over fields of positive characteristic

Supported by 'China Scholarship Council'.

Apr.2011 - Nov.2011 Zhejiang University, Hangzhou, China
Left symmetric algebras from DNA insertion

Supported by ZJNSF(No.Y6100148, Y610027) and Education Department of Zhejiang Province(No.201019063).

Apr.2011 - Oct. 2011 Mathematical Sciences Center, Tsinghua University, Beijing, China
Vertex algebras and Poisson vertex algebras with applications to Hamiltonian integrable systems

- Explored the connection between Lie conformal algebra and Lie algebra.
- Applied the vertex algebra to the Hamiltonian integrable systems.
- Deepened the understanding of between the lie algebras and other algebras by attending the Twelfth National Conference of Lie algebras at Huzhou University.

Sep.2010 - May 2011 Zhejiang University, Hangzhou, China
Infinite dimensional lie algebras

- Explored the Affine Algebras: the Normalized Invariant Form, the Root System, and the Weyl Group.
- Deepened the understanding of between the lie algebras and other algebras by attending the Twelfth National Conference of Lie algebras at Huzhou University.

Teaching experience

Sep 2010 - Sep 2011 Zhejiang University, China
Teaching Assistants in Algebra, Linear Algebra and Advanced Algebras

Honors and Awards

- **China Scholarship Council four years Scholarshop, China, 2012-2016**
- **Second-class Award of Honor for Graduate, ZJU,2011**
- **First Scholarship for Outstanding students, SDU, |2008|2009|2010**
- **Triple A Students for successive years, SDU 2006|2007|2008|2010**
- **Advanced Individual of College Student's Holiday Social Practice, SDU 2008**

Interests

- **I like travelling and have travelled 10 percent of the world.**
- **Dancing, Swimming**

Referees

Name	Prof. Dr. István Heckenberger	**Name**	Prof. Zhixiang Wu
Contact	heckenberger@mathematik.uni-marburg.de	**Contact**	wzx@zju.edu.cn
Tel	+49-06421 28-25486	**Tel**	+86-0571-87951609-8202
Addresse	Philipps-Universität Marburg	**Addresse**	Zhejiang University
	Fachbereich Mathematik und Informatik		Department of Mathematics
	Hans-Meerwein-Straße		College of Science
	D-35032 Marburg		Hangzhou 310027, Zhejiang