Philipps Universität Marburg

Analysis on the Oshima compactification of a Riemannian symmetric space of non-compact type

Dissertation

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Abstract

Consider a Riemannian symmetric space space $\mathbb{X} = G/K$ of non-compact type, where G is a connected, real, semi-simple Lie group, and K a maximal compact subgroup of G. Let $\widetilde{\mathbb{X}}$ be its Oshima compactification, and $(\pi, C(\widetilde{\mathbb{X}}))$ the regular representation of G on $\widetilde{\mathbb{X}}$. In this thesis, we examine the convolution operators $\pi(f)$, for rapidly decaying functions f on G, and characterize them within the framework of totally characteristic pseudo-differential operators, describing the singular nature of their Schwartz kernels. In particular, we obtain asymptotics for the heat and resolvent kernels associated to strongly elliptic operators on $\widetilde{\mathbb{X}}$. Based on the description of the Schwartz kernels we define a regularized trace for the operators $\pi(f)$, yielding a distribution on G. We then show a regularity result for this distribution, and in fact prove a fixed-point formula for it, analogous to the Atiyah-Bott fixed-point formula for parabolically induced representations. Finally, we make some preliminary computations that suggest a possible development of scattering theory on symmetric spaces, and in the light of results earlier in the thesis, indicate some lines along which this could be done.

Deutsche Zusammenfassung

Sei $\mathbb{X} = G/K$ ein Riemannscher symmetrischer Raum vom nicht-kompakten Typ, wobei G eine zusammenhängende, reelle, halb-einfache Lie-Gruppe und K eine maximal kompakte Untergruppe von G ist. Es bezeichne desweiteren $\widetilde{\mathbb{X}}$ die Oshima-Kompaktifizierung von $\mathbb{X}$ und $(\pi, C(\widetilde{\mathbb{X}}))$ die reguläre Darstellung von G auf $\widetilde{\mathbb{X}}$. In dieser Arbeit untersuchen wir Konvolutionsoperatoren der Form $\pi(f)$ für schnell-fallende Funktionen f auf G und charakterisieren diese Operatoren innerhalb der Theorie der total-charakteristischen Pseudodifferential-Operatoren. Dadurch sind wir in der Lage, die Singularitäten ihrer Schwartz-Kerne zu beschreiben. Insbesondere erhalten wir Asymptotiken für die Wärme- und Resolventenkerne von stark elliptischen Operatoren auf $\widetilde{\mathbb{X}}$. Ausgehend von der Beschreibung der Schwartz-Kerne definieren wir desweiteren eine regularisierte Spur für die Operatoren $\pi(f)$ und erhalten eine Distribution auf G. Wir zeigen dann ein Regularitätsergebnis für diese Distribution und beweisen eine Fixpunkt-Formel für dieselbe, welche analog zur Atiyah-Bott-Fixpunktformel für parabolisch induzierte Darstellungen ist. Schlieślich führen wir einige erste Ergebnisse an, welche die Möglichkeit der Entwicklung einer Streutheorie auf symmetrischen Räumen suggerieren und weisen im Lichte der in

dieser Arbeit erzielten Ergebnisse mögliche Forschungsrichtungen auf, längs derer dies erzielt werden könnte.

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CHAPTER 1

Introduction

Symmetry is a notion that is central to both mathematics and physics, and it is in this context that group actions have been studied extensively. The theory of transformation groups, as initiated by Sophus Lie, has led to the development of modern Lie theory with its local and global aspects, the rich geometry, the elaborate algebraic structure, the subtle and deep analytic features, and a strong interplay between these aspects. Against this backdrop, harmonic analysis on semi-simple Lie groups and homogeneous spaces has been of particular relevance and interest. A homogeneous space, for us, will be a smooth manifold M with a smooth, transitive action of a Lie group G. We can then identify M with the quotient space G/H, where H is the stabilizer of some chosen point on M.

Now, there is a natural representation π of G on some suitable space of functions $\mathcal{S}(M)$ on M called the *regular representation* defined by

$$(\pi(g)f)(x) = f(g \cdot x)$$

where $g \in G, x \in M, f \in \mathcal{S}(M)$ and $\cdot$ denotes the action of G on M. A central question in harmonic analysis on G/H is then to decompose π into irreducible representations of G. When G/H admits a G-invariant measure, we can take $\mathcal{S}(M)$ to be $\mathrm{L}^2(G/H)$. π is then a *unitary* representation and we can ask for a decomposition of π into unitary, irreducible representations. This is called the Plancherel formula for G/H. Note that the group G itself can be regarded as a homogeneous space, indeed a symmetric space, in the following manner. Let $G \times G$ act on G by $(g_1, g_2) \cdot g = g_1 g g_2^{-1}$, where $g_1, g_2, g \in G$. This is a transitive action with the stabilizer at the identity element e of G being the diagonal subgroup $H = \{(g, g) | g \in G\}$, and so $G \cong (G \times G)/H$. When G is a compact Lie group, the Plancherel formula is given by the Peter-Weyl decomposition. The case of complex semi-simple Lie groups is due to Gelfand and Naimark, while those of real semi-simple Lie groups and Riemannian symmetric spaces are due to Harish Chandra. Refer to [**War72a**], [**War72b**] for the details. For the more general case of reductive symmetric spaces, this is a result of van den Ban and Schlichtkrull. We refer to [**vdB05**] for a detailed survey. A consequence of the detailed structure theory of real reductive groups is that these Plancherel theorems are *explicit*.

In addition, when one has differential operators on M that commute with the action of G, and which are essentially self-adjoint operators on $\mathrm{L}^2(G/H)$, the spectral decomposition of these operators is preserved by G. So the spectral decompositions of these invariant differential operators lead to decompositions of the regular representation into sub-representations. The spectral theory of invariant differential operators is thus central to harmonic analysis.

It is natural, then, to ask for the formulation of analogous questions when the action of G on M is no more transitive, and to look for possible answers. One would expect the existence of various orbit types to necessitate a more careful analysis, taking into account some additional singularities from the orbits of lower dimensions. In this thesis we address certain aspects of this analysis in the case that G is a real semi-simple Lie group and $M = \widetilde{\mathbb{X}}$, the so-called Oshima compactification of a Riemannian symmetric space $\mathbb{X} = G/K$, where K is a maximal compact subgroup of G. Here $\widetilde{\mathbb{X}}$ carries an action of G with some copies of the symmetric space $\mathbb{X}$ occurring as open orbits, and several orbits of lower dimensions. The rest of this introductory chapter is devoted to describing in more detail, and with more precision, the type of questions that we answer, and the way we answer them. Significant parts of this thesis are contained in the two pre-prints [**PR11a**], [**PR11b**].

Let $\mathbb{X}$ be a Riemannian symmetric space of non-compact type. Then $\mathbb{X}$ is isomorphic to G/K, where G is a connected, real, semi-simple Lie group, and K a maximal compact subgroup. Consider further the Oshima compactification [**Osh78**] $\widetilde{\mathbb{X}}$ of $\mathbb{X}$, a closed, simply connected, real-analytic manifold on which G acts analytically. The orbital decomposition of $\widetilde{\mathbb{X}}$ is of normal crossing type, and the open orbits are isomorphic to G/K, the number of them being equal to 2^l, where l denotes the rank of G/K. In this thesis, we will be primarily concerned with the study of the integral operators

$$\pi(f) = \int_G f(g)\pi(g)d_G(g), \tag{1.1}$$

where π is the regular representation of G on the Banach space $\mathrm{C}(\widetilde{\mathbb{X}})$ of continuous functions on $\widetilde{\mathbb{X}}$, f a smooth, rapidly decreasing function on G, and d_G a Haar measure on G. Such operators play an important role in representation theory, as evidenced, for example, by the theory of the Arthur-Selberg trace formula (see [**Lap10**] for a nice exposition of this theory). One of the advantages of studying integral operators associated with differential operators, instead of the differential operators themselves, is that one can use the theory of compact operators to analyze them, avoiding the use of unbounded operators.

Our initial effort will be directed towards the elucidation of the microlocal structure of the operators $\pi(f)$ within the theory of pseudodifferential operators. Since the underlying group action on $\widetilde{\mathbb{X}}$ is not transitive, the operators $\pi(f)$ are not smooth, and the orbit structure of $\widetilde{\mathbb{X}}$ is reflected in the singular behaviour of their Schwartz kernels. As it turns out, the operators in question can be characterized as pseudodifferential operators belonging to a particular class, first introduced in

[**Mel82**] in connection with boundary problems. In fact, if $\widetilde{\mathbb{X}}_\Delta$ denotes a component in $\widetilde{\mathbb{X}}$ isomorphic to G/K, we prove that the restrictions

$$\pi(f)_{|\overline{\widetilde{\mathbb{X}}_\Delta}} : \mathrm{C}^\infty_{\mathrm{c}}(\overline{\widetilde{\mathbb{X}}_\Delta}) \longrightarrow \mathrm{C}^\infty(\overline{\widetilde{\mathbb{X}}_\Delta})$$

of the operators $\pi(f)$ to the manifold with corners $\overline{\widetilde{\mathbb{X}}_\Delta}$ are totally characteristic pseudodifferential operators of class $\mathrm{L}^{-\infty}_b$. A similar description of such integral operators on prehomogeneous vector spaces was obtained in [**Ram06**].

We then consider the holomorphic semigroup generated by a strongly elliptic operator Ω associated to the regular representation $(\pi, \mathrm{C}(\widetilde{\mathbb{X}}))$ of G, as well as its resolvent. Such operators were first studied by Langlands in his unpublished thesis [**Lan60**], and their study has seen quite some development since then. We refer to [**TER96**] for a brief overview, and further references. Since both the holomorphic semigroup associated to a strongly elliptic operator and the resolvent of such an operator can be characterized as operators of the form (1.1) they can be studied by the methods that we develop. Relying, in addition, on the theory of elliptic operators on Lie groups, as in [**Rob91**], we then obtain a description of the asymptotic behaviour of the semigroup and resolvent kernels on $\widetilde{\mathbb{X}}_\Delta \simeq \mathbb{X}$ at infinity. In the particular case of the Laplace-Beltrami operator on $\mathbb{X}$, these questions have been intensively studied before. While for the classical heat kernel on $\mathbb{X}$ precise upper and lower bounds were previously obtained in [**AJ99**] using spherical analysis, a detailed description of the analytic properties of the resolvent of the Laplace-Beltrami operator on $\mathbb{X}$ was given in [**MM87**], [**MV05**]. These results on the spectral theory suggest the possibility of studying the continuous spectrum of such operators, and one way to do this is to use scattering theory. On symmetric spaces such a theory was initiated in [**STS76**] and has been developed further in [**PS93**] and [**Hel98**]. It is worth noting here that compactifcations are natural to consider from the point of view of scattering theory as they relate the spectrum of a space to the boundary of its compactification. In this context, it is conceivable that the bijection between the algebra of G-invariant differential operators on the Oshima compactification $\widetilde{\mathbb{X}}$ with real analytic coefficients and the algebra of G-invariant differential operators on $\mathbb{X}$ would play a crucial role. We make some preliminary computations in this direction and sketch an outline for further work.

To motivate further results we begin with the observation that, in his early work on infinite dimensional representations of semi-simple Lie groups, Harish–Chandra [**HC54**] realized that the correct generalization of the character of a finite-dimensional representation was a distribution on the group given by the trace of a convolution operator on the representation space. This distribution character is given by a locally integrable function which is analytic on the set of regular elements, and satisfies character formulae analogous to the finite dimensional case. Later, Atiyah and Bott [**AB68**] gave a similar description of the character of a parabolically induced representation in their work on Lefschetz fixed point formulae for elliptic complexes. It is worth noting here that they used the theory of pseudodifferential operators to obtain their formula. More precisely, let H be a closed co-compact subgroup of G, and ϱ a representation of H on a finite dimensional vector space V. If $T(g) = (\iota_* \varrho)(g)$ is the representation of G induced by ϱ in the space of sections over G/H with values in the homogeneous vector bundle $G \times_H V$, then its distribution character is given by the distribution

$$\Theta_T : \mathrm{C}^\infty_\mathrm{c}(G) \ni f \longmapsto \mathrm{Tr}\, T(f), \qquad T(f) = \int_G f(g) T(g) d_G(g),$$

where d_G denotes a Haar measure on G. The point to be noted here is that as the action of G on G/H is transitive, $T(f)$ is a smooth operator, and since G/H is compact, it does have a well-defined trace. On the other hand, assume that $g \in G$ acts on G/H only with simple fixed points. In this case, a transversal trace $\mathrm{Tr}^\flat T(g)$ of $T(g)$ can be defined within the framework of pseudodifferential operators, which is given by a sum over fixed points of g. Atiyah and Bott then showed that, on an open set $G_T \subset G$,

$$\Theta_T(f) = \int_{G_T} f(g)\, \mathrm{Tr}^\flat T(g) d_G(g), \qquad f \in \mathrm{C}^\infty_\mathrm{c}(G_T).$$

This means that, on G_T, the character Θ_T of the induced representation T is represented by the locally integrable function $\mathrm{Tr}^\flat T(g)$, and its computation reduced to the evaluation of a sum over fixed points. When G is a p-adic reductive group defined over a non-Archimedean local field of characteristic zero, a similar analysis of the character of a parabolically induced representation was carried out in [**Clo84**].

In analogy with the above results, we associate a distribution on G to the regular representation coming from the G-action on the Oshima compactification of G/K and prove a corresponding regularity result. More precisely, as the G-action on $\widetilde{\mathbb{X}}$ is not transitive, the corresponding convolution operators $\pi(f)$, $f \in \mathrm{C}^\infty_\mathrm{c}(G)$, are not smooth, and therefore do not have a well-defined trace. Nevertheless they can be characterized as totally characteristic pseudodifferential operators of order $-\infty$. Using this fact, we are able to define a regularized trace $\mathrm{Tr}_{reg}\, \pi(f)$ for the operators $\pi(f)$, and in this way obtain a map

$$\Theta_\pi : \mathrm{C}^\infty_\mathrm{c}(G) \ni f \mapsto \mathrm{Tr}_{reg}(f) \in \mathbb{C},$$

which is shown to be a distribution on G. This distribution is defined to be the character of the representation π. We then show that, on a certain open set $G(\widetilde{\mathbb{X}})$ of transversal elements,

$$\mathrm{Tr}_{reg}\, \pi(f) = \int_{G(\widetilde{\mathbb{X}})} f(g)\, \mathrm{Tr}^\flat \pi(g) d_G(g), \qquad f \in \mathrm{C}^\infty_\mathrm{c}(G(\widetilde{\mathbb{X}})),$$

where, with the notation $\Phi_g(\tilde{x}) = g \cdot \tilde{x}$,

$$\operatorname{Tr}^\flat \pi(g) = \sum_{\tilde{x} \in \operatorname{Fix}(g)} \frac{1}{|\det(1 - d\Phi_{g^{-1}}(\tilde{x}))|},$$

the sum being over the (simple) fixed points of $g \in G(\widetilde{\mathbb{X}})$ on $\widetilde{\mathbb{X}}$. Thus, on the open set $G(\widetilde{\mathbb{X}})$, Θ_π is represented by the locally integrable function $\operatorname{Tr}^\flat \pi(g)$, which is given by a formula similar to the character of a parabolically induced representation. It is likely that similar distribution characters could be introduced for G-manifolds with a dense union of open orbits, or for spherical varieties, and that corresponding character formulae could be proved. This will be the subject of future work.

The thesis is organized in the following manner. In Chapter 2 we start by briefly recalling those parts of the structure theory of real semisimple Lie groups that are relevant to our purposes. We then describe the G-action on the homogeneous spaces $G/P_\Theta(K)$, where $P_\Theta(K)$ is a closed subgroup of G associated naturally to a subset Θ of the set of simple roots, and the corresponding fundamental vector fields. This leads to the definition of the Oshima compactification $\widetilde{\mathbb{X}}$ of the symmetric space $\mathbb{X} \simeq G/K$ in Section 2.2, together with a description of the orbital decomposition of $\widetilde{\mathbb{X}}$. Since this decomposition is of normal crossing type, it is well-suited for our analytic purposes. We also prove some results regarding the finer structure of the G-action on $\widetilde{\mathbb{X}}$. A thorough and unified description of the various compactifications of a symmetric space is given in [**BJ06**]. In Section 2.3, we write down the orbital decomposition of the Oshima compactification in the case that $\mathbb{X} = \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$. After recalling some results on fixed points on homogeneous spaces, we then prove that when $\operatorname{rank}(G) = \operatorname{rank}(K)$, any regular element in G acts transversally on G/K. Section 2.5 contains a summary of some of the basic facts in the theory pseudodifferential operators needed in the sequel. In particular, the class of totally characteristic pseudodifferential operators on a manifold with corners is introduced.

Chapter 3 is the core of this thesis in the sense that all the results that are proved subsequently depend on the results obtained in this chapter in a crucial way. In Section 3.1, after a brief description of Bochner integrals, we define a space $\mathcal{S}(G)$ of rapidly decreasing functions on G by making use of the bounds satisfied by general Banach representations. Taking into account the orbit structure for the G-action on $\widetilde{\mathbb{X}}$, we give a description, in local coordinates, of the operators $\pi(f)$ for functions f belonging to the space $\mathcal{S}(G)$. The analysis of the orbit structure enables us, in Section 3.2, to elucidate the microlocal structure of the convolution operators $\pi(f)$ in Theorem 2. As a corollary, we obtain a characterization of these operators as totally characteristic pseudodifferential operators on the manifold with corners $\overline{\widetilde{\mathbb{X}}_\Delta}$. This leads immediately to a description of the asymptotic behaviour of their Schwartz kernels at infinity on $\widetilde{\mathbb{X}}_\Delta \simeq \mathbb{X}$.

In Chapter 4, we consider the holomorphic semigroup S_τ generated by the closure $\overline{\Omega}$ of a strongly elliptic differential operator Ω associated to the representation π. Since $S_\tau = \pi(K_\tau)$, where $K_\tau(g)$ is a smooth and rapidly decreasing function on G, we can apply our previous results to describe the Schwartz kernel of S_τ. On the basis of L^1 and L^∞ bounds for the group kernels $K_t(g)$ we derive the asymptotics for S_τ. In Section 4.2, the Schwartz kernel of the resolvent $(\lambda\mathbf{1}+\overline{\Omega})^{-\alpha}$ is treated similarly, where $\alpha > 0$, and $\operatorname{Re}\lambda$ is sufficiently large, but it is subtler due to the singularity of the corresponding group kernel $R_{\alpha,\lambda}(g)$ at the identity. We finish the section with the observation that when we take $(\sigma, C(\widetilde{\mathbb{X}}))$ to be the regular representation coming from the action of $S = AN^-$ on $\widetilde{\mathbb{X}}$, and $\Omega = -d\sigma(C')$, where C' is a suitable modification of the Casimir operator, then the heat kernel on S associated to C' coincides with the heat kernel on $\mathbb{X}$ associated with the Laplace-Beltrami operator on $\mathbb{X}$, and that our bounds are in concordance with the classical estimates.

We begin Chapter 5 by defining a regularized trace $\mathrm{Tr}_{reg}\,\pi(f)$ for the convolution operators $\pi(f)$, using the explicit description of the singularities of the Schwartz kernels, obtained as a consequence of Theorem 2. We then show that the map $f \overset{\Theta_\pi}{\longmapsto} \mathrm{Tr}_{reg}\,\pi(f)$ is a distribution on G. The transversal trace of a pseudodifferential operator is introduced in Section 5.2, followed by a discussion of the global character of an induced representation. In Section 5.3, we prove that the distribution Θ_π is regular on the set of transversal elements $G(\widetilde{\mathbb{X}})$, and is given by a locally integrable function $\mathrm{Tr}^\flat\,\pi(g)$ which in turn is expressed as a sum over fixed points for the G-action on $\widetilde{\mathbb{X}}$, in analogy with the work of Atiyah and Bott on the Lefschetz fixed-point formula [**AB68**]. When $\mathrm{rank}(\mathbb{X}) = 1$ and $\mathrm{rank}_{\mathbb{R}}(G) = \mathrm{rank}_{\mathbb{R}}(K)$, we show that the open set $G(\widetilde{\mathbb{X}})$ contains the set G' of regular elements, and is therefore dense in G.

The final chapter, Chapter 6, intends to give an outlook at possible applications of our analysis on $\widetilde{\mathbb{X}}$ to scattering theory on symmetric spaces. At first we make some historical remarks on scattering theory on symmetric spaces. Then, in Section 6.2, we compute the invariant metric on $\mathbb{X}$ by looking at its embedding in the Oshima compactification. This enables us to compute the Laplacian on $\mathbb{X}$ in the coordinates of the embedding. We then look at the contribution from the boundary, which in the rank one case is the second-order ordinary differential operator $t^2\frac{d^2}{dt^2} + t\frac{d}{dt}$, and obtain a self-adjoint extension of this. In the final Section 6.4, we put the above computations into perspective, and outline further directions in the context of scattering theory, after surveying known results.

CHAPTER 2

Preliminaries

In this chapter, we write down some basic results that are required in subsequent chapters, and also fix notation. While a large part of the chapter deals with recalling classical and known facts, there are some new results as well.

2.1. Some structure theory of real semi-simple Lie groups

We begin by describing parts of the structure theory of real semi-simple Lie groups that are relevant to our purpose.

Let G be a connected, real, semi-simple Lie group with finite centre and with Lie algebra $\mathfrak{g}$. Define the *Cartan-Killing form* on $\mathfrak{g}$ by $\langle X, Y\rangle = \mathrm{tr}\,(\mathrm{ad}\, X \circ \mathrm{ad}\, Y)$ for $X, Y \in \mathfrak{g}$. Let θ be the Cartan involution of $\mathfrak{g}$, and

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

the Cartan decomposition of $\mathfrak{g}$ into the eigenspaces of θ, corresponding to the eigenvalues $+1$ and-1, respectively. Let K be the analytic subgroup of G corresponding to the Lie algebra $\mathfrak{k}$. For $X, Y \in \mathfrak{g}$, set

$$\langle X, Y\rangle_\theta := -\langle X, \theta Y\rangle.$$

Observe that as $\langle , \rangle$ is invariant under θ, the Cartan decomposition is orthogonal with respect to $\langle , \rangle_\theta$. Consider a maximal Abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. The dimension l of $\mathfrak{a}$ is called the *real rank of* G. Note that this is well-defined as all the maximal Abelian subspaces of $\mathfrak{p}$ are conjugate under the adjoint action of K, and hence have the same dimension. Now, $\mathrm{ad}\,(\mathfrak{a})$ is a commuting family of self-adjoint operators on $\mathfrak{g}$. Indeed, for $X, Y, Z \in \mathfrak{g}$, one computes that

$$\begin{aligned}\langle \mathrm{ad}\, X(Z), Y\rangle_\theta &= -\langle [X, Z], \theta Y\rangle = -\langle Z, [\theta Y, X]\rangle = -\langle Z, \theta[Y, \theta X]\rangle = \langle Z, [Y, \theta X]\rangle_\theta \\ &= \langle Z, -[\theta X, Y]\rangle_\theta = \langle Z, -\mathrm{ad}\,\theta X(Y)\rangle_\theta.\end{aligned}$$

So $-\mathrm{ad}\,\theta X$ is the adjoint of $\mathrm{ad}\, X$ with respect to $\langle , \rangle_\theta$. Therefore, if we take $X \in \mathfrak{a}$, then $\mathrm{ad}\, X$ is self-adjoint with respect to $\langle , \rangle_\theta$. Next, one defines for each $\alpha \in \mathfrak{a}^*$, the dual of $\mathfrak{a}$, the simultaneous eigenspaces $\mathfrak{g}^\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}$ of $\mathrm{ad}\,(\mathfrak{a})$. A functional $0 \neq \alpha \in \mathfrak{a}^*$ is called a *(restricted) root* of $(\mathfrak{g}, \mathfrak{a})$ if $\mathfrak{g}^\alpha \neq \{0\}$. Setting $\Sigma = \{\alpha \in \mathfrak{a}^* : \alpha \neq 0, \mathfrak{g}^\alpha \neq \{0\}\}$, we obtain the decomposition

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}^\alpha,$$

where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Note that this decomposition is orthogonal with respect to $\langle\cdot,\cdot\rangle_\theta$. and write A, N^+ and N^- for the analytic subgroups of G corresponding to $\mathfrak{a}$, $\mathfrak{n}^+$, and $\mathfrak{n}^-$, respectively. The *Iwasawa decomposition* of G is then given by

$$G = KAN^\pm.$$

Next, let $M = \{k \in K : \mathrm{Ad}\,(k)H = H \text{ for all } H \in \mathfrak{a}\}$ be the centralizer of $\mathfrak{a}$ in K and $M^* = \{k \in K : \mathrm{Ad}\,(k)\mathfrak{a} \subset \mathfrak{a}\}$, the normalizer of $\mathfrak{a}$ in K. The quotient $W = M^*/M$ is the *Weyl group* corresponding to the pair $(\mathfrak{g}, \mathfrak{a})$, and acts on $\mathfrak{a}$ as a group of linear transformations via the adjoint action. Alternatively, W can be characterized as follows. For each $\alpha_i \in \Delta$, define a reflection in $\mathfrak{a}^*$ with respect to the Cartan-Killing form $\langle\cdot,\cdot\rangle$ by

$$w_{\alpha_i} : \lambda \mapsto \lambda - 2\alpha_i\langle\lambda, \alpha_i\rangle/\langle\alpha_i, \alpha_i\rangle,$$

where $\langle\lambda, \alpha\rangle = \langle H_\lambda, H_\alpha\rangle$. Here H_λ is the unique element in $\mathfrak{a}$ corresponding to a given $\lambda \in \mathfrak{a}^*$, and determined by the non-degeneracy of the Cartan-Killing form, with $\lambda(H) = \langle H, H_\lambda\rangle$ for all $H \in \mathfrak{a}$. One can then identify the Weyl group W with the group generated by the reflections $\{w_{\alpha_i} : \alpha_i \in \Delta\}$. For a subset Θ of Δ, let W_Θ denote the subgroup of W generated by reflections corresponding to the elements in Θ. Define

$$P_\Theta = \bigcup_{w \in W_\Theta} P m_w P,$$

where m_w denotes a representative of w in M^*, and $P = MAN^+$ is a minimal parabolic subgroup. It is then a classical result in the theory of parabolic subgroups [**War72a**] that, as Θ ranges over the subsets of Δ, one obtains all the parabolic subgroups of G containing P. In particular, if $\Theta = \emptyset$, $P_\Theta = P$. Let us now introduce, for $\Theta \subset \Delta$, the subalgebras

$$\mathfrak{a}_\Theta = \{H \in \mathfrak{a} : \alpha(H) = 0 \text{ for all } \alpha \in \Theta\},$$

$$\mathfrak{a}(\Theta) = \{H \in \mathfrak{a} : \langle H, X\rangle_\theta = 0 \text{ for all } X \in \mathfrak{a}_\Theta\}.$$

Note that when restricted to the $+1$ or the -1 eigenspace of θ, the orthogonal complement of a subspace with respect to $\langle\cdot,\cdot\rangle$ is the same as its orthogonal complement with respect to $\langle\cdot,\cdot\rangle_\theta$. We further define

$$\mathfrak{n}_\Theta^+ = \sum_{\alpha \in \Sigma^+ \setminus \langle\Theta\rangle^+} \mathfrak{g}^\alpha, \qquad \mathfrak{n}_\Theta^- = \theta(\mathfrak{n}_\Theta^+),$$

$$\mathfrak{n}^+(\Theta) = \sum_{\alpha \in \langle\Theta\rangle^+} \mathfrak{g}^\alpha, \qquad \mathfrak{n}^-(\Theta) = \theta(\mathfrak{n}^+(\Theta)),$$

$$\mathfrak{m}_\Theta = \mathfrak{m} + \mathfrak{n}^+(\Theta) + \mathfrak{n}^-(\Theta) + \mathfrak{a}(\Theta), \qquad \mathfrak{m}_\Theta(K) = \mathfrak{m}_\Theta \cap \mathfrak{k},$$

where $\langle\Theta\rangle^+ = \Sigma^+ \cap \sum_{\alpha_i \in \Theta} \mathbb{R}\alpha_i$, and denote by $A_\Theta, A(\Theta)$, $N_\Theta^\pm, N^\pm(\Theta), M_{\Theta,0}$, and $M_\Theta(K)_0$ the corresponding connected analytic subgroups of G. Then one obtains the decompositions $A = A_\Theta A(\Theta)$ and $N^\pm = N_\Theta^\pm N(\Theta)^\pm$, the second being a semi-direct product. Next, let $M_\Theta = MM_{\Theta,0}$, and let $M_\Theta(K) = MM_\Theta(K)_0$. One has the *Iwasawa decompositions*

$$M_\Theta = M_\Theta(K)A(\Theta)N^\pm(\Theta),$$

and the *Langlands decompositions*

$$P_\Theta = M_\Theta A_\Theta N_\Theta^+ = M_\Theta(K)AN^+.$$

In particular, $P_\Delta = M_\Delta = G$, since $m_\Delta = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}^\alpha$, and $\mathfrak{a}_\Delta, \mathfrak{n}_\Delta^+$ are trivial. One then defines

$$P_\Theta(K) = M_\Theta(K)A_\Theta N_\Theta^+.$$

$P_\Theta(K)$ is a closed subgroup, and G is a union of the open and dense submanifold $N^- A(\Theta)P_\Theta(K) = N_\Theta^- P_\Theta$, and submanifolds of lower dimension, see [**Osh78**], Lemma 1. For $\Delta = \{\alpha_1, \dots, \alpha_l\}$, let $\{H_1, \dots, H_l\}$ be the basis of $\mathfrak{a}$ dual to Δ, i.e. $\alpha_i(H_j) = \delta_{ij}$ for $i \leq i, j \leq l$. Fix a basis $\{X_{\lambda,i} : 1 \leq i \leq m(\lambda)\}$ of $\mathfrak{g}^\lambda$ for each $\lambda \in \Sigma^+$. Clearly,

$$[H, -\theta X_{\lambda,i}] = -\theta[\theta H, X_{\lambda,i}] = -\lambda(H)(-\theta X_{\lambda,i}), \qquad H \in \mathfrak{a},$$

so that setting $X_{-\lambda,i} = -\theta(X_{\lambda,i})$, one obtains a basis $\{X_{-\lambda,i} : 1 \leq i \leq m(\lambda)\}$ of $\mathfrak{g}^{-\lambda} \subset \mathfrak{n}^-$. One now has the following lemma due to Oshima.

Lemma 1. *Fix an element $g \in G$, and identify $N^- \times A(\Theta)$ with an open dense submanifold of the homogeneous space $G/P_\Theta(K)$ by the map $(n, a) \mapsto gnaP_\Theta(K)$. For $Y \in \mathfrak{g}$, let $Y_{|G/P_\Theta(K)}$ be the fundamental vector field corresponding to the action of the one-parameter group $\exp(sY), s \in \mathbb{R}$, on $G/P_\Theta(K)$. Then, at any point $p = (n, a) \in N^- \times A(\Theta)$, we have*

$$\begin{aligned}(Y_{|G/P_\Theta(K)})_p = &\sum_{\lambda\in\Sigma^+} \sum_{i=1}^{m(\lambda)} c_{-\lambda,i}(g, n)(X_{-\lambda,i})_p + \sum_{\lambda\in\langle\Theta\rangle^+} \sum_{i=1}^{m(\lambda)} c_{\lambda,i}(g, n)e^{-2\lambda(\log a)}(X_{-\lambda,i})_p \\ &+ \sum_{\alpha_i\in\Theta} c_i(g, n)(H_i)_p\end{aligned}$$

with the identification $T_nN^- \bigoplus T_a(A(\Theta)) \simeq T_p(N^- \times A(\Theta)) \simeq T_{gnaP_\Theta(K)}G/P_\Theta(K)$. The coefficient functions $c_{\lambda,i}(g, n), c_{-\lambda,i}(g, n), c_i(g, n)$ are real-analytic, and are determined by the equation

$$\operatorname{Ad}^{-1}(gn)Y = \sum_{\lambda\in\Sigma^+} \sum_{i=1}^{m(\lambda)} (c_{\lambda,i}(g, n)X_{\lambda,i} + c_{-\lambda,i}(g, n)X_{-\lambda,i}) + \sum_{i=1}^{l} c_i(g, n)H_i \mod \mathfrak{m}. \tag{2.1}$$

Proof. Due to its importance, we give a detailed proof of the lemma, following the original proof given in [**Osh78**], Lemma 3. Let $s \in \mathbb{R}$, and assume that $|s|$

is small. Fix $g \in G$, and let $(n, a) \in N^- \times A(\Theta)$. According to the direct sum decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{a} \oplus \mathfrak{n}^+ \oplus \mathfrak{m}$, one has, for an arbitrary $Y \in \mathfrak{g}$,

$$(gn)^{-1} \exp(sY) gn = \exp N_1^-(s) \exp A_1(s) \exp N_1^+(s) \exp M_1(s), \tag{2.2}$$

where $N_1^-(s) \in \mathfrak{n}^-$, $A_1(s) \in \mathfrak{a}$, $N_1^+(s) \in \mathfrak{n}^+$, and $M_1(s) \in \mathfrak{m}$. The action of $\exp(sY)$ on the homogeneous space $G/P_\Theta(K)$ is therefore given by

$$\begin{aligned} \exp(sY) gnaP_\Theta(K) &= gn \exp N_1^-(s) \exp A_1(s) \exp N_1^+(s) \exp M_1(s) aP_\Theta(K) \\ &= gn \exp N_1^-(s) \exp A_1(s) \exp N_1^+(s) a \exp M_1(s) P_\Theta(K) \\ &= gn \exp N_1^-(s) \exp A_1(s) \exp N_1^+(s) aP_\Theta(K), \end{aligned}$$

since M is the centralizer of A in K, and $\exp M_1(s) \in MM_\Theta(K)_0 \subset P_\Theta(K)$. The Lie algebra of $P_\Theta(K)$ is $\mathfrak{m}_\Theta(K) \oplus \mathfrak{a}_\Theta \oplus \mathfrak{n}_\Theta^+$, which we shall henceforth denote by $\mathfrak{p}_\Theta(K)$. Using the decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{a}(\Theta) \oplus \mathfrak{p}_\Theta(K)$ we see that

$$a^{-1} \exp N_1^+(s) a = \exp N_2^-(s) \exp A_2(s) \exp P_2(s),$$

where $N_2^-(s) \in \mathfrak{n}^-$, $A_2(s) \in \mathfrak{a}(\Theta)$, and $P_2(s) \in \mathfrak{p}_\Theta(K)$. From this we obtain that

$$\begin{aligned} &gn \exp N_1^-(s) \exp A_1(s) \exp N_1^+(s) aP_\Theta(K) \\ &\qquad = gn \left(\exp N_1^-(s) \exp A_1(s) a \exp N_2^-(s)\right) \exp A_2(s) \exp P_2(s) P_\Theta(K) \\ &\qquad = gn \left(\exp N_1^-(s) \exp A_1(s) a \exp N_2^-(s) a^{-1}\right) a \exp A_2(s) P_\Theta(K). \end{aligned}$$

Noting that $[\mathfrak{a}, \mathfrak{n}^-] \subset \mathfrak{n}^-$ one deduces the equality

$$\exp N_1^-(s) \exp A_1(s) a \exp N_2^-(s) a^{-1} \exp A_1(s)^{-1} = \exp N_3^-(s) \in N^-,$$

and consequently

$$\exp N_1^-(s) \exp A_1(s) a \exp N_2^-(s) a^{-1} = \exp N_3^-(s) \exp A_1(s), \tag{2.3}$$

which in turn yields

$$\begin{aligned} &gn \exp N_1^-(s) \exp A_1(s) \exp N_1^+(s) aP_\Theta(K) \\ &\qquad = gn \exp N_3^-(s) \exp A_1(s) a \exp A_2(s) P_\Theta(K) \\ &\qquad = gn \exp N_3^-(s) a \exp(A_1(s) + A_2(s)) P_\Theta(K). \end{aligned}$$

The action of $\mathfrak{g}$ on $G/P_\Theta(K)$ can therefore be characterized as

$$\exp(sY) gnaP_\Theta(K) = gn \exp N_3^-(s) a \exp(A_1(s) + A_2(s)) P_\Theta(K). \tag{2.4}$$

Set $dN_i^-(s)/ds|_{s=0} = N_i^-$, $dN_1^+(s)/ds|_{s=0} = N_1^+$, $dA_i(s)/ds|_{s=0} = A_i$, and $dP_2(s)/ds|_{s=0} = P_2$, where $i = 1, 2$, or 3. By differentiating equations (2.2) - (2.3) at $s = 0$ one computes

$$\mathrm{Ad}^{-1}(gn) Y = N_1^- + A_1 + N_1^+ \mod \mathfrak{m}, \tag{2.5}$$

$$\mathrm{Ad}^{-1}(a) N_1^+ = N_2^- + A_2 + P_2, \tag{2.6}$$

$$N_1^- + \mathrm{Ad}(a) N_2^- = N_3^-. \tag{2.7}$$

In what follows, we express $N_1^\pm \in \mathfrak{n}^\pm$ in terms of the basis of $\mathfrak{n}^\pm$, and A_1 in terms of the one of $\mathfrak{a}$, as

$$\begin{aligned} N_1^\pm &= \sum_{\lambda\in\Sigma^+} \sum_{i=1}^{m(\lambda)} c_{\pm\lambda,i}(g,n) X_{\pm\lambda,i}, \\ A_1 &= \sum_{i=1}^{l} c_i(g,n) H_i = \sum_{\alpha_i \in \Theta} c_i(g,n) H_i \quad \mod \mathfrak{a}_\Theta. \end{aligned}$$

For a fixed $X_{\lambda,i}$ one has that $[H, X_{\lambda,i}] = \lambda(H) X_{\lambda,i}$ for all $H \in \mathfrak{a}$. Setting $H = -\log a$, $a \in A$, we get $\mathrm{ad}\,(-\log a) X_{\lambda,i} = -\lambda(\log a) X_{\lambda,i}$. Exponentiating, we obtain that $e^{\mathrm{ad}\,(-\log a)} X_{\lambda,i} = e^{-\lambda(\log a)} X_{\lambda,i}$, which, together with the relation $e^{\mathrm{ad}\,(-\log a)} = \mathrm{Ad}\,(\exp(-\log a))$, yields

$$\mathrm{Ad}^{-1}(a) X_{\lambda,i} = e^{-\lambda(\log a)} X_{\lambda,i}.$$

Analogously, one has $[H, X_{-\lambda,i}] = \theta[\theta H, -X_{\lambda,i}] = -\lambda(H) X_{-\lambda,i}$ for all $H \in \mathfrak{a}$, so that

$$\mathrm{Ad}^{-1}(a) X_{-\lambda,i} = e^{\lambda(\log a)} X_{-\lambda,i}. \tag{2.8}$$

We therefore arrive at

$$\begin{aligned} \mathrm{Ad}^{-1}(a) X_{\lambda,i} &= e^{-\lambda(\log a)} (X_{\lambda,i} - X_{-\lambda,i}) + e^{-\lambda(\log a)} X_{-\lambda,i} \\ &= e^{-\lambda(\log a)} (X_{\lambda,i} - X_{-\lambda,i}) + e^{-2\lambda(\log a)} \mathrm{Ad}^{-1}(a) X_{-\lambda,i}. \end{aligned}$$

Now, since $\theta(X_{\lambda,i} - X_{-\lambda,i}) = \theta(X_{\lambda,i}) - \theta(X_{-\lambda,i}) = -X_{-\lambda,i} - (-X_{\lambda,i}) = X_{\lambda,i} - X_{-\lambda,i}$, we see that $X_{\lambda,i} - X_{-\lambda,i} \in \mathfrak{k}$. Consequently, if λ is in $\langle\Theta\rangle^+$, one deduces that $X_{\lambda,i} - X_{-\lambda,i} \in (\mathfrak{m} + \mathfrak{n}^+(\Theta) + \mathfrak{n}^-(\Theta) + \mathfrak{a}(\Theta)) \cap \mathfrak{k} = \mathfrak{m}_\Theta(K)$. On the other hand, if λ is in $\Sigma^+ - \langle\Theta\rangle^+$, then $\mathrm{Ad}^{-1}(a) X_{\lambda,i} = e^{-\lambda(\log a)} X_{\lambda,i}$ belongs to $\mathfrak{n}_\Theta^+$. Collecting everything we obtain

$$\begin{aligned}
\mathrm{Ad}^{-1}(a)N_1^+ &= \sum_{\lambda\in\Sigma^+}\sum_{i=1}^{m(\lambda)} c_{\lambda,i}(g,n)\mathrm{Ad}^{-1}(a)X_{\lambda,i} \\
&= \sum_{\lambda\in\langle\Theta\rangle^+}\sum_{i=1}^{m(\lambda)} c_{\lambda,i}(g,n)\mathrm{Ad}^{-1}(a)X_{\lambda,i} \\
&\qquad + \sum_{\lambda\in\Sigma^+-\langle\Theta\rangle^+}\sum_{i=1}^{m(\lambda)} c_{\lambda,i}(g,n)\mathrm{Ad}^{-1}(a)X_{\lambda,i} \\
&= \sum_{\lambda\in\langle\Theta\rangle^+}\sum_{i=1}^{m(\lambda)} c_{\lambda,i}(g,n)\left(e^{-2\lambda(\log a)}\mathrm{Ad}^{-1}(a)X_{-\lambda,i}+\right. \\
&\left. e^{-\lambda(\log a)}(X_{\lambda,i}-X_{-\lambda,i})\right) + \sum_{\lambda\in\Sigma^+-\langle\Theta\rangle^+}\sum_{i=1}^{m(\lambda)} c_{\lambda,i}(g,n)e^{-\lambda(\log a)}X_{\lambda,i} \\
&= \sum_{\lambda\in\langle\Theta\rangle^+}\sum_{i=1}^{m(\lambda)} c_{\lambda,i}(g,n)e^{-2\lambda(\log a)}\mathrm{Ad}^{-1}(a)X_{-\lambda,i} \\
&+ \sum_{\lambda\in\langle\Theta\rangle^+}\sum_{i=1}^{m(\lambda)} c_{\lambda,i}(g,n)e^{-\lambda(\log a)}(X_{\lambda,i}-X_{-\lambda,i}) \\
&\qquad + \sum_{\lambda\in\Sigma^+-\langle\Theta\rangle^+}\sum_{i=1}^{m(\lambda)} c_{\lambda,i}(g,n)e^{-\lambda(\log a)}X_{\lambda,i}.
\end{aligned}$$

Comparing this with the expression (2.6) we had obtained earlier for $\mathrm{Ad}^{-1}(a)N_1^+$, we obtain that

$$A_2 = 0,$$

and $N_2^- = \sum_{\lambda\in\langle\Theta\rangle^+}\sum_{i=1}^{m(\lambda)} c_{\lambda,i}(g,n)e^{-2\lambda(\log a)}\mathrm{Ad}^{-1}(a)X_{-\lambda,i}$, since $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}^-$, and $\mathfrak{p}_\Theta(K)\cap\mathfrak{a}(\Theta) = \{0\}$. Therefore

$$\begin{gathered}
N_3^- = N_1^- + \mathrm{Ad}\,(a)N_2^- \\
= \sum_{\lambda\in\Sigma^+}\sum_{i=1}^{m(\lambda)} c_{-\lambda,i}(g,n)X_{-\lambda,i} + \sum_{\lambda\in\langle\Theta\rangle^+}\sum_{i=1}^{m(\lambda)} c_{\lambda,i}(g,n)e^{-2\lambda(\log a)}X_{-\lambda,i}, \\
A_1 + A_2 = \sum_{\alpha_i\in\Theta} c_i(g,n)H_i \mod \mathfrak{a}_\Theta.
\end{gathered}$$

As $N^- \times A(\Theta)$ can be identified with an open dense submanifold of the homogeneous space $G/P_\Theta(K)$, we have the isomorphisms $T_{gnaP_\Theta(K)}G/P_\Theta(K) \simeq T_p(N^- \times A(\Theta)) \simeq T_nN^- \bigoplus T_a(A(\Theta))$, where $p = (n,a) \in N^- \times A(\Theta)$. Therefore, by equation (2.4) and the expressions for N_3^- and $A_1 + A_2$, we finally deduce that the fundamental vector field $Y_{|G/P_\Theta(K)}$ at a point p corresponding to the action of $\exp(sY)$ on $G/P_\Theta(K)$ is given by

$$(Y_{|G/P_\Theta(K)})_p = \sum_{\lambda\in\Sigma^+}\sum_{i=1}^{m(\lambda)} c_{-\lambda,i}(g,n)(X_{-\lambda,i})_p + \sum_{\lambda\in\langle\Theta\rangle^+}\sum_{i=1}^{m(\lambda)} c_{\lambda,i}(g,n)e^{-2\lambda\log a}(X_{-\lambda,i})_p + \sum_{\alpha_i\in\Theta} c_i(g,n)(H_i)_p,$$

where $Y \in \mathfrak{g}$, and the coefficients are given by (2.1). $\square$

2.2. The Oshima compactification of a symmetric space

In this section, we briefly describe the construction, due to Oshima, of a compactification of a Riemannian symmetric space. With G and K as in the previous section, set $\mathbb{X} = G/K$. Then $\mathbb{X}$ is a symmetric space of non-compact type, and $l = \dim \mathfrak{a}$ is called the *rank of the symmetric space* $\mathbb{X}$. Note that l is also the real rank of G. The Oshima compactification $\widetilde{\mathbb{X}}$ of such a symmetric space is a closed real analytic manifold carrying a real analytic G-action, and containing the union of 2^l copies of $\mathbb{X}$ as an open dense subset. In addition, the closure of each copy of $\mathbb{X}$ contains a unique compact G-orbit which is isomorphic to G/P. Here P is the minimal parabolic subgroup determined by the Iwasawa decomposition of G.

We remark that, in this approach, one shows that the infinitesimal action of $\mathfrak{g}$ on the various orbits match to give analytic vector fields on the compactification $\widetilde{\mathbb{X}}$. This is in the spirit of Lie's original approach to the local theory of transformation groups. For a nice and detailed exposition of this we refer to Section 2.16, [**Var84**]. In his work, Lie considered the infinitesimal description of the action of an analytic group G on an analytic manifold M. Let X belong to the Lie algebra $\mathfrak{g}$ of G. For the action of the one parameter group $X \longmapsto \exp(tX)$ on G, he introduced a vector field $\tilde{X}$ on M whose integral curves are of the form $t \longmapsto \exp(-tX)\cdot m$, where $m \in M$ and $\cdot$ denotes the G-action on M. The first fundamental theorem of Lie says that the map $X \longmapsto \tilde{X}$ is a homomorphism of $\mathfrak{g}$ into the algebra of all analytic vector fields on M. We refer to $\tilde{X}$ as the *fundamental vector field* corresponding to $X \in \mathfrak{g}$ for the action of G on M. The second fundamental theorem of Lie asserts that any such infinitesimal $\mathfrak{g}$-action gives rise to at least an essentially unique local G-action on M , and which, under certain conditions extends to a global action, see Theorem 2.16.13, [**Var84**].

We begin a more detailed description of the Oshima compactification with the following lemma.

LEMMA 2. *Let $Y \in \mathfrak{n}^- \oplus \mathfrak{a}$ be given by $Y = \sum_{\lambda\in\Sigma^+} \sum_{i=1}^{m(\lambda)} c_{-\lambda,i} X_{-\lambda,i} + \sum_{j=1}^{l} c_j H_j$, and introduce the notation $t^\lambda = t_1^{\lambda(H_1)} \cdots t_l^{\lambda(H_l)}$. Then, via the identification of $N^- \times \mathbb{R}^l_+$ with $N^- A$ by $(n,t) \mapsto n \cdot exp(-\sum_{j=1}^{l} H_j \log t_j)$, the left invariant vector field on the Lie group $N^- A$ corresponding to Y is expressed as*

$$\tilde{Y}_{|N^-\times\mathbb{R}^l_+} = \sum_{\lambda\in\Sigma^+} \sum_{i=1}^{m(\lambda)} c_{-\lambda,i} t^\lambda X_{-\lambda,i} - \sum_{j=1}^{l} c_j t_j \frac{\partial}{\partial t_j},$$

and can analytically be extended to a vector field on $N^- \times \mathbb{R}^l$.

PROOF. The lemma is proved in Oshima, [**Osh78**], Lemma 8, but for greater clarity, we include an expanded proof of it here. Let $X_{-\lambda,i}$ be a fixed basis element of $\mathfrak{n}^-$. The corresponding left-invariant vector field on the Lie group $N^- A$ at the point na is given by

$$\begin{aligned} \frac{d}{ds} f(na \exp(sX_{-\lambda,i}))_{|s=0} &= \frac{d}{ds} f(n(a \exp(sX_{-\lambda,i}) a^{-1}) a)_{|s=0} \\ &= \frac{d}{ds} f(n\, \mathrm{e}^{sAd(a)X_{-\lambda,i}}\, a)_{|s=0}, \end{aligned}$$

where f is a smooth function on $N^- A$. Regarded as a left invariant vector field on $N^- \times \mathbb{R}^l_+$, it is therefore given as

$$\tilde{X}_{-\lambda,i|N^-\times\mathbb{R}^l_+} = \mathrm{Ad}\,(a) X_{-\lambda,i} = e^{-\lambda(\log a)} X_{-\lambda,i} = t^\lambda X_{-\lambda,i},$$

compare 2.8. Similarly, for a basis element H_i of $\mathfrak{a}$ the corresponding left invariant vector field on $N^- A$ reads

$$\begin{aligned} \frac{d}{ds} f(na \exp(sH_i))_{|s=0} &= \frac{d}{ds} f(n \exp(-\sum_{j=1}^{l} \log t_j H_j) \exp(sH_i))_{|s=0} \\ &= \frac{d}{ds} f\Big(n \exp(-\sum_{j=1}^{l} \log t_j H_j + sH_i)\Big)_{|s=0} \\ &= \frac{d}{ds} f\Big(n \exp(-\sum_{j\neq i} \log t_j H_j - \log(t_i e^{-s}) H_i)\Big)_{|s=0}. \end{aligned}$$

So with the identification $N^- A \simeq N^- \times \mathbb{R}^l_+$, we obtain

$$\tilde{H}_{i|N^-\times\mathbb{R}^l_+} = -t_i \frac{\partial}{\partial t_i}.$$

As there are no negative powers of t, $\tilde{Y}_{N^-\times\mathbb{R}^l_+}$ can be extended analytically to $N^- \times \mathbb{R}^l$, and the lemma follows. □

Similarly Lemma 1 implies, by the identification $G/K \simeq N^- \times A \simeq N^- \times \mathbb{R}^l_+$ via the mappings $(n,t) \mapsto n \cdot exp(-\sum_{i=1}^l H_i \log t_i) \cdot a \mapsto gnaK$, that the action on G/K of the fundamental vector field corresponding to $\exp(sY)$, $Y \in \mathfrak{g}$, is given by

$$Y_{|N^- \times \mathbb{R}^l_+} = \sum_{\lambda \in \Sigma^+} \sum_{i=1}^{m(\lambda)} (c_{\lambda,i}(g,n)t^{2\lambda} + c_{-\lambda,i}(g,n))X_{-\lambda,i} - \sum_{i=1}^{l} c_i(g,n) t_i \frac{\partial}{\partial t_i}, \tag{2.9}$$

where the coefficients are given by (2.1).

REMARK 1. The fundamental vector field corresponding to $H_i \in \mathfrak{a}$ is $-t_i \frac{\partial}{\partial t_i}$.

REMARK 2. Notice that the fundamental vector field (2.9) can be similarly extended analytically to $N^- \times \mathbb{R}^l$, but in contrast to the left invariant vector field $\tilde{Y}_{|N^- \times \mathbb{R}^l}$, $Y_{|N^- \times \mathbb{R}^l}$ does not necessarily vanish even if $t_1 = \dots t_l = 0$.

We come now to the description of the Oshima compactification of the Riemannian symmetric space G/K. For this, let $\hat{\mathbb{X}}$ be the product manifold $G \times N^- \times \mathbb{R}^l$. Take $\hat{x} = (g,n,t) \in \hat{\mathbb{X}}$, where $g \in G$, $n \in N^-$, $t = (t_1, \dots, t_l) \in \mathbb{R}^l$, and define an action of G on $\hat{\mathbb{X}}$ by $g' \cdot (g,n,t) := (g'g,n,t)$, $g' \in G$. For $s \in \mathbb{R}$, let

$$\operatorname{sgn} s = \begin{cases} s/|s|, & s \neq 0, \\ 0, & s = 0, \end{cases}$$

and put $\operatorname{sgn} \hat{x} = (\operatorname{sgn} t_1, \dots, \operatorname{sgn} t_l) \in \{-1,0,1\}^l$. We then define the subsets $\Theta_{\hat{x}} = \{\alpha_i \in \Delta : t_i \neq 0\}$. Also let $a(\hat{x}) = \exp(-\sum_{t_i \neq 0} H_i \log |t_i|) \in A(\Theta_{\hat{x}})$. Writing a for $a(\hat{x})$, the definition then gives us that

$$\log a = -\sum_{t_i \neq 0} H_i \log |t_i|,$$

hence that

$$\alpha_j(\log a) = -\sum_{t_i \neq 0} \alpha_j(H_i) \log |t_i| = -\delta_{ji} \log t_i,$$

and so $t_j = e^{-\alpha_j \log a}$. On $\hat{\mathbb{X}}$, define now an equivalence relation by declaring that

$$\hat{x} = (g,n,t) \sim \hat{x}' = (g', n,' t')$$

if and only if

(1) $\operatorname{sgn} \hat{x} = \operatorname{sgn} \hat{x}'$, and
(2) $g\, n\, a(\hat{x})\, P_{\Theta_{\hat{x}}}(K) = g'\, n'\, a(\hat{x}')\, P_{\Theta_{\hat{x}'}}(K)$.

Note that the condition $\operatorname{sgn} \hat{x} = \operatorname{sgn} \hat{x}'$ implies that $\hat{x}, \hat{x}'$ determine the same subset $\Theta_{\hat{x}}$ of Δ, and consequently the same group $P_{\Theta_{\hat{x}}}(K)$, as well as the same homogeneous space $G/P_{\Theta_{\hat{x}}}(K)$. Condition 2), therefore, makes sense. It says that $gna(\hat{x})$, $g'n'a(\hat{x}')$ are in the same $P_{\Theta_{\hat{x}}}(K)$ orbit on G, corresponding to the right action by $P_{\Theta_{\hat{x}}}(K)$ on G. We now define

$$\widetilde{\mathbb{X}} := \hat{\mathbb{X}} / \sim,$$

endowing it with the quotient topology, and denote by $\pi : \hat{\mathbb{X}} \to \widetilde{\mathbb{X}}$ the canonical projection. The action of G on $\hat{\mathbb{X}}$ is compatible with the equivalence relation $\sim$, yielding a G-action

$$g' \cdot \pi(g,n,t) := \pi(g'g,n,t)$$

on $\widetilde{\mathbb{X}}$. For each $g \in G$, one can show that the maps

$$\varphi_g : N^- \times \mathbb{R}^l \to \widetilde{U}_g : (n,t) \mapsto \pi(g,n,t), \qquad \widetilde{U}_g = \pi(\{g\} \times N^- \times \mathbb{R}^l), \tag{2.10}$$

are bijections. One has then the following

THEOREM 1. (1) *$\widetilde{\mathbb{X}}$ is a simply connected, compact, real-analytic manifold without boundary.*

(2) *$\widetilde{\mathbb{X}} = \cup_{w \in W} \widetilde{U}_{m_w} = \cup_{g \in G} \widetilde{U}_g$. For $g \in G$, $\widetilde{U}_g$ is an open submanifold of $\widetilde{\mathbb{X}}$ topologized in such a way that the coordinate map φ_g defined above is a real-analytic diffeomorphism. Furthermore, $\widetilde{\mathbb{X}} \setminus \widetilde{U}_g$ is the union of a finite number of submanifolds of $\widetilde{\mathbb{X}}$ whose codimensions in $\widetilde{\mathbb{X}}$ are not lower than 2.*

(3) *The action of G on $\widetilde{\mathbb{X}}$ is real-analytic. For a point $\hat{x} \in \hat{\mathbb{X}}$, the G-orbit of $\pi(\hat{x})$ is isomorphic to the homogeneous space $G/P_{\Theta_{\hat{x}}}(K)$, and for $\hat{x}, \hat{x}' \in \hat{\mathbb{X}}$ the G-orbits of $\pi(\hat{x})$ and $\pi(\hat{x}')$ coincide if and only if $\operatorname{sgn}\hat{x} = \operatorname{sgn}\hat{x}'$. Hence the orbital decomposition of $\widetilde{\mathbb{X}}$ with respect to the action of G is of the form*

$$\widetilde{\mathbb{X}} \simeq \bigsqcup_{\Theta \subset \Delta} 2^{\#\Theta}(G/P_\Theta(K)) \quad \textit{(disjoint union)}, \tag{2.11}$$

where $\#\Theta$ is the number of elements of Θ and $2^{\#\Theta}(G/P_\Theta(K))$ is the disjoint union of $2^{\#\Theta}$ copies of $G/P_\Theta(K)$.

PROOF. See Oshima, [**Osh78**], Theorem 5. □

We observe here that statement (3) shows, in particular, that there are 2^l copies of $\mathbb{X}$ as open G-orbits in $\widetilde{\mathbb{X}}$ while the one copy of G/P occurs as the unique closed orbit.

We will now prove some results concerning the G-action on $\widetilde{\mathbb{X}}$ that will be of use later. Let $\left\{(\widetilde{U}_{m_w}, \varphi_{m_w}^{-1})\right\}_{w \in W}$ be the finite atlas on the Oshima compactification $\widetilde{\mathbb{X}}$ defined above. For each point $\tilde{x} \in \widetilde{\mathbb{X}}$, choose open neighborhoods $\widetilde{W}_{\tilde{x}} \subsetneq \widetilde{W}'_{\tilde{x}}$ of $\tilde{x}$ contained in a chart $\widetilde{U}_{m_w(\tilde{x})}$. Since $\widetilde{\mathbb{X}}$ is compact, we can find a finite subcover of the cover $\left\{\widetilde{W}_{\tilde{x}}\right\}_{\tilde{x} \in \widetilde{\mathbb{X}}}$, and in this way obtain a finite atlas $\left\{(\widetilde{W}_\gamma, \varphi_\gamma^{-1})\right\}_{\gamma \in I}$ of $\widetilde{\mathbb{X}}$, where for simplicity we wrote $\widetilde{W}_\gamma = \widetilde{W}_{\tilde{x}_\gamma}$, $\varphi_\gamma = \varphi_{m_w(\tilde{x}_\gamma)}$. Further, let $\{\alpha_\gamma\}_{\gamma \in I}$ be a partition of unity subordinate to this atlas, and let $\{\bar{\alpha}_\gamma\}_{\gamma \in I}$ be another set of functions satisfying $\bar{\alpha}_\gamma \in \mathrm{C}^\infty_{\mathrm{c}}(\widetilde{W}'_\gamma)$ and $\bar{\alpha}_{\gamma|\widetilde{W}_\gamma} \equiv 1$. We now have the following result on the factorization of the G-action in the t-coordinates.

LEMMA 3. *For $\tilde{x} = \varphi_\gamma(n,t) \in \widetilde{W}_\gamma$, let $V_{\gamma,\tilde{x}}$ denote the set of $g \in G$ such that $g \cdot \tilde{x} \in \widetilde{W}_\gamma$. Then we have the power series expansion*

$$t_j(g \cdot \tilde{x}) = \sum_{\substack{\alpha,\beta \\ \beta_j \neq 0}} c^j_{\alpha,\beta}(g) n^\alpha(\tilde{x}) t^\beta(\tilde{x}), \qquad j = 1, \ldots, l, \tag{2.12}$$

where the coefficients $c^j_{\alpha,\beta}(g)$ depend real-analytically on $g \in V_{\gamma,\tilde{x}}$, and α, β are multi-indices.

PROOF. By Theorem 1, a G-orbit in $\widetilde{\mathbb{X}}$ is locally determined by the signature of any of its elements. In particular, for $\tilde{x} \in \widetilde{W}_\gamma$ and $g \in V_{\gamma,\tilde{x}}$ as above, we have $\operatorname{sgn} t_j(g \cdot \tilde{x}) = \operatorname{sgn} t_j(\tilde{x})$ for all $j = 1, \ldots, l$. Hence, $t_j(g \cdot \tilde{x}) = 0$ if and only if $t_j(\tilde{x}) = 0$. Now, due to the analyticity of the coordinates $(\varphi_\gamma, \widetilde{W}_\gamma)$, there is a power series expansion

$$t_j(g \cdot \tilde{x}) = \sum_{\alpha,\beta} c^j_{\alpha,\beta}(g) n^\alpha(\tilde{x}) t^\beta(\tilde{x}), \qquad \tilde{x} \in \widetilde{W}_\gamma,\ g \in V_{\gamma,\tilde{x}},$$

for every $j = 1, \ldots, l$, which can be rewritten as

$$t_j(g \cdot \tilde{x}) = \sum_{\substack{\alpha,\beta \\ \beta_j \neq 0}} c^j_{\alpha,\beta}(g) n^\alpha(\tilde{x}) t^\beta(\tilde{x}) + \sum_{\substack{\alpha,\beta \\ \beta_j = 0}} c^j_{\alpha,\beta}(g) n^\alpha(\tilde{x}) t^\beta(\tilde{x}). \tag{2.13}$$

Suppose $t_j(\tilde{x}) = 0$. Then the first summand of the last equation must vanish, as in each term of the summation a non-zero power of $t_j(\tilde{x})$ occurs. Also, $t_j(g \cdot \tilde{x}) = 0$. Therefore (2.13) implies that the second summand must vanish, too. But the latter is independent of t_j. So we conclude

$$\sum_{\substack{\alpha,\beta \\ \beta_j = 0}} c^j_{\alpha,\beta}(g) n^\alpha(\tilde{x}) t^\beta(\tilde{x}) \equiv 0$$

for all $\tilde{x} \in \widetilde{W}_\gamma$, $g \in V_{\gamma,\tilde{x}}$, and the assertion follows. $\square$

From Lemma 3 we deduce that

$$t_j(g \cdot \tilde{x}) = t_j^{q_j}(\tilde{x}) \chi_j(g, \tilde{x}), \qquad \tilde{x} \in \widetilde{W}_\gamma,\ g \in V_{\gamma,\tilde{x}},$$

where $\chi_j(g, \tilde{x})$ is a function that is real-analytic in g and in $\tilde{x}$, and $q_j \geq 1$ is the lowest power of t_j that occurs in the expansion (2.12). Furthermore, since $t_j(g \cdot \tilde{x}) = t_j(\tilde{x})$ for $g = e$, one has $q_1 = \cdots = q_l = 1$. A computation now shows that

$$1 = \chi_j(g^{-1}, g \cdot \tilde{x}) \cdot \chi_j(g, \tilde{x}) \qquad \forall\, \tilde{x} \in \widetilde{W}_\gamma, \quad g \in V_{\gamma,\tilde{x}},$$

where $g^{-1} \in V_{\gamma, g\tilde{x}}$. This implies

$$\chi_j(g, \tilde{x}) \neq 0 \qquad \forall\, \tilde{x} \in \widetilde{W}_\gamma, \quad g \in V_{\gamma,\tilde{x}}, \tag{2.14}$$

since $\chi_j(g^{-1}, g\cdot\tilde{x})$ is a finite complex number. Thus, for $\tilde{x} = \varphi_\gamma(x) \in \widetilde{W}_\gamma$, $x = (n,t)$, $g \in V_{\gamma,\tilde{x}}$, we have

$$\varphi_\gamma^g(x) = (n_1(g\cdot\tilde{x}), \dots, n_k(g\cdot\tilde{x}), t_1(\tilde{x})\chi_1(g,\tilde{x}), \dots, t_l(\tilde{x})\chi_l(g,\tilde{x})). \tag{2.15}$$

Note that similar formulae hold for $\tilde{x} \in \widetilde{U}_{m_w}$ and g sufficiently close to the identity. The following lemma describes the G-action on $\widetilde{\mathbb{X}}$ as far as the t-coordinates are concerned.

LEMMA 4. *Let $X_{-\lambda,i}$ and H_j be the basis elements for $\mathfrak{n}^-$ and $\mathfrak{a}$ introduced in Section 2.1, $w \in W$, and $\tilde{x} \in \widetilde{U}_{m_w}$. Then, for small $s \in \mathbb{R}$,*

$$\chi_j(\mathrm{e}^{sH_i}, \tilde{x}) = e^{-c_{ij}(m_w)s},$$

where the $c_{ij}(m_w)$ are the matrix coefficients of the adjoint representation of M^* on $\mathfrak{a}$, and are given by $\mathrm{Ad}\,(m_w^{-1})H_i = \sum_{j=1}^l c_{ij}(m_w)H_j$. Furthermore, when $\tilde{x} = \pi(e,n,t)$,

$$\chi_j(\mathrm{e}^{sX_{-\lambda,i}}, \tilde{x}) \equiv 1.$$

PROOF. Let $Y \in \mathfrak{g}$. As we saw in the proof of Lemma 1, the action of the one-parameter group $\exp(sY)$ on the homogeneous space $G/P_\Theta(K)$ is given by equation (2.4), where $N_3^-(s) \in \mathfrak{n}^-$, $A_1(s) \in \mathfrak{a}$, $A_2(s) \in \mathfrak{a}(\Theta)$. Denote the derivatives of $N_3^-(s)$, $A_1(s)$, and $A_2(s)$ at $s = 0$ by N_3^-, A_1, and A_2 respectively. The analyticity of the G-action implies that $N_3^-(s), A_1(s), A_2(s)$ are real-analytic functions in s. Furthermore, from (2.4) it is clear that $N_3^-(0) = 0$, $A_1(0) + A_2(0) = 0$, so that for small s we have

$$A_1(s) + A_2(s) = (A_1 + A_2)\,s + \frac{1}{2}\frac{d^2}{ds^2}(A_1(s) + A_2(s))|_{s=0}\, s^2 + \dots$$

$$N_3^-(s) = N_3^-\,s + \frac{1}{2}\frac{d^2}{ds^2}N_3^-(s)|_{s=0}\, s^2 + \dots.$$

Next, fix $m_w \in M^*$ and let $\Theta = \Delta$. The action of the one-parameter group corresponding to H_i at $\tilde{x} = \pi(m_w, n, t) \in \widetilde{U}_{m_w} \cap \widetilde{\mathbb{X}}_\Delta$ is given by

$$\exp(sH_i)m_w naK = m_w\left(m_w^{-1}\exp(sH_i)m_w\right)naK = m_w \exp(s\mathrm{Ad}\,(m_w^{-1})H_i)naK.$$

As m_w lies in M^*, $\exp(s\mathrm{Ad}\,(m_w^{-1})H_i)$ lies in A. Since A normalizes N^-, we conclude that $\exp(s\mathrm{Ad}\,(m_w^{-1})H_i)n\exp(-s\mathrm{Ad}\,(m_w^{-1})H_i)$ belongs to N^-. Writing

$$n^{-1}\exp(s\mathrm{Ad}\,(m_w^{-1})H_i)n\exp(-s\mathrm{Ad}\,(m_w^{-1})H_i) = \exp N_3^-(s)$$

we get

$$\exp(sH_i)m_w naK = m_w n \exp N_3^-(s) a \exp(s\mathrm{Ad}\,(m_w^{-1})H_i)K.$$

In the notation of (2.4) we therefore obtain $A_1(s) + A_2(s) = s\mathrm{Ad}\,(m_w^{-1})H_i$, and by writing $\mathrm{Ad}\,(m_w^{-1})H_i = \sum_{j=1}^l c_{ij}(m_w)H_j$ we arrive at

$$a\exp(A_1(s) + A_2(s)) = \exp\Big(\sum_{j=1}^{l}(c_{ij}(m_w)s - \log t_j)H_j\Big).$$

In terms of the coordinates this shows that $t_j(\exp(sH_i)\cdot\tilde{x}) = t_j(\tilde{x})e^{-c_{ij}(m_w)s}$ for $\tilde{x}\in \widetilde{U}_{m_w}\cap\widetilde{\mathbb{X}}_\Delta$, and by analyticity we obtain that $\chi_j(\,\mathrm{e}^{sH_i}\,,\tilde{x}) = e^{-c_{ij}(m_w)s}$ for arbitrary $\tilde{x}\in\widetilde{U}_{m_w}$. On the other hand, let $Y = X_{-\lambda,i}$, and $\tilde{x} = \varphi_e(n,t)\in\widetilde{U}_e\cap\widetilde{\mathbb{X}}_\Delta$. Then the action corresponding to $X_{-\lambda,i}$ at $\tilde{x}$ is given by

$$\exp(sX_{-\lambda,i})naK \exp N_3^-(s)aK,$$

where we wrote $\exp N_3^-(s) = s\mathrm{Ad}\,(n^{-1})\exp X_{-\lambda,i}$. In terms of the coordinates this implies that $t_j(\exp(sX_{-\lambda,i})\cdot\tilde{x}) = t_j(\tilde{x})$ showing that $\chi_j(\,\mathrm{e}^{sX_{-\lambda,i}}\,,\tilde{x})\equiv 1$ for $\tilde{x}\in\widetilde{U}_e\cap\widetilde{\mathbb{X}}_\Delta$, and, by analyticity, for general $\tilde{x}\in\widetilde{U}_e$, finishing the proof of the lemma. $\square$

Next, for $\hat{x} = (g,n,t)$ define the set $B_{\hat{x}} = \{(t_1'\dots t_l')\in\mathbb{R}^l : \operatorname{sgn} t_i = \operatorname{sgn} t_i', 1\leq i\leq l\}$. By analytic continuation, one can restrict the vector field (2.9) to $N^-\times B_{\hat{x}}$, and with the identifications $G/P_{\Theta_{\hat{x}}}(K)\simeq N^-\times A(\Theta_{\hat{x}})\simeq N^-\times B_{\hat{x}}$ via the maps

$$gnaP_{\Theta_{\hat{x}}}\leftarrow(n,a)\mapsto(n,\operatorname{sgn} t_1 e^{-\alpha_1(\log a)},\dots,\operatorname{sgn} t_l e^{-\alpha_l(\log a)}),$$

one actually sees that this restriction coincides with the vector field in Lemma 1. The action of the fundamental vector field on $\widetilde{\mathbb{X}}$ corresponding to $\exp sY, Y\in\mathfrak{g}$, is therefore given by the extension of (2.9) to $N^-\times\mathbb{R}^l$. Note that for a simply connected nilpotent Lie group N with Lie algebra $\mathfrak{n}$, the exponential $\exp:\mathfrak{n}\to N$ is a diffeomorphism. So, in our setting, we can identify N^- with $\mathbb{R}^k$. Thus, for every point in $\widetilde{\mathbb{X}}$, there exists a local coordinate system $(n_1,\dots,n_k,t_1,\dots,t_l)$ in a neighbourhood of that point such that two points $(n_1,\dots,n_k,t_1,\dots,t_l)$ and $(n_1',\dots,n_k',t_1',\dots,t_l')$ belong to the same G-orbit if, and only if, $\operatorname{sgn} t_j = \operatorname{sgn} t_j'$, for $j=1,\dots,l$. This means that the orbital decomposition of $\widetilde{\mathbb{X}}$ is of *normal crossing type*. In what follows, we shall identify the open G-orbit $\pi(\{\hat{x} = (e,n,t)\in\hat{\mathbb{X}} : \operatorname{sgn}\hat{x} = (1,\dots,1)\})$ with the Riemannian symmetric space G/K, and the orbit $\pi(\{\hat{x}\in\hat{\mathbb{X}} : \operatorname{sgn}\hat{x} = (0,\dots,0)\}$ of lowest dimension with its Martin boundary G/P. Note that the closure of a copy of $\mathbb{X}$ in the Oshima compactification is a compact manifold with corners.

A very important property of the Oshima compactification is the following. Let $\mathcal{D}(\widetilde{\mathbb{X}})$ denote the algebra of G-invariant differential operators on $\widetilde{\mathbb{X}}$ with real analytic coefficients, and let $\mathcal{D}(\mathbb{X})$ be, as usual, the algebra of G-invariant differential operators on $\mathbb{X}$. Then, the natural restriction $\mathcal{D}(\widetilde{\mathbb{X}})\to\mathcal{D}(\mathbb{X})$ is a bijection. Further, for any fixed homomorphism of algebras $\chi:\mathcal{D}(\widetilde{\mathbb{X}})\to\mathbb{C}$, the system of differential equations on $\widetilde{\mathbb{X}}$ given by

$$\mathfrak{m}_\chi : (D-\chi(D))u = 0, \qquad D\in\mathcal{D}(\widetilde{\mathbb{X}})$$

has regular singularity along the set of walls $\widetilde{\mathbb{X}}_i = \{t_i = 0\}$ with edge $\{t_1 = \dots = t_l = 0\}\cong G/P$. (Refer [**Osh78**], Theorem 5). This has to be seen in the context of the circle of ideas originating in the work of Oshima and Kashiwara on systems of differential equations with regular singularities, leading eventually, with others, to the settling of the Helgason conjecture. Recall that this conjecture gives a G-isomorphism between the two representation spaces of G, namely the joint eigenspace of all the invariant differential operators on $\mathbb{X}$ and the space of hyperfunction-valued sections of a certain line-bundle over G/P.

To close, we would like to mention that there is the following alternative construction of the Oshima compactification using the notion of self-gluing of a manifold with corners. We will sketch this briefly, and refer to the book of Borel-Ji, [**BJ06**], Part II for further details. Let M be a manifold with corners of dimension n. Then, every point $p \in M$ has a neighbourhood of the form $\mathbb{R}^{n-i} \times [0,\infty)^i$, $0 \leq i \leq n$, and i is called the *rank of* p or the *local codimension of* p. The maximum of such i is called the rank of M, denoted by $rk(M)$. M has a stratification such that the strata comprise of points of the same rank, and a connected component of a stratum is called an *open boundary face* of M. The closure of an open boundary face in M is called a *boundary face*. A boundary face of codimension 1 is called a *boundary hypersurface*. We make the following assumptions on the boundary hypersurfaces of M.

(1) All the boundary hypersurfaces are embedded i.e., for every point p of rank i belonging to the boundary of a boundary hypersurface H, there exist $i-1$ boundary hypersurfaces $H_1, \cdots, H_{i-1}$ different from H such that p belongs to the intersection $H \cap H_1 \cdots \cap H_{i-1}$ and the intersection has codimension i .
(2) The set $\mathcal{H}_M$ of boundary hypersurfaces of M is locally finite i.e. each point has a neighbourhood that intersects only finitely many of them.
(3) The set $\mathcal{H}_M$ admits a *finite partition* i.e.,

$$\mathcal{H}_M = \cup_{j=1}^{N} \mathcal{H}_{M,j}$$

with the elements of each $\mathcal{H}_{M,j}$ being disjoint for $1 \leq j \leq N$.

Notice that condition (1) is automatically satisfied if all the H_i's are distinct, while conditions (2) and (3) are satisfied if M is compact.

Remark 3. It is sometimes customary to call M a manifold with corners only if it satisfies condition (1) and we shall follow this custom henceforth.

For M satisfying the above conditions, one can construct a closed manifold $\widetilde{M}$ by gluing 2^N copies of M along boundary hypersurfaces. If M is real analytic, then so is $\widetilde{M}$. Further, any real analytic action of a Lie group G on M extends to a real analytic action on $\widetilde{M}$. See Borel-Ji [**BJ06**], Section II.1.

Remark 4. When $N = 1$, M is simply a manifold with boundary and the gluing procedure is the usual doubling of such manifolds.

Let M be the so-called maximal Satake compactification of the symmetric space $\mathbb{X} = G/K$. This is a compact real analytic manifold with corners of codimension $l = \operatorname{rank}(\mathbb{X})$ on which there is a real analytic action of G. The above gluing procedure then gives us a closed real analytic manifold $\widetilde{M}$, with a real analytic G-action containing 2^l copies of $\mathbb{X}$, which is isomorphic to $\widetilde{\mathbb{X}}$. In this approach, the hard part is to show the real-analyticity of the maximal Satake compactification M of $\mathbb{X}$, and in fact this is done, essentially, by using the arguments of Oshima sketched earlier.

Another way of showing the analyticity of M is as follows, see [**BJ06**], Section II.9. Let $\mathbf{G}$ and $\mathbf{K}$ denote the complexifications of G and K, respectively. Set $\mathbf{X} = \mathbf{G}/\mathbf{K}$. Let τ be the complexification of the Cartan involution on G, whence $\mathbf{G}^\tau = \mathbf{K}$. Choosing a set Δ_τ of simple τ-roots, for $\Theta \subset \Delta_\tau$, denote by $\mathbf{P}_{\tau,\boldsymbol{\Theta}}$ a certain closed subgroup of $\mathbf{G}$ such that $\mathbf{P}_{\tau,\boldsymbol{\Delta}_\tau} = \mathbf{K}$. Notice the similarity with the subgroups $P_\Theta(K)$ in the context of the Oshima compactification of $\mathbb{X}$. Set $\mathbf{O}_{\tau,\boldsymbol{\Theta}} = \mathbf{G}/\mathbf{P}_{\tau,\boldsymbol{\Theta}}$. Then there exists a compactification of the complex symmetric space $\mathbf{X}$ called the DeConcini-Procesi *wonderful compactification* and denoted by $\mathbf{X}^{\mathbf{W}}$ with the following properties. $\mathbf{X}^{\mathbf{W}}$ is a smooth complex projective variety on which $\mathbf{G}$ acts morphically, and is a disjoint union of 2^l orbits isomorphic to $\mathbf{O}_{\tau,\boldsymbol{\Theta}}$, $\Theta \subset \Delta_\tau$. In particular, $\mathbf{O}_{\tau,\boldsymbol{\Delta}_\tau} = \mathbf{G}/\mathbf{K}$ is an open, Zariski-dense $\mathbf{G}$-orbit in $\mathbf{X}^{\mathbf{W}}$. If we denote by $\mathbf{D}_{\tau,\boldsymbol{\Theta}}$ the Zariski-closures of $\mathbf{O}_{\tau,\boldsymbol{\Theta}}$, and set $\mathbf{D}_\tau^{\boldsymbol{\Theta}} = \mathbf{D}_{\tau,\boldsymbol{\Delta}_\tau-\boldsymbol{\Theta}}$, then $\mathbf{D}_\tau^{\boldsymbol{\Theta}}$ are smooth varieties of dimension $\#\Theta$. For $\alpha \in \Delta_\tau$, therefore, $D^{\{\alpha\}}$ are smooth divisors with normal crossings. If $\mathbb{X}^W := \mathbf{X}^{\mathbf{W}}(\mathbb{R})$ denotes the real locus of the wonderful compactification of $\mathbf{X}$, then $\mathbb{X}^W$ contains $\mathbb{X}$, and is a smooth real projective variety, being Zariski-dense in $\mathbf{X}^{\mathbf{W}}$. We have the following decomposition

$$\mathbb{X}^W \simeq \bigsqcup_{\Theta \subset \Delta_\tau} \mathbf{O}_{\tau,\boldsymbol{\Theta}}(\mathbb{R}).$$

Notice that $\dim_{\mathbb{C}} \mathbf{X}^{\mathbf{W}} = \dim_{\mathbb{R}} \mathbb{X}^W$. The closure of $\mathbb{X}$ in $\mathbb{X}^W$ is a compactification of $\mathbb{X}$ and is a real analytic manifold with corners. On the other hand, this closure can be shown to be isomorphic to the maximal Satake compactification of $\mathbb{X}$, thus showing the analytic structure on it.

2.3. The case $\mathbb{X} = \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$

In this section we describe in detail the Oshima compactification of the Riemannian symmetric space $\mathbb{X} = \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$, so that we can have an explicit example when $\operatorname{rank}\mathbb{X} > 1$. Thus, let $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{R})$ be the Lie algebra of G. A Cartan involution $\theta : \mathfrak{g} \to \mathfrak{g}$ is given by $X \mapsto -X^t$, where X^t denotes the transpose of X, and the corresponding Cartan decomposition of $\mathfrak{g}$ reads $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k} = \{X \in \mathfrak{sl}(3,\mathbb{R}) : X^t = -X\}$, and $\mathfrak{p} = \{X \in \mathfrak{sl}(3,\mathbb{R}) : X^t = X\}$. Next, let

$$\mathfrak{a} = \{D(a_1, a_2, a_3) : a_1, a_2, a_3 \in \mathbb{R}, a_1 + a_2 + a_3 = 0\},$$

where $D(a_1, a_2, a_3)$ denotes the diagonal matrix with diagonal elements a_1, a_2 and a_3. Then $\mathfrak{a}$ is a maximal Abelian subalgebra in $\mathfrak{p}$. Define $e_i : \mathfrak{a} \to \mathbb{R}$ by $D(a_1, a_2, a_3) \mapsto$

a_i, $i=1,2,3$. The set of roots Σ of $(\mathfrak{g},\mathfrak{a})$ is given by $\Sigma=\{\pm(e_i-e_j):1\leq i<j\leq 3\}$. We order the roots such that the positive roots are $\Sigma^+=\{e_1-e_2,e_2-e_3,e_1-e_3\}$, and obtain $\Delta=\{e_1-e_2,e_2-e_3\}$ as the set of simple roots. The root space corresponding to the root e_1-e_2 is given by

$$\mathfrak{g}^{e_1-e_2}=\left\{\begin{pmatrix}0&x&0\\0&0&0\\0&0&0\end{pmatrix}:x\in\mathbb{R}\right\},$$

and similar computations show that

$$\mathfrak{g}^{e_2-e_3}=\left\{\begin{pmatrix}0&0&0\\0&0&z\\0&0&0\end{pmatrix}:z\in\mathbb{R}\right\},\quad \mathfrak{g}^{e_1-e_3}=\left\{\begin{pmatrix}0&0&y\\0&0&0\\0&0&0\end{pmatrix}:y\in\mathbb{R}\right\}.$$

For a subset $\Theta\subset\Delta$, let $\langle\Theta\rangle$ denote those elements of Σ that are given as linear combinations of the roots in Θ. Write $\langle\Theta\rangle^{\pm}$ for $\Sigma^{\pm}\cap\langle\Theta\rangle$. Put $\mathfrak{n}^{\pm}(\Theta)=\sum_{\lambda\in\langle\Theta\rangle^{\pm}}\mathfrak{g}^{\lambda}$, and $\mathfrak{n}^+_{\Theta}=\sum_{\lambda\in\Sigma^+-\langle\Theta\rangle^+}\mathfrak{g}^{\lambda}$. Let $\mathfrak{n}^-_{\Theta}=\theta(\mathfrak{n}^+_{\Theta})$. Consider now the case $\Theta=\{e_1-e_2\}$. Then $\mathfrak{n}^+(e_1-e_2)=\mathfrak{g}^{e_1-e_2}$, and $\mathfrak{n}^+_{e_1-e_2}=\mathfrak{g}^{e_2-e_3}\oplus\mathfrak{g}^{e_1-e_3}$. In other words,

$$\mathfrak{n}^+_{e_1-e_2}=\left\{\begin{pmatrix}0&0&y\\0&0&z\\0&0&0\end{pmatrix}:y,z\in\mathbb{R}\right\}.$$

Exponentiating, we find that the corresponding analytic subgroups are given by

$$N^+(e_1-e_2)=\left\{\begin{pmatrix}1&x&0\\0&1&0\\0&0&1\end{pmatrix}:x\in\mathbb{R}\right\},\quad N^+_{e_1-e_2}=\left\{\begin{pmatrix}1&0&y\\0&1&z\\0&0&1\end{pmatrix}:y,z\in\mathbb{R}\right\}.$$

In a similar fashion, we obtain that

$$\mathfrak{n}^-(e_1-e_2)=\mathfrak{g}^{e_2-e_1}=\left\{\begin{pmatrix}0&0&0\\x&0&0\\0&0&0\end{pmatrix}:x\in\mathbb{R}\right\},$$

$$\mathfrak{n}^-_{e_1-e_2}=\theta(\mathfrak{n}^+_{e_1-e_2})=\left\{\begin{pmatrix}0&0&0\\0&0&0\\y&z&0\end{pmatrix}:y,z\in\mathbb{R}\right\},$$

and that the corresponding analytic subgroups read

$$N^-(e_1-e_2)=\left\{\begin{pmatrix}1&0&0\\x&1&0\\0&0&1\end{pmatrix}:x\in\mathbb{R}\right\},\quad N^-_{e_1-e_2}=\left\{\begin{pmatrix}1&0&0\\0&1&0\\y&z&1\end{pmatrix}:y,z\in\mathbb{R}\right\}.$$

The Cartan-Killing form $\langle\cdot,\cdot\rangle:\mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$ is given by $(X,Y)\mapsto\operatorname{Tr}(XY)$, and the modified Cartan-Killing form by $\langle X,Y\rangle_\theta:=-\operatorname{Tr}(X\theta(Y))=-\operatorname{Tr}(X(-Y^t))=\operatorname{Tr}(XY^t)$. Next, let $\mathfrak{a}(\Theta)=\sum_{\lambda\in\langle\Theta\rangle^+}\mathbb{R}Q_\lambda$, where $Q_\lambda=[\theta X,X]$ for $X\in\mathfrak{g}^{\lambda}$ such that $\langle X, X\rangle_\theta = 1$. Also, let $\mathfrak{a}_\Theta$ be the orthogonal complement of $\mathfrak{a}(\Theta)$ with respect to $\langle\cdot,\cdot\rangle_\theta$. Again, suppose that $\Theta = \{e_1 - e_2\}$. We find

$$Q_{e_1-e_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so that

$$\mathfrak{a}(e_1 - e_2) = \mathbb{R}Q_{e_1-e_2} = \left\{ \begin{pmatrix} r & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & 0 \end{pmatrix} : r \in \mathbb{R} \right\}.$$

This in turn gives us that

$$\mathfrak{a}_{e_1-e_2} = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -2a \end{pmatrix} : a \in \mathbb{R} \right\}.$$

Exponentiation then shows that the corresponding analytic subgroups are

$$A(e_1 - e_2) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} : a \in \mathbb{R}^+ \right\}, A_{e_1-e_2} = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a^{-2} \end{pmatrix} : a \in \mathbb{R}^+ \right\}.$$

Take $K = \mathrm{SO}(3)$ as a maximal compact subgroup of $\mathrm{SL}(3,\mathbb{R})$, and denote by $M_\Theta(K)$ the centralizer of $\mathfrak{a}_\Theta$ in K. Observing that the adjoint action of a matrix group G is just the matrix conjugation, we see that

$$M_{e_1-e_2}(K) = Z_K(\mathfrak{a}_{e_1-e_2}) = \begin{pmatrix} \mathrm{SO}(2) & 0 \\ 0 & 1 \end{pmatrix} \cup \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \mathrm{SO}(2) & 0 \\ 0 & 1 \end{pmatrix}.$$

Notice that $M_{e_1-e_2}(K)$ has 2 connected components. Put $M = Z_K(A)$, and let $P = MAN^+$ be the minimal parabolic subgroup given by the ordering of the roots of $(\mathfrak{g}, \mathfrak{a})$. For $G = \mathrm{SL}(3,\mathbb{R})$ one computes

$$M = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}.$$

As Θ varies over the subsets of Δ, we get all the parabolic subgroups P_Θ of G containing P, and we write $P_\Theta = M_\Theta(K)AN^+$. By definition, $P_\Theta(K) = M_\Theta(K)A_\Theta N_\Theta^+$

so that, in particular,

$$P_{e_1-e_2}(K) = \left(\begin{pmatrix} \mathrm{SO}(2) & 0 \\ 0 & 1 \end{pmatrix} \cup \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \mathrm{SO}(2) & 0 \\ 0 & 1 \end{pmatrix} \right) \cdot \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a^{-2} \end{pmatrix} : a \in \mathbb{R}^+ \right\} \cdot \left\{ \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : z \in \mathbb{R} \right\}.$$

The orbital decomposition of the Oshima compactification $\widetilde{\mathbb{X}}$ of $\mathbb{X} = \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ is therefore given by

$$\widetilde{\mathbb{X}} = G/P \sqcup 2(G/P_{e_1-e_2}(K)) \sqcup 2(G/P_{e_2-e_3}(K)) \sqcup 2^2(G/K).$$

2.4. Fixed points of group actions on homogeneous spaces

In this section, we write down some results on Lie group actions on homogeneous spaces. While some of these results are well-known, we also prove some new results. Let G be a Lie group with Lie algebra $\mathfrak{g}$, $H \subset G$ a closed subgroup with Lie algebra $\mathfrak{h}$, and $\pi : G \to G/H$ the canonical projection. For an element $g \in G$, consider the natural left action $l_g : G/H \to G/H$ given by $l_g(xH) = gxH$. Let Ad^G denote the adjoint action of G on $\mathfrak{g}$. We begin with two well-known lemmata, see e.g. [**AB68**], page 463.

LEMMA 5. *$l_{g^{-1}} : G/H \to G/H$ has a fixed point if and only if $g \in \bigcup_{x\in G} xHx^{-1}$. Moreover, to every fixed point xH one can associate a unique conjugacy class $h(g, xH)$ in H.*

PROOF. Clearly,

$$l_{g^{-1}}(xH) = xH \iff g^{-1}xH = xH \iff (g^{-1}x)^{-1}x \in H \iff x^{-1}gx = h(g,x),$$

where $h(g,x) \in H$. So $l_{g^{-1}}$ has a fixed point if, and only if, $g \in \bigcup_{x\in G} xHx^{-1}$. Now, if $y \in G$ is such that $xH = yH$, then $y = xh$ for some $h \in H$. This gives us that $h(g,y) = y^{-1}gy = (xh)^{-1}g(xh) = h^{-1}(x^{-1}gx)h = h^{-1}h(g,x)h$. Thus, as x varies over representatives of the coset xH, $h(g,x)$ varies over a conjugacy class $h(g,xH)$ in H. □

LEMMA 6. *Let xH be a fixed point of $l_{g^{-1}}$ and let $h \in h(g,xH)$. Then*

$$\det(\mathbf{1} - dl_{g^{-1}})_{xH} = \det(\mathbf{1} - \mathrm{Ad}_H^G(h)),$$

where $\mathrm{Ad}_H^G : H \to \mathrm{Aut}(\mathfrak{g}/\mathfrak{h})$ is the isotropy action of H on $\mathfrak{g}/\mathfrak{h}$.

PROOF. Let L_g and R_g be the left and right translations, respectively, of $g \in G$ on G. We begin with the observation that

$$\pi \circ L_{g^{-1}} = l_{g^{-1}} \circ \pi, \tag{2.16}$$

where π is the natural map from G to G/H. Let e be the identity in G, and $T_{\pi(e)}(G/H)$ the tangent space to G/H at the point $\pi(e)$. The derivative $d\pi : \mathfrak{g} \to T_{\pi(e)}(G/H)$ is a surjective linear map with kernel $\mathfrak{h}$, and therefore induces an isomorphism between $\mathfrak{g}/\mathfrak{h}$ and $T_{\pi(e)}(G/H)$, which we shall again denote by $d\pi$. Notice also that, for $h \in H$, $\mathrm{Ad}^G(h)$ leaves $\mathfrak{h}$ invariant and so induces a map

$$\mathrm{Ad}^G_H(h) : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}.$$

Now, let xH be a fixed point of $l_{g^{-1}}$, and take $h \in h(g, xH)$. Choose x in the coset xH such that $g^{-1}x = xh$. For $y \in G$ one computes

$$(\pi \circ L_{g^{-1}} \circ R_{h^{-1}})(y) = \pi(g^{-1}yh^{-1}) = g^{-1}yH = l_{g^{-1}}(yH) = (l_{g^{-1}} \circ \pi)(y),$$

so that

$$\pi \circ L_{g^{-1}} \circ R_{h^{-1}} = l_{g^{-1}} \circ \pi. \tag{2.17}$$

Observe, additionally, that $L_{g^{-1}} \circ R_{h^{-1}}$ fixes x. We therefore see that $L_{g^{-1}} \circ R_{h^{-1}} \circ L_x = L_x \circ L_h \circ R_{h^{-1}}$, which, together with equations (2.16) and (2.17), leads us to

$$l_x \circ \pi \circ L_h \circ R_{h^{-1}} = l_{g^{-1}} \circ l_x \circ \pi.$$

Differentiating this, and using the identification $dl_x \circ d\pi : \mathfrak{g}/\mathfrak{h} \to T_{\pi(x)}(G/H)$, we obtain the commutative diagram

$$\begin{array}{ccc} \mathfrak{g}/\mathfrak{h} & \xrightarrow{\mathrm{Ad}^G_H(h)} & \mathfrak{g}/\mathfrak{h} \\ {\scriptstyle dl_x \circ d\pi}\downarrow & & {\scriptstyle dl_x \circ d\pi}\downarrow \\ T_{\pi(x)}(G/H) & \xrightarrow{dl_{g^{-1}}} & T_{\pi(x)}(G/H) \end{array}$$

thus proving the lemma. □

Consider now the case when G is a connected, real, semi-simple Lie group with finite centre, θ a Cartan involution of $\mathfrak{g}$, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the corresponding Cartan decomposition. Further, let K be the maximal compact subgroup of G associated to $\mathfrak{k}$, and consider the corresponding Riemannian symmetric space $\mathbb{X} = G/K$ which is assumed to be of non-compact type. By definition, θ is an involutive automorphism of $\mathfrak{g}$ such that the bilinear form $\langle \cdot, \cdot \rangle_\theta$ is strictly positive definite. In particular, $\langle \cdot, \cdot \rangle_{\theta|\mathfrak{p}\times\mathfrak{p}}$ is a symmetric, positive-definite, bilinear form, yielding a left-invariant metric on G/K. Endowed with this metric, G/K becomes a complete, simply connected, Riemannian manifold with non-positive sectional curvature. Such manifolds are called *Hadamard manifolds*. Furthermore, for each $g \in G$, $l_{g^{-1}} : G/K \to G/K$ is an isometry on G/K with respect to this left-invariant metric. Note that Riemannian symmetric spaces of non-compact type are precisely the simply connected

Riemannian symmetric spaces with sectional curvature $\kappa \leq 0$ and with no Euclidean de Rham factor. We now have the following

LEMMA 7. *Let $g \in G$ be such that $l_{g^{-1}} : G/K \to G/K$ is transversal. Then $l_{g^{-1}}$ has a unique fixed point in G/K.*

PROOF. Let M be a Hadamard manifold, and φ an isometry on M that leaves two distinct points $x, y \in M$ fixed. By general theory, there is a unique minimal geodesic $\gamma : \mathbb{R} \to M$ joining x and y. Let $\gamma(0) = x$ and $\gamma(1) = y$, so that $\varphi \circ \gamma(0) = \varphi(x) = x$ and $\varphi \circ \gamma(1) = \varphi(y) = y$. Since isometries take geodesics to geodesics, $\varphi \circ \gamma$ is a geodesic in M, joining x and y. By the uniqueness of γ we therefore conclude that $\varphi \circ \gamma = \gamma$. This means that an isometry on a Hadamard manifold with two distinct fixed points also fixes the unique geodesic joining them point by point. Since, by assumption, $l_{g^{-1}} : G/K \to G/K$ has only isolated fixed points, the lemma follows. $\square$

In what follows, we shall call an element $g \in G$ *transversal relative to a closed subgroup H* if $l_{g^{-1}} : G/H \to G/H$ is transversal, and denote the set of all such elements by $G(H)$.

LEMMA 8. *Let $\mathfrak{h}$ be a Cartan subalgebra of a real semi-simple Lie algebra $\mathfrak{g}$. If H is the Cartan subgroup of G associated to $\mathfrak{h}$, then for any regular element h in H,* $\det(\mathrm{Ad}\,(h) - \mathbf{1})_{|\mathfrak{g}/\mathfrak{h}} \neq 0$.

PROOF. For $x \in G$, let $\mathfrak{g}_x$ denote the centralizer of x in $\mathfrak{g}$ i.e., $\mathfrak{g}_x = \{X \in \mathfrak{g} : \mathrm{Ad}\,(x)X = X\}$. Now, by definition, the Cartan subgroup H associated to $\mathfrak{h}$ is the centralizer of $\mathfrak{h}$ in G, and so is given by $H = \{g \in G : \mathrm{Ad}\,(g)X = X, \quad \forall \quad H \in \mathfrak{h}\}$. Therefore $\mathfrak{h} \subset \ker(\mathrm{Ad}\,(h) - \mathbf{1})$ for any $h \in H$. If now $h \in H' := H \cap G'$, the set of regular elements in H, then $\mathfrak{g}_h$ is a Cartan subalgebra of $\mathfrak{g}$. Since $\ker(\mathrm{Ad}\,(h) - \mathbf{1}) \supset \mathfrak{h}$, the fact that Cartan subalgebras are, in particular, maximal Abelian subalgebras, then gives us that $\mathfrak{g}_h = \mathfrak{h}$. $\mathrm{Ad}\,(h) - \mathbf{1}$, thus, induces a non-singular map on $\mathfrak{g}/\mathfrak{h}$ and the lemma follows. $\square$

PROPOSITION 1. *Let G be a connected, real, semi-simple Lie group with finite centre, and K a maximal compact subgroup of G. Suppose* $\mathrm{rank}(G) = \mathrm{rank}(K)$. *Then any regular element of G is transversal relative to K. In other words, $G' \subset G(K)$, where G' denotes the set of regular elements in G.*

PROOF. If a regular element g is such that $l_{g^{-1}} : G/K \to G/K$ has no fixed points, it is of course transversal. Let, therefore, $g \in G'$ be such that $l_{g^{-1}}$ has a fixed point x_0K. By Lemma 5, g must be conjugate to an element $k(g, x_0)$ in K. Consider now a maximal family of mutually non-conjugate Cartan subgroups $J_1, \dots, J_r$ in G, and put $J'_i = J_i \cap G'$ for $i \in \{1, \dots, r\}$. A result of Harish Chandra then implies that

$$G' = \bigcup_{i=1}^{r} \bigcup_{x \in G} x\, J'_i\, x^{-1},$$

see [**War72a**], Theorem 1.4.1.7. From this we deduce that

$$g = xk(g, x_0)x^{-1} = yjy^{-1} \quad \text{for some } x, y \in G, j \in J_i' \text{ for some } i.$$

Hence, $k(g, x_0)$ must be regular. Now, let T be a maximal torus of K. It is a Cartan subgroup of K, and the assumption that $\mathrm{rank}(G) = \mathrm{rank}(K)$ implies that that T is also Cartan in G. Let $k(g, x_0K)$ be the conjugacy class in K associated to x_0K, as in Lemma 5. As K is compact, the maximal torus T intersects every conjugacy class in K. Varying x_0 over the coset x_0K, we can therefore assume that $k(g, x_0) \in k(g, x_0K) \cap T$. Thus, we conclude that $k(g, x_0) \in T \cap G'$. Note that, in particular, we can choose $J_i = T$ by the maximality of the $J_1, \dots, J_r$. Now from Lemma 8, we know that for a regular element $h \in G$ belonging to a Cartan subgroup H one necessarily has $\det(\mathbf{1} - \mathrm{Ad}\,^G_H(h)) \neq 0$. Therefore $\det(\mathbf{1} - \mathrm{Ad}\,^G_T(k(g, x_0))) \neq 0$, and consequently, $\det(\mathbf{1} - \mathrm{Ad}\,^G_K(k(g, x_0))) \neq 0$. The assertion of the proposition now follows from Lemma 6. □

Corollary 1. *Let G be a connected, real, semi-simple Lie group with finite centre, K a maximal compact subgroup of G, and suppose that* $\mathrm{rank}\,(G) = \mathrm{rank}\,(K)$. *Then the set of transversal elements $G(K)$ is open and dense in G.*

Proof. Clearly, $G(K)$ is open. Since the set of regular elements G' is dense in G, the corollary follows from the previous proposition. □

Remark 5. Let us remark that with G as above, and P a parabolic subgroup of G, it is a classical result that $G' \subset G(P)$, see [**Clo84**], page 51.

2.5. Review of pseudodifferential operators

Generalities. This section is devoted to an exposition of some basic facts about pseudodifferential operators needed to formulate our main results in the sequel. For a detailed introduction to the field, the reader is referred to [**Hör85**] and [**Shu01**]. Consider first an open set U in $\mathbb{R}^n$, and let $x_1, \dots, x_n$ be the standard coordinates. For any real number l, we denote by $\mathrm{S}^l(U \times \mathbb{R}^n)$ the class of all functions $a(x, \xi) \in \mathrm{C}^\infty(U \times \mathbb{R}^n)$ such that, for any multi-indices α, β, and any compact set $\mathcal{K} \subset U$, there exist constants $C_{\alpha,\beta,\mathcal{K}}$ for which

$$|(\partial_\xi^\alpha \, \partial_x^\beta \, a)(x, \xi)| \leq C_{\alpha,\beta,\mathcal{K}} \, \langle \xi \rangle^{l - |\alpha|}, \qquad x \in \mathcal{K}, \quad \xi \in \mathbb{R}^n, \tag{2.18}$$

where $\langle \xi \rangle$ stands for $(1 + |\xi|^2)^{1/2}$, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We further put $\mathrm{S}^{-\infty}(U \times \mathbb{R}^n) = \bigcap_{l \in \mathbb{R}} \mathrm{S}^l(U \times \mathbb{R}^n)$. Note that, in general, the constants $C_{\alpha,\beta,K}$ also depend on $a(x, \xi)$. For any such $a(x, \xi)$ one then defines the continuous linear operator

$$A : \mathrm{C}^\infty_c(U) \longrightarrow \mathrm{C}^\infty(U)$$

by the formula

$$Au(x) = \int e^{ix \cdot \xi} a(x, \xi) \hat{u}(\xi) đ\xi, \tag{2.19}$$

where $\hat{u}$ denotes the Fourier transform of u, and $đ\xi = (2\pi)^{-n}\, d\xi$. [1] An operator A of this form is called a *pseudodifferential operator of order* l, and we denote the class of all such operators for which $a(x,\xi) \in \mathrm{S}^l(U \times \mathbb{R}^n)$ by $\mathrm{L}^l(U)$. The set $\mathrm{L}^{-\infty}(U) = \bigcap_{l\in\mathbb{R}} \mathrm{L}^l(U)$ consists of all operators with smooth kernel. They are called *smooth operators*. By inserting in (2.19) the definition of $\hat{u}$, we obtain for Au the expression

$$Au(x) = \int\int e^{i(x-y)\cdot\xi} a(x,\xi) u(y)\, dy\, đ\xi, \tag{2.20}$$

which has a suitable regularization as an oscillatory integral. The Schwartz kernel of A is a distribution $K_A \in \mathcal{D}'(U \times U)$ which is given the oscillatory integral

$$K_A(x,y) = \int e^{i(x-y)\cdot\xi} a(x,\xi)\, đ\xi. \tag{2.21}$$

It is a smooth function off the diagonal in $U \times U$. Consider next an n-dimensional paracompact C^∞ manifold $\mathbf{X}$, and let $\{(\kappa_\gamma, \widetilde{U}^\gamma)\}$ be an atlas for $\mathbf{X}$. Then a linear operator

$$A : \mathrm{C}^\infty_{\mathrm{c}}(\mathbf{X}) \longrightarrow \mathrm{C}^\infty(\mathbf{X}) \tag{2.22}$$

is called a *pseudodifferential operator on* $\mathbf{X}$ *of order* l if for each chart diffeomorphism $\kappa_\gamma : \widetilde{U}^\gamma \to U^\gamma = \kappa_\gamma(\widetilde{U}^\gamma)$, the operator $A^\gamma u = [A_{|\widetilde{U}^\gamma}(u \circ \kappa_\gamma)] \circ \kappa_\gamma^{-1}$ given by the diagram

$$\begin{tikzcd}
\mathrm{C}^\infty_{\mathrm{c}}(\widetilde{U}^\gamma) \arrow[r, "A_{|\widetilde{U}^\gamma}"] & \mathrm{C}^\infty(\widetilde{U}^\gamma) \\
\mathrm{C}^\infty_{\mathrm{c}}(U^\gamma) \arrow[r, "A^\gamma"] \arrow[u, "\kappa_\gamma^*"] & \mathrm{C}^\infty(U^\gamma) \arrow[u, "\kappa_\gamma^*"']
\end{tikzcd}$$

is a pseudodifferential operator on U^γ of order l, and its kernel K_A is smooth off the diagonal. In this case we write $A \in \mathrm{L}^l(\mathbf{X})$. Note that, since the $\widetilde{U}^\gamma$ are not necessarily connected, we can choose them in such a way that $\mathbf{X} \times \mathbf{X}$ is covered by the open sets $\widetilde{U}^\gamma \times \widetilde{U}^\gamma$. Therefore the condition that K_A is smooth off the diagonal can be dropped. Now, in general, if $\mathbf{X}$ and $\mathbf{Y}$ are two smooth manifolds, and

$$A : \mathrm{C}^\infty_{\mathrm{c}}(\mathbf{X}) \longrightarrow \mathrm{C}^\infty(\mathbf{Y}) \subset \mathcal{D}'(\mathbf{Y})$$

is a continuous linear operator, where $\mathcal{D}'(\mathbf{Y}) = (\mathrm{C}^\infty_{\mathrm{c}}(\mathbf{Y}, \Omega))'$ and $\Omega = |\Lambda^n(\mathbf{Y})|$ is the density bundle on $\mathbf{Y}$, its Schwartz kernel is given by the distribution section $K_A \in \mathcal{D}'(\mathbf{Y} \times \mathbf{X}, \mathbf{1} \boxtimes \Omega_{\mathbf{X}})$, where $\mathcal{D}'(\mathbf{Y} \times \mathbf{X}, 1 \boxtimes \Omega_{\mathbf{X}}) = (\mathrm{C}^\infty_{\mathrm{c}}(\mathbf{Y} \times \mathbf{X}, (\mathbf{1} \boxtimes \Omega_{\mathbf{X}})^* \otimes \Omega_{\mathbf{Y}\times\mathbf{X}}))'$. Observe that $\mathrm{C}^\infty_{\mathrm{c}}(\mathbf{Y}, \Omega_{\mathbf{Y}}) \otimes \mathrm{C}^\infty(\mathbf{X}) \simeq \mathrm{C}^\infty(\mathbf{Y} \times \mathbf{X}, (\mathbf{1} \boxtimes \Omega_{\mathbf{X}})^* \otimes \Omega_{\mathbf{Y}\times\mathbf{X}})$. In case that

[1] Here and in what follows we use the convention that, if not specified otherwise, integration is to be performed over the Euclidean space of relevant dimension.

$\mathbf{X} = \mathbf{Y}$ and $A \in \mathrm{L}^l(\mathbf{X})$, A is given locally by the operators A^γ, which can be written in the form

$$A^\gamma u(x) = \int\int e^{i(x-y)\cdot\xi} a^\gamma(x,\xi) u(y)\, dy đ\xi,$$

where $u \in \mathrm{C}^\infty_c(U^\gamma)$, $x \in U^\gamma$, and $a^\gamma(x,\xi) \in \mathrm{S}^l(U^\gamma, \mathbb{R}^n)$. The kernel of A is then determined by the kernels $K_{A^\gamma} \in \mathcal{D}'(U^\gamma \times U^\gamma)$. For $l < -\dim \mathbf{X}$, they are continuous, and given by absolutely convergent integrals. In this case, their restrictions to the respective diagonals in $U^\gamma \times U^\gamma$ define continuous functions

$$k^\gamma(m) = K_{A^\gamma}(\kappa_\gamma(m), \kappa_\gamma(m)), \qquad m \in \widetilde{U}^\gamma,$$

which, for $m \in \widetilde{U}^{\gamma_1} \cap \widetilde{U}^{\gamma_2}$, satisfy the relations

$$k^{\gamma_2}(m) = |\det{(\kappa_{\gamma_1} \circ \kappa_{\gamma_2}^{-1})'}| \circ \kappa_{\gamma_2}(m) k^{\gamma_1}(m),$$

and therefore define a density $k \in C(\mathbf{X}, \Omega)$ on $\Delta_{\mathbf{X}} \times \mathbf{X} \simeq \mathbf{X}$. If $\mathbf{X}$ is compact, this density can be integrated, yielding the trace of the operator A,

$$\operatorname{tr} A = \int_{\mathbf{X}} k = \sum_\gamma \int_{U^\gamma} (\alpha_\gamma \circ \kappa_\gamma^{-1})(x)\, K_{A^\gamma}(x,x)\, dx, \tag{2.23}$$

where $\{\alpha_\gamma\}$ denotes a partition of unity subordinate to the atlas $\{(\kappa_\gamma, \widetilde{U}^\gamma)\}$, and dx is Lebesgue measure in $\mathbb{R}^n$. Finally, if E and F are vector bundles over X trivialized by

$$\alpha_E : E_{|U^\gamma} \longrightarrow U^\gamma \times \mathbb{C}^e, \qquad \alpha_F : F_{|U^\gamma} \longrightarrow U^\gamma \times \mathbb{C}^f, \tag{2.24}$$

then a continuous linear operator

$$A : \Gamma_c(E) \longrightarrow \Gamma(F) \tag{2.25}$$

is called a *pseudodifferential operator between sections of E and F of order l*, if for any U^γ there is a $f \times e$-matrix of pseudodifferential operators $A_{ij} \in \mathrm{L}^l(U^\gamma)$ such that

$$(\alpha_F \circ (Au)_{|U^\gamma})_i = \sum_{ij} A_{ij}(\alpha_E \circ u)_j, \qquad u \in \Gamma_c(U^\gamma; E). \tag{2.26}$$

We shall write $A \in \mathrm{L}^l(X; E, F)$, or simply $\mathrm{L}^l(E,F)$. The notions of bounded sets and properly supported operators in $\mathrm{L}^l(U)$ are carried over to $\mathrm{L}^l(X)$ and $\mathrm{L}^l(E,F)$ in a natural way. If E and F are topological vector spaces, we will denote by $\mathcal{L}(E,F)$ the vector space of all continuous linear maps from E to F, endowed with the topology of uniform convergence on bounded sets.

Totally characteristic pseudodifferential operators. We introduce now a special class of pseudodifferential operators associated in a natural way to a C^∞ manifold $\mathbf{X}$ with boundary $\partial\mathbf{X}$. Our main reference will be [**Mel82**] in this case. Let $\mathrm{C}^\infty(\mathbf{X})$ be the space of functions on $\mathbf{X}$ which are C^∞ up to the boundary, and

$\dot{\mathrm{C}}^\infty(\mathbf{X})$ the subspace of functions vanishing to all orders on $\partial\mathbf{X}$. The standard spaces of distributions over $\mathbf{X}$ are

$$\mathcal{D}'(\mathbf{X}) = (\dot{\mathrm{C}}^\infty_c(\mathbf{X},\Omega))', \qquad \dot{\mathcal{D}}(\mathbf{X})' = (\mathrm{C}^\infty_c(\mathbf{X},\Omega))',$$

the first being the space of *extendible distributions*, whereas the second is the space of *distributions supported by* $\mathbf{X}$.

Let $Z = \overline{\mathbb{R}^+} \times \mathbb{R}^{n-1}$ be the standard manifold with boundary with the natural coordinates $x = (x_1, x')$, where x_1 is the boundary-defining function on Z. Then any differential operator in the algebra of differential operators generated by the vector fields $x_1\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_j}$, $1 < j \leq n$ is called a *totally characteristic differential operator*. Notice that the vector field $x_1\frac{\partial}{\partial x_1}$ is tangential to the boundary of Z. Consider now the translated partial Fourier transform of a symbol $a(x,\xi) \in \mathrm{S}^l(\mathbb{R}^n \times \mathbb{R}^n)$,

$$Ma(x,\xi';t) = \int e^{i(1-t)\xi_1} a(x,\xi_1,\xi')d\xi_1,$$

where we wrote $\xi = (\xi_1,\xi')$. $Ma(x,\xi';t)$ is C^∞ away from $t = 1$, and one says that $a(x,\xi)$ is *lacunary* if it satisfies the condition

$$Ma(x,\xi';t) = 0 \qquad \text{for } t < 0. \tag{2.27}$$

The subspace of lacunary symbols will be denoted by $\mathrm{S}^l_{la}(\mathbb{R}^n \times \mathbb{R}^n)$. In order to define on Z operators of the form (2.20), where now $a(x,\xi) = \widetilde{a}(x_1,x',x_1\xi_1,\xi')$ is a more general amplitude and $\widetilde{a}(x,\xi)$ is lacunary, one rewrites the formal adjoint of A by making a singular coordinate change. Thus, for $u \in \mathrm{C}^\infty_c(Z)$, one considers

$$A^*u(y) = \int\int e^{i(y-x)\cdot\xi}\overline{a}(x,\xi)u(x)\; dx đ\xi.$$

By putting $\lambda = x_1\xi_1$, $s = x_1/y_1$, this can be rewritten as

$$A^*u(y) = (2\pi)^{-n}\int\int\int\int e^{i(1/s-1,y'-x')\cdot(\lambda,\xi')}\overline{\widetilde{a}}(y_1 s,x',\lambda,\xi')u(y_1 s,x')d\lambda\frac{ds}{s}dx'd\xi'. \tag{2.28}$$

According to [**Mel82**], Propositions 3.6 and 3.9, for every $\widetilde{a} \in \mathrm{S}^{-\infty}_{la}(Z\times\mathbb{R}^n)$, the successive integrals in (2.28) converge absolutely and uniformly, thus defining a continuous bilinear form

$$\mathrm{S}^{-\infty}_{la}(Z\times\mathbb{R}^n) \times \mathrm{C}^\infty_c(Z) \longrightarrow \mathrm{C}^\infty(Z),$$

which extends to a separately continuous form

$$\mathrm{S}^{\infty}_{la}(Z\times\mathbb{R}^n) \times \mathrm{C}^\infty_c(Z) \longrightarrow \mathrm{C}^\infty(Z).$$

If $\widetilde{a} \in \mathrm{S}^\infty_{la}(Z\times\mathbb{R}^n)$ and $a(x,\xi) = \widetilde{a}(x_1,x',x_1\xi_1,\xi')$, one then defines the operator

$$A : \dot{\mathcal{E}}'(Z) \longrightarrow \dot{\mathcal{D}}'(Z), \tag{2.29}$$

written formally as (2.20), as the adjoint of A^*. In this way, the oscillatory integral (2.20) is identified with a separately continuous bilinear mapping

$$\mathrm{S}^{\infty}_{la}(Z \times \mathbb{R}^n) \times \dot{\mathcal{E}}'(Z) \longrightarrow \dot{\mathcal{D}}'(Z).$$

The space $\mathrm{L}^l_b(Z)$ of *totally characteristic pseudodifferential operators on* Z *of order* l consists of those continuous linear maps (2.29) such that for any $u, v \in \mathrm{C}^{\infty}_{\mathrm{c}}(Z)$, vAu is of the form (2.20) with $a(x,\xi) = \widetilde{a}(x_1, x', x_1\xi_1, \xi')$ and $\widetilde{a}(x,\xi) \in \mathrm{S}^l_{la}(Z \times \mathbb{R}^n)$. Similarly, a continuous linear map (2.22) on a smooth manifold $\mathbf{X}$ with boundary $\partial\,\mathbf{X}$ is said to be an element of the space $\mathrm{L}^l_b(\mathbf{X})$ of *totally characteristic pseudodifferential operators on* $\mathbf{X}$ *of order* l, if for a given atlas $(\kappa_\gamma, \widetilde{U}^\gamma)$ the operators $A^\gamma u = [A_{|\widetilde{U}^\gamma}(u \circ \kappa_\gamma)] \circ \kappa_\gamma^{-1}$ are elements of $\mathrm{L}^l_b(Z)$, where the $\widetilde{U}^\gamma$ are coordinate patches isomorphic to subsets in Z.

The main motivation for the definition of the lacunarity condition on the symbol is as follows. If A is a pseudodifferential operator defined as above from a lacunary symbol, then A preserves the conditions that the distributions u, on which it acts, has restrictions, or traces, of all orders to the boundary and

$$(Au)_{|\partial Z} = A_0(u_{|\partial Z})$$

for some pseudodifferential operator A_0 defined on ∂Z. This is the fundamental property that allows such operators to act on distributions satisfying differential or pseudodifferential equations with boundary conditions.

In an analogous way, it is possible to introduce the concept of a totally characteristic pseudodifferential operator on a manifold with corners. As the standard manifold with corners, consider

$$\mathbb{R}^{n,k} = [0,\infty)^k \times \mathbb{R}^{n-k}, \qquad 0 \leq k \leq n,$$

with coordinates $x = (x_1, \dots, x_k, x')$. A *totally characteristic pseudodifferential operator on* $\mathbb{R}^{n,k}$ *of order* l is locally given by an oscillatory integral (2.20) with $a(x,\xi) = \widetilde{a}(x, x_1\xi_1, \dots, x_k\xi_k, \xi')$, where now $\widetilde{a}(x,\xi)$ is a symbol of order l that satisfies the lacunary condition for each of the coordinates $x_1, \dots, x_k$, i.e.

$$\int e^{i(1-t)\xi_j} a(x,\xi)\, d\xi_j = 0 \qquad \text{for } t < 0 \text{ and } 1 \leq j \leq k.$$

In this case we write $\widetilde{a}(x,\xi) \in \mathrm{S}^l_{la}(\mathbb{R}^{n,k} \times \mathbb{R}^n)$. A continuous linear map (2.22) on a smooth manifold $\mathbf{X}$ with corners is then said to be an element of the space $\mathrm{L}^l_b(\mathbf{X})$ of *totally characteristic pseudodifferential operators on* $\mathbf{X}$ *of order* l, if for a given atlas $(\kappa_\gamma, \widetilde{U}^\gamma)$ the operators $A^\gamma u = [A_{|\widetilde{U}^\gamma}(u \circ \kappa_\gamma)] \circ \kappa_\gamma^{-1}$ are totally characteristic pseudodifferential operator on $\mathbb{R}^{n,k}$ of order l, where the $\widetilde{U}^\gamma$ are coordinate patches isomorphic to subsets in $\mathbb{R}^{n,k}$. For an extensive treatment, we refer the reader to [**Loy98**].

CHAPTER 3

Integral operators and the main structure theorem

This chapter is concerned with the integral operators that are the focus of our study. After a few preliminaries about their definition as Bochner integrals, we explore the micro-local structure of these operators taking into account the orbit structure for the G-action on the Oshima compactification of the symmetric space $\mathbb{X}$. This culminates in one of the central results of this thesis - Theorem 2, Section 3.2.

3.1. The integral operators $\pi(f)$

In this section, we study integral operators of the form

$$\pi(f) = \int_G f(g)\pi(g)d_G(g) \tag{3.1}$$

where π is the regular representation of G on the Banach space $C(\widetilde{\mathbb{X}})$ of continuous functions on $\mathbb{X}$, f a smooth, rapidly decreasing function on G, and d_G a Haar measure on G. These are defined as Bochner integrals. Such operators play an important role in representation theory, and our interest will be directed towards the elucidation of the microlocal structure of the operators $\pi(f)$.

Now we make a brief digression to consider Bochner integrals. Let $(\mathcal{M}, \Sigma, \mu)$ be a measure space, and $\mathcal{B}$ a Banach space over $\mathbb{R}$. The definition of the Bochner integral can be given, essentially in the spirit of the definition of the Lebesgue integral. A function $\psi : \mathcal{M} \to \mathcal{B}$ taking only a finite number of values, say $b_1, b_2, \ldots, b_n$, is called a simple function if $E_j = \psi^{-1}(b_j) \in \Sigma$ for each j. Then one has that

$$\psi = \sum_{j=1}^{n} \chi_{E_j} b_j,$$

where χ_{E_j} is the characteristic function of the set E_j. If $\mu(E_j) < \infty$ for each non-zero b_j, then ψ is called a simple function with finite support. Then the integral of such a $\mathcal{B}$-valued simple function is the vector $\int_{\mathcal{M}} \psi d\mu \in \mathcal{B}$ is defined as

$$\int_{\mathcal{M}} \psi d\mu = \sum_{j=1}^{n} \mu(E_j) b_j.$$

It can be shown that this integral is defined independently of the representation for ψ. The integral so defined is a linear operator from the space of simple functions with finite support to $\mathcal{B}$. As one would expect, if $E \in \Sigma$, then the integral $\int_E \psi d\mu$ of ψ over E is defined as

$$\int_E \psi d\mu = \int_{\mathcal{M}} \psi \chi_E d\mu.$$

A function $f : \mathcal{M} \to \mathcal{B}$ is said to be *strongly measurable* if there exists a sequence ψ_n of simple functions with finite support such that, for almost all $m \in \mathcal{M}$, we have $\lim_{n\to\infty} \|f(m) - \psi_n(m)\|_{\mathcal{B}} = 0$, where $\|.\|_{\mathcal{B}}$ is the norm on the Banach space $\mathcal{B}$. Clearly, simple functions with finite support are strongly measurable. If f is strongly measurable, then $\|f\| : \mathcal{M} \to \mathbb{R}$, defined by $\|f\|(m) := \|f(m)\|_{\mathcal{B}}$, is also strongly measurable. A strongly measurable function $f : \mathcal{M} \to \mathcal{B}$ is called *Bochner integrable* if there is a sequence ψ_n of simple functions with finite support such that the real-valued function $\|f - \psi_n\|$ is Lebesgue integrable for each n and

$$\lim_{n\to\infty} \int_{\mathcal{M}} \|f - \psi_n\|_{\mathcal{B}} \, d\mu = 0.$$

For such an f, we can , for $E \in \Sigma$, define the *Bochner integral over E* to be

$$\int_E f d\mu = \lim_{n\to\infty} \int_E \psi_n d\mu,$$

where the limit of the sequence of vectors $\int_E \psi_n$ is taken in the norm topology of $\mathcal{B}$. This is well-defined as any two sequences of simple functions approximating a given strongly measurable function have the same limit. It is in this fashion that we make sense of the integral operators $\pi(f)$.

Let now $\widetilde{\mathbb{X}}$ be the Oshima compactification of a Riemannian symmetric space $\mathbb{X} = G/K$ of non-compact type. As was already explained, G acts analytically on $\widetilde{\mathbb{X}}$, and the orbital decomposition is of normal crossing type. Consider the Banach space $\mathrm{C}(\widetilde{\mathbb{X}})$ of continuous, complex valued functions on $\widetilde{\mathbb{X}}$, equipped with the supremum norm, and let $(\pi, \mathrm{C}(\widetilde{\mathbb{X}}))$ be the corresponding continuous regular representation of G given by

$$\pi(g)\varphi(\tilde{x}) = \varphi(g \cdot \tilde{x}), \qquad \varphi \in \mathrm{C}(\widetilde{\mathbb{X}}).$$

The representation of the universal enveloping algebra $\mathfrak{U}$ of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ on the space of differentiable vectors $\mathrm{C}(\widetilde{\mathbb{X}})_\infty$ will be denoted by $d\pi$. We will also consider the regular representation of G on $\mathrm{C}^\infty(\widetilde{\mathbb{X}})$ which, equipped with the topology of uniform convergence, becomes a Fréchet space. This representation will be denoted by π as well. Let $(L, \mathrm{C}^\infty(G))$ be the left regular representation of G. With respect to the left-invariant metric on G given by $\langle , \rangle_\theta$, we define $d(g, h)$ as the distance between two points $g, h \in G$, and set $|g| = d(g, e)$, where e is the identity element of G. A function f on G is said to be of *at most of exponential growth*, if there exists a $\kappa > 0$ such that $|f(g)| \leq Ce^{\kappa|g|}$ for some constant $C > 0$. As before, denote a Haar measure on G by d_G. Consider next the space $\mathcal{S}(G)$ of rapidly decreasing functions on G introduced in [**Ram06**].

DEFINITION 1. *The space of rapidly decreasing functions on G, denoted by $\mathcal{S}(G)$, is given by all functions $f \in \mathrm{C}^\infty(G)$ satisfying the following conditions:*

i) *For every $\kappa \geq 0$, and $X \in \mathfrak{U}$, there exists a constant $C > 0$ such that*

$$|dL(X)f(g)| \leq Ce^{-\kappa|g|};$$

ii) *for every $\kappa \geq 0$, and $X \in \mathfrak{U}$, one has $dL(X)f \in \mathrm{L}^1(G, e^{\kappa|g|}d_G)$.*

For later purposes, let us recall the following integration formulae.

PROPOSITION 2. *Let $f_1 \in \mathcal{S}(G)$, and assume that $f_2 \in \mathrm{C}^\infty(G)$, together with all its derivatives, is at most of exponential growth. Let $X_1, \dots, X_d$ be a basis of $\mathfrak{g}$, and for $X^\gamma = X_{i_1}^{\gamma_1} \dots X_{i_r}^{\gamma_r}$ write $X^{\tilde{\gamma}} = X_{i_r}^{\gamma_r} \dots X_{i_1}^{\gamma_1}$, where γ is an arbitrary multi-index. Then*

$$\int_G f_1(g) dL(X^\gamma) f_2(g) d_G(g) = (-1)^{|\gamma|} \int_G dL(X^{\tilde{\gamma}}) f_1(g) f_2(g) d_G(g).$$

PROOF. See [**Ram06**], Proposition 1. □

To every $f \in \mathcal{S}(G)$ and $\varphi \in \mathrm{C}(\widetilde{\mathbb{X}})$, we then associate an element in $\mathrm{C}(\widetilde{\mathbb{X}})$ given by $\int_G f(g)\pi(g)\varphi \, d_G(g)$. It is defined as a Bochner integral, and the continuous linear operator on $\mathrm{C}(\widetilde{\mathbb{X}})$ obtained this way is denoted by (3.1). Its restriction to $\mathrm{C}^\infty(\widetilde{\mathbb{X}})$ induces a continuous linear operator

$$\pi(f) : \mathrm{C}^\infty(\widetilde{\mathbb{X}}) \longrightarrow \mathrm{C}^\infty(\widetilde{\mathbb{X}}) \subset \mathcal{D}'(\widetilde{\mathbb{X}}),$$

with Schwartz kernel given by the distribution section $\mathcal{K}_f \in \mathcal{D}'(\widetilde{\mathbb{X}} \times \widetilde{\mathbb{X}}, \mathbf{1} \boxtimes \Omega_{\widetilde{\mathbb{X}}})$. The properties of the Schwartz kernel $\mathcal{K}_f$ will depend on the analytic properties of f, as well as the orbit structure of the underlying G-action, and our main effort will be directed towards the elucidation of the structure of $\mathcal{K}_f$. For this, let us consider the orbital decomposition (2.11) of $\widetilde{\mathbb{X}}$, and remark that the restriction of $\pi(f)\varphi$ to any of the connected components isomorphic to $G/P_\Theta(K)$ depends only on the restriction of $\varphi \in \mathrm{C}(\widetilde{\mathbb{X}})$ to that component, so that we obtain the continuous linear operators

$$\pi(f)_{|\widetilde{\mathbb{X}}_\Theta} : \mathrm{C}^\infty_{\mathrm{c}}(\widetilde{\mathbb{X}}_\Theta) \longrightarrow \mathrm{C}^\infty(\widetilde{\mathbb{X}}_\Theta),$$

where $\widetilde{\mathbb{X}}_\Theta$ denotes a component in $\widetilde{\mathbb{X}}$ isomorphic to $G/P_\Theta(K)$. Let us now assume that $\Theta = \Delta$, so that $P_\Theta(K) = K$. Since G acts transitively on $\widetilde{\mathbb{X}}_\Delta$ one deduces that $\pi(f)_{|\widetilde{\mathbb{X}}_\Delta} \in \mathrm{L}^{-\infty}(\widetilde{\mathbb{X}}_\Delta)$, c.p. [**Ram06**], Section 4.

Now, we shall begin a microlocal description of the integral operators $\pi(f)$. Let $\left\{(\widetilde{U}_{m_w}, \varphi_{m_w}^{-1})\right\}_{w \in W}$ be the finite atlas on the Oshima compactification $\widetilde{\mathbb{X}}$ defined earlier. As before, for each point $\tilde{x} \in \widetilde{\mathbb{X}}$, we choose open neighbourhoods $\widetilde{W_{\tilde{x}}} \subsetneq \widetilde{W'_{\tilde{x}}}$ of $\tilde{x}$ contained in a chart $\widetilde{U}_{m_w(\tilde{x})}$. Since $\widetilde{\mathbb{X}}$ is compact, we can find a finite subcover of the cover $\left\{\widetilde{W_{\tilde{x}}}\right\}_{\tilde{x} \in \widetilde{\mathbb{X}}}$, and in this way obtain a finite atlas $\left\{(\widetilde{W}_\gamma, \varphi_\gamma^{-1})\right\}_{\gamma \in I}$ of $\widetilde{\mathbb{X}}$, where for simplicity we wrote $\widetilde{W}_\gamma = \widetilde{W}_{\tilde{x}_\gamma}$, $\varphi_\gamma = \varphi_{m_w(\tilde{x}_\gamma)}$. Further, let $\{\alpha_\gamma\}_{\gamma\in I}$ be a partition of unity subordinate to this atlas, and let $\{\bar{\alpha}_\gamma\}_{\gamma\in I}$ be another set of functions satisfying $\bar{\alpha}_\gamma \in \mathrm{C}^\infty_\mathrm{c}(\widetilde{W}'_\gamma)$ and $\bar{\alpha}_{\gamma|\widetilde{W}_\gamma} \equiv 1$. Consider now the localization of $\pi(f)$ with respect to the atlas above given by

$$A^\gamma_f u = [\pi(f)_{|\widetilde{W}_\gamma}(u\circ\varphi_\gamma^{-1})]\circ\varphi_\gamma, \qquad u \in \mathrm{C}^\infty_\mathrm{c}(W_\gamma),\, W_\gamma = \varphi_\gamma^{-1}(\widetilde{W}_\gamma) \subset \mathbb{R}^{k+l}.$$

Writing $\varphi^g_\gamma = \varphi_\gamma^{-1}\circ g\circ\varphi_\gamma$ and $x = (x_1,\dots,x_{k+l}) = (n,t) \in W_\gamma$ we obtain

$$A^\gamma_f u(x) = \int_G f(g)[(u\circ\varphi_\gamma^{-1})\bar{\alpha}_\gamma](g\cdot\varphi_\gamma(x))\, d_G(g) = \int_G f(g)c_\gamma(x,g)(u\circ\varphi^g_\gamma)(x)d_G(g),$$

where we put $c_\gamma(x,g) = \bar{\alpha}_\gamma(g\cdot\varphi_\gamma(x))$. Next, define the functions

$$\hat{f}_\gamma(x,\xi) = \int_G e^{i\varphi^g_\gamma(x)\cdot\xi}c_\gamma(x,g)f(g)dg, \qquad a^\gamma_f(x,\xi) = e^{-ix\cdot\xi}\hat{f}_\gamma(x,\xi).$$

Differentiating under the integral we see that $\hat{f}_\gamma(x,\xi), a^\gamma_f(x,\xi) \in \mathrm{C}^\infty(W_\gamma\times\mathbb{R}^{k+l})$.

Let now $x = (n,t) \in W_\gamma$, and let T_x be the diagonal $(l\times l)$-matrix with entries $x_{k+1},\dots,x_{k+l}$. Further, let $\chi_1,\dots,\chi_l$ be as in Lemma 4. We introduce the auxiliary symbol

$$\begin{aligned}\tilde{a}^\gamma_f(x,\xi) &= a^\gamma_f(x,(\mathbf{1}_k\otimes T_x^{-1})\xi) = e^{-i(x_1,\dots,x_k,1,\dots,1)\cdot\xi}\int_G \psi^\gamma_{\xi,x}(g)c_\gamma(x,g)f(g)d_G(g)\\ &= \int_G e^{i\Psi_\gamma(g,x)\cdot\xi}c_\gamma(x,g)f(g)d_G(g),\end{aligned} \tag{3.2}$$

where we put

$$\begin{aligned}\Psi_\gamma(g,x) &= [(\mathbf{1}_k\otimes T_x^{-1})(\varphi^g_\gamma(x)-x)]\\ &= (x_1(g\cdot\tilde{x})-x_1(\tilde{x}),\dots,x_k(g\cdot\tilde{x})-x_k(\tilde{x}),\chi_1(g,\tilde{x})-1,\dots,\chi_l(g,\tilde{x})-1),\end{aligned}$$

as well as

$$\psi^\gamma_{\xi,x}(g) = e^{i(x_1(g\cdot\tilde{x}),\dots,x_k(g\cdot\tilde{x}),\chi_1(g,\tilde{x}),\dots,\chi_l(g,\tilde{x}))\cdot\xi}.$$

Clearly, $\tilde{a}^\gamma_f(x,\xi) \in \mathrm{C}^\infty(W_\gamma\times\mathbb{R}^{k+l})$. Our next goal is to show that $\tilde{a}^\gamma_f(x,\xi)$ is a lacunary symbol. To do so, we need the following

PROPOSITION 3. *Let $(L, \mathrm{C}^\infty(G))$ be the left regular representation of G. Let $X_{-\lambda,i}, H_j$ be the basis elements of $\mathfrak{n}^-$ and $\mathfrak{a}$ introduced in Section 2.1, and $(\widetilde{W}_\gamma,\varphi_\gamma)$ an arbitrary chart. With $x = (n,t) \in W_\gamma$, $\tilde{x} = \varphi_\gamma(x) \in \widetilde{W}_\gamma$, $g \in V_{\gamma,\tilde{x}}$ one has*

$$\begin{pmatrix} dL(X_{-\lambda,1})\psi^\gamma_{\xi,x}(g)\\ \vdots \\ dL(H_l)\psi^\gamma_{\xi,x}(g)\end{pmatrix} = i\psi^\gamma_{\xi,x}(g)\Gamma(x,g)\xi, \tag{3.3}$$

with

$$\Gamma(x,g)=\begin{pmatrix}\Gamma_1 & \Gamma_2\\ \Gamma_3 & \Gamma_4\end{pmatrix}=\left(\begin{array}{c|c} dL(X_{-\lambda,i})n_{j,\tilde{x}}(g) & dL(X_{-\lambda,i})\chi_j(g,\tilde{x}) \\ \hline dL(H_i)n_{j,\tilde{x}}(g) & dL(H_i)\chi_j(g,\tilde{x})\end{array}\right) \tag{3.4}$$

belonging to $\mathrm{GL}(l+k,\mathbb{R})$, *where* $n_{j,\tilde{x}}(g)=n_j(g\cdot\tilde{x})$.

Proof. Fix a chart $(\widetilde{W}_\gamma,\varphi_\gamma^{-1})$, and let x, $\tilde{x}$, g be as above. For $X\in\mathfrak{g}$, one computes that

$$dL(X)\psi^\gamma_{\xi,x}(g)=\frac{d}{ds}e^{i(\mathbf{1}_k\otimes T_x^{-1})\varphi_\gamma^{\mathrm{e}^{-sX}\,g}(x)\cdot\xi}{}_{|s=0}=i\psi^\gamma_{\xi,x}(g)\Big[\sum_{i=1}^k\xi_i dL(X)n_{i,\tilde{x}}(g)$$
$$+\sum_{j=1}^l\xi_{k+j}dL(X)\chi_j(g,\tilde{x})\Big],$$

showing the first equality. To see the invertibility of the matrix $\Gamma(x,g)$, note that for small s

$$\chi_j(\,\mathrm{e}^{-sX}\,g,\tilde{x})=\chi_j(g,\tilde{x})\chi_j(\,\mathrm{e}^{-sX}\,,g\cdot\tilde{x}).$$

Lemma 4 then yields

$$dL(H_i)\chi_j(g,\tilde{x})=\chi_j(g,\tilde{x})\frac{d}{ds}\Big(e^{c_{ij}(m_{w_\gamma})s}\Big)_{|s=0}=\chi_j(g,\tilde{x})c_{ij}(m_{w_\gamma}).$$

This means that Γ_4 is the product of the matrix $\big(c_{ij}(m_{w_\gamma})\big)_{i,j}$ with the diagonal matrix whose j-th diagonal entry is $\chi_j(g,\tilde{x})$. Since $\big(c_{ij}(m_{w_\gamma})\big)_{i,j}$ is just the matrix realization of $\mathrm{Ad}\,(m_{w_\gamma}^{-1})$ relative to the basis $\{H_1,\dots,H_l\}$ of $\mathfrak{a}$, it is invertible. On the other hand, by (2.14), $\chi_j(g,\tilde{x})$ is non-zero for all $j\in\{1,\dots,l\}$ and arbitrary g and $\tilde{x}$. Therefore Γ_4, being the product of two invertible matrices, is invertible. Next, let us show that the matrix Γ_1 is non-singular. Its $(ij)^{th}$ entry reads

$$dL(X_{-\lambda,i})n_{j,\tilde{x}}(g)=\frac{d}{ds}n_{j,\tilde{x}}(\,\mathrm{e}^{-sX_{-\lambda,i}}\,\cdot g)_{|s=0}=(-X_{-\lambda,i|\widetilde{\mathbb{X}}})_{g\cdot\tilde{x}}(n_j).$$

For $\Theta\subset\Delta$, $q\in\mathbb{R}^l$, we define the k-dimensional submanifolds

$$\mathfrak{L}_\Theta(q)=\{\tilde{x}=\varphi_\gamma(n,q)\in\widetilde{W}_\gamma: q_i\neq 0\Leftrightarrow\alpha_i\in\Theta\}.$$

As g varies over G in Lemma 1, one deduces that $N^-\times A(\Theta)$ acts locally transitively on $\widetilde{\mathbb{X}}_\Theta$. In addition, $T_{g\cdot\tilde{x}}\mathfrak{L}_\Theta(q)$ is equal to the span of the vector fields $\{X_{-\lambda,i|\widetilde{\mathbb{X}}}\}$, which means that N^- acts locally transitively on $\mathfrak{L}_\Theta(q)$ for arbitrary Θ. Since the latter is parametrized by the coordinates $(n_1,\dots,n_k)$, one concludes that the matrix

$((X_{-\lambda,i|\widetilde{\mathbb{X}}})_{g\cdot\tilde{x}}(n_j))_{ij}$ has full rank. Thus, Γ_1 is non-singular. On the other hand, if $\tilde{x} = \pi(e,n,t) \in \tilde{U}_e$, Lemma 4 implies

$$dL(X_{-\lambda,i})\chi_j(g,\tilde{x}) = \chi_j(g,\tilde{x})\frac{d}{ds}\Big(\chi_j(e^{-sX_{-\lambda,i}}, g\cdot\tilde{x})\Big)_{|s=0} = 0,$$

showing that Γ_2 is identically zero, while Γ_4 is a non-singular diagonal matrix in this case. Geometrically, this amounts to the fact that the fundamental vector field corresponding to H_j is transversal to the hypersurface defined by $t_j = q \in \mathbb{R}\setminus\{0\}$, while the vector fields corresponding to the Lie algebra elements $X_{-\lambda,r}, H_i$, $i \neq j$, are tangential. We therefore conclude that $\Gamma(x,g)$ is non-singular if $\tilde{x} \in \widetilde{U}_e$, which is dense in $\widetilde{\mathbb{X}}$. For symmetry reasons, the same must hold if $\tilde{x}$ lies in one of the remaining charts $\widetilde{U}_{m_{w_\gamma}}$, and the assertion of the lemma follows.

□

3.2. Structure theorem for $\pi(f)$

The main goal of this section is to prove that the restrictions of the operators $\pi(f)$ to the manifolds with corners $\overline{\widetilde{\mathbb{X}}_\Delta}$ are totally characteristic pseudodifferential operators of class $\mathrm{L}_b^{-\infty}$. In what follows, $\{(\widetilde{W}_\gamma, \varphi_\gamma^{-1})\}_{\gamma\in I}$ will always denote the finite atlas of $\widetilde{\mathbb{X}}$ constructed above.

Theorem 2. *Let $\widetilde{\mathbb{X}}$ be the Oshima compactification of a Riemannian symmetric space $\mathbb{X} = G/K$ of non-compact type, and $f \in \mathcal{S}(G)$ a rapidly decaying function on G. Then the operators $\pi(f)$ are locally of the form*

$$A_f^\gamma u(x) = \int e^{ix\cdot\xi} a_f^\gamma(x,\xi)\hat{u}(\xi)d\xi, \qquad u \in \mathrm{C}_{\mathrm{c}}^\infty(W_\gamma), \tag{3.5}$$

where $a_f^\gamma(x,\xi) = \tilde{a}_f^\gamma(x,\xi_1,\dots,\xi_k, x_{k+1}\xi_{k+1},\dots,\xi_{k+l}x_{k+l})$, and $\tilde{a}_f^\gamma(x,\xi) \in \mathrm{S}_{\mathrm{la}}^{-\infty}(W_\gamma \times \mathbb{R}_\xi^{k+l})$ is given by (3.2). In particular, the kernel of the operator A_f^γ is determined by its restrictions to $W_\gamma^* \times W_\gamma^*$, where $W_\gamma^* = \{x = (n,t) \in W_\gamma : t_1\cdots t_l \neq 0\}$, and given by the oscillatory integral

$$K_{A_f^\gamma}(x,y) = \int e^{i(x-y)\cdot\xi} a_f^\gamma(x,\xi)d\xi. \tag{3.6}$$

As a consequence, we obtain the following

Corollary 2. *Let $\widetilde{\mathbb{X}}_\Delta$ be an open G-orbit in $\widetilde{\mathbb{X}}$ isomorphic to G/K. Then the continuous linear operators*

$$\pi(f)_{|\overline{\widetilde{\mathbb{X}}_\Delta}} : \mathrm{C}_{\mathrm{c}}^\infty(\overline{\widetilde{\mathbb{X}}_\Delta}) \longrightarrow \mathrm{C}^\infty(\overline{\widetilde{\mathbb{X}}_\Delta}),$$

are totally characteristic pseudodifferential operators of class $\mathrm{L}_b^{-\infty}$ on the manifolds with corners $\overline{\widetilde{\mathbb{X}}_\Delta}$.

□

Proof of Theorem 2. Our considerations will essentially follow the proof of Theorem 4 in [**Ram06**]. Let $\Gamma(x,g)$ be the matrix defined in (3.4), and consider its extension as an endomorphism in $\mathbb{C}^1[\mathbb{R}^{k+l}_\xi]$ to the symmetric algebra $\mathrm{S}(\mathbb{C}^1[\mathbb{R}^{k+l}_\xi]) \simeq \mathbb{C}[\mathbb{R}^{k+l}_\xi]$. By Proposition 3, $\Gamma(x,g)$ is invertible for $\tilde{x} \in \widetilde{W}_\gamma$, $g \in V_{\gamma,\tilde{x}}$. Therefore, its extension to $\mathrm{S}^N(\mathbb{C}^1[\mathbb{R}^{k+l}_\xi])$ is also an automorphism for any $N \in \mathbb{N}$. Regarding the polynomials $\xi_1, \dots, \xi_{k+l}$ as a basis in $\mathbb{C}^1[\mathbb{R}^{k+l}_\xi]$, let us denote the image of the basis vector ξ_j under the endomorphism $\Gamma(x,g)$ by $\Gamma\xi_j$, so that by (3.3)

$$\begin{aligned} \Gamma\xi_j &= -i\psi^\gamma_{-\xi,x}(g) dL(X_{-\lambda,j})\psi^\gamma_{\xi,x}(g), \qquad 1 \le j \le k, \\ \Gamma\xi_j &= -i\psi^\gamma_{-\xi,x}(g) dL(H_j)\psi^\gamma_{\xi,x}(g), \qquad k+1 \le j \le k+l. \end{aligned}$$

Every polynomial $\xi_{j_1} \otimes \cdots \otimes \xi_{j_N} \equiv \xi_{j_1} \dots \xi_{j_N}$ can then be written as a linear combination

$$\xi^\alpha = \sum_\beta \Lambda^\alpha_\beta(x,g)\Gamma\xi_{\beta_1} \cdots \Gamma\xi_{\beta_{|\alpha|}}, \tag{3.7}$$

where the $\Lambda^\alpha_\beta(x,g)$ are real-analytic functions given in terms of the matrix coefficients of $\Gamma(x,g)$. We need now the following

Lemma 9. *For arbitrary indices $\beta_1, \dots, \beta_r$, one has*

$$\begin{aligned} i^r\psi^\gamma_{\xi,x}(g)\Gamma\xi_{\beta_1}\cdots\Gamma\xi_{\beta_r} &= dL(X_{\beta_1}\cdots X_{\beta_r})\psi^\gamma_{\xi,x}(g) \\ &\quad + \sum_{s=1}^{r-1}\sum_{\alpha_1,\dots,\alpha_s} d^{\beta_1,\dots,\beta_r}_{\alpha_1,\dots,\alpha_s}(x,g) dL(X_{\alpha_1}\cdots X_{\alpha_s})\psi^\gamma_{\xi,x}(g), \end{aligned} \tag{3.8}$$

where the coefficients $d^{\beta_1,\dots,\beta_r}_{\alpha_1,\dots,\alpha_s}(x,g)$ are real-analytic functions given by the matrix coefficients of $\Gamma(x,g)$ which are at most of exponential growth in g, and independent of ξ.

Proof. The lemma is proved by induction. For $r=1$ one has $i\psi^\gamma_{\xi,x}(g)\Gamma\xi_p = dL(X_p)\psi^\gamma_{\xi,x}(g)$, where $1 \le p \le d$. Differentiating the latter equation with respect to X_j, and writing $\Gamma\xi_p = \sum_{s=1}^{k+l}\Gamma_{ps}(x,g)\,\xi_s$, we obtain with (3.7) the equality

$$-\psi^\gamma_{\xi,x}(g)\Gamma\xi_j\Gamma\xi_p = dL(X_jX_p)\psi^\gamma_{\xi,x}(g) - \sum_{s,r=1}^{k+l}(dL(X_j)\Gamma_{ps})(x,g)\Lambda^s_r(x,g)dL(X_r)\psi^\gamma_{\xi,x}(g).$$

Hence, the assertion of the lemma is correct for $r=1,2$. Now, assume that it holds for $r \le N$. Setting $r=N$ in equation (3.8), and differentiating with respect to X_p, yields for the left hand side

$$i^{N+1}\psi^\gamma_{\xi,x}(g)\Gamma\xi_p\Gamma\xi_{\beta_1}\cdots\Gamma\xi_{\beta_N}$$
$$+i^N\psi^\gamma_{\xi,x}(g)\Big(\sum_{s,q=1}^{k+l}(dL(X_p)\Gamma_{\beta_1 s})(x,g)\Lambda^s_q(x,g)\Gamma\xi_q\Big)\Gamma\xi_{\beta_2}\cdots\Gamma\xi_{\beta_N}+\dots .$$

By assumption, we can apply (3.8) to the products $\Gamma\xi_q\Gamma\xi_{\beta_2}\cdots\Gamma\xi_{\beta_N},\dots$ of at most N factors. Since

$$\|\pi(g)\|\leq ce^{\kappa|g|},\qquad g\in G, \tag{3.9}$$

for some constants $c\geq 1,\kappa\geq 0$, see [**Rob91**], page 12, the functions $n_{i,\tilde{x}}(g)$, $\chi_j(g,\tilde{x})$, and consequently the coefficients of $\Gamma(x,g)$, are at most of exponential growth in g, and the assertion of the lemma follows. □

End of proof of Theorem 2. Let us next show that $\tilde{a}^\gamma_f(x,\xi)\in \mathrm{S}^{-\infty}(W_\gamma\times\mathbb{R}^{k+l}_\xi)$. As already noted, $\tilde{a}^\gamma_f(x,\xi)\in \mathrm{C}^\infty(W_\gamma\times\mathbb{R}^{k+l}_\xi)$. While differentiation with respect to ξ does not alter the growth properties of $\tilde{a}^\gamma_f(x,\xi)$, differentiation with respect to x yields additional powers in ξ. Now, as an immediate consequence of equations (3.7) and (3.8), one computes for arbitrary $N\in\mathbb{N}$

$$\psi^\gamma_{\xi,x}(g)(1+|\xi|^2)^N=\sum_{r=0}^{2N}\sum_{|\alpha|=r}b^N_\alpha(x,g)dL(X^\alpha)\psi^\gamma_{\xi,x}(g), \tag{3.10}$$

where the coefficients $b^N_\alpha(x,g)$ are at most of exponential growth in g. Now, the derivate $(\partial^\alpha_\xi\,\partial^\beta_x\,\tilde{a}^\gamma_f)(x,\xi)$ is a finite sum of terms of the form

$$\xi^{\beta'}e^{-i(x_1,\dots,x_k,1,\dots,1)\cdot\xi}\int_G f(g)d^\alpha_{\beta'\beta''}(x,g)\psi^\gamma_{\xi,x}(g)(\partial^{\beta''}_x\,c_\gamma)(x,g)dg,$$

the functions $d^\alpha_{\beta'\beta''}(x,g)$ being at most of exponential growth in g. Making use of equation (3.10), and integrating according to Proposition 2, we finally obtain for arbitrary α,β the estimate

$$|(\partial^\alpha_\xi\,\partial^\beta_x\,\tilde{a}^\gamma_f)(x,\xi)|\leq\frac{1}{(1+\xi^2)^N}C_{\alpha,\beta,\mathcal{K}}\qquad x\in\mathcal{K},$$

where $\mathcal{K}$ denotes an arbitrary compact set in W_γ, and $N\in\mathbb{N}$. This proves that $\tilde{a}^\gamma_f(x,\xi)\in \mathrm{S}^{-\infty}(W_\gamma\times\mathbb{R}^{k+l}_\xi)$. Since equation (3.5) is an immediate consequence of the Fourier inversion formula, it remains to show that $\tilde{a}^\gamma_f(x,\xi)$ satisfies the lacunary condition (2.27) for each of the coordinates t_i. Now, it is clear that $a^\gamma_f(x,\xi)\in \mathrm{S}^{-\infty}(W^*_\gamma\times\mathbb{R}^{k+l}_\xi)$, since G acts transitively on each $\widetilde{\mathbb{X}}_\Delta$. As a consequence, the Schwartz kernel of the restriction of the operator $A^\gamma_f:\mathrm{C}^\infty_\mathrm{c}(W_\gamma)\to\mathrm{C}^\infty(W_\gamma)$ to W^*_γ

is given by the absolutely convergent integral

$$\int e^{i(x-y)\cdot\xi} a_f^\gamma(x,\xi) đ\xi \in \mathrm{C}^\infty(W_\gamma^* \times W_\gamma^*).$$

Next, let us write $W_\gamma = \bigcup_{\Theta\subset\Delta} W_\gamma^\Theta$, where $W_\gamma^\Theta = \{x=(n,t) : t_i \neq 0 \Leftrightarrow \alpha_i \in \Theta\}$. Since on W_γ^Θ the function $A_f^\gamma u$ depends only on the restriction of $u \in \mathrm{C}_\mathrm{c}^\infty(W_\gamma)$ to W_γ^Θ, one deduces that

$$\operatorname{supp} K_{A_f^\gamma} \subset \bigcup_{\Theta\subset\Delta} \overline{W_\gamma^\Theta} \times \overline{W_\gamma^\Theta}. \tag{3.11}$$

Therefore, each of the integrals

$$\int e^{i(x_j-y_j)\xi_j} \tilde a_f^\gamma(x,(\mathbf{1}_k \otimes T_x)\xi)\, d\xi_j, \qquad j=k+1,\dots,k+l,$$

which are smooth functions on $W_\gamma^* \times W_\gamma^*$, must vanish if x_j and y_j do not have the same sign. With the substitution $r_j = y_j/x_j - 1$, $\xi_j x_j = \xi_j'$ one finally arrives at the conditions

$$\int e^{-ir_j\xi_j} \tilde a_f^\gamma(x,\xi)\, d\xi_j = 0 \qquad \text{for } r_j < -1,\ x \in W_\gamma^*.$$

But since $\tilde a_f^\gamma$ is rapidly decreasing in ξ, the Lebesgue bounded convergence theorem implies that these conditions must also hold for $x \in W_\gamma$. Thus, the lacunarity of the symbol $\tilde a_f^\gamma$ follows. The fact that the kernel $K_{A_f^\gamma}$ must be determined by its restriction to $W_\gamma^* \times W_\gamma^*$, and hence by the oscillatory integral (3.6), is now a consequence of [**Mel82**], Lemma 4.1, completing the proof of Theorem 2. □

As a consequence of Theorem 2, we can describe the asymptotic behavior of the kernels $K_{A_f^\gamma}(x,y)$ as $|x_j| \to 0$ or $|y_j| \to 0$ for $k+1 \leq j \leq k+l$. Note that this corresponds to the asymptotic behavior of the kernel of $\pi(f)$ on $\widetilde{\mathbb{X}}_\Delta \simeq \mathbb{X}$ at infinity.

Corollary 3. *Let $k+1 \leq j \leq k+l$. Then $K_{A_f^\gamma}(x,y)$ is rapidly decreasing as $|x_j| \to 0$ or $|y_j| \to 0$, provided that $x_j \neq y_j$.*

Proof. According to Theorem 2, the kernel of $\pi(f)$ is locally given by

$$\begin{aligned} K_{A_f^\gamma}(x,y) &= \int e^{i(x-y)\cdot\xi} a_f^\gamma(x,\xi) đ\xi = \int e^{i(x-y)\cdot(\mathbf{1}_k\otimes T_x^{-1})\xi} \tilde a_f^\gamma(x,\xi) |\det(\mathbf{1}_k \otimes T_x^{-1})'(\xi)| đ\xi \\ &= \frac{1}{|x_{k+1}\cdots x_{k+l}|} \tilde A_f^\gamma\Big(x, x_1-y_1,\dots,1-\frac{y_{k+1}}{x_{k+1}},\dots\Big), \qquad x_{k+1}\cdots x_{k+l} \neq 0, \end{aligned}$$

where $\tilde A_f^\gamma(x,y)$ denotes the inverse Fourier transform of $\tilde a_f^\gamma(x,\xi)$,

$$\tilde A_f^\gamma(x,y) = \int e^{iy\cdot\xi} \tilde a_f^\gamma(x,\xi)\, đ\xi. \tag{3.12}$$

Since for $x \in W^\gamma$ the amplitude $\tilde{a}_f^\gamma(x,\xi)$ is rapidly falling in ξ, it follows that $\tilde{A}_f^\gamma(x,y) \in \mathcal{S}(\mathbb{R}_y^{k+l})$, the Fourier transform being an isomorphism on Schwartz space. Therefore $K_{A_f^\gamma}(x,y)$ is rapidly decreasing as $|x_j| \to 0$ if $x_j \neq y_j$ and $k+1 \leq j \leq k+l$. Furthermore, by the lacunarity of $\tilde{a}_f^\gamma$, $K_{A_f^\gamma}(x,y)$ is also rapidly decaying as $|y_j| \to 0$ if $x_j \neq y_j$ and $k+1 \leq j \leq k+l$. $\square$

CHAPTER 4

Asymptotics for strongly elliptic operators on symmetric spaces

In this chapter, we study the holomorphic semigroup generated by a strongly elliptic operator Ω associated to the regular representation $(\pi, \mathrm{C}(\widetilde{\mathbb{X}}))$ of G, as well as its resolvent. Both the holomorphic semigroup and the resolvent can be characterized as convolution operators of the type considered before, so that we can study them by the methods developed in the previous chapter. In particular, this will allow us to obtain a description of the asymptotic behaviour of the semigroup and resolvent kernels on $\widetilde{\mathbb{X}}_\Delta \simeq \mathbb{X}$ at infinity.

We begin by recalling some basic facts about elliptic operators and parabolic evolution equations on Lie groups, our main references being [**Rob91**] and [**TER96**]. The theory of strongly elliptic operators was initiated in the thesis of Langlands [**Lan60**] where it was shown that such operators generate semigroups with smooth universal kernels. On a Lie group, one constructs the semigroup kernels by locally approximating via the exponential map and using a uniform iteration method to construct a global kernel. The bounds on the local approximant and their derivatives give nice decay properties for the global kernel, and the fact that, unlike the usual time-dependent questions, the series expansion for the kernel is uniformly convergent for all time $t > 0$. We are interested in studying the properties of such operators in a Banach representation of the Lie group under consideration. We shall prove results concerning both the holomorphic semi-groups that such operators generate, as well as their resolvents. These correspond, respectively, to the heat kernel and Green function approaches in the theory of differential equations.

4.1. Holomorphic semigroups

Let $\mathcal{G}$ be a Lie group, and π a continuous representation of $\mathcal{G}$ on a Banach space $\mathcal{B}$. Let further $X_1, \dots, X_d$ be a basis of the Lie algebra $\mathrm{Lie}(\mathcal{G})$ of $\mathcal{G}$, and

$$\Omega = \sum_{|\alpha| \leq q} c_\alpha \, d\pi(X^\alpha)$$

a *strongly elliptic differential operator of order* q associated with π, meaning that for all $\xi \in \mathbb{R}^d$ one has the inequality $\mathrm{Re}\,(-1)^{q/2} \sum_{|\alpha|=q} c_\alpha \xi^\alpha \geq \kappa |\xi|^q$ for some $\kappa > 0$.

Remark 6. A strongly elliptic operator is necessarily of even order. This follows simply from observing that $(-\xi)^\alpha = (-1)^{|\alpha|}\xi^\alpha$, α any multi-index, and hence for a strongly elliptic operator of order q as above, $\sum_{|\alpha|=q} c_\alpha(-\xi)^\alpha = (-1)^q \sum_{|\alpha|=q} c_\alpha \xi^\alpha$ giving the evenness of q.

By the general theory of strongly continuous semigroups, the closure of a strongly elliptic operator generates a strongly continuous holomorphic semigroup of bounded operators given by

$$S_\tau = \frac{1}{2\pi i}\int_\Gamma e^{\lambda\tau}(\lambda \mathbf{1} + \overline{\Omega})^{-1} d\lambda,$$

where Γ is a appropriate path in $\mathbb{C}$ coming from infinity and going to infinity such that λ does not lie in the spectrum $\sigma(\overline{\Omega})$ of $\overline{\Omega}$ for $\lambda \in \Gamma$. Here $|\arg\tau| < \alpha$ for an appropriate $\alpha \in (0, \pi/2]$, and the integral converges uniformly with respect to the operator norm. Furthermore, the subgroup S_τ can be characterized by a convolution semigroup of complex measures μ_τ on $\mathcal{G}$ according to

$$S_\tau = \int_{\mathcal{G}} \pi(g) d\mu_\tau(g),$$

π being measurable with respect to the measures μ_τ. The measures μ_τ are absolutely continuous with respect to Haar measure $d_{\mathcal{G}}$ on $\mathcal{G}$, and denoting by $f_\tau(g) \in L^1(\mathcal{G}, d_{\mathcal{G}})$ the corresponding Radon-Nikodym derivative, one has

$$S_\tau = \pi(f_\tau) = \int_{\mathcal{G}} f_\tau(g)\pi(g) d_{\mathcal{G}}(g).$$

The function $f_\tau(g) \in \mathrm{L}^1(\mathcal{G}, d_{\mathcal{G}})$ is analytic in τ and g, and universal for all Banach representations. It satisfies the parabolic differential equation

$$\frac{\partial f_\tau}{\partial \tau}(g) + \sum_{|\alpha|\leq q} c_\alpha \, dL(X^\alpha) f_\tau(g) = 0, \qquad \lim_{\tau\to 0} f_\tau(g) = \delta(g),$$

where $(L, \mathrm{C}^\infty(\mathcal{G}))$ denotes the left regular representation of $\mathcal{G}$. As a consequence, f_τ must be supported on the identity component $\mathcal{G}_0$ of $\mathcal{G}$. We call it the *Langlands kernel* of the holomorphic semigroup S_τ, and it satisfies the following L^1- and L^∞-bounds.

THEOREM 3. *For each $\kappa \geq 0$, there exist constants $a, b, c > 0$, and $\omega \geq 0$ such that*

$$\int_{\mathcal{G}_0} |dL(X^\alpha)\,\partial_\tau^\beta f_\tau(g)| e^{\kappa|g|}\, d_{\mathcal{G}_0}(g) \leq a b^{|\alpha|} c^\beta |\alpha|!\,\beta!(1+\tau^{-\beta-|\alpha|/q})e^{\omega\tau}, \tag{4.1}$$

for all $\tau > 0$, $\beta = 0, 1, 2, \ldots$ and multi-indices α. Furthermore,

$$|dL(X^\alpha)\,\partial_\tau^\beta f_\tau(g)| \leq a b^{|\alpha|} c^\beta |\alpha|!\,\beta!(1+\tau^{-\beta-(|\alpha|+d+1)/q})e^{\omega\tau} e^{-\kappa|g|}, \tag{4.2}$$

for all $g \in \mathcal{G}_0$, where $d = \dim \mathcal{G}_0$, and q denotes the order of Ω.

□

A detailed exposition of these facts can be found in [**Rob91**], pages 30, 152, 166, and 167. Let now $\mathcal{G} = G$, and $(\pi, \mathcal{B})$ be the regular representation of G on $C(\widetilde{\mathbb{X}})$. Theorem 3 implies that the Langlands kernel f_τ belongs to the space $\mathcal{S}(G)$ of rapidly falling functions on G. As a consequence of the previous considerations we obtain

THEOREM 4. *Let Ω be a strongly elliptic differential operator of order q associated with the regular representation $(\pi, C(\widetilde{\mathbb{X}}))$, and $S_\tau = \pi(f_\tau)$ the holomorphic semigroup of bounded operators generated by $\overline{\Omega}$. Then the operators S_τ are locally of the form* (3.5) *with f being replaced by f_τ, and totally characteristic pseudodifferential operators of class $\mathrm{L}_b^{-\infty}$ on the manifolds with corners $\overline{\widetilde{\mathbb{X}}_\Delta}$. Furthermore, on $W_\gamma \times W_\gamma$, the kernel $S_\tau^\gamma(x, y)$ of S_τ is given by*

$$K_{A^\gamma_{f_\tau}}(x, y) = \int e^{i(x-y)\cdot\xi} a^\gamma_{f_\tau}(x, \xi) d\xi = \frac{1}{|x_{k+1} \cdots x_{k+l}|} \tilde{A}^\gamma_{f_\tau}(x, (\mathbf{1}_k \otimes T_x^{-1})(x - y))$$

where $x_{k+1} \cdots x_{k+l} \neq 0$, and $\tilde{A}^\gamma_{f_\tau}(x, y)$ was defined in (3.12)*. In particular, $S_\tau^\gamma(x, y)$ is rapidly falling as $|x_j| \to 0$, or $|y_j| \to 0$, as long as $x_j \neq y_j$, where $k+1 \leq j \leq k+l$. In addition,*

$$|\tilde{A}^\gamma_{f_\tau}(x, y)| \leq \begin{cases} c_1(1 + \tau^{-(l+k+1)/q}), & 0 < \tau \leq 1, \\ c_2 e^{\omega\tau}, & 1 < \tau, \end{cases} \tag{4.3}$$

uniformly on compact subsets of $W_\gamma \times W_\gamma$ for some constants $c_i > 0$.

PROOF. The first assertions are immediate consequences of Theorem 2, and its corollary. In order to prove (4.3), note that for large $N \in \mathbb{N}$ one computes with (3.2), (3.10), and (3.12)

$$\begin{aligned} |\tilde{A}^\gamma_{f_\tau}(x, y)| &\leq \int |\tilde{a}^\gamma_{f_\tau}(x, \xi)| \,đ\xi = \int \Big| \int_G \psi^\gamma_{\xi,x}(g) c_\gamma(x, g) f_\tau(g) d_G(g) \Big| đ\xi \\ &= \int (1 + |\xi|^2)^{-N} \Big| \int_G c_\gamma(g) f_\tau(g) \sum_{r=0}^{2N} \sum_{|\alpha|=r} b^N_\alpha(x, g) dL(X^\alpha) \psi^\gamma_{\xi,x}(g) d_G(g) \Big| đ\xi. \end{aligned}$$

If we now apply Proposition 2, and take into account the estimate (4.1) we obtain

$$\begin{aligned} |\tilde{A}^\gamma_{f_\tau}(x, y)| &\leq \int (1 + |\xi|^2)^{-N} \Big| \int_G \psi^\gamma_{\xi,x}(g) \sum_{r=0}^{2N} \sum_{|\alpha|=r} dL(X^{\tilde{\alpha}}) [b^N_\alpha(x, g) c_\gamma(x, g) f_\tau(g)] dg \Big| đ\xi \\ &\leq \begin{cases} c_1(1 + \tau^{-2N/q}), & 0 < \tau \leq 1, \\ c_2 e^{\omega\tau}, & 1 < \tau, \end{cases} \end{aligned}$$

for certain constants $c_i > 0$. Expressing $\xi_j^{k+l+1} \psi^\gamma_{\xi,x}(g)$ on $\{\xi \in \mathbb{R}^n : |\xi_i| \leq |\xi_j| \, \forall \, i\}$ as left derivatives of $\psi^\gamma_{\xi,x}(g)$ according to (3.7) and (3.8), and estimating the maximum norm by the usual norm, a similar argument shows that the last estimate is also valid for $N=(k+l+1)/2$, compare (4.9). The proof is now complete. $\square$

4.2. Resolvent kernels

Let us now turn to the resolvent of the closure of the strongly elliptic operator Ω. By (4.1) one has the bound $\|S_\tau\| \leq c e^{\omega\tau}$ for some constants $c \geq 1, \omega \geq 0$. For $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$, the resolvent of $\overline{\Omega}$ can be expressed by means of the Laplace transform according to

$$(\lambda \mathbf{1} + \overline{\Omega})^{-1} = \Gamma(1)^{-1} \int_0^\infty e^{-\lambda\tau} S_\tau \, d\tau,$$

where Γ is the Γ-function. More generally, one can consider for arbitrary $\alpha > 0$ the integral transforms

$$(\lambda \mathbf{1} + \overline{\Omega})^{-\alpha} = \Gamma(\alpha)^{-1} \int_0^\infty e^{-\lambda\tau} \tau^{\alpha-1} S_\tau \, d\tau.$$

As it turns out, the functions

$$r_{\alpha,\lambda}(g) = \Gamma(\alpha)^{-1} \int_0^\infty e^{-\lambda\tau} \tau^{\alpha-1} f_\tau(g) \, d\tau$$

are in $\mathrm{L}^1(G, e^{\kappa|g|} d_G)$, where $\kappa \geq 0$ is such that $\|\pi(g)\| \leq c e^{\kappa|g|}$ for some $c \geq 1$, see (3.9). This implies that the resolvent of $\overline{\Omega}$ can be expressed as the convolution operator

$$(\lambda \mathbf{1} + \overline{\Omega})^{-\alpha} = \pi(r_{\alpha,\lambda}) = \int_G r_{\alpha,\lambda}(g) \pi(g) \, d_G(g).$$

The resolvent kernels $r_{\alpha,\lambda}$ decrease exponentially as $|g| \to \infty$, but they are singular at the identity if $d \geq q\alpha$. More precisely, one has the following

THEOREM 5. *There exist constants $b, c, \lambda_0 > 0$, and $a_{\alpha,\lambda} > 0$, such that*

$$|dL(X^\delta) r_{\alpha,\lambda}(g)| \leq \begin{cases} a_{\alpha,\lambda} |g|^{-(d+|\delta|-q\alpha)} e^{-(b(\operatorname{Re}\lambda)^{1/q}-c)|g|}, & d > q\alpha, \\ a_{\alpha,\lambda}(1+|\log|g||) e^{-(b(\operatorname{Re}\lambda)^{1/q}-c)|g|}, & d = q\alpha, \\ a_{\alpha,\lambda} e^{-(b(\operatorname{Re}\lambda)^{1/q}-c)|g|}, & d < q\alpha \end{cases}$$

for each $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \lambda_0$.

$\square$

A proof of these estimates is given in [**Rob91**], pages 238 and 245. Our next aim is to understand the microlocal structure of the operators $\pi(r_{\alpha,\lambda})$ on the Oshima compactification $\widetilde{\mathbb{X}}$ of $\mathbb{X} = G/K$. Consider again the atlas $\left\{(\widetilde{W}_\gamma, \varphi_\gamma^{-1})\right\}_{\gamma \in I}$ of $\widetilde{\mathbb{X}}$ introduced in Section 3.1, and the local operators

$$A^\gamma_{r_{\alpha,\lambda}} u = [\pi(r_{\alpha,\lambda})_{|\widetilde{W}_\gamma}(u \circ \varphi_\gamma^{-1})] \circ \varphi_\gamma, \tag{4.4}$$

where $u \in \mathrm{C}_c^\infty(W_\gamma)$ and $W_\gamma = \varphi_\gamma^{-1}(\widetilde{W}_\gamma)$. By the Fourier inversion formula, $A^\gamma_{r_{\alpha,\lambda}}$ is given by the absolutely convergent integral

$$A^\gamma_{r_{\alpha,\lambda}} u(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} a^\gamma_{r_{\alpha,\lambda}}(x,\xi)\hat{u}(\xi) đ\xi, \tag{4.5}$$

where

$$a^\gamma_{r_{\alpha,\lambda}}(x,\xi) = \int_G e^{i(\varphi_\gamma^g(x)-x)\cdot\xi} c_\gamma(x,g) r_{\alpha,\lambda}(g) d_G(g),$$
$$\tilde{a}^\gamma_{r_{\alpha,\lambda}}(x,\xi) = \int_G e^{i\Psi_\gamma(g,x)\cdot\xi} c_\gamma(x,g) r_{\alpha,\lambda}(g) d_G(g)$$

are smooth functions on $W_\gamma \times \mathbb{R}^{k+l}$, since $r_{\alpha,\lambda} \in \mathrm{L}^1(G, e^{\kappa|g|}d_G)$. Moreover, in view of the L^1-bound (4.1), the functions $e^{-\lambda\tau}\tau^{\alpha-1}\tilde{a}^\gamma_{f_\tau}(x,\xi)$ and $e^{-\lambda\tau}\tau^{\alpha-1}a^\gamma_{f_\tau}(x,\xi)$ are integrable in τ over $(0,\infty)$, and by Fubini we obtain the equalities

$$a^\gamma_{r_{\alpha,\lambda}}(x,\xi) = \Gamma(\alpha)^{-1}\int_0^\infty e^{-\lambda\tau}\tau^{\alpha-1} a^\gamma_{f_\tau}(x,\xi)d\tau,$$
$$\tilde{a}^\gamma_{r_{\alpha,\lambda}}(x,\xi) = \Gamma(\alpha)^{-1}\int_0^\infty e^{-\lambda\tau}\tau^{\alpha-1} \tilde{a}^\gamma_{f_\tau}(x,\xi)d\tau.$$

In what follows, we shall describe the microlocal structure of the resolvent $(\lambda\mathbf{1}+\overline{\Omega})^{-\alpha}$ on $\widetilde{\mathbb{X}}$, and in particular, its kernel.

Proposition 4. *Let Q be the largest integer such that $Q < q\alpha$. Then $\tilde{a}^\gamma_{r_{\alpha,\lambda}}(x,\xi) \in \mathrm{S}^{-Q}_{la}(W_\gamma \times \mathbb{R}^{k+l})$. That is, for any compactum $\mathcal{K} \subset W_\gamma$, and arbitrary multi-indices β, ε there exist constants $C_{\mathcal{K},\beta,\varepsilon} > 0$ such that*

$$|(\partial_x^\varepsilon \partial_\xi^\beta \tilde{a}^\gamma_{r_{\alpha,\lambda}})(x,\xi)| \leq C_{\mathcal{K},\beta,\varepsilon}(1+|\xi|^2)^{(-Q-|\beta|)/2}, \qquad x \in \mathcal{K},\ \xi \in \mathbb{R}^{k+l}, \tag{4.6}$$

and $\tilde{a}^\gamma_{r_{\alpha,\lambda}}$ satisfies the lacunary condition (2.27) for each of the coordinates x_j, $k+1 \leq j \leq k+l$.

Proof. For a fixed chart $(\widetilde{W}_\gamma, \varphi_\gamma)$ of $\widetilde{\mathbb{X}}$ we write $x = (n,t) \in W_\gamma$, $\tilde{x} = \varphi_\gamma(x) \in \widetilde{W}_\gamma$, as usual. As a consequence of Proposition 3 and Lemma 9 one computes with (3.10) for arbitrary $N \in \mathbb{N}$

$$(\partial_\xi^{2\beta} \tilde{a}^\gamma_{r_{\alpha,\lambda}})(x,\xi) = \int_G e^{i\Psi_\gamma(g,x)\cdot\xi}[i\Psi_\gamma(g,x)]^{2\beta} c_\gamma(x,g) r_{\alpha,\lambda}(g) d_G(g)$$
$$= (1+|\xi|^2)^{-N} e^{-i(x_1,\dots,x_k,1,\dots,1)\cdot\xi} \sum_{r=0}^{2N}\sum_{|\delta|=r}\int_G b_\delta^N(x,g) dL(X^\delta)\psi^\gamma_{\xi,x}(g)$$
$$\cdot[i\Psi_\gamma(x,g)]^{2\beta} c_\gamma(x,g) r_{\alpha,\lambda}(g) d_G(g).$$

Now, $n_r(g\cdot\tilde{x}) \to n_r(\tilde{x})$ and $\chi_r(g,\tilde{x}) \to 1$ as $g \to e$, so that due to the analyticity of the G-action on $\widetilde{\mathbb{X}}$ one deduces

$$|\Psi_\gamma(g,x)| = |(n_1(g\cdot\tilde{x}) - n_1(\tilde{x}),\dots,\chi_1(g\cdot\tilde{x})-1,\dots)| \leq C_{\mathcal{K}}|g|, \qquad x \in \mathcal{K}, \tag{4.7}$$

for some constant $C_{\mathcal{K}}$. Indeed, let

$$(\zeta_1,\dots,\zeta_d) \mapsto e^{\zeta_1 X_1+\dots+\zeta_d X_d} = g$$

be canonical coordinates of the first type near the identity $e \in G$. We then have the power expansions

$$\chi_r(g,\tilde{x}) - 1 = \sum_{\alpha,\beta,\gamma} c^r_{\alpha,\beta,\gamma} n^\alpha t^\beta \zeta^\gamma, \qquad n_r(g\cdot\tilde{x}) - n_r(\tilde{x}) = \sum_{\alpha,\beta,\gamma} d^r_{\alpha,\beta,\gamma} n^\alpha t^\beta \zeta^\gamma, \tag{4.8}$$

where $c^r_{\alpha,\beta,\gamma}$, $d^r_{\alpha,\beta,\gamma} = 0$ if $|\gamma| = 0$. Hence,

$$|n_r(g\cdot\tilde{x}) - n_r(\tilde{x})|,\ |\chi_r(g,\tilde{x}) - 1| \leq C_1|\zeta| \leq C_2|g|,$$

compare [**Rob91**], pages 12-13, and we obtain (4.7). With Theorem 5, and, say, $d > q\alpha$, we therefore have the pointwise estimates

$$|\Psi_\gamma(g,x)^{\beta'} dL(X^{\delta'}) r_{\alpha,\lambda}(g)| \leq C_{\mathcal{K},\alpha,\lambda} |g|^{-(d+|\delta'|-q\alpha-|\beta'|)} e^{-(b(\operatorname{Re}\lambda)^{1/q}-c)|g|}$$

for some constant $C_{\mathcal{K},\alpha,\lambda} > 0$ uniformly on $\mathcal{K} \times V_{\gamma,\tilde{x}}$. Now, let $2\tilde{Q}$ be the largest even number strictly smaller than $q\alpha$. Applying the same reasoning as in the proof of Proposition 2, one obtains for $N = \tilde{Q} + |\beta|$

$$\begin{aligned}(\partial^{2\beta}_\xi \tilde{a}^\gamma_{r_{\alpha,\lambda}})(x,\xi) = (1+|\xi|^2)^{-\tilde{Q}-|\beta|} \sum_{r=0}^{2\tilde{Q}+2|\beta|} \sum_{|\delta|=r} (-1)^{|\delta|} \int_G e^{i\Psi_\gamma(g,x)\cdot\xi} \\ \cdot dL(X^{\tilde{\delta}}) \big[b^{\tilde{Q}+|\beta|}_\delta(x,g)[i\Psi_\gamma(g,x)]^{2\beta} c_\gamma(x,g) r_{\alpha,\lambda}(g)\big] d_G(g),\end{aligned}$$

since all the occuring combinations $\Psi_\gamma(g,x)^{\beta'} dL(X^{\delta'}) r_{\alpha,\lambda}(g)$ on the right hand side are such that $q\alpha + |\beta'| - |\delta'| > 0$, implying that the corresponding integrals over G converge. Equality then follows by the left-invariance of $d_G(g)$, and Lebesgue's Theorem on Dominated Convergence. To show the estimate (4.6) in general for $\varepsilon = 0$, let $x \in \mathcal{K}$, and $\xi \in \mathbb{R}^{k+l}$ be such that $|\xi| \geq 1$, and $|\xi|_{\max} = \max\{|\xi_r| : 1 \leq r \leq k+l\} = |\xi_j|$. Using (3.7) and (3.8) we can express $\xi_j^{Q+|\beta|}\psi^\gamma_{\xi,x}(g)$ as left derivatives of $\psi^\gamma_{\xi,x}(g)$, and repeating the previous argument we obtain the estimate

(4.9)

$$\begin{aligned}|(\partial^\beta_\xi \tilde{a}^\gamma_{r_{\alpha,\lambda}})(x,\xi)| = &|\xi_j|^{-Q-|\beta|} \Big| \sum_{r=0}^{Q+|\beta|} \sum_{|\delta|=r} \int_G b^j_\delta(x,g) dL(X^\delta) \psi^\gamma_{\xi,x}(g) \\ &\cdot [i\Psi_\gamma(x,g)]^\beta c_\gamma(x,g) r_{\alpha,\lambda}(g) d_G(g)\Big| \leq \tilde{C}_{\mathcal{K},\beta} \frac{1}{|\xi|^{Q+|\beta|}_{\max}} \leq C_{\mathcal{K},\beta} \frac{1}{|\xi|^{Q+|\beta|}},\end{aligned}$$

where the coefficients $b^j_\delta(x,g)$ are at most of exponential growth in g. But since $\tilde a^\gamma_{r_{\alpha,\lambda}}(x,\xi) \in \mathrm{C}^\infty(W_\gamma \times \mathbb{R}^{k+l})$, we obtain (4.6) for $\varepsilon = 0$. Let us now turn to the x-derivatives. We have to show that the powers in ξ that arise when differentiating $(\partial^\beta_\xi\, \tilde a^\gamma_{r_{\alpha,\lambda}})(x,\xi)$ with respect to x can be compensated by an argument similar to the previous considerations. Now, (4.8) clearly implies

$$\partial^\varepsilon_x\,(\chi_r(g,\tilde x)-1) = O(|g|), \qquad \partial^\varepsilon_x\,(n_r(g\cdot\tilde x)-n_r(\tilde x)) = O(|g|).$$

Thus, each time we differentiate the exponential $e^{i\Psi_\gamma(g,x)\cdot\xi}$ with respect to x, the result is of order $O(|\xi||g|)$. Therefore, expressing the ocurring powers $\xi^{\varepsilon'}\psi^\gamma_{\xi,x}(g)$ as left derivatives of $\psi^\gamma_{\xi,x}(g)$, we can repeat the preceding argument to absorb the powers in ξ, and (4.6) follows. Note that the previous argument also implies that $a^\gamma_{r_{\alpha,\lambda}}(x,\xi) \in \mathrm{S}^{-Q}(W^*_\gamma \times \mathbb{R}^{k+l}_\xi)$, where we wrote $W^*_\gamma = \{x=(n,t) \in W_\gamma : t_1\cdots t_l \neq 0\}$, the G-action being transitive on each $\widetilde{\mathbb{X}}_\Delta$. The Schwartz kernel $K_{A^\gamma_{r_{\alpha,\lambda}}}$ of the restriction of the operator (4.4) to W^*_γ is therefore given by the oscillatory integral

$$\int e^{i(x-y)\cdot\xi} a^\gamma_{r_{\alpha,\lambda}}(x,\xi) đ\xi \in \mathcal{D}'(W^*_\gamma \times W^*_\gamma),$$

which is C^∞ off the diagonal. As in (3.11) we have $\operatorname{supp} K_{A^\gamma_{r_{\alpha,\lambda}}} \subset \bigcup_{\Theta\subset\Delta} \overline{W^\Theta_\gamma} \times \overline{W^\Theta_\gamma}$, so that each of the integrals

$$\int e^{i(x_j-y_j)\xi_j}\,\tilde a^\gamma_{r_{\alpha,\lambda}}(x,(\mathbf{1}_k\otimes T_x)\xi)\,d\xi_j, \qquad j=k+1,\dots,k+l,$$

must vanish if x_j and y_j do not have the same sign. Hence,

$$\int e^{-ir_j\xi_j}\,\tilde a^\gamma_{r_{\alpha,\lambda}}(x,\xi)\,d\xi_j = 0 \qquad \text{for } r_j<-1,\ x\in W^*_\gamma.$$

Since $\tilde a^\gamma_{r_{\alpha,\lambda}}(x,\xi) \in \mathrm{S}^{-Q}(W_\gamma \times \mathbb{R}^{k+l}_\xi)$, these integrals are absolutely convergent for $r_j \neq 0$. Lebesgue's Theorem on Bounded Convergence then implies that these conditions must also hold for $x \in W_\gamma$. The proof of the proposition is now complete. $\square$

Remark 7. One would actually expect $\tilde a^\gamma_{r_{\alpha,\lambda}}(x,\xi)$, being the local symbol of the resolvent $(\lambda\mathbf{1}+\overline{\Omega})^{-\alpha}$, to belong to $\mathrm{S}^{-q\alpha}_{la}(W_\gamma\times\mathbb{R}^{k+l})$. However, the general estimates of Theorem 5 for the resolvent kernels $r_{\alpha,\lambda}$, which correctly reflect the singular behaviour at the identity, are not sufficient to show this, and more information about them is required. Indeed, $dL(X^\beta)r_{\alpha,\lambda} \in L_1(G, d_G(g))$ only holds if $0 < q\alpha - |\beta|$.

We are now in a position to describe the microlocal structure of the resolvent $(\lambda\mathbf{1}+\overline{\Omega})^{-\alpha}$.

Theorem 6. *Let Ω be a strongly elliptic differential operator of order q associated with the representation $(\pi, C(\widetilde{\mathbb{X}}))$ of G. Let $\omega \geq 0$ be given by Theorem 3, and $\lambda \in \mathbb{C}$ be such that $\operatorname{Re}\lambda > \omega$. Let further $\alpha > 0$, and denote by Q the largest integer such that $Q < q\alpha$. Then $(\lambda\mathbf{1}+\overline{\Omega})^{-\alpha} = \pi(r_{\alpha,\lambda})$ is locally of the form (4.5), where*

$a^\gamma_{r_{\alpha,\lambda}}(x,\xi) = \tilde{a}^\gamma_{r_{\alpha,\lambda}}(x,(\mathbf{1}_k \otimes T_x)\xi)$, *and* $\tilde{a}^\gamma_{r_{\alpha,\lambda}}(x,\xi) \in \mathrm{S}^{-Q}_{la}(W_\gamma \times \mathbb{R}^{k+l})$. *In particular,* $(\lambda\mathbf{1}+\overline{\Omega})^{-\alpha}$ *is a totally characteristic pseudodifferential operators of class* L^{-Q}_b *on the manifolds with corners* $\overline{\widetilde{\mathbb{X}}_\Delta}$. *Furthermore, its kernel is locally given by the oscillatory integral*

$$R^\gamma_{\alpha,\lambda}(x,y) = \int e^{i(x-y)\xi} a^\gamma_{r_{\alpha,\lambda}}(x,\xi) d\xi = \frac{1}{|x_{k+1}\cdots x_{k+l}|} \int e^{i(\mathbf{1}_k \otimes T_x^{-1})(x-y)\cdot\xi} \tilde{a}^\gamma_{r_{\alpha,\lambda}}(x,\xi) d\xi,$$

where $x_{k+1}\cdots x_{k+l} \neq 0$, $x,y \in W_\gamma$. $R^\gamma_{\alpha,\lambda}(x,y)$ *is smooth off the diagonal, and rapidly falling as* $|x_j| \to 0$, *or* $|y_j| \to 0$, *as long as* $x_j \neq y_j$, *where* $k+1 \leq j \leq k+l$.

Proof. The assertions of the theorem are direct consequences of our previous considerations, except for the behaviour of $R^\gamma_{\alpha,\lambda}(x,y)$ at infinity. Let $k+1 \leq j \leq k+l$. While the behaviour as $|y_j| \to 0$ is a direct consequence of the lacunarity of $\tilde{a}^\gamma_{R_{\alpha,\gamma}}$, the behaviour as $|x_j| \to 0$ is a direct consequence of the fact that, as oscillatory integrals,

$$\int e^{i(\mathbf{1}_k \otimes T_x^{-1})(x-y)\cdot\xi} \tilde{a}^\gamma_{r_{\alpha,\lambda}}(x,\xi) d\xi = \frac{1}{|(\mathbf{1}_k \otimes T_x^{-1})(x-y)|^{2N}} \int e^{i(x-y)\cdot\xi} \Delta^N_\xi \tilde{a}^\gamma_{r_{\alpha,\lambda}}(x,\xi) d\xi,$$

where $\Delta_\xi = \partial^2_{\xi_1} + \cdots + \partial^2_{\xi_{k+l}}$, $x \neq y$, and N is arbitrarily large. $\square$

Remark 8. The singular behaviour of $r_{\alpha,\lambda}(g)$ at the identity corresponds to the fact that, as a pseudodifferential operator of class L^{-Q}_b, $(\lambda\mathbf{1}+\overline{\Omega})^{-\alpha}$ has a kernel which is singular at the diagonal.

To conclude, let us say some words about the classical heat kernel on a Riemannian symmetric space of non-compact type. Consider thus the regular representation $(\sigma, \mathrm{C}(\widetilde{\mathbb{X}}))$ of the solvable Lie group $S = AN^- \simeq \mathbb{X} = G/K$ on the Oshima compactification $\widetilde{\mathbb{X}}$ of $\mathbb{X}$, and associate to every $f \in \mathcal{S}(S)$ the corresponding convolution operator

$$\int_S f(g)\sigma(g)\, d_S(g).$$

Its restriction to $\mathrm{C}^\infty(\widetilde{\mathbb{X}})$ induces again a continuous linear operator

$$\sigma(f) : \mathrm{C}^\infty(\widetilde{\mathbb{X}}) \longrightarrow \mathrm{C}^\infty(\widetilde{\mathbb{X}}) \subset \mathcal{D}'(\widetilde{\mathbb{X}}),$$

and an examination of the arguments in Section 2.4 shows that an analogous analysis applies to the operators $\sigma(f)$. In particular, Theorem 2 holds for them, too. Let ϱ be the half sum of all positive roots, and

$$C = \sum_j H_j^2 - \sum_j Z_j^2 - \sum_j [X_j\theta(X_j) + \theta(X_j)X_j] \equiv \sum_j H_j^2 - 2\varrho + 2\sum X_j^2 \mod \mathfrak{U}(\mathfrak{g})\mathfrak{k}$$

be the Casimir operator in $\mathfrak{U}(\mathfrak{g})$, where $\{H_j\}$, $\{Z_j\}$, and $\{X_j\}$ are orthonormal basis of $\mathfrak{a}$, $\mathfrak{m}$, and $\mathfrak{n}^-$, respectively, and put $C' = \sum_j H_j^2 - 2\varrho + 2\sum X_j^2$. Though $-d\pi(C')$ is not a strongly elliptic operator in the sense defined above, $\Omega = -d\sigma(C')$

certainly is. Consequently, if $f'_\tau(g) \in \mathcal{S}(S)$ denotes the corresponding Langlands kernel, Theorems 4 and 6 yield descriptions of the Schwartz kernels of $\sigma(f'_\tau)$ and $(\lambda \mathbf{1} + \overline{\Omega})^{-\alpha}$ on $\widetilde{\mathbb{X}}$. On the other hand, denote by Δ the Laplace-Beltrami operator on $\mathbb{X}$. Then

$$\Delta\varphi(gK) = \varphi(g : C) = \varphi(g : C'), \qquad \varphi \in \mathrm{C}^\infty(\mathbb{X}),$$

and the associated heat kernel $h_\tau(g)$ on $\mathbb{X}$ coincides with the heat kernel on S associated to C'. But the latter is essentially given by the Langlands kernel $f'_\tau(g)$, being the solution of the parabolic equation

$$\frac{\partial f'_\tau}{\partial \tau}(g) - dL(C')f'_\tau(g) = 0, \qquad \lim_{\tau \to 0} f'_\tau(g) = \delta(g)$$

on S. In this particular case, optimal upper and lower bounds for h_τ and the Bessel-Green-Riesz kernels were given in [**AJ99**] using spherical analysis under certain restrictions coming from the lack of control in the Trombi-Varadarajan expansion for spherical functions along the walls. Our asymptotics for the kernels of $\sigma(f'_\tau)$ and $(\lambda \mathbf{1} + \overline{\Omega})^{-\alpha}$ on $\widetilde{\mathbb{X}}_\Delta \simeq \mathbb{X}$ are free of restrictions, and in concordance with those of [**AJ99**], though, of course, less explicit. A detailed description of the resolvent of Δ on $\mathbb{X}$ was given in [**MV05**] and [**Str05**], where an analytic continuation of the resolvent of the Laplacian were given. The former uses mainly some analytic methods coming from N-body scattering while the latter uses the theory of spherical functions. [**MW99**] and [**HP09**] give a precise description of the poles of this meromorphic continuation in the rank one case.

CHAPTER 5

Regularized traces and a fixed-point formula

This chapter deals with the definition of a regularized trace $\mathrm{Tr}_{reg}\,\pi(f)$ for the operators $\pi(f)$, using our understanding of the singularities of their Schwartz kernels in the previous chapter. After recalling in Section 5.2 the work of Atiyah and Bott on transversal traces, we prove in Section 5.3 a fixed-point formula for the mapping $f \longmapsto \mathrm{Tr}_{reg}\,\pi(f)$, analogous to the Atiyah-Bott fixed-point formula for the character of an induced representation.

5.1. Regularized traces

We shall, in this section, define a regularized trace for the convolution operators $\pi(f)$ introduced in Section 3.1. We start with a little discussion on the regularization of certain integrals, following, essentially, the discussion on distributions with algebraic singularities in Chapter 1, Section 3, [**GS58**].

Define, for $x \in \mathbb{R}$, and $s \in \mathbb{C}$ with $\mathrm{Re}\, s > -1$,

$$x_+^s = \begin{cases} x^s & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

Since $\int_0^t x^s dx$ converges absolutely if $\mathrm{Re}\, s > -1$, x_+^s is a locally integrable function when $\mathrm{Re}\, s > -1$. Therefore, it defines a distribution in that region. We write, for $\mathrm{Re}\, s > -1$ and $u \in \mathrm{C}_c^\infty(\mathbb{R})$,

$$\begin{aligned} \int_0^\infty x^s u(x) dx &= \int_0^1 x^s u(x) dx + \int_1^\infty x^s u(x) dx \\ &= \int_0^1 x^s [u(x) - u(0)] dx + \int_1^\infty x^s u(x) dx + \frac{u(0)}{s+1}. \end{aligned}$$

Notice that the first term on the right-hand side is defined for $\mathrm{Re}\, s > -2$, the second term for all $s \in \mathbb{C}$, while the third term is defined when $s \neq -1$. Thus in this region the right-hand side exists, and defines a regularization of the integral on the left. So the distribution x_+^s defined by the left-hand side of the above equation can be analytically continued to $\mathrm{Re}\, s > -2$, $s \neq -1$. Proceeding in a similar fashion we have, for $\operatorname{Re} s > -n-1$, and $s \neq -1, -2, \ldots$,

$$\int_0^\infty x^s u(x)dx = \int_0^1 x^s[u(x) - u(0) - xu'(0) - \cdots - \frac{x^{n-1}}{(n-1)!}u^{(n-1)}(0)]dx + \int_1^\infty x^s u(x)dx + \sum_{k=1}^{n} \frac{u^{(k-1)}(0)}{(k-1)!(s+k)}.$$

The right-hand side regularizes the integral on the left and defines the distribution x_+^s in the region $\operatorname{Re} s > -n-1$ with poles at $s = -1, -2, \ldots - n$. The above equation also shows that the residue of $\left(x_+^s, u\right)$ considered as a function of s at $s = -k$ is $\frac{u^{(k-1)}(0)}{(k-1)!}$, where $(,)$ was the duality bracket. Now, the equality $u^{(k-1)}(0) = (-1)^{k-1}\left(\delta^{k-1}(x), u(x)\right)$, allows us to say that the distribution x_+^s itself has a simple pole at $s = -k$ where it has a residue $\frac{(-1)^{(k-1)}}{(k-1)!}\delta^{k-1}(x)$.

Later, using Hironaka's theorem on resolution of singularities, Bernshtein and Gel'fand [**BG69**], and, independently, Atiyah [**Ati70**] generalized these considerations to analytic manifolds. Thus, let M be a real analytic manifold and f a non-zero, real analytic function on M. Then $|f|^s$, which is locally integrable for $\operatorname{Re} s > 0$, extends analytically to a distribution on M which is a meromorphic function of s in the whole complex plane. The poles are located at the negative rational numbers, and their order does not exceed the dimension of M. From this, one deduces that if $f : M \to \mathbb{C}$ is a non-zero analytic function, then there exists a distribution S on M such that $fS = 1$. This is the Hörmander-Lojasiewicz theorem on the division of distributions, and implies the existence of temperate fundamental solutions for constant-coefficient partial differential equations. Now, we have the following result.

Proposition 5. *Let $u \in \mathrm{C}_{\mathrm{c}}^\infty(\mathbb{R}^n)$. Then the function*

$$\int_{\mathbb{R}^n} |x_1|^{s_1} \ldots |x_n|^{s_n} u(x_1, \ldots, x_n)dx_1 \ldots dx_n$$

can be analytically continued to all values of $s_1, \ldots, s_n$ as a meromorphic function of $s_1, \ldots, s_n$ and its poles lie on hypersurfaces of the form $s_i + m = 0$, where m is an odd natural number.

Proof. The proof is essentially iterated integration to reduce it to the case discussed above. See Lemma 2, [**BG69**]. □

Remark 9. There are other ways to normalize such integrals, and more generally, the distributions defined by such integrals. We refer to [**Hör83**], Section 3.2 for a detailed discussion on the methods due to M.Riesz and Hadamard. However, as pointed out in [**GS58**], the specific method used to arrive at such a regularization is only of auxiliary interest. The method is only the means, whereas the definition is the end which has a meaning in and of itself, independent of the method used to arrive at it.

Now, recall that, as a consequence of Theorem 2, we can write the kernel of $\pi(f)$ locally in the form

(5.1)
$$K_{A_f^\gamma}(x,y) = \int e^{i(x-y)\cdot\xi} a_f^\gamma(x,\xi) đ\xi = \int e^{i(x-y)\cdot(\mathbf{1}_k \otimes T_x^{-1})\xi} \tilde{a}_f^\gamma(x,\xi) |\det(\mathbf{1}_k \otimes T_x^{-1})'(\xi)| đ\xi$$
$$= \frac{1}{|x_{k+1}\cdots x_{k+l}|} \tilde{A}_f^\gamma(x, x_1 - y_1, \dots, 1 - \frac{y_{k+1}}{x_{k+1}}, \dots), \qquad x_{k+1}\cdots x_{k+l} \neq 0,$$

where $\tilde{A}_f^\gamma(x,y)$ denotes the inverse Fourier transform of the lacunary symbol $\tilde{a}_f^\gamma(x,\xi)$ given by (3.12). Consider now the partition of unity $\{\alpha_\gamma\}$ subordinate to the atlas $\{(\widetilde{W}_\gamma, \varphi_\gamma^{-1})\}$. By equation (5.1), the restriction of the kernel of A_f^γ to the diagonal is given by

$$K_{A_f^\gamma}(x,x) = \frac{1}{|x_{k+1}\cdots x_{k+l}|} \tilde{A}_f^\gamma(x,0), \qquad x_{k+1}\cdots x_{k+l} \neq 0.$$

These restrictions yield a family of smooth functions $k_f^\gamma(\tilde{x}) = K_{A_f^\gamma}(\varphi_\gamma^{-1}(\tilde{x}), \varphi_\gamma^{-1}(\tilde{x}))$ which define a density k_f on

$$2^{\#l}(G/K) \subset \widetilde{\mathbb{X}}.$$

Nevertheless, the functions $k_f^\gamma(\tilde{x})$ are not locally integrable on the entire compactification $\widetilde{\mathbb{X}}$, so that we cannot define a trace of $\pi(f)$ by integrating the density k_f over the diagonal $\Delta_{\widetilde{\mathbb{X}}\times\widetilde{\mathbb{X}}} \simeq \widetilde{\mathbb{X}}$ as in (2.23).

The following criterion of Schwartz tells us when a linear map $T : \mathrm{C_c^\infty}(M) \to \mathbb{C}$ is a distribution, see [**Sch57**], page 85.

Criterion: A linear $T : \mathrm{C_c^\infty}(M) \to \mathbb{C}$ is a distribution if and only if given any open relatively compact subset ω of a manifold M there exist finitely many differential operators $D_1, \dots, D_r$ on M such that

$$|T(f)| \leq \sum_{i=1}^r \sup |D_i f|, \qquad f \in \mathrm{C_c^\infty}(\omega).$$

We are now in a position to state the following result.

PROPOSITION 6. *Let $f \in \mathcal{S}(G)$, $s \in \mathbb{C}$, and define for* $\mathrm{Re}\, s > 0$

$$\mathrm{Tr}_s\, \pi(f) = \sum_\gamma \int_{W_\gamma} (\alpha_\gamma \circ \varphi_\gamma)(x) |x_{k+1}\cdots x_{k+l}|^s \widetilde{A}_f^\gamma(x,0) dx$$
$$= \left\langle |x_{k+1}\cdots x_{k+l}|^s, \sum_\gamma (\alpha_\gamma \circ \varphi_\gamma) \widetilde{A}_f^\gamma(\cdot, 0) \right\rangle.$$

Then $\mathrm{Tr}_s\, \pi(f)$ *can be continued analytically to a meromorphic function in s with at most poles at $-1, -3, \dots$. Furthermore, for $s \in \mathbb{C} - \{-1, -3, \dots\}$,*

$$\Theta_\pi^s : \mathrm{C_c^\infty}(G) \ni f \mapsto \mathrm{Tr}_s\, \pi(f) \in \mathbb{C} \tag{5.2}$$

defines a distribution density on G.

PROOF. The fact that $\operatorname{Tr}_s \pi(f)$ can be continued meromorphically is a consequence of the analytic continuation of $|x_{k+1}\cdots x_{k+l}|^s$ as a distribution in $\mathbb{R}^{k+l}$, see Proposition 5 above. One even has that

$$\langle |x_{k+1}|^{s_1}\cdots|x_{k+l}|^{s_l}, u\rangle, \qquad u\in \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^{k+l}),$$

can be continued meromorphically in the variables $s_1,\dots,s_l$ to $\mathbb{C}^l$ with poles $s_i = -1,-3,\dots$. To see that (5.2) is a distribution density, note that $\Theta^s_\pi : \mathrm{C}^\infty_{\mathrm{c}}(G)\to\mathbb{C}$ is certainly linear. Since $|x_{k+1}\cdots x_{k+l}|^s$ is a distribution, for any open, relatively compact subset $\omega\subset\mathbb{R}^{k+l}$ there exist $C_\omega>0$ and $B_\omega\in\mathbb{N}$ such that

$$|\langle |x_{k+1}\cdots x_{k+l}|^s, u\rangle| \leq C_\omega \sum_{|\beta|\leq B_\omega} \sup|\partial^\beta u|, \qquad u\in\mathrm{C}^\infty_{\mathrm{c}}(\omega). \tag{5.3}$$

Let now $\mathcal{O}\subset G$ be an arbitrary open, relatively compact subset, and $f\in\mathrm{C}^\infty_{\mathrm{c}}(\mathcal{O})$. With equation (3.12) one has

$$\begin{aligned}\operatorname{Tr}_s \pi(f) &= \left\langle |x_{k+1}\cdots x_{k+l}|^s, \sum_\gamma (\alpha_\gamma\circ\varphi_\gamma)\int \tilde{a}^\gamma_f(\cdot,\xi)\,d\!\!\!{}^-\xi\right\rangle\\ &= \sum_\gamma \int_{W_\gamma}\int (\alpha_\gamma\circ\varphi_\gamma)(x)|x_{k+1}\cdots x_{k+l}|^s \tilde{a}^\gamma_f(x,\xi)\,d\!\!\!{}^-\xi\,dx. \end{aligned}\tag{5.4}$$

By equation (3.10), one computes for arbitrary $N\in\mathbb{N}$ that

$$e^{i\Psi_\gamma(g,x)\cdot\xi} = \frac{1}{(1+|\xi|^2)^N}\sum_{r=0}^{2N}\sum_{|\alpha|=r} b^N_\alpha(x,g)\,dL(X^\alpha)\Big[e^{i\Psi_\gamma(g,x)\cdot\xi}\Big],$$

where the coefficients $b^N_\alpha(x,g)$ are smooth, and at most of exponential growth in g. By (3.2) and Proposition 2 we therefore obtain, for $\tilde{a}^\gamma_f(x,\xi)$, the expression

$$\tilde{a}^\gamma_f(x,\xi) = \frac{1}{(1+|\xi|^2)^N}\int_G e^{i\Psi_\gamma(g,x)\cdot\xi}\sum_{r=0}^{2N}\sum_{|\alpha|=r}(-1)^r dL(X^{\tilde\alpha})\Big[b^N_\alpha(x,g)c_\gamma(x,g)f(g)\Big]dg.$$

Inserting this in (5.4), and taking N sufficiently large, we obtain with (5.3) that

$$\begin{aligned}|\operatorname{Tr}_s\pi(f)| \leq{}& \sum_\gamma\int_{W_\gamma}\int_G\sum_{r=0}^{2N}\sum_{|\alpha|=r}\Big|(\alpha_\gamma\circ\varphi_\gamma)(x)|x_{k+1}\cdots x_{k+l}|^s\\ & dL(X^{\tilde\alpha})\Big[b^N_\alpha(x,g)c_\gamma(x,g)f(g)\Big]\Big|\cdot d_G(g)\,dx\int(1+|\xi|^2)^{-N}\,d\!\!\!{}^-\xi\\ \leq{}& C_{\mathcal{O}}\sum_{|\beta|\leq B_{\mathcal{O}}}\sup|dL(X^\beta)f|\end{aligned}$$

for suitable $C_{\mathcal{O}} > 0$ and $B_{\mathcal{O}} \in \mathbb{N}$. Since the universal enveloping algebra $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$ can be identified with the algebra of invariant differential operators on G, the assertion now follows from the above-mentioned criterion. $\square$

Consider next the Laurent expansion of $\Theta^s_\pi(f)$ at $s = -1$. For this, let $u \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^{k+l})$ be a test function, and consider the expansion

$$\langle |x_{k+1} \cdots x_{k+l}|^s, u\rangle = \sum_{j=-q}^{\infty} S_j(u)(s+1)^j,$$

where $S_k \in \mathcal{D}'(\mathbb{R}^{k+l})$. Since $|x_{k+1} \cdots x_{k+l}|^{s+1}$ has no pole at $s = -1$, we necessarily must have

$$|x_{k+1} \cdots x_{k+l}| \cdot S_j = 0 \quad \text{for } \ j < 0, \qquad |x_{k+1} \cdots x_{k+l}| \cdot S_0 = 1$$

as distributions. Therefore $S_0 \in \mathcal{D}'(\mathbb{R}^{k+l})$ represents a distributional inverse of $|x_{k+1} \cdots x_{k+l}|$. Using the same reasoning as the proof of Proposition 6 we arrive at the following

PROPOSITION 7. *For $f \in \mathcal{S}(G)$, let the regularized trace of the operator $\pi(f)$ be defined by*

$$\mathrm{Tr}_{reg}\, \pi(f) = \left\langle S_0, \sum_\gamma (\alpha_\gamma \circ \varphi_\gamma) \tilde{A}^\gamma_f(\cdot, 0) \right\rangle.$$

Then $\Theta_\pi : \mathrm{C}^\infty_{\mathrm{c}}(G) \ni f \mapsto \mathrm{Tr}_{reg}\, \pi(f) \in \mathbb{C}$ constitutes a distribution density on G, which is called the character of the representation *π.*

$\square$

REMARK 10. The regularized trace defined in [**Loy98**] using the calculus of b-pseudo differential operators developed by Melrose, applied to this context, would yield a similar regularized trace $\mathrm{Tr}_{reg}\, \pi(f)$. For a detailed description, the reader is referred to [**Loy98**], Section 6.

In what follows, we shall identify distributions with distribution densities on G via the Haar measure d_G. Our next aim is to understand the distributions Θ^s_π and Θ_π in terms of the G-action on $\widetilde{\mathbb{X}}$. We shall actually show that on a certain open set of transversal elements, they are represented by locally integrable functions given in terms of fixed points. Similar expressions where derived by Atiyah and Bott for the global character of an induced representation of G. Their work is based on the concept of transversal trace of a pseudodifferential operator, and will be explained in the next section.

5.2. Transversal trace and characters of induced representations

In [**AB67**], Atiyah and Bott extended the classical Lefschetz fixed point theorem to geometric endomorphisms on elliptic complexes. Their work relies on the concept of transversal trace of a smooth operator, and its extension by continuity to pseudodifferential operators. The Lefschetz theorem then follows by showing that the Lefschetz number of a geometric endomorphism is given by an alternating sum of transversal traces, and extending an analogous alternating sum formula for smooth endomorphisms. To explain the notion of transversal trace of a pseudodifferential operator, let us introduce the following

Definition 2. *Let M be a smooth manifold. A fixed point x_0 of a smooth map $f : M \to M$ is said to be* simple *if* $\det(\mathbf{1} - df_{x_0}) \neq 0$*, where df_{x_0} denotes the differential of f at x_0. The map f is called* transversal *if it has only simple fixed points.*

Note that the non-vanishing condition on the determinant is equivalent to the requirement that the graph of f intersects the diagonal transversally at $(x_0, x_0) \in M \times M$, and hence the terminology. In particular, a simple fixed point is an isolated fixed point.

Let now U be an open subset of $\mathbb{R}^n$, V an open subset of U, and consider a smooth map $\alpha : V \to U$ with a simple fixed point at x_0. We choose V so small, that $x \mapsto x - \alpha(x)$ defines a diffeomorphism of V onto its image, by the inverse function theorem. Let

$$\Lambda : V \to U \times U$$

be the map defined by $\Lambda(x) = (\alpha(x), x)$, which then induces a map

$$\Lambda^* : \mathrm{C}^\infty(U \times U) \to \mathrm{C}^\infty(V)$$

that is defined, for $g \in \mathrm{C}^\infty(U \times U)$ and $x \in V$, by

$$\begin{aligned}(\Lambda^* g)(x) &= (g \circ \Lambda)(x) \\ &= g(\alpha(x), x).\end{aligned}$$

Suppose that $A \in \mathrm{L}^{-\infty}(U)$ is a smooth operator with symbol $a(x, \xi)$. The kernel K_A of A is a smooth function on $U \times U$, and its restriction $\Lambda^* K_A$ to the graph of α defines a distribution on V according to

$$\begin{aligned}\langle \Lambda^* K_A, v\rangle &= \int\int e^{i(\alpha(x)-x)\cdot\xi} a(\alpha(x), \xi) v(x)\, đ\xi\, dx \\ &= \int\int e^{-iy\cdot\xi} \frac{a(\alpha(x(y)), \xi) v(x(y))}{|\det(\mathbf{1} - d\alpha(x(y)))|}\, dy\, đ\xi, \qquad v \in \mathrm{C}^\infty_{\mathrm{c}}(V),\end{aligned} \tag{5.5}$$

where we made the substitution $y = x - \alpha(x)$, and the change in order of integration is permissible because $a(x, \xi) \in \mathrm{S}^{-\infty}(U)$. Differentiating the inner integral with respect to ξ and integrating by parts with respect to y, we obtain the estimate

$$|\partial_\xi^\gamma \int e^{-iy\cdot\xi} a(\alpha(x(y)),\xi) \frac{v(x(y))}{|\det(\mathbf{1} - d\alpha(x(y)))|} dy| \leq C \langle \xi \rangle^q$$

for arbitrary $q \in \mathbb{R}$, where the constant C depends on $\mathrm{supp}(v)$ as well the constant that arises in the estimate for $a(x,\xi)$. Now, for $a(x,\xi) \in \mathrm{S}^l(U)$, the same procedure gives us the estimate

$$|\partial_\xi^\gamma \int e^{-iy\cdot\xi} a(\alpha(x(y)),\xi) \frac{v(x(y))}{|\det(\mathbf{1} - d\alpha(x(y)))|} dy| \leq C \langle \xi \rangle^{l-|\beta|}$$

for arbitrary multi-indices γ and β and some constant $C > 0$. Thus, as an oscillatory integral, the last expression in (5.5) defines a distribution on V for any $a(x,\xi) \in \mathrm{S}^l(U)$. The distribution $\Lambda^* K_A$ is called *the transversal trace of* $A \in \mathrm{L}^l(U)$. If, in particular, $a(x,\xi) = a(x)$ is a polynomial of degree zero in ξ, one computes that

$$\Lambda^* K_A = \frac{a(x_0)\delta_{x_0}}{|\det(1 - d\alpha(x_0))|}. \tag{5.6}$$

Remark 11. Notice that, in particular, the above estimate shows that the inner integral in (5.5) is a Schwartz function in the ξ variable, i.e., it belongs to $\mathcal{S}_\xi(\mathbb{R}^n)$.

This diccussion can be globalized. Let $\mathbf{X}$ be a smooth manifold, E a vector bundle over $\mathbf{X}$, $\alpha : \mathbf{X} \to \mathbf{X}$ a C^∞-map with only simple fixed points, and

$$A : \Gamma_c(\alpha^* E) \longrightarrow \Gamma(E)$$

a pseudodifferential operator of order l between smooth sections. Denote the density bundle on $\mathbf{X}$ by Ω, put $F = \alpha^* E$, and define $F' = F^* \otimes \Omega$. The kernel K_A is then a distributional section of $E \boxtimes F'$. In other words, $K_A \in \mathcal{D}'(E \boxtimes F') = \mathcal{D}'(\mathbf{X} \times \mathbf{X}, E \boxtimes F')$. By the definition of distributions and the identifications made above, K_A gives a continuous function $K_A : \Gamma_c(E' \boxtimes F) \to \mathbb{C}$. Therefore, if $\psi \in \Gamma_c(E')$ and $\varphi \in \Gamma_c(F)$, then the value of K_A on $\psi \otimes \varphi$ is given by

$$K_A \cdot \psi \otimes \varphi = \langle \psi, A\varphi \rangle,$$

where $\langle , \rangle$ are the usual duality brackets. Similarly, one has $K_{\alpha^* A} \in \mathcal{D}'(\mathbf{X} \times \mathbf{X}, F \boxtimes F')$, where $\alpha^* A$ denotes the composition

$$\alpha^* A : \Gamma_c(F) \xrightarrow{A} \Gamma(E) \xrightarrow{\alpha^*} \Gamma(F).$$

If $A \in \mathrm{L}^{-\infty}(F,E)$, K_A is a smooth section on $\mathbf{X} \times \mathbf{X}$, and $K_A(\tilde{x},\tilde{y}) \in E_{\tilde{x}} \otimes F'_{\tilde{y}}$. In this case, $K_{\alpha^* A}(\tilde{x},\tilde{y}) = K_A(\alpha(\tilde{x}),\tilde{y})$, so that one deduces $K_{\alpha^* A}(\tilde{x},\tilde{x}) \in E_{\alpha(\tilde{x})} \otimes F'_{\tilde{x}} = F_{\tilde{x}} \otimes (F^* \otimes \Omega)_{\tilde{x}} \simeq \mathcal{L}(F_{\tilde{x}}, F_{\tilde{x}}) \otimes \Omega_{\tilde{x}}$, where $\mathcal{L}(F_{\tilde{x}}, F_{\tilde{x}})$ denotes the space of all linear continuous maps from $F_{\tilde{x}}$ to itself. As a consequence, $\mathrm{Tr}\, K_{\alpha^* A}(\tilde{x},\tilde{x})$ becomes a section of Ω, where Tr denotes the bundle homomorphism

$$\mathrm{Tr} : F \otimes F' \longrightarrow \Omega, \tag{5.7}$$

that is given on each section by the natural pairing

$$(\quad): F_{\tilde{x}} \times F'_{\tilde{x}} \to \Omega_{\tilde{x}}.$$

Hence, if $\mathbf{X}$ is compact, and $A \in \mathrm{L}^{-\infty}(F, E)$, one can define the trace of $\alpha^* A$ as the distribution density

$$\langle \operatorname{Tr} \alpha^* A, u \rangle = \int_{\mathbf{X}} u(\tilde{x}) \operatorname{Tr} K_{\alpha^* A}(\tilde{x}, \tilde{x})\, d\tilde{x}, \qquad u \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbf{X}).$$

This we denote in short by

$$\operatorname{Tr} \alpha^* A = \int_{\mathbf{X}} \operatorname{Tr} K_{\alpha^* A}(\tilde{x}, \tilde{x}).$$

This trace can be extended to arbitrary $A \in \mathrm{L}^l(\mathbf{X})$. Indeed, let Δ be the diagonal in $\mathbf{X} \times \mathbf{X}$, and denote the canonical isomorphism $\Delta \simeq \mathbf{X}$ also by Δ. The foregoing local considerations, in essence, imply the following

Proposition 8. *The map $\Theta : \mathcal{L}(\mathcal{E}'(F), \Gamma(E)) \to \Gamma(F \otimes F')$ given by $A \mapsto \Delta^* K_{\alpha^* A} = K_{\alpha^* A}(\tilde{x}, \tilde{x})$ has an extension*

$$\Theta : \mathrm{L}^l(F, E) \longrightarrow \mathcal{D}'(F \otimes F')$$

which is continuous with respect to the strong operator topology on bounded sets of $\mathrm{L}^l(F, E)$.

Proof. See [**AB67**], Proposition 5.3. □

Since the bundle homomorphism (5.7) induces continuous linear maps

$$\operatorname{Tr} : \Gamma(F \otimes F') \longrightarrow \Gamma(\Omega), \qquad \operatorname{Tr} : \mathcal{D}'(F \otimes F') \longrightarrow \mathcal{D}'(\Omega),$$

where $\mathcal{D}'(\Omega) = \mathcal{D}'(\mathbf{X}, \Omega) = \Gamma_c(\Omega^* \otimes \Omega)' = \Gamma_c(1)' = \mathrm{C}^\infty_{\mathrm{c}}(\mathbf{X})'$ is the space of distribution densities on $\mathbf{X}$, we see that $\operatorname{Tr} \Theta(A)$ can be defined for any $A \in L^l(F, E)$ in a unique way. Consequently, for compact $\mathbf{X}$, the map $\mathrm{L}^{-\infty}(F, E) \longrightarrow \mathbb{C}$ given by

$$A \longmapsto \operatorname{Tr} \alpha^* A$$

has a unique continuous extension to

$$\operatorname{Tr}_\alpha : \mathrm{L}^l(F, E) \longrightarrow \mathbb{C}$$

that is given by

$$A \longmapsto \operatorname{Tr}_\alpha A = \langle \operatorname{Tr} \Theta(A), 1 \rangle$$

called the *transversal trace of* A. In the case that A is induced by a bundle homomorphism φ, it follows from (5.6) that

$$\operatorname{Tr}_\alpha A = \sum_{\tilde{x} \in \operatorname{Fix}(\alpha)} \nu_{\tilde{x}}(A), \qquad \nu_{\tilde{x}}(A) = \frac{\operatorname{Tr} \varphi_{\tilde{x}}}{|\det(\mathbf{1} - d\alpha(\tilde{x}))|}, \tag{5.8}$$

the sum being over the fixed points of α on $\mathbf{X}$, see [**AB67**], Corollary 5.4.

In the context of representation theory, this trace was employed by Atiyah and Bott in [**AB68**] to compute the global character of an induced representation. Thus, let G be a Lie group, H a closed subgroup of G, and ϱ a representation of H on a finite dimensional vector space V. The representation of G induced by ϱ is a geometric endomorphism in the space of sections over G/H with values in the homogeneous vector bundle $G\times_H V$, and shall be denoted by $T(g) = (\iota_*\varrho)(g)$. Assume that G/H is compact, and let d_G be a Haar measure on G. Consider a compactly supported smooth function $f \in \mathrm{C^\infty_c}(G)$, and the corresponding convolution operator $T(f) = \int_G f(g)T(g)d_G(g)$. It is a smooth operator, and, since G/H is compact, has a well defined trace. Consequently, the map

$$\Theta_T : \mathrm{C^\infty_c}(G) \ni f \longmapsto \operatorname{Tr} T(f) \in \mathbb{C}$$

defines a distribution on G called the *distribution character of the induced representation* T. On the other hand, assume that $g\in G$ is such that $l_{g^{-1}} : G/H \to G/H, xH \mapsto g^{-1}xH$, has only simple fixed points. In this case, a flat trace $\operatorname{Tr}^\flat T(g)$ of $T(g)$ can be defined according to

$$\operatorname{Tr}^\flat T(g) = \operatorname{Tr}_{l_{g^{-1}}}(\Gamma(\varphi_g)),$$

where $\varphi_g : l^*_{g^{-1}}(G\times_H V) \to G\times_H V$ is the endomorphism associated to $T(g)$ such that

$$T(g) = \varphi_g \circ l^*_{g^{-1}},$$

and $\Gamma(\varphi_g) : \Gamma(l^*_{g^{-1}}(G\times_H V)) \to \Gamma(G\times_H V)$. $\operatorname{Tr}^\flat T(g)$ is given by a sum over fixed points of g, and one can show that, on an open set $G_T \subset G$,

$$\Theta_T(f) = \int_{G_T} f(g) \operatorname{Tr}^\flat T(g) d_G(g), \qquad f\in \mathrm{C^\infty_c}(G_T).$$

Thus, the distribution character of a parabolically induced representation of a Lie group G is represented on G_T by the flat trace of the corresponding geometric endomorphism. If G is compact, the Hermann–Weyl formula can be recovered from the Lefschetz theorem. It can be interpreted as expressing the character of a finite dimensional representation as an alternating sum of characters of infinite dimensional representations.

5.3. A Fixed-point formula

We are now in a position to describe the distributions Θ^s_π and Θ_π introduced in Section 5.1. In what follows, we shall prove fixed-point formulae for these distributions, similar to the one of Atiyah and Bott described in the previous section. Thus, let $(\pi, C(\widetilde{\mathbb{X}}))$ be the regular representation of G on the Oshima compactification $\widetilde{\mathbb{X}}$ of the Riemannian symmetric space $\mathbb{X}=G/K$ of non-compact type, and denote by $\Phi_g(\tilde x) = g\cdot \tilde x$ the G-action on $\widetilde{\mathbb{X}}$. Let further $G(\widetilde{\mathbb{X}}) \subset G$ be the set of elements g in G acting transversally on $\widetilde{\mathbb{X}}$.

REMARK 12. The set $G(\widetilde{\mathbb{X}})$ is open. Corollary 1 and Remark 5 imply that $G(\widetilde{\mathbb{X}})$ is dense if $\operatorname{rank}(G/K) = 1$.

THEOREM 7. *Let $f \in \mathrm{C}_c^\infty(G)$ have support in $G(\widetilde{\mathbb{X}})$, and $s \in \mathbb{C}$, $\operatorname{Re} s > -1$. Then*

(5.9)
$$\operatorname{Tr}_s \pi(f) = \int_{G(\widetilde{\mathbb{X}})} f(g) \left(\sum_{\tilde{x} \in \operatorname{Fix}(\widetilde{\mathbb{X}}, g)} \sum_\gamma \frac{\alpha_\gamma(\tilde{x}) |x_{k+1}(\kappa_\gamma^{-1}(\tilde{x})) \cdots x_{k+l}(\kappa_\gamma^{-1}(\tilde{x}))|^{s+1}}{|\det(\mathbf{1} - d\Phi_g(\tilde{x}))|} \right) d_G(g)$$

the notation being as before, and where $\operatorname{Fix}(\widetilde{\mathbb{X}}, g)$ denotes the set of fixed points of g on $\widetilde{\mathbb{X}}$. In particular, $\Theta_\pi^s : \mathrm{C}_c^\infty(G) \ni f \to \operatorname{Tr}_s \pi(f) \in \mathbb{C}$ is regular on $G(\widetilde{\mathbb{X}})$.

PROOF. By Proposition 3,
$$\operatorname{Tr}_s \pi(f) = \sum_\gamma \int_{W_\gamma} (\alpha_\gamma \circ \varphi_\gamma)(x) |x_{k+1} \cdots x_{k+l}|^s \widetilde{A}_f^\gamma(x, 0) dx$$
is a meromorphic function in s with possible poles at $-1, -3, \dots$. Assume that $\operatorname{Re} s > -1$. Since $\alpha_\gamma \in \mathrm{C}_c^\infty(\widetilde{W}_\gamma)$, and $\widetilde{A}_f^\gamma(x,0) = \int \tilde{a}_f^\gamma(x,\xi) đ\xi$, where $\tilde{a}_f^\gamma(x,\xi) \in \mathrm{S}_{la}^{-\infty}(W_\gamma \times \mathbb{R}^{k+l})$ is rapidly decaying in ξ by Theorem 2, we can interchange the order of integration to obtain
$$\operatorname{Tr}_s \pi(f) = \sum_\gamma \int \int_{W_\gamma} (\alpha_\gamma \circ \varphi_\gamma)(x) |x_{k+1} \cdots x_{k+l}|^s \tilde{a}_f^\gamma(x, \xi) dx\, đ\xi.$$
Let $\chi \in \mathrm{C}_c^\infty(\mathbb{R}^{k+l}, \mathbb{R}^+)$ be equal 1 in a neighbourhood of 0, and $\varepsilon > 0$. Then, by Lebesgue's theorem on bounded convergence,
$$\operatorname{Tr}_s \pi(f) = \lim_{\varepsilon \to 0} I_\varepsilon,$$
where we defined
$$I_\varepsilon = \sum_\gamma \int \int_{W_\gamma} (\alpha_\gamma \circ \varphi_\gamma)(x) |x_{k+1} \cdots x_{k+l}|^s \tilde{a}_f^\gamma(x, \xi) \chi(\varepsilon \xi)\, dx\, đ\xi.$$
Taking into account (3.2), and interchanging the order of integration once more, one sees that
$$I_\varepsilon = \int_G f(g) \sum_\gamma \int \int_{W_\gamma} e^{i\Psi_\gamma(g,x)\cdot\xi} c_\gamma(x,g) (\alpha_\gamma \circ \varphi_\gamma)(x) |x_{k+1} \cdots x_{k+l}|^s \chi(\varepsilon\xi) dx\, đ\xi\, d_G(g),$$
everything in sight being absolutely convergent. Let us now set
$$I_\varepsilon(g) = f(g) \sum_\gamma \int \int_{W_\gamma} e^{i\Psi_\gamma(g,x)\cdot\xi} c_\gamma(x,g) (\alpha_\gamma \circ \varphi_\gamma)(x) |x_{k+1} \cdots x_{k+l}|^s \chi(\varepsilon\xi) dx\, đ\xi,$$
so that $I_\varepsilon = \int_G I_\varepsilon(g)\, d_G(g)$. We would like to pass to the limit under the integral, for which we are going to show that $\lim_{\varepsilon \to 0} I_\varepsilon(g)$ is an integrable function on G. For

this, let us fix an arbitrary $g \in G(\widetilde{\mathbb{X}})$. By definition, g acts only with simple fixed points on $\widetilde{\mathbb{X}}$. Since each of them is isolated, g can have at most finitely many fixed points on $\widetilde{\mathbb{X}}$. Consider therefore a cut–off function $\beta_g \in \mathrm{C}^\infty(\widetilde{\mathbb{X}}, \mathbb{R}^+)$ which is equal 1 in a small neighbourhood of each fixed point of g, and whose support decomposes into a disjoint union of connected components, each of which contains only one fixed point of g. By choosing the support of β_g sufficiently close to the fixed points we can, in addition, assume that

$$\det(\mathbf{1} - d\Phi_g(\tilde{x})) \neq 0 \quad \text{on } \operatorname{supp} \beta_g. \tag{5.10}$$

Since the action of G is real analytic, we obtain a family of functions $\beta_g(\tilde{x})$ depending analytically on $g \in G(\widetilde{\mathbb{X}})$. Multiplying the integrand of $I_\varepsilon(g)$ with $\beta_g \circ \varphi_\gamma(x)$, and $1 - \beta_g \circ \varphi_\gamma(x)$, respectively, we obtain the decomposition

$$I_\varepsilon(g) = I_\varepsilon^{(1)}(g) + I_\varepsilon^{(2)}(g).$$

Let us first examine what happens away from the fixed points. Integrating by parts $2N$ times with respect to ξ yields

$$\begin{aligned} I_\varepsilon^{(2)}(g) &= f(g) \sum_\gamma \int \int_{W_\gamma} e^{i\Psi_\gamma(g,x)\cdot\xi} c_\gamma(x,g) \alpha_\gamma (1-\beta_g)\varphi_\gamma(x) |x_{k+1}\cdots x_{k+l}|^s \chi(\varepsilon\xi) dx\, đ\xi \\ &= f(g) \sum_\gamma \int \int_{W_\gamma} \frac{e^{i\Psi_\gamma(g,x)\cdot\xi}}{|\Psi_\gamma(g,x)|^{2N}} \Delta_\xi^N \chi(\varepsilon\xi) c_\gamma(x,g)\alpha_\gamma(1-\beta_g)\varphi_\gamma(x)|x_{k+1}\cdots x_{k+l}|^s dx\, đ\xi \end{aligned}$$

where $\Delta_\xi = \partial_{\xi_1}^2 + \cdots + \partial_{\xi_{k+l}}^2$. Now, for arbitrary N,

$$|\Delta_\xi^N[\chi(\varepsilon\xi)]| \leq C_N(1+|\xi|^2)^{-N},$$

where C_N does not depend on ε for $0 < \varepsilon \leq 1$. Furthermore, there exists a constant $M_f > 0$ such that $|\Psi_\gamma(g,x)|^{2N} \geq M_f$ on the support of $1 - \beta_g \circ \varphi_\gamma$ for all $g \in \operatorname{supp} f$ and γ. By Lebesgue's theorem, we may therefore pass to the limit under the integral, and obtain

$$\lim_{\varepsilon \to 0} I_\varepsilon^{(2)}(g) = 0.$$

Hence, as $\varepsilon \to 0$, the main contributions to $I_\varepsilon(g)$ originate from the fixed points of g. To examine these contributions, note that condition (5.10) implies that $x \mapsto \varphi_\gamma^g(x) - x$ defines a diffeomorphism on each of the connected components of $\operatorname{supp}(\alpha_\gamma\beta_g) \circ \varphi_\gamma$ onto their respective images. Performing the change of variables $y = x - \varphi_\gamma^g(x)$ we get

$$\begin{aligned}
I_\varepsilon^{(1)}(g) &= f(g)\sum_\gamma \int\int_{W_\gamma} e^{i\Psi_\gamma(g,x)\cdot\xi} c_\gamma(x,g)(\alpha_\gamma\beta_g)(\varphi_\gamma(x))|x_{k+1}\cdots x_{k+l}|^s \chi(\varepsilon\xi)dx\, đ\xi \\
&= f(g)\sum_\gamma \int\int e^{-i(\mathbf{1}_k\otimes T^{-1}_{x(y)})y\cdot\xi}|x_{k+1}(y)\cdots x_{k+l}(y)|^s \frac{(\alpha_\gamma\beta_g)(\varphi_\gamma x(y))c_\gamma(x(y),g)}{|\det(\mathbf{1}-d\varphi_\gamma^g(x(y)))|}\chi(\varepsilon\xi)dy\, đ\xi \\
&= f(g)\sum_\gamma \int |x_{k+1}(y)\cdots x_{k+l}(y)|^s c_\gamma(x(y),g)\frac{(\alpha_\gamma\beta_g)(\varphi_\gamma(x(y)))\hat\chi((\mathbf{1}_k\otimes T^{-1}_{x(y)})y/\varepsilon)}{(2\pi)^{k+l}\varepsilon^{k+l}|\det(\mathbf{1}-d\varphi_\gamma^g(x(y)))|}dy \\
&= f(g)\sum_\gamma \int |x_{k+1}(\varepsilon y)\cdots x_{k+l}(\varepsilon y)|^s c_\gamma(x(\varepsilon y),g)\frac{(\alpha_\gamma\beta_g)(\varphi_\gamma(x(\varepsilon y)))\hat\chi(\mathbf{1}_k\otimes T^{-1}_{x(\varepsilon y)}y)}{(2\pi)^{k+l}|\det(\mathbf{1}-d\varphi_\gamma^g(x(\varepsilon y)))|}dy.
\end{aligned}$$

Since in a neighbourhood of a fixed point $\tilde x$ of g, the Jacobian of the singular change of coordinates $z=(\mathbf{1}_k\otimes T^{-1}_{x(\varepsilon y)})y$ converges, as $\varepsilon\to 0$, to the expression $|x_{k+1}(\kappa_\gamma^{-1}(\tilde x))\cdots x_{k+l}(\kappa_\gamma^{-1}(\tilde x))|^{-1}$, and $(2\pi)^{-k-l}\int\hat\chi(y)dy=\chi(0)=1$, we finally obtain that

$$\begin{aligned}
\lim_{\varepsilon\to0} I_\varepsilon^{(1)}(g) = \lim_{\varepsilon\to0} f(g)\sum_\gamma\int & |x_{k+1}(\varepsilon y(z))\cdots x_{k+l}(\varepsilon y(z))|^s c_\gamma(x(\varepsilon y(z)),g) \\
& \frac{(\alpha_\gamma\beta_g)(\varphi_\gamma(x(\varepsilon y(z))))|\,\partial y/\,\partial z|}{(2\pi)^{k+l}|\det(\mathbf{1}-d\varphi_\gamma^g(x(\varepsilon y(z))))|}\hat\chi(z)dz \\
& = f(g)\sum_{\tilde x\in\mathrm{Fix}(\widetilde{\mathbb{X}},g)}\sum_\gamma \frac{\alpha_\gamma(\tilde x)|x_{k+1}(\kappa_\gamma^{-1}(\tilde x))\cdots x_{k+l}(\kappa_\gamma^{-1}(\tilde x))|^{s+1}}{|\det(\mathbf{1}-d\Phi_g(\tilde x))|},
\end{aligned}$$

since $\bar\alpha_\gamma\equiv1$ on $\operatorname{supp}\alpha_\gamma$, and $\beta_g(\tilde x)=1$. The limit function $\lim_{\varepsilon\to0}I_\varepsilon(g)$ is therefore clearly integrable on G for $\operatorname{Re}s>-1$, so that by passing to the limit under the integral one computes

$$\begin{aligned}
\mathrm{Tr}_s\,\pi(f) &= \lim_{\varepsilon\to0} I_\varepsilon = \lim_{\varepsilon\to0}\int_G I_\varepsilon(g)\,d_G(g) = \int_G \lim_{\varepsilon\to0}\big(I_\varepsilon^{(1)}+I_\varepsilon^{(2)}\big)(g)d_G(g) \\
&= \int_G f(g)\sum_{\tilde x\in\mathrm{Fix}(\widetilde{\mathbb{X}},g)}\sum_\gamma \frac{\alpha_\gamma(\tilde x)|x_{k+1}(\kappa_\gamma^{-1}(\tilde x))\cdots x_{k+l}(\kappa_\gamma^{-1}(\tilde x))|^{s+1}}{|\det(1-d\Phi_g(\tilde x))|}\,d_G(g),
\end{aligned}$$

yielding the desired description of Θ_π^s. □

As an immediate consequence of the previous theorem we see that, for f belonging to $\mathrm{C_c^\infty}(G(\widetilde{\mathbb{X}}))$, $\mathrm{Tr}_s\,\pi(f)$ is not singular at $s=-1$. This observation leads to the following

COROLLARY 4. *Let $f\in\mathrm{C_c^\infty}(G)$ have support in $G(\widetilde{\mathbb{X}})$. Then*

$$\mathrm{Tr}_{reg}\,\pi(f) = \mathrm{Tr}_{-1}\,\pi(f) = \int_{G(\widetilde{\mathbb{X}})} f(g)\sum_{\tilde x\in\mathrm{Fix}(\widetilde{\mathbb{X}},g)}\frac{1}{|\det(1-d\Phi_g(\tilde x))|}\,d_G(g).$$

In particular, the distribution $\Theta_\pi : f \to \mathrm{Tr}_{reg}(f)$ is regular on $G(\widetilde{\mathbb{X}})$.

PROOF. Consider the Laurent expansion of $\Theta^s_\pi(f)$ at $s = -1$ given by

$$\begin{aligned}\mathrm{Tr}_s\,\pi(f) &= \Big\langle |x_{k+1}\cdots x_{k+l}|^s, \sum_\gamma (\alpha_\gamma \circ \varphi_\gamma)\widetilde{A}^\gamma_f(\cdot,0)\Big\rangle \\ &= \sum_{j=-q}^{\infty} S_j\Big(\sum_\gamma (\alpha_\gamma \circ \varphi_\gamma)\widetilde{A}^\gamma_f(\cdot,0)\Big)(s+1)^j,\end{aligned}$$

where $S_k \in \mathcal{D}'(\mathbb{R}^{k+l})$. Since by (5.9), $\mathrm{Tr}_s\,\pi(f)$ has no pole at $s=-1$, we necessarily must have

$$S_j\Big(\sum_\gamma (\alpha_\gamma \circ \varphi_\gamma)\widetilde{A}^\gamma_f(\cdot,0)\Big) = 0 \qquad \text{for } j<0,$$

so that

$$\mathrm{Tr}_{-1}\,\pi(f) = \Big\langle S_0, \sum_\gamma (\alpha_\gamma \circ \varphi_\gamma)\widetilde{A}^\gamma_f(\cdot,0)\Big\rangle = \mathrm{Tr}_{reg}\,\pi(f).$$

The assertion now follows with the previous theorem. □

In particular, Corollary 4 implies that $\mathrm{Tr}_{reg}\,\pi(f)$ is invariantly defined. Now, interpreting $\pi(g)$ as a geometric endomorphism on the trivial bundle $E = \widetilde{\mathbb{X}} \times \mathbb{C}$ over the Oshima compactification $\widetilde{\mathbb{X}}$, a flat trace $\mathrm{Tr}^\flat\,\pi(g)$ of $\pi(g)$ can be defined according to

$$\mathrm{Tr}^\flat\,\pi(g) = \mathrm{Tr}_{\Phi_g}(\Gamma(\varphi_g)),$$

where $\varphi_g : \Phi_g^* E \to E$ is the associated bundle homomorphism which identifies the fiber $E_{\Phi_g(\tilde{x})}$ with $E_{\tilde{x}}$, and satisfies $(\mathrm{Tr}\,\varphi_g)_{|\tilde{x}} = 1$ at each fixed point $\tilde{x}$ of g. Taking into account (5.8), the previous corollary can be reformulated, and we finally deduce the following character formula for the distribution character of π.

THEOREM 8. *On the set of transversal elements $G(\widetilde{\mathbb{X}})$, the distribution $\Theta_\pi : f \to \mathrm{Tr}_{reg}\,\pi(f)$ is given by*

$$\mathrm{Tr}_{reg}\,\pi(f) = \int_{G(\widetilde{\mathbb{X}})} f(g)\,\mathrm{Tr}^\flat\,\pi(g) d_G(g), \qquad f \in \mathrm{C}^\infty_{\mathrm{c}}(G(\widetilde{\mathbb{X}})),$$

where

$$\mathrm{Tr}^\flat\,\pi(g) = \sum_{\tilde{x}\in \mathrm{Fix}(\widetilde{\mathbb{X}},g)} \frac{1}{|\det(1-d\Phi_g(\tilde{x}))|},$$

the sum being over the (simple) fixed points of $g \in G(\widetilde{\mathbb{X}})$ on $\widetilde{\mathbb{X}}$.

□

CHAPTER 6

Some aspects of scattering theory on symmetric spaces

The intention of this final chapter is to provide some preliminary computations towards a possible development of scattering theory on symmetric spaces, in the light of what we have done earlier in the thesis. Towards this, we compute the invariant metric on G/K in the Oshima coordinates, and the Laplacian of this metric. We then show, in Section 6.3, a self-adjoint extension of a second order differential operator which arises as the contribution from the boundary of the compactification, when $\operatorname{rank}\mathbb{X} = 1$. Finally, we close the chapter by mentioning some plausible lines for further development.

Let us begin with some historical remarks. Let $G = KAN$ be the Iwasawa decomposition of G. Let M be the centralizer of A in K, and $P = MAN$ the corresponding minimal parabolic subgroup. Let $\mathfrak{a}$ be the Lie algebra of A, and $H : G \to \mathfrak{a}$ denote the Iwasawa projection defined by $g = k \exp H(g) n$. For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, the functions

$$\varphi_\lambda(x) = \int_K e^{(\lambda-\varrho)H(xk)} dk$$

are precisely all the elementary spherical functions on G, where ϱ is the half-sum of positive roots with multiplicities. These are the matrix coefficients of the principal series representation induced from the trivial representation of M defined by the function 1. The idea is to investigate φ_λ through the differential equations they satisfy. The key observation here is that these differential equations are, on the positive Weyl chamber A^+, perturbations of certain constant coefficient equations on A^+ close to the walls of the Weyl chamber. One then uses the *method of descent* - the leading term of an eigenfunction on G at infinity in the neighbourhood of a parabolic subgroup Q is an eigenfunction on the corresponding Levi component M_0. By working in a slightly larger class of groups, the so-called Harish-Chandra class, an inductive analysis becomes possible. The perturbation analysis indicated above therefore focusses attention on two parabolic subgroups P and Q; the matrix coefficient belongs to the representation induced from P, while the constant term of this matrix coefficient is regarded in the direction of Q. The main asymptotic result is that given $b \in \mathfrak{U}(\mathfrak{g})$, there exist constants $C = C(b) > 0$ and $s = s(b) \geq 0$ such that for a in a conical region in $\overline{A^+}$ and $\lambda \in i\mathfrak{a}^*$,

$$|e^{\varrho(\log a)}\varphi_\lambda(a; b) - e^{\varrho_0(\log a)}\psi_\lambda(a; b)| \leq C(1 + |\lambda|)^s(1 + |\log a|)^s e^{-2\tau_0(\log a)},$$

where $\psi_\lambda(\cdot)$ is the constant term in the direction of the parabolic Q, ϱ_0 the half-sum of positive roots of M_0, and τ_0 a functional on $\mathfrak{a}$ determined by the roots of G and M_0. This can be thought of as the statement that the matrix coefficient is well-approximated by a *plane wave*. The inducing data specify a plane wave in the direction of P. So the asymptotics can be interpreted as providing a scattering operator from P to Q. Thus a scattering theory for symmetric spaces is implicit in the work of Harish-Chandra, and the goal would be to make this more explicit. We refer to [**GV88**] for the above discussion.

6.1. The invariant metric on G/K

Let $X = G/K$ be a Riemannian symmetric space of non-compact type, and of rank l. With notation as in Section 2.1, let $\langle X, Y\rangle_\theta := -\langle X, \theta Y\rangle$. The restriction $\langle\cdot,\cdot\rangle_{\theta|\mathfrak{p}\times\mathfrak{p}}$ is a symmetric, positive-definite, bilinear form, and thus gives a metric on $\mathbb{X}$. Indeed, note that $\mathfrak{p}$ is $Ad_G(K)$ invariant. This can be seen as follows. Let Θ be the unique analytic involutive automorphism of G fixing K such that $d\Theta = \theta$. For $X \in \mathfrak{p}, k \in K$, we have

$$\Theta(\exp Ad(k)tX) = \Theta(k\exp(tX)k^{-1}) = k\Theta(\exp tX)k^{-1} = k\exp(-tX)k^{-1}.$$

Differentiating this expression at $t = 0$, we get that

$$\theta Ad(k)X = -Ad(k)X = Ad(k)\theta X$$

ie., $Ad(k)_{|\mathfrak{p}}$ commutes with θ. This along with the fact that the Cartan-Killing form is invariant under Lie algebra automorphisms gives us that $\langle\cdot,\cdot\rangle_{\theta|\mathfrak{p}\times\mathfrak{p}}$ is K-invariant.

Next, let $\pi : G \to G/K$ be the canonical map. Then $d\pi : \mathfrak{g} \to T_oX$ is surjective with kernel $\mathfrak{k}$ where $o = \pi(e)$ and so $d\pi_{|\mathfrak{p}} : \mathfrak{p} \to T_{(o)}X$ is an isomorphism. We write $d\pi$ instead of $(d\pi)_e$, for simplicity, where e is the identity element in G. We observe that for $k \in K, X \in \mathfrak{p}, \pi(\exp Ad(k)tX) = \pi(k\exp(tX)k^{-1}) = \tau(k)\pi(\exp tX)$ where, for $g \in G, \tau(g) : G/K \to G/K$ is the analytic diffeomorphism defined by $xK \mapsto gxK$. Again, differentiating at $t = 0$ we have $d\pi Ad(k)X = d\tau(k)d\pi(X)$ giving that the isomorphism $d\pi : \mathfrak{g} \to T_oX$ commutes with the action of K. Therefore we have that the bilinear form Q_o on T_oX defined by

$$Q_o(X_o, Y_o) = \langle d\pi^{-1}(X_o), d\pi^{-1}(Y_o)\rangle_{\theta|\mathfrak{p}\times\mathfrak{p}}, \qquad X_o, Y_o \in T_oX,$$

is symmetric and positive definite and is invariant under all the mappings $d\tau(k)$, $k \in K$. For each $p \in G/K$ define the bilinear form Q_p on $T_pX \times T_pX$ by

$$Q_p(d\tau(g)X_o, d\tau(g)Y_o) = Q_o(X_o, Y_o), \qquad X_o, Y_o \in T_oX$$

where $g \in G$ is chosen such that $g \cdot o = p$. Now if $g_1, g_2 \in G$ are such that $g_1 \cdot o = p = g_2 \cdot o$, that is, if $\tau(g_1)\pi(e) = \tau(g_2)\pi(e)$ then one has, by the fact that τ and π commute, that $\pi(g_1) = \pi(g_2)$. This implies that $g_1^{-1}g_2 \in K$ and so we have

$$
\begin{aligned}
Q_p(d\tau(g_2)X_o, d\tau(g_2)Y_o) &= Q_p(d\tau(g_1k')X_o, d\tau(g_1k')Y_o)\\
&= Q_p(d\tau(g_1)d\tau(k')X_o, d\tau(g_1)d\tau(k')Y_o)\\
&= Q_o(d\tau(k')X_o, d\tau(k')Y_o)\\
&= Q_o(X_o, Y_o)
\end{aligned}
$$

where $k' = g_1^{-1}g_2 \in K$. The invariance of Q_o under the mappings $d\tau(k), k \in K$ remarked above gives the last equality. Thus Q_p is well-defined.

Let $p \in X$ and let $g \in G$. Choose $g_0 \in G$ such that $g_0 \cdot o = p$. Now since $(gg_0) \cdot o = g \cdot p$ we have for $X_o, Y_o \in T_oX$,

$$
\begin{aligned}
Q_{g\cdot p}(d\tau(gg_0)X_o, d\tau(gg_0)Y_o) &= Q_o(X_o, Y_o) \qquad \text{(by definition)}\\
&= Q_p(d\tau(g_0)X_o, d\tau(g_0)Y_o).
\end{aligned}
$$

This, together with the fact that each $\tau(g), g \in G$, is an analytic diffeomorphism of G/K onto itself, gives us that $p \mapsto Q_p$ is an analytic G-invariant Riemannian structure on G/K. This invariant metric, induced by the Cartan-Killing form, we shall denote by g_L.

Remark 13. Conversely, each G-invariant Riemannian structure on G/K arises from a K-invariant form on $\mathfrak{p}$.

6.2. The Laplacian on $\mathbb{X}$ in Oshima coordinates

Let $\left(\widetilde{U}_g, \varphi_g^{-1}\right)_{g\in G}$ be the atlas on the Oshima compactification $\widetilde{X}$ of G/K as earlier. For $g \in G$, we have the maps

$$G/K \xleftarrow{\sim} N^-A \xrightarrow{\sim} N^- \times \mathbb{R}_+^l \hookrightarrow N^- \times \mathbb{R}^l \longleftarrow \widetilde{U}_g$$

given by

$$gnaK \longleftarrow n \cdot a \longrightarrow (n, e^{-\alpha_1 \log a}, \dots, e^{-\alpha_l \log a}) \to (n, t) \longleftarrow \pi(g, n, t).$$

Thus, the coordinates for G/K in the Oshima compactification are given by $\psi : N^- \times \mathbb{R}_+^l \longrightarrow G/K$ defined by $(n,t) \mapsto gnaK$ where $a = exp(-\sum_{i=1}^l H_i \log t_i)$. For convenience, we enumerate the positive roots in some fixed order as $\lambda_1, \dots, \lambda_r$. With the basis $\{H_i\}_{i=1}^l$ of $\mathfrak{a}$ and $\{X_{-\lambda_{i,j}}, 1 \le i \le r, 1 \le j \le m(\lambda_i)\}$ of $\mathfrak{n}^-$ as in Section 2.1, and the coordinates on N^- coming from the diffeomorphism $\exp : n^- \longrightarrow N^-$, we have by Lemma 2,

$$\left(\psi_*^{-1}\right)_{gnaK}\left(X_{-\lambda_{i,j}}\right) = \left(X_{-\lambda_{i,j}|N^-\times\mathbb{R}_+^l}\right)_{(n,t)} = \left(t^{\lambda_i}\frac{\partial}{\partial n_{i,j}}\right)_{(n,t)}$$

i.e.,

$$(\psi_*)_{(n,t)}\left(\frac{\partial}{\partial n_{i,j}}\right) = \frac{1}{t^{\lambda_i}}\left(X_{-\lambda_{i,j}}\right)_{gnaK},$$

and

$$\left(\psi_*^{-1}\right)_{gnaK}(H_i) = \left(H_{i|N^-\times\mathbb{R}_+^l}\right)_{(n,t)} = -\left(t_i\frac{\partial}{\partial t_i}\right)_{(n,t)},$$

and therefore,

$$(\psi_*)_{(n,t)}\left(\frac{\partial}{\partial t_i}\right) = -\frac{1}{t_i}(H_i)_{gnaK}.$$

The invariant metric on G/K in the Oshima coordinates is then the pull-back of g_L by ψ. So we have

$$\begin{aligned}\psi^* g_L\left(\frac{\partial}{\partial n_{i,j}}, \frac{\partial}{\partial n_{p,q}}\right) &= g_L\left(\psi_*(\frac{\partial}{\partial n_{i,j}}), \psi_*(\frac{\partial}{\partial n_{p,q}})\right) = g_L\left(\frac{1}{t^{\lambda_i}}X_{-\lambda_i,j}, \frac{1}{t^{\lambda_p}}X_{-\lambda_p,q}\right)\\ &= \frac{1}{t^{\lambda_i+\lambda_p}}\delta_{ip}\delta_{jq}.\end{aligned}$$

Similarly, we have that

$$\psi^* g_L\left(\frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j}\right) = g_L\left(\psi_*(\frac{\partial}{\partial t_i}), \psi_*(\frac{\partial}{\partial t_j})\right) = g_L\left(\frac{-1}{t_j}H_i, \frac{-1}{t_j}H_j\right) = \frac{1}{t_i t_j}\delta_{ij}$$

and

$$\psi^* g_L\left(\frac{\partial}{\partial t_i}, \frac{\partial}{\partial n_{j,k}}\right) = g_L\left(\psi_*(\frac{\partial}{\partial t_i}), \psi_*(\frac{\partial}{\partial n_{j,k}})\right) = g_L\left(-\frac{1}{t_j}H_i, \frac{1}{t^{\lambda_j}}X_{-\lambda_j,k}\right) = 0$$

(as $\mathfrak{n}$ and $\mathfrak{a}$ are orthogonal to each other). Denoting $\psi^* g_L$ by s, we have that

$$ds^2 = \sum_{j=1}^{r}\sum_{k=1}^{m(\lambda_i)}\frac{dn_{j,k}^2}{t^{2\lambda_j}} + \sum_{i=1}^{l}\frac{dt_i^2}{t_i^2}. \tag{6.1}$$

We also have $\overline{s} = |\det(s_{ij})| = t^{-2m(\lambda_1)\lambda_1}\ldots t^{-2m(\lambda_r)\lambda_r}t_1^{-2}\ldots t_l^{-2}$ and let the inverse of (s_{ij}) be denoted by (s^{ij}). So the Laplacian Δ_s with respect to the metric $s = \psi^* g_L$ acting on a smooth function f is given by

$$\begin{aligned}\Delta_s(f) = &\frac{1}{\sqrt{\overline{s}}}\sum_{j=1}^{r}\sum_{k=1}^{m(\lambda_j)}\frac{\partial}{\partial n_{j,k}}\left(\sqrt{\overline{s}}s^{(j,k)(j,k)}\frac{\partial}{\partial n_{j,k}}(f)\right)\\ &+\frac{1}{\sqrt{\overline{s}}}\sum_{i=1}^{l}\frac{\partial}{\partial t_i}\left(\sqrt{\overline{s}}s^{(r,m(\lambda_r)+i(r,m(\lambda_r)+i}\frac{\partial}{\partial t_i}(f)\right).\end{aligned}$$

We have

$$\begin{aligned}\frac{1}{\sqrt{\overline{s}}}\sum_{j=1}^{r}\sum_{k=1}^{m(\lambda_j)}\frac{\partial}{\partial n_{j,k}}\left(\sqrt{\overline{s}}s^{(j,k)(j,k)}\frac{\partial}{\partial n_{j,k}}\right) &= \frac{1}{\sqrt{\overline{s}}}\sum_{j=1}^{r}\sum_{k=1}^{m(\lambda_j)}\frac{\partial}{\partial n_{j,k}}\left(\sqrt{\overline{s}}t^{2\lambda_j}\frac{\partial}{\partial n_{j,k}}\right)\\ &= \sum_{j=1}^{r}\sum_{k=1}^{m(\lambda_j)}\frac{\partial}{\partial n_{j,k}}\left(t^{2\lambda_j}\frac{\partial}{\partial n_{j,k}}\right) = \sum_{j=1}^{r}t^{2\lambda_j}\sum_{k=1}^{m(\lambda_j)}\frac{\partial^2}{\partial n_{j,k}^2}\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\sqrt{s}}\frac{\partial}{\partial t_i}\,(t^{-\sum_{j=1}^r m(\lambda_j)\lambda_j}\; t_1^{-1}\dots t_l^{-1} t_i^2 \frac{\partial}{\partial t_i})\\
&= \frac{1}{\sqrt{s}}\frac{\partial}{\partial t_i}\left(t_1^{-\sum_{j=1}^r m(\lambda_j)\lambda_j(H_1)}\dots t_l^{-\sum_{j=1}^r m(\lambda_j)\lambda_j(H_l)} t_1^{-1}\dots t_l^{-1} t_i^2 \frac{\partial}{\partial t_i}\right)\\
&= \frac{1}{\sqrt{s}}\frac{\partial}{\partial t_i}\left(t_1^{-\sum_{j=1}^r m(\lambda_j)\lambda_j(H_1)}..\hat{t_i}..t_l^{-\sum_{j=1}^r m(\lambda_j)\lambda_j(H_l)} t_1^{-1}..\hat{t_i}..t_l^{-1} t_i^{(1-\sum_{j=1}^r m(\lambda_j)\lambda_j(H_i))}\frac{\partial}{\partial t_i}\right)\\
&= t_i^{(1+\sum_{j=1}^r m(\lambda_j)\lambda_j(H_i))}\frac{\partial}{\partial t_i}\left(t_i^{(1-\sum_{j=1}^r m(\lambda_j)\lambda_j(H_i))}\frac{\partial}{\partial t_i}\right)\\
&= t_i^{(1+\sum_{j=1}^r m(\lambda_j)\lambda_j(H_i))}\left[t_i^{-\sum_{j=1}^r m(\lambda_j)\lambda_j(H_i)}\frac{\partial}{\partial t_i} + t_i^{(1+\sum_{j=1}^r m(\lambda_j)\lambda_j(H_i))}\frac{\partial^2}{\partial t_i^2}\right]\\
&= t_i^2\frac{\partial^2}{\partial t_i^2} + t_i\frac{\partial}{\partial t_i}.
\end{aligned}$$

Therefore, we have the following lemma.

LEMMA 10. *The Laplacian for the invariant metric on $\mathbb{X}$ in the coordinates of its embedding in the Oshima compactification has the form*

$$\Delta_s = \sum_{j=1}^r t^{2\lambda_j}\left(\sum_{k=1}^{m(\lambda_j)}\frac{\partial^2}{\partial n_{j,k}^2}\right) + \sum_{i=1}^l\left(t_i^2\frac{\partial^2}{\partial t_i^2} + t_i\frac{\partial}{\partial t_i}\right) \quad \square. \tag{6.2}$$

When $\mathbb{X} = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ is the upper-half plane, the metric in these coordinates takes the form $ds^2 = \frac{dn^2}{t^{2\alpha}} + \frac{dt^2}{t^2}$, and consequently the Laplacian looks like

$$\Delta_s = t^{2\alpha}\frac{d^2}{dn^2} + \left(t^2\frac{d^2}{dt^2} + t\frac{d}{dt}\right).$$

The second-order ordinary differential operator $t^2\frac{d^2}{dt^2} + t\frac{d}{dt}$ can be thought of as the contribution from the boundary, as $t \to 0$, and we shall call it the comparison operator. The Riemannian measure on G/K with the metric $s = \psi^* g_L$ (where $G/K \hookrightarrow \widetilde{\mathbb{X}}$ via the identifications given before) is given, for f a compactly supported smooth function, by

$$\begin{aligned}
\mu(f) &= \sum\left|\int_{\varphi_g^{-1}(\widetilde{U_g})}\left(f\circ\varphi_g^{-1}\right)(n_{(1,1)},\dots,n_{(r,m(\lambda_r))},t_1,\dots,t_l)\sqrt{s}\,dn_1\dots dn_k dt_1\dots dt_l\right|\\
&= \sum\left|\int_{\varphi_g^{-1}(\widetilde{U_g})}\left(f\circ\varphi_g^{-1}\right)(n_1,\dots,n_k,t_1,\dots,t_l)\frac{dn_1\dots dn_k}{t^{m(\lambda_1)\lambda_1}\dots t^{m(\lambda_r)\lambda_r}}\frac{dt_1\dots dt_l}{t_1\dots t_l}\right|\\
&= \sum\left|\int_{\varphi_g^{-1}(\widetilde{U_g})}\left(f\circ\varphi_g^{-1}\right)(n_1,\dots,n_k,t_1,\dots,t_l)\frac{dn_1\dots dn_k dt_1\dots dt_l}{\prod_{i=1}^l t_i^{(1+\sum_{j=1}^r m(\lambda_j)\lambda_j(H_i))}}\right|
\end{aligned}$$

6.3. Self-adjoint extensions of the comparison operator

We now consider the case where G/K is of rank one and show the symmetry of the singular second-order ordinary differential operator $t^2\frac{d^2}{dt^2}+t\frac{d}{dt}$ with domain $\left(\mathrm{C}_{\mathrm{c}}^{\infty}\left((\varepsilon,\infty)\right),\frac{dt}{t}\right)$, $\varepsilon>0$. For $f,g\in\mathrm{C}_{\mathrm{c}}^{\infty}((\varepsilon,\infty))$ we have, integrating by parts,

$$\begin{aligned}
\int_{\varepsilon}^{\infty} t^2\frac{d^2}{dt^2}\left(f(t)\right)g(t)\frac{dt}{t} &= \int_{\varepsilon}^{\infty}\frac{d^2}{dt^2}\left(f(t)\right)tg(t)dt\\
&= tg(t)\frac{d}{dt}f(t)|_{\varepsilon}^{\infty} - \int_{\varepsilon}^{\infty}\frac{d}{dt}f(t)\frac{d}{dt}\left(tg(t)\right)dt\\
&= -\varepsilon f'(\varepsilon)g(\varepsilon) - \int_{\varepsilon}^{\infty}\frac{d}{dt}f(t)\left(g(t)+t\frac{d}{dt}g(t)\right)dt\\
&= -\varepsilon f'(\varepsilon)g(\varepsilon) - \int_{\varepsilon}^{\infty}\frac{d}{dt}(f(t))g(t)dt - tf(t)\frac{d}{dt}g(t)|_{\varepsilon}^{\infty}\\
&\quad + \int_{\varepsilon}^{\infty} f(t)\frac{d}{dt}\left(t\frac{d}{dt}g(t)\right)dt\\
&= -\varepsilon f'(\varepsilon)g(\varepsilon)+\varepsilon g'(\varepsilon)f(\varepsilon) - \int_{\varepsilon}^{\infty}\frac{d}{dt}(f(t))(g(t)dt\\
&\quad + \int_{\varepsilon}^{\infty} f(t)\left(\frac{d}{dt}g(t)+t\frac{d^2}{dt^2}g(t)\right)dt\\
&= -\int_{\varepsilon}^{\infty} t\frac{d}{dt}(f(t))g(t)\frac{dt}{t} + \int_{\varepsilon}^{\infty} f(t)\left(t^2\frac{d^2}{dt^2}g(t)+t\frac{d}{dt}g(t)\right)\frac{dt}{t}\\
&\quad + \varepsilon g'(\varepsilon)f(\varepsilon)-\varepsilon f'(\varepsilon)g(\varepsilon)
\end{aligned}$$

i.e., we have

$$\begin{aligned}
\int_{\varepsilon}^{\infty}\left(t^2\frac{d^2}{dt^2}+t\frac{d}{dt}\right)\left(f(t)\right)g(t)\frac{dt}{t} = &\int_{\varepsilon}^{\infty} f(t)\left(t^2\frac{d^2}{dt^2}+t\frac{d}{dt}\right)\left(g(t)\right)\frac{dt}{t}\\
&+\left[\varepsilon g'(\varepsilon)f(\varepsilon)-\varepsilon f'(\varepsilon)g(\varepsilon)\right],
\end{aligned}$$

where the last term on the right-hand side is the boundary term. The boundary term measures how far the operator is from being symmetric. So when $f,g\in\mathrm{C}_{\mathrm{c}}^{\infty}\left((\varepsilon,\infty)\right)$ are such that the boundary term vanishes, we obtain the symmetry of the operator $t^2\frac{d^2}{dt^2}+t\frac{d}{dt}$. In general, imposing a boundary condition of the form $f'(\varepsilon)=\chi\varepsilon^m f(\varepsilon)$, where $\chi\in[0,\infty)$ and $m\in\mathbb{Z}$, we get a symmetric operator defined on the domain $\mathcal{D}=\{f\in\mathrm{C}_{\mathrm{c}}^{\infty}([\varepsilon,\infty))\}|f'(\varepsilon)=\chi\varepsilon^m f(\varepsilon)$. Thus we have the following proposition.

PROPOSITION 9. *The operator $t^2\frac{d^2}{dt^2}+t\frac{d}{dt}$ with domain $\mathrm{C}_{\mathrm{c}}^{\infty}((\varepsilon,\infty))$, then, has a self-adjoint extension, denoted by A_1, to $\mathrm{L}^2\left([\varepsilon,\infty),\frac{dt}{t}\right)$.*

PROOF. Compare [**Mül83**], page 252. $\square$

6.4. Outlook

In this section, after recalling known results in scattering theory on symmetric spaces, we shall put the computations done earlier in the chapter into perspective, and indicate some lines for further development. In studying the spectral theory of the Laplacian, or other operators of interest, on non-compact objects one has, in general, to deal with a continuous spectrum. Then one would like to know a parametrization of it, together with a precise knowledge of the poles of its resolvent. Scattering theory, in a way, can be thought of as the study of the continuous spectrum. One way of dealing with this question is to use a suitable compactification and analyze the question in terms of this embedding. Compactifcations are natural to consider from the point of view of scattering theory as they relate the spectrum of a space to the boundary of its compactification. Our interest is in scattering theory on a Riemannian symmetric space $\mathbb{X} = G/K$, in relation to its Oshima compactification, and in view of the results in the previous chapters.

A Lax-Phillips scattering theory (see [**LP67**]) for higher rank symmetric spaces, where the time variable becomes multi-dimensional, was initiated in [**STS76**], and developed in the works of Phillips-Shahshahani [**PS93**] and Helgason [**Hel98**]. We describe what is known, briefly, referring mostly to the latter. When $\mathbb{X}$ is a Riemannian symmetric space a natural analog to the wave equation was defined by Semenov-Tian-Shansky in [**STS76**] using the algebra of invariant differential operators. Let $I(\mathfrak{a})$ denote the space of W-invariant polynomials in the symmetric algebra $S(\mathfrak{a})$, and let $H(\mathfrak{a})$ denote the space of harmonic polynomials on $\mathfrak{a}$. Let $\Gamma : \mathcal{D}(G/K) \longrightarrow I(\mathfrak{a})$ be the surjective isomorphism coming from the Iwasawa decomposition. Fix a real homogeneous basis $p_i = 1, \dots, p_{|W|}$ of $H(\mathfrak{a})$, where $|W|$ is the order of the Weyl group of $(\mathfrak{g}, \mathfrak{a})$. For fixed $f_i \in \mathrm{C}^\infty(\mathbb{X})$, $1 \leq i \leq |W|$, one considers the following Cauchy problem

$$Du = \partial(\Gamma(D))u, \qquad D \in \mathcal{D}(G/K) \tag{6.3}$$

for $u \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{X} \times \mathfrak{a})$ with the initial conditions

$$(\partial(p_i)u)(x, 0) = f_i(x), \qquad 1 \leq i \leq |W|. \tag{6.4}$$

The above system of equations is the analog to the wave equation on a symmetric space. Uniqueness for the solutions for the above Cauchy problem follows by using results from invariant theory on Lie algebras, under the assumption that for each $H \in \mathfrak{a}$ the functions $u(x, H)$ have compact support in the x-variable. Invoking the theory of the Fourier transform on $\mathbb{X}$, an explicit expression for the solutions is then given as a convolution $u(x, H) = \sum_{i=1}^{|W|}(f_i * S^i_H)(x)$, where for each i, $H \in \mathfrak{a}$, S^i_H belongs to the space $\mathcal{E}'(\mathbb{X})$ of compactly supported distributions on $\mathbb{X}$. An energy form is then defined on the space of solutions for the system (6.3 -6.4) which is positive and is constant in time $H \in \mathfrak{a}$. For each $w \in W$, a map $\mathcal{E}^w$ is defined on $\mathcal{D}(\mathbb{X}) \times \cdots \times \mathcal{D}(\mathbb{X})$ in terms of the solutions of the above system, and is then shown to be an injective norm-preserving map onto a dense subspace of $\mathrm{L}^2(\mathfrak{a}^* \times$

$B, d\lambda db/|\pi(\lambda)c(\lambda)|^2)$, where $B = K/M$, c is the Harish-Chandra c-function, and π the product of positive indivisible roots, see Theorem 6.4, [**Hel98**]. This spectral representation theorem, in the sense of the Lax-Phillips theory, can be seen as an extension of the Plancherel theorem for the Fourier transform on $\mathbb{X}$.

For $w \in W$, and $\Theta \subset \Delta$, the operators W_w^Θ defined in equation 14, [**STS76**], relating a solution of the system of equations (6.3-6.4) to its asymptotics, seem to be the analogues of wave operators when the time is multi-dimensional. We recall here, that in the time-dependent approach to scattering theory wave operators are the central objects, see [**Kat76**], [**Mül83**]. Given self-adjoint operators A_0 and A_1 on Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively, and a unitary operator $U : \mathcal{H}_0 \longrightarrow \mathcal{H}_1$, the wave operators $W_\pm(A_0, A_1; U)$ are said to exist if the strong limit

$$W_\pm(A_0, A_1; U) = s - \lim_{t \to \pm\infty} e^{itA_0} U e^{-itA_1} P_{ac}(A_1)$$

exists, where $P_{ac}(A_1)$ denotes the orthogonal projection onto the absolutely continuous subspace of A_1. If the wave operators exist and are complete, they define a unitary equivalence between the absolutely continuous parts of A_0 and A_1. Further, the scattering operator $S = W_+^* \circ W_-$ can be the defined. In this context, it looks reasonable from the analysis done in Section 6.3 that, to start with, we can consider an embedding of a rank-one symmetric space $\mathbb{X}$ in the Oshima compactification, look at the Laplace-Beltrami operator in the coordinates of the embedding, take A_0 to be its self-adjoint extension and A_1, as in Proposition 9, the differential operator giving the boundary contribution. Now, the crucial question, and one that we are unable to deal with at the moment, is that because the Riemannian measure on the closure $\overline{\widetilde{\mathbb{X}}_\Delta}$ of a copy of $\mathbb{X}$ becomes singular at the boundary we have to find some suitable spaces in which to compare the above operators. Once this can be dealt with, the idea then would be to prove the existence of the corresponding wave operator and thereafter show its completeness. One way to do this could be to see if one operator is a trace-class perturbation of the other in a suitable L^2 space, and then use the Birman-Kato theory, see [**Kat76**], Chapter X, Section 3. Perhaps, one can also prove the general limit results for the operators W_w^Θ that appear in [**STS76**]. See also Theorem 6.6, [**Hel98**].

Carrying out the above sketch of ideas, one expects, at the very least, to be able to recover some known results on the spectrum of the Laplacian on symmetric spaces in this fashion, while it is possible that we get a more detailed description coming from our understanding of the analysis at infinity. It will also be of interest to see the relation of this approach with the Lax-Phillips theory using the bijection of the algebra of G-invariant differential operators on the Oshima compactification $\widetilde{\mathbb{X}}$ with real analytic coefficients and the algebra of G-invariant differential operators on $\mathbb{X}$, see Section 2.2.

On the other hand, in the stationary approach to scattering theory the object of focus is the resolvent with emphasis on an analytic continuation of the resolvent and a precise description of its poles. There are two possible aspects here that would be of interest. One is the meromorphic continuation of the resolvent of strongly elliptic operators on symmetric spaces on the basis of the results in Section 4.2, and a subsequent understanding of the poles of this continuation. It should also be interesting to try to use the above-mentioned property of invariant differential operators on the Oshima compactification to obtain a description of the poles of the meromorphic continuation of the resolvent of the Laplacian in the higher-rank setting. The results for the rank-one case can be found in [**MW99**],[**HP09**].

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