Algebraic theory of affine monoids

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Deutsche Zusammenfassung

Diese Arbeit behandelt verschiedene Aspekte der Theorie der affinen Monoide. Ein *affines Monoid* ist hierbei eine endlich erzeugte Unterhalbgruppe der freien abelschen Gruppe $\mathbb{Z}^N$ für eine natürliche Zahl $N \in \mathbb{N}$. Zu einem affinen Monoid Q betrachten wir immer auch die Monoidalgebra $\mathbb{K}[Q]$ (für einen Körper $\mathbb{K}$).

Affine Monoide Die Theorie der affinen Monoide berührt viele verschiedene Gebiete der reinen und angewandten Mathematik. So bildet die Lösungsmenge eines Systems von linearen diophantischen Gleichungen und Ungleichungen ein (normales) affines Monoid. Dies ist der Ausgangspunkt der Ehrharttheorie über die Anzahl der Gitterpunkte in rationalen Polyedern. Eine schöne Einführung in dieses Gebiet ist das Buch von Beck und Robins [BR07]. Weiterhin sind die Koordinatenringe vieler klassischer Varietäten Monoidalgebren. Dies gilt zum Beispiel für die neilsche Parabel oder die Veronesefläche. Solche Varietäten enthalten immer einen algebraischen Torus $(\mathbb{K}^*)^n$ als dichte Teilmenge und werden daher als "torisch" bezeichnet. Im Wesentlichen sind dies jene Varietäten die sich durch Monome parametrisieren lassen, wie zum Beispiel die neilsche Parabel mit der Parametrisierung $t \mapsto (t^2, t^3)$. Torische Varietäten werden ausführlich in der Monographie [CLS11] behandelt. Andererseits lassen sich normale Monoidalgebren auch als invariante Unterringe des Polynomrings unter einer Toruswirkung beschreiben. In diesem Kontext wurde von Hochster bewiesen, dass normale Monoidalgebren Cohen-Macaulay sind [Hoc72]. Schließlich finden affine Monoide Anwendung in der Theorie hypergeometrischer Funktionen [Iye+07, Kapitel 24].

Auch in der angewandten Mathematik treten affine Monoide auf. So gibt es einen Algorithmus zur ganzzahligen linearen Optimierung von Conti und Traverso [CT91], der in die Theorie der affinen Monoide eingeordnet werden kann. Weiterhin wurde von Diaconis und Sturmfels [DS98] eine Anwendung in der algebraischen Statistik gefunden. Diese und weitere Anwendungen werden im Buch von Sturmfels [Stu96] dargestellt.

Das torische Ideal eines affinen Monoides ist der Kern der Abbildung $\mathbb{K}[X_1, \dots, X_n] \to \mathbb{K}[Q], X_i \mapsto x_i$, wobei $x_1, \dots, x_n$ ein minimales Erzeugendensytem von Q ist. Für beide oben genannten Anwendungen benötigt man explizite Erzeugendensysteme von torischen Idealen. Diese Erzeugendensysteme (genannt Markovbasen) können sehr umfangreich und aufwändig zu berechnen sein. Die Software 4ti2 [4ti2] wurde zu ihrer Berechnung entwickelt.

Inhalt dieser Arbeit Diese Arbeit ist in drei Teile gegliedert. Im ersten Teil werden die Grundlagen der Theorie der affinen Monoide und ihrer Algebren zusammengestellt. Der zweite und dritte Teil bilden den Kern dieser Arbeit.

Der zweite Teil beschäftigt sich mit allgemeinen affinen Monoiden. Im ersten Kapitel des zweiten Teils wird die Menge der Löcher in einem affinen Monoid Q untersucht. Dies ist die Menge der Elemente der Normalisierung $\overline{Q}$ von Q, die nicht in Q selbst liegen. Wir geben eine geometrische Beschreibung dieser Menge und setzen sie mit algebraischen Eigenschaften der Monoidalgebra $\mathbb{K}[Q]$, insbesondere mit ihrer lokalen Kohomologie, in Beziehung. Zum Beispiel erlaubt die Kenntnis der Menge $\overline{Q} \setminus Q$ eine Abschätzung der Tiefe von $\mathbb{K}[Q]$ und in manchen Fällen sogar deren explizite Berechnung. Diese Methode zur Bestimmung der Tiefe kann deutlich einfacher sein als die alternative Herangehensweise über Gröbner Basen, wie wir an einem Beispiel exemplarisch vorführen.

Wir wenden die Ergebnisse dieses Kapitels anschließend auf zwei spezielle Klassen von affinen Monoiden an, nämlich auf simpliziale und auf seminormale affine Monoide. Verschiedene bekannte Ergebnisse über dieses Klassen von affinen Monoiden können damit von uns neu bewiesen und verallgemeinert werden. Weiterhin verwenden wir unsere Ergebnisse um die Abhängigkeit der lokalen Kohomologie von $\mathbb{K}[Q]$ vom Grundkörper $\mathbb{K}$ näher zu untersuchen. So können wir beispielsweise zeigen, dass Serres Bedingung (S_3) für $\mathbb{K}[Q]$ nur von Q und nicht von $\mathbb{K}$ abhängt.

Nehmen wir nun an, dass $\mathbb{K}[Q]$ homogen ist. In diesem Falle kann man aus der oben erwähnten Korrespondenz zwischen lokaler Kohomologie und den Löchern Aussagen über die Castelnuovo-Mumford Regularität gewinnen. Insbesondere können wir auf diese Weise einen Spezialfall der Eisenbud-Goto Vermutung [EG84] beweisen.

Im letzten Kapitel des zweiten Teils konstruieren wir eine spezielle Familie von affinen Monoiden. Viele bekannte Beispiele von nicht-normalen Gitterpolytopen haben die Eigenschaft, dass die Löcher in der Nähe des Randes liegen. Außerdem ist bekannt, dass ein affines Monoid, das eine Familie von Löchern mit Kodimension 1 besitzt, immer schon eine solche Familie im Gitterabstand ≤ 1 von einer Facette hat. Im Gegensatz dazu konstruieren wir eine Familie von nicht-normalen Gittersimplizes, bei denen die Löcher beliebig weit im Inneren liegen.

Im dritten Teil dieser Arbeit betrachten wir zwei spezielle Klassen von affinen Monoiden. Zunächst wenden wir uns torischen Kantenringen (*toric edge rings*, [OH98]) zu. Dies sind die Monoidalgebren, die von quadratfreien Monomen vom Grad 2 erzeugt werden. Torische Kantenringe können durch Graphen beschrieben werden. Die Theorie torischer Kantenringe ist einfacher als für allgemeine affine Monoidalgebren, aber viele Phänomene treten bereits in dieser kleineren Klasse auf. In dieser Arbeit wird ein Kriterium für Serres Bedingung (R_1) für torische Kantenringe erarbeitet. Dies ist eine Zusammenarbeit mit Prof. Hibi, die in [HK12] veröffentlicht wurde. Weiterhin zeigen wir, dass die torischen Kantenringe einer bestimmten Klasse von Graphen seminormal sind. Hieraus lässt sich ein Spezialfall einer Vermutung von Hibi et al. [Hib+11] über die Tiefe torischer Kantenringe folgern.

Schließlich betrachten wird das affine Monoid, das vom linearen Ordnungspolytop (*Linear ordering Polytope*) erzeugt wird. Dieses Polytop spielt eine wichtige Rolle in der kombinatorischen Optimierung [Rei85; MR11]. Andererseits entspricht die zugehörige torische Varietät dem Babington-Smith Modell, einem statistischem Modell aus dem Kontext der Analyse von statistischen Ordnungen [Mar95]. Die algebraische Untersuchung dieses Objektes wurde in [SW12] begonnen. Im Kontext von torischen Modellen ist es von Interesse, eine Markovbasis, das ist ein Erzeugendensystem des zugehörigen torischen Ideals, zu bestimmen. In der vorliegenden Arbeit bestimmen wir die Elemente von Grad 2 im torischen Ideal des linearen Ordnungspolytopes. Dieses Problem ist äquivalent zu einer kombinatorischen Fragestellung über Permutationen, die wir mit Methoden aus der Graphentheorie beantworten können.

Im Anhang an die eigentliche Arbeit stellen wir einige algebraische Aussagen über graduierte Ringe zusammen, die in der Literatur nur schwer zu finden sind. Außerdem wird im Anhang ein Beweis eines graphentheoretischen Satzes gegeben, den wir bei der Untersuchung des linearen Ordnungspolytopes benötigen.

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Contents

Part I.

Preliminaries

1. General Preliminaries

1.1. Notation

We use the letters $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ to denote the sets of the integers, the rational numbers and the real numbers, respectively. The symbol $\mathbb{N} := \{1, 2, \dots\}$ denotes the positive integers, while $\mathbb{N}_0 := \{0, 1, \dots\}$ denotes the non-negative integers. Moreover, $\mathbb{R}_+$ denotes the set of non-negative reals. We use the symbols "$\subset$" and "$\subseteq$" synonymously to denote inclusion or equality among sets. For a natural number $n \in \mathbb{N}$ we write $[n] := \{1, \dots, n\}$. If S is a set and $n \in \mathbb{N}_0$, then $\binom{S}{n}$ denotes the set of all n-element subsets of S, and 2^S denotes the set of all subsets of S.

1.2. The basics of affine monoids

1.2.1. Polyhedral cones

A set $P \subset \mathbb{R}^d$ is called *polyhedron* if it is the intersection of finitely many halfspaces. A polyhedron is called a *polytope* if it is bounded and it is called a *(polyhedral) cone* if it is closed under addition. More general, a *convex cone* is a convex subset of $\mathbb{R}^d$ that is closed under multiplication with non-negative real numbers. However, we only consider polyhedral cones. Therefore we use the convention that every cone is understood to be polyhedral and we drop the adjective. For a set $S \subset \mathbb{R}^d$, we write $\operatorname{conv}(S)$ for its convex hull and $\mathbb{R}_+S$ for the set of real, non-negative linear combinations of the elements of S. Further, we write $\operatorname{aff}(S)$ for the affine hull of S and we call $\dim S := \operatorname{aff}(S)$ the *dimension* of S. There are equivalent descriptions in terms of generators for polytopes and cones:

Theorem 1.2.1 (Theorem 1.1, [Zie95])**.** *A set $P \subset \mathbb{R}^d$ is a polytope if and only if it is the convex hull* $\operatorname{conv}(S)$ *of a finite set S.*

Theorem 1.2.2 (Theorem 1.3, [Zie95])**.** *A set $P \subset \mathbb{R}^d$ is a cone if and only if it is of the form $\mathbb{R}_+S$ for a finite set S.*

Definition 1.2.3. A hyperplane $\mathcal{H}$ is called a *support hyperplane* of a polyhedron P if P is contained in one of the two closed halfspaces defined by $\mathcal{H}$ and $P \cap \mathcal{H} \neq \emptyset$. If $\mathcal{H}$ is a supporting hyperplane, the a linear form σ defining $\mathcal{H}$ is called a *support form* of P. The intersection $P \cap \mathcal{H}$ is called a *face* of P. Moreover, P itself and $\emptyset$ are called the *improper faces* of P; in particular, they are considered as faces.

The faces of dimension zero and one are called *vertices* and *rays*, resp. The faces of maximal dimension are called the *facets* of P. Every face is itself a polyhedron, and faces of faces of P are again faces of P.

If $C \subset \mathbb{R}^d$ is a cone, then we call the set of elements $c \in C$ such that $-c \in C$ the *lineality space* of C and denote it by $\operatorname{lin}(C)$. A cone C is *pointed* if $\operatorname{lin}(C) = \{ 0 \}$. Every cone can be decomposed as $C = C' + \operatorname{lin}(C)$, where C' is a pointed cone and $C' \cap \operatorname{lin}(C) = \{ 0 \}$. The cone over a polytope $P \subset \mathbb{R}^{d-1}$ is defined as $C(P) := \mathbb{R}_+ \left\{ (p,1) \in \mathbb{R}^d \,\middle|\, p \in P \right\}$. This is a pointed cone, and every pointed cone can be written as cone over a polytope. For a given pointed cone C, a polytope P such that $C = C(P)$ can be obtained by intersecting C with a suitable hyperplane. We call this polytope P a *cross section polytope* of C, it is uniquely determined up to affine transformations.

1.2.2. Affine monoids

Recall that a *monoid* is a set Q with an operation $+ : Q \times Q \to Q$ (additively written) which is associative and has a neutral element 0.

Definition 1.2.4. A monoid Q is *affine* if it is commutative, cancellative, torsionfree and finitely generated.

Here, *cancellative* means that $x + y = x + z$ implies $y = z$ for elements $x, y, z \in Q$. Moreover, *torsionfree* means that $nx := \underbrace{x + \cdots + x}_{n} \neq 0$ for every $n \in \mathbb{N}$ and $x \in Q \setminus \{ 0 \}$. A finitely generated monoid is affine if and only if it can be embedded into $\mathbb{Z}^N$ for some N. For general information about affine monoids see [BG09] or [MS05]. If $Q \subset \mathbb{Z}^N$ is an affine monoid, we write $\mathbb{Z}Q$ for the subgroup of $\mathbb{Z}^N$ generated by the elements of Q. In particular, $\mathbb{Z}Q$ is a free abelian group. This group can be characterized intrinsically by the following universal property: There exists an injection $\iota : Q \hookrightarrow \mathbb{Z}Q$ and every monoid homomorphism $Q \to G$ into a group G factors uniquely through ι.

Definition 1.2.5. A *face* $F \subseteq Q$ of an affine monoid Q is a subset such that for $a, b \in Q$ the following holds:

$$a + b \in F \qquad \Longleftrightarrow \qquad a, b \in F$$

Note that a face is again an affine monoid. The intersection of faces is again a face, so there exists a unique minimal face F_0. To an affine monoid Q we associate the cone $\mathbb{R}_+Q$ in $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}Q \cong \mathbb{R}^d$.

Lemma 1.2.6 (Lemma 7.12, [MS05])**.** *The map $F \mapsto \mathbb{R}_+F$ is a bijection from the faces of Q to the faces of the cone $\mathbb{R}_+Q$.*

The preceding lemma implies that Q has only finitely many faces. For every facet $\mathcal{F}$ of Q there exists a linear form σ defining $\mathcal{F}$. It is unique only up to scalar multiple. However, we can make it unique by the further restriction that it takes integer values on $\mathbb{Z}Q$ and that it is not a multiple of another integer-valued support form.

Definition 1.2.7. Let $\mathcal{F}$ be a facet of Q. The *support form* $\sigma_{\mathcal{F}} : \mathbb{Z}Q \to \mathbb{Z}$ associated to $\mathcal{F}$ is the unique linear form such that $\sigma_{\mathcal{F}}(Q) = \mathbb{N}_0$ and $\sigma_{\mathcal{F}}(q) = 0$ for $q \in Q$ if and only if $q \in \mathcal{F}$. For an element $q \in Q$, we call $\sigma_{\mathcal{F}}(q)$ the *lattice height* of q above $\mathcal{F}$.

A *unit* is an element in $u \in Q$, such that $-u \in Q$. The set of units forms a face, and in fact it is the unique minimal face F_0. The *dimension* of a face F is the rank of the free abelian group $\mathbb{Z}F$ generated by F. It is convenient to consider a normalized version of the dimension.

Definition 1.2.8. The **dimension* of an affine monoid Q is ${}^*\dim Q := \dim Q - \dim F_0$. Further, for a face $F \subset Q$, we define ${}^*\dim F := \dim F - \dim F_0$.

For every element $q \in Q$, there exists a unique minimal face F containing q. We say that q is an *interior point* of F and write $\operatorname{int} F$ for the set of interior points of F. Note that by definition $0 \in \operatorname{int} F_0$.

Proposition 1.2.9 (Sect. 5, [Swa92])**.** *Let Q be an affine monoid, $F \subset Q$ a face and $q \in F$. Then q is an interior point of F if and only if for every $q_1 \in F$ we can find a $q_2 \in F$ and an $k \in \mathbb{N}$, such that $q_1 + q_2 = kq$.*

1.2.3. Monoid algebras

For a field $\mathbb{K}$, we write $\mathbb{K}[Q]$ for the monoid algebra of Q. Further, for an element $q \in Q$, we write $\mathbf{x}^q \in \mathbb{K}[Q]$ for the corresponding monomial. For a face F we define $\mathfrak{p}_F \subseteq \mathbb{K}[Q]$ to be the vector space generated by those monomials $\mathbf{x}^q$ such that $q \in Q \setminus F$. Then $\mathfrak{p}_F$ is a monomial prime ideal of $\mathbb{K}[Q]$ and all monomial prime ideals are of this type. Moreover, $\mathbb{K}[Q]/\mathfrak{p}_F \cong \mathbb{K}[F]$. $\mathbb{K}[Q]$ carries a natural $\mathbb{Z}Q$-grading. With respect to this grading, the homogeneous ideals $\mathbb{K}[Q]$ are exactly the monomial ideals. Thus the ideal $\mathfrak{p}_{F_0}$ associated to the minimal face is the unique maximal graded ideal of $\mathbb{K}[Q]$. We will sometimes write $\mathfrak{m}$ for this ideal. Its height equals the maximal length of a descending chain of faces of Q, so $(\mathbb{K}[Q], \mathfrak{m})$ is a *local ring of *dimension ${}^*\dim Q$. More general, the height of $\mathfrak{p}_F$ equals ${}^*\dim F$ for every face F.

1.2.4. Localization of affine monoids

For a face F of Q, we denote by

$$Q_F := \{\, q - f \mid q \in Q, f \in F \,\}$$

the *localization* of Q at F. Note that $Q = Q_F$ if and only if F is the minimal face F_0. We say that a localization is *proper* if F is not the minimal face F_0. It holds that $\mathbb{K}[Q_F] = \mathbb{K}[Q]_{(\mathfrak{p}_F)}$, where the later is the homogeneous localization of $\mathbb{K}[Q]$ at $\mathfrak{p}_F$. The following lemma collects useful properties of the localization. Recall that a *vertex figure* of a polytope P at a vertex v is a polytope obtained by intersecting P with an affine hyperplane that cuts off the vertex v. The vertex figure depends on the choice of the cutting hyperplane, but its face lattice does not. In fact, the face lattices of iterated vertex figures of P correspond the upper intervals in the face lattice of P, see [Zie95, Theorem 2.7].

Lemma 1.2.10. *Let $F \subset Q$ a face.*

1. *Localizing at F is the same as inverting a single element from the interior of F: $Q_F = \{ q - kv \mid q \in Q, k \in \mathbb{N}_0 \}$ for every $v \in \operatorname{int} F$.*
2. *The face lattice of Q_F is isomorphic to the upper interval $[F, Q]$ in the face lattice of Q. In particular, the cross section polytope of Q_F is an iterated vertex figure of the cross section polytope of Q.*

Proof. The first part follows easily from the characterization of the interior given in Lemma 1.2.9

The second statement is clear from the corresponding statement about the homogeneous prime ideals in $\mathbb{K}[Q]$ resp. $\mathbb{K}[Q_F]$. □

1.2.5. Modules over affine monoids

A set M is called a *Q-module* if there is an operation $+ : M \times Q \to M$ (additively written) of Q on M, such that $(q + p) + m = q + (p + m)$ and $0 + m = m$ for $q, p \in Q, m \in M$. If M is a Q-module, then the vector space $\mathbb{K}\{M\}$ with basis given by the elements of M is naturally a $\mathbb{Z}Q$-graded $\mathbb{K}[Q]$-module. If $M \subset \mathbb{Z}Q$, then we define the localization $M_F := \{ m - f \mid m \in M, f \in F \}$ of M at a face F. One may also consider the localization for general modules, but we only need this special case. Note that if $U \subset M \subset \mathbb{Z}Q$ are modules, then $\mathbb{K}\{M_F\}/\mathbb{K}\{U_F\} = \mathbb{K}\{M\}/\mathbb{K}\{U\}_{(\mathfrak{p}_F)}$.

For a $\mathbb{Z}Q$-graded $\mathbb{K}[Q]$-module N (in the algebraic sense), the *support* of N, $\operatorname{Supp} N$, is defined to be the set of those $q \in \mathbb{Z}Q$, such that there exists an element of degree q in N. If M is a Q-module and $U \subset M$ a submodule, then $\operatorname{Supp} \mathbb{K}\{M\}/\mathbb{K}\{U\} = M \setminus U$.

1.3. Special classes of affine monoids

1.3.1. Normal affine monoids

Definition 1.3.1. Let Q be an affine monoid. The *normalization* of Q is $\overline{Q} := \mathbb{Z}Q \cap \mathbb{R}_+ Q$. We call Q *normal* if $Q = \overline{Q}$. Equivalently, Q is normal if and only every element $q \in \mathbb{Z}Q$ with $kq \in Q$ for a $k \in \mathbb{N}$ is itself in Q. We call the elements of $\overline{Q} \setminus Q$ the *holes* of Q.

There is an algebraic counterpart of normality. Recall that a domain is called *normal* if it is integrally closed in its field of fractions. The terminology is explained by the following proposition.

Proposition 1.3.2 (Theorem 4.40, [BG09])**.** *An affine monoid Q is normal if and only if the monoid algebra $\mathbb{K}[Q]$ is normal.*

One important property of normal affine monoids is given by the following theorem.

Theorem 1.3.3 (Hochster, Theorem 6.10, [BG09])**.** *Let Q be a normal affine monoid. Then $\mathbb{K}[Q]$ is Cohen-Macaulay for every field $\mathbb{K}$.*

The faces of Q are in bijection with the faces of $\overline{Q}$, but as sets they may be different. Therefore, for a face F of Q, we write $F^{\overline{Q}} := \left\{ q \in \overline{Q} \;\middle|\; \exists n \in \mathbb{N} : nq \in F \right\}$ for the corresponding face of $\overline{Q}$.

Proposition 1.3.4. *Normalization and localization commute. More precisely, if $F \subset Q$ is a face, then it holds that $\overline{(Q_F)} = (\overline{Q})_F$. Moreover, it makes no difference if we localize $\overline{Q}$ as a Q-module or as an affine monoid on its own: $(\overline{Q})_F = (\overline{Q})_{F^{\overline{Q}}}$.*

Proof. The equality $\overline{(Q_F)} = (\overline{Q})_F$ follows from the corresponding algebraic statement, see [Eis95, Prop. 4.13]. Further, $(\overline{Q})_F = (\overline{Q})_{F^{\overline{Q}}}$, because F contains an interior point of $F^{\overline{Q}}$. □

Definition 1.3.5. We call Q *locally normal* if every proper localization of Q is normal.

Since localizations of normal affine monoids are again normal, it is enough to consider faces of *dimension 1. Intuitively, an affine monoid is locally normal if the set of holes $\overline{Q} \setminus Q$ is small. This is made precise in the following proposition.

Proposition 1.3.6. *An affine monoid Q is locally normal if and only if $\overline{Q} \setminus Q$ is a finite union of translates of F_0.*

Another way of stating this is that Q is locally normal if and only if there are only finitely many holes up to units. This will be proven below as a part of Lemma 2.3.2.

Definition 1.3.7. An affine monoid Q is called *seminormal* if $2q, 3q \in Q$ implies $q \in Q$ for $q \in \mathbb{Z}Q$. Equivalently, for every $q \in \overline{Q} \setminus Q$, the set $\{ k \in \mathbb{N}_0 \mid kq \in Q \}$ is contained in a proper subgroup of $\mathbb{Z}$.

A characterization of seminormality can be found in [BG09, p. 66f]. Geometrically, Q is seminormal if the holes are contained in the proper faces of Q. We will prove the precise statement below in Lemma 2.3.11.

1.3.2. Simplicial and regular affine monoids

Definition 1.3.8. Let Q be an affine monoid.

1. Q is called *simplicial* if its cross section polytope is a simplex. Equivalently, Q is simplicial if its face lattice is a boolean lattice.

2. Q is called *locally simplicial* if every proper localization of Q is simplicial. Equivalently, Q is locally simplicial if its cross section polytope is simple.

Some authors require simplicial affine monoids to be positive, but we allow non-positive simplicial affine monoids. Recall that a polytope is simple if and only if every vertex figure is a simplex, cf. [Zie95, prop. 2.16]

Definition 1.3.9. Let Q be an affine monoid.

1. Q is called *regular* if the monoid algebra $\mathbb{K}[Q]$ is a regular ring.

2. Q is called *locally regular* if every proper localization of Q is regular.

The next proposition shows that regularity is a very restrictive condition.

Proposition 1.3.10 (Prop. 4.45, [BG09])**.** *Q is regular if and only if $Q \cong \mathbb{Z}^m \oplus \mathbb{N}_0^n$ for suitable $m, n \in \mathbb{N}_0^n$. In particular, this property does not depend on the field $\mathbb{K}$.*

Note that by the preceding proposition, every regular affine monoid is normal and simplicial. Consequently, every locally regular affine monoid is locally normal and locally simplicial.

1.3.3. Positive, homogeneous and polytopal affine monoids

Definition 1.3.11. An affine monoid Q is called *positive* if 0 is the only unit of Q. Equivalently, Q is positive if the cone $\mathbb{R}_+Q$ is pointed.

Note that for positive Q the dimension and the *dimension coincide. Moreover, the maximal graded ideal $\mathfrak{m}$ of $\mathbb{K}[Q]$ is a maximal ideal (in the ungraded sense) if and only if Q is positive. Localization inevitably destroys positivity since it creates units. Sometimes one can split Q into a positive part and a group of units:

Proposition 1.3.12. *Let Q be an affine monoid.*

1. *If Q is normal, then $Q = F_0 \oplus Q'$ for a positive affine monoid Q'.*

2. *Let F be a face of Q and let H be a hyperplane such that $F = H \cap Q$. Then Q_F can be written as a direct sum $\mathbb{Z}F \oplus Q'$ for a positive affine monoid Q' if and only if $\mathbb{Z}F = H \cap \mathbb{Z}Q$.*

The second part gives a precise condition under which the units of the localization can be split off. In particular, this is not always possible.

Proof. The first statement can be found in Proposition 2.26, [BG09]. The proof given in [BG09] uses only that $\mathbb{Z}F_0$ is a direct summand of $\mathbb{Z}Q$. But our hypothesis of the second statement, $\mathbb{Z}F = H \cap \mathbb{Z}Q$, is equivalent to the statement that $\mathbb{Z}F$ is a direct summand of $\mathbb{Z}Q$. Therefore, even if Q_F is not be normal, the proof of that proposition can be applied in our situation. $\square$

Definition 1.3.13. Q is called *homogeneous* if it admits a generating set such that all generators lie in a common hyperplane not passing through 0.

A homogeneous affine monoid is positive and it admits a $\mathbb{Z}$-grading giving every generator degree 1.

Let $P \subset \mathbb{R}^{N-1}$ be a *lattice polytope*, i.e. a polytope whose vertices have integral coordinates. Then $Q(P) \subset \mathbb{Z}^N$ if defined to be the affine monoid generated by the set

$$\left\{ (p, 1) \in \mathbb{Z}^N \,\middle|\, p \in P \cap \mathbb{Z}^{N-1} \right\} .$$

Note that $\mathbb{R}_+ Q(P) = C(P)$, so P is a cross section polytope for Q.

Definition 1.3.14. An affine monoid Q is called *polytopal*, if it coincides with $Q(P)$ for some lattice polytope P.

Every polytopal affine monoid is homogeneous. There is an intrinsic characterization of polytopal affine monoids.

Proposition 1.3.15 (Proposition 2.28, [BG09])**.** *A homogeneous affine monoid Q is polytopal if and only if it coincides with its normalization in degree* 1.

Definition 1.3.16. Let $P \subset \mathbb{R}^{N-1}$ be a lattice polytope. P is called

- *normal* if $Q(P)$ is normal,
- *very ample* if $Q(P)$ is locally normal, and
- *smooth* if $Q(P)$ is locally regular.

Note that every smooth polytope is simple, because every locally regular affine monoid is locally simplicial. In the literature, very ample polytopes are sometimes defined by requiring that the set $\overline{Q(P)} \setminus Q(P)$ is finite. This is equivalent to our definition by Lemma 1.3.6.

1.3.4. Serre's conditions

We recall the definition of Serre's conditions.

Definition 1.3.17. Let R be a Noetherian ring.

1. R satisfies *Serre's condition* (R_ℓ), if $R_\mathfrak{p}$ is a regular local ring for all $\mathfrak{p} \in \operatorname{Spec} R$ of height at most ℓ.
2. R satisfies *Serre's condition* (S_ℓ), if $\operatorname{depth} R_\mathfrak{p} \geq \min(\ell, \dim R_\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec} R$.

If R is graded, then for both properties it is enough to consider homogeneous localizations at homogeneous prime ideals, see Lemma A.1.22 and Lemma A.1.24. An integral domain always satisfies (R_0) and (S_1).

Theorem 1.3.18 (Serre, Theorem 2.2.22, [BH98]). *A Noetherian ring is normal if and only if it satisfies (R_1) and (S_2).*

A combinatorial condition for a semigroup ring to satisfy Serre's condition (R_ℓ) is given in [Vit09, Theorem 2.7], but for our purposes we only need the case $\ell = 1$.

Proposition 1.3.19 ([Vit09]). *Let Q be an affine monoid. The monoid algebra $\mathbb{K}[Q]$ satisfies Serre's condition (R_1) if and only if every facet $\mathcal{F}$ of Q satisfies the following two conditions:*

(i) There exists $x \in Q$ such that $\sigma_{\mathcal{F}}(x) = 1$.

(ii) $\mathbb{Z}\mathcal{F} = \mathbb{Z}Q \cap \mathcal{H}$, where $\mathcal{H}$ is the supporting hyperplane of $\mathcal{F}$;

The situation is more complicated for (S_ℓ). A combinatorial characterization is known only for (S_2).

Proposition 1.3.20 ([SS90], [Ish88]). *The monoid algebra $\mathbb{K}[Q]$ satisfies Serre's condition (S_2) if and only if*

$$Q = \bigcap_{F \text{ facet of } Q} Q_F \,.$$

In particular, (S_2) does not depend on the field $\mathbb{K}$. We will see in Lemma 2.4.3 that the same is true for (S_3). However, for $\ell \geq 4$ the validity of (S_ℓ) may indeed depend on the field, so we might not expect a combinatorial characterization of this property.

1.4. Toric ideals

Let Q be an affine monoid. By definition Q is finitely generated, so there are finitely many elements $q_1, \ldots, q_r \in Q$ such that every element of Q can be written as a non-negative integer linear combination of these elements.

Proposition 1.4.1 ([BG09], p.55f). *Every affine monoid Q has a minimal generating set that is unique up to units.*

If Q is positive, then its minimal generating set is unique. In this case, the set of minimal generators is called the *Hilbert basis* of Q. In general, every choice of a generating set of Q with, say, n generators gives rise to a presentation

$$\pi : \mathbb{N}_0^n \to Q$$

of Q. This presentation can be lifted to a presentation of the monoid algebra $\mathbb{K}[Q]$:

$$\hat{\pi} : \mathbb{K}[x_1, \dots, x_n] \to \mathbb{K}[Q], \quad x_i \mapsto \mathbf{x}^{q_i}$$

The kernel of $\hat{\pi}$ is called the *toric ideal* of Q. We denote it by I_Q. Note that I_Q depends on a choice of generators of Q. The following theorem characterizes toric ideals.

Theorem 1.4.2 (Theorem 4.32, [BG09])**.** *Let $I \subset \mathbb{K}[x_1, \dots, x_n]$ be an ideal. Then I is a toric ideal on an affine monoid if and only if I is prime and generated by binomials $\prod_i x_i^{a_i} - \prod_i x_i^{b_i}$.*

In general, ideals generated by binomials are called *binomial ideals*. See [ES96] for structural results about this class of ideals. The binomials in I_Q encode the linear relations among the generators of Q, in the sense that a binomial $\prod_i x_i^{a_i} - \prod_i x_i^{b_i}$ lies in I_Q if and only if $\sum_i a_i q_i = \sum_i b_i q_i$. However, it is not sufficient to consider a basis of the kernel of π to generate I_Q. See Chapter 12 of [Stu96] or Section 1.3 of [DSS08] for a discussion of the generators of I_Q.

The polynomial ring $\mathbb{K}[x_1, \dots, x_n]$ carries a $\mathbb{Z}Q$-grading by setting $\deg x_i := q_i$. With respect to this grading, the toric ideal I_Q is homogeneous and the induced grading on the quotient $\mathbb{K}[x_1, \dots, x_n]/I_Q \cong \mathbb{K}[Q]$ coincides with the natural $\mathbb{Z}Q$-grading on $\mathbb{K}[Q]$. There is a useful criterion for normality in terms of the toric ideal I_Q.

Proposition 1.4.3 (Proposition 13.13, [Stu96])**.** *Let Q be a homogeneous affine monoid. If I_Q has a squarefree initial ideal $\mathrm{in}_\prec(I_Q)$ for some term order $\prec$, then Q is normal.*

Part II.

General affine monoids

2. The holes of a non-normal affine monoid

Let Q be an affine monoid. In this chapter, we give a geometric description of the set of holes $\overline{Q} \setminus Q$ in Q and relate it to properties of Q. Our main result in this direction is the following.

Theorem (Lemma 2.1.8)**.** *Let Q be an affine monoid. There exists a (not necessarily disjoint) decomposition*

$$\overline{Q} \setminus Q = \bigcup_{i=1}^{l} (q_i + \mathbb{Z}F_i) \cap \mathbb{R}_+ Q \tag{2.1}$$

with $q_i \in \overline{Q}$ and faces F_i of Q. If the union is irredundant (i.e. no $q_i + \mathbb{Z}F_i$ can be omitted), then the decomposition is unique.

We call a set $q_i + \mathbb{Z}F_i$ from (2.1) a j-**dimensional family of holes*, where j is the *dimension of F. There is an algebraic interpretation of the sets appearing in (2.1). Let $\mathbb{K}$ be a field and $\mathbb{K}[Q]$ be the monoid algebra of Q. Then the faces in (2.1) correspond to the associated primes of the quotient $\mathbb{K}\{\overline{Q}\}/\mathbb{K}[Q]$. The same face may appear several times in (2.1), in fact, the number of times a face appears equals the multiplicity of the corresponding prime.

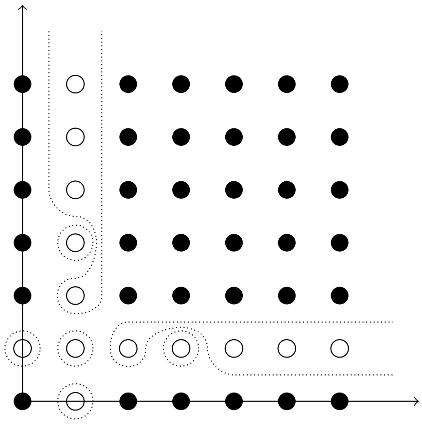
In [BG09, Prop. 2.35] a different decomposition of the holes is considered. It is shown in [BG09] that one can always find a decomposition of $\overline{Q} \setminus Q$ into a disjoint union of translates of faces of Q:

$$\overline{Q} \setminus Q = \bigcup_{i=1}^{l} q_i + F_i \tag{2.2}$$

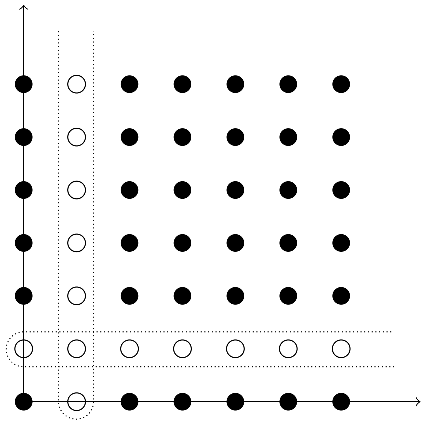
In fact, this statement and its proof have been the motivation for proving Lemma 2.1.8. Figure 2.1 shows an example of both kinds of decompositions. The decomposition given in (2.2) is disjoint, but far from being unique. On the other hand, our decomposition is (in general) not disjoint, but it is unique in the sense that the *sets* $q_i + \mathbb{Z}F_i$ are uniquely determined up to reordering. Moreover, we show in Lemma 2.1.10 that it behaves nicely under localization.

In the second section of this chapter, we consider the local cohomology of $\mathbb{K}[Q]$ with support on the *maximal ideal $\mathfrak{m}$. There is a close relation to the families of holes that is summarized in the next result.

Theorem (Lemma 2.2.4 and Lemma 2.2.6)**.** *Let $q \in \mathbb{Z}Q$ such that $q \notin -\operatorname{int} \overline{Q}$. If $H^{i+1}_{\mathfrak{m}}(\mathbb{K}[Q])_q \neq 0$ for some i, then q is contained in a family of holes of *dimension at*



(a) The decomposition of [BG09]



(b) Our decomposition

Figure 2.1.: Different decomposition of the holes of a 2-dimensional affine monoid

*least i. On the other hand, every i-*dimensional family of holes contains an element $q \in \mathbb{Z}Q$ such that $H^{i+1}_{\mathfrak{m}}(\mathbb{K}[Q])_q \neq 0$.*

Several ring-theoretic properties of $\mathbb{K}[Q]$ can be described in terms of the families of holes.

Theorem (Lemma 2.3.2). *Let Q be an affine monoid of *dimension d. The following holds:*

- *If $d \geq 2$, then $^*\operatorname{depth} \mathbb{K}[Q] = 1$ if and only if there is a 0-*dimensional family of holes.*
- *Q is locally normal if and only if there is no family of holes of positive *dimension.*
- *$\mathbb{K}[Q]$ satisfies Serre's condition (R_1) if and only if there is no family of holes of *dimension $d-1$.*
- *$\mathbb{K}[Q]$ satisfies Serre's condition (S_2) if and only if every family of holes has *dimension $d-1$.*

Note, this implies that if Q is locally normal, but not normal, then $^*\operatorname{depth} \mathbb{K}[Q] = 1$. The first item of the preceding theorem generalizes to an upper bound on the *depth of $\mathbb{K}[Q]$:

Theorem (Lemma 2.3.3). *If Q has an i-*dimensional family of holes, then the *depth of $\mathbb{K}[Q]$ is at most $i+1$.*

This theorem states that a non-normal affine monoid with "few" holes has a low *depth. This is somewhat counterintuitive, because Hochster's Theorem (Lemma 1.3.3) states that the absence of holes, i.e. normality, implies maximal *depth. In small examples, it is often not too difficult to determine the bound given by this theorem geometrically. This can be easier than to compute the actual *depth algebraically. In general, the *depth may be strictly smaller than the bound given by Lemma 2.3.3. However, in Lemma 2.3.4, Lemma 2.3.8 and Lemma 2.3.19 we identify some special cases where equality holds. See Lemma 2.3.6 for an application.

In the further parts of Section 2.3, we apply our results to simplicial and seminormal affine monoids. For simplicial affine monoids, the characterization of the Cohen-Macaulay property of [GSW76] is extended to the non-positive case. For seminormal affine monoids, we give a new proof of the cohomological characterization of seminormality of [BLR06]. While our proof is not actually simpler than the original one, we believe that it offers a new, more geometric perspective. Moreover, we extend this and some other results of [BLR06] to the non-positive case.

In Section 2.4, we consider the dependence of local cohomology on the characteristic of $\mathbb{K}$ in the case of affine monoid algebras. It is known that in general $^*\operatorname{depth} \mathbb{K}[Q]$ can depend on the characteristic [TH86]. However, we will see in Lemma 2.4.1 that certain parts of the local cohomology do not depend on the field. In particular, it turns out that Serre's condition (S_3) is independent of the characteristic. Moreover, if $^*\dim Q \leq 5$ then the support of $H^i_{\mathfrak{m}}(\mathbb{K}[Q])$ also does not depend on $\operatorname{char} \mathbb{K}$. These results are best possible in the sense that the statements are wrong for (S_4) and for $^*\dim Q = 6$. We see in Section 2.3 that if Q is either simplicial or seminormal, locally simplicial and positive, then the Cohen-Macaulayness of $\mathbb{K}[Q]$ is independent of $\mathbb{K}$. Somewhat surprisingly, this does not hold for the *depth. We construct a simplicial seminormal affine monoid whose *depth varies with $\mathbb{K}$ in Lemma 2.4.4.

2.1. The structure of the set of holes

In this section, we describe the structure of the set of holes $\overline{Q} \setminus Q$. Following an idea from [BG09, p. 139], we consider a more general situation. Let M be a finitely generated Q-submodule of $\mathbb{Z}Q$ and let $U \subset M$ be a submodule of M. We are interested in the structure of the difference $M \setminus U$. Clearly, in the case $M = \overline{Q}$ and $U = Q$ this corresponds to the holes $\overline{Q} \setminus Q$. While for our purpose it would actually suffice to consider this case, we believe that the additional generality makes the exposition more clear. Another case of potential interest is $N = Q$ and $U \subset Q$ a submodule. This corresponds to a monomial ideal in $\mathbb{K}[Q]$. As noted above, the set $M \setminus U$ can be encoded as the support of the quotient $\mathbb{K}\{M\}/\mathbb{K}\{U\}$. The following simple observation is the key idea: Consider an $m \in M \setminus U$ and a $q \in Q$. Let $\mathbf{x}^m$ and $\mathbf{x}^q$ denote the corresponding monomials in

$\mathbb{K}\{M\}/\mathbb{K}\{U\}$ resp. in $\mathbb{K}[Q]$. Then

$$q + m \in U \qquad \Longleftrightarrow \qquad \mathbf{x}^q \mathbf{x}^m = 0 \,.$$

Now let F be a face of Q. It holds that $m \in M_F$, because $M \subset M_F$. However, $m \in U_F$ if and only if $\mathbf{x}^m$ goes to zero when localizing the quotient $\mathbb{K}\{M\}/\mathbb{K}\{U\}$ at $\mathfrak{p}_F$. This is the case if and only if the annihilator of $\mathbf{x}^m$ is not contained in $\mathfrak{p}_F$, i.e. if there is a $q \in F$ such that $q + m \in U$. Consider the case that $m \notin U_F$ and F is maximal with this property. By this we mean that $m \in U_G$ for all faces $G \supsetneq F$. Because $\mathfrak{p}_G \subset \mathfrak{p}_F$, this is equivalent to $\mathfrak{p}_F$ being a minimal prime over the annihilator of $\mathbf{x}^m$. We summarize what we have proven:

Lemma 2.1.1. *Let F be a face of Q, $m \in M \setminus U$ and $\mathbf{x}^m$ be the corresponding monomial in $\mathbb{K}\{M\}/\mathbb{K}\{U\}$. Then $F + m \subseteq M \setminus U$ if and only if $\mathfrak{p}_F$ contains the annihilator of $\mathbf{x}^m$. Moreover, F is a maximal face with this property if and only if $\mathfrak{p}_F$ is a minimal prime of the annihilator of $\mathbf{x}^m$*

In view of our objective to find an irredundant decomposition of the set $M \setminus U$, it seems natural to take the largest possible pieces. Therefore, we consider the family of sets

$$\mathcal{F}(M) = \{\, \mathbb{Z}F + m \subset \mathbb{Z}Q \mid m \in M \setminus U, \mathfrak{p}_F \text{ is a minimal prime over } \operatorname{Ann} \mathbf{x}^m \,\} \,.$$

These sets will yield the desired decomposition. Note that for $m, n \in M \setminus U$ with $m - n \in \mathbb{Z}F$, it holds that $\mathfrak{p}_F$ is a minimal prime over $\operatorname{Ann} \mathbf{x}^m$ if and only if it is minimal over $\operatorname{Ann} \mathbf{x}^n$. So we are free in choosing representatives of the sets in $\mathcal{F}(M)$. We first show that their union comprises all of $M \setminus U$:

Lemma 2.1.2. *It holds that*

$$M \setminus U = \bigcup_{S \in \mathcal{F}(M)} S \cap M \tag{2.3}$$

Proof. Every monomial $\mathbf{x}^m \in \mathbb{K}\{M\}/\mathbb{K}\{U\}$ has at least one minimal prime over its annihilator, so the left-hand side of (2.3) is clearly contained in the right-hand side. On the other hand, consider $n \in (\mathbb{Z}F + m) \cap M$ for $\mathbb{Z}F + m \in \mathcal{F}(M)$. There exist f_1, f_2 in F such that $f_1 + m = f_2 + n$. It follows that $n \notin U$, because $F + m \subset M \setminus U$ by Lemma 2.1.1, and hence $n \in M \setminus U$. □

Next, we consider the behaviour of $\mathcal{F}(M)$ under localization.

Lemma 2.1.3. *Let $F \subseteq G$ be faces of Q and let $m \in \mathbb{Z}Q$. Then $\mathbb{Z}G + m \in \mathcal{F}(M)$ if and only if $\mathbb{Z}G + m \in \mathcal{F}(M_F)$.*

Proof. If $m \in M_F \setminus U_F$, then there exists an $f \in F$ such that $f + m \in M \setminus U$ and, since $F \subset G$, it holds that $\mathbb{Z}G + m = \mathbb{Z}G + f + m$. So we can assume that $m \in M \setminus U$. In this case, $\mathbb{Z}G + m \in \mathcal{F}(M)$ if and only if $\mathfrak{p}_G$ is minimal over the annihilator of $\mathbf{x}^m \in \mathbb{K}\{M\}/\mathbb{K}\{U\}$. But this property is preserved under localization with $\mathfrak{p}_F \supseteq \mathfrak{p}_G$. Hence the claim follows. □

Using the preceding lemma, we prove the finiteness of our decomposition:

Lemma 2.1.4. *For every face F of Q, the number of sets of the form $\mathbb{Z}F + m \in \mathcal{F}(M)$ equals the multiplicity of $\mathfrak{p}_F$ on $\mathbb{K}\{M\}/\mathbb{K}\{U\}$. In particular, $\mathcal{F}(M)$ is finite. Moreover, a face F appears in $\mathcal{F}(M)$ if and only if $\mathfrak{p}_F$ is an associated prime of $\mathbb{K}\{M\}/\mathbb{K}\{U\}$.*

Proof. Consider the $\mathbb{K}[Q]$-module

$$N := H^0_{\mathfrak{p}_F}(\mathbb{K}\{M\}/\mathbb{K}\{U\}) = \{\, \mathbf{x} \in \mathbb{K}\{M\}/\mathbb{K}\{U\} \mid \mathfrak{p}_F^n \mathbf{x} = \text{ for } n \gg 0 \,\} \ .$$

The multiplicity of $\mathfrak{p}_F$ in $\mathbb{K}\{M\}/\mathbb{K}\{U\}$ is defined as the length of the localization $N_{\mathfrak{p}_F}$, see [Eis95, p. 102]. Note that $N = \bigcup_{n>0}(0 :_{\mathbb{K}\{M\}/\mathbb{K}\{U\}} \mathfrak{p}_F^n)$ is $\mathbb{Z}Q$-graded. Therefore, by Lemma A.1.13 the length of $N_{\mathfrak{p}_F}$ equals the *length of

$$N_{(\mathfrak{p}_F)} = H^0_{\mathfrak{p}_F}((\mathbb{K}\{M\}/\mathbb{K}\{U\})_{(\mathfrak{p}_F)}) = H^0_{\mathfrak{p}_F}(\mathbb{K}\{M_F\}/\mathbb{K}\{U_F\})$$

Here, the first equality is a standard result about the localization of local cohomology, cf. [Iye+07, Prop 7.15]. From the formula above it is obvious that the multiplicity is invariant under localization at F. By Lemma 2.1.3, the same holds for the number of sets of the form $\mathbb{Z}F + m \in \mathcal{F}(M)$. So we may assume that F is the minimal face of Q and thus $N = N_{(\mathfrak{p}_F)}$ is already of finite *length.

We consider a *composition series of N, i.e. a chain $0 \subsetneq N_1 \subsetneq N_2 \subsetneq \cdots \subsetneq N_r = N$ of graded modules, such that each quotient N_i/N_{i-1} is a *simple graded module. But every *simple graded module is of the form $\mathbb{K}[Q]/\mathfrak{p}_F(m) = \mathbb{K}[F](m)$, where $(\cdot)(m)$ indicates a shift in the grading by $m \in \mathbb{Z}Q$. Hence $\operatorname{Supp} N = \bigcup_i \operatorname{Supp} N_i/N_{i+1} = \bigcup_i \mathbb{Z}F + m_i$. Moreover, since every graded component of N has vector space dimension 1 over $\mathbb{K}$, this union is disjoint.

On the other hand, we claim that $\operatorname{Supp} N$ is the union of the sets $\mathbb{Z}F + m \in \mathcal{F}(M)$ (for our fixed F). Indeed, $m \in M \setminus U$ is contained in $\operatorname{Supp} N$ if and only $\mathbf{x}^m$ is annihilated by some power of $\mathfrak{p}_F$. Since $\mathfrak{p}_F$ is the maximal graded ideal of $\mathbb{K}[Q]$, this is equivalent to saying that $\mathfrak{p}_F$ is a minimal prime over the annihilator of $\mathbf{x}^m$ (cf. Lemma A.1.14). Thus, the number of sets of the form $\mathbb{Z}F + m$ in $\mathcal{F}(M)$ equals the length r of the *composition series. □

We now turn to proving the irredundancy and the uniqueness of (2.3). This has a more geometric flavour than the preceding algebraic arguments. First, we give a variant of the well-known fact that a vector space over an infinite field cannot be written as a union of finitely many subspaces.

Lemma 2.1.5. *Let V be a vector space over $\mathbb{Q}$ and $C \subseteq V$ be a convex cone. If C contains a generating set of V, then it is not contained in any finite union of proper subspaces of V.*

Proof. Assume on the contrary that the cone C is contained in the union of finitely many proper subspaces $V_1, \dots, V_l$ of V. We may further assume that none of the subspaces is contained in the union of the others and that every V_i has a non-empty intersection with C. We have certainly at least two subspaces, because C contains a generating set of V. Hence we can choose elements $x_1, x_2 \in C$, such that $x_i \in V_i \setminus \bigcup_{j \neq i} V_j$ for $i = 1, 2$. For every $i \geq 2$ there exists at most one $\lambda \in \mathbb{Q}$ with $\lambda x_1 + x_2 \in V_i$. Indeed, if we had $\lambda x_1 + x_2 \in V_i$ and $\lambda' x_1 + x_2 \in V_i$ for two different $\lambda, \lambda' \in \mathbb{Q}$, then $(\lambda - \lambda')x_1 \in V_i$, a contradiction to our choice of x_1. Since there are infinitely many non-negative rational numbers and it holds $\lambda x_1 + x_2 \in C$ for every such $\lambda \geq 0$, we conclude that there exists a $\lambda \in \mathbb{Q}$ such that $\lambda x_1 + x_2 \in V_1$. But now $x_2 = (\lambda x_1 + x_2) - \lambda x_1 \in V_1$, a contradiction. $\square$

We have defined only polyhedral cones, but the preceding result actually holds for any convex cone, i.e any subset $C \subset V$ such that for $v, w \in C$ and $\lambda, \mu \geq 0$ it follows $\lambda v + \mu w \in C$. Next we prepare a discrete analogue of the preceding lemma.

Lemma 2.1.6. *Let $q, p_1, \dots, p_l \in \mathbb{Z}Q$ be lattice points and let $F, G_1, \dots, G_l$ be (not necessarily distinct) faces of Q. If $F + q$ is contained in the union $\bigcup_i \mathbb{Z}G_i + p_i$, then it is already contained in one of the sets $\mathbb{Z}G_i + p_i$.*

Note that this Lemma does not hold for arbitrary subgroups of $\mathbb{Z}^N$, for example $\mathbb{Z} = 2\mathbb{Z} \cup (2\mathbb{Z} + 1)$.

Proof. We may assume that $F + q$ has a non-empty intersection with every $\mathbb{Z}G_i + p_i$ for $1 \leq i \leq l$. If $F \subseteq G_i$ for any i, then $F + q \subset \mathbb{Z}F + q' \subset \mathbb{Z}G_i + q' = \mathbb{Z}G_i + p_i$ for $q' \in F + q \cap \mathbb{Z}G_i + p_i$. Thus in this case our claim holds. We will show that there exists always an i such that $F \subseteq G_i$.

Assume that $F \nsubseteq G_i$ for every i. As a notation, for a subset $S \subset \mathbb{Q}^N$, we write $\mathbb{Q}S$ for the $\mathbb{Q}$-subspace generated by S. Then, $\mathbb{Q}(\mathbb{Z}F \cap \mathbb{Z}G_i) \subsetneq \mathbb{Q}F$ for every i. Indeed, it holds that $\mathbb{Q}(\mathbb{Z}F \cap \mathbb{Z}G_i) \subseteq \mathbb{Q}F \cap \mathbb{Q}G_i \subseteq \mathbb{Q}F$. The second inclusion is strict except in the case that $\mathbb{Q}F \subseteq \mathbb{Q}G_i$. But this would imply that $F \subseteq G_i$, because $F = \mathbb{Q}F \cap Q$ and $G_i = \mathbb{Q}G_i \cap Q$. Here we use that F and G_i are faces of a common affine monoid.

By Lemma 2.1.5, we can find an element $\hat{p}$ in the cone generated by F that is not contained in any $\mathbb{Q}(\mathbb{Z}F \cap \mathbb{Z}G_i)$. By multiplication with a positive scalar, we can assume $\hat{p} \in F$. For every non-negative integer λ, it holds that $\lambda\hat{p} + q \in F + q \subset \bigcup_i \mathbb{Z}G_i + p_i$. Since the union is finite, there exists an index i and two different integers $\lambda, \lambda' \in \mathbb{Z}$ such that $\lambda\hat{p} + q, \lambda'\hat{p} + q \in \mathbb{Z}G_i + p_i$. But now it follows that $(\lambda - \lambda')\hat{p} \in \mathbb{Z}F \cap \mathbb{Z}G_i$ and thus $\hat{p} \in \mathbb{Q}(\mathbb{Z}F \cap \mathbb{Z}G_i)$, a contradiction to our choice of $\hat{p}$. $\square$

Now we are ready to prove that our decomposition is in fact irredundant and unique.

Lemma 2.1.7. *Consider a finite decomposition*

$$M \setminus U = \bigcup_i (\mathbb{Z}G_i + m_i) \cap M$$

of $M \setminus U$ with $m_i \in M \setminus U$ and faces G_i of Q. Then every set in $\mathcal{F}(M)$ appears in this decomposition. Thus, (2.3) defines the unique irredundant finite decomposition of $M \setminus U$.

Proof. Let $m \in M \setminus U$ and F a face of Q such that $\mathbb{Z}F + m \in \mathcal{F}(M)$. By Lemma 2.1.6, there exists an index i such that $F + m \subseteq \mathbb{Z}G_i + m_i$. In particular, $\mathbb{Z}G_i + m = \mathbb{Z}G_i + m_i$. Hence, $F \subseteq \mathbb{Z}G_i$ and therefore $F \subseteq \mathbb{Z}G_i \cap Q = G_i$. On the other hand, by Lemma 2.1.1, F is a maximal face of Q such that $F + m \subset M \setminus U$. But $G_i + m \subset \mathbb{Z}G_i + m \cap M \subset M \setminus U$, so we conclude that $G_i = F$. Whence $\mathbb{Z}F + m = \mathbb{Z}G_i + m_i$. ☐

This completes the proof of our theorem:

Theorem 2.1.8. *Let Q be an affine monoid, let $M \subset \mathbb{Z}Q$ be a module and let $U \subset M$ be a submodule. Then there exists a unique irredundant finite (non-disjoint) decomposition*

$$M \setminus U = \bigcup_i (\mathbb{Z}F_i + m_i) \cap M\,. \tag{2.4}$$

The number of times F appears in (2.4) equals the multiplicity of $\mathfrak{p}_F$ on $\mathbb{K}\{M\}/\mathbb{K}\{U\}$. In particular, a face F appears in (2.4) if and only if $\mathfrak{p}_F$ is an associated prime of $\mathbb{K}\{M\}/\mathbb{K}\{U\}$. Geometrically, a set $\mathbb{Z}F_i + m_i$ appears in (2.4) if and only if F_i is a maximal face such that $F_i + m_i \subset M \setminus U$.

From now on, we specialize to the case $M = \overline{Q}$ and $U = Q$. For the ease of reference, call a face F *associated to* Q if it appears in (2.4). The following is immediate:

Corollary 2.1.9. *Q has a j-*dimensional family of holes if and only if there is a j-*dimensional associated face of Q.*

We get a description of the holes of the localization Q_F from Lemma 2.1.3:

Proposition 2.1.10. *Let F be a face of Q. The families of holes of Q_F are exactly those families of holes $\mathbb{Z}G + q$ of Q which satisfy $F \subset G$. In particular, Q_F is normal if and only if no associated face contains F.*

We would like to point our another special case of Lemma 2.1.8. Set $Q = M = \mathbb{N}_0^n$ for some $n \in \mathbb{N}$ and let $U \subset \mathbb{N}_0^n$ be a module generated by vectors $v_1, \ldots, v_r \in \mathbb{N}_0^n$, such that every entry of v_i is either 0 or 1 for every i. Then $\mathbb{K}\{U\}$ is a squarefree monomial ideal in the polynomial ring $\mathbb{K}[Q] = \mathbb{K}[x_1, \ldots, x_n]$ and thus the Stanley-Reisner ideal of some simplicial complex $\Delta \subset 2^{[n]}$. Then (2.4) corresponds to the well-known primary decomposition of Stanley-Reisner ideals, see [MS05, Theorem 1.7]. In particular, the faces appearing in (2.4) correspond to the facets of Δ. Moreover, the dimension of $\mathbb{K}\{M\}/\mathbb{K}\{U\}$ equals the maximal dimension of the faces in (2.4), and thus one plus the dimension of Δ.

2.2. Local cohomology and holes

In this section, we consider the local cohomology of the monoid algebra $\mathbb{K}[Q]$ with support on the maximal graded ideal $\mathfrak{m} := \mathfrak{p}_{F_0}$. Recall that the local cohomology can be computed by the Ishida complex [Ish88] as follows: Consider the $\mathbb{Z}Q$-graded complex

$$\mho_Q : 0 \to \mathbb{K}[Q] \to \bigoplus_{F \in \mathcal{F}_1} \mathbb{K}[Q_F] \to \bigoplus_{F \in \mathcal{F}_2} \mathbb{K}[Q_F] \to \cdots \to \bigoplus_{F \in \mathcal{F}_{d-1}} \mathbb{K}[Q_F] \to \mathbb{K}[\mathbb{Z}Q] \to 0$$

where $\mathcal{F}_i$ denotes the set of i-*dimensional faces of Q. The maps are given by δ_i : $\mathbb{K}[Q_F] \ni \mathbf{x}^q \mapsto \sum_{G \supset F} \epsilon(F,G)\mathbf{x}^q$ via the canonical inclusion $\mathbb{K}[Q_F] \to \mathbb{K}[Q_G]$ for $F \subset G$, and $\epsilon(F,G)$ is an appropriate sign function. See [MS05, Section 13.3] for the exact definition. The cohomological degrees are chosen such that the modules $\mathbb{K}[Q]$ and $\mathbb{K}[\mathbb{Z}Q]$ sit in degree 0 respectively d.

Theorem 2.2.1 (Thm. 13.24, [MS05])**.** *The local cohomology of any $\mathbb{K}[Q]$-module M supported on $\mathfrak{m}$ is the cohomology of the Ishida complex tensored with M:*

$$H^i_{\mathfrak{m}}(M) \cong H^i(M \otimes \mho_Q)$$

The isomorphism respects the $\mathbb{Z}Q$-grading.

We use the Ishida complex to relate the local cohomology of $\mathbb{K}[Q]$ to the local cohomology of $\mathbb{K}\{\overline{Q}\}/\mathbb{K}[Q]$.

Theorem 2.2.2. *Let Q be an affine monoid of *dimension d, $i \leq d$ and integer and let $q \in \mathbb{Z}Q$. Then the following holds:*

1. *If $i < d$, then*
$$H^i_{\mathfrak{m}}(\mathbb{K}[Q]) \cong H^{i-1}_{\mathfrak{m}}(\mathbb{K}\{\overline{Q}\}/\mathbb{K}[Q])\,.$$
2. *If $i = d$ and $q \notin -\operatorname{int}\overline{Q}$, then*
$$H^i_{\mathfrak{m}}(\mathbb{K}[Q])_q \cong H^{i-1}_{\mathfrak{m}}(\mathbb{K}\{\overline{Q}\}/\mathbb{K}[Q])_q$$
a $\mathbb{K}$-vector spaces.
3. *If $q \in -\operatorname{int}\overline{Q}$, then*
$$H^i_{\mathfrak{m}}(\mathbb{K}[Q])_q = \begin{cases} \mathbb{K} & \textit{if } i = d\,, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. First, we compute the local cohomology of $\mathbb{K}\{\overline{Q}\}$. For this we compare the Ishida complex of $\mathbb{K}[\overline{Q}]$ with the complex $\mathbb{K}\{\overline{Q}\} \otimes \mho_Q$ over $\mathbb{K}[Q]$. It holds that $(\overline{Q})_F = (\overline{Q})_{F\overline{Q}}$ (Lemma 1.3.4) for every face F of Q. Therefore, for every $q \in \mathbb{Z}Q$, the degree q part of $\mho_{\overline{Q}}$ coincides with the degree q part of $\mathbb{K}\{\overline{Q}\} \otimes \mho_Q$ (as a complex of $\mathbb{K}$-vector spaces). Hence, the support of the local cohomology of $\mathbb{K}[\overline{Q}]$ (as a ring) equals the support of the local cohomology of $\mathbb{K}\{\overline{Q}\}$ (as a $\mathbb{K}[Q]$-module). In particular, $H^i_{\mathfrak{m}}(\mathbb{K}\{\overline{Q}\})_q = 0$ for $i < d$, because $\mathbb{K}[\overline{Q}]$ is Cohen-Macaulay (as a ring). Moreover, $H^d_{\mathfrak{m}}(\mathbb{K}\{\overline{Q}\})_q = 0$ for $q \notin -\operatorname{int}\overline{Q}$. For this, note that $H^d_{\mathfrak{m}}(\mathbb{K}[\overline{Q}])_q \neq 0$ if and only if q is not in the image of the map δ_{d-1} in $\mho_{\overline{Q}}$. Since $\overline{Q}$ is normal, this is equivalent to $\sigma_F(q) < 0$ for every facet, hence $q \in -\operatorname{int}\overline{Q}$.

Next, we consider the short exact sequence

$$0 \to \mathbb{K}[Q] \to \mathbb{K}\{\overline{Q}\} \to \mathbb{K}\{\overline{Q}\}/\mathbb{K}[Q] \to 0$$

Form the corresponding long exact sequence in cohomology we see that $H^i_{\mathfrak{m}}(\mathbb{K}[Q]) \cong H^{i-1}_{\mathfrak{m}}(\mathbb{K}\{\overline{Q}\}/\mathbb{K}[Q])$ for $i < d$.

For $i = d$, we make a case distinction. If $q \notin -\operatorname{int}\overline{Q}$, then $H^i_{\mathfrak{m}}(\mathbb{K}\{\overline{Q}\})_q = 0$ by the discussion above. Hence one can read off from the long exact sequence that $H^d_{\mathfrak{m}}(\mathbb{K}[Q])_q \cong H^{d-1}_{\mathfrak{m}}(\mathbb{K}\{\overline{Q}\}/\mathbb{K}[Q])_q$, because the maps are homogeneous. On the other hand, if $q \in -\operatorname{int}\overline{Q}$, then $q \notin Q_F$ for any face F of Q. So the degree q part of $\mho_Q$ is just $0 \to \mathbb{K} \to 0$ with the $\mathbb{K}$ in cohomological degree d. □

Corollary 2.2.3. *If Q is not normal, then* ${}^*\operatorname{depth}\mathbb{K}[Q] = {}^*\operatorname{depth}\mathbb{K}\{\overline{Q}\}/\mathbb{K}[Q] + 1$.

Proof. This is immediate from Lemma 2.2.2 using the graded version of Grothendieck's Non-Vanishing Theorem (cf. Lemma A.1.26). □

We give a condition on which graded components of the local cohomology of $\mathbb{K}[Q]$ can be nonzero:

Corollary 2.2.4. *Let $q \in \mathbb{Z}Q$ such that $q \notin -\operatorname{int}\overline{Q}$. If $H^{i+1}_{\mathfrak{m}}(\mathbb{K}[Q])_q \neq 0$ for some i, then q is contained in a family of holes of *dimension at least i.*

Proof. By Lemma 2.2.2, we have $H^{i+1}_{\mathfrak{m}}(\mathbb{K}[Q])_q \cong H^i_{\mathfrak{m}}(\mathbb{K}\{\overline{Q}\}/\mathbb{K}[Q])_q$. We consider the i^{th} module in $\mho_Q \otimes \mathbb{K}\{\overline{Q}\}/\mathbb{K}[Q]$. It is

$$\bigoplus_{F \in \mathcal{F}_i} \mathbb{K}[Q_F] \otimes \mathbb{K}\{\overline{Q}\}/\mathbb{K}[Q] = \bigoplus_{F \in \mathcal{F}_i} \mathbb{K}\{\overline{Q}_F\}/\mathbb{K}[Q_F]$$

If $H^{i+1}_{\mathfrak{m}}(\mathbb{K}[Q])_q \neq 0$, then there is an element of degree q in this module. Hence there is a face F of *dimension i such that $q \in \overline{Q}_F \setminus Q_F$. Now our description of the holes in the localization Q_F (cf. Lemma 2.1.10) implies that q is contained in a family of holes $\mathbb{Z}G + p$ of Q with $F \subset G$. In particular, ${}^*\dim G \geq {}^*\dim F = i$. □

Our next goal is a partial converse to Lemma 2.2.4. For this, we take a closer look at the Ishida complex. In this we follow [MS05, Section 12.2]. Fix an element $q \in \mathbb{Z}Q$. The part of $\mho_Q$ in degree q is determined by the faces $F \subset Q$ such that $q \in Q_F$. Therefore, we consider the set $\nabla(q) := \{\, F \subseteq Q \mid q \in Q_F, F \text{ a face} \,\}$. This set is clearly closed under going up in the face lattice of Q. Now let $\mathcal{P}$ be a cross-section polytope of $\mathbb{R}_+Q$ and let $\mathcal{P}^\vee$ be the polar polytope of $\mathcal{P}$. Then the face lattice of $\mathcal{P}^\vee$ equals the order dual of the face lattice of Q (i.e. the face lattice of $\mathcal{P}$ turned upside down). Hence the images $\nabla(q)^\vee$ of the faces in $\nabla(q)$ in the face lattice of $\mathcal{P}^\vee$ form a set that is closed under going down. In other words, $\nabla(q)^\vee$ is a polyhedral subcomplex of the boundary complex of $\mathcal{P}^\vee$. Because $\nabla(q)$ corresponds to the part of $\mho_Q$ in degree q, we can reinterpret this part as an (augmented) polyhedral chain complex for $\nabla(q)^\vee$, while reversing the cohomological degrees. So the reduced homology of the polyhedral cell complex $\nabla(q)^\vee$ gives us the local cohomology of $\mathbb{K}[Q]$ in degree q ([MS05, p. 258]):

$$H^i_{\mathfrak{m}}(\mathbb{K}[Q])_q = \tilde{H}_{d-1-i}(\nabla(q)^\vee, \mathbb{K}) \tag{2.5}$$

Here, $d = {}^*\!\dim Q = \dim \mathcal{P} + 1$. Using this formula, we can explicitly compute part of the local cohomology of $\mathbb{K}[Q]$:

Proposition 2.2.5. *Let Q be an affine monoid and let $\mathbb{Z}F + p$ be an j-*dimensional family of holes. Let $q \in \mathbb{Z}F + p$ an element that lies beyond every facet G not containing F. By this we mean that $\sigma_G(q) < 0$, where σ_G is the supporting linear form of G. Then*

$$H^i_{\mathfrak{m}}(\mathbb{K}[Q])_q = \begin{cases} \mathbb{K} & \textit{if } i = j+1\,, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. We prove that $\nabla(q) = \{\, G \mid G \supsetneq F \,\}$. Thus $\nabla(q)^\vee$ is the boundary complex of the face $\bar{F}$ corresponding to F in the polar polytope $\mathcal{P}^\vee$. This is a sphere of dimension $\dim \bar{F} - 1 = \dim \mathcal{P} - 1 - ({}^*\!\dim F - 1) - 1 = d - 2 - j$. So by (2.5) it follows

$$H^i_{\mathfrak{m}}(\mathbb{K}[Q])_q = \tilde{H}_{d-1-i}(S^{d-2-j}, \mathbb{K}) = \begin{cases} \mathbb{K} & \text{if } i = j+1\,, \\ 0 & \text{otherwise.} \end{cases}$$

To compute $\nabla(q)$, we first consider a face G that does not contain F. For such a G we can find a facet $G' \supset G$ that does not contain F. By our assumption, q lies beyond G' and hence $q \notin Q_{G'}$. Thus $q \notin Q_G$ and therefore $G \notin \nabla(q)$. Next, by our choice of q, it holds that $q \in \overline{Q}_F \setminus Q_F$. In particular $F \notin \nabla(q)$. Moreover, $q \in \overline{Q}_G$ for every $G \supset F$, because $\overline{Q}_G \supset \overline{Q}_F$. It remains to show that $q \in Q_G$ for every $G \supsetneq F$. So assume on the contrary that $q \in \overline{Q}_G \setminus Q_G$ for such a G. There exists an element f from the interior of F to get $q + f \in \overline{Q}_G \setminus Q_G \cap \overline{Q}$. But this implies $G + q + f \subset \overline{Q} \setminus Q$, which contradicts our choice $q \in \mathbb{Z}F + p$, by Lemma 2.1.8. $\square$

This gives a partial converse to Lemma 2.2.4:

Corollary 2.2.6. *Every i-*dimensional family of holes contains an element $q \in \mathbb{Z}Q$, such that $H^{i+1}_{\mathfrak{m}}(\mathbb{K}[Q])_q \cong \mathbb{K}$. If $i > 0$, then there are in fact infinitely many such elements (even up to units).*

Proof. Let $\mathbb{Z}F + p$ be a family of holes and let $q' \in \operatorname{int} F$. For every facet $G \not\supseteq F$ it holds that $\sigma_G(q') > 0$. Hence, $p - mq'$ satisfies the hypothesis of Lemma 2.2.5 for every sufficiently large $m \in \mathbb{N}$. This yields infinitely many non-vanishing graded components of $H^{i+1}_{\mathfrak{m}}(\mathbb{K}[Q])$. If $i > 0$, then $F \neq F_0$ and thus $q' \notin F_0$. So these components are $\mathbb{K}[F_0]$-linearly independent. □

The preceding proof shows in particular that $H^{i+1}_{\mathfrak{m}}(\mathbb{K}[Q]) \cong H^{i}_{\mathfrak{m}}(\mathbb{K}\{\overline{Q}\}/\mathbb{K}[Q])$ is not finitely generated if $\mathbb{K}\{\overline{Q}\}/\mathbb{K}[Q]$ has an associated prime of dimension $i > 0$. This is true for general finitely generated modules over an (ungraded) local ring which is a homomorphic image of a local Gorenstein ring, see [BS98, Corollary 11.3.3]. For later use, we give a criterion for the vanishing of certain parts of the local cohomology.

Lemma 2.2.7. *Let $q_1, q_2, \dots$ be a sequence of elements in $\mathbb{Z}Q$ such that $H^i_{\mathfrak{m}}(\mathbb{K}[Q])_{q_j} \cong H^i_{\mathfrak{m}}(\mathbb{K}[Q])_{q_1}$ for every j. Assume further that there is a facet F of Q such that $\sigma_F(q_j) < \sigma_F(q_{j+1})$ for every j. Then $H^i_{\mathfrak{m}}(\mathbb{K}[Q])_{q_j} = 0$ for every j.*

Proof. Assume to the contrary that $H^i_{\mathfrak{m}}(\mathbb{K}[Q])_{q_j} \neq 0$. Consider the submodules $\mathcal{H}_l \subseteq H^i_{\mathfrak{m}}(\mathbb{K}[Q])$ generated by $\left\{ H^i_{\mathfrak{m}}(\mathbb{K}[Q])_{q_j} \,\middle|\, j \geq l \right\}$. Clearly $\mathcal{H}_{l+1} \subseteq \mathcal{H}_l$. By our hypothesis, $\sigma_F(q_l) < \sigma_F(q_j)$ for every $j > l$. Therefore, q_l is not contained in the Q-submodule of $\mathbb{Z}Q$ generated by the q_j for $j > l$. This implies that $\mathcal{H}_{l+1} \subsetneq \mathcal{H}_l$, so we get an infinite descending chain of submodules. This contradicts the fact that $H^i_{\mathfrak{m}}(\mathbb{K}[Q])$ is *Artinian, see Lemma A.1.25. □

We give an example to demonstrate the geometric meaning of the results in this section.

Example 2.2.8. Consider the affine monoid $Q \subset \mathbb{Z}^3$ generated by $(0,0,1)$, $(1,0,1)$, $(0,2,1)$, $(1,2,1)$, $(0,3,1)$ and $(1,3,1)$. It is shown in the left part of Figure 2.2. This example is taken from [TH86]. The holes of Q form a "wall" parallel to the xz-plane. Hence, nontrivial local cohomology of $\mathbb{K}[Q]$ can only appear in the degrees of this wall. The right part of Figure 2.2 shows this wall and the intersections with the facet defining hyperplanes. In each region, $\nabla(.)$ and thus $H^i_{\mathfrak{m}}(\mathbb{K}[Q])$ is constant. In the shaded unbounded region pointing downwards, we have $H^3_{\mathfrak{m}}(\mathbb{K}[Q]) \neq 0$ by Lemma 2.2.5. All other unbounded regions do not support local cohomology by Lemma 2.2.7 (just take the points on any ray reaching outwards). So the only part of the local cohomology that is not classified so far is the lattice point q in the small shaded triangle. In fact, one may compute directly that $\dim_{\mathbb{K}} H^2_{\mathfrak{m}}(\mathbb{K}[Q])_q = 1$.

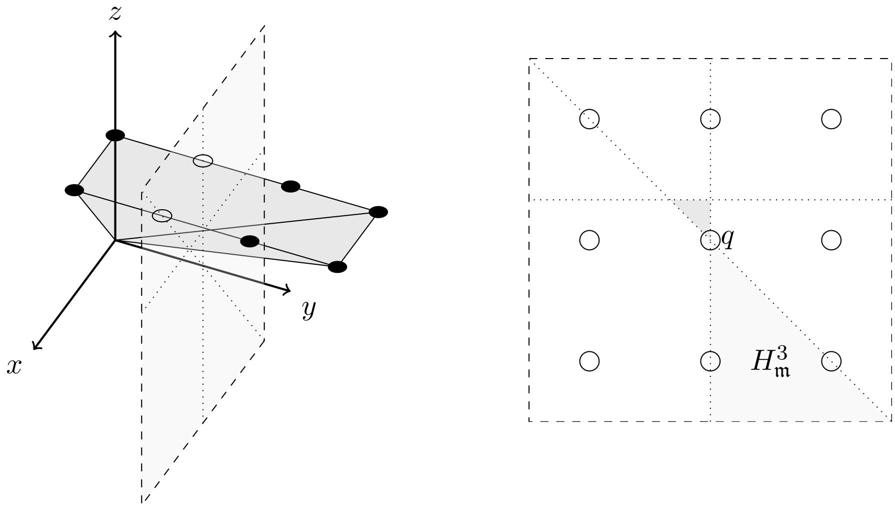


Figure 2.2.: The example of Trung and Hoa

2.3. Applications

2.3.1. Special configurations of holes

In this section, we show that various ring-theoretical properties of $\mathbb{K}[Q]$ correspond to special configurations of the holes in Q. For positive Q, the next proposition appeared as Corollary 5.3 in [SS90].

Proposition 2.3.1. *Let Q be an affine monoid with* $^*\dim Q \geq 2$. *Then* $H^1_{\mathfrak{m}}(\mathbb{K}[Q])_q \neq 0$ *if and only if q is contained in a zero-**dimensional family of holes. In this case,* $H^1_{\mathfrak{m}}(\mathbb{K}[Q])_q = \mathbb{K}$ *and* $H^i_{\mathfrak{m}}(\mathbb{K}[Q])_q = 0$ *for* $i \neq 1$.

Proof. By Lemma 2.2.2, we have $H^1_{\mathfrak{m}}(\mathbb{K}[Q]) \cong H^0_{\mathfrak{m}}(\mathbb{K}\{\overline{Q}\}/\mathbb{K}[Q])$. In the proof of Lemma 2.1.4, we have already verified that the support of $H^0_{\mathfrak{m}}(\mathbb{K}\{\overline{Q}\}/\mathbb{K}[Q])$ is the union of the zero-*dimensional families of holes.

For the second claim, note that $H^0_{\mathfrak{m}}(\mathbb{K}\{\overline{Q}\}/\mathbb{K}[Q])$ is a submodule of $\mathbb{K}\{\overline{Q}\}/\mathbb{K}[Q]$ and every nontrivial homogeneous component of $\mathbb{K}\{\overline{Q}\}/\mathbb{K}[Q]$ has dimension 1 over $\mathbb{K}$. Moreover, if $H^i_{\mathfrak{m}}(\mathbb{K}[Q])_q \neq 0$ for some $i > 1$, then by Lemma 2.2.4 q has to be contained in a family of holes $\mathbb{Z}G + p$ of *dimension $i - 1 > 0$. But then the zero-*dimensional family of holes containing q is also contained in $\mathbb{Z}G + p$ (because the minimal face F_0 is contained in G), which is absurd. □

Theorem 2.3.2. *Let Q be an affine monoid of *dimension d. The following holds:*

- *If $d \geq 2$, then* $^*\mathrm{depth}\,\mathbb{K}[Q] = 1$ *if and only if there is a* 0-**dimensional family of holes.*

- *Q is locally normal if and only if there is no family of holes of positive *dimension.*
- *$\mathbb{K}[Q]$ satisfies Serre's condition (R_1) if and only if there is no family of holes of *dimension $d-1$.*
- *$\mathbb{K}[Q]$ satisfies Serre's condition (S_2) if and only if every family of holes has *dimension $d-1$.*

Proof. We start by proving the criterion for $^*\operatorname{depth} \mathbb{K}[Q] = 1$. If $^*\dim Q \geq 2$, then $^*\operatorname{depth} \mathbb{K}[Q] = 1$ if and only if $H^1_{\mathfrak{m}}(\mathbb{K}[Q]) \neq 0$. This is equivalent to the existence of a zero-*dimensional family of holes by Lemma 2.3.1.

Next, we prove the criterion for local normality. Recall that $\mathbb{K}[Q]$ is locally normal if its localizations at all 1-*dimensional faces are normal. By Lemma 2.1.10, this is clearly equivalent to the statement that there are no families of holes of positive *dimension.

To prove the criterion for Serre's condition (R_1), recall $\mathbb{K}[Q]$ satisfies R_1 if and only if $\mathbb{K}[Q_F]$ is regular for every facet F. On the other hand, by Lemma 2.1.10, Q_F is normal for every facet if and only if there is no $(d-1)$-dimensional family of holes. Every 1-*dimensional normal affine monoid is isomorphic to $\mathbb{Z}^m \oplus \mathbb{N}_0$ for some $m \in \mathbb{N}_0$. But these are exactly the 1-*dimensional regular affine monoids by Lemma 1.3.10.

Finally, we prove the criterion for Serre's condition (S_2). By above discussion and Lemma 2.1.10, it holds for any face F of Q that $^*\operatorname{depth} \mathbb{K}[Q_F] = 1$ if and only if F is associated to Q. On the other hand, it holds that $^*\dim \mathbb{K}[Q_F] = 1$ if and only if F is a facet of Q. Therefore, Serre's condition (S_2) is satisfied if and only if every associated face of Q is a facet. $\square$

Note that the preceding theorem implies that Q is normal if and only if it satisfies (R_1) and (S_2). Hence as a corollary we obtain Serre's Theorem (Lemma 1.3.18) for affine monoid algebras. Further, if $^*\dim Q \geq 2$, then Q is normal if and only if it is locally normal and $^*\operatorname{depth} \mathbb{K}[Q] \geq 2$. It follows from Serre's Theorem that this statement also holds for general Noetherian rings. The first part of the Lemma 2.3.2 can be generalized to an upper bound on the *depth:

Theorem 2.3.3. *If Q has an i-*dimensional family of holes, then the *depth of $\mathbb{K}[Q]$ is at most $i+1$.*

Proof. If Q has an i-*dimensional family of holes, then $H^{i+1}_{\mathfrak{m}}(\mathbb{K}[Q]) \neq 0$ by Lemma 2.2.6. Hence $^*\operatorname{depth} \mathbb{K}[Q] \leq i+1$. Alternatively, by Lemma 2.2.3 we can consider the depth of $\mathbb{K}\{\overline{Q}\}/\mathbb{K}[Q]$. The families of holes of Q correspond to the associated primes of $\mathbb{K}\{\overline{Q}\}/\mathbb{K}[Q]$ (cf. Lemma 2.1.8), so the claim follows from the general fact that the *depth of a module is bounded above by the dimensions of its associated primes, cf. [BH98, Prop 1.2.13] and Lemma A.1.18. $\square$

We identify a special case where equality holds:

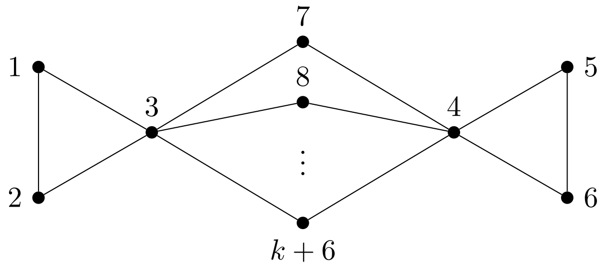


Figure 2.3.: The graph G_{k+6}

Proposition 2.3.4. *If $\overline{Q} \setminus Q = q + F$ for an element $q \in \overline{Q} \setminus Q$ and a face F of Q, then* ${}^*\mathrm{depth}\,\mathbb{K}[Q] = 1 + {}^*\dim F$.

Note that it is not sufficient to require that there is only one family of holes, see Lemma 2.2.8. Before we can prove Lemma 2.3.4, we need another lemma:

Lemma 2.3.5. *Let $q \in Q$ and F a face of Q, such that $F+q \subseteq \overline{Q}\setminus Q$. Then $\overline{F}+q \subseteq \overline{Q}\setminus Q$.*

Proof. Assume the contrary. Then there exists an element $f \in \overline{F}$ such that $f + q \in Q$. We can write $f = f_1 - f_2$ with $f_1, f_2 \in F$. But $f + f_2 + q = f_1 + q \in \overline{Q} \setminus Q$ by assumption, a contradiction. □

Proof of Lemma 2.3.4. By Lemma 2.1.1, our hypothesis is equivalent to the statement that the module $\mathbb{K}\{\overline{Q}\}/\mathbb{K}[Q]$ is cyclic with annihilator $\mathfrak{p}_F$. Hence $\mathbb{K}\{\overline{Q}\}/\mathbb{K}[Q] \cong \mathbb{K}[Q]/\mathfrak{p}_F = \mathbb{K}[F]$ (The isomorphism shifts the grading). Next, recall that the *depth of a module M over a ring R equals its *depth over $R/\operatorname{Ann} M$. Together with Lemma 2.2.3 this yields

$$ {}^*\mathrm{depth}_{\mathbb{K}[Q]}\,\mathbb{K}[Q] = 1 + {}^*\mathrm{depth}_{\mathbb{K}[F]}\,\mathbb{K}[F]\,. $$

Now we use Lemma 2.3.5 to conclude that $\overline{Q} \setminus Q = F + q \subseteq \overline{F} + q \subseteq \overline{Q} \setminus Q$, so $F = \overline{F}$. Hence $\mathbb{K}[F]$ is normal and the result follows from Hochster's Theorem. □

We give an example how one can effectively compute the *depth using Lemma 2.3.4.

Example 2.3.6. For positive $k \in \mathbb{N}$ consider the graph G_{k+6} in Figure 2.3. In [Hib+11] the *depth of the toric edge ring of this family of graphs is computed. See Chapter 4 below for the definition of toric edge rings. We will show that these edge rings satisfy the assumption of Lemma 2.3.4, and thus give an alternative computation of the *depth.

First, it is known that $\mathbb{K}[\overline{Q}]$ is generated as a $\mathbb{K}[Q]$-module by $x_1x_2x_3x_4x_5x_6$, i.e. the monomial corresponding to the vector $q \in \overline{Q} \subseteq \mathbb{R}^{k+6}$ which assigns 1 to the vertices $1, \ldots, 6$. If we add one of the "middle" edges, e.g. $\{3, 8\}$, to q, then it is easy to see that the result lies in Q. On the other hand, if we add any combination of edges from the triangles to q, then the result will always be in $\overline{Q} \setminus Q$. To see this, note that the sum over the vertices of each triangle is always odd. This implies that $\overline{Q} \setminus Q = F + q$, where F is the face spanned by the six edges in the triangles. The dimension of F is 6, so by Lemma 2.3.4 it follows that depth $\mathbb{K}[Q] = 1 + 6 = 7$.

We generalize this computation to show that every toric edge ring can be realized as a combinatorial pure subring of a toric edge ring of *depth at most 7. The construction is as follows: To a given graph G, add two triangles on six (in total) new vertices. Then connect every vertex of G with every new vertex. Obviously, the toric edge ring of G is a combinatorial pure subring of the edge ring of this bigger graph, because G is an induced subgraph of the later. Then it is not difficult to see that the face spanned by the six edges in the triangles is associated. Its dimension is six, so the *depth of the ring is at most seven. In [Hib+11] it was conjectured that every toric edge ring has a *depth of at least seven. So we consider it as likely that the toric edge ring we constructed has a *depth of exactly seven.

2.3.2. Simplicial affine monoids

A well-known result by Goto, Suzuki and Watanabe [GSW76] states that if Q is simplicial, positive and satisfies Serre's condition (S_2), then $\mathbb{K}[Q]$ is Cohen-Macaulay. We give a proof of this result without the positivity assumption using our description of Serre's condition (S_2).

Proposition 2.3.7. *Let Q be a simplicial affine monoid. If Q satisfies Serre's condition (S_2), then $\mathbb{K}[Q]$ is Cohen-Macaulay.*

We also identify another case where the upper bound on the *depth given in Lemma 2.2.3 is tight.

Proposition 2.3.8. *Let Q be a simplicial affine monoid. If the families of holes of Q are pairwise disjoint, then the *depth of $\mathbb{K}[Q]$ equals one plus the smallest *dimension of a family of holes.*

Both results depend on the following lemma. Recall that we defined $\nabla(q)$ to be the set of faces of Q such that $q \in Q_F$ for $q \in \mathbb{Z}Q$. The complex $\nabla(q)^\vee$ is defined by turning the face poset of $\nabla(q)$ upside down. It is a subcomplex of the polytope $\mathcal{P}^\vee$ polar to the cross-section polytope $\mathcal{P}$ of $\mathbb{R}_+Q$. Since $\mathcal{P}$ is a simplex, the same holds for $\mathcal{P}^\vee$. So $\nabla(q)^\vee$ is a simplicial complex whose vertices correspond to the facets of Q. A *minimal non-face* of a simplicial complex Δ is a minimal face of the ambient simplex that is not a face of Δ.

Lemma 2.3.9. *Let Q be a simplicial affine monoid, $q \in \mathbb{Z}Q$ and $i \geq 2$. The $(i-1)$-dimensional minimal non-faces of $\nabla(q)^\vee$ correspond to the $(d-i)$-*dimensional families of holes containing q.*

Proof. For $q \in \mathbb{Z}Q$ let $\overline{\nabla}(q)$ denote the set of faces F of Q such that $q \in \overline{Q}_F$. Obviously it holds that $\nabla(q) \subset \overline{\nabla}(q)$. Hence $\nabla(q)^\vee$ is a simplicial subcomplex of the simplicial complex $\overline{\nabla}(q)^\vee$ contained in the boundary complex of $\mathcal{P}^\vee$. Every minimal non-face of $\nabla(q)^\vee$ that is not contained in $\overline{\nabla}(q)^\vee$ is also a minimal non-face of the latter.

So we start by computing the minimal non-faces of $\overline{\nabla}(q)^\vee$. For this we claim that $\overline{\nabla}(q)$ has a unique minimal element. We write $\mathcal{F}_\geq$ for the set of facets F of Q such that $\sigma_F(q) \geq 0$ and we write $\mathcal{F}_<$ for the set of facets F such that $\sigma_F(q) < 0$. Our candidate for the unique minimal element is the intersection G of the facets in $\mathcal{F}_\geq$. This is indeed a face because Q is simplicial. If $p \in \operatorname{int} G$ is an interior element, then by construction $\sigma_F(p) > 0$ for all F in $\mathcal{F}_<$. Therefore, $q + mp \in \overline{Q}$ for $m \gg 0$ and hence $q \in \overline{Q}_G$. On the other hand, let G' be a face such that $q \in \overline{Q}_{G'}$. Then there exists an element $g \in G'$ such that $q + g \in \overline{Q}$. It follows that $\sigma_F(g) > 0$ for all $F \in \mathcal{F}_<$. Hence G is not contained in any facet in $\mathcal{F}_<$ and can therefore be written as an intersection of facets in $\mathcal{F}_\geq$. It follows that $F \subset G$. We now return the face lattice of $\overline{\nabla}(q)^\vee$. Because $\overline{\nabla}(q)$ has a unique minimal element, $\overline{\nabla}(q)^\vee$ is isomorphic to the complex of faces of a simplex. In particular, the minimal non-faces of $\overline{\nabla}(q)^\vee$ are only vertices.

Next, we consider the minimal non-faces of $\nabla(q)^\vee$ that are contained in $\overline{\nabla}(q)^\vee$. This minimal non-faces correspond to the maximal faces F such that $q \in \overline{Q}_F \setminus Q_F$. But these are exactly the families of holes containing q. So the minimal non-faces of $\nabla(q)^\vee$ correspond either to the families of holes containing q or they are vertices (these come from the minimal non-faces of $\overline{\nabla}(q)^\vee$). □

Proof of Lemma 2.3.7. By Lemma 2.3.2, Serre's condition (S_2) implies that all families of holes have *dimension $d-1$. So $\nabla(q)^\vee$ is a simplicial complex with only 0-dimensional minimal non-faces for every $q \in \mathbb{Z}Q$. In other words, $\nabla(q)^\vee$ is either a simplex or empty. So the only possible nontrivial (reduced) homology lies in degree -1. By (2.5), this amounts to saying that $H^i_\mathfrak{m}(\mathbb{K}[Q]) = 0$ for $i < d$, so $\mathbb{K}[Q]$ is Cohen-Macaulay. □

Proof of Lemma 2.3.8. Every $q \in \mathbb{Z}Q$ is contained in at most one family of holes. Hence $\nabla(q)^\vee$ is a simplicial complex with only one minimal non-face of positive dimension. This is either a ball of a sphere. Evaluating (2.5) then yields the result. □

Since we allow non-positive affine monoids in Lemma 2.3.7 we immediately obtain a local version.

Corollary 2.3.10. *Let Q be a locally simplicial affine monoid. If Q satisfies Serre's condition (S_2), then $\mathbb{K}[Q]$ is locally Cohen-Macaulay for every field $\mathbb{K}$.*

2.3.3. Seminormal affine monoids

In this subsection, we apply our results to seminormal affine monoids. This way we reprove and extend some results of [BG09]. First, we give a geometric characterization of seminormality that is similar in spirit to the characterizations given in [BG09, p. 66f].

Proposition 2.3.11. *Let Q be an affine monoid. Q is seminormal if and only if for every family of holes $\mathbb{Z}F + q$ it holds that $q \in \mathbb{Q}F$.*

Here, $\mathbb{Q}F$ denotes the $\mathbb{Q}$-subspace of $\mathbb{Q}Q$ generated by F.

Proof. First, assume that the condition in the statement is satisfied. Consider a family of holes $\mathbb{Z}F + q$. Since $q \in \overline{Q} \setminus Q$, there exists an $m \in \mathbb{N}$ such that $mq \in Q$. By our assumption, it holds that $mq \in F$ and therefore $jmq + q \in \mathbb{Z}F + q \cap \overline{Q} \subset \overline{Q} \setminus Q$ for every $j \in \mathbb{N}_0$. It follows that either $2q \notin Q$ or $3q \notin Q$. Thus, Q is seminormal.

On the other hand, assume there is a family of holes $\mathbb{Z}F + q$ such that $q \notin \mathbb{Q}F$. Then there exists an element $p \in \mathbb{Z}F + q$ such that $p \in \operatorname{int} \overline{G}$ and $p \notin G$ for some face $G \supset F$. Thus Q is not seminormal by [BG09, Proposition 2.40]. $\square$

Corollary 2.3.12. *Localizations of seminormal affine monoids are again seminormal.*

Proof. This follows from Lemma 2.3.11 and the description of the families of holes of a localization given in Lemma 2.1.10. $\square$

Corollary 2.3.13 (Corollary 5.4,[BLR06])**.** *Let Q be a seminormal positive affine monoid of dimension at least 2. Then* $\operatorname{depth} \mathbb{K}[Q] \geq 2$*.*

Proof. If Q is positive, then the minimal face F_0 contains only the origin $0 \in \mathbb{Z}Q$. By Lemma 2.3.11 every 0-dimensional family of holes would be contained in $\mathbb{Q}F_0 = \{ 0 \} \subset Q$, so there is no 0-dimensional family of holes. Hence the claim follows from Lemma 2.3.2. $\square$

This result is not valid if one omits the requirement that Q is positive. For example, consider $Q \subset \mathbb{Z}^3$ defined by

$$Q = \left\{ (x, y, z) \in \mathbb{Z}^3 \,\middle|\, x, y \geq 0, z \text{ even or } x > 0 \text{ or } y > 0 \right\}$$

This monoid is seminormal, has *dimension 2 and has a 0-*dimensional family of holes, namely the odd points on the z-axis. So it has $^*\operatorname{depth} \mathbb{K}[Q] = 1$ by Lemma 2.3.2.

Next we give a preliminary characterization of seminormality. Geometrically, we show that the graded components of the local cohomology of a seminormal affine monoid are, in a certain sense, constant on rays from the origin.

Lemma 2.3.14. *An affine monoid Q is seminormal if and only if it satisfies the following condition: For every $q \in \mathbb{Z}Q$ there exists a positive $m \in \mathbb{N}$ such that for every $j \in \mathbb{N}_0$ and every $i \in \mathbb{N}_0$ it holds that $H^i_{\mathfrak{m}}(\mathbb{K}[Q])_q \cong H^i_{\mathfrak{m}}(\mathbb{K}[Q])_{(1+mj)q}$ (as $\mathbb{K}$-vector space).*

Proof. Assume that Q is seminormal and fix an element $q \in \mathbb{Z}Q$. We will find an $m \in \mathbb{N}$ such that $\nabla(q) = \nabla((1 + mj)q)$ for every $j \in \mathbb{N}_0$. This implies our claim by (2.5). First, note that $q \in Q_F$ implies $mq \in Q_F$ for every $m \in \mathbb{N}$. Similarly, $q \notin \overline{Q}_F$ implies $mq \notin \overline{Q}_F$ for every $m \in \mathbb{N}$. So it remains to show the following: There exists an $m \in \mathbb{N}$, such that for every face F with $q \in \overline{Q}_F \setminus Q_F$ and every $j \in \mathbb{N}_0$ it holds that $(1+jm)q \in \overline{Q}_F \setminus Q_F$. For this, consider a face F of Q such that $q \in \overline{Q}_F \setminus Q_F$. By Lemma 2.3.12, the localization Q_F is seminormal and thus the set $\{ m \in \mathbb{N}_0 \mid mq \in Q_F \}$ is contained in a proper subgroup of $\mathbb{Z}$. Since there are only finitely many such faces, we can choose an m in the intersection of these subgroups (for example, the product of the generators). Then $1 + jm$ is not contained in any of these subgroups for every $j \in \mathbb{N}_0$. Whence our claim follows.

For the converse, assume that Q is not seminormal. Let $\mathbb{Z}F + q$ be a family of holes such that $q \notin \mathbb{Q}F$. Then there exists a facet $G \supset F$ of Q such that $\sigma_G(q) > 0$. By Lemma 2.2.5, we can find an $p \in \mathbb{Z}F + q$ such that $H^i_{\mathfrak{m}}(\mathbb{K}[Q])_p \neq 0$ for $i = {}^*\dim F + 1$. Now the sequence $p_j := (1+mj)p$ for $j = 0, 1, \ldots$ satisfies the hypothesis of Lemma 2.2.7, so we conclude that $H^i_{\mathfrak{m}}(\mathbb{K}[Q])_p = 0$, a contradiction. □

With a little more care, one can show that the m in the preceding lemma can be chosen independently of q. We extend the characterization of seminormality given in Theorem 4.7 of [BLR06].

Theorem 2.3.15. *Let Q be an affine monoid. The following statements are equivalent:*

1. *Q is seminormal.*
2. *$H^i_{\mathfrak{m}}(\mathbb{K}[Q])_q = 0$ for all $q \in \mathbb{Z}Q$ such that $q \notin -\overline{Q}$ and all i.*
3. *$H^i_{\mathfrak{m}}(\mathbb{K}[Q])_q = 0$ for all $q \in \mathbb{Z}Q$ such that $q \notin -\overline{Q}$ and all i such that Q has an $(i+1)$-*dimensional family of holes.*

Note that the third condition generalizes Theorem 4.9 in [BLR06].

Proof. 1) $\Rightarrow$ 2) Let Q be seminormal and let $q \in \mathbb{Z}Q$. By Lemma 2.3.14, there exists a positive integer m such that $H^i_{\mathfrak{m}}(\mathbb{K}[Q])_q \cong H^i_{\mathfrak{m}}(\mathbb{K}[Q])_{(1+mj)q}$ for every i and every $j \in \mathbb{N}_0$. If $q \notin -\overline{Q}$, then the sequence $q_j := (1 + mj)q$ satisfies the condition of Lemma 2.2.7, so we conclude that $H^i_{\mathfrak{m}}(\mathbb{K}[Q])_q = 0$.

2) $\Rightarrow$ 3) This is obvious.

3) $\Rightarrow$ 1) Assume that Q is not seminormal. Then, by Lemma 2.3.11, there is a family of holes $\mathbb{Z}F + q$ of Q such that $q \notin \mathbb{Q}F$. There exists a facet G containing F such that $\sigma_G(q) > 0$. By Lemma 2.2.6, there exists an element $p \in \mathbb{Z}F + q$ such that $H^i_{\mathfrak{m}}(\mathbb{K}[Q])_p \neq 0$ where $i = {}^*\dim F + 1$. The linear form σ_G is constant on $\mathbb{Z}F + q$, so $\sigma_G(p) > 0$ and hence $p \notin -\overline{Q}$. □

Our next results extend Proposition 4.15 of [BLR06]. In the non-positive case we may have nontrivial local cohomology in the degrees $F_0^{\overline{Q}} := \overline{Q} \cap (-\overline{Q}) = \mathbb{Q}F_0 \cap \mathbb{Z}Q$. Note that if Q is positive, then $F_0^{\overline{Q}} = \{ 0 \} \subset Q$, so there can be no local cohomology supported in $F_0^{\overline{Q}}$. Moreover, if $H^i_{\mathfrak{m}}(\mathbb{K}[Q])_q \neq 0$ for some $q \in \mathbb{Z}Q$ then $q \in -\overline{Q}$ by the preceding theorem. So in this case, $q \notin F_0^{\overline{Q}}$ if and only if $q \notin \overline{Q}$. We will see in Lemma 2.4.5 below that it is indeed necessary to consider this graded components separately.

Proposition 2.3.16 (Prop. 4.14, [BLR06])**.** *Let Q be a seminormal affine monoid. If $H^i_{\mathfrak{m}}(\mathbb{K}[Q])_q \neq 0$ for some $q \in \mathbb{Z}Q$, $q \notin F_0^{\overline{Q}}$, then $H^i_{\mathfrak{m}}(\mathbb{K}[Q])$ is not finitely generated.*

Proof. Assume to the contrary that $H^i_{\mathfrak{m}}(\mathbb{K}[Q])$ is finitely generated, say, in degrees $p_1, \ldots, p_l \in \mathbb{Z}Q$. We assumed that $q \notin \overline{Q}$, so there exists a facet F such that $\sigma_F(q) < 0$. By Lemma 2.3.14, there is an $m \in \mathbb{N}$ such that $H^i_{\mathfrak{m}}(\mathbb{K}[Q])_{(1+mj)q} \neq 0$ for every $j \in \mathbb{N}_0$. For sufficiently large $j \in \mathbb{N}_0$, it holds that $\sigma_F((1+mj)q) < \sigma_F(p_k)$ for every k, so the graded component in this degree cannot be generated by our supposed set of generators, a contradiction. □

Corollary 2.3.17. *Let Q be a seminormal affine monoid such that $H^i_{\mathfrak{m}}(\mathbb{K}[Q])_q = 0$ for every $q \in F_0^{\overline{Q}}$ and every $i < {}^*\!\dim Q$. Then*

$$^*\mathrm{depth}\, \mathbb{K}[Q] = \min \left\{ {}^*\mathrm{depth}\, \mathbb{K}[Q_F] + 1 \mid F \text{ a face, } {}^*\!\dim F = 1 \right\} .$$

In particular, $\mathbb{K}[Q]$ is Cohen-Macaulay if and only if it is locally Cohen-Macaulay.

Proof. Our hypothesis implies that all the non-vanishing local cohomology modules of $\mathbb{K}[Q]$ are not finitely generated. Therefore, the claim is immediate from Lemma A.1.28. □

There are analogues of Lemma 2.3.7 and Lemma 2.3.8 for the seminormal case. The first of the following results appeared in [BLR06, Corollary 5.6] for positive Q.

Proposition 2.3.18. *Let Q be a seminormal locally simplicial affine monoid such that $H^i_{\mathfrak{m}}(\mathbb{K}[Q])_q = 0$ for every $q \in F_0^{\overline{Q}}$ and every $i < {}^*\!\dim Q$. Then $\mathbb{K}[Q]$ is Cohen-Macaulay if and only if Q satisfies Serre's condition (S_2).*

We will see below in Lemma 2.4.5 that the hypothesis on the local cohomology cannot be dropped.

Proposition 2.3.19. *Let Q be a seminormal affine monoid. If its the families of holes are pairwise disjoint, then the *depth of $\mathbb{K}[Q]$ equals one plus the smallest *dimension of a family of holes.*

The proofs proceed similarly to the simplicial case, once we have shown the following fact:

Lemma 2.3.20. *Let Q be an seminormal affine monoid, $q \in -\overline{Q}$. Let $\overline{\nabla}(q)$ denote the set of faces of Q such that $q \in \overline{Q}_F$. Then $\overline{\nabla}(q)$ has a unique minimal element.*

Proof. The claim is a statement about $\overline{Q}$, so we may assume that $Q = \overline{Q}$. There exists a unique face F such that $q \in -\operatorname{int} F$. Evidently $q \in \overline{Q}_F$. We show that this F is the unique minimal element. So let G be a facet of Q that does not contain F. Then

$\sigma_F(q) < 0$ and hence $q \notin Q_G$. The same holds then for every face contained in G. It follows that every face $G \in \overline{\nabla}(q)$ is contained only in those facets that contain F. Hence, F is the unique minimal element of $\overline{\nabla}(q)$. $\square$

Proof of Lemma 2.3.18. Q is locally simplicial, so the cross-section polytope $\mathcal{P}$ of $\mathbb{R}_+Q$ is simple. Hence its polar $\mathcal{P}^\vee$ is a simplicial polytope.

Let $q \in -\overline{Q}$. By the preceding lemma, $\overline{\nabla}(q)^\vee$ is a face of $\mathcal{P}^\vee$. We only need to consider those q such that $H^i_\mathfrak{m}(\mathbb{K}[Q])_q \neq 0$ for some $i < {}^*\!\dim Q$. In this case, by assumption, it holds that $q \notin \overline{Q}$. Equivalently, $\overline{\nabla}(q)^\vee$ is a proper face of $\mathcal{P}^\vee$ and thus a simplex. Now the proof is analogous to the proof of Lemma 2.3.7. $\square$

We would like to point out an alternative way to derive Lemma 2.3.18 from Lemma 2.3.7. Assume that Q is locally simplicial, seminormal, satisfies (S_2) and $H^i_\mathfrak{m}(\mathbb{K}[Q])_q = 0$ for every $q \in F_0^{\overline{Q}}$. Then $\mathbb{K}[Q]$ is locally Cohen-Macaulay by Lemma 2.3.10 and hence Cohen-Macaulay by Lemma 2.3.17.

Proof of Lemma 2.3.19. Let $q \in -\overline{Q}$. We have seen in the preceding proof that $\overline{\nabla}(q)^\vee$ is a face of the polytope $\mathcal{P}^\vee$, though now $\mathcal{P}^\vee$ is not assumed to be simplicial and $\overline{\nabla}(q)^\vee$ might not be a proper face. However, by our hypothesis $\nabla(q)^\vee$ is obtained from $\overline{\nabla}(q)^\vee$ by removing just one face. Hence, as in the proof of Lemma 2.3.8, $\nabla(q)^\vee$ is either a ball or a sphere and the claim follows from (2.5). $\square$

2.4. Dependence on the characteristic

In this section, we show that the local cohomology of low-dimensional affine monoids algebras does not depend on the field. Moreover, we give two constructions of affine monoid algebras with certain properties whose *depth does depend on $\mathbb{K}$.

In the following proposition, the case $i = 0$ is actually trivial and the cases $i = 1$ and $i = d$ follow from the description of $H^1_\mathfrak{m}(\mathbb{K}[Q])$ and $H^d_\mathfrak{m}(\mathbb{K}[Q])$ given in [SS90], at least for positive Q.

Proposition 2.4.1. *Let Q be an affine monoid of *dimension d and let $q \in \mathbb{Z}Q$. The vector space dimension $\dim_\mathbb{K} H^i_\mathfrak{m}(\mathbb{K}[Q])_q$ of the local cohomology modules does not depend on the characteristic of the field $\mathbb{K}$ for $i \in \{0, 1, 2, d-1, d\}$.*

Proof. We use (2.5) to compute $\dim_\mathbb{K} H^i_\mathfrak{m}(\mathbb{K}[Q])_q$. If i equals d or $d-1$, then this amounts to the -1^{st} and 0^{th} Betti number of the polyhedral cell complex $\nabla(q)^\vee$, and these numbers do not depend on the characteristic.

For the other values of i, we use Alexander duality. Recall that $\nabla(q)^\vee$ is a subcomplex of the boundary complex of the polytope $\mathcal{P}^\vee$, which is a $(d-2)$-sphere. Indeed, if $\nabla(q)^\vee$

contains the interior of $\mathcal{P}^\vee$, then $q \in Q$, so the local cohomology vanished in degree q by Lemma 2.2.4. By Alexander duality (cf. [Hat02, Theorem 3.44]), it holds that

$$\dim_{\mathbb{K}} H^i_{\mathfrak{m}}(\mathbb{K}[Q])_q = \dim_{\mathbb{K}} \tilde{H}_{d-i-1}(\nabla(q)^\vee, \mathbb{K}) = \dim_{\mathbb{K}} H^{i-2}(S^{d-2} \setminus \nabla(q)^\vee, \mathbb{K}) \,.$$

As in the case above, this number does not depend on $\mathbb{K}$ for $i \leq 2$, because the -1^{st} and the 0^{th} Betti numbers of $S^{d-2} \setminus \nabla(q)^\vee$ are independent of $\mathbb{K}$. □

Corollary 2.4.2. *If* $^*\!\dim Q \leq 5$*, then* $\dim_{\mathbb{K}} H^i_{\mathfrak{m}}(\mathbb{K}[Q])_q$ *is independent of* $\mathbb{K}$ *for any* i *and any* $q \in \mathbb{Z}Q$*.*

Proof. If $^*\!\dim Q \leq 4$, this follows at once from Lemma 2.4.1. For $^*\!\dim Q = 5$ we only need to consider $H^3_{\mathfrak{m}}(\mathbb{K}[Q])_q = \tilde{H}_1(\nabla(q)^\vee, \mathbb{K})$. Again, we use that $\nabla(q)^\vee$ is a subcomplex of a 3-sphere, so by [Hat02, Cor. 4.45], $\tilde{H}_1(\nabla(q)^\vee, \mathbb{Z})$ is torsionfree. On the other hand, the 0^{th} reduced homology of a complex is always torsionfree, so by the universal coefficient theorem (cf. [Hat02, Cor. 3A.6]) the dimension $\dim_{\mathbb{K}} \tilde{H}_1(\nabla(q)^\vee, \mathbb{K})$ does not depend on $\mathbb{K}$. □

Corollary 2.4.3. *Serre's condition* (S_3) *does not depend on the characteristic of the field.*

Proof. By Lemma 2.4.1, the property of $\mathbb{K}[Q]$ having *depth at least 3 does not depend on the characteristic, so the same holds for (S_3). □

There are 6-*dimensional affine monoids Q such that $^*\!\operatorname{depth} \mathbb{K}[Q]$ does depend on the characteristic of the field. An example is given in [TH86, p.165]. The affine monoid Q constructed in that paper has *dimension 6 and $H^3_{\mathfrak{m}}(\mathbb{K}[Q])$ vanishes if and only of $\operatorname{char} \mathbb{K} \neq 2$. This shows that the results above cannot be improved.

We have seen in that if Q is either simplicial or seminormal, locally simplicial and positive, then the Cohen-Macaulayness of $\mathbb{K}[Q]$ does not depend on the field $\mathbb{K}$. In the second part of this subsection, we give two examples showing how these results cannot be extended. First, we construct a seminormal, simplicial and positive affine monoid whose *depth does depend on the characteristic. Secondly, we construct a seminormal, locally simplicial, non-positive affine monoid whose Cohen-Macaulayness depends on $\mathbb{K}$. This shows that the assumption on the local cohomology in Lemma 2.3.17 and Lemma 2.3.18 is necessary. As a notation, for $q = (q_1, \dots, q_d) \in \mathbb{N}_0^d$, we write $\operatorname{Supp} q = \{\, i \in [d] \mid q_i \neq 0 \,\}$ and $\deg q = \sum_i q_i$. For background information on Stanley-Reisner rings, see Chapter 1 of [MS05].

Proposition 2.4.4. *Let* Δ *be an simplicial complex on the vertex set* $[d]$ *with Stanley-Reisner ring* $\mathbb{K}[\Delta]$*. Then there exists a seminormal simplicial positive affine monoid*

$Q = Q(\Delta)$ of dimension d such that

$$H^i_{\mathfrak{m}}(\mathbb{K}[Q])_q = \begin{cases} H^{i-1}_{\mathfrak{m}}(\mathbb{K}[\Delta])_q & \text{if } \deg q \text{ is odd and } \operatorname{Supp} q \in \Delta\,; \\ 0 & \text{otherwise} \end{cases}$$

for $1 \le i \le d-1$. If Δ is acyclic, then $\operatorname{depth} \mathbb{K}[Q] = \operatorname{depth} \mathbb{K}[\Delta] + 1$.

If Δ is the cone over the triangulated projective plane, then this yields an example of a seminormal simplicial affine monoid whose *depth depends on the characteristic.

Proof. We construct Q as a submonoid of $\mathbb{N}_0^d$. Let Q be the set of all elements $q \in \mathbb{N}_0^d$ such that either $\operatorname{Supp} q \notin \Delta$ or $\deg q$ is even. One can verify directly that this is a seminormal simplicial positive affine monoid. We identify the faces of Q with the subsets of $[d]$. The families of holes then correspond to the facets of Δ.

We compute the local cohomology. Let $q \in \mathbb{Z}Q$ such that $H^i_{\mathfrak{m}}(\mathbb{K}[Q])_q \neq 0$. By Lemma 2.2.4 it follows that $\deg q$ is odd and $\operatorname{Supp} q \in \Delta$. In particular, $q \in -\operatorname{int} F$ for a unique face $F \in \Delta$. The set $\nabla(q)$ contains the faces containing F that are not in Δ, in other words $\nabla(q) = \{\, G \subset [d] \mid F \subset G, G \setminus F \notin \operatorname{lk}_\Delta F \,\}$. Here $\operatorname{lk}_\Delta F = \{\, G \subset [d] \mid G \cap F = \emptyset, F \cup G \in \Delta \,\}$ denotes the *link* of F in Δ. It follows that $\nabla(q)^\vee$ equals the Alexander dual of the link of F. Using (2.5), Alexander duality ([MS05, Theorem 5.6]) and Hochster's Formula ([HH11, Theorem A.7.3]), we compute

$$\begin{aligned} H^i_{\mathfrak{m}}(\mathbb{K}[Q])_q &= \tilde{H}_{d-1-i}(\nabla(q)^\vee, \mathbb{K}) \\ &= \tilde{H}^{d-|F|-(d-i)-2}(\operatorname{lk}_\Delta F, \mathbb{K}) \\ &= H^{i-1}_{\mathfrak{m}}(\mathbb{K}[\Delta])_q\,. \end{aligned}$$

Here, $|F|$ denotes the number of vertices of F, so $d - |F|$ is the cardinality of the ground set of $\operatorname{lk}_\Delta F$. In particular, it follows that $\operatorname{depth} \mathbb{K}[Q] \ge 1 + \operatorname{depth} \mathbb{K}[\Delta]$.

Now assume that Δ is acyclic. If $H^{i-1}_{\mathfrak{m}}(\mathbb{K}[\Delta]) \neq 0$, then by Hochster's formula there exists a face $F \subset [d]$, such that $\tilde{H}^{i-|F|-2}(\operatorname{lk}_\Delta F, \mathbb{K}) \neq 0$. Because Δ is acyclic, it holds that $F \neq \emptyset$. So we can find an element $q \in -\operatorname{int} F$ with odd degree. By above computation, it holds that $H^i_{\mathfrak{m}}(\mathbb{K}[Q])_q = \tilde{H}^{i-|F|-2}(\operatorname{lk}_\Delta F, \mathbb{K}) \neq 0$ and thus $\operatorname{depth} \mathbb{K}[Q] \le 1 + \operatorname{depth} \mathbb{K}[\Delta]$. $\square$

Proposition 2.4.5. *There exists a seminormal locally simplicial affine monoid Q satisfying (S_2), such that $\mathbb{K}[Q]$ is Cohen-Macaulay if and only if $\operatorname{char} \mathbb{K} \neq 2$.*

Proof. Let Δ be a simplicial complex on d vertices. We consider Δ as a subcomplex of the full simplex Γ on the d vertices. Assume that $\Delta \subsetneq \Gamma$, so Δ is in fact a subcomplex of the boundary complex of Γ. Next, we pass to the barycentric subdivision of Δ and Γ. This way, we obtain an inclusion $\operatorname{sd}(\Delta) \subset \operatorname{sd}(\Gamma)$, where $\operatorname{sd}(\Delta)$ is homeomorphic to Δ and $\operatorname{sd}(\Delta)$ is a *vertex-induced subcomplex* of $\operatorname{sd}(\Gamma)$. This means that there is a subset

V of the vertices of $\operatorname{sd}(\Gamma)$, such that $\operatorname{sd}(\Delta)$ is the restriction of $\operatorname{sd}(\Gamma)$ to V. We now consider the dual $\operatorname{sd}(\Gamma)^\vee$ of $\operatorname{sd}(\Gamma)$. We can realize $\operatorname{sd}(\Gamma)^\vee$ as the boundary complex if a (necessarily) simple polytope P. Indeed, P is just the polar of the barycentric subdivision of a simplex.

Let P' be the cone over P with apex v. Note that $\dim P' = d$. We embed P' as a lattice polytope into $\mathbb{R}^n \times \{1\} \subset \mathbb{R}^{n+1}$ and we write $C \subset \mathbb{R}^{n+1}$ for the cone generated by P' and Q_1 for the affine monoid generated by the lattice points in C. Note that we have a bijective correspondence between the vertices of $\operatorname{sd}(\Gamma)$ and the facets of Q_1 that contain v. We define $Q_2 \subset Q_1$ as the subset of all elements of even degree, and those elements q of odd degree, such that every facet containing q and v corresponds to a vertex in the set V defined above. Q_2 is seminormal, because we restricted Q_1 to subgroups in faces, and it satisfies (S_2), because we did the restriction facetwise. Let $F = Q_2 \cap \mathbb{R}_+ v$ be the one-dimensional face of Q_2 corresponding to the ray through v.

Finally, we set $Q := Q_{2,F}$. Then Q inherits seminormality and (S_2) from Q_2. Moreover, Q is simple, because its faces are in bijection with the faces of the simple polytope P and ${}^*\!\dim Q = d$. Let us compute the local cohomology of Q. Fix a $q \in -\overline{Q}$. If $q \notin \overline{Q}$, then by the proof of Lemma 2.3.18 if holds that $H^i_{\mathfrak{m}}(\mathbb{K}[Q])_q = 0$ for all $i < {}^*\!\dim Q$. On the other hand, if $q \in \overline{Q} \cap (-\overline{Q})$ and $q \notin Q$, then by construction $\nabla(q)^\vee = \operatorname{sd}(\Delta)$. Hence in this case $H^i_{\mathfrak{m}}(\mathbb{K}[Q])_q = \tilde{H}_{d-1-i}(\Delta, \mathbb{K})$. So if we choose Δ to be triangulated projective plane, then $\mathbb{K}[Q]$ is Cohen-Macaulay if and only if $\operatorname{char} \mathbb{K} \neq 2$. □

We would like to point out that instead of the construction in the preceding proof one can also consider the affine monoid M constructed in [BLR06, Theorem 7.4]. The localization of M at the vertex v (in the notation of [BLR06]) yields another affine monoid satisfying the claim of Lemma 2.4.5. We chose to present the new construction because we consider it as simpler than the one given in [BLR06].

Finally, note that the constructions given in Lemma 2.4.4 and Lemma 2.4.5 are optimal in the following sense. Every example of a monoid algebra depending on the characteristic needs to have at least *dimension 6. Since the triangulated projective plane has 6 vertices, the construction of Lemma 2.4.5 has the minimal possible *dimension. The cone of the triangulated projective plane has 7 vertices, so the *dimension of the affine monoid constructed in Lemma 2.4.4 has *dimension 7. But by Lemma 2.3.17, the *depth of $\mathbb{K}[Q]$ is determined by its 6-*dimensional localizations, so again the *dimension is minimal.

2.5. Additional results

In this last section of Chapter 2 some additional results are listed. We give a direct proof that our criterion for Serre's condition (S_2) is equivalent to the one given in [Ish88; SS90]. Further, an interpretation of the greatest *dimension of a family of holes is given. We also obtain a curious new characterization of normal affine monoids. Finally, we give a bound on the Castelnuovo-Mumford regularity of seminormal homogeneous affine monoids and prove a special case of the Eisenbud-Goto conjecture (Lemma 2.5.7).

2.5.1. Intersection of localizations

In this section, we give a direct proof that our criterion for Serre's condition (S_2) in Lemma 2.3.2 is equivalent to the classical criterion given in Lemma 1.3.20. This is the content of the next result.

Proposition 2.5.1. *Let Q be an affine monoid of *dimension d. Then Q satisfies*

$$Q = \bigcap_{F \text{ facet of } Q} Q_F \,. \tag{2.6}$$

*if and only if every family of holes of Q has *dimension $d-1$.*

This follows from the more general

Proposition 2.5.2. *For $0 \leq i \leq d-2$ it holds that*

$$\bigcap_{{}^*\dim G = i} Q_G \subseteq \bigcap_{{}^*\dim G = i+1} Q_G \,, \tag{2.7}$$

*where the intersection runs over the faces of Q of the indicated *dimension. The inclusion is strict if and only if there exists a family of holes of *dimension i.*

Proof. The inclusion is obvious, because $F \subset G$ implies $Q_F \subset Q_G$. So we need only to prove the case of equality.

If the inclusion is strict, then we can choose an element q from the difference of the right and left hand side of (2.7). For this q, there exists a i-*dimensional face F with $q \notin Q_F$, but $q \in Q_G$ for every face $G \supsetneq F$. By Lemma 2.1.1, this implies that $\mathfrak{p}_F$ is a minimal prime over the annihilator of $\mathbf{x}^q$. Thus is associated by Lemma 2.1.8.

On the other hand, assume there is an i-*dimensional associated face F. Then there exists a monomial $\mathbf{x}^q \in \mathbb{K}\{\overline{Q}\}/\mathbb{K}[Q]$ with annihilator $\mathfrak{p}_F$, in particular $q \notin Q_F$. Since $\mathfrak{p}_F \nsubseteq \mathfrak{p}_G$ for all faces G of *dimension $i+1$, it follows from Lemma 2.1.1 that $G + q \nsubseteq \overline{Q} \setminus Q$ and thus $q \in Q_G$ for all $(i+1)$-*dimensional faces G. Hence, q is contained in the right-hand side, but not in the left-hand side of (2.7). $\square$

Note that $Q = Q_{F_0} = \bigcap_{{}^*\dim F = 0} Q_F$. Therefore, Q satisfies (2.6) if and only if all associated faces are facets. Let us add some remarks here. There is a chain of inclusions

$$Q \subseteq \bigcap_{F \in \mathcal{F}_1} Q_F \subseteq \bigcap_{F \in \mathcal{F}_2} Q_F \subseteq \ldots \subseteq \bigcap_{F \in \mathcal{F}_{d-1}} Q_F \subseteq \overline{Q} \,. \tag{2.8}$$

This chain of inclusions gives rise to a similar chain on the algebra $\mathbb{K}[Q]$ and also on the quotients modulo $\mathbb{K}[Q]$. It yields a filtration of $\mathbb{K}\{\overline{Q}\}/\mathbb{K}[Q]$ that turns out to be the dimension filtration, as defined in [Sch99]. It follows that if the $\mathbb{K}[Q]$-module $\mathbb{K}\{\overline{Q}\}/\mathbb{K}[Q]$ is sequentially Cohen-Macaulay, then *depth of $\mathbb{K}\{\overline{Q}\}/\mathbb{K}[Q]$ equals the smallest non-zero component in the filtration. In view of Lemma 2.2.3, this means that the *depth of $\mathbb{K}[Q]$ is one more than the smallest *dimension of a family of holes.

2.5.2. The biggest family of holes

We have seen that the smallest *dimension of a family of holes gives an upper bound for the *depth. Moreover, in some cases, equality holds. As a supplement to this, we give an interpretation of the greatest *dimension of a family of holes.

Proposition 2.5.3. *Let Q be an non-normal affine monoid. The following numbers are equal:*

1. *The maximal *dimension of a family of holes of Q.*
2. *The minimal *dimension j, such that all localizations Q_F at faces of *dimension strictly greater than j are normal.*

If Q is homogeneous, then this number equals the degree of the difference between the Hilbert polynomials of $\mathbb{K}[\overline{Q}]$ and $\mathbb{K}[Q]$. If Q satisfies (R_1), then this number equals the maximal $i < d-1$ such that $H^{i+1}_{\mathfrak{m}}(\mathbb{K}[Q]) \neq 0$.

See [Stu96, Theorem 13.12] for a variant of this result (stated without proof).

Proof. The first claim is immediate from Lemma 2.1.10. For the statement about the Hilbert polynomials, note that the mentioned difference is just the Hilbert polynomial of $\mathbb{K}\{\overline{Q}\}/\mathbb{K}[Q]$. The statement about the local cohomology follows from Lemma 2.2.4 and Lemma 2.2.5, or more directly from the exact sequence in the proof of Lemma 2.2.3. □

2.5.3. A criterion for normality

From our proof of Lemma 2.2.2 we can extract a curious characterization of normal affine monoids.

Proposition 2.5.4. *An affine monoid Q is normal if and only if $\mathbb{K}\{\overline{Q}\}$ has finite projective dimension over $\mathbb{K}[Q]$.*

Proof. If Q is normal, then $\mathbb{K}[Q] = \mathbb{K}\{\overline{Q}\}$ is free and thus has projective dimension 0. On the other hand, assume that $\mathbb{K}\{\overline{Q}\}$ has finite projective dimension. The proof of Lemma 2.2.2 shows in particular that $\mathbb{K}\{\overline{Q}\}$ is a Cohen-Macaulay $\mathbb{K}[Q]$-module. Therefore, it holds that $^*\mathrm{depth}\,\mathbb{K}\{\overline{Q}\} = {}^*\mathrm{dim}\,\mathbb{K}\{\overline{Q}\} = {}^*\mathrm{dim}\,\mathbb{K}[Q]$. Hence, by the graded

Auslander-Buchsbaum Formula (Lemma A.1.20), we have $^*\operatorname{depth} \mathbb{K}\{\overline{Q}\} + \operatorname{pd} \mathbb{K}\{\overline{Q}\} = {}^*\operatorname{depth} \mathbb{K}[Q] \leq {}^*\dim \mathbb{K}[Q] = {}^*\operatorname{depth} \mathbb{K}\{\overline{Q}\}$. It follows that the projective dimension of $\mathbb{K}\{\overline{Q}\}$ over $\mathbb{K}[Q]$ is zero, thus $\mathbb{K}\{\overline{Q}\}$ is a projective $\mathbb{K}[Q]$-module. But $\mathbb{K}[Q]$ is a *local ring, so graded projective modules are free.

It remains to show that $\mathbb{K}\{\overline{Q}\}$ hat rank 1 over $\mathbb{K}[Q]$. Assume the contrary. Then we can find two $\mathbb{K}[Q]$-linearly independent elements $\mathbf{x}^{p_1}, \mathbf{x}^{p_2} \in \mathbb{K}\{\overline{Q}\}$. There exist an element $q \in Q$ such that $p_1 + q, p_2 + q \in Q$, cf. [BG09, Prop 2.33]. Thus $\mathbf{x}^{p_1+q}\mathbf{x}^{p_2} - \mathbf{x}^{p_2+q}\mathbf{x}^{p_1} = 0$ with $\mathbf{x}^{p_1+q}, \mathbf{x}^{p_2+q} \in \mathbb{K}[Q]$, a contradiction to our assumption. □

2.5.4. Regularity of seminormal affine monoids

Let Q be an affine monoid which is *homogeneous*, i.e. it admits a generating set such that all generatorsr are contained in a common affine hyperplane. In this case $\mathbb{K}[Q]$ carries a natural $\mathbb{Z}$-grading such that all generators are of degree 1. The *Castelnuovo-Mumford regularity* is defined as

$$\operatorname{reg} \mathbb{K}[Q] := \max \left\{ i + j \;\middle|\; H^i_{\mathfrak{m}}(\mathbb{K}[Q])_j \neq 0 \right\}$$

If $q \in Q$ is an element in the interior of Q, then it is known that $H^d_{\mathfrak{m}}(\mathbb{K}[Q])_{-q} \neq 0$. This gives us a lower bound on the regularity: If $m \in \mathbb{N}$ denotes the smallest degree of an interior element of Q, then $\operatorname{reg} \mathbb{K}[Q] \geq d - m$.

In the case that Q is seminormal, it was already noted in [HR76, Remark 5.34] that the local cohomology vanishes in positive degrees. This implies that $\operatorname{reg} \mathbb{K}[Q] \leq d$. We get a slightly stronger bound from Theorem 4.7 of [BLR06] (resp. Lemma 2.3.15), namely the local cohomology vanishes in all non-negative degrees. Thus we get the following bound on the regularity:

Proposition 2.5.5. *Let Q be a homogeneous seminormal affine monoid of dimension d. Then $\operatorname{reg} \mathbb{K}[Q] \leq d - 1$. If Q contains an element of degree 1 in its interior, then equality holds.*

This generalizes the bound for the normal case in [Stu96, Theorem 13.14] and the bound for the seminormal simplicial case in [Nit12, Theorem 3.14]. A famous open conjecture in commutative algebra is the Eisenbud-Goto conjecture:

Conjecture 2.5.6 ([EG84])**.** *Let S be the polynomial ring with the standard grading and let $I \subset S$ be a homogeneous prime ideal. Then*

$$\operatorname{reg} S/I \leq \operatorname{mult} S/I - \operatorname{codim} S/I$$

where mult *is the multiplicity and* codim *is the codimension.*

Theorem 2.5.7. *Let Q be a homogeneous affine monoid. If Q is seminormal and contains an inner point in degree 1, then Lemma 2.5.6 holds for $\mathbb{K}[Q]$.*

Proof. Let d be the dimension of Q. By the discussion above, we know that the regularity of $\mathbb{K}[Q]$ is $d-1$. We may assume that $Q \subset \mathbb{Z}^d$ and $\mathbb{Z}Q = \mathbb{Z}^d$. Let $\mathcal{P}$ be the convex hull of the elements of degree 1 of Q. So $\mathcal{P}$ is a $(d-1)$-dimensional convex polytope. The multiplicity of $\mathbb{K}[Q]$ can be computed as the normalized volume of $\mathcal{P}$ (cf. [BG09, Theorem 6.54]). Finally, the codimension of $\mathbb{K}[Q]$ is $n-d$, where n is the number of generators of Q. Since every generator of Q has degree 1, n is bounded above by the number of lattice points in $\mathcal{P}$. So the claim follows from the following geometric proposition. □

Proposition 2.5.8. *Let $\mathcal{P} \subset \mathbb{R}^{d-1}$ be a polytope with integral vertices that has a lattice point in its interior. Let N be the number of all lattice points in $\mathcal{P}$. Then the normalized volume of $\mathcal{P}$ is at least $N-1$.*

Proof. Let p be an inner lattice point in $\mathcal{P}$. By Carathéodory's Theorem, p lies in the convex hull of d other lattice points of $\mathcal{P}$. Every $(d-1)$-subset of these lattice points together with p forms an lattice simplex. Since every lattice simplex has normalized volume of at least 1, the convex hull of the d lattice points has normalized volume of at least d. Now we add the other lattice points of $\mathcal{P}$, one after the other. Every time, we get at least one new simplex in the convex hull, to the normalized volume increases by 1. If the number of lattice points in $\mathcal{P}$ is N, then the normalized volume is at least $d + (N-d-1) \cdot 1 = N-1$. □

There is another proof of Lemma 2.5.8 using the δ-vector (sometimes called h^*-vector) $\delta = (\delta_0, \ldots, \delta_{d-1})$ of the polytope, see [Hib92, p.101]. This is a vector with d non-negative integer entries which sum up to the normalized volume of $\mathcal{P}$. The last entry δ_{d-1} counts the number of interior lattice points of $\mathcal{P}$ and $\delta_1 + d$ equals the total number of lattice points in $\mathcal{P}$. If $\mathcal{P}$ has an interior lattice point, then $1 \leq \delta_1 \leq \delta_i$ for $2 \leq i \leq d-2$ (cf. [Hib92, Theorem 36.1]). From this, we compute that

$$\sum_{i=0}^{d-1} \delta_i - (\delta_1 + d - 1) \geq d - 1 + \delta_1 - (\delta_1 + d - 1) \geq 0.$$

But this is exactly the claim of Lemma 2.5.8.

The conclusion of Lemma 2.5.8 does not hold without the assumption on the existence of an inner point. For example, the triangle with vertices $(0,0),(1,0),(0,2)$ in the plane has four lattice points, but the normalized volume is only 2. So this approach cannot be used to prove Lemma 2.5.6 for more general seminormal affine monoids.

3. Polytopal affine monoids with holes deep inside

Let $P \subset \mathbb{R}^n$ be a lattice polytope. In general, it is a difficult question if the polytopal affine monoid $Q(P)$ is normal. Even though there seems to be no simple characterization of this in terms of P, some sufficient criteria are known. For example, the main result of [Gub12] is that if the edges of P are sufficiently "long", then $Q(P)$ is normal. This suggests that the normality of $Q(P)$ is somehow determined by the boundary of P. Therefore, it seems natural to ask if it is enough to consider normality "near the boundary". We measure the distance to the boundary by the lattice height over the facets of Q. For example, it is enough to consider elements of lattice height at most 1 in $\overline{Q}$ to detect families of holes of dimension $d-1$ (Lemma 1.3.19). The main result of this chapter is that this observation does not generalize to higher codimension.

Theorem 3.0.9. *For every natural number $k \in \mathbb{N}$, there exists a 3-simplex $P = P(k)$, such that the polytopal affine monoid $Q(P)$ is not normal, and every hole $q \in \overline{Q(P)} \setminus Q(P)$ has a lattice height of at least k above each facet of $Q(P)$.*

In other words, there are polytopes P such that all holes of the monoid $Q(P)$ are "deep inside". So it is not sufficient to look for holes near the boundary. Note that this result is trivial if one considers more general affine monoids that are not polytopal. One may just take a big normal polytope P and remove a point from its far interior to obtain a homogeneous affine monoid with the desired property.

3.1. Rectangular Simplices

We will construct the simplices in Lemma 3.0.9 as a special case of the *rectangular simplices* introduced in [BG99]. In this section we recall the construction. Let $d \in \mathbb{N}$ be a positive integer and set $r := d - 1$. Let $\mathbf{e}_i \in \mathbb{R}^d$ denote the i^{th} unit vector. Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r)$

be a vector of positive integers. We consider the r-simplex $\Delta = \Delta(\boldsymbol{\lambda}) \subset \mathbb{R}^d$ with vertices

$$
\begin{aligned}
\mathbf{v}_1 &:= (\lambda_1, 0, \dots, 0, 0, 1) && = \lambda_1 \mathbf{e}_1 + \mathbf{e}_d, \\
\mathbf{v}_2 &:= (0, \lambda_2, \dots, 0, 0, 1) && = \lambda_2 \mathbf{e}_2 + \mathbf{e}_d, \\
&\vdots && \vdots \\
\mathbf{v}_r &:= (0, 0, \dots, 0, \lambda_r, 1) && = \lambda_r \mathbf{e}_r + \mathbf{e}_d, \\
\mathbf{v}_d &:= (0, 0, \dots, 0, 0, 1) && = \mathbf{e}_d \,.
\end{aligned}
$$

Write $Q = Q(\boldsymbol{\lambda})$ for the polytopal affine monoid generated by the $\mathbf{v}_1, \dots, \mathbf{v}_d$. Note that $\mathbb{Z}Q = \mathbb{Z}^d$, because $\mathbf{e}_d, \mathbf{e}_1 + \mathbf{e}_d, \dots, \mathbf{e}_r + \mathbf{e}_d \in Q$ for a basis of $\mathbb{Z}^d$. There are two kinds of facets of Q:

- The coordinate hyperplanes define facets of Q. We denote the facet corresponding to the i^{th} coordinate hyperplane by F_i. The lattice height σ_i above F_i is given by the i^{th} coordinate of a point $q \in \mathbb{Z}Q$.
- There is one "skew" facet spanned by the vertices $\lambda_i \mathbf{e}_i + \mathbf{e}_d, 1 \le i \le r$. Let us denote this facet by $F_{\boldsymbol{\lambda}}$. The lattice height above this facet is given by the linear form
$$\sigma_{\boldsymbol{\lambda}}(z) := L z_d - \sum_{i=1}^{r} \frac{L}{\lambda_i} z_i \,,$$
where $L := \operatorname{lcm}(\lambda_1, \dots, \lambda_r)$ is the least common multiple of $\lambda_1, \dots, \lambda_r$.

3.2. Reduction to the skew facet

In this section, we prove the following result that allows us to restrict our attention to the facet $F_{\boldsymbol{\lambda}}$.

Proposition 3.2.1. *Let k be a positive integer. Assume that $Q(\boldsymbol{\lambda})$ is not normal and every hole has lattice height at least k above $F_{\boldsymbol{\lambda}}$. Assume further that $Q(\boldsymbol{\lambda})$ has no holes in its boundary. Then there exists a $\boldsymbol{\lambda}'$ such that $Q(\boldsymbol{\lambda}')$ is not normal and its holes have lattice height at least k above every facet.*

The idea of the proof is taken from [BG99, Theorem 1.6]. For a fixed index $1 \le i \le r$, set $\ell = \operatorname{lcm}(\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_r)$. We define $\boldsymbol{\lambda}' = (\lambda'_1, \dots, \lambda'_r)$ by

$$\lambda'_j = \begin{cases} \lambda_j & \text{if } j \neq i \,; \\ \lambda_j + \ell & \text{if } j = i \,. \end{cases}$$

Theorem 1.6 of [BG99] states that in this situation $Q(\boldsymbol{\lambda})$ is normal if and only if $Q(\boldsymbol{\lambda}')$ is normal. We modify the argument given in [BG99] to obtain the following result.

Lemma 3.2.2. *Use the notation as above. Assume that $Q(\boldsymbol{\lambda})$ has no holes in its boundary. Then there is a bijective linear map $\alpha : \mathbb{Z}^d \to \mathbb{Z}^d$, such that the preimage of every hole in $Q(\boldsymbol{\lambda}')$ is a hole in $Q(\boldsymbol{\lambda})$ (i.e. α is surjective on holes). Moreover, α strictly increases the lattice height of every hole above the facet F_i, and it preserves all other lattice heights.*

We can iterate this construction to increase the lattice height of the holes above every facet except $F_{\boldsymbol{\lambda}}$. This proves Lemma 3.2.1. The map α is taken from the proof of Theorem 1.6 in [BG99]; we give its definition below. For the proof of Lemma 3.2.2, we need the following lemma.

Lemma 3.2.3. *Let $z \in Q(\boldsymbol{\lambda})$, $\tilde{z} \in \mathbb{Z}^d$ with $0 \le \tilde{z}_i \le z_i$ for $1 \le i \le r$ and $\tilde{z}_d = z_d$. Then $\tilde{z} \in Q(\boldsymbol{\lambda})$.*

Proof. We first note that the statement holds if $z_d = 1$. This follows from the definition of the simplex $\Delta(\boldsymbol{\lambda})$. In general, z can be written as a sum of elements of degree 1. For each summand, we may decrease its components without leaving $Q(\boldsymbol{\lambda})$. This way, we obtain a representation of $\tilde{z}$ as a sum of degree 1 elements. Hence, $\tilde{z} \in Q(\boldsymbol{\lambda})$. ☐

Proof of Lemma 3.2.2. Set $L = \operatorname{lcm}(\lambda_1, \dots, \lambda_r)$ and $L' = \operatorname{lcm}(\lambda'_1, \dots, \lambda'_r)$. Recall that

$$\sigma_{\boldsymbol{\lambda}}(z) = L z_d - \sum_{i=1}^{r} \frac{L}{\lambda_i} z_i$$

and an analogous formula holds for $\boldsymbol{\lambda}'$. We consider the linear form

$$\beta(z) := \frac{\ell}{L}\left(\sigma_{\boldsymbol{\lambda}}(z) + \frac{L}{\lambda_i}\sigma_i(z)\right) = \ell z_d - \sum_{\substack{j=1\\ j\neq i}}^{r} \frac{\ell}{\lambda_j} z_j\,.$$

defined on $\mathbb{Z}^d$. Note that β takes non-negative integer values on $Q(\boldsymbol{\lambda})$. The map α mentioned above can then be defined by $\alpha(z) := z + \beta(z)\mathbf{e}_i$.

One directly verifies that $\sigma_{\boldsymbol{\lambda}'}(\alpha(z)) = \sigma_{\boldsymbol{\lambda}}(z)$ for every $z \in \mathbb{Z}^d$. It follows that α preserves the height above every facet except F_i. Since $Q(\boldsymbol{\lambda})$ has no holes in its boundary, every hole z has $\sigma_{\boldsymbol{\lambda}}(z) > 0$, so the height of $\alpha(z)$ above F_i is strictly larger than the height of z.

It remains to show that α is surjective on holes. As a preparation, we show that $\alpha(Q(\boldsymbol{\lambda})) \subset Q(\boldsymbol{\lambda}')$. We first note that it follows from the discussion above that $\alpha(\overline{Q}(\boldsymbol{\lambda})) \subset \overline{Q}(\boldsymbol{\lambda}')$. Next, consider an element $w \in Q(\boldsymbol{\lambda})$. It can be written as a sum of elements of degree 1. Since α preserves the degree, this yields a representation of its image $\alpha(w)$ as a sum of degree 1 elements of $\overline{Q}(\boldsymbol{\lambda}')$. But $Q(\boldsymbol{\lambda}')$ coincides with $\overline{Q}(\boldsymbol{\lambda}')$ in degree 1, hence $\alpha(w) \in Q(\boldsymbol{\lambda}')$.

Let $z' \in \overline{Q}(\boldsymbol{\lambda}') \setminus Q(\boldsymbol{\lambda}')$ be a hole and set $z := \alpha^{-1}(z')$. We need to show that z is a hole of $Q(\boldsymbol{\lambda})$. It is immediate that $z \notin Q(\boldsymbol{\lambda})$, because otherwise $z' = \alpha(z) \in Q(\boldsymbol{\lambda}')$. It remains to show that $z \in \overline{Q}(\boldsymbol{\lambda})$. Assume the contrary. Then $z_i < 0$, or equivalently, $z'_i < \beta(z)$. Let $\tilde{z}' := z' + (\beta(z) - z'_i)\mathbf{e}_i$. The linear form β does not depend on z_i nor on λ_i, therefore

$$\beta(z') = \beta(z) = \frac{\ell}{L'}\left(\sigma_{\boldsymbol{\lambda}'}(z') + \frac{L'}{\lambda'_i}\sigma_i(z')\right)$$

Using this, we compute

$$\begin{aligned}\sigma_{\boldsymbol{\lambda}'}(\tilde{z}') &= \sigma_{\boldsymbol{\lambda}'}(z') + \frac{L'}{\lambda'_i}(z'_i - \beta(z')) \\ &= \left(\frac{L'}{\ell} - \frac{L'}{\lambda'_i}\right)\beta(z') \\ &\geq 0\end{aligned}$$

Here we used that $\lambda'_i = \lambda_i + \ell > \ell$. It follows that $\tilde{z}' \in \overline{Q}(\boldsymbol{\lambda}')$.

Set $\tilde{z} := \alpha^{-1}(\tilde{z}')$. By construction, $\tilde{z}_i = 0$ and $\tilde{z} \in \overline{Q}(\boldsymbol{\lambda})$. But by assumption $Q(\boldsymbol{\lambda})$ has no holes in its boundary, thus $\tilde{z} \in Q(\boldsymbol{\lambda})$. It follows that $\tilde{z}' = \alpha(\tilde{z}) \in \alpha(Q(\boldsymbol{\lambda})) \subset Q(\boldsymbol{\lambda}')$. Now Lemma 3.2.3 implies that $z' \in Q(\boldsymbol{\lambda}')$, a contradiction. □

3.3. Good triples

In this section, we present our choice of the parameters $\boldsymbol{\lambda}$. First, we show that for 3-dimensional rectangular simplices the hypothesis of Lemma 3.2.1 is always satisfied.

Lemma 3.3.1. *A 3-dimensional rectangular simplex $Q(\lambda_1, \lambda_2, \lambda_3)$ has no holes in its boundary.*

Proof. The facets are 2-dimensional polytopal affine monoids. Thus, they are normal and even integrally closed in the ambient lattice $\mathbb{Z}^4$ (cf. [BG09, Corollary 2.54]). Hence, $Q(\lambda_1, \lambda_2, \lambda_3)$ has no holes in its boundary. □

It is now sufficient to find $(\lambda_1, \lambda_2, \lambda_3)$ such that the distance of the holes to the facet $F_{\boldsymbol{\lambda}}$ is bounded below. This is achieved with the following class of triples.

Definition 3.3.2. Let $\lambda_1 \leq \lambda_2 \leq \lambda_3$ be positive integers and let $\boldsymbol{\delta} := (-1, 2, -1, 0) \in \mathbb{Z}^4$. We call $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ a *good triple* if the following conditions are met:

1. λ_1, λ_2 and λ_3 are pairwise coprime;
2. $\sigma_{\boldsymbol{\lambda}}(\boldsymbol{\delta}) = 2$, i.e. $\lambda_2\lambda_3 - 2\lambda_1\lambda_3 + \lambda_1\lambda_2 = 2$;
3. $\lambda_1 + 2 < \lambda_2$.

The following can be verified directly.

Proposition 3.3.3. *Let $\lambda_1 \geq 5$ be an odd positive integer. Then $(\lambda_1, 2\lambda_1 - 1, 2\lambda_1^2 - \lambda_1 - 2)$ is a good triple.*

Next, we show that good triples yield examples of simplices satisfying our needs. So the next proposition completes the proof of Lemma 3.0.9.

Proposition 3.3.4. *Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ be a good triple. Then $Q(\boldsymbol{\lambda})$ is not normal and every hole has lattice height at least $\lambda_1 + 2$ over $F_{\boldsymbol{\lambda}}$.*

We prepare two lemmata before we prove this proposition.

Lemma 3.3.5. *Let $\lambda_1, \ldots, \lambda_r$ be pairwise coprime. For every positive integer $s > 0$, there exists at most one element $\mathbf{q} \in \overline{Q}(\boldsymbol{\lambda})$ with $\sigma_{\boldsymbol{\lambda}}(\mathbf{q}) = s$ and $\sigma_i(\mathbf{q}) < \lambda_i$ for every $1 \leq i \leq n$.*

Proof. This follows easily from the observation that $\ker \sigma_{\boldsymbol{\lambda}}$ is generated as a group by $\mathbf{v}_1, \ldots, \mathbf{v}_r$. □

We note that the proof of Lemma 3.3.5 is inspired by the proof of Proposition 1.3 in [BG99].

Lemma 3.3.6. *Let $\lambda_1, \ldots, \lambda_r$ be pairwise coprime and let s be a positive integer. If for every positive integer $t \leq s$, there exists an element $\mathbf{p}_t \in Q(\boldsymbol{\lambda})$ with $\sigma_{\boldsymbol{\lambda}}(\mathbf{p}_t) = t$, then every hole $\mathbf{q} \in \overline{Q}(\boldsymbol{\lambda}) \setminus Q(\boldsymbol{\lambda})$ has $\sigma_{\boldsymbol{\lambda}}(\mathbf{q}) > s$.*

Proof. We may assume that $\sigma_i(\mathbf{p}_t) < \lambda_i$ and $\sigma_i(\mathbf{q}) < \lambda_i$ for every t and every i, because otherwise we can replace $\mathbf{p}_t$ by $\mathbf{p}_t - \mathbf{v}_i$ respectively $\mathbf{q}$ by $\mathbf{q} - \mathbf{v}_i$ Now the claim is immediate from the preceding Lemma 3.3.5. □

Proof of Lemma 3.3.4. First, we show that both λ_1 and λ_3 are odd. For this assume to the contrary that $\lambda_1 = 2\lambda_1'$ for an integer λ_1'. Then $\lambda_2\lambda_3 = 2(1 + 2\lambda_1'\lambda_3 - \lambda_1'\lambda_2)$, thus either λ_2 or λ_3 are even, violating the coprimeness assumption. The proof that λ_3 is odd is analogous.

Next, consider the vector

$$\begin{aligned}\mathbf{p} &:= \frac{1}{2}(\mathbf{v}_1 + \mathbf{v}_3 + \boldsymbol{\delta}) \\ &= \left(\frac{\lambda_1 - 1}{2}, 1, \frac{\lambda_3 - 1}{2}, 1\right).\end{aligned}$$

It is easy to see that $\mathbf{p} \in Q(\boldsymbol{\lambda})$ and that $\sigma_{\boldsymbol{\lambda}}(\mathbf{p}) = 1$. For $0 \leq k \leq \frac{\lambda_1 - 1}{2}$ it holds that $\mathbf{p} + k\boldsymbol{\delta} \in Q(\boldsymbol{\lambda})$ and $\sigma_{\boldsymbol{\lambda}}(\mathbf{p} + k\boldsymbol{\delta}) = 1 + 2k$. Moreover, $2\mathbf{p} + k\boldsymbol{\delta} \in Q(\boldsymbol{\lambda})$ and $\sigma_{\boldsymbol{\lambda}}(2\mathbf{p} + k\boldsymbol{\delta}) = 2 + 2k$. Thus, by Lemma 3.3.6, there exists no hole with lattice height less than $\lambda_1 + 2$ above $F_{\boldsymbol{\lambda}}$.

Let

$$\begin{aligned}\mathbf{q} &:= \mathbf{p} + \left(\frac{\lambda_1 - 1}{2} + 1\right)\boldsymbol{\delta} + \mathbf{v}_1 \\ &= \left(\lambda_1 - 1,\, \lambda_1 + 2,\, \frac{\lambda_3 - \lambda_1}{2} - 1,\, 2\right).\end{aligned}$$

The components of $\mathbf{q}$ are non-negative and $\sigma_{\boldsymbol{\lambda}}(\mathbf{q}) = \lambda_1 + 2$, hence $\mathbf{q} \in \overline{Q}(\boldsymbol{\lambda})$. We claim that $\mathbf{q} \notin Q(\boldsymbol{\lambda})$. This clearly implies that $Q(\boldsymbol{\lambda})$ is not normal. So assume that $\mathbf{q} = \mathbf{q}_1 + \mathbf{q}_2$ for $\mathbf{q}_1, \mathbf{q}_2 \in Q(\boldsymbol{\lambda})$. Since λ_1, λ_2 and λ_3 are pairwise coprime, the only elements of $Q(\boldsymbol{\lambda})$ in $F_{\boldsymbol{\lambda}}$ of degree 1 are $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$. But $\lambda_1 - 1 < \lambda_1$, $\lambda_1 + 2 < \lambda_2$ (by assumption) and $\frac{\lambda_3 - \lambda_1}{2} - 1 < \lambda_3$, so $\mathbf{q} - \mathbf{v}_i$ has a negative component. It follows that $\sigma_{\boldsymbol{\lambda}}(\mathbf{q}_1), \sigma_{\boldsymbol{\lambda}}(\mathbf{q}_2) > 0$. Since $\sigma_{\boldsymbol{\lambda}}(\mathbf{q}) = \lambda_1 + 2$ is odd, one of $\sigma_{\boldsymbol{\lambda}}(\mathbf{q}_1)$ and $\sigma_{\boldsymbol{\lambda}}(\mathbf{q}_2)$ is odd. Without loss of generality assume $\sigma_{\boldsymbol{\lambda}}(\mathbf{q}_1)$. By Lemma 3.3.5, all elements $\mathbf{v}$ of $Q(\boldsymbol{\lambda})$ of degree 1 with $1 \le \sigma_{\boldsymbol{\lambda}}(\mathbf{v}) \le \lambda_1$ and $\sigma_{\boldsymbol{\lambda}}(\mathbf{v})$ odd are of the form $\mathbf{p} + k\boldsymbol{\delta}$ for $0 \le k \le \frac{\lambda_1 - 1}{2}$. But $\mathbf{q} - (\mathbf{p} + k\boldsymbol{\delta}) = \mathbf{v}_1 + \left(\frac{\lambda_1 - 1}{2} + 1 - k\right)\boldsymbol{\delta}$ has a negative third component. Thus $\mathbf{q}$ cannot be written as a sum of elements of degree 1 in $Q(\boldsymbol{\lambda})$. $\square$

Part III.

Special affine monoids

4. Toric edge rings

In this chapter, we consider a special class of affine monoids, the *edge monoids* of graphs. Let us first fix some notation. A *graph* G is a pair (V, E) consisting of a finite set V, the *vertices* of G and a set $E \subset \binom{V}{2}$, the *edges* of G. We write $E(G)$ and $V(G)$ for the sets of edges and vertices of G. Note that our graphs are finite, undirected, and have neither multiple edges nor loops. For two vertices $v, w \in V$, we set $vw := \{ v, w \}$. For a non-empty subset $U \subset V$ we denote by G_U the subgraph induced by G on U.

Definition 4.0.7. Let G be a graph on the vertex set $[d] = \{ 1, \dots, d \}$ and let $\mathbf{e}_1, \dots, \mathbf{e}_d$ denote the unit vectors of $\mathbb{R}^d$. The *edge monoid* $Q(G) \subset \mathbb{Z}^d$ is the affine monoid generated by the vectors $\rho(vw) := \mathbf{e}_v + \mathbf{e}_w$ for $vw \in E(G)$.

This is a polytopal affine monoid and its polytope is called the *edge polytope* $\mathcal{P}_G \subset \mathbb{R}^d$; it is the convex hull of the set $\{ \rho(vw) \mid vw \in E(G) \}$. The *toric edge ring* of G is the affine monoid algebra $\mathbb{K}[G] = \mathbb{K}[Q(G)]$ of $Q(G)$. Note that toric edge rings are exactly the affine monoid algebras that are generated by squarefree monomials of degree 2.

Toric edge rings were first considered in [SVV94] in connection with the Rees algebra of the more familiar edge ideals. Later, they were studied systematically in [OH98] and in [OH99b]. In [OH98] and independently in [SVV98] a characterization of the normality of $\mathbb{K}[G]$ was given, which we state in Lemma 4.1.4 below. In several subsequent publications [OH99a; OH99b; OH08], various algebraic properties of $\mathbb{K}[G]$ are characterized in terms of the graph G.

The outline of this chapter is as follows. In Section 4.1, we collect some basic facts about edge monoids, namely we describe the group $\mathbb{Z}Q(G)$, the facets of $Q(G)$ and the minimal generators of the normalization of $Q(G)$. In Section 4.2 we characterize for which graphs G the edge monoid $Q(G)$ satisfies Serre's condition (R_1). The results of that section are published in [HK12]. In Section 4.3, we give a necessary condition on G for $Q(G)$ to be seminormal. This yields a partial result on a conjectured lower bound for the depth of $\mathbb{K}[G]$ from [Hib+11].

4.1. General facts about toric edge rings

In this section, we collect some general facts about toric edge rings. If G is not connected, then $Q(G)$ decomposes as a direct sum of the edge monoids of the connected components. Therefore, we only consider the case that G is connected. Let us introduce some notation.

An *cycle* C in G is a sequence of pairwise distinct vertices $C = (v_1, \dots, v_m)$ of G, such that $v_i v_{i+1} \in E(G)$ for $1 \le i \le m-1$ and $v_m v_1 \in E(G)$. We call a cycle *minimal* if it has no chord, that is $v_i v_j \in E(G)$ only if $j = i+1$ or $\{ i, j \} = \{ 1, m \}$. We sometimes identify a cycle with its set of vertices, to allow to write $v \in C$. A cycle is called *odd* if is has an odd number of vertices. A graph G is called *bipartite* of its set of vertices can be partitioned into to disjoint subsets $V(G) = V_1 \cup V_2$ such that every edge of G has one endpoint in each subset. It is well-known that a graph is bipartite if and only if it contains no odd cycle.

4.1.1. Group and dimension

Lemma 4.1.1. *Let G be a connected graph on the vertex set $[d]$.*

- *If G is bipartite, then let $[d] = V_1 \cup V_2$ be the corresponding partition of $[d]$. Then $\mathbb{Z}Q(G) \subseteq \mathbb{Z}^d$ is the set of all integer vectors $q = (q_1, \dots, q_d)$ with $\sum_{i \in V_1} q_i = \sum_{i \in V_2} q_i$.*
- *If G is not bipartite, then $\mathbb{Z}Q(G)$ is the set of all integer vectors in $\mathbb{Z}^d$ with an even coordinate sum.*

This was remarked on page 426 of [OH98]. However, the distinction between the bipatite and the nonbipartite case was not made in that paper, so we give a proof for completeness.

Proof. Choose a vertex v and a spanning tree G' of G. To a vector $q \in \mathbb{Z}^d$ we can add and subtract edge vectors $\mathbf{e}_v + \mathbf{e}_w, vw \in E(G)$ of G' to obtain an element q' whose only non-zero component is on the vertex v. Clearly $q \in \mathbb{Z}Q(G)$ if and only if $q' \in \mathbb{Z}Q(G)$.

Now assume that G is bipartite and let $[d] = V_1 \cup V_2$ be the corresponding partition of $[d]$. For $p \in \mathbb{Z}^d$ we write $\tau(p) := \sum_{i \in V_1} p_i - \sum_{i \in V_2} p_i$. Every edge of G has one endpoint in each set, hence every $q \in \mathbb{Z}Q(G)$ satisfies $\tau(q) = 0$. On the other hand, if $\tau(q) = 0$ for a $q \in \mathbb{Z}^d$, then also $\tau(q') = 0$. But this implies that $q' = 0$ and hence $q \in \mathbb{Z}Q(G)$.

Next, we assume that G has an odd cycle. Assume further that the vertex v is contained in this cycle. Since every generator of $Q(G)$ has an even coordinate sum, this holds also for every element of $\mathbb{Z}Q(G)$. On the other hand, $q \in \mathbb{Z}Q$ has an even coordinate sum if and only if q' is an even multiple of $\mathbf{e}_v$. So it remains to prove that $2\mathbf{e}_v \in \mathbb{Z}Q(G)$. For this, assume the edges $e_1, \dots, e_\ell$ form an odd cycle of G and let v be the common vertex of e_1 and e_ℓ. Then

$$2\mathbf{e}_v = \sum_{j=1}^{\ell} (-1)^{j+1} \rho(e_j) \in \mathbb{Z}Q(G)\,.$$

$\square$

Since the rank of $\mathbb{Z}Q(G)$ equals the dimension of $Q(G)$, we immediately get a formula for the later.

Corollary 4.1.2 (Proposition 1.3, [OH98]). *Let G be a connected graph on d vertices. Then*

$$^{*}\dim Q(G) = \begin{cases} d-1 & \textit{if } G \textit{ is bipartite;} \\ d & \textit{if } G \textit{ is not bipartite.} \end{cases}$$

4.1.2. Facets

Next, we recall a description of the facets of $Q(G)$. We need this only for nonbipartite graphs, so we restrict to this case. The bipartite case is similar and can be found in [OH98, Theorem 1.7].

To every vertex i we associate the linear form $\sigma_i : \mathbb{R}^d \to \mathbb{R}$ which projects onto the i^{th} component. Moreover, we set $\mathcal{H}_i = \left\{ x \in \mathbb{R}^d \,:\, \sigma_i(x) = 0 \right\}$ and $\mathcal{F}_i = Q(G) \cap \mathcal{H}_i$. A nonempty subset $T \subset [d]$ is called *independent* if $ij \notin E(G)$ for all $i, j \in T$ with $i \neq j$. If T is an independent set, then we write $N(G;T)$ for the set of vertices $j \in [d]$ with $ij \in E(G)$ for some $i \in T$. To every independent set T we associate the linear form

$$\sigma_T : \mathbb{R}^d \ni (x_1, \ldots, x_d) \mapsto \sum_{j \in N(G;T)} x_j - \sum_{i \in T} x_i$$

and we set $\mathcal{H}_T = \left\{ x \in \mathbb{R}^d \,:\, \sigma_T(x) = 0 \right\}$ and $\mathcal{F}_T = Q(G) \cap \mathcal{H}_T$. Clearly, every element $q \in \overline{Q(G)}$ satisfies $\sigma_i(q) \geq 0$ and $\sigma_T(q) \geq 0$ for every vertex i and every independent set T. Not all these inequalities define a facet, so we need additional conditions.

If T is independent, then the *bipartite graph induced by* T is defined to be the bipartite graph having the vertex set $\tilde{T} := T \cup N(G;T)$ and consisting of all edges $ij \in E(G)$ with $i \in T$ and $j \in N(G;T)$. We say that an independent subset $T \subset [d]$ is *fundamental* if

- the bipartite graph induced by T is connected;
- either $\tilde{T} = [d]$ or every connected component of the induced subgraph $G_{[d] \setminus \tilde{T}}$ has at least one odd cycle.

Moreover, we call a vertex $i \in [d]$ *regular* if every connected component of $G_{[d] \setminus i}$ has at least one odd cycle. Note that a regular vertex is not the same as a fundamental set with one element.

Proposition 4.1.3 (Theorem 1.7, [OH98]). *Let G be a connected graph which has an odd cycle. Then the facets of $Q(G)$ are exactly the sets $\mathcal{F}_i$ and $\mathcal{F}_T$ for all regular vertices i and all fundamental sets T.*

4.1.3. Normalization

If C and C' are vertex-disjoint cycles, then the we call an edge $vw \in E(G)$ with $v \in C$ and $w \in C'$ a *bridge* between C and C'. Two vertex disjoint minimal odd cycles C, C' are called *exceptional* if there is no bridge between them.

Theorem 4.1.4 ([OH98; SVV98])**.** *Let G be a connected graph. The edge monoid $Q(G)$ is normal if and only if G has no pair of exceptional cycles.*

This condition is called the *odd cycle condition* in [FHM65]. Note that this condition is always satisfied for bipartite graphs. This is the reason why we restrict to graphs containing an odd cycle.

To every odd cycle C, we associate the vector $\mathbf{e}_C := \sum_{v \in C} \mathbf{e}_v$. It was shown in [OH99b, Theorem 2.2] that $\overline{Q(G)}$ is generated as a $Q(G)$-module by $0 \in Q(G)$ and by all vectors of the form $\sum_i (\mathbf{e}_{C_i} + \mathbf{e}_{C_i'})$, where C_i, C_i' are two exceptional odd cycles for every i, and all cycles in the sum are disjoint. However, this set of generators is not minimal. We give a minimal generating set in the next proposition.

Proposition 4.1.5. *$\overline{Q(G)}$ is minimally generated as an $Q(G)$-module by $0 \in Q(G)$ and the vectors $\sum_{C \in \mathcal{C}} \mathbf{e}_C$, where $\mathcal{C}$ runs over all even collections of pairwise exceptional odd cycles.*

Proof. First note that the vectors $\sum_{C \in \mathcal{C}} \mathbf{e}_C$ are contained in $\overline{Q(G)}$, because they can be written as a sum of vectors of the form $\mathbf{e}_C + \mathbf{e}_{C'}$. Next, we show that they form a generating set. Let $q \in \overline{Q(G)}$. By [OH99b, Theorem 2.2], we can write $q = q_1 + p$ with $p \in Q(G)$ and q_1 is of the form $\sum_i (\mathbf{e}_{C_i} + \mathbf{e}_{C_i'})$, as described above. Assume there is a bridge between two of the cycles, say between C and C'. Then $\mathbf{e}_C + \mathbf{e}_{C'} \in Q(G)$, because we can write it as a sum of edges. So we consider $q_2 := q_1 - \mathbf{e}_C - \mathbf{e}_{C'}$. This vector is in $\overline{Q(G)}$, because it is still a sum of an even number of cycles. If we iterate this reduction as long as possible, then we either get 0 or a vector $\sum_{C \in \mathcal{C}} \mathbf{e}_C$, where the occurring cycles are pairwise exceptional.

It remains to show that this set of generators is minimal. So assume that one generator $q = \sum_{C \in \mathcal{C}} \mathbf{e}_C$ can be omitted. That is, there exist an edge $e \in E(G)$, such that $q' := q - \rho(\mathbf{e}_e) \in \overline{Q(G)}$. The two vertices of e are neighbours in one of the cycles in $\mathcal{C}$, say in C. So we can write $C = (v_1, \dots, v_m)$ and $e = \{ v_{m-1}, v_m \}$. The vertices $v_1, v_3, \dots, v_{q-2}$ form an independent set in G, because C has no chord. The sum over the corresponding components of q' is $(m-1)/2$. The sum over the set of neighbours of this set is $(m-1)/2 - 1$, because the only neighbours that correspond to non-zero components in q' are the vertices $v_2, v_4, \dots, v_{m-3}$. But this contradicts the fact that the sum over an independent set never exceeds the sum over its set of neighbours. $\square$

4.2. Serre's condition (R_1) for toric edge rings

The main result of this section of the following characterization of Serre's condition (R_1).

Theorem 4.2.1 ([HK12])**.** *Let G be a connected nonbipartite graph on $[d]$. Then the edge ring $\mathbb{K}[G]$ of G satisfies Serre's condition (R_1) if and only if the following conditions are satisfied:*

(i) *For every regular vertex* $i \in [d]$, *the induced subgraph* $G_{[d]\setminus i}$ *is connected;*

(ii) *For every fundamental set* $T \subset [d]$, *one has either* $\tilde{T} = [d]$ *or the induced subgraph* $G_{[d]\setminus \tilde{T}}$ *is connected.*

We use the criterion of Lemma 1.3.19 to prove Lemma 4.2.1. Recall that by Lemma 1.3.19 $Q(G)$ satisfies (R_1) if and only every facet $\mathcal{F}$ of $Q(G)$ satisfies the following two conditions:

(A) There exists an $x \in Q(G)$ such that $\sigma_{\mathcal{F}}(x) = 1$.

(B) $\mathbb{Z}\mathcal{F} = \mathbb{Z}Q(G) \cap \mathcal{H}$, where $\mathcal{H}$ is the supporting hyperplane of $\mathcal{F}$.

In the sequel, let G be a connected nonbipartite graph on $[d]$. We start with condition (A).

Lemma 4.2.2. *Every facet of* $Q(G)$ *satisfies condition (A).*

Proof. First, let $i \in [d]$ be a regular vertex. Since G is connected, there exists an edge $ij \in E(G)$ to another vertex j. Then $\sigma_i(\rho(ij)) = 1$.

Second, let $T \subset [d]$ be a fundamental set. If $\tilde{T} \subsetneq [d]$, then there exists an edge $ij \in E(G)$ such that $i \in N(G;T)$ and $j \in [d] \setminus \tilde{T}$. This edge satisfies $\sigma_T(\rho(ij)) = 1$. If instead $\tilde{T} = [d]$, then every edge of G has either both endpoints in $N(G;T)$, or one in $N(G;T)$ and one in T. Hence $\sigma_T(e) \in \{0, 2\}$ for every edge e of G. In this case $\frac{1}{2}\sigma_T$ satisfies condition (A). □

Next, we consider condition (B) for facets coming from regular vertices.

Lemma 4.2.3. *Let* $i \in [d]$ *be a regular vertex of* G. *Then* $\mathcal{F}_i$ *satisfies condition (B) if and only if* $G_{[d]\setminus i}$ *is connected.*

Proof. Recall that $\mathcal{F}_i = Q(G_{[d]\setminus i})$. We denote the connected components of $G_{[d]\setminus i}$ by G'_j, $j = 1, \ldots, l$. Then it is easy to see that $\mathcal{F}_i = \bigoplus_j Q(G'_j)$ and hence $\mathbb{Z}\mathcal{F}_i = \bigoplus_j \mathbb{Z}Q(G'_j)$. Since every G'_j is connected and contains an odd cycle, we can apply Lemma 4.1.1 to describe $\mathbb{Z}Q(G'_j)$. If $G_{[d]\setminus i}$ is connected, then $\mathbb{Z}\mathcal{F}_i$ and $\mathbb{Z}Q(G) \cap \mathcal{H}_i$ are both the set of integer vectors in $\mathbb{Z}^d$ with even coordinate sum and i^{th} coordinate equal to zero, thus these sets coincide.

We now consider the case that $G_{[d]\setminus i}$ has at least two different connected components G'_1, G'_2. Then we can choose a vector $x \in \mathbb{Z}^d$ such that (i) its coordinate sum is even, (ii) $\sigma_i(x) = 0$, and (iii) the restricted coordinate sum over the vertices in G'_1 is odd. This x is contained in $\mathbb{Z}Q(G) \cap \mathcal{H}_i$, but not in $\mathbb{Z}\mathcal{F}_i$, thus $\mathcal{F}_i$ violates the condition. □

Finally, we consider condition (B) for those faces that come from fundamental sets.

Lemma 4.2.4. *Let* $T \subset [d]$ *be a fundamental set of* G. *Then* $\mathcal{F}_T$ *satisfies condition (B) if and only if one has either* $\tilde{T} = [d]$ *or the induced subgraph* $G_{[d]\setminus \tilde{T}}$ *is connected.*

Proof. Again, we denote the connected components of $G_{[d]\setminus\tilde{T}}$ by G'_j, $j = 1, \ldots, l$. We claim that

$$\mathbb{Z}\mathcal{F}_T = \bigoplus_j \mathbb{Z}Q(G'_j) \oplus \left\{ x \in \mathbb{Z}^d \;\middle|\; \operatorname{Supp}(x) \subset \tilde{T}, \sigma_T(x) = 0 \right\}. \tag{4.1}$$

Here, Supp(.) denotes the support of a vector. The sum is direct, because the supports of the summands are disjoint. $\mathcal{F}_T$ (and thus $\mathbb{Z}\mathcal{F}_T$) is generated by the set $\{ \rho(e) \mid e \in E(G), \sigma_T(\rho(e)) = 0 \}$. For an edge $e \in E(G)$, it holds that $\sigma_T(\rho(e)) = 0$ if and only if either both endpoints lie in $\tilde{T}$ or both are not contained in this set. Thus, a set of generators for the left side of (4.1) is contained in the right side of the equation, and hence one inclusion follows. Furthermore $\bigoplus_j \mathbb{Z}Q(G'_j) \subset \mathbb{Z}\mathcal{F}_T$. Thus it remains to show that

$$\left\{ x \in \mathbb{Z}^d \;\middle|\; \operatorname{Supp}(x) \subset \tilde{T}, \sigma_T(x) = 0 \right\} \subset \mathbb{Z}\mathcal{F}_T.$$

For this we consider a spanning tree of the induced bipartite graph on $\tilde{T}$. Its edges form a $\mathbb{Z}$-basis for the left set, hence it is contained in $\mathbb{Z}\mathcal{F}_T$. Finally, we note that

$$\begin{aligned} \mathbb{Z}Q(G) \cap \mathcal{H}_T = &\left\{ x \in \mathbb{Z}^d \;\middle|\; \operatorname{Supp}(x) \cap \tilde{T} = \emptyset,\ \sum x_i \text{ even} \right\} \\ &\oplus \left\{ x \in \mathbb{Z}^d \;\middle|\; \operatorname{Supp}(x) \subset \tilde{T},\ \sigma_T(x) = 0 \right\}. \end{aligned}$$

Now the reasoning is completely analogous to the proof of Lemma 4.2.3. □

4.3. Seminormality

In this section we show the following necessary criterion for an edge monoid to be seminormal.

Theorem 4.3.1. *Let G be a graph that does not contain three pairwise exceptional cycles. Then the edge semigroup $Q(G)$ is seminormal.*

In [Hib+11, Conjecture 0.1] it was conjectured that the depth of $\mathbb{K}[G]$ is at least 7, provided the obviously necessary condition that $d \geq 7$. We obtain a special case of this conjecture as a corollary of above result.

Corollary 4.3.2. *Let G be a graph that does not contain three pairwise exceptional cycles. Moreover, assume that $\dim \mathbb{K}[G] \geq 7$. Then the depth of $\mathbb{K}[G]$ is at least 7.*

Proof. For an affine monoid Q we let $n(Q)$ be the maximal integer i, such that all faces of *dimension at most i are normal. If Q is positive and seminormal, then by [BLR06, Theorem 5.3] it holds that depth $\mathbb{K}[Q] \geq \min\{ n(Q) + 1, \dim Q \}$.

If G does not contain three pairwise exceptional cycles, then $Q(G)$ is seminormal by Lemma 4.3.1. Moreover, $Q(G)$ is positive, so we can apply [BLR06, Theorem 5.3]. Note that every face F of $Q(G)$ is again an edge monoid by the form of its generators. It follows from Lemma 4.1.4 that $\mathbb{K}[G]$ is normal if G has 6 or less vertices. Hence $n(Q(G))$ is at least 6 by Lemma 4.1.2, and the claim follows. $\square$

The remainder of this section is devoted to the proof of Lemma 4.3.1. For this, we study the structure of the holes $\overline{Q(G)} \setminus Q(G)$ in the edge monoid. In the sequel, let G be a connected graph. We do not (yet) assume that G contains no three pairwise exceptional cycles. Let us introduce some notation. A *walk* in G is a sequence

$$\Gamma := (\{ v_1, v_2 \}, \{ v_2, v_3 \}, \dots, \{ v_m, v_{m+1} \})$$

of edges $\{ v_i, v_{i+1} \} \in E(G)$ of G. We call Γ *closed* if $v_1 = v_{m+1}$, and we call Γ *even* or *odd*, if m is even or odd. Note that the vertices are labelled according to their position in Γ, so we allow that the same vertex appears several times in the same walk. If $\Gamma = (e_1, \dots, e_m)$ is a walk, then we call the sequence $\Gamma^{(o)} := (e_1, e_3, \dots, e_{2i-1}, \dots)$ the *odd part* of Γ, and $\Gamma^{(e)} := (e_2, e_4, \dots, e_{2i}, \dots)$ the *even part*. A *parity-respecting subwalk* Γ' of Γ is a walk, such that

- either the even part of Γ' is a subset of the even part of Γ and the odd part of Γ' is a subset of the odd part of Γ,
- or the even part of Γ' is a subset of the odd part of Γ and the odd part of Γ' is a subset of the even part of Γ.

We do not require the subsets to respect the induced ordering. We will write pr-subwalk for a parity-respecting subwalk.

Example 4.3.3. Let $\Gamma = (e_1, \dots, e_m)$ is a walk in G. Then all subwalks obtained by skipping some edges in the beginning or in the end of Γ are parity-preserving.

Example 4.3.4. Let $\Gamma = (\{ v_1, v_2 \}, \{ v_2, v_3 \}, \dots, \{ v_m, v_{m+1} \})$ be a walk that visits the same vertex twice, say $v_i = v_j$ with $i < j$. Then we can consider the subwalk Γ' defined by skipping all edges between v_i and v_j, i.e.

$$\Gamma' = (\{ v_1, v_2 \}, \dots, \{ v_{i-1}, v_i \}, \{ v_j, v_{j+1} \}, \dots, \{ v_m, v_{m+1} \}) .$$

if $j - i$ is even, then the subwalk Γ' is parity-respecting.

Let C and C' be two exceptional odd cycles and let $q := \mathbf{e}_C + \mathbf{e}_{C'}$. For every element $f \in Q(G)$, it holds that $q + f \in \overline{Q(G)}$. In the next lemma, we classify those $f \in Q(G)$, such that $q + f \in Q(G)$. This is the technical heart of our proof of Lemma 4.3.1. However, it is more convenient to consider f as a sum of edges. More precisely, let $\mathbb{N}_0^E$ denote the set of functions $f : E \to \mathbb{N}_0$. Then

$$\rho(f) = \sum_{e \in E} f(e) \rho(e) \in Q(G)$$

is the linear extension of ρ to $\mathbb{N}_0^E$. We say that a walk Γ in G *connects* C and C', if its first vertex is in C, its last vertex is in C', and all other vertices are neither in C nor in C'. To such a walk, we associate the vectors

$$\mathbf{e}_{\Gamma,e} := \sum_{e\in\Gamma^{(e)}} \mathbf{e}_e \text{ and}$$
$$\mathbf{e}_{\Gamma,o} := \sum_{e\in\Gamma^{(o)}} \mathbf{e}_e$$

in $\mathbb{N}_0^E$. Moreover, let "$\leq$" denote the natural partial order on $\mathbb{N}_0^E$ by componentwise comparison.

Lemma 4.3.5. *Let C and C' be two exceptional odd cycles, $q := \mathbf{e}_C + \mathbf{e}_{C'}$ and $f \in \mathbb{N}_0^E$. It holds that $q + \rho(f) \in Q(G)$ if and only if there exists a walk Γ in G connecting C and C', such that either $\mathbf{e}_{\Gamma,e} \leq f$, or $\mathbf{e}_{\Gamma,o} \leq f$.*

Proof. The sufficiency is simple. Write $f = \mathbf{e}_{\Gamma,e} + f'$ resp. $f = \mathbf{e}_{\Gamma,o} + f'$ for some $f' \in \mathbb{N}_0^E$. Then $q + \rho(\mathbf{e}_{\Gamma,e})$ resp. $q + \rho(\mathbf{e}_{\Gamma,o})$ can be written as a sum of edges of G. It follows $q + \rho(f) \in Q(G)$.

Now we prove the necessity. Pick any two vertices $v_1 \in C, v_2 \in C'$. Consider the graph G' which is constructed from G by adding the edge $b := \{ v_1, v_2 \}$. Then $q \in Q(G')$. Indeed, it can be written as a sum of edges in the cycles C, C' and the bridge b. So there is a vector $q' \in \mathbb{N}_0^{E\cup\{b\}}$ such that $\rho(q') = q$.

On the other hand, $q + \rho(f) \in Q(G)$, so it can be written as a sum of edges of G, i.e. not using b. Hence there exists another vector $q'' \in \mathbb{N}_0^E$ with $\rho(q'') = q$. So we have $\rho(q') = \rho(q'')$, but $q' \neq q''$, the later because $\mathbf{e}_b \leq q'$. In this situation, by the proof of [OH99b, Lemma 1.1], we can find an even closed walk $\Gamma' = (e_1, \dots, e_l)$ in G', such that $e_1 = b$, and $\mathbf{e}_{\Gamma',e} \leq q''$. Since q'' does not contain b, we are almost done. However, it may happen that Γ' contains edges from the cycles. To avoid this we follow the walk from v_1 to v_2. Set v_1' to be the last vertex of C that appears in Γ'. Further, set v_2' to be the first vertex of C' we encounter after we have passed v_1'. Now the pr-subwalk of Γ' connecting v_1' and v_2' has the desired properties. □

Let Γ be a walk connecting C and C' and let Γ' be a pr-subwalk, which also connects C and C'. For any $f \in \mathbb{N}_0^E$, the condition $\mathbf{e}_{\Gamma,e} \leq f$ or $\mathbf{e}_{\Gamma,o} \leq f$ implies the same statement for Γ'. Therefore, we can restrict our attention to those walks that are minimal with respect to pr-subwalks connecting C and C'. In the next lemma, we give two reductions of this kind.

Lemma 4.3.6. *Let Γ be a walk connecting C and C' that contains the edge e at least twice. If*

- *Γ passes e twice in the same direction, or*

- Γ *passes* e *in different directions, and* e *appears in both the odd part and the even part of* Γ,

then there is a pr-subwalk of $\Gamma \neq \Gamma$ *connecting* C *and* C'*. In particular, if* $q = \mathbf{e}_C + \mathbf{e}_{C'}$ *and* $f \in \mathbb{N}_0^E$ *satisfies* $q + \rho(f) \in \overline{Q(G)} \setminus Q(G)$ *and* $q + 2\rho(f) \in \overline{Q(G)} \setminus Q(G)$*, then* $q + k\rho(f) \in \overline{Q(G)} \setminus Q(G)$ *for every* $k \in \mathbb{Z}_{>0}$*.*

Proof. Let us write $\Gamma = (e_1, \dots, e_m)$ and $e_i = \{ v_i, v_{i+1} \}$ for $1 \leq i \leq m$. Assume $e = e_i = e_j$ for $i < j$. If Γ passes e_i and e_j in the same direction, i.e. $v_i = v_j, v_{i+1} = v_{j+1}$, and $j - i$ is even, then we can skip all edges between e_i and e_j, as in Lemma 4.3.4. Similarly, if Γ passes the edges in different directions, i.e. $v_i = v_{j+1}$, and $e_i \in \Gamma^{(o)}, e_j \in \Gamma^{(e)}$ (or vice versa), then $j - i$ is odd, so $j + 1 - i$ is even. Hence we can apply the same argument.

It remains to consider the case that Γ passes through e_i and e_j in the same direction and $j - i$ is odd. In this situation, Γ goes around an odd cycle, and the idea is to go around it in the other direction. Thus, we follow Γ up to e_{i-1}. Then we continue on e_{j-1} and follow Γ backwards up to e_{i+1}, and then we continue on e_{j+1}. We show that this is indeed a pr-subwalk as follows: e_i is either in the even or in the odd part of Γ, say, in the even part. Then e_j is in the odd part. Thus $e_{i-1} = \{ v_{i-1}, v_i \}$ is in the odd part, $e_{j-1} = \{ v_{j-1}, v_j \}$ is in the even part, and $v_i = v_j$. So we can go from e_{i-1} to e_{j-1}. The same argument applies to the transition from e_{i+1} to e_{j+1}. This is a proper subwalk, because it is two edges shorter than Γ.

For the second statement, note that by our discussion no edge is needed more than twice in Γ. Thus, the coefficients in $\mathbf{e}_{\Gamma,o}$ and $\mathbf{e}_{\Gamma,e}$ are at most two. Hence, if $2\rho(f)$ does not satisfy the criterion of Lemma 4.3.5, then neither does $k\rho(f)$ for any $k \in \mathbb{Z}_{>0}$. $\square$

Now we specialize to the case that G has no three pairwise exceptional cycles. It follows from Lemma 4.1.5 that under this assumption every minimal generator of $\overline{Q(G)}$ is of the form $\mathbf{e}_C + \mathbf{e}_{C'}$ for two exceptional odd cycles C and C'. For this to hold, it would actually be sufficient to require that G has no four pairwise exceptional cycles. However, the stronger assumption is necessary for Lemma 4.3.8 to hold.

Before we can proceed, we need another technical lemma.

Lemma 4.3.7. *Every odd closed walk contains a pr-subwalk that walks around an odd cycle.*

Proof. Let Γ be an odd closed walk. Let i be the smallest index of a vertex in Γ, such that there is an index $j < i$ with $v_i = v_j$. If $i - j$ is odd, then Γ passes through an odd number of edges from v_l to v_i, so this is the odd cycle we are looking for. Otherwise, we consider the pr-subwalk Γ' of Γ, created by skipping all the edges between v_j and v_i. Since G has no loop, iterating this procedure will finally yield an odd cycle as a pr-subwalk. $\square$

Note that the odd cycle in the preceding lemma need not to be minimal. We can now prove a stronger version of Lemma 4.3.6:

Lemma 4.3.8. *Assume that G has no three pairwise exceptional cycles. Let C, C' be two exceptional cycles, $f \in \mathbb{N}_0^E$ and Γ be a walk in G connecting C and C'. If $\mathbf{e}_{\Gamma,o} \leq f$ or $\mathbf{e}_{\Gamma,e} \leq f$, then there exists a walk Γ' connecting C and C' with the same property, such that Γ' visits every edge at most once. In particular, if $q \in \overline{Q(G)} \setminus Q(G)$ and $f \in \mathbb{N}_0^E$, such that $q + \rho(f) \in \overline{Q(G)} \setminus Q(G)$, then $q + k\rho(f) \in \overline{Q(G)} \setminus Q(G)$ for every $k \in \mathbb{Z}_{>0}$.*

The walk Γ' will not be a pr-subwalk of Γ. However, if $\mathbf{e}_{\Gamma,o} \leq f$, then the even or the odd part of Γ' will be a subset of the odd part of Γ. Similarly, if $\mathbf{e}_{\Gamma,e} \leq f$, then the even or the odd part of Γ will be a subset of th even part of Γ.

Proof. The only case left open in Lemma 4.3.6 is when Γ passes the same edge twice in different directions, and both passes are either in the odd or in the even part of Γ. So we consider this case. Let $e_i = \{ v_i, v_{i+1} \}, e_j = \{ v_j, v_{j+1} \}$ be the two edges with $i < j, j - i$ even, $v_i = v_{j+1}$ and $v_{i+1} = v_j$. Note that the part of Γ from v_{i+1} to v_j is an odd closed walk. Therefore, by Lemma 4.3.7 it contains a pr-subwalk that walks around an odd cycle C_1. If this cycle is not minimal, then there is a subset C_2 of its vertices forming a minimal odd cycle.

By our assumption, there is a bridge b between C_2 and C or C', and thus between C_1 and C or C'. Assume that the bridge is between C_1 and C. The other case is similar. Let v_l be the vertex in Γ where the bridge ends. We want to add the bridge to the walk while preserving the statement about f. Now, one of the edges e_{l-1} and e_l in Γ is in the even part, and the other one is in the odd part. Let us assume that $\mathbf{e}_{\Gamma,o} \leq f$. The other case is similar. The we construct Γ' by first traversing b, and then walking along the odd edge among e_{l-1} and e_l. This way, b is added to the part of Γ that is not compared with f. If we added e_l, then we follow Γ to v_j and further to its end. If we added e_{l-1} instead, then we walk backwards along Γ, until we reach e_i. Then we continue on $e_{j+1} = \{ v_i, v_{j+2} \}$ to the end of Γ.

To prove the second statement, write $q = \mathbf{e}_C + \mathbf{e}_{C'} + q'$ and $q' \in Q(G)$. Choose a preimage q'' of q' in $\mathbb{N}_0^E$. We will apply Lemma 4.3.5 to $\mathbf{e}_C + \mathbf{e}_{C'} + \rho(q'' + f)$. By the first part of this Lemma, we can assume that the coefficients of $\mathbf{e}_{\Gamma,o}$ and $\mathbf{e}_{\Gamma,e}$ are at most one. So the criterion of Lemma 4.3.5 gives the same answer for $q'' + f$ and $q'' + kf$ for every $k \in \mathbb{Z}_{>0}$. $\square$

Using our preparations, it is now not difficult to give the proof of Lemma 4.3.1.

Proof of Lemma 4.3.1. Let $q \in \overline{Q(G)} \setminus Q(G)$. We have to show that $2q \in \overline{Q(G)} \setminus Q(G)$ or $3q \in \overline{Q(G)} \setminus Q(G)$. By our hypothesis and Lemma 4.1.5, q can be written as $p + f$ for $f \in Q(G)$ and $p = \mathbf{e}_C + \mathbf{e}_{C'}$, where C and C' are two exceptional odd cycles. Obviously $2p \in Q(G)$, so we only consider $3q = p + 2p + 3f$.

We claim that $p + 3f \in \overline{Q(G)} \setminus Q(G)$. Indeed, if $p + 3f \in Q(G)$, then by Lemma 4.3.5 and Lemma 4.3.8 it follows that $q = p + f \in Q(G)$, a contradiction. But now also

$3q = p + 3f + 2p \in \overline{Q(G)} \setminus Q(G)$, because $2p$ contains only edges in the cycles and the criterion of Lemma 4.3.5 does not depend on those edges. $\square$

5. The linear ordering polytope

In this chapter, we study a special family of affine monoids, namely the polytopal affine monoid defined by the *linear ordering polytope*. For a natural number $n \in \mathbb{N}$, we denote by S_n the group of permutations of the set $[n]$.

Definition 5.0.9. Let $n \in \mathbb{N}$. Set $k := n(n-1)/2$ and consider $\mathbb{R}^k$ as the vector space with basis $\mathbf{e}_{ij}$ for $1 \le i < j \le n$. To every permutation $\pi \in S_n$ we associate a vector $v_\pi \in \mathbb{R}^k$ by setting

$$(v_\pi)_{ij} = \begin{cases} 1 & \text{if } \pi(i) > \pi(j), \\ 0 & \text{otherwise.} \end{cases}$$

The n^{th} *linear ordering polytope* P_n is defined to be the convex hull of these vectors.

The linear ordering polytope is a $\binom{n}{2}$-dimensional 0/1-polytope. It is an important and well-studied object in combinatorial optimization, see for example Chapter 6 of [MR11] and the references therein.

The linear ordering polytope also appears in a different context. In [SW12], toric statistical ranking models are considered. Every toric statistical ranking model is defined by a positive affine monoid. The polytopal affine monoid of the linear ordering polytope defines one of these models, namely the *inversion model*. The inversion model is also known as *Babington-Smith Model* in the statistics literature, see [Mar95].

The main focus of this chapter lies on the toric ideal I_{LOP} of the linear ordering polytope. Recall that this ideal can be defined as the kernel of the map

$$\begin{aligned} k[X_\pi \mid \pi \in S_n] &\to k[X_{ij} \mid 1 \le i < j \le n] \\ X_\pi &\mapsto \prod_{\substack{i<j \\ \pi(i)>\pi(j)}} X_{ij} \end{aligned}$$

However, I_{LOP} turns out to be a rather large and complex object, for example the authors of [SW12] found using the software 4ti2 [4ti2] that for $n = 6$ there are as many as 130377 quadratic generators and there are also generators of higher degree. Therefore, as a first step in understanding this object, we study its quadratic generators for all n. The ideal I_{LOP} is invariant under the action of the S_n (see below), so if $m := X_{\pi_1}X_{\pi_2} - X_{\tau_1}X_{\tau_2} \in I_{\mathrm{LOP}}$, then also $\pi_1 m = X_{id_n}X_{\pi_2\pi_1^{-1}} - X_{\tau_1\pi_1^{-1}}X_{\tau_2\pi_1^{-1}} \in I_{\mathrm{LOP}}$, where $id_n \in S_n$ is the identity permutation. Therefore we can restrict our attention to binomials of the form

$$X_{id_n} X_\pi - X_{\tau_1} X_{\tau_2} \, .$$

These binomials can be described in terms of the *inversion set*

$$T(\pi) := \left\{ \{ i, j \} \in \binom{[n]}{2} \,\middle|\, i < j, \pi(i) > \pi(j) \right\}$$

of a permutation $\pi \in S_n$.

Proposition 5.0.10. *A binomial $X_{id_n} X_\pi - X_{\tau_1} X_{\tau_2}$ lies in I_{LOP} if and only if π, τ_1 and τ_2 satisfy*

$$\begin{aligned} &T(\tau_1) \cup T(\tau_2) = T(\pi) \, , \\ &T(\tau_1) \cap T(\tau_2) = \emptyset \quad \textit{and} \\ &\tau_1, \tau_2 \neq id_n \, . \end{aligned} \tag{5.1}$$

The proof is immediate from the definitions. This observation leads to the following problem:

Problem 5.0.11. For a given permutation $\pi \in S_n$, give a description of all $\tau_1, \tau_2 \in S_n$ that satisfy (5.1).

This is the main problem we going to address in Section 5.2.

In the recent preprint [Dew+11], the following closely related question is considered: Let $\omega_{0,n} \in S_n$ denote the permutation of maximal length (i.e. the one mapping $i \mapsto n+1-i$).

Problem 5.0.12. Give a description of all sets $\{ \tau_1, \dots, \tau_l \} \subset S_n$ such that $T(\omega_{0,n}) = \bigcup_i T(\tau_i)$ and $T(\tau_i) \cap T(\tau_j) = \emptyset$ for $i \neq j$.

The motivation and the methods employed by the authors of [Dew+11] are different from ours, but some intermediate results of this paper were also found independently there. In particular, Lemma 5.3.1 and part of Lemma 5.2.9 resemble Proposition 2.2 and Proposition 3.14 in [Dew+11]. Another question about the linear ordering polytope turns out to be related to our problem:

Problem 5.0.13 ([You78])**.** Which permutations $\pi \in S_n$ are neighbours of the identity permutation in the graph of the linear ordering polytope?[1]

In [You78], a characterization of these permutations is obtained, but as we show after Lemma 5.2.9 there is a gap in the proof. Nevertheless, the result from [You78] is correct. We extend it and provide a proof in Lemma 5.2.9. It turns out that a permutation has a decomposition as in (5.1) if and only if it is not a neighbour of the identity permutation

[1] The linear ordering polytope is called the "permutation polytope" in [You78].

in the graph of the linear ordering polytope. However, we are interested in a description of all possible decompositions of type (5.1).

This chapter is divided into four sections. In Section 5.1 we collect some basic facts about the linear ordering polytope and its symmetry group. Moreover, we review the concept of modular decomposition for graphs, the characterization of inversion sets of permutations and we discuss blocks of permutations. In Section 5.2, we prove our main results. In Lemma 5.2.3, we give an answer to Lemma 5.0.11 in terms of the modular decomposition of the inversion graph of π. Moreover, we consider a modification of (5.1), where we impose the further restriction that $\pi = \tau_1\tau_2$. We show in Lemma 5.2.6 that if π admits a solution of (5.1), then it also admits a solution satisfying $\pi = \tau_1\tau_2$. Since Lemma 5.0.11 is formulated without referring to graphs, in Lemma 5.2.13 we give a reformulation of Lemma 5.2.3 which avoids notions from graph theory. In Section 5.3, we show that the problem of decomposing an inversion set into three or more inversion sets can be reduced to (5.1). From this we deduce that the cubic generators in any reverse lexicographic initial ideal of I_{LOP} are squarefree. We further consider the linear ordering polytope for small values of n. A computational result of [SW12] is that $Q(P_n)$ is normal and Gorenstein for $n \leq 5$. We give a simple theoretical proof for this.

5.1. Preliminaries

5.1.1. Notation

For $\pi \in S_n$ we denote by $T(\pi)$ the *inversion set*

$$\left\{ \{ i, j \} \in \binom{[n]}{2} \;\middle|\; i < j, \pi(i) > \pi(j) \right\} .$$

This set can be considered as the edge set of an undirected graph $G(\pi) = ([n], T(\pi))$, the *inversion graph* of π. We consider this graph without the natural labelling on its vertices, therefore in general $G(\pi)$ does not uniquely determine π. The graphs arising this way are called *permutation graphs*, see [BSL99]. By abuse of notation, we write $ij \in T(\pi)$ (resp. $ij \in G(\pi)$) if $\{ i, j \}$ is an inversion of π. For two subsets $A, B \subset [n]$, we write $A < B$ if $a < b$ for every $a \in A, b \in B$.

5.1.2. The symmetry group of the linear ordering polytope

Recall that we defined P_n as a subset of $\mathbb{R}^k$ with $k = n(n-1)/2$. Let $\mathbf{e}_{ij}$ for $1 \leq i < j \leq n$ denote the unit basis vectors of $\mathbb{R}^k$. Further, let $\mathbb{I} \in \mathbb{R}^k$ be the vector with all entries equal to 1. Every permutation $\tau \in S_n$ defines an affine linear map E_τ on $\mathbb{R}^k$ as follows:

$$E_\tau(\mathbf{e}_{ij}) := \begin{cases} \mathbf{e}_{\tau(i)\tau(j)} & \text{if } \tau(i) < \tau(j) \\ \mathbb{I} - \mathbf{e}_{\tau(j)\tau(i)} & \text{if } \tau(i) > \tau(j) \end{cases}$$

Moreover, we define the map $D : \mathbb{R}^k \to \mathbb{R}^k$ by setting $D(\mathbf{e}_{ij}) := \mathbb{I} - \mathbf{e}_{ij}$. A direct calculation shows that $E_\tau(v_\pi) = v_{\pi\tau^{-1}}$ and $D(v_\pi) = v_{\omega_{0,n}\pi}$. So the maps E_τ for $\tau \in S_n$ and D form a group $S_n \times S_2$ of affine linear automorphisms of P_n. Note that the group is vertex-transitive on P_n.

The action of $S_n \times S_2$ can be carried over to the polynomial ring $\mathbb{K}[X_\pi \mid \pi \in S_n]$ by setting $E_\tau(X_\pi) := X_{\pi\tau^{-1}}$ and $D(X_\pi) := X_{\omega_{0,n}\pi}$. It follows from the invariance of P_n that the toric ideal I_{LOP} is also invariant under this action. In fact, the symmetry group of the linear ordering polytope P_n is actually larger. In [Fio01] it is shown that the full symmetry group of P_n is $S_{n+1} \times S_2$. However, we will only use the restricted group $S_n \times S_2$.

5.1.3. Modular decomposition of graphs

In this subsection we review the modular composition for graphs, see [BSL99, Chapter 1.5] for a reference. Let $G = (V, E)$ be a graph.

Definition 5.1.1 ([BSL99])**.**

1. A set $M \subset V$ is called a *module* of G if for $m_1, m_2 \in M$ and $v \in V \setminus M$ it holds that $vm_1 \in G$ if and only if $vm_2 \in G$.
2. A module M is called *strong* if for every other module N either $M \cap N = \emptyset$, $M \subset N$ or $N \subset M$ holds.

In [BSL99, p. 14] it is shown that for every module there is a unique minimal strong module containing it. A graph is called *prime* if V and its vertices are its only modules. We denote by $\overline{G}$ the complementary graph $\overline{G} = (V, \binom{V}{2} \setminus E)$ of G. For a subset $U \subset V$, we denote by G_U the induced subgraph of G on U.

Theorem 5.1.2 (Theorem 1.5.1, [BSL99])**.** *Let $G = (V, E)$ be a graph with at least two vertices. Then the maximal strong submodules (*m.s.s.*) of G form a partition of V and exactly one of the following conditions hold:*

Parallel case: *G is not connected. Then its m.s.s. are its connected components.*

Serial case: *$\overline{G}$ is not connected. Then the m.s.s. of G are the connected components of $\overline{G}$.*

Prime case: *Both G and $\overline{G}$ are connected. Then there is a subset $U \subset V$ such that*

1. $\#U > 3$,

2. *G_U is a maximal prime subgraph of G,*
3. *and every m.s.s. M of G has $\#M \cap U = 1$.*

We call a module M of G *parallel*, *serial* or *prime* corresponding to which condition of the above theorem is satisfied by G_M. As a convention, we consider single vertices as parallel modules. By the following lemma, we do not need to distinguish between modules of G contained in a module M and modules of G_M.

Lemma 5.1.3. *Let G be a graph, M a module of G and $U \subset M$ a subset. Then U is a module of G if and only if it is a module of G_M. Moreover, U is a strong module of G if and only if it is strong as a module of G_M.*

Proof. The first statement is immediate from the definitions. For the second statement, first assume that U is not strong as a module in G_M. We say that a module N overlaps U if $N \cap U \neq \emptyset$, $N \nsubseteq U$ and $U \nsubseteq N$ holds. So by our assumption, there is a module $N \subset M$ of G_M overlapping U. But N is also a module of G, hence U is not strong as a module of G. On the other hand, if U is not strong as a module of G, then there is a module N of G overlapping U. Now, $M \setminus N$ is a module of G ([BSL99, Prop 1.5.1 (ii)]), and thus a module of G_M. But $M \setminus N$ overlaps U, so U is not strong as a module of G_M. □

If M and N are two disjoint modules of G, then one of the following holds:

1. Either every vertex of M is connected to every vertex of N. Then we call M and N connected in G and we write MN for the set of edges between vertices of M and N.

2. Otherwise no vertex of M is connected to any vertex of N.

The edges connecting the m.s.s. of a module M are called *external edges* of M. So M is parallel if and only if it has no external edges. Note that every edge of G is an external edge for exactly one strong module. We close this section by giving a description of the non-strong modules of G:

Lemma 5.1.4. *Let G be a graph and let M be a module which is not strong. Then M is the union of some m.s.s. of a parallel or serial strong module. On the other hand, any union of m.s.s. of a parallel or serial strong module is a module.*

Proof. Let N be the smallest strong module containing M. The m.s.s. of N partition it, so M is a union of some of them. If N is prime, then consider the set U in Lemma 5.1.2. Since M is not strong, it is a union of at least two but not of all m.s.s. of N. So $M \cap U$ is a nontrivial submodule of G_U, contradicting Lemma 5.1.2. Hence N is either serial of parallel.

For the converse, let M be a union of m.s.s. of a serial or parallel strong module N. By Lemma 5.1.3, it suffices to prove that M is a module of G_N. Let $x, y \in M$ and $m \in N \setminus M$. The edges xm, ym are both external in N. But if N is serial, it has all possible external edges and if it is parallel, it has none at all. In both cases, the claim is immediate. □

5.1.4. Inversion sets and blocks

We recall the characterization of those sets that can arise as inversion sets of a permutation.

Proposition 5.1.5 (Proposition 2.2 in [YO69], see also [BW91])**.** *Let $T \subset \binom{[n]}{2}$ be a subset. The following conditions are equivalent:*

1. *There exists a permutation $\pi \in S_n$ with $T = T(\pi)$.*
2. *For every $1 \leq i < j < k \leq n$ it holds that:*
   - *If $ij, jk \in T$, then $ik \in T$.*
   - *If $ik \in T$, then at least one of ij and jk lies in $T(\pi)$.*

If a subset $T \subset \binom{[n]}{2}$ satisfies the conditions of above proposition, say $T = T(\pi)$, then so does its complement by $\binom{[n]}{2} \setminus T = T(\omega_{0,n}\pi)$. We now take a closer look at the modules of the inversion graph of a permutation $\pi \in S_n$. Let us call a set $I \subset [n]$ of consecutive integers an *interval*.

Definition 5.1.6 ([Bri10])**.**

1. A π-*block* is an interval $I \subset [n]$ such that its image $\pi(I)$ is again an interval.
2. A π-block is called *strong* if for every other π-block J either $I \cap J = \emptyset$, $I \subset J$ or $J \subset I$ holds.

The importance of π-blocks for our purpose stems from the following theorem:

Theorem 5.1.7. *Let $I \subset [n]$ and $\pi \in S_n$. The following implications hold:*

1. *I is a π-block $\Longrightarrow$ I is a module of $G(\pi)$*
2. *I is a strong π-block $\Longleftrightarrow$ I is a strong module of $G(\pi)$*

In particular, every strong module of $G(\pi)$ is an interval.

The first part of this theorem is relatively easy to prove and is mentioned in [You78]. Its converse fails for trivial reasons: By Lemma 5.1.4, the non-strong modules of $G(\pi)$ are exactly the unions of m.s.s. of parallel or serial strong modules of $G(\pi)$. But such a union is not necessarily an interval. A complete proof of Lemma 5.1.7 is included in the appendix. We call a π-block parallel, serial or prime if it is a module of this type.

5.2. Main results

In this section we prove our main results concerning Lemma 5.0.11. Fix a permutation $\pi \in S_n$. For $\tau_1, \tau_2 \in S_n$, we will write $\pi = \tau_1 \sqcup \tau_2$ to indicate that the three permutations satisfy (5.1). We call $\tau_1 \sqcup \tau_2$ an *inv-decomposition* of π. If an inv-decomposition of π exists, we call π *inv-decomposable*.

5.2.1. Inversion decomposition

In this subsection, we describe all possible inv-decompositions of π. We start with an elementary observation:

Lemma 5.2.1. *Let $i, j, k \in [n]$ such that $ij, ik \in G(\pi)$ and $jk \notin G(\pi)$. Assume that $\pi = \tau_1 \sqcup \tau_2$ for $\tau_1, \tau_2 \in S_n$. Then ij, ik are both either in $G(\tau_1)$ or in $G(\tau_2)$.*

Proof. We consider the different relative orders of i, j and k separately, but we may assume $j < k$.

$i < j < k$: The edge ik is contained either in $T(\tau_1)$ or in $T(\tau_2)$, say in $T(\tau_1)$. By assumption $jk \notin T(\tau_1)$, therefore by Lemma 5.1.5 we have $ij \in T(\tau_1)$.

$j < i < k$: This case is excluded by Lemma 5.1.5.

$j < k < i$: Analogous to the first case.

□

Note that there is no assumption on the relative order of i, j and k, so this is really a statement about the inversion graph of π. Lemma 5.2.1 gives rise to a partition of the edges of $G(\pi)$: Two edges ij, $ik \in G(\pi)$ with a common endpoint are in the same *edge class* if $jk \notin G(\pi)$, and our partition is the transitive closure of this relation. Thus by Lemma 5.2.1 two edges in the same class always stay together when we distribute the inversions of π on τ_1 and τ_2. In [Gal67] edge classes are considered for a different motivation. In that paper the following description is given[2].

Proposition 5.2.2 ([Gal67])**.** *Let $G = (V, E)$ be a graph with at least two vertices. Then there are two kinds of edge classes:*

1. *For two m.s.s. $M_1, M_2 \subset M$ of a serial module M, the set M_1M_2 is an edge class.*
2. *The set of external edges of a prime module forms an edge class.*

Every edge class is of one of the above types.

[2] Note that what we call module is called “geschlossene Menge” (closed set) in [Gal67].

Edge classes are also considered in [Gol80, Chapter 5] under the name "colour classes" and in [You78] as the connected components of a certain graph Γ_π. Theorem 1 in the latter reference gives a different characterization of edge classes. Now we can state our main result. We give a description of all ways of partitioning $T(\pi)$ into two sets satisfying (5.1).

Theorem 5.2.3. *Consider a partition $T(\pi) = T_1 \dot\cup T_2$ of the inversion set of π into nonempty subsets $T_1, T_2 \subset T(\pi)$. For such a partition, the following conditions are equivalent:*

1. *There exist permutations $\tau_1, \tau_2 \in S_n$ such that $T_i = T(\tau_i)$ for $i = 1, 2$. In particular, $\pi = \tau_1 \sqcup \tau_2$.*
2. *For every strong prime module of $G(\pi)$, all its external edges are either in T_1 or in T_2. For every strong serial module of $G(\pi)$ with p maximal strong submodules $M_1 < \ldots < M_p$ there exists a permutation $\sigma \in S_p$, such that for each pair $1 \le i < j \le p$ it holds that $M_iM_j \subset T_1$ if and only if $ij \in T(\sigma)$.*

Proof. (1) ⇒ (2): Every edge of $G(\pi)$ is an external edge of a module M that is either prime or serial. If M is a prime module, then its external edges form an edge class, hence they all are in T_1 or T_2. If M is a serial module with m.s.s. $M_1, \ldots, M_p$, then the sets M_iM_j are edge classes. For every M_i, choose a representative $a_i \in M_i$. We construct a permutation $\sigma \in S_p$ as follows: Order the images $\tau_1(a_i), i = 1, \ldots, p$ in the natural order. Then $\sigma(i)$ is the position of $\tau_1(a_i)$ in this order. Thus, for $i < j$ we have

$$\begin{aligned} M_iM_j \subset T(\tau_1) &\iff \tau_1(a_i) > \tau_1(a_j) \\ &\iff \sigma(i) > \sigma(j) \\ &\iff ij \in T(\sigma) \end{aligned}$$

(2) ⇒ (1): By symmetry, we only need to show the existence of τ_1. For this, we verify conditions of Lemma 5.1.5. This is a condition for every three numbers $1 \le i < j < k \le n$, so let us fix them. Note that our hypothesis on T_1 and T_2 implies that every edge class of $T(\pi)$ is contained either in T_1 or T_2.

Let M be the smallest strong module containing these three numbers. It holds that i and k are in different m.s.s. of M, because every strong module containing both would also contain j, since it is an interval by Lemma 5.1.7. Now we distinguish two cases: Either, i and j are in the same m.s.s. of M, or all three numbers are in different m.s.s..

In the first case, let $i, j \in M_a$ and $k \in M_b$. Then $ik, jk \in M_aM_b$ belong to the same edge class, so either both or neither of them are in T_1. This is sufficient to prove that the criterion is satisfied.

In the second case, the edges ij, jk, ik are all external to M. Hence, if M is prime, either none of them is in T_1 or all that are also in $T(\pi)$. Since $T(\pi)$ is the inversion set of a permutation, the criterion of Lemma 5.1.5 is clearly satisfied in this case. If M is serial, then the edges correspond to inversions of σ: Let $i \in M_a, j \in M_b, k \in M_c$, then $M_a M_b \subset T_1$ if and only if $ab \in T(\sigma)$ and similarly for the other edges. Since σ is a permutation, the criterion is again satisfied. $\square$

As a corollary, we can count the number of inv-decompositions of π:

Corollary 5.2.4. *Let m be the number of strong prime modules and let k_i be the number of strong serial modules with i maximal strong submodules, $2 \leq i \leq n$. The number of inv-decompositions of π is*

$$\frac{1}{2} 2^m \prod_{i=2}^{n} (i!)^{k_i} - 1$$

In particular, the number of inv-decompositions depends only on the inversion graph $G(\pi)$.

We exclude the trivial inv-decomposition $\pi = \pi \sqcup id_n$, therefore the "-1" in above formula. The factor $\frac{1}{2}$ is there because we identify $\tau_1 \sqcup \tau_2 = \tau_2 \sqcup \tau_1$.

5.2.2. Multiplicative decompositions

A notable special case of an inv-decomposition is the following:

Definition 5.2.5. We call an inv-decomposition $\pi = \tau_1 \sqcup \tau_2$ *multiplicative* if $\pi = \tau_1\tau_2$ or $\pi = \tau_2\tau_1$ (multiplication as permutations).

This kind of inv-decomposition is surprisingly common. For the statement of the next theorem, we recall that a *decreasing subsequence* of a permutation $\pi \in S_n$ is a set of indices $1 \leq a_1 < \ldots < a_l \leq n$ such that $\pi(a_1) > \ldots > \pi(a_l)$. In this subsection, we prove the following theorem.

Theorem 5.2.6. *Every inv-decomposable permutation has a multiplicative inv-decomposition. Moreover, if a permutation π has a non-multiplicative inv-decomposition and $G(\pi)$ is connected, then π has a decreasing subsequence of size* 4.

The assumption that $G(\pi)$ is connected is needed to avoid a rather trivial case. It follows from Lemma 5.2.7 below that $G(\pi)$ is disconnected if and only if π maps a lower interval $[k] \subsetneq [n]$ to itself. So in this case, π is the product of a permutation π_1 on $[k]$ and a permutation π_2 on $\{ k+1, \ldots, n \}$. If we have multiplicative inv-decompositions $\pi_1 = \tau_{11}\tau_{12}$ and $\pi_2 = \tau_{21}\tau_{22}$, then $\pi = \tau_{11}\tau_{22} \sqcup \tau_{21}\tau_{12}$ is in general not multiplicative. Before we prove Lemma 5.2.6, we prepare two lemmata.

Lemma 5.2.7. *If $C \subset [n]$ is the set of vertices of a connected component of $G(\pi)$, then $\pi(C) = C$.*

Proof. Consider $i \in [n] \setminus C$ and $c \in C$. If $i < c$, then $\pi(i) < \pi(c)$ and the same is true for ">", thus the claim follows from bijectivity. □

Lemma 5.2.8. *Assume $\pi = \tau_1 \sqcup \tau_2$. If every connected component of $G(\tau_2)$ is an induced subgraph of $G(\pi)$, then $\pi = \tau_1\tau_2$.*

Proof. We will prove that $T(\tau_1\tau_2) = T(\pi)$. Let $M_1, \dots, M_s$ be the vertex sets of the connected components of $G(\tau_2)$. Note that $[n] = \bigcup M_k$. By [BB05, Ex 1.12] it holds that $T(\tau_1\tau_2)$ is the symmetric difference of $T(\tau_2)$ and $\tau_2^{-1}T(\tau_1)\tau_2$. However, we observe that in our situation the two sets are disjoint, thus the symmetric difference is actually a disjoint union. To see this, note that every edge of $G(\tau_2)$ has both endpoints in the same M_k for a $1 \leq k \leq s$, and every edge of $G(\tau_1)$ has its endpoints in different sets.Since by Lemma 5.2.7 it holds that $\tau_2(M_k) = M_k$ for every k, this property is preserved under the conjugation with τ_2. Hence, the sets are disjoint.

Next, we prove that every M_k is a $G(\pi)$-module. So fix a k. If M_k has only one element, then it is trivially a $G(\pi)$-module, so assume that M_k has more than one element. Let M' be the smallest strong module of $G(\pi)$ containing M_k and let G_k be the subgraph of $G(\pi)$ induced by M_k. Because $\pi = \tau_1 \sqcup \tau_2$ is a valid decomposition, G_k is a union of edge classes. Thus, if M' is prime, we conclude that $M_k = M'$ and we are done. If M' is parallel, then G_k cannot be connected, thus we only need to consider the case that M' is serial. But in this case, M_k is a union of m.s.s. of M' because of the form of the edge classes, given by Lemma 5.2.2. By Lemma 5.1.4, we conclude that M_k is indeed a module of $G(\pi)$. Moreover, it follows that M_k is also a module of $G(\tau_1)$, because $G(\pi)$ and $G(\tau_1)$ differ only inside the M_k. Finally, consider the set

$$\tau_2^{-1}T(\tau_1)\tau_2 = \bigcup_{k,l} \{\, \{\, \tau_2(i), \tau_2(j) \,\} \mid \{\, i, j \,\} \in T(\tau_1), i \in M_k, j \in M_l \,\} \,.$$

Because $\tau_2(M_k) = M_k$ and M_k is a module of $G(\tau_1)$ for all k, it holds that

$$\{\, \{\, \tau_2(i), \tau_2(j) \,\} \mid \{\, i, j \,\} \in T(\tau_1), i \in M_k, j \in M_l \,\} = \\ \{\, \{\, i, j \,\} \in T(\tau_1) \mid i \in M_k, j \in M_l \,\} \,.$$

Hence $\tau_2^{-1}T(\tau_1)\tau_2 = T(\tau_1)$ and the claim follows. □

Proof of Lemma 5.2.6. For the first statement, assume that π is inv-decomposable. Then by Lemma 5.2.4 there are either at least two non-parallel strong π-blocks $I_1, I_2 \subset [n]$, or at least one serial strong π-block I_3 with at least three m.s.s..

In the first case, we may assume $I_1 \not\subseteq I_2$. We set T_2 to be the set of edges in the induced subgraph of $G(\pi)$ on I_2. In the second case, we set T_2 to be the set of edges in the induced subgraph of G on the union of the two first m.s.s. of I. In both cases, we set

$T_1 = T(\pi) \setminus T_2$. By Lemma 5.2.3, this is a valid inv-decomposition, and by Lemma 5.2.8 it is multiplicative.

For the second statement, we will prove that $G(\pi)$ contains a complete subgraph on 4 vertices. Let $\pi = \tau_1 \sqcup \tau_2$ be a non-multiplicative inv-decomposition. Consider a minimal path from 1 to n in $G(\pi)$. If i and j are two vertices in this path that are not adjacent in this path, then they are not adjacent in $G(\pi)$, because otherwise we had a shortcut. Thus by Lemma 5.2.1 we conclude that every edge in this path lies in the same edge class. Hence either $G(\tau_1)$ or $G(\tau_2)$ contains a path connecting 1 with n, say $G(\tau_1)$. By Lemma 5.2.1 this implies that $G(\tau_1)$ has no isolated vertices.

By our hypothesis and by Lemma 5.2.8, there exists a connected component of $G(\tau_2)$ that is not an induced subgraph of $G(\pi)$. Then there exist $1 \leq i, j \leq n, i \neq j$ such that $ij \in G(\tau_1)$ and there is a minimal path $i, i', \ldots, j$ connecting i and j in $G(\tau_2)$. By Lemma 5.2.1 we have $i'j \in G(\pi)$. We also want to make sure that $i'j \in G(\tau_2)$. If this is not the case, then replace i by i'. Then the corresponding statements still hold, but the minimal path is shorter. Thus, by induction we may assume $i'j \in G(\tau_2)$. Since $G(\tau_1)$ has no isolated vertices, there is a vertex k such that $i'k \in G(\tau_1)$. Again by Lemma 5.2.1 we conclude that $ik, jk \in G(\pi)$. Thus $G(\pi)$ contains the complete subgraph on i, i', j and k. □

5.2.3. Characterization of inv-decomposability

We use the results we have proven so far to derive a characterization of inv-decomposability. This also provides an answer to Lemma 5.0.13.

Theorem 5.2.9. *For $\pi \in S_n$ the following statements are equivalent:*

1. *There exist $\tau_1, \tau_2 \in S_n \setminus \{ id_n \}$ such that $T(\pi) = T(\tau_1) \dot\cup T(\tau_2)$ and $\pi = \tau_1\tau_2$, i.e. π has a multiplicative inv-decomposition.*
2. *There exist $\tau_1, \tau_2 \in S_n \setminus \{ id_n \}$ such that $T(\pi) = T(\tau_1) \dot\cup T(\tau_2)$, i.e. π is inv-decomposable.*
3. *v_π is not a neighbour of the identity in the graph of the linear ordering polytope.*
4. *There are at least two edge classes of $G(\pi)$.*
5. *There are at least two (not necessarily strong) non-trivial non-parallel π-blocks. (By a non-trivial π-block, we mean a π-block that is neither a singleton nor $[n]$)*

In [You78], the implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (2) are proven, although the condition (2) is not explicitly mentioned. As indicated in the introduction of this chapter, there a gap in the proof. Indeed on page 4 of [You78], in the proof of the implication (3) $\Rightarrow$ (4) the following argument is used. If v_π is not a neighbour of v_{id_n}, then there is a point on the line between the points that can be written as a convex combination of other vertices, e.g. $\lambda v_{id_n} + (1-\lambda)v_\pi = \sum \lambda_i v_{\tau_i}$ for $\lambda, \lambda_i \in [0,1]$ and the λ_i sum up to 1. Considering the support set of the vectors on the left and right-hand side of this equation we obtain an expression $T(\pi) = \bigcup T(\tau_i)$. Note, that in general this union is not disjoint. In [You78], the existence of this expression, together with the assumption that $G(\pi)$ has only one edge class leads to a contradiction, proving (3) $\Rightarrow$ (4). But $T(2413) = \{13, 23, 24\} = T(2314) \cup T(1423)$ and $G(2413)$ has only one edge class, providing a counterexample to above argument. Since the notation of [You78] is different from ours, we provide a full proof of the implications for the convenience of the reader.

Proof. $1 \Leftrightarrow 2:$ Lemma 5.2.6.

$2 \Rightarrow 3:$ If $T(\pi) = T(\tau_1) \dot\cup T(\tau_2)$, then the midpoint of the line connecting v_{id_n} and v_π is also the midpoint of the line connecting v_{τ_1} and v_{τ_2}, thus it cannot be an edge.

$3 \Rightarrow 4:$ If v_π is not a neighbour of v_{id_n}, then we can write $\lambda v_{id_n} + (1-\lambda)v_\pi = \sum \lambda_i v_{\tau_i}$ for $\lambda, \lambda_i \in [0,1]$ and $\tau_i \neq \pi$ for every i. We clear denominators to make the coefficients integral. The important observation is that every non-zero component of the right-hand side has the same value.

Consider $a, b, c \in [n]$ such that $ab, bc \in T(\pi)$ and $ac \notin T(\pi)$. Then b cannot lie between a and c because of Lemma 5.1.5. There remain four possible relative orders of a, b and c. We assume $b < a < c$, the other cases follow analogously. Every τ_i with $bc \in T(\tau_i)$ has also $ba \in T(\tau_i)$, again by Lemma 5.1.5. But the number of τ_i having the inversion bc equals the number of those having ba. Hence, every τ_i has either both or none of the inversions. It follows that if $T(\tau_i)$ contains an inversion, then it already contains the whole edge class of it. Thus if $G(\pi)$ has only one edge class, then for every i either $\tau_i = \pi$ or $\tau_i = id_n$, which is absurd.

$4 \Rightarrow 5:$ This follows from the description of the edge classes, Lemma 5.2.2.

$5 \Rightarrow 2:$ Under our hypothesis, the formula in Lemma 5.2.4 cannot evaluate to zero.

□

5.2.4. Substitution decomposition

We give a reformulation of Lemma 5.2.3 avoiding notions from graph theory. For this, we employ the concept of *substitution decomposition*, which was introduced in [AA05], see [Bri10] for a survey. We start by giving an explicit description of the three types of π-blocks.

Proposition 5.2.10. *Let $I \subset [n]$ be a π-block with at least two elements and let $I_1 < \ldots < I_l$ be its maximal strong submodules.*

1. *I is parallel if and only if $\pi(I_1) < \pi(I_2) < \ldots < \pi(I_l)$.*

2. *I is serial if and only if $\pi(I_1) > \pi(I_2) > \ldots > \pi(I_l)$.*

3. *Otherwise I is prime.*

Proof. This is a consequence of Lemma 5.1.7. I is parallel if and only if it has no external edges. This translates to the statement that the relative order of the I_i is preserved. Similarly, I is serial if and only if it has all possible external edges. Again, this translates to the statement that the relative order of the I_i is reversed. □

In the remainder of this section, we consider permutations as words $\pi = \pi_1\pi_2 \ldots \pi_n$. The *size* of a permutation is the number of letters in its word[3]. The special word $id_n := 12 \ldots (n-1)n$ is called an *identity*. If $\pi = \pi_1\pi_2 \ldots \pi_n$ is a permutation, we call $\check{\pi} := \pi_n\pi_{n-1} \ldots \pi_1$ the *reversal* of π. The word $\omega_{0,n} := \check{id}_n = n(n-1) \ldots 21$ is called *reverse identity*. Two finite sequences $a_1, \ldots, a_q$ and $b_1, \ldots, b_q$ of natural numbers are called *order isomorphic* whenever $a_i < a_j$ if and only if $b_i < b_j$. Given a permutation $\pi \in S_m$ and m further permutations $\sigma_1, \ldots, \sigma_m$ of not necessarily the same size, we define the *inflation* $\pi[\sigma_1, \ldots, \sigma_m]$ by replacing the value $\pi(i)$ by an interval order isomorphic to σ_i. For a more detailed treatment of the inflation operation see [Dew+11]. A permutation π is called *simple* if there are no other π-blocks than $[n]$ and the singletons. Note that by Lemma 5.1.7 a permutation π is simple if and only if its inversion graph $G(\pi)$ is prime.

Proposition 5.2.11. *Every permutation π can be uniquely expressed as an iterated inflation, such that every permutation appearing in this expression is either an identity, a reverse identity or a simple permutation, and no identity or reverse identity is inflated by a permutation of the same kind.*

We call this the substitution decomposition of π. It is slightly different from the decomposition in [Bri10]. The existence of our decomposition follows from the existence of the decomposition given in that paper, but we consider it to be instructive for our discussion to give a proof nevertheless.

Proof. Let $I_1 < I_2 < \ldots < I_l$ be the maximal strong π-subblocks of $[n]$. Define a permutation $\alpha \in S_l$ by requiring $\alpha(i) < \alpha(j) \Leftrightarrow \pi(I_i) < \pi(I_j)$ for $1 \leq i, j \leq l$. Moreover, let σ_i be the permutation order isomorphic to $\pi(I_i)$ for $1 \leq i \leq l$. Then $\pi = \alpha[\sigma_1, \ldots, \sigma_l]$. By Lemma 5.1.2 the π-block $[n]$ is either parallel, serial or prime. Hence by Lemma 5.2.10 we conclude that α is either an identity, a reverse identity or simple. By applying this procedure recursively to the σ_i, we get the claimed decomposition.

The last claim follows also from Lemma 5.1.2, because it implies that no serial module has a maximal strong submodule which is again serial, and the same for parallel modules. This is just the statement that connected components of a graph are connected. □

[3] This is called "length" in [Bri10] but we reserve that notion for the number of inversions.

The proof gives a correspondence between the strong π-blocks and the permutations appearing in the substitution decomposition. The strong parallel, serial and prime π-blocks correspond to the identities, reverse identities and simple permutations, respectively. Now we can reformulate Lemma 5.2.3 in terms of inflations:

Construction 5.2.12. Let π be a permutation. Define two new permutations τ_1, τ_2 in the following way: Write down two copies of the substitution decomposition of π. For every simple permutation α in it, replace α in one of the copies by an identity. For every reverse identity, replace it in one copy by an arbitrary permutation σ of the same size and in the other by the reverse $\check{\sigma}$. Then let τ_1 and τ_2 be the permutations defined by these iterated inflations.

Theorem 5.2.13. *Let π, τ_1, τ_2 be permutations as above and assume that $\tau_1, \tau_2 \neq id_n$. Then $\pi = \tau_1 \sqcup \tau_2$ and every pair (τ_1, τ_2) satisfying this condition can be found this way.*

Proof. This is immediate from Lemma 5.2.3 using the correspondence described above. $\square$

5.3. Further results

5.3.1. Squarefree cubic relations

First, we consider the generalization of (5.1) to more than two components. It turns out that this case can easily be reduced to the case of two components, as the next proposition shows.

Proposition 5.3.1. *Let $\pi, \tau_1, \dots, \tau_l \in S_n$ be permutations such that $T(\pi) = \bigcup T(\tau_i)$ and $T(\tau_i) \cap T(\tau_j) = \emptyset$ for $i \neq j$. Then for every $1 \leq i, j \leq l$ there exists a $\tau_{ij} \in S_n$ such that $T(\tau_{ij}) = T(\tau_i) \dot\cup T(\tau_j)$.*

Proof. We show that $T := T(\tau_i) \cup T(\tau_2)$ satisfies the condition of Lemma 5.1.5. Fix $1 \leq a_1 < a_2 < a_3 \leq n$. Note that $\binom{[n]}{2} \setminus T = \binom{[n]}{2} \setminus T(\tau_i) \cap \binom{[n]}{2} \setminus T(\tau_j)$, so if $a_1a_2 \notin T$ and $a_2a_3 \notin T$, then $a_1a_3 \notin T$. On the other hand, if $a_1a_2, a_2a_3 \in T$, then $a_1a_3 \in T(\pi)$ and thus $a_1a_3 \in T(\tau_k)$ for some k. But then $T(\tau_k)$ contains also a_1a_2 or a_2a_3, therefore k equals i or j. It follows that $a_1a_3 \in T$. $\square$

We apply the preceding result to show that the minimal generators in degree 3 of I_{LOP} and every reverse lexicographic initial ideal of I_{LOP} are squarefree. Here, we say that a binomial in I_{LOP} is a *minimal generator* if it is part of a minimal generating system of I_{LOP} (which may not be unique). Note that every binomial of degree 2 in I_{LOP} is squarefree. Indeed, if $X_{\pi_1}^2 - X_{\pi_2}X_{\pi_3} \in I_{\mathrm{LOP}}$, then $2v_{\pi_1} = v_{\pi_2} + v_{\pi_3}$, but this is impossible for the vertices of a 0/1-polytope. More generally, no binomial of the form $X_\tau^d - \prod_k X_{\pi_k}$ for $\tau \neq \pi_k, 1 \leq k \leq d$ is contained in I_{LOP}.

Corollary 5.3.2. *Every minimal generator of I_{LOP} of degree 3 is squarefree.*

Proof. Let $X_{\pi_1}^2 X_{\pi_2} - X_{\pi_3} X_{\pi_4} X_{\pi_5}$ be a binomial in I_{LOP}. So simplify notation, we write $X_i := X_{\pi_i}$. By symmetry, we may assume that $\pi_1 = id_n$. Then it holds that $T(\pi_2) = T(\pi_3) \dot\cup T(\pi_4) \dot\cup T(\pi_5)$, so by Lemma 5.3.1 there exists a $\tau \in S_n$ such that $T(\tau) = T(\pi_3) \dot\cup T(\pi_4)$. It follows that $X_1 X_2 - X_\tau X_5 \in I_{\mathrm{LOP}}$ and $X_1 X_\tau - X_3 X_4 \in I_{\mathrm{LOP}}$, therefore

$$X_1^2 X_2 - X_3 X_4 X_5 = X_1(X_1 X_2 - X_\tau X_5) - X_5(X_1 X_\tau - X_3 X_4)$$

is not a minimal generator. □

Corollary 5.3.3. *Let $\prec$ be a reverse lexicographic term order on $\mathbb{K}[X_\pi \mid \pi \in S_n]$ and let J be the initial ideal of I_{LOP} with respect to $\prec$. Then every minimal generator of degree 3 of J is squarefree.*

Proof. Assume on the contrary that there is a minimal generator of J of degree 3. By the argument above, it is of the form $X_{\pi_0}^2 X_{\pi_1}$ of J. Then there is a binomial of the form $X_{\pi_0}^2 X_{\pi_1} - X_{\pi_2} X_{\pi_3} X_{\pi_4} \in I_{\mathrm{LOP}}$ and $X_{\pi_0}^2 X_{\pi_1} \succ X_{\pi_2} X_{\pi_3} X_{\pi_4}$. As above, we write $X_i := X_{\pi_i}$ for short. By Lemma 5.3.2, there exist $\pi_{23}, \pi_{24}, \pi_{34} \in S_n$ such that the following binomials are in I_{LOP}:

$$X_0 X_1 - X_{23} X_4, \quad X_0 X_1 - X_{24} X_3, \quad X_0 X_1 - X_{34} X_2$$

Here, we set $X_{ij} := X_{\pi_{ij}}$. We assumed that $X_0^2 X_1 \succ X_2 X_3 X_4$. Because $X_0^2 X_1$ is a minimal generator, it holds that $X_0 X_1 \prec \min\{ X_{23} X_4, X_{24} X_3, X_{34} X_2 \}$. Hence, because $\prec$ is reverse lexicographic, it holds that $\min\{ X_0, X_1 \} \prec \min\{ X_2, X_3, X_4 \}$. It follows $X_0^2 X_1 \prec X_2 X_3 X_4$, a contradiction. □

5.3.2. Small linear ordering polytopes

In this subsection, we study the linear ordering polytope P_n for small values of n. For $n \leq 5$, the facet structure of P_n is quite simple. There is only one symmetry class of facets [MR11, p. 130], namely

$$x_{ij} \geq 0 \quad \text{for } 1 \leq i < j \leq n\,.$$

These facets are called *trivial* and 3-*cycle* facets[4]. It follows at once that every vertex of P_n has lattice height at most one over every facet. This property is known to characterize *compressed* polytopes [Sul06, Theorem 2.4], so P_n is compressed. By the definition of compressed, every reverse lexicographical Gröbner basis of the toric ideal of P_n is squarefree, so in particular P_n is normal for $n \leq 5$. Moreover, the element $v_{id_n} + v_{\omega_{0,n}}$ of the affine monoid generated by P_n has lattice height exactly 1 over every facet. The

[4]The trivial facets and the 3-cycle facets lie in different orbits with respect to the S_n-symmetry, therefore the different names. They are symmetric under the bigger symmetry group S_{n+1}.

existence of such an element characterizes the Gorenstein property for normal affine monoids [BG09, Ex. 2.13], so $\mathbb{K}[Q(P_n)]$ is Gorenstein.

From $n = 6$ on, the situation is more complicated. To begin with, there are more classes of facets. In particular, for every $n \leq 6$, there are the so-called 3-*fence* inequalities [Mar95, Theorem 6.9]. If v_π is contained in a 3-fence facet, then $v_{\omega_{0,n}\pi}$ has lattice height ≥ 2 over that facet. Thus, P_n is no longer compressed. Moreover, the element $v_{id_n} + v_{\omega_{0,n}}$ is still the unique element having lattice height 1 over the trivial and the 3-cycle facets, but is has lattice height ≥ 2 over the 3-fence facets. Therefore, there exist no element in $Q(P_n)$ having lattice height 1 over every facet. Thus, if P_n is normal then $\mathbb{K}[Q(P_n)]$ cannot be Gorenstein. Finally, a result by Sullivant [Sul06, Corollary 2.5] states that if P is a non-compressed polytope with a vertex-transitive symmetry group, then no reverse lexicographical Gröbner basis of its toric ideal is squarefree. On the other hand, every generator if a reverse lexicographical initial ideal of degree at most 3 is squarefree, by Lemma 5.3.3. Hence, every reverse lexicographical Gröbner basis of I_{LOP} for $n \geq 6$ has generators of degree at least 4. We summarize what we have proven:

Proposition 5.3.4. *The linear ordering polytope is compressed if and only if $n \leq 5$. For $n \leq 5$, the monoid algebra $\mathbb{K}[Q(P_n)]$ is normal and Gorenstein. For $n \geq 6$, no reverse lexicographic Gröbner basis of its toric ideal is squarefree. Moreover, if P_n is normal for $n \geq 6$, then $\mathbb{K}[Q(P_n)]$ is not Gorenstein.*

The normality and Gorensteinness for $n \leq 5$ has also been obtained computationally in [SW12, Theorem 6.1] using the software normaliz [BIS10]. Moreover, in the same paper, it is shown computationally that P_6 is normal (and thus not Gorenstein). Further, the toric ideal of P_n is generated quadratically for $n \leq 5$, but from $n = 6$ on, there are generators of higher degree. The normality for $n \geq 7$ and a description of the generators of the toric ideal for $n \geq 6$ are still open problems.

A. Appendix

A.1. Graded commutative algebra

In this appendix, we give some general results on (commutative) graded rings. Many textbooks on commutative algebra like [Eis95] or [BS98] consider mainly local rings or rings with a $\mathbb{Z}$-grading over a field. But this turns out to be too restrictive for our applications. An affine monoid algebra $\mathbb{K}[Q]$ is a $\mathbb{Z}Q$-graded ring, and if Q is not positive, then it is not possible to specialize the grading to $\mathbb{Z}$ in a reasonable way. $\mathbb{Z}Q$ is a finitely generated free abelian group and thus isomorphic to $\mathbb{Z}^n$ for some $n \in \mathbb{N}$. Therefore, we provide formulations of some classical results for $\mathbb{Z}^n$-graded rings. It is possible to generalize even further by allowing torsion or not finitely generated grading groups. We will not consider this, but the interested reader may consult [Kam95] or [Joh12]. With view on our intended application, we make the convention that *all rings are commutative and Noetherian*.

A.1.1. Basic properties

We start by recalling the basic definitions.

Definition A.1.1. 1. A ring R is called $\mathbb{Z}^n$*-graded* if it has a decomposition

$$R = \bigoplus_{g \in \mathbb{Z}^n} R_g$$

into abelian groups $R_g, g \in \mathbb{Z}^n$ such that $R_g R_h \subset R_{g+h}$ for all $g, h \in R$.

2. An R-module M is called $\mathbb{Z}^n$*-graded* if it has a decomposition

$$M = \bigoplus_{g \in \mathbb{Z}^n} M_g$$

and $R_g M_h \subset M_{g+h}$ for all $g, h \in \mathbb{Z}^n$. The elements $m \in M_g$ for some $g \in \mathbb{Z}^n$ are called *homogeneous*.

3. A submodule $N \subset M$ of a $\mathbb{Z}^n$-graded R-module is called *homogeneous* if it can be generated by homogeneous elements of M. Equivalently, N is homogeneous if it is $\mathbb{Z}^n$-graded and the grading is compatible with the grading on M, in the sense that $N_g = N \cap M_g$ for every $g \in \mathbb{Z}^n$.

4. A homomorphism $\varphi : M \to N$ between $\mathbb{Z}^n$-graded R-modules M and N is called *homogeneous of degree d* if $\varphi(M_g) \subset N_{g+d}$. A homogeneous homomorphism of degree 0 is just called *homogeneous*.

5. If M is a $\mathbb{Z}^n$-graded R-module and $g \in \mathbb{Z}^n$, then $M(g)$ is the graded module given by $M(g)_h = M_{g+h}$.

The only kind of grading we consider is the $\mathbb{Z}^n$-grading. Therefore, in the sequel we will omit the prefix and just write *graded* for *$\mathbb{Z}^n$-graded*. The kernel and the image of a homogeneous homomorphism are homogeneous. In particular, the annihilators of homogeneous elements in a $\mathbb{Z}^n$-graded module are homogeneous. As usual, $\operatorname{Spec} R$ denotes the set of prime ideals in R. Extending this, we write ${}^*\operatorname{Spec} R$ for the set of homogeneous prime ideals in R.

Let $\mathfrak{p}$ be a prime ideal in the graded ring R. We write $\mathfrak{p}_*$ for the ideal generated by the homogeneous elements in $\mathfrak{p}$. The ideals $\mathfrak{p}_*$ are an important tool to carry results from the local to the graded setting.

Lemma A.1.2 (Lemma 4.9,[BG09])**.** *Let R be a graded ring.*

1. If $\mathfrak{p} \subset R$ is a prime ideal, then $\mathfrak{p}_*$ is also prime.

2. Let M be a graded R-module.

   a) If $\mathfrak{p} \in \operatorname{Supp} M$, then $\mathfrak{p}_* \in \operatorname{Supp} M$.

   b) Every associated prime of M is homogeneous and the annihilator of a homogeneous element of M.

Definition A.1.3. An homogeneous ideal $\mathfrak{m}$ in a graded ring R is called **maximal*, if every homogeneous ideal properly containing $\mathfrak{m}$ is the whole ring R. A graded ring R is called **local* if it has a unique *maximal ideal.

A *maximal ideal does not need to be maximal. However, every *maximal ideal $\mathfrak{m}$ is prime, because $\mathfrak{m} = \mathfrak{p}_*$ for every maximal ideal $\mathfrak{p}$ containing $\mathfrak{m}$. Note that a local ring is *local with respect to the trivial grading, therefore this notion generalizes local rings. We give a simple criterion for detecting *local rings. Quite surprisingly, the "if"-part does not seem to be in the literature.

Proposition A.1.4. *A graded ring R is *local if and only if R_0 is local.*

Proof. If $\mathfrak{m}$ is the unique *maximal ideal in R, then $\mathfrak{m}_0 := \mathfrak{m} \cap R_0$ is the unique maximal ideal in R_0. On the other hand, assume that R_0 is local with maximal ideal $\mathfrak{m}_0$. Define $\mathfrak{m} \subset R$ as the ideal generated by all homogeneous non-units in R. Clearly, $\mathfrak{m}$ is homogeneous and contains every homogeneous ideal. So it remains to prove that $\mathfrak{m}$ is a proper ideal. Assume on the contrary that $1 \in \mathfrak{m}$. Then there is an expression

$1 = \sum_i f_i g_i$, where $f_i \in R$ and g_i are homogeneous non-units of R. We may assume that every f_i is homogeneous and we may further assume that every term $f_i g_i$ has degree 0. Since g_i is not a unit it holds that $f_i g_i$ is a non-unit of degree 0 and thus contained in $\mathfrak{m}_0$. Hence $1 \in \mathfrak{m}_0$, a contradiction. □

Corollary A.1.5 (Proposition 1.1.7 [GW78])**.** *If R is a graded ring such that R_0 is a field, then R is *local.*

In particular, affine monoid algebra are *local. We give another very useful lemma.

Lemma A.1.6. *Let R be a graded ring and let M be a graded R-module. If $M_\mathfrak{m} = 0$ for all *maximal ideals $\mathfrak{m}$ of R, then $M = 0$.*

Proof. If $M \neq 0$, then there exists a homogeneous non-zero element $m \in M$. The annihilator of m is a homogeneous ideal and thus contained in a *maximal ideal $\mathfrak{p}$. Therefore, m does not go to zero in the localization $M_\mathfrak{p}$ and thus $M_\mathfrak{p} \neq 0$. □

The special case of a *local ring is central, so we state it for the ease of reference. It is well-known in the $\mathbb{Z}$-graded case.

Corollary A.1.7. *If $(R, \mathfrak{m})$ is a *local ring and M a graded R-module, then $M = 0$ if and only if $M_\mathfrak{m} = 0$.*

Let R be a graded ring and $\mathfrak{p}$ a homogeneous prime ideal of R. We write $R_{(\mathfrak{p})}$ for the *homogeneous localization* at $\mathfrak{p}$, that is the localization at the multiplicatively closed set of all homogeneous elements of $R \setminus \mathfrak{p}$.

A.1.2. Generators and minimal free resolutions

Definition A.1.8. Let $(R, \mathfrak{m})$ be a *local ring and let M be a finitely generated graded R-module. A *minimal graded free resolution* of M is an exact sequence

$$\mathcal{F} : \ldots \to F_1 \overset{\varphi_1}{\to} F_0 \overset{\varphi_0}{\to} M \to 0$$

of free graded R-modules F_i with homogeneous maps φ_i, such that $\Im \varphi_i \subset \mathfrak{m} F_i$.

Proposition A.1.9. *Every finitely generated graded R-module has a minimal graded free resolution.*

Proof. This can be proven as in the $\mathbb{Z}$-graded case (cf. [BH98, p. 36]). The idea is as follows: Let $g_1, \ldots, g_r$ be a minimal set of homogeneous generators of M. Set $F_0 = \bigoplus_i R(-\deg g_i)$ and define $\varphi_0(e_i) := g_i$, where e_i is the generator of the i^{th} summand of F_0. Then one shows that $\ker \varphi_0 \subset \mathfrak{m} F_0$. Iterating this construction yields the minimal graded free resolution. □

Definition A.1.10. The *projective dimension* of M, $\operatorname{pd} M$, is the minimal length of a graded free resolution of M.

Recall that the localization at the *maximal ideal is an exact functor. Therefore, the localization of a graded free resolution $\mathcal{F}$ of M is a free resolution $\mathcal{F}_{\mathfrak{m}}$ of $M_{\mathfrak{m}}$. Moreover, $\mathcal{F}$ is minimal if and only if $\mathcal{F}_{\mathfrak{m}}$ is minimal. For the next proposition, recall that it makes no difference if we compute Tor in the graded category or in the category of all R-modules (cf. [BG09, p.207]).

Proposition A.1.11. *Every minimal graded free resolutions of M has the same length $\operatorname{pd} M$. Further, $\operatorname{pd} M$ is the minimal $i \in \mathbb{N}_0$, such that $\operatorname{Tor}^R_{i+1}(R/\mathfrak{m}, M) = 0$.*

Proof. This can be proven as in the ungraded case, see [Eis95, Corollary 19.5]. Alternatively, it follows from the corresponding result in the ungraded case by localization at $\mathfrak{m}$. For this, recall that $\operatorname{Tor}^R_{i+1}(R/\mathfrak{m}, M)_{\mathfrak{m}} \cong \operatorname{Tor}^{R_{\mathfrak{m}}}_{i+1}(R_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}}, M_{\mathfrak{m}})$, cf. [Rot09, Prop 7.17]. Moreover, minimal graded free resolutions localize to minimal free resolution. □

Corollary A.1.12. *Let $(R, \mathfrak{m})$ be a *local ring and let M be a finitely generated graded R-module. Then*

$$\operatorname{pd}_R M = \operatorname{pd}_{R_{\mathfrak{m}}} M_{\mathfrak{m}}$$

Proof. This is immediate from the foregoing. □

A.1.3. Modules of finite length and *Artinian modules

Let R be a graded ring. A graded R-module is called **simple* if it has no proper graded submodules. A **composition series* for a graded R-module M is a descending chain of graded submodules of M

$$M = M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq \cdots \supsetneq M_r = 0$$

such that all successive quotients M_i/M_{i+1} are *simple. If M has a *composition series of length r, then we call r the **length* of M and denote it by $^*\ell(M)$. As in the ungraded case, one proves that the *length does not depend on the choice of the *composition series.

Proposition A.1.13. *Let $(R, \mathfrak{m})$ be a *local ring and let M be a graded R-module. Then $^*\ell(M) = \ell(M_{\mathfrak{m}})$. In particular, M has finite *length over R if and only if $M_{\mathfrak{m}}$ has finite length over $R_{\mathfrak{m}}$, and M is *simple if and only if $M_{\mathfrak{m}}$ is simple.*

Proof. Consider a strictly descending chain of graded submodules of M:

$$M = M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq \dots \tag{A.1}$$

Apply Lemma A.1.7 to the successive quotients M_i/M_{i+1} to see that localization at $\mathfrak{m}$ gives a strictly descending chain of submodules of $M_\mathfrak{m}$. Hence ${}^*\ell(M) \leq \ell(M_\mathfrak{m})$. If ${}^*\ell(M) = \infty$, then we are done. Otherwise, assume that (A.1) is a *composition series of M. The quotients M_i/M_{i+1} are *simple and thus isomorphic to $R/\mathfrak{m}(q_i)$ for some $q_i \in \mathbb{Z}^n$. Localizing with $\mathfrak{m}$ yields $R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}$, a field, and thus a simple $R_\mathfrak{m}$-module. So the localization of (A.1) is a composition series for $M_\mathfrak{m}$. If follows that ${}^*\ell(M) = \ell(M_\mathfrak{m})$. □

Corollary A.1.14. *Let R be a graded ring, $M \neq 0$ be a finitely generated graded R-module and $\mathfrak{p} \in \operatorname{Supp} M$ homogeneous. Then the following are equivalent:*

1. *$\mathfrak{p}$ is a minimal prime over $\operatorname{Ann} M$.*
2. *$M_{(\mathfrak{p})}$ is a $R_{(\mathfrak{p})}$-module of finite *length.*
3. *A power of $\mathfrak{p}_{(\mathfrak{p})}$ annihilates $M_{(\mathfrak{p})}$ over $R_{(\mathfrak{p})}$.*

Proof. This can be proven as in the ungraded case, see Corollary 2.18 and Corollary 2.19 in [Eis95]. Alternatively, it can be deduced from the ungraded case by localization at $\mathfrak{p}$. For this, it is enough to note that all three conditions are invariant under localization at $\mathfrak{p}$. The first condition is obviously invariant the second and third conditions are invariant by Lemma A.1.13 and Lemma A.1.7. □

We say that a graded R-module M is **Artinian* if every descending chain of submodules stabilizes. Note that *Artinian modules need not to be Artinian. For example $\mathbb{K}[x, x^{-1}]$ with $\deg x = 1$ is a *simple $\mathbb{Z}$-graded ring and thus *Artinian, but not Artinian. As in the ungraded setting, one can show that M has finite *length if and only if M is *Artinian and Noetherian. Therefore, over a *local ring $(R, \mathfrak{m})$, a finitely generated graded module M is *Artinian if and only if $M_\mathfrak{m}$ is Artinian. We generalize this to not necessarily finitely generated modules.

Lemma A.1.15. *Let $(R, \mathfrak{m})$ be a *local ring. A graded R-module M is *Artinian if and only if $M_\mathfrak{m}$ is Artinian.*

Proof. Suppose that $M_\mathfrak{m}$ is Artinian. Let $M = M_1 \supseteq M_2 \supseteq M_3 \supseteq \dots$ be a descending chain of graded submodules of M. Localizing at $\mathfrak{m}$ gives a descending chain of submodules of $M_\mathfrak{m}$. By our hypothesis, this chain stabilizes, so there is an n_0 such that $0 = M_{i,\mathfrak{m}}/M_{i+1,\mathfrak{m}} = (M_i/M_{i+1})_\mathfrak{m}$ for all $i \geq n_0$. But a graded module vanishes if and only if its localization at the *maximal ideal vanishes, hence the descending sequence of M stabilizes.

Now suppose that M is *Artinian. The converse is more difficult, because we cannot lift a decreasing chain in $M_\mathfrak{m}$ to a decreasing chain of homogeneous submodules of M. Instead, we use the following criterion: $M_\mathfrak{m}$ is Artinian if and only if it is $\mathfrak{m}$-torsion (i.e. $M_\mathfrak{m} = H^0_{\mathfrak{m}_\mathfrak{m}}(M_\mathfrak{m})$) and its socle has finite rank as a $R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}$-vector space, cf. [Iye+07, Theorem A.33]. For the first condition, it is enough to show that $M = H^0_\mathfrak{m}(M)$, because

$H^0_{\mathfrak{m}}(M)_{\mathfrak{m}} \cong H^0_{\mathfrak{m}_{\mathfrak{m}}}(M_{\mathfrak{m}})$. Moreover, M is generated by its homogeneous elements, therefore it suffices to show that every homogeneous element of M is contained in $H^0_{\mathfrak{m}}(M)$. For this, let $m \in M$ be a homogeneous element. The descending chain $Rm \supset \mathfrak{m}m \supset \mathfrak{m}^2 m \supset \dots$ stabilizes, so there exists a $t \in \mathbb{N}$ such that $\mathfrak{m}^t m = \mathfrak{m}^{t+1} m$. By Nakayama's lemma it follows that $\mathfrak{m}^t m = 0$ and thus $m \in H^0_{\mathfrak{m}}(M)$.

For the second condition consider the socle $\operatorname{soc}_R M := \{\, m \in M \mid \mathfrak{m}m = 0 \,\}$ of M. One easily shows that this is a homogeneous submodule of M and that $(\operatorname{soc}_R M)_{\mathfrak{m}} = \operatorname{soc}_{R_{\mathfrak{m}}} M_{\mathfrak{m}}$. Moreover, it holds that $\operatorname{rank}_{R_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}}} \operatorname{soc}_{R_{\mathfrak{m}}} M_{\mathfrak{m}} = \ell(\operatorname{soc}_{R_{\mathfrak{m}}} M_{\mathfrak{m}}) = {}^*\ell(\operatorname{soc}_R M) = \operatorname{rank}_{R/\mathfrak{m}} \operatorname{soc}_R M$, where we used Lemma A.1.13. But $\operatorname{soc}_R M$ is *Artinian (because it is a submodule of M) as an R-module and thus as an $R/\mathfrak{m}$-module. Hence its rank is finite. □

A.1.4. *Dimension and *Depth

Let R be a graded ring and let M be a graded R-module. Recall that the *dimension* of M is defined as the supremum over the lengths of strictly increasing chains of primes in $\operatorname{Supp} M$. If M is finitely generated, then this equals the dimension of $R/\operatorname{Ann} M$. The graded analogue of the dimension can be defined as follows:

Definition A.1.16. The **dimension* if M is the supremum over the lengths of strictly increasing chains of homogeneous primes in $\operatorname{Supp} M$.

In general, the *dimension of M may be strictly smaller that the dimension of M. For example, consider $R = \mathbb{K}[x, x^{-1}]$ for a field $\mathbb{K}$ and an indeterminate t of degree 1. Then ${}^*\!\dim R = 0$, but $\dim R = 1$. Now we turn to the concept of *depth for *local rings.

Definition A.1.17. Let $(R, \mathfrak{m})$ be a *local ring and let M be a finitely generated graded R-module. We define the **depth* of M as the maximal length of an M-sequence in $\mathfrak{m}$, in other words ${}^*\!\operatorname{depth} M := \operatorname{grade}(\mathfrak{m}, M)$.

Note that we do not require the M-sequence to consist of homogeneous elements. Indeed, if we have a $\mathbb{Z}$-grading, then we can always find a homogeneous M-sequence of length ${}^*\!\operatorname{depth} M$ (cf. [BH98, Prop. 1.5.11]), but in general this is not possible. The classical example is the $\mathbb{Z}^2$-graded ring $R = \mathbb{K}[x, y]/(xy)$. It holds that ${}^*\!\operatorname{depth} R = 1$, but every homogeneous element in R is a zerodivisor.

The main result for comparing the *dimension and *depth of a module with its ungraded counterparts is the following.

Theorem A.1.18. *Let R be a graded ring, M a graded R-module and $\mathfrak{p} \in \operatorname{Supp} M$. Let $d(\mathfrak{p}) := \operatorname{ht} \mathfrak{p}/\mathfrak{p}_*$.*

1. *It holds that* $\dim M_{\mathfrak{p}} = {}^*\!\dim M_{(\mathfrak{p})} + d(\mathfrak{p})$.
2. *Assume additionally that M is finitely generated. Then it holds that* $\operatorname{depth} M_{\mathfrak{p}} = {}^*\!\operatorname{depth} M_{(\mathfrak{p})} + d(\mathfrak{p})$.

Proof. This proof is mainly a collection of known results.

1. We start with the formula for the dimension. By [GW78, Prop. 1.2.2], it holds that $\dim M_\mathfrak{p} = \dim M_{\mathfrak{p}_*} + d(\mathfrak{p})$. So we may assume that $\mathfrak{p}$ is homogeneous and it remains to prove that $\dim M_\mathfrak{p} = {}^*\!\dim M_{(\mathfrak{p})}$. Every strictly increasing chain of homogeneous prime ideals in $\operatorname{Supp} M_{(\mathfrak{p})}$ localizes to a strictly increasing chain of primes in $\operatorname{Supp} M_\mathfrak{p}$, hence ${}^*\!\dim M_{(\mathfrak{p})} \leq \dim M_\mathfrak{p}$.

   On the other hand, consider a strictly increasing chain of prime ideals in $\operatorname{Supp} M_\mathfrak{p}$ of maximal length. Its preimage in $\operatorname{Supp} M_{(\mathfrak{p})}$ is again a strictly increasing chain of prime ideals, but the ideals may not be homogeneous. However, the maximal ideal in this chain is $\mathfrak{p}$ and therefore homogeneous. Moreover, the minimal ideal $\mathfrak{p}_0$ is also homogeneous, because otherwise we could extend the chain by $(\mathfrak{p}_0)_*$, which lies in the support of M by Lemma A.1.2. But now [Uli10, Lemma 1.7] allows us to find a chain of homogeneous prime ideals from $\mathfrak{p}_0$ to $\mathfrak{p}$ of the same the length as the original one. Hence ${}^*\!\dim M_{(\mathfrak{p})} \geq \dim M_\mathfrak{p}$.

2. It holds that $\operatorname{depth} M_\mathfrak{p} = \operatorname{depth} M_{\mathfrak{p}_*} + d(\mathfrak{p})$ by [GW78, Corollary 1.2.4], so we may again assume that $\mathfrak{p}$ is homogeneous. Moreover, we may pass to the homogeneous localization at $\mathfrak{p}$ and thus assume that R is *local with unique *maximal ideal $\mathfrak{p}_*$. By the graded version of Nakayama's lemma ([Joh12, Prop. 2.30]), we have $M \neq \mathfrak{p}M$, because $\mathfrak{p} \in \operatorname{Supp} M$. So we can apply Rees's theorem ([BH98, Theorem 1.2.5]), which states that all maximal M-sequences in $\mathfrak{p}$ have the same length, and this length can be computed as

   $$\operatorname{grade}(\mathfrak{p}, M) = \min\left\{ i \in \mathbb{N}_0 \,\middle|\, \operatorname{Ext}^i_R(R/\mathfrak{p}, M) \neq 0 \right\}$$

   Since R is Noetherian and M is finitely generated, the Ext-modules carry a natural grading, cf. [GW78, p.243]. Moreover, it holds that $\operatorname{Ext}^i_R(R/\mathfrak{p}, M)_\mathfrak{p} \cong \operatorname{Ext}^i_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}, M_\mathfrak{p})$, see [Rot09, Prop. 7.39]. Therefore, the claim follows from Lemma A.1.7.

□

For a local ring R, it holds that $\dim R = \operatorname{ht} \mathfrak{p}$. So the following corollary is immediate.

Corollary A.1.19. *If $(R, \mathfrak{m})$ is *local, then ${}^*\!\dim R = \operatorname{ht} \mathfrak{m}$.*

Corollary A.1.20 (Graded Auslander-Buchsbaum formula)**.** *Let $(R, \mathfrak{m})$ be a *local ring and let M be a finitely generated graded R-module of finite projective dimension. Then the following holds:*

$$^*\!\operatorname{depth} M + \operatorname{pd} M = {}^*\!\operatorname{depth} R$$

Proof. All quantities are invariant under localization at $\mathfrak{m}$ and the local case is classical, cf. [Eis95, Theorem 19.9] □

We apply Lemma A.1.18 to give a graded version of Serre's condition (S_ℓ). Let us recall the definition.

Definition A.1.21. A finitely generated module M over a Noetherian ring R satisfies *Serre's condition* (S_ℓ) if

$$\operatorname{depth} M_\mathfrak{p} \geq \min\{\,\ell, \dim M_\mathfrak{p}\,\}$$

for every $\mathfrak{p} \in \operatorname{Spec} R$.

Proposition A.1.22. *Let R be a graded ring and M be a finitely generated graded module. Then M satisfies (S_ℓ) if and only if*

$$^*\!\operatorname{depth} M_{(\mathfrak{p})} \geq \min\left\{\,\ell,\ ^*\!\dim M_{(\mathfrak{p})}\,\right\}$$

for every homogeneous $\mathfrak{p} \in \operatorname{Spec} R$.

Proof. This follows easily from Lemma A.1.18, as noticed in [SS90, Lemma 6.2]. So assume that the condition on homogeneous prime ideals is satisfied and let $\mathfrak{p} \in \operatorname{Spec} R$. Set $d := \operatorname{ht} \mathfrak{p}/\mathfrak{p}_*$. Then the following holds:

$$\begin{aligned}
\operatorname{depth} M_\mathfrak{p} &= {}^*\!\operatorname{depth} M_{(\mathfrak{p})} + d \\
&\geq \min\left\{\,\ell,\ ^*\!\dim M_{(\mathfrak{p})}\,\right\} + d \\
&= \min\{\,\ell, \dim M_\mathfrak{p} - d\,\} + d \\
&\geq \min\{\,\ell, \dim M_\mathfrak{p}\,\}
\end{aligned}$$

□

A.1.5. Serre's condition (R_ℓ)

Let us recall the definition of Serre's condition (R_ℓ).

Definition A.1.23. A Noetherian ring R satisfies *Serre's condition* (R_ℓ) if $R_\mathfrak{p}$ is regular for every $\mathfrak{p} \in \operatorname{Spec} R$ of height at most ℓ.

Similarly to (S_ℓ), there is a graded version of (R_ℓ).

Proposition A.1.24. *If R is graded, then R satisfies Serre's condition (R_ℓ) if and only if $R_{(\mathfrak{p})}$ is regular for every homogeneous $\mathfrak{p} \in {}^*\!\operatorname{Spec} R$ of height at most ℓ.*

Proof. First assume that R satisfies (R_ℓ) and let $\mathfrak{p} \in {}^*\!\operatorname{Spec} R$ with $\operatorname{ht} \mathfrak{p} \leq \ell$. Then $R_\mathfrak{p}$ is regular and we need to show that $R_{(\mathfrak{p})}$ also regular. For this let $\mathfrak{q}$ be a maximal ideal in $R_{(\mathfrak{p})}$. Then $\mathfrak{q}_* \subset \mathfrak{p}$, so $R_{\mathfrak{q}_*}$ is regular. By [GW78, Prop. 1.2.5], it follows that $R_\mathfrak{q}$ and thus $R_{(\mathfrak{p})}$ are regular.

On the other hand, assume that $R_{(\mathfrak{p})}$ is regular for every $\mathfrak{p} \in {}^*\mathrm{Spec}\, R$ with $\mathrm{ht}\,\mathfrak{p} \le \ell$. Let $\mathfrak{q} \in \mathrm{Spec}\, R$ with $\mathrm{ht}\,\mathfrak{q} \le \ell$. Since $\mathrm{ht}\,\mathfrak{q}_* \le \mathrm{ht}\,\mathfrak{q}$, we have by assumption that $R_{(\mathfrak{q}_*)} = R_{(\mathfrak{q})}$ is regular. Now $R_\mathfrak{q}$ is a further localization of $R_{(\mathfrak{q})}$ and thus regular. □

A.1.6. Properties of the graded local cohomology

In the last part of this section, we collect several results about the local cohomology of *local rings with support at the *maximal ideal.

Theorem A.1.25. *Let $(R, \mathfrak{m})$ be a *local ring and let M be a finitely generated graded R-module. Then the local cohomology modules $H^i_\mathfrak{m}(M)$ are *Artinian for all $i \in \mathbb{N}_0$.*

Proof. We localize at the *maximal ideal and use that $H^i_\mathfrak{m}(M)_\mathfrak{m} \cong H^i_{\mathfrak{m}_\mathfrak{m}}(M_\mathfrak{m})$ (cf. [BS98, p. 4.3.3]). By Lemma A.1.15, this reduces the claim to the local case. But in the local case the result is well-known, see [BS98, p. 7.1.3]. □

Next, we give a graded version of Grothendieck's Vanishing and Non-vanishing Theorems.

Theorem A.1.26. *Let $(R, \mathfrak{m})$ be a *local ring and let M be a finitely generated graded R-module. Then*

$$ {}^*\mathrm{depth}\, M = \min \left\{ i \in \mathbb{N}_0 \;\middle|\; H^i_\mathfrak{m}(M) \neq 0 \right\} $$

and

$$ {}^*\mathrm{dim}\, M = \max \left\{ i \in \mathbb{N}_0 \;\middle|\; H^i_\mathfrak{m}(M) \neq 0 \right\} $$

Proof. Again, the local case is well-known (cf. [BS98, p. 6.2.8]) and the *local generalization follows by localizing at $\mathfrak{m}$. □

Finally, we give a graded version of Grothendieck's Finiteness Theorem. Let R be a graded ring, $\mathfrak{a} \subset R$ be a homogeneous ideal and M a finitely generated graded R-module. Following [BS98], we define

$$ f_\mathfrak{a}(M) := \min \left\{ i \in \mathbb{N}_0 \;\middle|\; H^i_\mathfrak{m}(M) \text{ is not finitely generated} \right\}. $$

Theorem A.1.27 (Grothendieck's Finiteness Theorem, 13.1.17, [BS98])**.** *Let R be a graded ring that is the homomorphic image of a regular ring. Let $\mathfrak{a}$ be a homogeneous ideal of R and let M be a finitely generated graded R-module. Then*

$$ f_\mathfrak{a}(M) = \inf \left\{ {}^*\mathrm{depth}\, M_{(\mathfrak{p})} + \mathrm{ht}(\mathfrak{a} + \mathfrak{p})/\mathfrak{p} \;\middle|\; \mathfrak{p} \in {}^*\mathrm{Spec}\, R \setminus (\mathrm{Supp}\, R/\mathfrak{a}) \right\}. \tag{A.2} $$

The proof given in [BS98] covers only the $\mathbb{Z}$-graded case, therefore we give an adapted proof for completeness.

Proof. By the ungraded version of Grothendieck's Finiteness Theorem [BS98, p. 9.5.2] it holds that

$$f_{\mathfrak{a}}(M) = \inf \{ \operatorname{depth} M_{\mathfrak{p}} + \operatorname{ht}(\mathfrak{a}+\mathfrak{p})/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R \setminus (\operatorname{Supp} R/\mathfrak{a}) \} .$$

Since ${}^*\!\operatorname{depth} M_{(\mathfrak{p})} = \operatorname{depth} M_{\mathfrak{p}}$ for every $\mathfrak{p} \in {}^*\!\operatorname{Spec} R$, we need to prove that the infimum is attained at the set ${}^*\!\operatorname{Spec} R \setminus (\operatorname{Supp} R/\mathfrak{a})$. For this, let $\mathfrak{p} \in \operatorname{Spec} R \setminus (\operatorname{Supp} R/\mathfrak{a})$ be a non-homogeneous prime ideal. By Lemma A.1.18, it holds that $\operatorname{depth} M_{\mathfrak{p}} = {}^*\!\operatorname{depth} M_{(\mathfrak{p})} + \operatorname{ht} \mathfrak{p}/\mathfrak{p}_*$. R is the image of a regular ring and thus catenary, therefore $\operatorname{ht}(\mathfrak{p}+\mathfrak{a})/\mathfrak{p} + \operatorname{ht} \mathfrak{p}/\mathfrak{p}_* = \operatorname{ht}(\mathfrak{p}+\mathfrak{a})/\mathfrak{p}_* \geq \operatorname{ht}(\mathfrak{p}_*+\mathfrak{a})/\mathfrak{p}_*$. We conclude that

$$\operatorname{depth} M_{\mathfrak{p}} + \operatorname{ht}(\mathfrak{a}+\mathfrak{p})/\mathfrak{p} \geq {}^*\!\operatorname{depth} M_{(\mathfrak{p}_*)} + \operatorname{ht}(\mathfrak{a}+\mathfrak{p}_*)/\mathfrak{p}_* .$$

Hence the infimum is attained at a homogeneous prime ideal and the claim follows. □

The following corollary is the graded analogue of Exercise 9.5.4 (i) in [BS98]. We use it in Lemma 2.3.17 and thus provide a proof for completeness.

Corollary A.1.28. *Let $(R, \mathfrak{m})$ be a *local ring that is the homomorphic image of a regular ring, and let M be a finitely generated graded R-module. Then*

$$f_{\mathfrak{m}}(M) = \inf \left\{ {}^*\!\operatorname{depth} M_{(\mathfrak{p})} + 1 \,\middle|\, \mathfrak{p} \in {}^*\!\operatorname{Spec} R,\ {}^*\!\dim R/\mathfrak{p} = 1 \right\}$$

Proof. We need to show that the infimum in (A.2) is attained at a prime $\mathfrak{p}$ with ${}^*\!\dim R/\mathfrak{p} = 1$. By Lemma A.1.27, there exists a $\mathfrak{p} \in {}^*\!\operatorname{Spec} R$ such that $f_{\mathfrak{m}}(M) = {}^*\!\operatorname{depth} M_{(\mathfrak{p})} + \operatorname{ht} \mathfrak{m}/\mathfrak{p}$. A priori, the condition ${}^*\!\dim R/\mathfrak{p} = 1$ may not be satisfied. To remedy this we choose a homogeneous prime ideal $\mathfrak{q}$ containing $\mathfrak{p}$ such that ${}^*\!\dim R/\mathfrak{q} = 1$. We compute

$$\begin{aligned} f_{\mathfrak{m}}(M) &\leq {}^*\!\operatorname{depth} M_{(\mathfrak{q})} + \operatorname{ht} \mathfrak{m}/\mathfrak{q} \\ &\leq f_{\mathfrak{q}}(M_{(\mathfrak{q})}) + \operatorname{ht} \mathfrak{m}/\mathfrak{q} \\ &\leq {}^*\!\operatorname{depth} M_{(\mathfrak{p})} + \operatorname{ht} \mathfrak{q}/\mathfrak{p} + \operatorname{ht} \mathfrak{m}/\mathfrak{q} \\ &= f_{\mathfrak{m}}(M) \end{aligned}$$

Here we used Lemma A.1.27 for the first and the third inequality. Moreover, for the last inequality we used that R is catenary and thus $\operatorname{ht} \mathfrak{q}/\mathfrak{p} + \operatorname{ht} \mathfrak{m}/\mathfrak{q} = \operatorname{ht} \mathfrak{m}/\mathfrak{p}$. Now $\mathfrak{q}$ is an ideal with ${}^*\!\dim R/\mathfrak{q} = 1$ attaining the infimum in (A.2), so the proof is complete. □

A.2. Blocks and modules

In this section, we prove the following theorem:

Theorem (Lemma 5.1.7)**.** *Let $I \subset [n]$ and $\pi \in S_n$. The following implications hold:*

1. *I is a π-block $\Longrightarrow$ I is a module of $G(\pi)$*
2. *I is a strong π-block $\Longleftrightarrow$ I is a strong module of $G(\pi)$*

For the rest of this section, let $\pi \in S_n$ denote a fixed permutation. For brevity, we write block for π-block and modules are to be understood as modules of $G(\pi)$. Recall that a block is an interval whose image under π is again an interval. The first statement of Lemma 5.1.7 is a direct consequence of the following lemma.

Proposition A.2.1. *Let $I \subset [n]$ be an interval. Then I is a module if and only if it is a block.*

Proof.

$$\begin{aligned} I \text{ module } &\Leftrightarrow \forall i \in [n] \setminus I : [\exists j \in I : ij \in G(\pi) \Rightarrow \forall j \in I : ij \in G(\pi)] \\ &\Leftrightarrow \forall i \in [n] \setminus I : [\exists j \in I : \pi(i) < \pi(j) \Rightarrow \forall j \in I : \pi(i) < \pi(j)] \\ &\Leftrightarrow \forall i \in [n] \setminus I : \pi(i) < \pi(I) \text{ or } \pi(i) > \pi(I) \\ &\Leftrightarrow I \text{ block} \end{aligned}$$

□

We split the proof of the second part of Lemma 5.1.7 into three lemmata. For a set $S \subset [n]$ we define $S_< := \{ x \in [n] \mid x < S \}$ and similarly $S_>$. We also define $S_{><} := [n] \setminus (S_< \cup S \cup S_>) = \{ x \in [n] \mid \exists a, b \in S : a < x < b, x \notin S \}$.

Lemma A.2.2. *Let M be a module. Then $\pi(M_< \cup M_>) = \pi(M)_< \cup \pi(M)_>$ and $\pi(M_{><}) = \pi(M)_{><}$.*

Proof. Let i be in $M_<$. If $ij \in G(\pi)$ for all $j \in M$, then $\pi(i) \in \pi(M)_>$. Otherwise $ij \notin G(\pi)$ for all $j \in M$ and $\pi(i) \in \pi(M)_<$. A similar argument for $i \in M_>$ proves that $\pi(M_< \cup M_>) \subset \pi(M)_< \cup \pi(M)_>$. For $i \in M_{><}$ there exist $j, k \in M$ with $j < i < k$. If $ij \in G(\pi)$, then also $ik \in G(\pi)$ and therefore $\pi(j) > \pi(i) > \pi(k)$. Otherwise $\pi(j) < \pi(i) < \pi(k)$. Hence $\pi(M_{><}) \subset \pi(M)_{><}$. Equality follows for both inclusions because π is bijective. □

Lemma A.2.3. *Every strong module is a strong block.*

Proof. Let M be a strong module. Assume that M is not an interval. We write $M \cup M_{><} = M_1 \cup M_2 \cup \ldots \cup M_l$ where the M_i are the interval components of M and $M_{><}$ and $M_1 < M_2 < \ldots < M_l$. We proceed by proving the following list of claims:

1. $M \cup M_{><}$ is a module.
2. $M_{><}$ is a module.

3. Either $\pi(M_1) < \pi(M_2) < \ldots < \pi(M_l)$ or $\pi(M_1) > \pi(M_2) > \ldots > \pi(M_l)$.

4. $M_1 \cup M_2$ is a module.

Once the claim are proven, the last claim yields a contradiction to the assumption that M is strong, because $M_1 \subset M$ and $M_2 \cap M = \emptyset$. Hence M must be an interval. By Lemma A.2.1 we conclude that it is a block. Every other block is also a module, hence the strongness as a block follows from the strongness as a module. We prove the claims one after the other:

1. From Lemma A.2.2 we know $\pi(M_{><}) = \pi(M)_{><}$ and hence $\pi(M \cup M_{><}) = \pi(M) \cup \pi(M)_{><}$. Thus this set is a block and the claim follows from Lemma A.2.1.

2. Because $M \cup M_{><}$ is a module, by Lemma 5.1.3 it suffices to prove that $M_{><}$ is a module of $M \cup M_{><}$. Let $i, j \in M_{><}, k \in M$ and $ik \in G(\pi)$. We need to prove $jk \in G(\pi)$. Choose $k_1, k_2 \in M$ such that $k_1 < i, j < k_2$. Because $ik \in G(\pi)$ and M is a module we know that $k_1 i, ik_2 \in G(\pi)$. Now we use Lemma 5.1.5 to conclude:

$$\begin{aligned} k_1 i, ik_2 \in G(\pi) &\Rightarrow k_1 k_2 \in G(\pi) \\ &\Rightarrow k_1 j, jk_2 \in G(\pi) \\ &\Rightarrow jk \in G(\pi) \end{aligned}$$

3. It suffices to prove for every $1 < i < l$: Either $\pi(M_{i-1}) < \pi(M_i) < \pi(M_{i+1})$ holds or the corresponding statement with '>' holds. If this is wrong, there are $x_k \in M_k, k \in \{ i-1, i, i+1 \}$ with $\pi(x_{i-1}) > \pi(x_i) < \pi(x_{i+1})$ or $\pi(x_{i-1}) < \pi(x_i) > \pi(x_{i+1})$. Assume that the first alternative holds, the other case is similar. Then $x_{i-1}x_i \in G(\pi)$ and $x_i x_{i+1} \notin G(\pi)$. But both edges are in $MM_{><}$, so this is a contradiction to the previous claim.

4. Since $M_1 \cup M_2$ is an interval, by Lemma A.2.1 it suffices to prove that $\pi(M_1 \cup M_2)$ is also an interval. For $x \in [n] \setminus (M_1 \cup M_2)$, it holds that either $x \in M_< \cup M_>$ or $x \in M_3 \cup \ldots \cup M_l$. In the first case we know by Lemma A.2.2 that $\pi(x) \in \pi(M)_< \cup \pi(M)_> \subset \pi(M_1 \cup M_2)_< \cup \pi(M_1 \cup M_2)_>$. For $x \in M_3 \cup \ldots \cup M_l$ it follows from the previous claim that $\pi(x) \in \pi(M_1 \cup M_2)_< \cup \pi(M_1 \cup M_2)_>$. Therefore, $\pi([n] \setminus (M_1 \cup M_2)) \subset \pi(M_1 \cup M_2)_< \cup \pi(M_1 \cup M_2)_>$. Because $M_1 \cup M_2$ is an interval we can conclude from this that $\pi(M_1 \cup M_2)_{><} = \emptyset$, thus the claim follows.

□

Lemma A.2.4. *Every strong block is a strong module.*

Proof. Suppose $I \subset [n]$ is a strong block. By Lemma A.2.1 I is a module. Thus it remains to prove that it is strong, so assume the contrary. By Lemma 5.1.4 it is the union of m.s.s. of a strong module M'. Write $M' = M_1 \cup \ldots \cup M_l$, where the M_i are the m.s.s. We have already proven in Lemma A.2.3 that they are intervals. Choose two consecutive ones M_i, M_{i+1} such that $M_i \subset I$ and $M_{i+1} \cap I = \emptyset$. Then $M_i \cup M_{i+1}$ is an interval by construction and a module by Lemma 5.1.4. Therefore, it is a block by Lemma A.2.1. But this is a contradiction to the hypothesis that I is strong. $\square$

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